Centralizers in the Full Transformation Semigroup

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Abstract

For an arbitrary set X (finite or infinite), denote by $T(X)$ the semigroup of full transformations on X. For $\alpha \in T(X)$, let $C(\alpha) = \{\beta \in T(X) : \alpha\beta = \beta\alpha\}$ be the centralizer of α in $T(X)$. The aim of this paper is to characterize the elements of $C(\alpha)$. The characterization is obtained by decomposing α as a join of connected partial transformations on X and analyzing the homomorphisms of the directed graphs representing the connected transformations. The paper closes with a number of open problems and suggestions of future investigations.

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1 Introduction

For a semigroup S and an element $a \in S$, the *centralizer* $C(a)$ of a in S is defined by $C(a) = \{x \in S : ax = xa\}$. It is clear that $C(a)$ is a subsemigroup of S. Let X be a set. We denote by $P(X)$ the semigroup of partial transformations on X, that is, the set of all functions $\alpha : A \to X$, where $A \subseteq X$, with function composition as multiplication. The semigroup $T(X)$ of full transformations on X is the subsemigroup of $P(X)$ consisting of all elements of $P(X)$ whose domain is X. Both $P(X)$ and $T(X)$ have the symmetric group $\mathrm{Sym}(X)$ of permutations of X as their group of units.

A significant amount of research has been devoted to studying centralizers in subsemigroups S of $P(X)$ in the case when X is finite. For example, for various S: the elements of centralizers have been described in [14], [28], [31], [34], [35], and [39]; Green's relations and regularity have been determined in [23], [24], and [25]; and some representation theorems have been obtained in [29], [30], and [37]. See also [1] for the semigroup generated by the idempotents of regular centralizers; and [2] for some centralizers related to maps preserving digraphs.

For an infinite X, the centralizers of idempotent transformations in $T(X)$ have been studied in [4], [5], and [38]. The cardinalities of $C(\alpha)$, for certain types of $\alpha \in T(X)$, have been established for a countable X in [20], [21], and [22]. The second author has investigated the centralizers of transformations in the semigroup $\Gamma(X)$ of injective elements of $T(X)$ [26], [27].

These investigations have been motivated by the fact that if S is a subsemigroup of $P(X)$ that contains the identity id_X, then for any $\alpha \in S$, the centralizer $C(\alpha)$ is a generalization of S in the sense that $S = C(\mathrm{id}_X)$. It is therefore of interest to find out which ideas, approaches, and techniques used to study S can be extended to the centralizers of its elements, and how these centralizers differ as semigroups from S.

Another reason to study centralizers is that semigroups are nothing but families of commuting maps. We say that two families of maps in $T(X)$, $(\overline{x} : X \to X)_{x \in X}$ and $(\underline{x} : X \to X)_{x \in X}$, are *linked* if for all $x, y \in X$ we have

$$(y)\underline{x} = (x)\overline{y}.$$

Linked families of maps induce naturally a groupoid (or *magma*) $S = (X, \cdot)$ with multiplication defined by

$$xy = (y)\underline{x} = (x)\overline{y}.$$

Now we have the following *folklore* result.

Theorem 1.1. *For a non-empty set X, let $(\overline{x} : X \to X)_{x \in X}$ and $(\underline{x} : X \to X)_{x \in X}$ be two linked families of maps. Then $(X, \cdot)$, the natural groupoid induced by these families of maps, is a semigroup if and only if*

$$(\forall x, y \in X) \;\; \underline{x} \in C(\overline{y}).$$

Conversely, every semigroup S induces a pair of linked maps $(\overline{s} : S^1 \to S^1)_{s \in S}$ and $(\underline{s} : S^1 \to S^1)_{s \in S}$ (the images of S under the left and right regular representations [16, page 7]) *such that every $\underline{s}$ commutes with every $\overline{t}$ ($s, t \in S$).*

Centralizers of transformations also attract some attention in various areas of mathematical research, for example, in the study of endomorphisms of unary algebras [19], [36]; in the study of commuting graphs [3], [10], [18]; and in the study of automorphism groups of semigroups [2], [6], and [7].

The first step in studying the centralizers in any transformation semigroup is to characterize their elements. A characterization theorem provides a foundation for all subsequent investigations. Such theorems have been provided for some special transformations, for example, for idempotent transformations $\varepsilon \in T(X)$ [5], [24], and for injective transformations [26]. The purpose of this paper is to provide a description of $C(\alpha)$ for a general $\alpha \in T(X)$, where X is an arbitrary set (finite or infinite). The paper will serve as a reference for future research on centralizers of transformations. The reason is that characterization theorems for transformations of special types can easily be obtained as consequences of either our general theorem or various lemmas (see Section 3) that lead to the theorem.

To obtain a characterization of the elements of $C(\alpha)$ for $\alpha \in T(X)$, we first, in Section 2, express any $\alpha \in P(X)$ as a join of connected elements of $P(X)$, which we will call the connected components of α. Then we assume that $\alpha \in T(X)$ and further decompose each connected component of α by expressing it as a join of certain basic injective elements of $P(X)$, which we will call cycles, rays, and chains. It turns out that, for $\alpha \in T(X)$, there are three types of connected components, depending on the types of basic partial transformations that occur in their decomposition. In Section 3, we represent a transformation $\alpha \in T(X)$ as a directed graph $D(\alpha)$. For given connected components γ and δ of α, we characterize digraph homomorphisms ϕ from $D(\gamma)$ to $D(\delta)$, where $D(\gamma)$ and $D(\delta)$ are the subgraphs of $D(\alpha)$ that represent γ and δ, respectively. In Section 4, we use the results of Sections 2 and 3 to characterize the elements of $C(\alpha)$ for an arbitrary $\alpha \in T(X)$. Finally, in Section 6, we outline a research program aimed at generalizing, for the centralizers of some particular transformations, many of the structure results proved for $T(X)$.

2 Decomposition of Full Transformations

In this section, we introduce the notion of the connected partial transformation on X and prove that every $\alpha \in P(X)$ can be decomposed uniquely as a join of connected $\gamma \in P(X)$ (called connected components of α). We then introduce the concept of the basic partial transformation and prove that each connected component of $\alpha \in T(X)$ can be further decomposed (although not uniquely) as a join of basic partial transformations. Depending on this decomposition, each connected component of α will be of one of three distinguished types.

Let $\gamma \in P(X)$. We denote the domain of γ by $\mathrm{dom}(\gamma)$ and the image of γ by $\mathrm{im}(\gamma)$. The union $\mathrm{dom}(\gamma) \cup \mathrm{im}(\gamma)$ will be called the *span* of γ and denoted $\mathrm{span}(\gamma)$.

We will write mappings on the right and compose from left to right; that is, for $f : A \to B$ and $g : B \to C$, we will write xf, rather than $f(x)$, and $x(fg)$, rather than $g(f(x))$.

Notation 2.1. From now on, we will fix a nonempty set X and an element $\diamond$ and assume that $\diamond \notin X$. For $\gamma \in P(X)$ and $x \in X$, we will write $x\gamma = \diamond$ if and only if $x \notin \mathrm{dom}(\gamma)$. We will also assume that $\diamond\gamma = \diamond$. With this notation, it will make sense to write $x\gamma = y\delta$ or $x\gamma \neq y\delta$ ($\gamma, \delta \in P(X)$, $x, y \in X$) even when $x \notin \mathrm{dom}(\gamma)$ or $y \notin \mathrm{dom}(\delta)$. We will denote by 0 the partial transformation on X that has empty set as its domain.

Definition 2.2. An element $\gamma \in P(X)$ is called *connected* if $\gamma \neq 0$ and for all $x, y \in \mathrm{span}(\gamma)$, $x\gamma^k = y\gamma^m \neq \diamond$ for some integers $k, m \geq 0$ (where $\gamma^0 = \mathrm{id}_X$).

Definition 2.3. Let $\gamma, \delta \in P(X)$. We say that γ and δ are *compatible* if $x\gamma = x\delta$ for all $x \in \mathrm{dom}(\gamma) \cap \mathrm{dom}(\delta)$; they are *disjoint* if $\mathrm{dom}(\gamma) \cap \mathrm{dom}(\delta) = \emptyset$; and they are *completely disjoint* if $\mathrm{span}(\gamma) \cap \mathrm{span}(\delta) = \emptyset$.

Definition 2.4. Let C be a set of pairwise compatible elements of $P(X)$. The *join* of the elements of C, denoted $\bigsqcup_{\gamma \in C} \gamma$, is an element of $P(X)$ defined by

$$x\Big(\bigsqcup_{\gamma \in C} \gamma\Big) = \begin{cases} x\gamma & \text{if } x \in \mathrm{dom}(\gamma) \text{ for some } \gamma \in C, \\ \diamond & \text{otherwise.} \end{cases}$$

If $C = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ is finite, we may write $\bigsqcup_{\gamma \in C} \gamma$ as $\gamma_1 \sqcup \gamma_2 \sqcup \cdots \sqcup \gamma_k$.

Let $\gamma \in P(X)$. We will write $x \xrightarrow{\gamma} y$ to mean that $x \in \mathrm{dom}(\gamma)$ and $x\gamma = y$. For $\delta \in P(X)$, we say that δ is *contained* in γ (or γ *contains* or *has* δ), and write $\delta \sqsubset \gamma$, if $\mathrm{dom}(\delta) \subseteq \mathrm{dom}(\gamma)$ and $x\delta = x\gamma$ for every $x \in \mathrm{dom}(\delta)$.

For a mapping $f : A \to B$ and $A_1 \subseteq A$, we denote by $f|_{A_1}$ the restriction of f to A_1, and by $A_1 f$ the image of A_1 under f.

Proposition 2.5. *Let $\alpha \in P(X)$ with $\alpha \neq 0$. Then there exists a unique set C of pairwise completely disjoint, connected elements of $P(X)$ such that $\alpha = \bigsqcup_{\gamma \in C} \gamma$.*

Proof. Define a relation ρ on $\operatorname{dom}(\alpha)$ by: $(x,y)\in\rho$ if $x\alpha^k = y\alpha^m \neq \diamond$ for some integers $k,m\geq 0$. It is clear that ρ is an equivalence relation on $\operatorname{dom}(\alpha)$. Let J be a complete set of representatives of the equivalence classes of ρ. For every $x\in J$, let $\gamma_x = \alpha|_{x\rho}$, where $x\rho$ is the ρ-equivalence class of x. By the definition of ρ, each such γ_x is connected, and γ_x and γ_y are completely disjoint for all $x,y\in J$ with $x\neq y$. Then the set $C=\{\gamma_x : x\in J\}$ consists of pairwise completely disjoint, connected transformations contained in α, and $\alpha=\bigsqcup_{\gamma\in C}\gamma$.

Suppose D is any set of pairwise completely disjoint, connected transformations contained in α such that $\alpha=\bigsqcup_{\delta\in D}\delta$. Let $\delta\in D$ and let $y\in\operatorname{dom}(\delta)$. Then $y\in x\rho$ for some $x\in J$. We want to prove that $\delta=\gamma_x$. Let $z\in\operatorname{dom}(\delta)$. Since δ is connected, $y\delta^k = z\delta^m\neq\diamond$ for some $k,m\geq 0$. But then, since δ is contained in α, we have $y\alpha^k = z\alpha^m\neq\diamond$. Hence $(y,z)\in\rho$, and so $z\in y\rho = x\rho = \operatorname{dom}(\gamma_x)$. We proved that $\operatorname{dom}(\delta)\subseteq\operatorname{dom}(\gamma_x)$.

Suppose to the contrary that $\operatorname{dom}(\gamma_x)$ is not included in $\operatorname{dom}(\delta)$, that is, that there is $w\in\operatorname{dom}(\gamma_x)$ such that $w\notin\operatorname{dom}(\delta)$. Since γ_x is connected, $w\gamma_x^p = x\gamma_x^q\neq\diamond$ for some $p,q\geq 0$. Let $y_i = y\gamma_x^i = y\alpha^i$ and $w_j = w\gamma_x^j = w\alpha^j$ for $i=0,1,\dots,p$ and $j=0,1,\dots,q$. Then $y_p = w_q$ and let $u=y_p=w_q$. With this notation, we have

$$y=y_0\xrightarrow{\alpha}y_1\xrightarrow{\alpha}\cdots\xrightarrow{\alpha}y_p=u \text{ and } w=w_0\xrightarrow{\alpha}w_1\xrightarrow{\alpha}\cdots\xrightarrow{\alpha}w_q=u.$$

Since $w\in\operatorname{dom}(\gamma_x)\subseteq\operatorname{dom}(\alpha)$, there is $\delta_1\in D$ such that $w\in\operatorname{dom}(\delta_1)$. We claim that $\{y_0,y_1,\dots,y_{p-1}\}\subseteq\operatorname{dom}(\delta)$. If not, then, since $y_0=y\in\operatorname{dom}(\delta)$, there would be some $i\in\{0,1,\dots,p-2\}$ such that $y_i\in\operatorname{dom}(\delta)$ and $y_{i+1}\notin\operatorname{dom}(\delta)$. But $y_{i+1}\in\operatorname{dom}(\alpha)$, and so $y_{i+1}\in\operatorname{dom}(\delta_2)$ for some $\delta_2\in D$. We would then have $\delta\neq\delta_2$ and $y_{i+1}\in\operatorname{span}(\delta)\cap\operatorname{span}(\delta_2)$, which is impossible since δ and δ_2 are completely disjoint. The claim has been proved. By the same argument applied to δ_1 and $\{w_0,w_1,\dots,w_{q-1}\}$, we obtain $\{w_0,w_1,\dots,w_{q-1}\}\subseteq\operatorname{dom}(\delta_1)$. Thus

$$y_{p-1}\delta = y_{p-1}\alpha = y_p = u = w_q = w_{q-1}\alpha = w_{q-1}\delta_1.$$

Thus we have $\delta\neq\delta_1$ with $u\in\operatorname{im}(\delta)\cap\operatorname{im}(\delta_1)$, which is a contradiction since δ and δ_1 are completely disjoint. We proved that $\operatorname{dom}(\gamma_x)\subseteq\operatorname{dom}(\delta)$, and so $\operatorname{dom}(\delta)=\operatorname{dom}(\gamma_x)$. Now for all $v\in\operatorname{dom}(\delta)=\operatorname{dom}(\gamma_x)$, we have $v\delta = v\alpha = v\gamma_x$, and so $\delta=\gamma_x\in C$. We proved that $D\subseteq C$.

For the reverse inclusion, let γ_x be an arbitrary element of C. Select $y\in\operatorname{dom}(\gamma_x)$. Then, there is $\delta\in D$ such that $y\in\operatorname{dom}(\delta)$. By the foregoing argument, we have $\delta=\gamma_x$, and so $\gamma_x\in D$. Hence $C\subseteq D$, and so $D=C$. We proved that the set C is unique, which completes the proof. □

Let $\alpha\in T(X)$. The elements of the set C from Proposition 2.5 will be called the *connected components* of α. This use of graph theory terminology is intentional since α can be represented by the directed graph $D(\alpha)=(X,\alpha)$, where (x,y) is an arc in $D(\alpha)$ if and only if $x\alpha=y$. (See Section 3 for details.) Then the connected components of α correspond to the connected components of the underlying undirected graph of $D(\alpha)$.

Regarding directed graphs, we will adopt the convention that the arrows will be deleted with the understanding that the arrow goes up along the edge, to the right if the edge is horizontal, and the arrows go counter-clockwise along a cycle. For example, the digraph in Figure 2.1 represents the transformation

$$\alpha=\begin{pmatrix}1&2&3&4&5&6&7&8&9&\dots\\2&3&1&1&1&5&8&9&10&\dots\end{pmatrix}\in T(X),$$

where $X=\{1,2,3,\dots\}$.

The connected components of $\alpha\in T(X)$ further decompose into basic elements in $P(X)$.

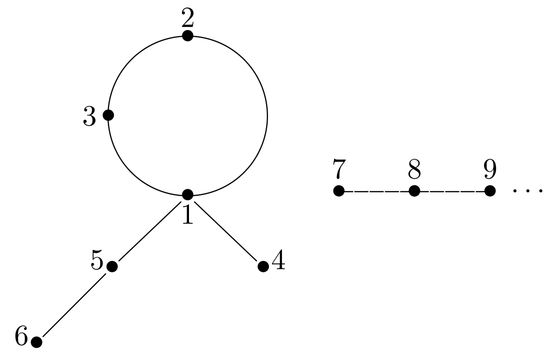

Figure 2.1: The digraph of a transformation.

Definition 2.6. Let $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ be pairwise distinct elements of X. The following elements of $P(X)$ will be called *basic* partial transformations on X (see Figure 2.2).

- A *cycle* of length k ($k \geq 1$), written $(x_0\, x_1 \ldots x_{k-1})$, is an element $\sigma \in P(X)$ with $\mathrm{dom}(\sigma) = \{x_0, x_1, \ldots, x_{k-1}\}$, $x_i\sigma = x_{i+1}$ for all $0 \leq i < k-1$, and $x_{k-1}\sigma = x_0$.
- A *right ray*, written $[x_0\, x_1\, x_2 \ldots\rangle$, is an element $\eta \in P(X)$ with $\mathrm{dom}(\eta) = \{x_0, x_1, x_2, \ldots\}$ and $x_i\eta = x_{i+1}$ for all $i \geq 0$.
- A *double ray*, written $\langle \ldots x_{-2}\, x_{-1}\, x_0\, x_1\, x_2 \ldots\rangle$, is an element $\omega \in P(X)$ with $\mathrm{dom}(\omega) = \{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$ and $x_i\omega = x_{i+1}$ for all i.
- A *left ray*, written $\langle \ldots x_2\, x_1\, x_0]$, is an element $\lambda \in P(X)$ with $\mathrm{dom}(\lambda) = \{x_1, x_2, x_3, \ldots\}$ and $x_i\lambda = x_{i-1}$ for all $i > 0$.
- A *chain* of length k ($k \geq 1$), written $[x_0\, x_1 \ldots x_k]$, is an element $\tau \in P(X)$ with $\mathrm{dom}(\tau) = \{x_0, x_1, \ldots, x_{k-1}\}$ and $x_i\tau = x_{i+1}$ for all $0 \leq i \leq k-1$.

By a *ray* we will mean a double, right, or left ray.

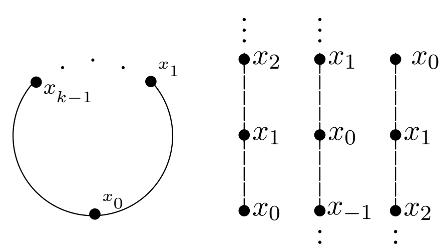

Figure 2.2: Basic partial transformations.

We note the following:

(i) All basic partial transformations are connected and injective.

(ii) The span of a basic partial transformation is exhibited by the notation. For example, the span of the right ray $[1\,2\,3 \ldots\rangle$ is $\{1, 2, 3, \ldots\}$.

(iii) The left bracket in "$\varepsilon = [x \ldots$" indicates that $x \notin \mathrm{im}(\varepsilon)$; while the right bracket in "$\varepsilon = \ldots x]$" indicates that $x \notin \mathrm{dom}(\varepsilon)$. For example, for the chain $\tau = [1\,2\,3\,4]$, $\mathrm{dom}(\tau) = \{1, 2, 3\}$ and $\mathrm{im}(\tau) = \{2, 3, 4\}$.

(iv) A cycle $(x_0\, x_1 \ldots x_{k-1})$ differs from the corresponding cycle in the symmetric group of permutations on X in that the former is undefined for every $x \in X - \{x_0, x_1, \ldots, x_{k-1}\}$ while the latter is fixed for every such x.

We will now analyze which combinations of basic transformations can occur in a connected component of $\alpha \in T(X)$.

Definition 2.7. A right ray $\eta = [x_0\, x_1\, x_2\, \ldots\rangle$ contained in $\alpha \in T(X)$ is called a *maximal right ray* in α if $x_0 \notin \mathrm{im}(\alpha)$.

For example, consider $\alpha = [4\,5\,6\,7\ldots\rangle \sqcup [1\,2\,3\,6] \in T(\mathbb{N})$, where $\mathbb{N}$ is the set of positive integers. Then α contains infinitely many right rays, for example $[3\,6\,7\,8\ldots\rangle$ and $[7\,8\,9\,10\ldots\rangle$, but only two of them, namely $[4\,5\,6\,7\ldots\rangle$ and $[1\,2\,3\,6\,7\,8\ldots\rangle$ are maximal. Note also that α is connected.

Lemma 2.8. *Let γ be a connected component of $\alpha \in T(X)$. Then:*

(1) *If γ has a cycle $(x_0\, x_1 \ldots x_{k-1})$, then for every $x \in \mathrm{dom}(\gamma)$, $x\gamma^m = x_0$ for some $m \geq 0$.*

(2) *If γ has a right ray $[x_0\, x_1\, x_2 \ldots\rangle$ or a double ray $\langle \ldots x_{-1}\, x_0\, x_1 \ldots\rangle$, then for every $x \in \mathrm{dom}(\gamma)$, $x\gamma^m = x_i$ for some $m, i \geq 0$.*

Proof. Suppose γ has a cycle $(x_0\, x_1 \ldots x_{k-1})$ and let $x \in \mathrm{dom}(\gamma)$. Since γ is connected, $x\gamma^p = x_0\gamma^q$ for some $p, q \geq 0$. Since x_0 lies on the cycle $(x_0\, x_1 \ldots x_{k-1})$, we may assume that $0 \leq q \leq k-1$. Thus for $m = p + k - q$, we have

$$x\gamma^m = x\gamma^{p+k-q} = (x\gamma^p)\gamma^{k-q} = (x_0\gamma^q)\gamma^{k-q} = x^q\gamma^{k-q} = x_0.$$

Suppose γ has a right ray $[x_0\, x_1\, x_2 \ldots\rangle$ and let $x \in \mathrm{dom}(\gamma)$. Since γ is connected, $x\gamma^m = x_0\gamma^i = x_i$ for some $m, i \geq 0$. A proof in the case of a double ray is the same. □

Lemma 2.9. *Let γ be a connected component of $\alpha \in T(X)$ and let $x \in \mathrm{im}(\gamma)$ such that x does not lie on a cycle in γ. Then γ contains a left ray $\langle \ldots y_3\, y_2\, y_1\, x]$ or a chain $[y_k\, y_{k-1} \ldots y_1\, x]$ ($k \geq 1$) with $y_k \notin \mathrm{im}(\gamma)$.*

Proof. Since $x \in \mathrm{im}(\gamma)$, there is $y_1 \in X$ such that $y_1\gamma = x_0$. If $y_1 \in \mathrm{im}(\gamma)$, then $y_2\gamma = y_1$ for some $y_2 \in X$. Continuing this way, we either arrive at $y_k \in X$ such that $y_k\gamma = y_{k-1}$ and $y_k \notin \mathrm{im}(\gamma)$ or the process of constructing $y_1, y_2, y_3, \ldots$ will go on forever. Note that there will be no repetition in the sequence $\langle y_i \rangle$ (finite or infinite) since x does not lie on a cycle in γ. Hence, either γ contains a left ray $\langle \ldots y_3\, y_2\, y_1\, x]$ or a desired chain. □

Proposition 2.10. *Let γ be a connected component of $\alpha \in T(X)$. Then:*

(1) *If γ has a cycle, then the cycle is unique and γ does not have any double or right rays.*

(2) *If γ does not have a cycle, then γ is a join of the double rays and maximal right rays contained in γ.*

Proof. Suppose that γ has a cycle, say $\sigma = (x_0\, x_1 \ldots x_{k-1})$. Let $\theta = (y_0\, y_1 \ldots y_{m-1})$ be any cycle in γ. We want to prove that $\sigma = \theta$. We may assume that $k \leq m$. By Lemma 2.8, $y_0\gamma^p = x_0$ for some $p \geq 0$. On the other hand, $y_0\gamma^p = y_j$ for some $j \in \{0, \ldots, m-1\}$, and so $x_0 = y_j$. Since we can rewrite θ as $(y_j\, y_{j+1} \ldots y_{j-1})$, we may assume that $y_j = y_0$, so $x_0 = y_0$. But then $x_i = x_0\gamma^i = y_0\gamma^i = y_i$ for every $i \in \{0, \ldots, k-1\}$ and $y_{k-1}\gamma = x_{k-1}\gamma = x_0 = y_0$. It follows that $k = m$ and $\sigma = \theta$. We proved that a cycle in γ is unique.

Suppose that γ with a cycle $(x_0\, x_1 \ldots x_{k-1})$ also has a double ray, say $\langle \ldots y_{-1}\, y_0\, y_1 \ldots \rangle$. By Lemma 2.8, $y_0\gamma^m = x_0$ for some $m \geq 0$. But then $y_0\gamma^{m+k} = (y_0\gamma^m)\gamma^k = x_0\gamma^k = x_0 = y_0$, which is a contradiction since $y_0\gamma^{m+k} = y_{m+k} \neq y_0$ (since $m \geq 0$ and $k \geq 1$). Thus γ does not have a double ray. Similarly, γ cannot have a right ray. We have proved (1).

To prove (2), suppose γ does not have a cycle. Let R be the set of all double and maximal right rays contained in γ. Then clearly $\bigsqcup_{\varepsilon \in R} \varepsilon \sqsubset \gamma$. (Note that if δ_1, δ_2 are contained in γ, then δ_1 and δ_2 are compatible since for every $x \in \mathrm{dom}(\delta_1) \cap \mathrm{dom}(\delta_2)$, $x\delta_1 = x\gamma = x\delta_2$.) Select any $x \in \mathrm{dom}(\gamma)$. Then $x_0 = x$, $x_1 = x\gamma$, $x_2 = x\gamma^2, \ldots$ are pairwise distinct since otherwise γ would have a cycle. Consider the right ray $\eta = [x_0\, x_1\, x_2 \ldots \rangle$ in γ. If η is not maximal (that is, $x_0 \in \mathrm{im}(\gamma)$), then γ contains a left ray $\langle \ldots y_3\, y_2\, y_1\, x_0]$ or a chain $[y_k\, y_{k-1} \ldots y_1\, x_0]$ $(k \geq 1)$ with $y_k \notin \mathrm{im}(\gamma)$ (by Lemma 2.9). In the former case, γ has a double ray $\langle \ldots y_2\, y_1\, x_0\, x_1\, x_2 \ldots \rangle$; and in the latter case γ has a maximal right ray $[y_k \ldots y_1\, x_0\, x_1\, x_2 \ldots \rangle$. Thus $x = x_0 \in \mathrm{dom}(\varepsilon)$ for some $\varepsilon \in R$, and it follows that $\gamma = \bigsqcup_{\varepsilon \in R} \varepsilon$. □

Definition 2.11. Let γ be a connected component of $\alpha \in T(X)$. By Proposition 2.10, exactly one of the following three conditions holds (see Figures 2.3, 2.4, and 2.5):

(i) γ contains a unique cycle;

(ii) γ contains a double ray;

(iii) γ does not contain a double ray and γ is the join of its maximal right rays.

If γ satisfies (iii), we will say that γ is of type *rro* ("right rays only").

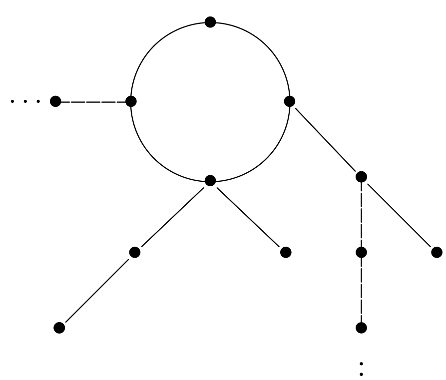

Figure 2.3: A connected component with a cycle.

Let $\varepsilon \in P(X)$ be a basic partial transformation. If $x \in \mathrm{span}(\varepsilon)$, we will say that x *lies* on ε.

Definition 2.12. Let γ be a connected component of $\alpha \in T(X)$.

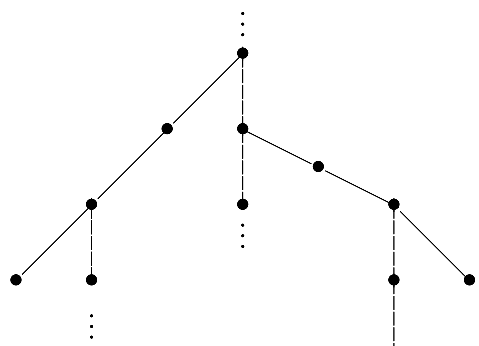

Figure 2.4: A connected component with a double ray.

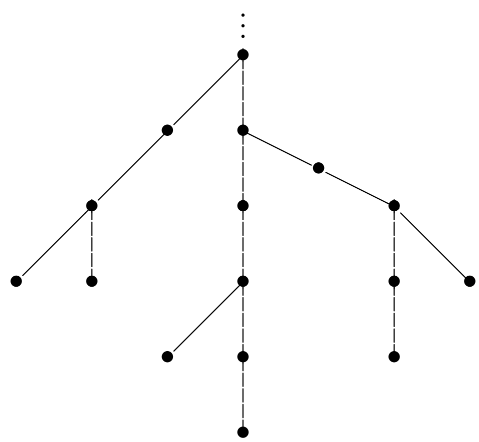

Figure 2.5: A connected component of type rro.

- A chain $[y_0\, y_1\, \dots y_m]$ in γ is called a *finite branch* of a cycle σ [double ray ω, right ray η] in γ if $y_0 \notin \mathrm{im}(\gamma)$, y_m lies on σ [ω, η], and y_{m-1} does not lie on σ [ω, η].
- A left ray $\langle \dots y_2\, y_1\, y_0]$ in γ is called an *infinite branch* of a cycle σ [double ray ω] in γ if y_0 lies on σ [ω] and y_1 does not lie on σ [ω].

By a *branch* we will mean a finite or infinite branch. We will use the notation $(\dots y_2\, y_1\, y_0]$ for a branch that is finite or infinite (but we do not know which).

If $\sigma = (x_0\, x_1 \dots x_{k-1})$ is a cycle in γ and ε is a branch of σ with the terminal point x_i, we will say that ε is a branch of σ at x_i. (If ε is a left ray or a chain, then the terminal point of ε is the element $x \in X$ such that $x \in \mathrm{im}(\varepsilon) - \mathrm{dom}(\varepsilon)$.) We will use a similar language for branches of a double ray ω and a right ray η. Note that all branches of a right ray η in γ are finite by definition.

For example, the cycle in Figure 2.3 has two infinite branches (at the same point) and four finite branches (at three different points). The transformation whose digraph is presented in Figure 2.4 has two double rays. The vertical double ray has one infinite branch and three finite branches (at two different points). The transformation from Figure 2.5 is connected of type rro with seven maximal right rays. The vertical maximal right ray has six (finite) branches (at three different points).

Proposition 2.13. *Let γ be a connected component of $\alpha \in T(X)$. Then:*

(1) *If γ has a (unique) cycle σ, then γ is the join of σ and its branches.*

(2) *If γ has a double ray ω, then γ is the join of ω and its branches.*

(3) *If γ is of type rro with a maximal right ray η, then γ is the join of η and its (finite) branches.*

Proof. Suppose γ has a cycle σ. Let $x \in \mathrm{dom}(\gamma)$ be such that x does not lie on σ. Let t be the smallest positive integer such that $x\gamma^t$ lies on σ (such a t exists by Lemma 2.8). If $x \notin \mathrm{im}(\gamma)$, then $[x\, x\gamma \dots x\gamma^t]$ is a finite branch of σ. If $x \in \mathrm{im}(\gamma)$, then γ contains a left ray $\langle \dots y_3\, y_2\, y_1\, x]$ or a chain $[y_k\, y_{k-1} \dots y_1\, x]$ ($k \geq 1$) with $y_k \notin \mathrm{im}(\gamma)$ (by Lemma 2.9). In the former case, $\langle \dots y_2\, y_1\, x\, x\gamma \dots x\gamma^t]$ is an infinite branch of σ; and in the latter case $[y_k \dots y_1\, x\, x\gamma \dots x\gamma^t]$ is a finite branch of σ. Thus every element of $\mathrm{dom}(\gamma)$ either lies on σ or on one of the branches of σ, which proves (1). The proofs of (2) and (3) are similar. We note that the foregoing argument applied to η in (3) will not produce and infinite branch since γ of type *rro* does not contain a double ray. □

3 Digraph Homomorphisms

In this section, we represent $\alpha \in T(X)$ and its connected components as directed graphs. For given connected components γ and δ of α, we investigate digraph homomorphisms from $D(\gamma)$ to $D(\delta)$. This approach is justified by Proposition 3.1 below.

A *directed graph* (or a *digraph*) is a pair $D = (X, \rho)$ where X is a non-empty set (not necessarily finite) and ρ is a binary relation on X. Any element $x \in X$ is called a *vertex* of D, and any pair $(x, y) \in \rho$ is called an *arc* of D. Let $D_1 = (X_1, \rho_1)$ and $D_2 = (X_2, \rho_2)$ be digraphs. A mapping $\phi : X_1 \to X_2$ is called a *homomorphism* from D_1 to D_2 if it preserves edges, that is, for all $x, y \in X_1$, if $(x, y) \in \rho_1$, then $(x\phi, y\phi) \in \rho_2$ [13]. We say that D_1 is *homomorphic* to D_2 if there is a homomorphism from D_1 to D_2.

Let $\alpha \in T(X)$. Then α can be represented by the directed graph $D(\alpha) = (X, \alpha)$, where α is viewed as a binary relation on X. In other words, for all $x, y \in X$, (x, y) is an arc in $D(\alpha)$ if and only if $x\alpha = y$. Let γ be a connected component of α. By $D(\gamma)$ we will mean the directed subgraph of $D(\alpha)$ induced by $\mathrm{dom}(\gamma)$. That is, $\mathrm{dom}(\gamma)$ is the set of vertices of $D(\gamma)$ and for all $x, y \in \mathrm{dom}(\gamma)$, (x, y) is an arc in $D(\gamma)$ if and only if (x, y) is an arc in $D(\alpha)$. (The latter is equivalent to $x\gamma = y$.) If (x, y) is an arc in $D(\gamma)$, we will write $x \xrightarrow{\gamma} y$ (or $x \to y$ if no ambiguity arises). The same convention will apply to the digraph $D(\alpha)$.

The following proposition provides a link between centralizers of elements of $T(X)$ and digraph homomorphisms.

Proposition 3.1. *Let $\alpha, \beta \in T(X)$. Then $\beta \in C(\alpha)$ if and only if β is a homomorphism from $D(\alpha)$ to $D(\alpha)$.*

Proof. Suppose $\beta \in C(\alpha)$. Let $x \to y$ be an arc in $D(\alpha)$, that is, $y = x\alpha$. Then, since $\alpha\beta = \beta\alpha$, we have $(x\beta)\alpha = x(\beta\alpha) = x(\alpha\beta) = (x\alpha)\beta = y\beta$, and so $x\beta \to y\beta$. Hence β is a homomorphism from $D(\alpha)$ to $D(\alpha)$.

Conversely, suppose that β is a homomorphism from $D(\alpha)$ to $D(\alpha)$. Let $x \in X$. Then $x\beta \to (x\alpha)\beta$ since $x \to x\alpha$ and β preserves edges. But $x\beta \to (x\alpha)\beta$ means that $(x\beta)\alpha = (x\alpha)\beta$, which implies $x(\alpha\beta) = x(\beta\alpha)$. Hence $\alpha\beta = \beta\alpha$, and so $\beta \in C(\alpha)$. □

For the remainder of this section, we will work on characterizing digraph homomorphisms ϕ from $D(\gamma)$ to $D(\delta)$, where γ and δ are connected components of $\alpha \in T(X)$. It will be convenient to introduce the following definitions. We agree that if $\theta = (y_0 \dots y_{m-1})$ is a cycle and i is an integer, then y_i means y_r where $r \equiv i \pmod{m}$ and $0 \le r < m$.

Definition 3.2. Let $\phi \in P(X)$.

- Let $\sigma = (x_0 \dots x_{k-1})$ and $\theta = (y_0 \dots y_{m-1})$ be cycles such $\mathrm{dom}(\sigma) \subseteq \mathrm{dom}(\phi)$. We say that ϕ *wraps* σ around θ at y_t if $x_i\phi = y_{t+i}$ for all $0 \le i \le k-1$. (See Figure 3.1.)
- Let $\tau = [z_0 \dots z_p]$ be a chain and $\theta = (y_0 \dots y_{m-1})$ be a cycle such that $\mathrm{span}(\tau) \subseteq \mathrm{dom}(\phi)$. We say that ϕ *wraps* τ around θ at y_t if $z_i\phi = y_{t+i}$ for all $0 \le i \le p$. (See Figure 3.2.)
- Let $\lambda = \langle \dots z_2\, z_1\, z_0]$ be a left ray and $\theta = (y_0 \dots y_{m-1})$ be a cycle such that $\mathrm{span}(\lambda) \subseteq \mathrm{dom}(\phi)$. We say that ϕ *counter-wraps* λ around θ at y_t if $z_i\phi = y_{t-i}$ for all $i \ge 0$. (See Figure 3.3.)
- Let $\omega = \langle \dots x_{-1}\, x_0\, x_1 \dots \rangle$ be a double ray and $\theta = (y_0 \dots y_{m-1})$ be a cycle such that $\mathrm{dom}(\omega) \subseteq \mathrm{dom}(\phi)$. We say that ϕ *double-wraps* ω around θ at y_t if $x_i\phi = y_{t+i}$ for all i. (See Figure 3.4.)
- Let $\eta = [x_0\, x_1\, x_3 \dots \rangle$ be a right ray and $\theta = (y_0 \dots y_{m-1})$ be a cycle such that $\mathrm{dom}(\eta) \subseteq \mathrm{dom}(\phi)$. We say that ϕ *wraps* η around θ at y_t if $x_i\phi = y_{t+i}$ for all $i \ge 0$. (See Figure 3.5.)

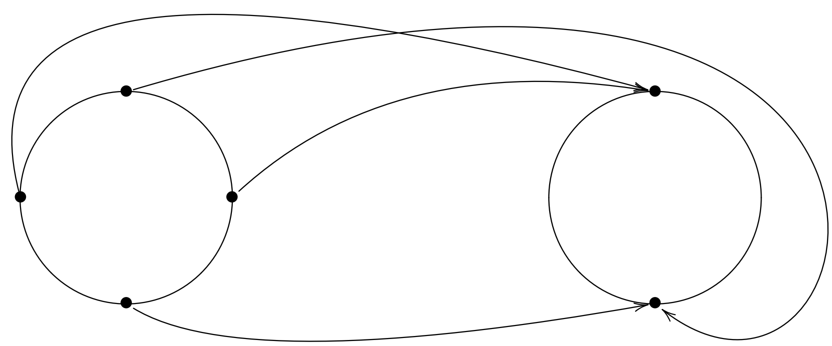

Figure 3.1: A cycle wrapped around a cycle.

Definition 3.3. Let $\phi \in P(X)$.

- Let $\tau = [z_0 \dots z_p]$ and $\kappa = [w_0 \dots w_p]$ be chains of the same length such that $\mathrm{span}(\tau) \subseteq \mathrm{dom}(\phi)$. We say that ϕ *maps* τ *onto* κ if $z_i\phi = w_i$ for all $0 \le i \le p$.
- Let $\lambda = \langle \dots z_2\, z_1\, z_0]$ and $\mu = \langle \dots w_2\, w_1\, w_0]$ be left rays such that $\mathrm{span}(\lambda) \subseteq \mathrm{dom}(\phi)$. We say that ϕ *maps* λ *onto* μ if $z_i\phi = w_i$ for all $i \ge 0$.

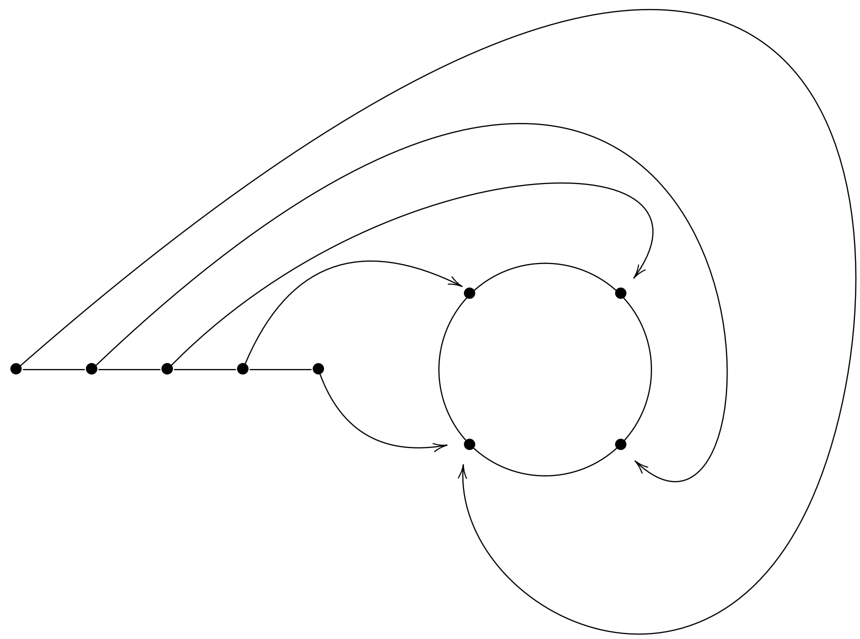

Figure 3.2: A chain wrapped around a cycle.

- Let $\omega = \langle \ldots\, x_{-1}\, x_0\, x_1 \ldots \rangle$ and $\pi = \langle \ldots\, y_{-1}\, y_0\, y_1 \ldots \rangle$ be double rays such that $\mathrm{dom}(\omega) \subseteq \mathrm{dom}(\phi)$. We say that ϕ *maps* ω *onto* π at y_t if $x_i\phi = y_{t+i}$ for all i.

- Let $\eta = [x_0\, x_1\, x_3 \ldots \rangle$ and $\mu = [y_0\, y_1\, y_3 \ldots \rangle$ be right rays such that $\mathrm{dom}(\eta) \subseteq \mathrm{dom}(\phi)$. We say that ϕ *maps* η *onto* μ if $x_i\phi = y_i$ for all $i \geq 0$.

We begin with the case when γ has a cycle. If $D(\gamma)$ is homomorphic to $D(\delta)$, then δ must also have a cycle.

Lemma 3.4. *Let γ and δ be connected components of $\alpha \in T(X)$ such that γ has a cycle $\sigma = (x_0 \ldots\, x_{k-1})$. Let $\phi : \mathrm{dom}(\gamma) \to \mathrm{dom}(\delta)$. Then ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$ if and only if δ has a cycle $\theta = (y_0 \ldots\, y_{m-1})$ and the following conditions are satisfied:*

(1) *m divides k;*

(2) *ϕ wraps σ around θ at some y_t;*

(3) *If $\tau = [z_0 \ldots\, z_p = x_i]$ is a finite branch of σ, then exactly one of the following holds:*

  (a) *ϕ wraps τ around θ at y_{t+i-p}; or*

  (b) *There is a branch $\mu = (\ldots\, w_q \ldots\, w_0 = y_j]$ of θ with $1 \leq q \leq p$ and $j + p - q \equiv t + i \pmod m$ such that ϕ maps $[z_0 \ldots\, z_q]$ onto $[w_q \ldots\, w_0 = y_j]$ and if $q < p$ then ϕ wraps $[z_q \ldots\, z_p = x_i]$ around θ at y_j;*

(4) *If $\lambda = \langle \ldots\, z_2\, z_1\, z_0 = x_i]$ is an infinite branch of σ, then exactly one of the following holds:*

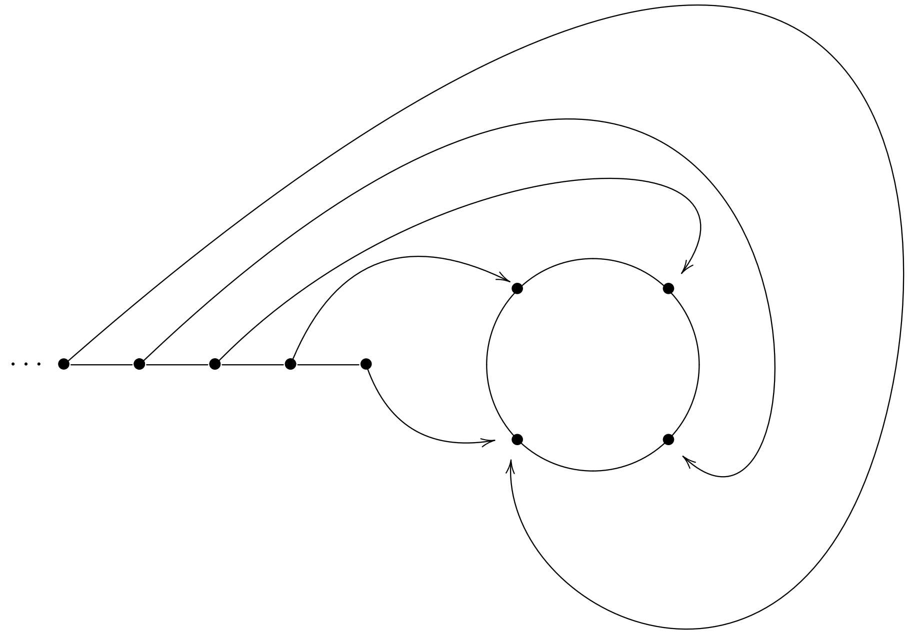

Figure 3.3: A left ray counter-wrapped around a cycle.

(a) *ϕ counter-wraps λ around θ at y_{t+i}; or*

(b) *There is an infinite branch $\mu = \langle \dots w_2\, w_1\, w_0 = y_j]$ of θ and there is $s \geq 0$ with $j + s \equiv t + i \pmod{m}$ such that ϕ maps $\langle \dots z_{s+2}\, z_{s+1}\, z_s]$ onto μ and if $s > 0$ then ϕ wraps $[z_s \dots z_1\, z_0 = x_i]$ around θ at y_j.*

Proof. Suppose ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$. Since σ is a cycle, we have

$$x_0 \to x_1 \to \cdots \to x_{k-1} \to x_0,$$

in $D(\gamma)$, and so, since ϕ preserves edges,

$$x_0\phi \to x_1\phi \to \cdots \to x_{k-1}\phi \to x_0\phi \tag{3.1}$$

in $D(\delta)$. If $x_0\phi, x_1\phi, \dots, x_{k-1}\phi$ are pairwise distinct, then $\theta = (x_0\phi\, x_1\phi \dots x_{k-1}\phi)$ is a cycle in δ of length k and ϕ wraps σ around θ at $x_0\phi$. Otherwise, let s be the smallest element of $\{0, \dots, k-1\}$ such that $x_s\phi = x_j\phi$ for some $j \in \{s+1, \dots, k-1\}$, and let m be the smallest positive integer such that $x_s\phi = x_{s+m}\phi$. Then $\theta = (x_s\phi\, x_{s+1}\phi \dots x_{s+m-1}\phi)$ is a cycle in δ. Moreover, since (3.1) can be rewritten as

$$x_s\phi \to x_{s+1}\phi \to \cdots \to x_{s+k-1}\phi \to x_s\phi,$$

it follows that m divides k and ϕ wraps σ around θ at some $x_{s+t}\phi$, where $0 \leq t < m$. We have proved that δ has a cycle $\theta = (y_0 \dots y_{m-1})$ and that (1) and (2) are satisfied.

To prove (3), let $\tau = [z_0 \dots z_p = x_i]$ be a finite branch of σ. Since $x_0\phi = y_t$ and ϕ wraps σ around θ, we have $x_i\phi = y_{t+i}$. Suppose $z_0\phi = y_j$ lies on θ. Then

$$z_0\phi = y_j \to z_1\phi = y_{j+1} \to \cdots \to z_p\phi = y_{j+p}.$$

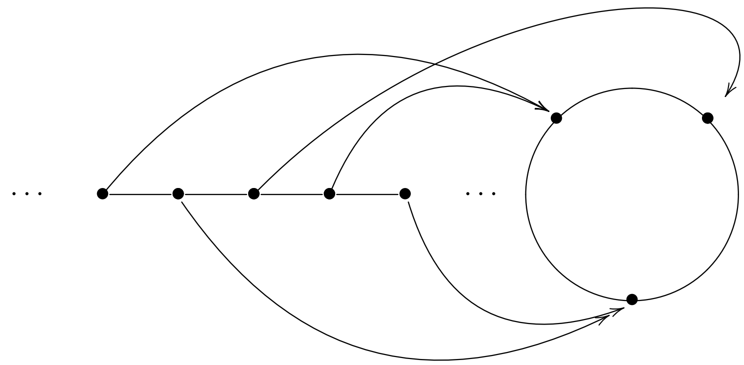

Figure 3.4: A double ray double-wrapped around a cycle.

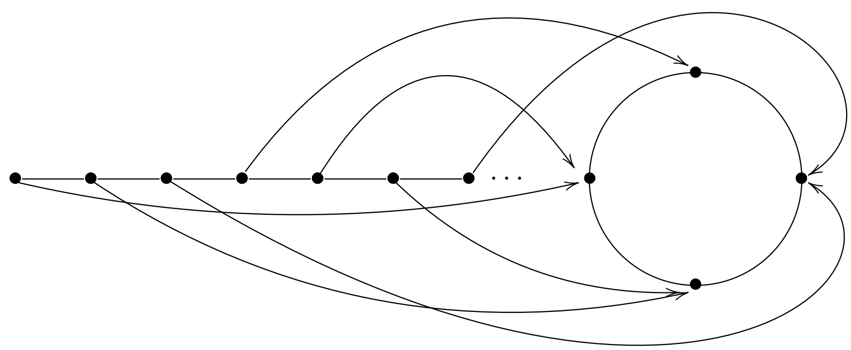

Figure 3.5: A right ray wrapped around a cycle.

Since $y_{j+p} = z_p\phi = x_i\phi = y_{t+i}$, we have $j + p \equiv t + i \pmod m$. Thus $y_j = y_{t+i-p}$, and so ϕ wraps τ around θ at y_{t+i-p}.

Suppose $z_0\phi$ does not lie on θ. Then, by Proposition 2.13, $z_0\phi$ lies on some branch (finite or infinite) $\mu = (\dots\, w_2\, w_1\, w_0 = y_j]$ of θ. Let $z_0\phi = w_q$. Then $q \geq 1$ (since $z_0\phi \notin \mathrm{dom}(\theta)$) and

$$z_0\phi = w_q \to z_1\phi = w_{q-1} \to \cdots \to z_q\phi = w_0 = y_j \to z_{q+1}\phi = y_{j+1} \to \cdots \to z_p\phi = y_{j+p-q}.$$

Thus $q \leq p$, ϕ maps $[z_0 \dots z_q]$ onto $[w_q \dots w_0 = y_j]$, and if $q < p$ then ϕ wraps $[z_q \dots z_p = x_i]$ around θ at y_j. Further, $y_{j+p-q} = z_p\phi = x_i\phi = y_{t+i}$, and so $j + p - q \equiv t + i \pmod m$.

Hence (3a) or (3b) holds. Thus exactly one of them holds since (3a) and (3b) are mutually exclusive. We have proved (3). The proof of (4) is similar.

Conversely, suppose (1)–(4) are satisfied. Let $x \to z$ be an edge in $D(\gamma)$. If $x \in \mathrm{dom}(\sigma)$, then $x_i = x \to z = x_{i+1}$ for some i, and so $x\phi = x_i\phi = y_{t+i} \to y_{t+i+1} = x_{i+1}\phi = z\phi$ by (2). If $x \notin \mathrm{dom}(\sigma)$, then $x \to z$ is an edge of some branch of σ (by Proposition 2.13), and so $x\phi \to z\phi$ in $D(\delta)$ by (3) and (4). $\square$

Figure 3.6 illustrates 3(b) of Lemma 3.4. Extending the finite branches in Figure 3.6 to infinite branches of the cycles, we obtain an illustration of 4(b) of Lemma 3.4.

Suppose γ has a double ray and $D(\gamma)$ is homomorphic to $D(\delta)$. Then the situation is more complicated since δ may have either a cycle or a double ray.

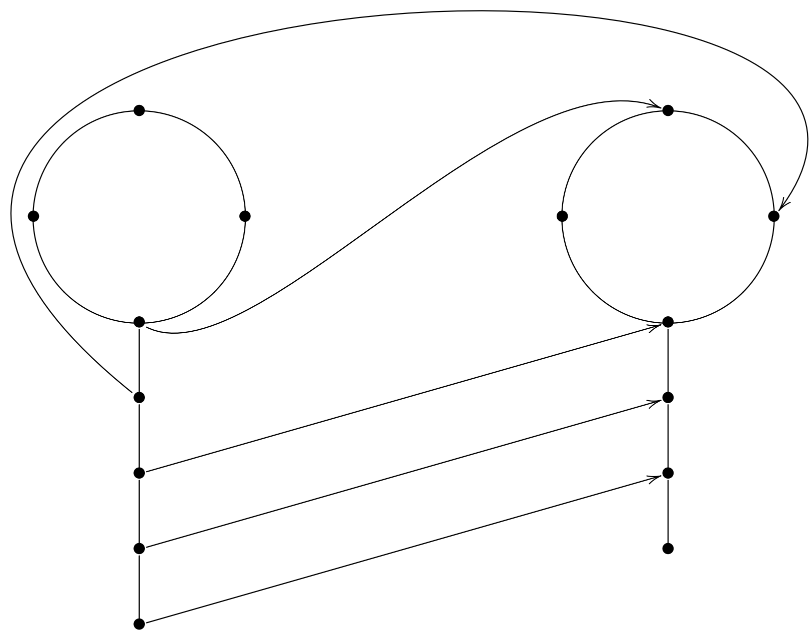

Figure 3.6: A homomorphism acting on a finite branch of a cycle in Lemma 3.4.

Lemma 3.5. *Let γ and δ be connected components of $\alpha \in T(X)$ such that γ has a double ray $\omega = \langle \dots\, x_{-1}\, x_0\, x_1 \dots \rangle$. Suppose ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$. Then exactly one of the following holds:*

(a) *δ has a cycle $\theta = (y_0 \dots\, y_{m-1})$ and either ϕ double wraps ω around θ at some y_t or there is u such that ϕ wraps $[x_u\, x_{u+1}\, x_{u+2} \dots \rangle$ around θ at some y_t and $x_{u-1} \notin \mathrm{dom}(\theta)$; or*

(b) *δ has a double ray π such that ϕ maps ω onto π.*

Proof. Since ϕ preserves edges, we have

$$\cdots \to x_{-1}\phi \to x_0\phi \to x_1\phi \to \cdots . \tag{3.2}$$

Suppose δ contains a cycle $\theta = (y_0 \dots\, y_{m-1})$. Since $x_0\phi, y_0 \in \mathrm{dom}(\delta)$ and δ is a connected component of α, there are integers $p, q \geq 0$ such that $(x_0\phi)\alpha^p = y_0\alpha^q$. By (3.2), $(x_0\phi)\alpha^p = x_p\phi$, and so $x_p\phi$ lies on θ. Suppose every $x_i\phi$ lies on θ and let $x_0\phi = y_t$. Then ϕ double-wraps ω around θ at y_t by (3.2). Suppose not every $x_i\phi$ lies on θ. Since $x_p\phi$ lies on θ, there is u such that $x_u\phi$ lies on θ but $x_{u-1}\phi$ does not. Let $x_u\phi = y_t$. Then, by (3.2), ϕ wraps $[x_u\, x_{u+1}\, x_{u+2} \dots \rangle$ around θ at y_t and $x_{u-1} \notin \mathrm{dom}(\theta)$.

Suppose δ does not contain a cycle. Then the vertices in (3.2) must be pairwise distinct. Thus $\pi = \langle \dots\, x_{-1}\phi\, x_0\phi\, x_1\phi \dots \rangle$ is a double ray in δ and ϕ maps ω onto π.

We have proved that (a) or (b) holds. By Proposition 2.10, (a) and (b) are mutually exclusive, so exactly one of them holds. □

We now analyze what happens in each of the cases exhibited by Lemma 3.5.

Lemma 3.6. *Let γ and δ be connected components of $\alpha \in T(X)$ such that γ has a double ray $\omega = \langle \dots x_{-1}\, x_0\, x_1 \dots \rangle$ and δ has a cycle $\theta = (y_0 \dots y_{m-1})$. Let $\phi : \mathrm{dom}(\gamma) \to \mathrm{dom}(\delta)$ be such that ϕ double wraps ω around θ at some y_t. Then ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$ if and only if the following conditions are satisfied:*

(1) *If $\tau = [z_0 \dots z_p = x_i]$ is a finite branch of ω, then exactly one of the following holds:*

   (a) *ϕ wraps τ around θ at y_{t+i-p}; or*

   (b) *There is a branch $\mu = (\dots w_q \dots w_0 = y_j]$ of θ with $1 \le q \le p$ and $j + p - q \equiv t + i \pmod m$ such that ϕ maps $[z_0 \dots z_q]$ onto $[w_q \dots w_0 = y_j]$ and if $q < p$ then ϕ wraps $[z_q \dots z_p = x_i]$ around θ at y_j;*

(2) *If $\lambda = \langle \dots z_2\, z_1\, z_0 = x_i]$ is an infinite branch of ω, then exactly one of the following holds:*

   (a) *ϕ counter-wraps λ around θ at y_{t+i}; or*

   (b) *There is an infinite branch $\mu = \langle \dots w_2\, w_1\, w_0 = y_j]$ of θ and there is $s \ge 0$ with $j + s \equiv t + i \pmod m$ such that ϕ maps $\langle \dots z_{s+2}\, z_{s+1}\, z_s]$ onto μ and if $s > 0$ then ϕ wraps $[z_s \dots z_0 = x_i]$ around θ at y_j.*

Proof. Suppose ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$. To prove (1), follow the proof of Lemma 3.4(3) (almost verbatim). The proof of (2) is similar.

Conversely, suppose (1) and (2) are satisfied. Let $x \to z$ be an edge in $D(\gamma)$. If $x \in \mathrm{dom}(\omega)$, then $x_i = x \to z = x_{i+1}$ for some i, and so $x\phi = x_i\phi = y_{t+i} \to y_{t+i+1} = x_{i+1}\phi = z\phi$ in $D(\delta)$ since ϕ double-wraps ω around θ at y_t. If $x \notin \mathrm{dom}(\omega)$, then $x \to z$ is an edge of some branch of ω (by Proposition 2.13), and so $x\phi \to z\phi$ in $D(\delta)$ by (1) and (2). $\square$

Lemma 3.7. *Let γ and δ be connected components of $\alpha \in T(X)$ such that γ has a double ray $\omega = \langle \dots x_{-1}\, x_0\, x_1 \dots \rangle$ and δ has a cycle $\theta = (y_0 \dots y_{m-1})$. Let $\phi : \mathrm{dom}(\gamma) \to \mathrm{dom}(\delta)$ be such that for some u, ϕ wraps $[x_u\, x_{u+1}\, x_{u+2} \dots \rangle$ around θ at some y_t and $x_{u-1} \notin \mathrm{dom}(\theta)$. Then ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$ if and only if the following conditions are satisfied:*

(1) *There is an infinite branch $\mu = \langle \dots v_2\, v_1\, v_0 = y_t]$ of θ such that ϕ maps $\langle \dots x_{u-2}\, x_{u-1}\, x_u]$ onto μ.*

(2) *If $\tau = [z_0 \dots z_p = x_i]$ is a finite branch of ω with $i \ge u$, then exactly one of the following holds:*

   (a) *ϕ wraps τ around θ at $y_{t+i-u-p}$; or*

   (b) *There is a branch $(\dots w_q \dots w_0 = y_j]$ of θ with $1 \le q \le p$ and $j + p - q \equiv t + i - u \pmod m$ such that ϕ maps $[z_0 \dots z_q]$ onto $[w_q \dots w_0 = y_j]$ and if $q < p$ then ϕ wraps $[z_q \dots z_p = x_i]$ around θ at y_j;*

(3) *If $\tau = [z_0 \dots z_p = x_i]$ is a finite branch of ω with $i < u$, then exactly one of the following holds:*

   (a) *ϕ maps τ onto $[v_{u-i-p} \dots v_{u-i}]$; or*

   (b) *There is a branch $(\dots w_q \dots w_1\, w_0 = v_j \dots v_0 = y_t]$ of θ with $1 \le q \le p$, $j + p - q = u - i$, and $w_1 \notin \mathrm{dom}(\mu)$ such that ϕ maps $[z_0 \dots z_q]$ onto $[w_q \dots w_0 = v_j]$ and if $q < p$ then ϕ maps $[z_q \dots z_p = x_i]$ onto $[v_j \dots v_{u-i}]$;*

(4) *If $\lambda = \langle \dots z_2\, z_1\, z_0 = x_i]$ is an infinite branch of ω with $i \geq u$, then exactly one of the following holds:*

   (a) *ϕ counter-wraps λ around θ at y_{t+i-u}; or*

   (b) *There is an infinite branch $\mu_1 = \langle \dots y_j]$ of θ and there is $s \geq 0$ with $j + s \equiv t + i - u \pmod m$ such that ϕ maps $\langle \dots z_{s+2}\, z_{s+1}\, z_s]$ onto μ_1 and if $s > 0$ then ϕ wraps $[z_s \dots z_0 = x_i]$ around θ at y_j;*

(5) *If $\lambda = \langle \dots z_2\, z_1\, z_0 = x_i]$ is an infinite branch of ω with $i < u$, then exactly one of the following holds:*

   (a) *ϕ maps λ onto $\langle \dots v_{u-i-2}\, v_{u-i-1}\, v_{u-i}]$; or*

   (b) *There is an infinite branch $\langle \dots w_2\, w_1\, w_0 = v_j \dots v_0 = y_t]$ of θ such that w_1 does not lie on μ, and there is $s \geq 0$ with $j + s = u - i$ such that ϕ maps $\langle \dots z_{s+2}\, z_{s+1}\, z_s]$ onto $\langle \dots w_2\, w_1\, w_0 = v_j]$ and if $s > 0$ then ϕ maps $[z_s \dots z_0 = x_i]$ onto $[v_j \dots v_{u-i}]$.*

Proof. Suppose ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$. Then, since ϕ wraps $[x_u\, x_{u+1}\, x_{u+2} \dots\rangle$ around θ at y_t, we have

$$\cdots \to x_{u-2}\phi \to x_{u-1}\phi \to x_u\phi = y_t \to x_{u+1}\phi = y_{t+1} \to \cdots .$$

Thus, since $x_{u-1}\phi \notin \mathrm{dom}(\theta)$, $\mu = \langle \dots x_{u-2}\phi\, x_{u-1}\phi\, x_u\phi = y_t]$ is an infinite branch of θ and ϕ maps $\langle \dots x_{u-2}\, x_{u-1}\, x_u]$ onto μ. We have proved (1).

To prove (2), we follow the proof of Lemma 3.4(3). Let $\tau = [z_0 \dots z_p = x_i]$ be a finite branch of ω with $i \geq u$. Since $x_u\phi = y_t$ and ϕ wraps $[x_u\, x_{u+1}\, x_{u+2} \dots\rangle$ around θ, we have $x_i\phi = y_{t+i-u}$. Suppose $z_0\phi = y_j$ lies on θ. Then

$$z_0\phi = y_j \to z_1\phi = y_{j+1} \to \cdots \to z_p\phi = y_{j+p}.$$

Since $y_{j+p} = x_i\phi = y_{t+i-u}$, we have $j + p \equiv t + i - u \pmod m$. Thus $y_j = y_{t+i-u-p}$, and so ϕ wraps τ around θ at $y_{t+i-u-p}$.

Suppose $z_0\phi$ does not lie on θ. Then, by Proposition 2.13, $z_0\phi$ lies on some branch $\mu = \langle \dots w_2\, w_1\, w_0 = y_j]$ of θ. Let $z_0\phi = w_q$. Then $q \geq 1$ (since $z_0\phi \notin \mathrm{dom}(\theta)$) and

$$z_0\phi = w_q \to z_1\phi = w_{q-1} \to \cdots \to z_q\phi = w_0 = y_j \to z_{q+1}\phi = y_{j+1} \to \cdots \to z_p\phi = y_{j+p-q}.$$

Thus $q \leq p$, ϕ maps $[z_0 \dots z_q]$ onto $[w_q \dots w_0 = y_j]$, and if $q < p$ then ϕ wraps $[z_q \dots z_p = x_i]$ around θ at y_j. Further, $y_{j+p-q} = z_p\phi = x_i\phi = y_{t+i-u}$, and so $j + p - q \equiv t + i - u \pmod m$.

Hence (2a) or (2b) holds. Thus exactly one of them holds since (2a) and (2b) are mutually exclusive. We have proved (2). The proofs of (3)–(5) are similar.

Conversely, suppose (1)–(5) are satisfied. Then ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$ by an argument similar to the one used in the proof of Lemma 3.6. □

Lemma 3.8. *Let γ and δ be connected components of $\alpha \in T(X)$ such that γ has a double ray $\omega = \langle \dots x_{-1}\, x_0\, x_1 \dots\rangle$ and δ has a double ray $\pi = \langle \dots y_{-1}\, y_0\, y_1 \dots\rangle$. Let $\phi : \mathrm{dom}(\gamma) \to \mathrm{dom}(\delta)$ be such that ϕ maps ω onto π at some y_t. Then ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$ if and only if the following conditions are satisfied:*

(1) *If $\tau = [z_0 \dots z_p = x_i]$ is a finite branch of ω, then exactly one of the following holds:*

(a) *ϕ maps τ onto $[y_{t+i-p} \dots y_{t+i}]$; or*

(b) *There is a branch $\mu = (\dots w_q \dots w_0 = y_{t+i-p+q}]$ of π with $1 \le q \le p$ such that ϕ maps $[z_0 \dots z_q]$ onto $[w_q \dots w_0 = y_{t+i-p+q}]$ and if $q < p$ then ϕ maps $[z_q \dots z_p = x_i]$ onto $[y_{t+i-p+q} \dots y_{t+i}]$;*

(2) *If $\lambda = \langle \dots z_2\, z_1\, z_0 = x_i]$ is an infinite branch of ω, then exactly one of the following holds:*

(a) *ϕ maps λ onto $\langle \dots y_{t+i-2}\, y_{t+i-1}\, y_{t+i}]$; or*

(b) *There is an infinite branch $\mu = \langle \dots w_2\, w_1\, w_0 = y_{t+i-s}]$ of π with $s \ge 0$ such that ϕ maps $\langle \dots z_{s+2}\, z_{s+1}\, z_s]$ onto μ and if $s > 0$ then ϕ maps $[z_s \dots z_0 = x_i]$ onto $[y_{t+i-s} \dots y_{t+i}]$.*

Proof. Suppose ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$. Let $\tau = [z_0 \dots z_p = x_i]$ be a finite branch of ω. Since ϕ preserves edges, we have

$$z_0\phi \to z_1\phi \to \cdots \to z_p\phi = x_i\phi = y_{t+i}, \tag{3.3}$$

where the last equality is true since $x_0\phi = y_t$ and ϕ maps ω onto π. Suppose $z_0\phi$ lies on π. Then, by (3.3), $z_0\phi = y_{t+i-p}$ and ϕ maps τ onto $[y_{t+i-p} \dots y_{t+i}]$.

Suppose $z_0\phi$ does not lie on π. Then, by Proposition 2.13, $z_0\phi$ lies on some branch $\mu = (\dots w_2\, w_1\, w_0 = y_j]$ of θ. Let $z_0\phi = w_q$. Then $q \ge 1$ (since $z_0\phi$ does not lie on π) and

$$z_0\phi = w_q \to \cdots \to z_q\phi = w_0 = y_j \to z_{q+1}\phi = y_{j+1} \to \cdots \to z_p\phi = y_{j+p-q}. \tag{3.4}$$

Since $y_{j+p-q} = z_p\phi = x_i\phi = y_{t+i}$, we have $j = t + i - p + q$. Thus, by (3.4), ϕ maps $[z_0 \dots z_q]$ onto $[w_q \dots w_0 = y_{t+i-p+q}]$, and if $q < p$ then ϕ maps $[z_q \dots z_p = x_i]$ onto $[y_{t+i-p+q} \dots y_{t+i}]$. Hence (1a) or (1b) holds. Thus exactly one of them holds since (1a) and (1b) are mutually exclusive. We have proved (1). The proof of (2) is similar.

Conversely, suppose (1) and (2) are satisfied. Then ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$ by an argument similar to the one used in the proof of Lemma 3.6. □

Figure 3.7 illustrates 2(b) of Lemma 3.8, where the assumption is that $\omega = \langle \dots x_{-1}\, x_0\, x_1 \dots \rangle$ and $\pi = \langle \dots y_{-1}\, y_0\, y_1 \dots \rangle$ are vertical double rays in the figure.

Finally, if γ is of type rro (see Definition 2.11) and $D(\gamma)$ is homomorphic to $D(\delta)$, then δ may have a cycle, or a double ray, or be of type rro.

Lemma 3.9. *Let γ and δ be connected components of $\alpha \in T(X)$ such that γ is of type rro, and let $\eta = [x_0\, x_1\, x_2 \dots \rangle$ be a maximal right ray in γ. Suppose ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$. Then exactly one of the following holds:*

(a) *δ has a cycle $\theta = (y_0 \dots y_{m-1})$ and either ϕ wraps η around θ at some y_t, or there is $u \ge 1$ such that ϕ wraps $[x_u\, x_{u+1}\, x_{u+2} \dots \rangle$ around θ at some y_t and $x_{u-1} \notin \mathrm{dom}(\theta)$;*

(b) *δ has a double ray $\pi = \langle \dots y_{-1}\, y_0\, y_1 \dots \rangle$ such that either ϕ maps η onto $[y_t\, y_{t+1} \dots \rangle$ for some t, or there are t and $u \ge 1$ such that ϕ maps $[x_u\, x_{u+1} \dots \rangle$ onto $[y_t\, y_{t+1} \dots \rangle$, $x_{u-1} \notin \mathrm{dom}(\pi)$, and x_0 does not lie on an infinite branch of π; or*

(c) *δ is of type rro and has a maximal right ray $\mu = [y_0\, y_1\, y_2 \dots \rangle$ such that ϕ maps η onto $[y_t\, y_{t+1} \dots \rangle$ for some $t \ge 0$.*

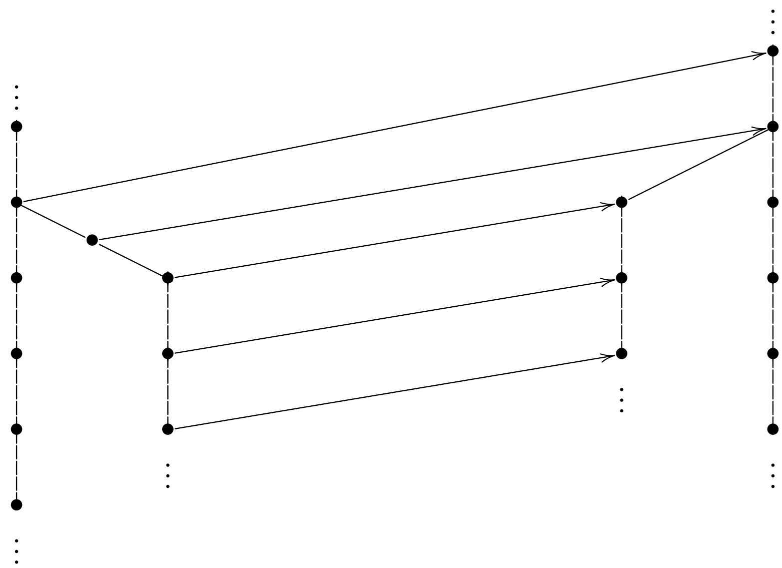

Figure 3.7: A homomorphism acting on an infinite branch of a double ray in Lemma 3.8.

Proof. Since ϕ preserves edges, we have

$$x_0\phi \to x_1\phi \to x_2\phi \to \cdots . \tag{3.5}$$

If δ contains a cycle $\theta = (y_0 \dots y_{m-1})$, then (a) holds by an argument similar to the proof of Lemma 3.5(a).

Suppose δ contains a double ray. Then the vertices in (3.5) must be pairwise distinct since otherwise δ would also contain a cycle, which is impossible by Proposition 2.10. Suppose there are vertices $w_0, w_1, w_2, \dots$ of $D(\delta)$ such that

$$\cdots \to w_2 \to w_1 \to w_0 \to x_0\phi \to x_1\phi \to x_2\phi \to \cdots . \tag{3.6}$$

Then δ has a double ray $\pi = \langle \dots y_{-1}\, y_0\, y_1 \dots \rangle$, namely $\pi = \langle \dots w_1\, w_0\, x_0\phi\, x_1\phi \dots \rangle$ such that ϕ maps η onto $[y_t\, y_{t+1} \dots \rangle$ for some t. Suppose there are no vertices $w_0, w_1, w_2, \dots$ of $D(\delta)$ such that (3.6) holds. Let $\pi = \langle \dots y_{-1}\, y_0\, y_1 \dots \rangle$ be any double ray in δ. Let u be the smallest non-negative integer such that $x_u\phi = y_t$ for some t. (Such u must exist since δ is a connected component of α.) Then $u \geq 1$ since if $u = 0$ then (3.6) would hold for $w_0 = y_{t-1}, w_1 = y_{t-2}, \dots$. Moreover, ϕ maps $[x_u\, x_{u+1} \dots \rangle$ onto $[y_t\, y_{t+1} \dots \rangle$ (since $x_u\phi = y_t$), $x_{u-1} \notin \mathrm{dom}(\pi)$ (by the choice of u), and $x_0\phi$ does not lie on an infinite branch of π (since if it did then (3.6) would hold for the vertices $w_0, w_1, \dots$ preceding $x_0\phi$ on that branch).

Suppose δ does not contain a cycle or a double ray. Then δ is of type rro by Proposition 2.10. Let $w_0, \dots, w_q$, $q \geq 0$, be vertices in $D(\delta)$ such that

$$w_0 \to \cdots \to w_q = x_0\phi \to x_1\phi \to x_2\phi \to \cdots$$

and $w_0 \notin \mathrm{im}(\alpha)$. (Such vertices must exist since otherwise γ would have a double ray.) Then $\mu = [w_0 \dots w_q = x_0\phi\, x_1\phi\, x_2\phi \dots \rangle$ is a desired maximal right ray in δ from (c).

We have proved that at least one of (a)–(c) holds. By Proposition 2.10, (a), (b), (c) are pairwise mutually exclusive, so exactly one of them holds. □

We now analyze what happens in each of the cases exhibited by Lemma 3.9. Recall that, by Proposition 2.13, if γ is of type rro and η is a maximal right ray in γ, then all branches of η are finite.

Lemma 3.10. *Let γ and δ be connected components of $\alpha \in T(X)$ such that γ is of type rro and has a maximal right ray $\eta = [x_0\, x_1\, x_2 \ldots\rangle$, and δ has a cycle $\theta = (y_0 \ldots\, y_{m-1})$. Let $\phi : \mathrm{dom}(\gamma) \to \mathrm{dom}(\delta)$ be such that ϕ wraps η around θ at some y_t. Then ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$ if and only if for every (finite) branch $\tau = [z_0 \ldots\, z_p = x_i]$ of η, exactly one of the following holds:*

(a) *ϕ wraps τ around θ at y_{t+i-p}; or*

(b) *There is a branch $\mu = (\ldots\, w_q \ldots\, w_0 = y_j]$ of θ with $1 \leq q \leq p$ and $j + p - q \equiv t + i \pmod m$ such that ϕ maps $[z_0 \ldots\, z_q]$ onto $[w_q \ldots\, w_0 = y_j]$ and if $q < p$ then ϕ wraps $[z_q \ldots\, z_p = x_i]$ around θ at y_j.*

Proof. Similar to the proof of Lemma 3.7(2). □

Lemma 3.11. *Let γ and δ be connected components of $\alpha \in T(X)$ such that γ is of type rro and has a right ray $\eta = [x_0\, x_1\, x_2 \ldots\rangle$, and δ has a cycle $\theta = (y_0 \ldots\, y_{m-1})$. Let $\phi : \mathrm{dom}(\gamma) \to \mathrm{dom}(\delta)$ be such that for some $u \geq 1$, ϕ wraps $[x_u\, x_{u+1}\, x_{u+2} \ldots\rangle$ around θ at some y_t and $x_{u-1} \notin \mathrm{dom}(\theta)$. Then ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$ if and only if the following conditions are satisfied:*

(1) *There is a branch (finite or infinite) $\mu = (\ldots\, v_2\, v_1\, v_0 = y_t]$ of θ such that the length of μ is at least $u + 1$ and ϕ maps $[x_0 \ldots\, x_u]$ onto $[v_u \ldots\, v_0]$.*

(2) *If $\tau = [z_0 \ldots\, z_p = x_i]$ is a branch of η with $i \geq u$, then exactly one of the following holds:*

   (a) *ϕ wraps τ around θ at $y_{t+i-u-p}$; or*

   (b) *There is a branch $(\ldots\, w_q \ldots\, w_0 = y_j]$ of θ with $1 \leq q \leq p$ and $j + p - q \equiv t + i - u \pmod m$ such that ϕ maps $[z_0 \ldots\, z_q]$ onto $[w_q \ldots\, w_q = y_j]$ and if $q < p$ then ϕ wraps $[z_q \ldots\, z_p = x_i]$ around θ at y_j;*

(3) *If $\tau = [z_0 \ldots\, z_p = x_i]$ is a branch of η with $i < u$, then exactly one of the following holds:*

   (a) *ϕ maps τ onto $[v_{u-i+p} \ldots\, v_{u-i}]$; or*

   (b) *There is a branch $(\ldots\, w_q \ldots\, w_1\, w_0 = v_j \ldots\, v_0 = y_t]$ of θ with $1 \leq q \leq p$, $j - p + q = u - i$ and $w_1 \notin \mathrm{dom}(\mu)$ such that ϕ maps $[z_0 \ldots\, z_q]$ onto $[w_q \ldots\, w_0 = v_j]$ and if $q < p$ then ϕ maps $[z_q \ldots\, z_p = x_i]$ onto $[v_j \ldots\, v_{u-i}]$.*

Proof. Suppose ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$. Then (1) and (2) follow by an argument similar to the proof of Lemma 3.7(1)(2).

To prove (3), let $\tau = [z_0 \ldots\, z_p = x_i]$ be a branch of η with $i < u$. Since ϕ maps $[x_0 \ldots\, x_u]$ onto $[v_u \ldots\, v_0]$ (by (1)) and $z_p = x_i$ with $i < u$, we have $z_p\phi = x_i\phi = v_{u-i}$.

Suppose $z_0\phi \in \mathrm{dom}(\mu)$, that is, $z_0\phi = v_s$ for some $s \geq 1$. Then

$$z_0\phi = v_s \to z_1\phi = v_{s-1} \to \cdots \to z_p\phi = v_{s-p}. \tag{3.7}$$

We have $v_{s-p} = z_p\phi = v_{u-i}$, and so $s - p = u - i$. Thus $s = u - i + p$, and so ϕ maps τ onto $[v_{u-i+p} \ldots\, v_{u-i}]$ by (3.7).

Suppose $z_0\phi \notin \mathrm{dom}(\mu)$. Let q be the smallest integer in $\{1, \dots, p\}$ such that $z_q\phi = v_j$ for some j. (Such a q exists since $z_p\phi = v_{u-i}$.) Then

$$z_0\phi \to \cdots \to z_q\phi = v_j \to z_{q+1}\phi = v_{j-1} \to \cdots \to z_p\phi = v_{j-(p-q)} = v_{u-i}.$$

Then, for the branch $(\dots\, w_q = z_0\phi \dots\, w_0 = z_q\phi = v_j \dots\, v_0 = y_t]$ of θ, ϕ maps $[z_0 \dots\, z_q]$ onto $[w_q \dots\, w_0 = v_j]$, ϕ maps $[z_q \dots\, z_p = x_i]$ onto $[v_j \dots\, v_{u-i}]$ (if $q < p$), and $j-p+q = u-i$ (since $v_{j-(p-q)} = v_{u-i}$).

Thus (3a) or (3b) holds, and so exactly one of them holds since (3a) and (3b) are mutually exclusive.

Conversely, suppose (1)–(3) are satisfied. Then ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$ by an argument similar to the one used in the proof of Lemma 3.6. □

Figure 3.8 illustrates how the maximal right ray $\eta = [x_0\, x_1\, x_2 \dots\rangle$ from Lemma 3.11 is mapped by a homomorphism.

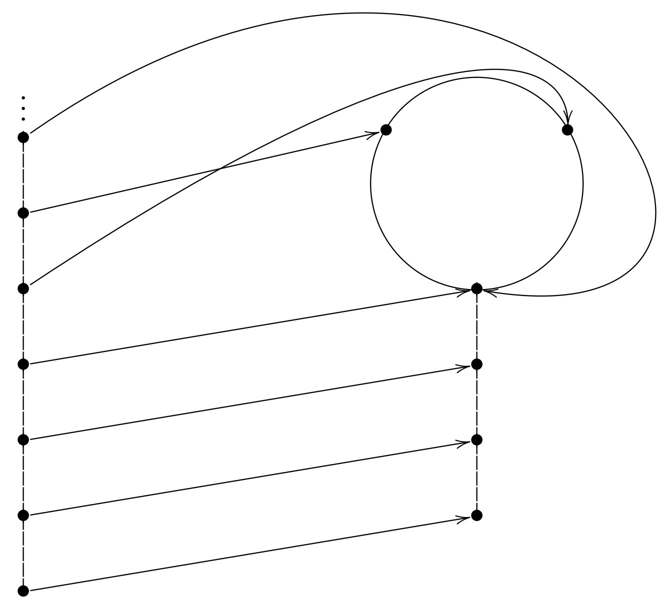

Figure 3.8: A homomorphism acting on a maximal right ray in Lemma 3.11.

Lemma 3.12. *Let γ and δ be connected components of $\alpha \in T(X)$ such that γ is of type rro with a maximal right ray $\eta = [x_0\, x_1\, x_2 \dots\rangle$, and δ has a double ray $\pi = \langle \dots\, y_{-1}\, y_0\, y_1 \dots\rangle$. Let $\phi : \mathrm{dom}(\gamma) \to \mathrm{dom}(\delta)$ such that ϕ maps η onto $[y_t\, y_{t+1} \dots\rangle$ for some t. Then ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$ if and only if for every branch $\tau = [z_0 \dots\, z_p = x_i]$ of η, exactly one of the following holds:*

(a) *ϕ maps τ onto $[y_{t+i-p} \dots\, y_{t+i}]$; or*

(b) *There is a branch $\mu = (\dots\, w_q \dots\, w_0 = y_{t+i-p+q}]$ of π with $1 \leq q \leq p$ such that ϕ maps $[z_0 \dots\, z_q]$ onto $[w_q \dots\, w_0 = y_{t+i-p+q}]$ and if $q < p$ then ϕ maps $[z_q \dots\, z_p = x_i]$ onto $[y_{t+i-p+q} \dots\, y_{t+i}]$.*

Proof. To prove (⇒), follow the proof of Lemma 3.8(1). The converse follows easily from the fact that γ is the join of η and its (necessarily finite) branches (see Proposition 2.13). □

Lemma 3.13. *Let γ and δ be connected components of $\alpha \in T(X)$ such that γ is of type rro and has a right ray $\eta = [x_0\, x_1\, x_2 \ldots\rangle$, and δ has a double ray $\pi = \langle \ldots\, y_{-1}\, y_0\, y_1 \ldots\rangle$ such that for some t and $u \geq 1$, ϕ maps $[x_u\, x_{u+1} \ldots\rangle$ onto $[y_t\, y_{t+1} \ldots\rangle$, $x_{u-1} \notin \mathrm{dom}(\pi)$, and $x_0\phi$ does not lie on an infinite branch of π. Then ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$ if and only if the following conditions are satisfied:*

(1) *There is a finite branch $\kappa = [\ldots\, v_0 \ldots\, v_u = y_t]$ of π such that ϕ maps $[x_0 \ldots\, x_u]$ onto $[v_0 \ldots\, v_u]$;*

(2) *If $\tau = [z_0 \ldots\, z_p = x_i]$ is a branch of η with $i \geq u$, then exactly one of the following holds:*

  (a) *ϕ maps τ onto $[y_{t+i-u-p} \cdots\, y_{t+i-u}]$; or*

  (b) *There is a branch $(\ldots\, w_q \ldots\, w_0 = y_{t+i-u-p+q}]$ of π with $1 \leq q \leq p$ such that ϕ maps $[z_0 \ldots\, z_q]$ onto $[w_q \ldots\, w_0 = y_{t+i-u-p+q}]$ and if $q < p$ then ϕ maps $[z_q \ldots\, z_p = x_i]$ onto $[y_{t+i-u-p+q} \cdots\, y_{t+i-u}]$;*

(3) *If $\tau = [z_0 \ldots\, z_p = x_i]$ is a branch of η with $i < u$, then exactly one of the following holds:*

  (a) *ϕ maps τ onto $[v_{i-p} \ldots\, v_i]$; or*

  (b) *There is a branch $(\ldots\, w_q \ldots\, w_1\, w_0 = v_{i-p+q} \ldots\, v_u = y_t]$ of π with $1 \leq q \leq p$ and $w_1 \notin \mathrm{dom}(\kappa)$ such that ϕ maps $[z_0 \ldots\, z_q]$ onto $[w_q \ldots\, w_0 = v_{i-p+q}]$ and if $q < p$ then ϕ maps $[z_q \ldots\, z_p = x_i]$ onto $[v_{i-p+q} \ldots\, v_i]$.*

Proof. Suppose ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$. Then (1) is satisfied since $x_0\phi$ lies on some finite branch of π (by Proposition 2.13 and the assumption about x_0).

Let $\tau = [z_0 \ldots\, z_p = x_i]$ be a branch of η with $i \geq u$. Since ϕ preserves edges, we have

$$z_0\phi \to z_1\phi \to \cdots \to z_p\phi = x_i\phi = y_{t+i-u}, \tag{3.8}$$

where the last equality is true since $x_u\phi = y_t$ and ϕ maps $[x_u\, x_{u+1} \ldots\rangle$ onto $[y_t\, y_{t+1} \ldots\rangle$. Suppose $z_0\phi$ lies on π. Then, by (3.8), $z_0\phi = y_{t+i-u-p}$ and ϕ maps τ onto $[y_{t+i-u-p} \cdots\, y_{t+i-u}]$.

Suppose $z_0\phi$ does not lie on π. Then, by Proposition 2.13, $z_0\phi$ lies on some branch $\mu = (\ldots\, w_2\, w_1\, w_0 = y_j]$ of π. Let $z_0\phi = w_q$. Then $q \geq 1$ (since $z_0\phi$ does not lie on π) and

$$z_0\phi = w_q \to \cdots \to z_q\phi = w_0 = y_j \to z_{q+1}\phi = y_{j+1} \to \cdots \to z_p\phi = y_{j+p-q}. \tag{3.9}$$

Since $y_{j+p-q} = z_p\phi = x_i\phi = y_{t+i-u}$, we have $j = t + i - u - p + q$. Thus, by (3.9), ϕ maps $[z_0 \ldots\, z_q]$ onto $[w_q \ldots\, w_0 = y_{t+i-u-p+q}]$, and if $q < p$ then ϕ maps $[z_q \ldots\, z_p = x_i]$ onto $[y_{t+i-u-p+q} \cdots\, y_{t+i-u}]$. Hence (2a) or (2b) holds. Thus exactly one of them holds since (2a) and (2b) are mutually exclusive. We have proved (2). The proof of (3) is similar.

Conversely, suppose (1)–(3) are satisfied. Then ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$ by an argument similar to the one used in the proof of Lemma 3.6. □

Figure 3.9 illustrates how the maximal right ray $\eta = [x_0\, x_1\, x_2 \ldots\rangle$ from Lemma 3.13 and its branch are mapped by a homomorphism, where we assume that η is the vertical maximal right ray in the figure and $i \geq u$ (see (2) of Lemma 3.13).

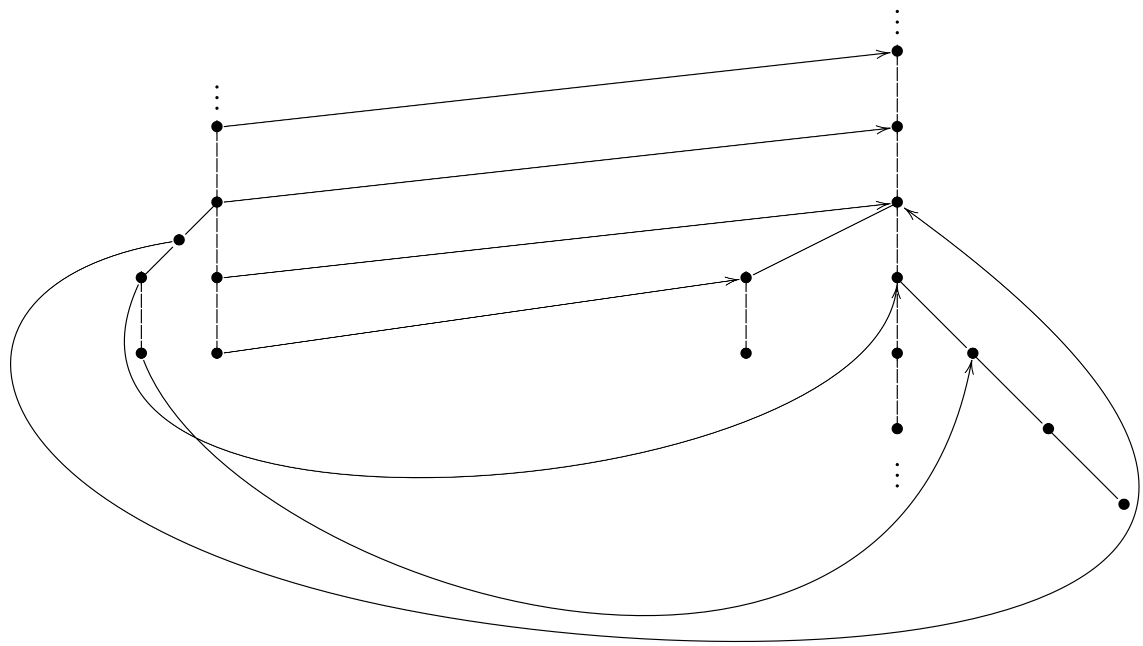

Figure 3.9: A homomorphism acting on a maximal right ray and its branch in Lemma 3.13.

Lemma 3.14. *Let γ and δ be connected components of $\alpha \in T(X)$ of type rro such that γ has a maximal right ray $\eta = [x_0\, x_1\, x_2 \ldots\rangle$ and δ has a maximal right ray $\xi = [y_0\, y_1\, y_2 \ldots\rangle$. Let $\phi : \mathrm{dom}(\gamma) \to \mathrm{dom}(\delta)$ such that ϕ maps η onto $[y_t\, y_{t+1} \ldots\rangle$ for some $t \geq 0$. Then ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$ if and only if for every branch $\tau = [z_0 \ldots z_p = x_i]$ of η, exactly one of the following holds:*

(a) *ϕ maps τ onto $[y_{t+i-p} \ldots y_{t+i}]$; or*

(b) *There is a branch $\kappa = [\ldots w_0 \ldots w_q = y_{t+i-p+q}]$ of ξ with $1 \leq q \leq p$ such that ϕ maps $[z_0 \ldots z_q]$ onto $[w_0 \ldots w_q = y_{t+i-p+q}]$ and if $q < p$ then ϕ maps $[z_q \ldots z_p = x_i]$ onto $[y_{t+i-p+q} \ldots y_{t+i}]$.*

Proof. Suppose ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$. Let $\tau = [z_0 \ldots z_p = x_i]$ be a branch of η. Since ϕ preserves edges, we have

$$z_0\phi \to z_1\phi \to \cdots \to z_p\phi = x_i\phi = y_{t+i}, \tag{3.10}$$

where the last equality is true since $x_0\phi = y_t$ and ϕ maps η onto $[y_t\, y_{t+1} \ldots\rangle$. Suppose $z_0\phi$ lies on ξ. Then, by (3.10), $z_0\phi = y_{t+i-p}$ and ϕ maps τ onto $[y_{t+i-p} \ldots y_{t+i}]$.

Suppose $z_0\phi$ does not lie on ξ. Then there exists a branch $\kappa = [\ldots w_0 \ldots w_q = y_j]$ of ξ (see by Proposition 2.13) such that $q \geq 1$ and $z_0\phi = w_0$. Hence

$$z_0\phi = w_0 \to \cdots \to z_q\phi = w_q = y_j \to z_{q+1}\phi = y_{j+1} \to \cdots \to z_p\phi = y_{j+p-q}. \tag{3.11}$$

Since $y_{j+p-q} = z_p\phi = x_i\phi = y_{t+i}$, we have $j = t + i - p + q$. Thus, by (3.11), ϕ maps $[z_0 \ldots z_q]$ onto $[w_0 \ldots w_q = y_{t+i-p+q}]$, and if $q < p$ then ϕ maps $[z_q \ldots z_p = x_i]$ onto $[y_{t+i-p+q} \ldots y_{t+i}]$. Hence (a) or (b) holds. Thus exactly one of them holds since (a) and (b) are mutually exclusive.

The converse follows easily from the fact that γ is the join of η and its (necessarily finite) branches (see Proposition 2.13). $\square$

Figure 3.10 illustrates how the maximal right ray $\eta = [x_0\, x_1\, x_2 \ldots\rangle$ from Lemma 3.14 and its branch are mapped by a homomorphism, where we assume that η and δ are the vertical maximal right rays in the figure.

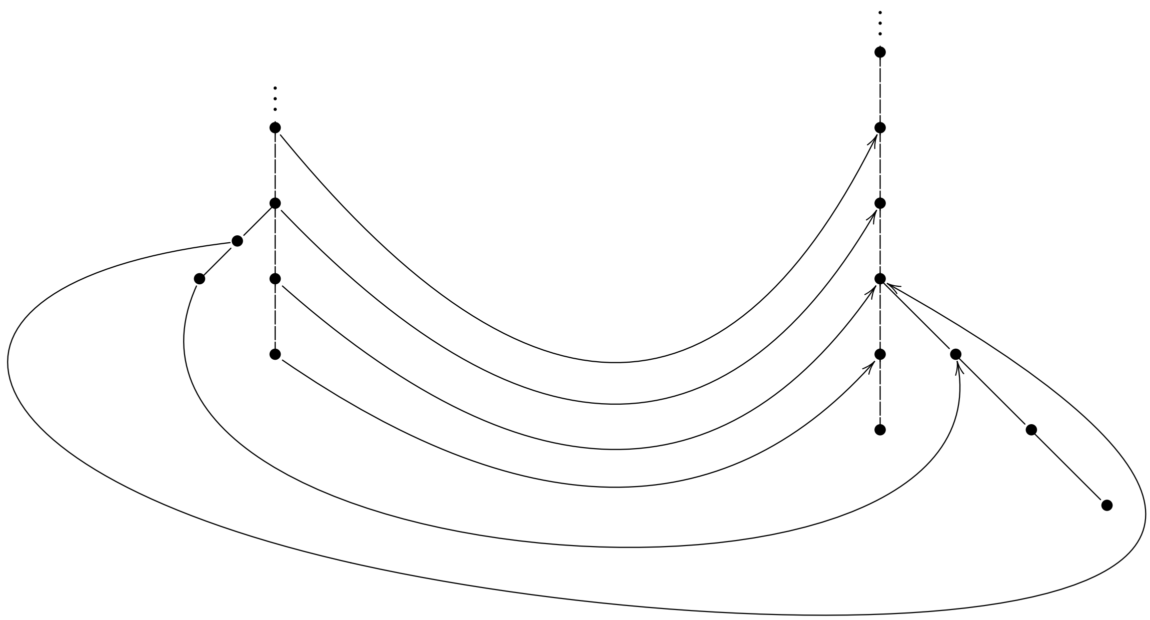

Figure 3.10: A homomorphism acting on a maximal right ray and its branch in Lemma 3.14.

4 The Characterization Theorem

The description of the elements $\beta \in C(\alpha)$, where $\alpha \in T(X)$, reduces to the description of the graph homomorphisms from $D(\gamma)$ to $D(\delta)$, where γ and δ are connected components of α. This follows from Proposition 4.1 below.

Proposition 4.1. *Let $\alpha, \beta \in T(X)$. Then $\beta \in C(\alpha)$ if and only if for every connected component γ of α, there exists a connected component δ of α such that $\beta|_{\mathrm{dom}(\gamma)}$ is a graph homomorphism from $D(\gamma)$ to $D(\delta)$.*

Proof. Suppose $\beta \in C(\alpha)$. Then, by Proposition 3.1, β is a homomorphism from $D(\alpha)$ to $D(\alpha)$. Let γ be a connected component of α and let $x \in \mathrm{dom}(\gamma)$. Then, by Proposition 2.5, $x\beta \in \mathrm{dom}(\delta)$ for some connected component δ of α. We claim that $(\mathrm{dom}(\gamma))\beta \subseteq \mathrm{dom}(\delta)$. Let $z \in \mathrm{dom}(\gamma)$. Since γ is connected, $x\alpha^k = x\gamma^k = z\gamma^m = z\alpha^m$ for some integers $k, m \geq 0$. Since $\beta \in C(\alpha)$, we have $\alpha\beta = \beta\alpha$, and so $(z\beta)\alpha^m = (z\alpha^m)\beta = (x\alpha^k)\beta = (x\beta)\alpha^k$, which implies that $z\beta$ and $x\beta$ are in the domain of the same connected component of α, that is, $z\beta \in \mathrm{dom}(\delta)$. The claim has been proved. Then $\beta|_{\mathrm{dom}(\gamma)}$ is a homomorphism from $D(\gamma)$ to $D(\delta)$ by the claim and the fact that β is a homomorphism from $D(\alpha)$ to $D(\alpha)$.

Conversely, suppose that the given condition is satisfied. Suppose $y \xrightarrow{\alpha} z$. Then $y, z \in \mathrm{dom}(\gamma)$ for some connected component γ of α. It is given that $\phi = \beta|_{\mathrm{dom}(\gamma)}$ is a homomorphism from $D(\gamma)$ to $D(\delta)$ for some connected component δ of α. Thus $y\beta = y\phi \xrightarrow{\delta} z\phi = z\beta$, implying $y\beta \xrightarrow{\alpha} z\beta$. Hence β is a homomorphism from $D(\alpha)$ to $D(\alpha)$, and so $\beta \in C(\alpha)$ by Proposition 3.1. □

In view of Proposition 4.1, we can now characterize the elements of $C(\alpha)$ using the results of Section 3.

Theorem 4.2. *Let $\alpha, \beta \in T(X)$, where X is an arbitrary set. Then $\beta \in C(\alpha)$ if and only if for every connected component γ of α, there exists a connected component δ of α such that the following conditions are satisfied for $\phi = \beta|_{\mathrm{dom}(\gamma)}$:*

(1) $\mathrm{im}(\phi) \subseteq \mathrm{dom}(\delta)$*;*

(2) *If γ has a cycle $\sigma = (x_0 \dots x_{k-1})$, then δ has a cycle $\theta = (y_0 \dots y_{m-1})$ and* (1)–(4) *of* Lemma 3.4 *are satisfied;*

(3) *If γ has a double ray $\omega = \langle \dots x_{-1}\, x_0\, x_1 \dots \rangle$, then exactly one of the following holds:*

   (a) *δ has a cycle $\theta = (y_0 \dots y_{m-1})$, ϕ double-wraps ω around θ at some y_t, and* (1) and (2) *of* Lemma 3.6 *are satisfied;*

   (b) *δ has a cycle $\theta = (y_0 \dots y_{m-1})$, there exists u such that ϕ wraps $[x_u\, x_{u+1} \dots \rangle$ around θ at some y_t and $x_{u-1} \notin \mathrm{dom}(\theta)$, and* (1)–(5) *of* Lemma 3.7 *are satisfied; or*

   (c) *δ has a double ray $\pi = \langle \dots y_{-1}\, y_0\, y_1 \dots \rangle$ such that ϕ maps ω onto π at some y_t and* (1) and (2) *of* Lemma 3.8 *are satisfied;*

(4) *If γ is of type rro and has a maximal right ray $\eta = [x_0\, x_1\, x_2 \dots \rangle$, then exactly one of the following holds:*

   (a) *δ has a cycle $\theta = (y_0 \dots y_{m-1})$, ϕ wraps η around θ at some y_t, and for every branch $\tau = [z_0 \dots z_p = x_i]$ of η,* (a) *or* (b) *of* Lemma 3.10 *holds;*

   (b) *δ has a cycle $\theta = (y_0 \dots y_{m-1})$, there exists $u \geq 1$ such that ϕ wraps $[x_u\, x_{u+1} \dots \rangle$ around θ at some y_t and $x_{u-1} \notin \mathrm{dom}(\theta)$, and* (1)–(3) *of* Lemma 3.11 *are satisfied;*

   (c) *δ has a double ray $\pi = \langle \dots y_{-1}\, y_0\, y_1 \dots \rangle$ such that ϕ maps η onto $[y_t\, y_{t+1} \dots \rangle$ for some t and for every branch $\tau = [z_0 \dots z_p = x_i]$ of η,* (a) *or* (b) *of* Lemma 3.12 *holds;*

   (d) *δ has a double ray $\pi = \langle \dots y_{-1}\, y_0\, y_1 \dots \rangle$ such that for some t and $u \geq 1$, ϕ maps $[x_u\, x_{u+1} \dots \rangle$ onto $[y_t\, y_{t+1} \dots \rangle$, $x_{u-1} \notin \mathrm{dom}(\pi)$, x_0 does not lie on an infinite branch of π, and* (1)–(3) *of* Lemma 3.13 *are satisfied; or*

   (e) *δ is of type rro and has a maximal right ray $\xi = [y_0\, y_1\, y_2 \dots \rangle$ such that ϕ maps η onto $[y_t\, y_{t+1} \dots \rangle$ for some $t \geq 0$ and for every branch $\tau = [z_0 \dots z_p = x_i]$ of η,* (a) *or* (b) *of* Lemma 3.14 *holds.*

Proof. Suppose $\beta \in C(\alpha)$. Let γ be a connected component of α and let $\phi = \beta|_{\mathrm{dom}(\gamma)}$. Then, by Proposition 4.1, there is a connected component δ of α such that ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$, and so (1) is satisfied. Further, (2) is satisfied by Lemma 3.4; (3) by Lemmas 3.6–3.8; and (4) by Lemma 3.10–3.14.

Conversely, suppose that for every connected component γ of α, there exists a connected component δ of α such that (1)–(4) are satisfied for $\phi = \beta|_{\mathrm{dom}(\gamma)}$. Let γ be a connected component of

α. Then there is a connected component δ of α such that (1)–(4) are satisfied for $\phi = \beta|_{\mathrm{dom}(\gamma)}$. Then $\phi : \mathrm{dom}(\gamma) \to \mathrm{dom}(\delta)$ by (1). We want to prove that ϕ is a homomorphism from $D(\gamma)$ to $D(\delta)$. If γ has a cycle, then ϕ is a homomorphism by (2) and Lemma 3.4. If γ has a double ray, then ϕ is a homomorphism by (3) and Lemmas 3.5–3.8. Suppose γ does not have a cycle or a double ray. Then γ is of type *rro* by Proposition 2.10 and Definition 2.11. Thus ϕ is a homomorphism by (3) and Lemmas 3.9–3.14. Hence $\beta \in C(\alpha)$ by Proposition 4.1. □

5 Special Cases

Theorem 4.2 can be applied to the case when X is finite and to particular types of transformations in $T(X)$. In this section, we provide some examples of these applications.

The finite case.

Suppose X is finite and let $\alpha \in T(X)$. Then α cannot have any rays, and so every connected component of $\alpha \in T(X)$ is the join of its unique cycle and the finite branches of the cycle. This gives the following corollary to Theorem 4.2 (see [31, Theorem 2]).

Corollary 5.1. *Let $\alpha, \beta \in T(X)$, where X is a finite set. Then $\beta \in C(\alpha)$ if and only if for every connected component γ of α with cycle $\sigma = (x_0 \dots x_{k-1})$, there exists a connected component δ of α with cycle $\theta = (y_0 \dots y_{m-1})$ such that the following conditions are satisfied for $\phi = \beta|_{\mathrm{dom}(\gamma)}$:*

(1) $\mathrm{im}(\phi) \subseteq \mathrm{dom}(\delta)$;

(2) *m divides k;*

(3) *ϕ wraps σ around θ at some y_t;*

(4) *If $\tau = [z_0 \dots z_p = x_i]$ is a finite branch of σ, then exactly one of the following holds:*

   (a) *ϕ wraps τ around θ at y_{t+i-p}; or*

   (b) *There is a finite branch $\mu = [\dots w_q \dots w_0 = y_j]$ of θ with $1 \le q \le p$ and $j + p - q \equiv t + i \pmod m$ such that ϕ maps $[z_0 \dots z_q]$ onto $[w_q \dots w_0 = y_j]$ and if $q < p$ then ϕ wraps $[z_q \dots z_p = x_i]$ around θ at y_j.*

Idempotents.

Let $\varepsilon \in T(X)$ (where X is arbitrary) be an idempotent. Then for every $x \in X$ and $y = x\varepsilon$, $y\varepsilon = (x\varepsilon)\varepsilon) = x(\varepsilon\varepsilon) = x\varepsilon = y$. It easily follows that if γ is a connected component of ε, then $\mathrm{im}(\gamma) = \{y\}$, for some $y \in X$, and γ is the joint of the 1-cycle (y) and some (possibly none) finite branches $[x\,y]$ of length 2 (see Figure 5.1).

Let γ and δ be connected components of an idempotent ε, with cycles (y) and (z), respectively. If $[x\,y]$ is a finite branch of (y), then $\beta \in T(X)$ satisfies (3b) of Lemma 3.4 if and only if $y\beta = z$ and $x\beta \in \mathrm{dom}(\delta)$. Therefore, Theorem 4.2 applied to an idempotent ε gives the following corollary (see [5, Lemma 2.2] and [24, Theorem 2.1]).

Corollary 5.2. *Let $\varepsilon, \beta \in T(X)$, where ε is an idempotent. Then $\beta \in C(\varepsilon)$ if and only if for every connected component γ of ε with cycle (y), there exists a connected component δ of ε with cycle (z) such that $y\beta = z$ and $(\mathrm{dom}(\gamma))\beta \subseteq \mathrm{dom}(\delta)$.*

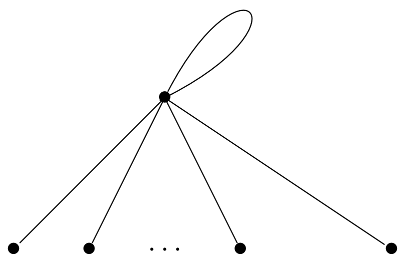

Figure 5.1: A connected component of an idempotent.

Injective transformations.

Any connected component of an injective transformations $\alpha \in T(X)$ is a cycle, a double ray, or a right ray (see Figure 5.2).

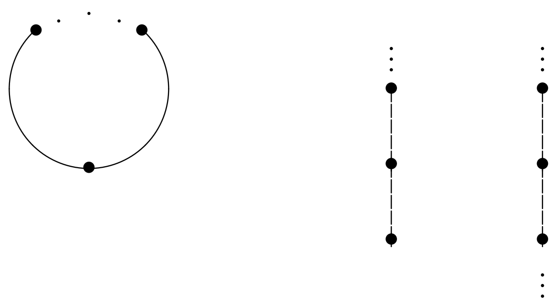

Figure 5.2: Connected components of an injective transformation.

Applying Theorem 4.2 to an injective transformation, we obtain the following corollary. (See [26, Theorem 3.9] where the centralizer of an injective α is described relative to the semigroup of injective transformations on X.)

Corollary 5.3. *Let $\alpha, \beta \in T(X)$ such that α is injective. Then $\beta \in C(\alpha)$ if and only if conditions are satisfied:*

(1) *For every cycle $\sigma = (x_0 \dots x_{k-1})$ in α, there is a cycle $\theta = (y_0 \dots y_{m-1})$ in α such that m divides k and β wraps σ around θ at some y_t;*

(2) *For every a double ray $\omega = \langle \dots x_{-1}\, x_0\, x_1 \dots \rangle$ in α, exactly one of the following holds:*

  (a) *There is a cycle $\theta = (y_0 \dots y_{m-1})$ in α such that β double-wraps ω around θ at some y_t;*

  (b) *There is a double ray $\pi = \langle \dots y_{-1}\, y_0\, y_1 \dots \rangle$ in α such that β maps ω onto π at some y_t;*

(3) *For every maximal right ray $\eta = [x_0\, x_1\, x_2 \dots \rangle$ in α, exactly one of the following holds:*

  (a) *There is a cycle $\theta = (y_0 \dots y_{m-1})$ in α such that β wraps η around θ at some y_t;*

  (b) *There is a double ray $\pi = \langle \dots y_{-1}\, y_0\, y_1 \dots \rangle$ in α such that β maps η onto $[y_t\, y_{t+1} \dots \rangle$ for some t;*

  (c) *There is a maximal right ray $\xi = [y_0\, y_1\, y_2 \dots \rangle$ in α such that β maps η onto $[y_t\, y_{t+1} \dots \rangle$ for some $t \geq 0$.*

Permutations.

Since any connected component of a permutation α on X is a cycle, Corollary 5.3 implies the following result.

Corollary 5.4. *Let $\alpha, \beta \in T(X)$ such that α is a permutation. Then $\beta \in C(\alpha)$ if and only if for every cycle $\sigma = (x_0 \ldots x_{k-1})$ in α, there is a cycle $\theta = (y_0 \ldots y_{m-1})$ in α such that m divides k and β wraps σ around θ at some y_t.*

6 Problems

In the previous sections, we described the centralizer of an arbitrary $\alpha \in T(X)$. Of course, this is just the first step towards a description of the structure (from a semigroup theoretical point of view) of $C(\alpha)$. Given the complexity of Theorem 4.2, however, such a description in all generality does not appear to be feasible. What can be done, though, is to provide such descriptions for particular types of transformations. For various particular types, it should be possible to obtain results similar to those contained in [4] and [5], and hence provide generalizations to $C(\alpha)$ of many results originally proved for the special case of $T(X) = C(\mathrm{id}_X)$. More specifically, given a transformation $\alpha \in T(X)$ of a certain pre-defined type, we would like to see the following program fulfilled:

1. Describe the automorphisms of $C(\alpha)$. (This has been done for the idempotents [4]; the most natural candidates to consider next are the injective transformations [26].)

2. Describe Green's relations in $C(\alpha)$.

3. Let T be one of Green's relations. Characterize the transformations $\alpha \in T(X)$ (of the given type) such that if $\beta, \gamma \in C(\alpha)$ and β, γ are T-related in $T(X)$, then β, γ are T-related in $C(\alpha)$.

4. Characterize the transformations $\alpha \in T(X)$ (of the given type) such that $\mathcal{D} = \mathcal{J}$ in $C(\alpha)$.

5. Characterize the transformations $\alpha \in T(X)$ (of the given type) such that the partial order of J-classes in $C(\alpha)$ is a chain.

6. For $\alpha \in T(X)$ (of the given type), describe the regular elements in $C(\alpha)$, and characterize those α for which $C(\alpha)$ is regular.

7. For $\alpha \in T(X)$ (of the given type), describe the semigroup generated by the idempotents of $C(\alpha)$.

8. Repeat problems 1–7 for $C_S(\alpha)$, where $\alpha \in T(X)$ is a transformation of a given type, S is a subsemigroup of $T(X)$, and $C_S(\alpha) = \{\beta \in S : \alpha\beta = \beta\alpha\}$ is the centralizer of α relative to S.

The program outlined above has been carried out for the idempotents and, in part, for the injective transformations [26]. We introduce some other types of transformations that appear to be especially interesting and promising.

Howie's transformations.

Let X be an infinite set. For $\alpha \in T(X)$ define the following sets:

$$S(\alpha) = \{x \in X : x\alpha \neq x\}, \quad Z(\alpha) = X \setminus \mathrm{im}(\alpha), \quad Cl(\alpha) = \bigcup \left\{ t\alpha^{-1} : \left| t\alpha^{-1} \right| \geq 2 \right\}.$$

By Howie's celebrated result [15], we know that the semigroup generated by the idempotents of $T(X)$ consists of the identity of $T(X)$ together with the elements of the following sets:

$$F(X) = \{\alpha \in T(X) : |S(\alpha)| < \aleph_0 \text{ and } |Z(\alpha)| > 0\},$$
$$Q(X) = \{\alpha \in T(X) : |S(\alpha)| = |Z(\alpha)| = |Cl(\alpha)| \geq \aleph_0\}.$$

Problem. Carry out (part of) the program outlined at the beginning of this section for the transformations in $F(X)$ [$Q(X)$], where X is infinite [$X = \mathbb{N}$].

Transformations with stabilizers.

Following [40], we define the *stabilizer* of $\alpha \in T(X)$ as the smallest integer $s \geq 0$ such that $\mathrm{im}(\alpha^s) = \mathrm{im}(\alpha^{s+1})$. If such an s does not exist, we say that α has no stabilizer. The transformations that have the stabilizer have been described in terms of their digraphs in [8].

Problem. Carry out (part of) the program outlined at the beginning of this section for the transformations with stabilizer, for an infinite X [$X = \mathbb{N}$].

Cayley functions.

A transformation $\alpha \in T(X)$ is called a *Cayley function* if there is a binary operation $*$ on X such that $(X, *)$ is a semigroup and α is an inner translation of the semigroup; that is, there exists $a \in X$ such that for every $x \in X$, $x\alpha = x * a$. The algebraic description of the Cayley functions has been given in [40]. The digraphs of the Cayley functions have been characterized in [8].

Problem. Carry out (part of) the program outlined at the beginning of this section for the Cayley functions [Cayley functions with stabilizer], for an infinite X [$X = \mathbb{N}$].

Transformations with large contraction index and collapse.

Let X be an infinite set and let $\alpha \in T(X)$. The *kernel* of α is the relation $\{(x, y) \in X \times X : x\alpha = y\alpha\}$. By $X/\ker(\alpha)$ we denote the partition of X induced by $\ker(\alpha)$, that is, $X/\ker(\alpha) = \{x\alpha^{-1} : x \in X\}$. The *defect* $d(\alpha)$ of α is the cardinal $|X \setminus \mathrm{im}(\alpha)|$; the *contraction index* $k(\alpha)$ of α is the number of classes in $X/\ker(\alpha)$ of size $|X|$; and the *collapse* $c(\alpha)$ of α is the cardinal $|X \setminus T_\alpha|$, where T_α is a cross-section of the partition $X/\ker(\alpha)$.

The importance of these parameters comes from the following two results proved in [17] (see also [9]), where it has been established when the symmetric group $\mathrm{Sym}(X)$ [the set of idempotents $E(X)$ in $T(X)$], with two extra elements, generates $T(X)$.

Theorem 6.1. ([17, Theorem 4.1]) *Let X be an infinite set such that $|X|$ is a regular cardinal. Then, for all $\mu, \nu \in T(X)$, $\langle \mathrm{Sym}(X), \mu, \nu \rangle = T(X)$ if and only if $\{\mu, \nu\}$ consists of an injection of defect $|X|$ and a surjection of contraction index $|X|$.*

Theorem 6.2. ([17, Theorem 6.1]) *Let X be an infinite set and let $E(X)$ denote the set of idempotents of $T(X)$. Then, for all $\mu, \nu \in T(X)$, $\langle E(X), \mu, \nu \rangle = T(X)$ if and only if $\{\mu, \nu\}$ consists of an injection of defect $|X|$ and a surjection of collapse $|X|$.*

Problem. Carry out (part of) the program outlined at the beginning of this section for the surjective transformations $\alpha \in T(X)$ such that every class in $X/\ker(\alpha)$ has size $|X|$ [α has collapse $|X|$].

Transformations associated with maximal subsemigroups of $T(X)$

We say that $S \leq T(X)$ is a maximal subsemigroup of $T(X)$ if $S \neq T(X)$ and for all $\alpha \in T(X) \setminus S$, S together with α generate $T(X)$. For $\alpha \in T(X)$, let $d(\alpha)$, $c(\alpha)$, and $k(\alpha)$ be, respectively, the defect, collapse, and contraction index of α (as defined above). For an infinite set X of regular cardinality, all maximal subsemigroups of $T(X)$ containing $\mathrm{Sym}(X)$ have been described in [12]:

Theorem 6.3. *Let X be any infinite set such that $|X|$ is a regular cardinal, and let M be a subsemigroup of $T(X)$ containing the symmetric group $\mathrm{Sym}(X)$. Then M is maximal if and only if M is one of the following semigroups:*

(1) $\{\alpha \in T(X) : c(\alpha) < \mu$ *or* $d(\alpha) \geq \mu\}$*, for some infinite cardinal* $\mu \leq |X|$*;*

(2) $\{\alpha \in T(X) : c(\alpha) = 0$ *or* $d(\alpha) > 0\}$*;*

(3) $\{\alpha \in T(X) : k(\alpha) < |X|\}$*;*

(4) $\{\alpha \in T(X) : c(\alpha) \geq \mu$ *or* $d(\alpha) < \mu\}$*, for some infinite cardinal* $\mu \leq |X|$*;*

(5) $\{\alpha \in T(X) : c(\alpha) > 0$ *or* $d(\alpha) = 0\}$.

Problem. Carry out (part of) the program outlined at the beginning of this section for the transformations in each one of the sets defined in Theorem 6.3.

Endomorphisms.

For a mathematical structure $\mathcal{A}$ with universe X, let $\mathrm{End}(\mathcal{A})$ denote the monoid of endomorphisms of $\mathcal{A}$ (see [6] and [7]). Then $\mathrm{End}(\mathcal{A})$ is a subsemigroup of $T(X)$. For example, if $\mathcal{A} = (X, \rho, R)$, where X is a set, ρ is an equivalence relation on X, and R is a cross-section of X/ρ, then the elements of $\mathrm{End}(\mathcal{A})$ are the maps in $T(X)$ that commute with the unique idempotent in $T(X)$ that has image R and kernel ρ (see [4] and [5]).

Problem. (Suggested by J.D. Mitchell.) Carry out (part of) the program outlined at the beginning of this section for the endomorphisms of various mathematical structures $\mathcal{A}$ with universe X, for example for the endomorphisms of a given partial order on X or the endomorphisms of a given graph with X as the set of vertices. (See [32].)

Surjective transformations.

It is easy to see that a transformation $\alpha \in T(X)$ is surjective if and only if it does not have any maximal right rays or finite branches.

Problem. Carry out (part of) the program outlined at the beginning of this section for the surjective transformations, for an infinite X [$X = \mathbb{N}$]. (See also [33].

Acknowledgments

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