The Commuting Graph of the Symmetric Inverse Semigroup

João Araújo
Universidade Aberta, R. Escola Politécnica, 147
1269-001 Lisboa, Portugal
&
Centro de Álgebra, Universidade de Lisboa
1649-003 Lisboa, Portugal, jaraujo@ptmat.fc.ul.pt

Wolfram Bentz
Centro de Álgebra, Universidade de Lisboa
1649-003 Lisboa, Portugal, wfbentz@fc.ul.pt

Janusz Konieczny
Department of Mathematics, University of Mary Washington
Fredericksburg, Virginia 22401, USA, jkoniecz@umw.edu

Abstract

The commuting graph of a finite non-commutative semigroup S, denoted $\mathcal{G}(S)$, is a simple graph whose vertices are the non-central elements of S and two distinct vertices x, y are adjacent if $xy = yx$. Let $\mathcal{I}(X)$ be the symmetric inverse semigroup of partial injective transformations on a finite set X. The semigroup $\mathcal{I}(X)$ has the symmetric group $\mathrm{Sym}(X)$ of permutations on X as its group of units. In 1989, Burns and Goldsmith determined the clique number of the commuting graph of $\mathrm{Sym}(X)$. In 2008, Iranmanesh and Jafarzadeh found an upper bound of the diameter of $\mathcal{G}(\mathrm{Sym}(X))$, and in 2011, Dolžan and Oblak claimed that this upper bound is in fact the exact value.

The goal of this paper is to begin the study of the commuting graph of the symmetric inverse semigroup $\mathcal{I}(X)$. We calculate the clique number of $\mathcal{G}(\mathcal{I}(X))$, the diameters of the commuting graphs of the proper ideals of $\mathcal{I}(X)$, and the diameter of $\mathcal{G}(\mathcal{I}(X))$ when $|X|$ is even or a power of an odd prime. We show that when $|X|$ is odd and divisible by at least two primes, then the diameter of $\mathcal{G}(\mathcal{I}(X))$ is either 4 or 5. In the process, we obtain several results about semigroups, such as a description of all commutative subsemigroups of $\mathcal{I}(X)$ of maximum order, and analogous results for commutative inverse and commutative nilpotent subsemigroups of $\mathcal{I}(X)$. The paper closes with a number of problems for experts in combinatorics and in group or semigroup theory.

2010 *Mathematics Subject Classification*. 05C25, 20M20, 20M14, 20M18.

Keywords: Commuting graphs of semigroups; symmetric inverse semigroup; commutative semigroups; inverse semigroups; nilpotent semigroups; clique number; diameter.

1 Introduction

The commuting graph of a finite non-abelian group G is a simple graph whose vertices are all non-central elements of G and two distinct vertices x, y are adjacent if $xy = yx$. Commuting graphs of various groups have been studied in terms of their properties (such as connectivity or diameter), for example in [8, 10, 20, 35]. They have also been used as a tool to prove group theoretic results, for example in [9, 33, 34].

For the particular case of the commuting graph of the finite symmetric group $\mathrm{Sym}(X)$, it has been proved [20] that its diameter is ∞ when $|X|$ or $|X|-1$ is a prime, and is at most 5 otherwise. It has been claimed [13] that if neither $|X|$ nor $|X|-1$ is a prime, then the diameter of $\mathcal{G}(\mathrm{Sym}(X))$ is exactly 5. The claim is correct but the proof contains a gap (see the end of Section 6). The clique number of $\mathcal{G}(\mathrm{Sym}(X))$ follows from the classification of the maximum order abelian subgroups of $\mathrm{Sym}(X)$ [11, 27]. In addition, the conjecture that there exists a common upper bound of the diameters of the (connected) commuting graphs of finite groups has recently been proved false [18].

The concept of the commuting graph carries over to semigroups. Suppose S is a finite non-commutative semigroup with center $Z(S)=\{a\in S : ab=ba \text{ for all } b\in S\}$. The *commuting graph* of S, denoted $\mathcal{G}(S)$, is the simple graph (that is, an undirected graph with no multiple edges or loops) whose vertices are the elements of $S-Z(S)$ and whose edges are the sets $\{a,b\}$ such that a and b are distinct vertices with $ab=ba$.

In 2011, Kinyon and the first and third author [6] initiated the study of the commuting graphs of (non-group) semigroups. They calculated the diameters of the ideals of the semigroup $T(X)$ of full transformations on a finite set X [6, Theorems 2.17 and 2.22], and for every natural number n, constructed a semigroup of diameter n [6, Theorem 4.1]. (The latter result shows that, in analogy with groups [18], there is no common upper bound of the diameters of the (connected) commuting graphs of finite semigroups.) Finally, the study of the commuting graphs of semigroups led to the solution of a longstanding open problem in semigroup theory [6, Proposition 5.3].

The goal of this paper is to extend to the finite symmetric inverse semigroups part of the research already carried out for the finite symmetric groups. The *symmetric inverse semigroup* $\mathcal{I}(X)$ on a set X is the semigroup whose elements are the partial injective transformations on X (one-to-one functions whose domain and image are included in X) and whose multiplication is the composition of functions. We will write functions on the right (xf rather than $f(x)$)) and compose from left to right ($x(fg)$ rather than $f(g(x))$. The semigroup $\mathcal{I}(X)$ is universal for the class of inverse semigroups since every inverse semigroup can be embedded in some $\mathcal{I}(X)$ [19, Theorem 5.1.7]. This is analogous to the fact that every group can be embedded in some symmetric group $\mathrm{Sym}(X)$ of permutations on X. We note that $\mathcal{I}(X)$ contains an identity (the transformation that fixes every element of X) and a zero (the transformation whose domain and image are empty). The class of inverse semigroups is arguably the second most important class of semigroups, after groups, because inverse semigroups have applications and provide motivation in other areas of study, for example, differential geometry and physics [28, 31].

Various subsemigroups of the finite symmetric inverse semigroup $\mathcal{I}(X)$ have been studied. One line of research in this area has been the determination of subsemigroups of $\mathcal{I}(X)$ of a given type that are either maximal (with respect to inclusion) or largest (with respect to order). (See, for example, [3, 15, 38, 39].)

In 1989, Burns and Goldsmith [11] obtained a complete classification of the abelian subgroups of maximum order of the symmetric group $\mathrm{Sym}(X)$, where X is a finite set. These abelian subgroups are of three different types depending on the value of n modulo 3, where $n=|X|$. We extend this result to the commutative subsemigroups of $\mathcal{I}(X)$ of maximum order (Theorem 5.3). We also determine the maximum order commutative inverse subsemigroups of $\mathcal{I}(X)$ (Theorem 3.2) and the maximum order commutative nilpotent subsemigroups of $\mathcal{I}(X)$ (Theorem 4.19). As a corollary of Theorem 5.3, we obtain the clique number of the commuting graph of $\mathcal{I}(X)$ (Corollary 6.1).

We also find the diameters of the commuting graphs of the proper ideals of $\mathcal{I}(X)$ (Theorem 6.7), the diameter of $\mathcal{G}(\mathcal{I}(X))$ when $n=|X|$ is even (Theorem 6.12) and when n is a power of an odd prime (Theorem 6.17), and establish that the diameter of $\mathcal{G}(\mathcal{I}(X))$ is 4 or 5 when n is odd and divisible by at least two distinct primes (Proposition 6.13). The diameter results extend to $\mathcal{G}(\mathcal{I}(X))$ the results obtained for $\mathcal{G}(\mathrm{Sym}(X))$ by Iranmanesh and Jafarzadeh [20] and Dolžan and Oblak [13]. (However, see our discussion at the end of Section 6 regarding a problem with Dolžan and Oblak's proof.) We conclude the paper with some problems that we believe will be of interest for mathematicians working in combinatorics and semigroup or group theory (Section 7).

The concept of the commuting graph of a transformation semigroup is central for associative algebras since, in a sense, the study of associativity is the study of commuting transformations and centralizers [7]. This paper builds upon the results on centralizers of transformations in general and of partial injective transformations in particular [2, 4, 5, 21, 22, 23, 24, 25, 26, 29].

Throughout this paper, we fix a finite set X and reserve n to denote the cardinality of X. To simplify the language, we will sometimes say "semigroup in $\mathcal{I}(X)$" to mean "subsemigroup of $\mathcal{I}(X)$." We will denote the identity in $\mathcal{I}(X)$ by 1 and the zero in $\mathcal{I}(X)$ by 0.

2 Commuting Elements of $\mathcal{I}(X)$

In this section, we collect some results about commuting transformations in $\mathcal{I}(X)$ that will be needed in the subsequent sections.

Let S be a semigroup with zero. An element $a \in S$ is called a *nilpotent* if $a^p = 0$ for some positive integer p; the smallest such p is called the *index* of a. We say that S is a *nilpotent semigroup* if every element of S is a nilpotent. A special type of a nilpotent semigroup is a *null semigroup* in which $ab = 0$ for all $a, b \in S$. Note that every nonzero nilpotent in a null semigroup has index 2. We say that S is a *null monoid* if it contains an identity 1 and $ab = 0$ for all $a, b \in S$ such that $a, b \neq 1$. Clearly, all null semigroups and all null monoids are commutative.

For $\alpha \in \mathcal{I}(X)$, we denote by $\mathrm{dom}(\alpha)$ and $\mathrm{im}(\alpha)$ the domain and image of α, respectively. The *rank* of α is the cardinality of $\mathrm{im}(\alpha)$ (which is the same as the cardinality of $\mathrm{dom}(\alpha)$ since α is injective). The union $\mathrm{span}(\alpha) = \mathrm{dom}(\alpha) \cup \mathrm{im}(\alpha)$ will be called the *span* of α.

Let $\alpha, \beta \in \mathcal{I}(X)$. We say that β is *contained* in α (or α *contains* β) if $\mathrm{dom}(\beta) \subseteq \mathrm{dom}(\alpha)$ and $x\beta = x\alpha$ for all $x \in \mathrm{dom}(\beta)$. We say that α and β in $\mathcal{I}(X)$ are *completely disjoint* if $\mathrm{span}(\alpha) \cap \mathrm{span}(\beta) = \emptyset$. Let $M = \{\gamma_1, \ldots, \gamma_k\}$ be a set of pairwise completely disjoint elements of $\mathcal{I}(X)$. The *join* of the elements of M, denoted $\gamma_1 \sqcup \cdots \sqcup \gamma_k$, is the element α of $\mathcal{I}(X)$ whose domain is $\mathrm{dom}(\gamma_1) \cup \ldots \cup \mathrm{dom}(\gamma_k)$ and whose values are defined by $x\alpha = x\gamma_i$, where γ_i is the (unique) element of M such that $x \in \mathrm{dom}(\gamma_i)$. If $M = \emptyset$, we define the join to be 0. Let $x_0, x_1, \ldots, x_k$ be pairwise distinct elements of X.

- A *cycle* of length k ($k \geq 1$), written $(x_0\, x_1 \ldots x_{k-1})$, is an element $\rho \in \mathcal{I}(X)$ with $\mathrm{dom}(\rho) = \{x_0, x_1, \ldots, x_{k-1}\}$, $x_i\rho = x_{i+1}$ for all $0 \leq i < k-1$, and $x_{k-1}\rho = x_0$.
- A *chain* of length k ($k \geq 1$), written $[x_0\, x_1 \ldots x_k]$, is an element $\tau \in \mathcal{I}(X)$ with $\mathrm{dom}(\tau) = \{x_0, x_1, \ldots, x_{k-1}\}$ and $x_i\tau = x_{i+1}$ for all $0 \leq i \leq k-1$.

The following decomposition result is given in [29, Theorem 3.2].

Proposition 2.1. *Let $\alpha \in \mathcal{I}(X)$ with $\alpha \neq 0$. Then there exist unique sets $\Gamma = \{\rho_1, \ldots, \rho_k\}$ of cycles and $\Omega = \{\tau_1, \ldots, \tau_m\}$ of chains such that the transformations in $\Gamma \cup \Omega$ are pairwise completely disjoint and $\alpha = \rho_1 \sqcup \cdots \sqcup \rho_k \sqcup \tau_1 \sqcup \cdots \sqcup \tau_m$.*

Let $\alpha = \rho_1 \sqcup \cdots \sqcup \rho_k \sqcup \tau_1 \sqcup \cdots \sqcup \tau_m$ as in Proposition 2.1. Note that every ρ_i and every τ_j is contained in α. Moreover, for every integer $p > 0$, $\alpha^p = \rho_1^p \sqcup \cdots \sqcup \rho_k^p \sqcup \tau_1^p \sqcup \cdots \sqcup \tau_m^p$. For example, if

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & - \end{pmatrix} = (1\,2\,3\,4) \sqcup [5\,6\,7\,8] \in \mathcal{I}(\{1, 2, \ldots, 8\}),$$

then $\alpha^2 = (1\,3) \sqcup (2\,4) \sqcup [5\,7] \sqcup [6\,8]$, $\alpha^3 = (1\,4\,3\,2) \sqcup [5\,8]$, and $\alpha^4 = (1) \sqcup (2) \sqcup (3) \sqcup (4)$.

Let $\alpha \in \mathcal{I}(X)$. Then:

- $\alpha \in \mathrm{Sym}(X)$ if and only if $\alpha = \rho_1 \sqcup \cdots \sqcup \rho_k$ is a join of cycles and $\cup_{i=1}^k \mathrm{dom}(\rho_i) = X$. The join $\alpha = \rho_1 \sqcup \cdots \sqcup \rho_k$ is equivalent to the cycle decomposition of α in group theory. Note that a cycle $(x_0\, x_1 \ldots x_{t-1})$ differs from the corresponding cycle in $\mathrm{Sym}(X)$ in that the former is undefined for every $x \in X - \{x_0, x_1, \ldots, x_{t-1}\}$, while the latter fixes every such x.
- α is a nilpotent if and only if $\alpha = \tau_1 \sqcup \cdots \sqcup \tau_m$ is a join of chains; and $\alpha^2 = 0$ if and only if $\alpha = [x_1\, y_1] \sqcup \cdots \sqcup [x_m, y_m]$ is a join of chains of length 1, where we agree that $\alpha = 0$ if $m = 0$.

The following proposition has been proved in [29, Theorem 10.1].

Proposition 2.2. *Let $\alpha, \beta \in \mathcal{I}(X)$. Then $\alpha\beta = \beta\alpha$ if and only if the following conditions are satisfied:*

(1) *If $\rho = (x_0\, x_1 \ldots x_{k-1})$ is a cycle in α such that some $x_i \in \mathrm{dom}(\beta)$, then every $x_j \in \mathrm{dom}(\beta)$ and there exists a cycle $\rho' = (y_0\, y_1 \ldots y_{k-1})$ in α (of the same length as ρ) such that*

$$x_0\beta = y_j,\ x_1\beta = y_{j+1}, \ldots, x_{k-1}\beta = y_{j+k-1},$$

where $j \in \{0, 1, \ldots, k-1\}$ and the subscripts on the y_is are calculated modulo k;

(2) *If $\tau = [x_0\, x_1 \ldots x_k]$ is a chain in α such that some $x_i \in \mathrm{dom}(\beta)$, then either $\mathrm{dom}(\beta) \cap \{x_0, x_1, \ldots, x_k\} = \{x_0\}$ and $x_0\beta \notin \mathrm{span}(\alpha)$ or there are $p \in \{0, 1, \ldots, k\}$ and a chain $\tau' = [y_0\, y_1 \ldots y_m]$ in α, with $m \geq p$, such that $\mathrm{dom}(\beta) \cap \{x_0, x_1, \ldots, x_k\} = \{x_0, x_1, \ldots, x_p\}$ and*

$$x_0\beta = y_{m-p},\ x_1\beta = y_{m-p+1}, \ldots, x_p\beta = y_m;$$

(3) *If $x \notin \mathrm{span}(\alpha)$ and $x \in \mathrm{dom}(\beta)$, then either $x\beta \notin \mathrm{span}(\alpha)$ or there exists a chain $\tau' = [y_0\, y_1 \ldots y_m]$ in α such that $x\beta = y_m$.*

The way to remember Proposition 2.2 is that $\alpha\beta = \beta\alpha$ if and only if β maps cycles in α onto cycles in α of the same length, and it maps initial segments of chains in α onto terminal segments of chains in α.

An element $\varepsilon \in \mathcal{I}(X)$ is an idempotent ($\varepsilon\varepsilon = \varepsilon$) if and only if $\varepsilon = (x_1) \sqcup (x_2) \sqcup \cdots \sqcup (x_k)$ is a join of cycles of length 1; and $\sigma \in \mathcal{I}(X)$ is a permutation on X if and only if $\mathrm{dom}(\sigma) = X$ and σ is a join of cycles. For a function $f : A \to B$ and $A_0 \subseteq A$, we denote by $f|_{A_0}$ the restriction of f to A_0.

The following lemma will be important in our inductive arguments in Sections 3 and 5.

Lemma 2.3. *Suppose $\gamma \in \mathcal{I}(X)$ is either an idempotent such that $\gamma \notin \{0, 1\}$, or a permutation on X such that not all cycles in γ have the same length. Then there is a partition $\{A, B\}$ of X such that $\beta|_A \in \mathcal{I}(A)$ and $\beta|_B \in \mathcal{I}(B)$ for all $\beta \in \mathcal{I}(X)$ such that $\gamma\beta = \beta\gamma$.*

Proof. Suppose $\gamma = (x_1) \sqcup (x_2) \sqcup \cdots \sqcup (x_k) \in \mathcal{I}(X)$ is an idempotent such that $\gamma \notin \{0, 1\}$. Let $A = \mathrm{dom}(\gamma) = \{x_1, x_2, \ldots, x_k\}$ and $B = X - A$. Then $A \neq \emptyset$ (since $\gamma \neq 0$), $B \neq \emptyset$ (since $\gamma \neq 1$), and $A \cap B = \emptyset$. Note that $B = X - \mathrm{span}(\gamma)$. Let $\beta \in \mathcal{I}(X)$ be such that $\gamma\beta = \beta\gamma$. Let $x_i \in A$ and $y \in B$ be such that $x_i, y \in \mathrm{dom}(\beta)$. Then $x_i\beta = x_j \in A$ by (1) of Proposition 2.2, and $y\beta \in B$ by (3) of Proposition 2.2. Hence $\beta|_A \in \mathcal{I}(A)$ and $\beta|_B \in \mathcal{I}(B)$.

Suppose $\gamma \in \mathcal{I}(X)$ is a permutation on X such that not all cycles in γ have the same length. Select any cycle ρ in γ and let k be the length of ρ. Let

$$A = \{x \in X : x \in \mathrm{span}(\rho') \text{ for some cycle } \rho' \text{ in } \gamma \text{ of length } k\}$$

and let $B = X - A$. Then $A \neq \emptyset$ (since ρ is a cycle in γ of length k), $B \neq \emptyset$ (since not all cycles in γ have length k), and $A \cap B = \emptyset$. Let $\beta \in \mathcal{I}(X)$ be such that $\gamma\beta = \beta\gamma$. Let $x \in A$ and $y \in B$ be such that $x, y \in \mathrm{dom}(\beta)$. Then $x\beta \in A$ and $y\beta \in B$ by (1) of Proposition 2.2. Hence $\beta|_A \in \mathcal{I}(A)$ and $\beta|_B \in \mathcal{I}(B)$. □

It is straightforward to prove the following lemma.

Lemma 2.4. *Let $\{A, B\}$ be a partition of X. Suppose $\alpha, \beta \in \mathcal{I}(X)$ are such that $\alpha|_A, \beta|_A \in \mathcal{I}(A)$ and $\alpha|_B, \beta|_B \in \mathcal{I}(B)$. Then:*

(1) *$(\alpha\beta)|_A = (\alpha|_A)(\beta|_A)$ and $(\alpha\beta)|_B = (\alpha|_B)(\beta|_B)$.*

(2) *$\alpha\beta = \beta\alpha$ if and only if $(\alpha|_A)(\beta|_A) = (\beta|_A)(\alpha|_A)$ and $(\alpha|_B)(\beta|_B) = (\beta|_B)(\alpha|_B)$.*

We conclude this section with a lemma that is an immediate consequence of the definition of commutativity.

Lemma 2.5. *For all $\alpha, \beta \in \mathcal{I}(X)$, if $\alpha\beta = \beta\alpha$, then $(\mathrm{im}\,\alpha)\beta \subseteq \mathrm{im}(\alpha)$ and $(\mathrm{dom}(\alpha))\beta^{-1} \subseteq \mathrm{dom}(\alpha)$.*

3 The Largest Commutative Inverse Semigroup in $\mathcal{I}(X)$

In this section, we will prove that the maximum order of a commutative inverse subsemigroup of $\mathcal{I}(X)$ is 2^n, and that the semilattice $E(\mathcal{I}(X))$ of idempotents is the unique commutative inverse subsemigroup of $\mathcal{I}(X)$ of the maximum order (Theorem 3.2).

An element a of a semigroup S is called *regular* if $a = axa$ for some $x \in S$. If all elements of S are regular, we say that S is a *regular semigroup*. An element $a' \in S$ is called an *inverse* of $a \in S$ if $a = aa'a$ and $a' = a'aa'$. Since regular elements are precisely those that have inverses (if $a = axa$ then $a' = xax$ is an inverse of a), we may define a regular semigroup as a semigroup in which each element has an inverse [19, p. 51]. The most extensively studied subclass of the regular semigroups has been the class of inverse semigroups (see [32] and [19, Chapter 5]). A semigroup S is called an *inverse semigroup* if every element of S has exactly one inverse [32, Definition II.1.1]. An alternative definition is that S is an inverse semigroup if it is a regular semigroup and its idempotents (elements $e \in S$ such that $ee = e$) commute [19, Theorem 5.1.1].

A *semilattice* is a commutative semigroup consisting entirely of idempotents. A semilattice can also be defined as a partially ordered set $(S, \leq)$ such that the greatest lower bound $a \wedge b$ exists for all $a, b \in S$. Indeed, if S is a semilattice, then $(S, \leq)$, where $\leq$ is a relation on S defined by $a \leq b$ if $a = ab$, is a poset with $a \wedge b = ab$ for all $a, b \in S$. Conversely, if $(S, \leq)$ is a poset such that $a \wedge b$ exists for all $a, b \in S$, then S with multiplication $ab = a \wedge b$ is a semilattice. (See [19, Proposition 1.3.2].) For a semigroup S, denote by $E(S)$ the set of idempotents of S. The set $E(\mathcal{I}(X))$ is a semilattice, which, viewed as a poset, is isomorphic to the poset $(\mathcal{P}(X), \subseteq)$ of the power set $\mathcal{P}(X)$ under inclusion.

For semigroups S and T, we will write $S \cong T$ to mean that S is isomorphic to T.

Lemma 3.1. *Let S be a commutative semigroup in $\mathcal{I}(X)$. Suppose there is a partition $\{A, B\}$ of X such that $\alpha|_A, \beta|_A \in \mathcal{I}(A)$ and $\alpha|_B, \beta|_B \in \mathcal{I}(B)$ for all $\alpha, \beta \in S$. Let $S_A = \{\alpha|_A : \alpha \in S\}$ and $S_B = \{\alpha|_B : \alpha \in S\}$. Then:*

(1) *S_A is a commutative semigroup in $\mathcal{I}(A)$ and S_B is a commutative semigroup in $\mathcal{I}(B)$.*

(2) *If S is an inverse semigroup, then S_A and S_B are inverse semigroups.*

(3) *If S is a maximal commutative semigroup in $\mathcal{I}(X)$, then $S \cong S_A \times S_B$.*

Proof. To prove (1), first note that S_A is a subset of $\mathcal{I}(A)$. It is closed under multiplication since for all $\alpha, \beta \in S$, we have $\alpha\beta \in S$, and so, by Lemma 2.4, $(\alpha|_A)(\beta|_A) = (\alpha\beta)|_A \in S_A$. Finally, S_A is commutative by Lemma 2.4 and the fact that S is commutative. The proof for S_B is the same.

Statement (2) follows from the well-known fact that homomorphic images of inverse semigroups are inverse semigroups themselves.

To prove (3), suppose that S is a maximal commutative semigroup in $\mathcal{I}(X)$. Define a function $\phi : S \to S_A \times S_B$ by $\alpha\phi = (\alpha|_A, \alpha|_B)$. Then ϕ is a homomorphism since for all $\alpha, \beta \in S$,

$$(\alpha\beta)\phi = ((\alpha\beta)|_A, (\alpha\beta)|_B) = ((\alpha|_A)(\beta|_A), (\alpha|_B)(\beta|_B)) = (\alpha|_A, \alpha|_B)(\beta|_A, \beta|_B) = (\alpha\phi)(\beta\phi).$$

Further, for all $\alpha, \beta \in S$, $(\alpha|_A, \alpha|_B) = (\beta|_A, \beta|_B)$ implies $\alpha = \beta$ (since $\{A, B\}$ is a partition of X). Thus ϕ is one-to-one. Let $(\sigma, \mu) \in S_A \times S_B$. Then $\sigma = \alpha|_A$ and $\mu = \beta|_B$ for some $\alpha, \beta \in S$. Define $\gamma \in \mathcal{I}(X)$ by $\gamma|_A = \alpha|_A$ and $\gamma|_B = \beta|_B$. Let $\delta \in S$. Then $\alpha\delta = \delta\alpha$ and $\beta\delta = \delta\beta$, and so, by Lemma 2.4, $(\gamma|_A)(\delta|_A) = (\alpha|_A)(\delta|_A) = (\delta|_A)(\alpha|_A) = (\delta|_A)(\gamma|_A)$ and $(\gamma|_B)(\delta|_B) = (\beta|_B)(\delta|_B) = (\delta|_B)(\beta|_B) = (\delta|_B)(\gamma|_B)$. Hence $\gamma\delta = \delta\gamma$, which implies that $\gamma \in S$ since S is a maximal commutative semigroup in $\mathcal{I}(X)$. Thus $\gamma\phi = (\gamma|_A, \gamma|_B) = (\alpha|_A, \beta|_B) = (\sigma, \mu)$, and so ϕ is onto. $\square$

A subgroup G of $\mathrm{Sym}(X)$ is called *semiregular* if the identity is the only element of G that fixes any point of X [37]. It is easy to see that G is semiregular if and only if for every $\sigma \in G$, all cycles in σ have the same length. If G is a semiregular subgroup of $\mathrm{Sym}(X)$ with $n = |X|$, then the order of G divides n [37, Proposition 4.2], and so $|G| \leq n$.

We can now prove our main theorem in this section.

Theorem 3.2. *Let X be a finite set with $n \geq 1$ elements. Then:*

(1) *If S is a commutative inverse subsemigroup of $\mathcal{I}(X)$, then $|S| \le 2^n$.*

(2) *The semilattice $E(\mathcal{I}(X))$ is the unique commutative inverse subsemigroup of $\mathcal{I}(X)$ of order 2^n.*

Proof. We will prove (1) and (2) simultaneously by induction on n. The statements are certainly true for $n = 1$. Let $n \ge 2$ and suppose that (1) and (2) are true for every symmetric inverse semigroup on a set with cardinality less than n.

Let S be a maximal commutative inverse semigroup in $\mathcal{I}(X)$. Let $G = S \cap \mathrm{Sym}(X)$ and $T = S - G$. If G is a semiregular subgroup of $\mathrm{Sym}(X)$ and $T = \{0\}$, then $|S| = |G| + 1 \le n + 1 < 2^n$ (since $n \ge 2$).

Suppose G is not semiregular or $T \ne \{0\}$. In the former case, G (and so S) contains a permutation σ such that not all cycles of σ are of the same length. Suppose $T \ne \{0\}$. Let $0 \ne \alpha \in T$ and let α' be the inverse of α in S. Then $\alpha = \alpha\alpha'\alpha$ and $\varepsilon = \alpha\alpha'$ is an idempotent. Note that $\varepsilon \ne 1$ (since $\alpha \notin \mathrm{Sym}(X)$) and $\varepsilon \ne 0$ (since $\alpha = \varepsilon\alpha$ and $\alpha \ne 0$).

Thus, in either case, by Lemmas 2.3 and 3.1, there is a partition $\{A, B\}$ of X such that $S \cong S_A \times S_B$, where S_A is a commutative inverse semigroup in $\mathcal{I}(A)$ and S_B is a commutative inverse semigroup in $\mathcal{I}(B)$. Let $k = |A|$ and $m = |B|$. Then $1 \le k, m < n$ with $k + m = n$, and so, by the inductive hypothesis, $|S| = |S_A| \cdot |S_B| \le 2^k \cdot 2^m = 2^{k+m} = 2^n$.

Suppose that $S \ne E(\mathcal{I}(X))$. Then, since S is a maximal commutative inverse semigroup in $\mathcal{I}(X)$, S is not included in $E(\mathcal{I}(X))$, and so it is not a semilattice. It follows that $S_A \ne E(\mathcal{I}(A))$ or $S_B \ne E(\mathcal{I}(B))$ (since $S \cong S_A \times S_B$ and the direct product of two semilattices is a semilattice). We may assume that $S_A \ne E(\mathcal{I}(A))$. By the inductive hypothesis again, $\mathcal{I}(A) < 2^k$, and so $|S| = |S_A| \cdot |S_B| < 2^k \cdot 2^m = 2^{k+m} = 2^n$.

We have proved that $|S| \le 2^n$ and if $S \ne E(\mathcal{I}(X))$ then $|S| < 2^n$. Statements (1) and (2) follow. $\square$

4 The Largest Commutative Nilpotent Semigroups in $\mathcal{I}(X)$

In this section, we consider nilpotent semigroups in $\mathcal{I}(X)$, that is, the semigroups whose every element is a nilpotent. We determine the maximum order of a commutative nilpotent semigroup in $\mathcal{I}(X)$, and describe the commutative nilpotent semigroups in $\mathcal{I}(X)$ of the maximum order (Theorem 4.19).

Definition 4.1. Let X be a set with $n \ge 2$ elements and let $\{K, L\}$ be a partition of X. Denote by $S_{K,L}$ the subset of $\mathcal{I}(X)$ consisting of all nilpotents of the form $[x_1\, y_1] \sqcup \cdots \sqcup [x_r\, y_r]$, where $x_i \in K$, $y_i \in L$, and $0 \le r \le \min\{|K|, |L|\}$.

For example, let $n = 4$, $X = \{1, 2, 3, 4\}$, $K = \{1, 2\}$, and $L = \{3, 4\}$. Then

$$S_{K,L} = \{0, [1\,3], [1\,4], [2\,3], [2\,4], [1\,3] \sqcup [2\,4], [1\,4] \sqcup [2\,3]\}.$$

Lemma 4.2. *Any set $S_{K,L}$ from* Definition 4.1 *is a null semigroup of order $\sum_{r=0}^{m} \binom{m}{r}\binom{n-m}{r} r!$, where $m = \min\{|K|, |L|\}$.*

Proof. Let $\alpha, \beta \in S_{K,L}$ and suppose $x \in \mathrm{dom}(\alpha)$. Then $x\alpha \notin \mathrm{dom}(\beta)$ (since $x\alpha \in L$), and so $x \notin \mathrm{dom}(\alpha\beta)$. It follows that $\alpha\beta = 0$.

Let $m = \min\{|K|, |L|\}$. Suppose $m = |K|$, so $|L| = n - m$. Let $\alpha = [x_1\, y_1] \sqcup \cdots \sqcup [x_r\, y_r]$ be a transformation in $S_{K,L}$ of rank r. Then, clearly, $0 \le r \le m$. The domain of α can be selected in $\binom{m}{r}$ ways, the image in $\binom{n-m}{r}$ ways, and the domain can be mapped to the image in $r!$ ways. It follows that $S_{K,L}$ contains $\binom{m}{r}\binom{n-m}{r} r!$ transformations of rank r, and so $|S| = \sum_{r=0}^{m} \binom{m}{r}\binom{n-m}{r} r!$. The result is also true when $m = |L|$ since $S_{K,L}$ has the same order as $S_{L,K}$. $\square$

Definition 4.3. A null semigroup $S_{K,L}$ from Definition 4.1 such that $|K| = \lfloor \frac{n}{2} \rfloor$ and $L = n - \lfloor \frac{n}{2} \rfloor$, or vice versa, will be called a *balanced null semigroup*. By Lemma 4.2, any balanced null semigroup has order

$$\lambda_n = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor}{r} \binom{n - \lfloor \frac{n}{2} \rfloor}{r} r!. \tag{4.1}$$

If $S_{K,L}$ is a balanced null semigroup, then the monoid $S_{K,L} \cup \{1\}$ will be called a *balanced null monoid*.

Note that λ_n from (4.1) is also defined for $n = 1$, and that $\lambda_1 = 1$ is the order of the trivial nilpotent semigroup $S = \{0\}$.

Our objective is to prove that the maximum order of a commutative nilpotent subsemigroup of $\mathcal{I}(X)$ is λ_n, and that, if $n \notin \{1, 3\}$, the balance null semigroups $S_{K,L}$ are the only commutative nilpotent subsemigroups of $\mathcal{I}(X)$ of order λ_n (Theorem 4.19). We will need some combinatorial lemmas, which we present now.

Lemma 4.4. *For every $n \geq 4$, $\lambda_n = \lambda_{n-1} + \lfloor \frac{n}{2} \rfloor \lambda_{n-2}$.*

Proof. Let $m = \lfloor \frac{n}{2} \rfloor$. Consider a balanced null semigroup $S_{K,L}$, where $|K| = n - m$ and $|L| = m$. Then $\lambda_n = |S_{K,L}|$. Fix $x \in K$. Then $S_{K,L} = S_1 \cup S_2$, where $S_1 = \{\alpha \in S_{K,L} : x \notin \mathrm{dom}(\alpha)\}$ and $S_2 = \{\alpha \in S_{K,L} : x \in \mathrm{dom}(\alpha)\}$. Then $S_1 = S_{K-\{x\},L} \subseteq \mathcal{I}(X - \{x\})$ with $|K - \{x\}| = \lfloor \frac{n-1}{2} \rfloor$ and $|L| = (n-1) - \lfloor \frac{n-1}{2} \rfloor$. Thus $|S_1| = \lambda_{n-1}$.

Let $\alpha \in S_2$. Then $\alpha = [x\,y] \sqcup \beta$, where $y \in L$ and $\beta \in S_{K-\{x\},L-\{y\}} \subseteq \mathcal{I}(X - \{x, y\})$ with $|K - \{x\}| = (n-2) - \lfloor \frac{n-2}{2} \rfloor$ and $|L| = \lfloor \frac{n-2}{2} \rfloor$. For a fixed $y \in L$, the mapping $\alpha = [x\,y] \sqcup \beta \to \beta$ is a bijection from $\{\alpha \in S_2 : x\alpha = y\}$ to $S_{K-\{x\},L-\{y\}}$. Thus, since there are $|L| = m$ choices for y, we have $|S_2| = m|S_{K-\{x\},L-\{y\}}| = m\lambda_{n-2}$. Hence

$$\lambda_n = |S_{K,L}| = |S_1| + |S_2| = \lambda_{n-1} + \left\lfloor \frac{n}{2} \right\rfloor \lambda_{n-2}$$

since $m = \lfloor \frac{n}{2} \rfloor$. $\square$

Lemma 4.5. *Let a, b be integers such that $1 \leq a, b \leq n$, $a < \lfloor \frac{n}{2} \rfloor$, and $b = n - a$. Then*

$$\sum_{r=0}^{a} \binom{a}{r}\binom{b}{r} r! < \sum_{r=0}^{a+1} \binom{a+1}{r}\binom{b-1}{r} r!.$$

Proof. Since $a < \lfloor \frac{n}{2} \rfloor$, and $b = n - a$, we have $a < b$ and hence $a + 1 \leq b$. Let $0 \leq r \leq a$. Then

$$\begin{aligned}
-b \leq -a - 1 &\Rightarrow -br \leq -ar - r \\
&\Rightarrow ba + b - br \leq ba + b - ar - r \\
&\Rightarrow b(a + 1 - r) \leq (b - r)(a + 1) \\
&\Rightarrow \frac{b}{b - r} \leq \frac{(a + 1)}{(a + 1 - r)} \\
&\Rightarrow \frac{b}{(b - r)(a - r)!(b - r - 1)!} \leq \frac{(a + 1)}{(a + 1 - r)(a - r)!(b - r - 1)!} \\
&\Rightarrow \frac{b}{(b - r)!(a - r)!} \leq \frac{(a + 1)}{(a + 1 - r)!(b - r - 1)!} \\
&\Rightarrow \frac{a!(b - 1)!b}{(b - r)!(a - r)!} \leq \frac{a!(b - 1)!(a + 1)}{(a + 1 - r)!(b - r - 1)!} \\
&\Rightarrow \frac{a!b!}{r!r!(b - r)!(a - r)!} \leq \frac{(b - 1)!(a + 1)!}{r!r!(a + 1 - r)!(b - r - 1)!} \\
&\Rightarrow \frac{a!}{r!(a - r)!}\frac{b!}{r!(b - r)!} \leq \frac{(a + 1)!}{r!(a + 1 - r)!}\frac{(b - 1)!}{r!(b - r - 1)!} \\
&\Rightarrow \binom{a}{r}\binom{b}{r} \leq \binom{a+1}{r}\binom{b-1}{r} \\
&\Rightarrow \binom{a}{r}\binom{b}{r} r! \leq \binom{a+1}{r}\binom{b-1}{r} r!
\end{aligned}$$

Hence $\sum_{r=0}^{a} \binom{a}{r}\binom{b}{r} r! \leq \sum_{r=0}^{a} \binom{a+1}{r}\binom{b-1}{r} r!$, and so $\sum_{r=0}^{a} \binom{a}{r}\binom{b}{r} r! < \sum_{r=0}^{a+1} \binom{a+1}{r}\binom{b-1}{r} r!$. $\square$

Lemma 4.6. *Let $n > 10$. Then:*

(1) $\lambda_n + 1 > 2(\lambda_{n-1} + 1)$.

(2) *For every positive integer k such that $k \geq 10$ and $n - k \geq 10$,*

$$\lambda_n + 1 > (\lambda_k + 1)(\lambda_{n-k} + 1).$$

Proof. To prove (1), fix $a \in X$ and consider a partition $\{A, B\}$ of $X - \{a\}$ such that $|A| = \left\lfloor \frac{n-1}{2} \right\rfloor$ and $|B| = (n-1) - |A|$. Note that $\lambda_n = |S_{A\cup\{a\},B}|$ and $\lambda_{n-1} = |S_{A,B}|$. We will consider two cases.

Case 1. n is even.

In this case $|B| = |A| + 1$, hence for every $\alpha \in S_{A,B}$, we can select an element $y_\alpha \in B - \mathrm{im}(\alpha)$. Then the mapping $\phi : S_{A,B} \to S_{A\cup\{a\},B}$ defined by $\alpha\phi = \alpha \sqcup [a\, y_\alpha]$ is one-to-one with $\mathrm{im}(\phi) \subseteq S_{A\cup\{a\},B} - S_{A,B}$. Since $n > 10$, we can select $y_1, y_2 \in B$ such that $y_1, y_2 \neq y_\alpha$ where $\alpha = 0$. Then $[a\, y_1], [a\, y_2] \in S_{A\cup\{a\},B} - (S_{A,B} \cup \mathrm{im}(\phi))$, which implies

$$\lambda_n = |S_{A\cup\{a\},B}| \geq |S_{A,B}| + |\,\mathrm{im}(\phi)| + |\{[a\, y_1], [a\, y_2]\}| = \lambda_{n-1} + \lambda_{n-1} + 2 > 2\lambda_{n-1} + 1.$$

Case 2. n is odd.

Let $m = |A| = |B| = \frac{n-1}{2}$. By direct calculations, $\lambda_{11} = 4051$ and $2\lambda_{10} + 1 = 3093$. So (1) is true for $n = 11$. Suppose $n \geq 13$ and note that $m \geq 6$. Denote by J_{m-2} the set of transformations of $S_{A,B}$ of rank at most $m - 2$ and note that

$$|J_{m-2}| = \lambda_{n-1} - m! - \binom{m}{1}^2 (m-1)! = \lambda_{n-1} - (m+1)m!$$

(since $S_{A,B}$ has $m!$ transformations of rank m, and $\binom{m}{1}^2(m-1)!$ transformations of rank $m - 1$). For every $\alpha \in J_{m-2}$, select two distinct elements $y_\alpha, z_\alpha \in B - \mathrm{im}(\alpha)$ (possible since $|B| = m$ and $\mathrm{rank}(\alpha) \leq m - 2$). Then the mappings $\phi, \psi : J_{m-2} \to S_{A\cup\{a\},B}$ defined by $\alpha\phi = \alpha \sqcup [a\, y_\alpha]$ and $\alpha\psi = \alpha \sqcup [a\, z_\alpha]$ are one-to-one with $\mathrm{im}(\phi) \cup \mathrm{im}(\psi) \subseteq S_{A\cup\{a\},B} - S_{A,B}$ and $\mathrm{im}(\phi) \cap \mathrm{im}(\psi) = \emptyset$. Therefore,

$$\begin{aligned}
\lambda_n &= |S_{A\cup\{a\},B}| \geq |S_{A,B}| + |\,\mathrm{im}(\phi)| + |\,\mathrm{im}(\psi)| \\
&= \lambda_{n-1} + 2(\lambda_{n-1} - (m+1)m!) = 2\lambda_{n-1} - 2(m+1)m! + \lambda_{n-1} \\
&> 2\lambda_{n-1} - 2(m+1)m! + \left(\binom{m}{0}^2 m! + \binom{m}{1}^2 (m-1)! + \binom{m}{2}^2 (m-2)! \right) \\
&= 2\lambda_{n-1} - 2(m+1)m! + \left(m! + m \cdot m! + \frac{m(m-1)}{4} m! \right) \\
&= 2\lambda_{n-1} + m! \left(-2m - 2 + 1 + m + \frac{m(m-1)}{4} \right) \\
&= 2\lambda_{n-1} + \frac{m!}{4} \left(m^2 - 5m - 4 \right) > 2\lambda_{n-1} + 1,
\end{aligned}$$

where the first strong inequality follows from the fact that $\lambda_{n-1} = |S_{A,B}|$, $m \geq 6$, and the expression $\binom{m}{0}^2 m! + \binom{m}{1}^2 (m-1)! + \binom{m}{2}^2 (m-2)!$ only counts the transformations in $S_{A,B}$ of ranks m, $m - 1$, and $m - 2$; and the last strong inequality inequality follows from the fact that for $m \geq 6$, $\frac{m!}{4} \geq 180$ and $m^2 - 5m - 4 \geq 2$.

To prove (2), suppose $k \geq 10$ and $n - k \geq 10$. We may assume that $k \leq n - k$. Consider a partition $\{A, B, C, D\}$ of X such that

$$|A| = \left\lfloor \frac{n-k}{2} \right\rfloor, \ |B| = (n-k) - |A|, \ |D| = \left\lfloor \frac{k}{2} \right\rfloor, \ |C| = k - |D|.$$

Then, either $|A| + |C| = \left\lfloor \frac{n}{2} \right\rfloor$ or $|B| + |D| = \left\lfloor \frac{n}{2} \right\rfloor$, and so $\lambda_n = |S_{A\cup C, B\cup D}|$, $\lambda_{n-k} = |S_{A,B}|$ and $\lambda_k = |S_{C,D}|$.

Let S be the subsemigroup of $S_{A\cup C,B\cup D}$ consisting of all α such that $\alpha|_A \in S_{A,B}$ and $\alpha|_C \in S_{C,D}$. We can construct a bijection between S and $S_{A,B}\times S_{C,D}$ as in the proof of Lemma 3.1, hence $|S| = \lambda_{n-k}\lambda_k$. Since the inequality in (2) is equivalent to $\lambda_n > \lambda_k\lambda_{n-k} + \lambda_k + \lambda_{n-k}$, it suffices to construct more then $2\lambda_{n-k} \geq \lambda_{n-k} + \lambda_k$ elements of $S_{A\cup C,B\cup D} - S$. We will consider two cases.

Case 1. $n-k$ is odd.

In this case $|B| = |A|+1$, so for each $\alpha \in S_{A,B}$, we can select an element $b_\alpha \in B - \operatorname{im}(\alpha)$. Now, for any pair $(c,\alpha) \in C \times S_{A,B}$, let $\alpha_c = \alpha \sqcup [c\, b_\alpha]$. It is clear that $\alpha_c \in S_{A\cup C,B\cup D} - S$ and that the mapping $(\alpha, c) \to \alpha_c$ is one-to-one. Since $k \geq 10$, we have $|C| = k - \lfloor \frac{k}{2} \rfloor \geq 5$. Thus, we have constructed $|C|\cdot|S_{A,B}| \geq 5\lambda_{n-k} > 2\lambda_{n-k}$ elements in $S_{A\cup C,B\cup D} - S$.

Case 2. $n-k$ is even.

Let $m = \frac{n-k}{2}$. Note that for any $\alpha \in S_{A,B}$ of rank smaller then m, we can find $b_\alpha \in B - \operatorname{im}(\alpha)$ and define α_c as in Case 1. This construction yields $|C|(\lambda_{n-k} - m!) \geq 5(\lambda_{n-k} - m!)$ distinct elements of $S_{A\cup C,B\cup D} - S$. Since $m \geq 5$, we have

$$\lambda_{n-k} = |S_{A,B}| > \binom{m}{0}\binom{m}{0}m! + \binom{m}{1}\binom{m}{1}(m-1)! > 2m!,$$

where the first inequality follows from the fact that $\binom{m}{0}\binom{m}{0}m! + \binom{m}{1}\binom{m}{1}(m-1)!$ only counts the elements of $S_{A,B}$ of rank m and $m-1$. Thus $3\lambda_{n-k} > 6m!$, and so

$$|C|(\lambda_{n-k} - m!) \geq 5(\lambda_{n-k} - m!) = 3\lambda_{n-k} + 2\lambda_{n-k} - 5m! > 6m! + 2\lambda_{n-k} - 5m! > 2\lambda_{n-k}.$$

The result follows. $\square$

Lemma 4.7. *If $n \geq 6$, then $\lambda_n > 2\lambda_{n-1}$.*

Proof. If $n > 10$, then $\lambda_n > 2\lambda_{n-1} + 1 > 2\lambda_{n-1}$ by Lemma 4.6. If $6 \leq n \leq 10$, then the result can be checked by direct calculations:

n	2	3	4	5	6	7	8	9	10
λ_n	2	3	7	13	34	73	209	501	1546
$2\lambda_{n-1}$	2	4	6	14	26	68	146	418	1002

$\square$

We begin the proof of Theorem 4.19 with introducing the following notation.

Notation 4.8. Let S be any commutative nilpotent subsemigroup of $\mathcal{I}(X)$. We define the following subset $C = C(S)$ of X:

$$C = \{c \in X : c \in \operatorname{dom}(\alpha) \cap \operatorname{im}(\beta) \text{ for some } \alpha, \beta \in S\}. \tag{4.2}$$

For a fixed $c \in C$, we define

$$A_c = \{a \in X : a\alpha = c \text{ for some } \alpha \in S\},$$
$$B_c = \{b \in X : c\alpha = b \text{ for some } \alpha \in S\}.$$

Note that A_c and B_c are not empty (by the definition of C) and that $A_c \cap B_c = \emptyset$. (Indeed, if $a \in A_c \cap B_c$, then $a\alpha = c$ and $c\beta = a$ for some $\alpha, \beta \in S$, that is, $\alpha = [\ldots a\, c \ldots] \sqcup \cdots$ and $\beta = [\ldots c\, a \ldots] \sqcup \cdots$. It then follows from Proposition 2.2 that $\alpha\beta \neq \beta\alpha$, which is a contradiction.)

In the following lemmas, S is a commutative nilpotent subsemigroup of $\mathcal{I}(X)$ and C is the subset of S defined by (4.2). Our immediate objective is to obtain certain bounds on $|A_c|$ and $|B_c|$ (see Lemma 4.11).

Lemma 4.9. *Let $c \in C$, $a \in A_c$, and $b \in B_c$. Then:*

(1) *There is a unique* $q = q(c, a, b) \in C$ *such that for all* $\alpha \in S$*, if* $a\alpha = c$*, then* $q\alpha = b$.

(2) *For all* $\beta \in S$*, if* $c\beta = b$*, then* $a\beta = q$*, where* $q = q(c, a, b)$ *is the unique element from* (1).

Proof. To prove (1), suppose $\alpha \in S$ with $a\alpha = c$, that is, $\alpha = [\dots a\, c \dots] \sqcup \cdots$. Since $b \in B_c$, $c\beta = b$ for some $\beta \in S$. Since $c \in \mathrm{dom}(\beta)$, Proposition 2.2 implies that $a \in \mathrm{dom}(\beta)$. Let $q = a\beta$. Then $q\alpha = (a\beta)\alpha = (a\alpha)\beta = c\beta = b$. Let $\alpha' \in S$ be such that $a\alpha' = c$. By the foregoing argument, there exists $q' \in X$ such that $a\beta = q'$ and $q'\alpha' = b$. But then $q = a\beta = q'$, so q is unique. Moreover, $q \in C$ since $q \in \mathrm{dom}(\alpha) \cap \mathrm{im}(\beta)$.

To prove (2), suppose $\beta \in S$ with $c\beta = b$. Since $a \in A_c$, $a\alpha = c$ for some $\alpha \in S$. But then, by the proof of (1), $a\beta = q$. $\square$

Lemma 4.10. *Let* $c \in C$*,* $a, a_1, a_2 \in A_c$*, and* $b, b_1, b_2 \in B_c$*. Then:*

(a) *If* $q(c, a, b_1) = q(c, a, b_2)$*, then* $b_1 = b_2$.

(b) *If* $q(c, a_1, b) = q(c, a_2, b)$*, then* $a_1 = a_2$.

Proof. To prove (1), let $q = q(c, a, b_1) = q(c, a, b_2)$. Since $a \in A_c$, there is $\alpha \in S$ such that $a\alpha = c$. But then, by Lemma 4.9, $b_1 = q\alpha = b_2$. The proof of (2) is similar. $\square$

We can now prove the lemma concerning the sizes of A_c and B_c.

Lemma 4.11. *Suppose* $C \neq \emptyset$*. Then, there exists* $c \in C$ *such that one of the following conditions holds:*

(a) $|A_c| \geq 2$ *and* $|B_c| \geq 2$*;*

(b) $|A_c| = 1$ *and* $|B_c| \leq \left\lfloor \frac{n}{2} \right\rfloor$*; or*

(c) $|B_c| = 1$ *and* $|A_c| \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. Suppose to the contrary that for every $c \in C$, none of (a)–(c) holds. Let $c \in C$. Then, since (a) does not hold for c, $|A_c| = 1$ or $|B_c| = 1$.

Suppose $|A_c| = 1$, say $A_c = \{a\}$. Then, since (b) does not hold for c, $|B_c| > \left\lfloor \frac{n}{2} \right\rfloor$. Let $b \in B_c$. We claim that $b \notin C$. Suppose to the contrary that $b \in C$. Construct elements $b_0, b_1, b_2, \dots$ in $C \cap B_c$ as follows. Set $b_0 = b$. Suppose $b_i \in C \cap B_c$ has been constructed ($i \geq 0$). Let b_{i+1} be any element of B_c such that $b_{i+1}\gamma_i = b_i$ for some $\gamma_i \in S$. Then $b_{i+1} \in C$ as $b_{i+1} \in \mathrm{dom}(\gamma_i) \cap B_c = \mathrm{dom}(\gamma_i) \cap \{c\}S$. If such an element b_{i+1} does not exist, stop the construction. Note that the construction must stop after finitely many steps. (Indeed, otherwise, since X is finite, we would have $b_k = b_j$ with $k > j \geq 0$. But then $b_j\gamma = b_k\gamma = b_j$ for $\gamma = \gamma_{k-1}\gamma_{k-2} \cdots \gamma_j \in S$, which is impossible since S consists of nilpotents.) Thus, there exists $i \geq 0$ such that $b_i \in C \cap B_c$ and no element of B_c is mapped to b_i by some transformation in S.

Let $b' = b_i$ and note that $A_{b'} \subseteq X - B_c$. Since $A_c = \{a\}$, $a\alpha = c$ for some $\alpha \in S$. Since $b' \in B_c$, $c\beta = b'$ for some $\beta \in S$. Let $q = q(c, a, b')$. Then, by Lemma 4.9, $q\alpha = b'$, and so $\{c, q\} \subseteq A_{b'}$. If $c \neq q$, then $|A_{b'}| \geq 2$. Suppose $c = q$. Then $a(\alpha\alpha) = c\alpha = q\alpha = b'$, and so $\{c, a\} \subseteq A_{b'}$. But $a \neq c$ (since $a\alpha = c$ and α is a nilpotent), and we again have $|A_{b'}| \geq 2$. On the other hand, since $A_{b'} \subseteq X - B_c$ and $|B_c| > \left\lfloor \frac{n}{2} \right\rfloor$, we have $|A_{b'}| \leq \left\lfloor \frac{n}{2} \right\rfloor$. But $b' \in C$ with $2 \leq |A_{b'}| \leq \left\lfloor \frac{n}{2} \right\rfloor$ contradicts our assumption (see the first sentence of the proof).

The claim has been proved. Hence, no element of B_c is in C, that is, $C \subseteq X - B_c$. Now, by Lemma 4.9, for each $b_i \in B_c$, there exists $q_i = q(c, a, b_i) \in C$ such that $a \in A_{q_i}$ and $b_i \in B_{q_i}$. Moreover, by Lemma 4.10, $q_i \neq q_j$ if $i \neq j$. But this is a contradiction since $|B_c| > \left\lfloor \frac{n}{2} \right\rfloor > |X - B_c| \geq |C|$.

If $|B_c| = 1$, we obtain a contradiction in a similar way. This concludes the proof. $\square$

We continue the proof of Theorem 4.19 by considering two cases. First, we suppose that S is a commutative semigroup of nilpotents such that $C = \emptyset$, that is, there is no $c \in X$ such that $c \in \mathrm{dom}(\alpha) \cap \mathrm{im}(\beta)$ for some $\alpha, \beta \in S$. Note that this implies that each nonzero element of S is a nilpotent of index 2.

Proposition 4.12. *Let X be a set with $n \geq 2$ elements and let $m = \lfloor \frac{n}{2} \rfloor$. Let S be a commutative nilpotent subsemigroup of $\mathcal{I}(X)$ with $C = \emptyset$. Suppose $S \neq S_{K,L}$ for every balanced null semigroup $S_{K,L}$ (see* Definition 4.3*). Then $|S| < \sum_{r=0}^{m} \binom{m}{r}\binom{n-m}{r} r!$.*

Proof. Let $A = \{x \in X : x \in \mathrm{dom}(\alpha)$ for some $\alpha \in S\}$ and $B = X - A$. Since $C = \emptyset$, we have $A \cap \{y \in X : y \in \mathrm{im}(\beta)$ for some $\beta \in S\} = \emptyset$, and so $S \subseteq S_{A,B}$.

Suppose $|A| = m$. Then $S \neq S_{A,B}$ by the assumption, and so, by Lemma 4.2, $|S| < |S_{A,B}| = \sum_{r=0}^{m} \binom{m}{r}\binom{n-m}{r} r!$. Suppose $|A| < m$. Let $a = |A|$ and $b = |B| = n - a$. By Lemma 4.2 again,

$$|S| \leq |S_{A,B}| = \sum_{r=0}^{a} \binom{a}{r}\binom{b}{r} r!.$$

Applying Lemma 4.5 $m - a$ times, we obtain

$$|S| \leq \sum_{r=0}^{a} \binom{a}{r}\binom{b}{r} r! < \sum_{r=0}^{m} \binom{m}{r}\binom{n-m}{r} r!.$$

Suppose $|A| > m$. Consider the semigroup $S' = \{\alpha^{-1} : \alpha \in S\}$ and note that S' is a nilpotent commutative semigroup with $C = C(S') = \emptyset$ and the corresponding set A' included in the original set B. Since $|A'| \leq m$, $|S| = |S'| < \sum_{r=0}^{m} \binom{m}{r}\binom{n-m}{r} r!$ by the foregoing argument. ∎

Second, we suppose that S is a commutative nilpotent subsemigroup of $\mathcal{I}(X)$ such that $C \neq \emptyset$. Note that this is possible only if $n \geq 3$. Fix $c \in C$ that satisfies one of the conditions (1)–(3) from Lemma 4.11. Our objective is to prove that for all $n \geq 3$,

$$|S| \leq \lambda_n = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor}{r}\binom{n - \lfloor \frac{n}{2} \rfloor}{r} r!. \tag{4.3}$$

We will proceed by strong induction on $n = |X|$. Let $n = 3$. Then the maximal commutative nilpotent semigroups in $\mathcal{I}(X)$ are the balanced null semigroups $\{0, [i\,j], [i\,k]\}$ and $\{0, [i\,k], [j\,k]\}$, and the cyclic semigroups $\{0, [i\,j\,k], [i\,k]\}$, where i, j, k are fixed, pairwise distinct, elements of X. Thus (4.3) is true for $n = 3$.

Inductive Hypothesis. Let $n \geq 4$ and suppose that (4.3) is true whenever $3 \leq |X| < n$.

Consider the following subset of S:

$$S_c = \{\alpha \in S : c \in \mathrm{span}(\alpha)\}. \tag{4.4}$$

Then $S - S_c$ is a commutative nilpotent subsemigroup of $\mathcal{I}(X - \{c\})$. If there is no $d \in X - \{c\}$ such that $d \in \mathrm{dom}(\alpha) \cap \mathrm{im}(\beta)$ for some $\alpha, \beta \in S - S_c$, then $|S - S_c| \leq \lambda_{n-1}$ by Proposition 4.12. If such a $d \in X - \{c\}$ exists, then $|S - S_c| \leq \lambda_{n-1}$ by the inductive hypothesis. Thus, at any rate,

$$|S - S_c| \leq \lambda_{n-1}. \tag{4.5}$$

We now want to find a suitable upper bound for the size of S_c (Lemma 4.17). To this end, we will map S_c onto a commutative subset S_c^* of $\mathcal{I}(X - \{c\})$ and analyze the preimages of the elements of S_c^*.

Definition 4.13. For $\alpha \in S_c$ with $c \in \mathrm{im}(\alpha)$, let U_α be the smallest subset of X containing $c\alpha^{-1}$ and closed under all transformations γ^{-1} and $\alpha\delta\alpha^{-1}$, where $\gamma, \delta \in S_c$.

For $\alpha \in S_c$ with $c \in \mathrm{dom}(\alpha)$, let D_α be the smallest subset of X containing c and closed under all transformations γ^{-1} and $\alpha\delta\alpha^{-1}$, where $\gamma, \delta \in S_c$.

For $\alpha \in S_c$, define $\alpha^* \in \mathcal{I}(X - \{c\})$ as follows:

$$\alpha^* = \begin{cases} \alpha|_{X - U_\alpha} & \text{if } c \in \mathrm{im}(\alpha) - \mathrm{dom}(\alpha), \\ \alpha|_{X - D_\alpha} & \text{if } c \in \mathrm{dom}(\alpha) - \mathrm{im}(\alpha), \\ \alpha|_{X - (D_\alpha \cup U_\alpha)} & \text{if } c \in \mathrm{dom}(\alpha) \cap \mathrm{im}(\alpha). \end{cases}$$

Let $S_c^* = \{\alpha^* : \alpha \in S_c\}$ and note that S_c^* is a subset of $\mathcal{I}(X - \{c\})$.

We will need the following lemma about the sets U_α and D_α.

Lemma 4.14. *Let $\alpha, \beta \in S_c$. Then:*

(1) *If $c \in \mathrm{im}(\alpha)$, then $U_\alpha \subseteq \mathrm{dom}(\alpha)$. Moreover, if $c \in \mathrm{im}(\beta)$ and $c\alpha^{-1} = c\beta^{-1}$, then $U_\alpha = U_\beta$ and $x\alpha = x\beta$ for all $x \in U_\alpha$.*

(2) *If $c \in \mathrm{dom}(\alpha)$, then $D_\alpha \subseteq \mathrm{dom}(\alpha)$. Moreover, if $c \in \mathrm{dom}(\beta)$ and $c\alpha = c\beta$, then $D_\alpha = D_\beta$ and $x\alpha = x\beta$ for all $x \in D_\alpha$.*

(3) *If $c \in \mathrm{dom}(\alpha) \cap \mathrm{im}(\alpha)$, then $D_\alpha = U_\alpha$. Moreover, if $c \in \mathrm{im}(\beta)$ and $c\alpha^{-1} = c\beta^{-1}$, then $c \in \mathrm{dom}(\beta)$, $U_\alpha = U_\beta = D_\beta$, and $x\alpha = x\beta$ for all $x \in U_\alpha$. If $c \in \mathrm{dom}(\beta)$ and $c\alpha = c\beta$, then $c \in \mathrm{im}(\beta)$, $U_\alpha = U_\beta = D_\beta$, and $x\alpha = x\beta$ for all $x \in U_\alpha$.*

Proof. To prove (1), suppose $c \in \mathrm{im}(\alpha)$ and let $a = c\alpha^{-1}$. Then clearly $a \in \mathrm{dom}(\alpha)$. By Lemma 2.5, $\mathrm{dom}(\alpha)$ is closed under γ^{-1} for all $\gamma \in S_c$. Let $x \in \mathrm{dom}(\alpha)$ and $\delta \in S_c$ be such that $x(\alpha\delta\alpha^{-1})$ is defined. Since $x\alpha \in \mathrm{im}(\alpha)$, we have $(x\alpha)\delta \in \mathrm{im}(\alpha)$ by Lemma 2.5, and so $x(\alpha\delta\alpha^{-1}) = ((x\alpha)\delta)\alpha^{-1} \in \mathrm{dom}(\alpha)$. Thus $\mathrm{dom}(\alpha)$ is also closed under $\alpha\delta\alpha^{-1}$ for all $\delta \in S_c$. It follows that $U_\alpha \subseteq \mathrm{dom}(\alpha)$.

Suppose $c \in \mathrm{im}(\beta)$ and $c\alpha^{-1} = c\beta^{-1}$. Let $a = c\alpha^{-1} = c\beta^{-1}$. Let $x \in U_\beta$. We will prove that $x \in U_\alpha$ and $x\alpha = x\beta$ by induction on the minimum number of steps needed to generate x from a.

If $x = a$, then $x \in U_\alpha$ and $x\alpha = x\beta$ since $x = a = c\beta^{-1} = c\alpha^{-1}$. Suppose $x = y\gamma^{-1}$ for some $y \in U_\beta$ and $\gamma \in S_c$. Then $y \in U_\alpha$ and $y\alpha = y\beta$ by the inductive hypothesis. Then $x = y\gamma^{-1} \in U_\alpha$ by the definition of U_α. Further, $y \in C$ (since $y \in \mathrm{dom}(\alpha) \cap \mathrm{im}(\gamma)$), $x \in A_y$ (since $x\gamma = y$), and $y\alpha \in B_y$. Since we also have $y\beta = y\alpha$, Lemma 4.9 implies

$$x\alpha = q(y, x, y\alpha) = q(y, x, y\beta) = x\beta.$$

Finally, suppose $x = y(\beta\delta\beta^{-1})$ for some $y \in U_\beta$ and $\delta \in S_c$. Then $y \in U_\alpha$ and $y\alpha = y\beta$ by the inductive hypothesis. Let $p = y(\alpha\delta)$. Then $y\alpha \in C$ (since $y\alpha \in \mathrm{dom}(\delta) \cap \mathrm{im}(\alpha))$), $y \in A_{y\alpha}$, and $p \in B_{y\alpha}$ (since $(y\alpha)\delta = p$). Again, since $y\beta = y\alpha$, Lemma 4.9 implies

$$p\alpha^{-1} = q(y\alpha, y, p) = q(y\beta, y, p) = p\beta^{-1}.$$

Then $x = y(\beta\delta\beta^{-1}) = (y(\alpha\delta))\beta^{-1} = p\beta^{-1} = p\alpha^{-1} = y(\alpha\delta\alpha^{-1})$. It follows that $x \in U_\alpha$ and $x\alpha = p = x\beta$. We have proved that $U_\beta \subseteq U_\alpha$ and $x\alpha = x\beta$ for all $x \in U_\beta$. By symmetry, $U_\alpha \subseteq U_\beta$ and $x\alpha = x\beta$ for all $x \in U_\alpha$. We have proved (1). The proof of (2) is similar.

To prove (3), suppose $c \in \mathrm{dom}(\alpha) \cap \mathrm{im}(\alpha)$, say $c\alpha = b$ and $a\alpha = c$. Then $a = c\alpha^{-1} \in D_\alpha$ and $c = a\alpha\alpha\alpha^{-1} \in U_\alpha$. Hence $U_\alpha = D_\alpha$ by the definitions of U_α and D_α. The remaining claims in (3) follow from (1) and (2). □

Lemma 4.15. *Any two transformations in S_c^* commute.*

Proof. Let $\alpha, \beta \in S_c$. We want to prove that $\alpha^*\beta^* = \beta^*\alpha^*$. Let $x \in X - \{c\}$. Since $\alpha\beta = \beta\alpha$, both $\alpha\beta$ and $\beta\alpha$ are either defined at x or undefined at x. In the latter case, both $\alpha^*\beta^*$ and $\beta^*\alpha^*$ are undefined at x.

So suppose that $x(\alpha\beta) = x(\beta\alpha)$ exists. If both $\alpha^*\beta^*$ and $\beta^*\alpha^*$ are defined at x, then $x(\alpha^*\beta^*) = x(\alpha\beta) = x(\beta\alpha) = x(\beta^*\alpha^*)$. Hence, it suffices to show that

$$x(\alpha^*\beta^*) \text{ is undefined} \Leftrightarrow x(\beta^*\alpha^*) \text{ is undefined.}$$

By symmetry, we may suppose that that $x(\alpha^*\beta^*)$ is undefined. We consider two possible cases.

Case 1. $x\alpha^*$ is undefined.

Since we are working under the assumption that $x(\alpha\beta)$ exists (and so $x\alpha$ exists), it follows from Definition 4.13 and Lemma 4.14 that $x \in K$, where $K = U_\alpha$ or $K = D_\alpha$. Since $x(\alpha\beta)$ exists, it is in $\mathrm{im}(\alpha)$ by Lemma 2.5, so $x(\alpha\beta\alpha^{-1})$ exists. Hence $x(\alpha\beta\alpha^{-1}) \in K$ by the definitions of U_α and D_α. We have $x(\alpha\beta\alpha^{-1}) = (x\beta\alpha)\alpha^{-1} = x\beta$, and so $x\beta \in K$. Thus $(x\beta)\alpha^*$ is undefined, and so $x(\beta^*\alpha^*)$ is undefined.

Case 2. $x\alpha^*$ is defined and $(x\alpha^*)\beta^*$ is undefined.

This can only happen when $x\alpha^* = x\alpha$ is in K, where $K = U_\beta$ or $K = D_\beta$. By the definitions of U_β and D_β, $x = (x\alpha)\alpha^{-1} \in K$ as well. But then $x\beta^*$ is undefined, and hence $x(\beta^*\alpha^*)$ is also undefined. ☐

Lemma 4.16. *Let $\alpha \in S_c$. Then:*

(1) *If $c \in \operatorname{im}(\alpha)$, then $\operatorname{span}(\alpha^*) \cap B_c = \emptyset$.*

(2) *If $c \in \operatorname{dom}(\alpha)$, then $\operatorname{span}(\alpha^*) \cap A_c = \emptyset$.*

Proof. To prove (1), let $c \in \operatorname{im}(\alpha)$ and $b \in B_c$, that is, $c\gamma = b$ for some $\gamma \in S_c$. Note that $b \in \operatorname{im}(\alpha)$ by Lemma 2.5. Then, since $c\alpha^{-1} \in U_\alpha$, we have $b\alpha^{-1} = (c\alpha^{-1})(\alpha\gamma\alpha^{-1}) \in U_\alpha$. Thus $b\alpha^{-1} \notin \operatorname{dom}(\alpha^*)$, and so $b \notin \operatorname{im}(\alpha^*)$. If $b \notin \operatorname{dom}(\alpha)$, then clearly $b \notin \operatorname{dom}(\alpha^*)$. Suppose $b \in \operatorname{dom}(\alpha)$. We have already established that $b\alpha^{-1} \in U_\alpha$. Thus $b = (b\alpha^{-1})(\alpha\alpha\alpha^{-1}) \in U_\alpha$, and so $b \notin \operatorname{dom}(\alpha^*)$. We have proved (1). The proof of (2) is similar. ☐

We can now obtain an upper bound for the size of S_c.

Lemma 4.17. *Let $p = |A_c|$ and $t = |B_c|$. Then*

$$|S_c| \leq (p-1)\lambda_{n-t-1} + (t-1)\lambda_{n-p-1} + 2\lambda_{n-p-t-1}.$$

Proof. Let $A = A_c$, $B = B_c$, and consider the following subsets of S_c^*:

$$\begin{aligned} F_A &= \{\alpha^* \in S_c^* : \operatorname{span}(\alpha^*) \cap A \neq \emptyset\}, \\ F_B &= \{\alpha^* \in S_c^* : \operatorname{span}(\alpha^*) \cap B \neq \emptyset\}, \\ F_0 &= \{\alpha^* \in S_c^* : \operatorname{span}(\alpha^*) \cap (A \cup B) = \emptyset\}. \end{aligned}$$

Suppose $\alpha^* \in F_A$. Then, by Lemma 4.16, $c \in \operatorname{im}(\alpha) - \operatorname{dom}(\alpha)$ and $\operatorname{span}(\alpha^*) \cap B = \emptyset$. Hence $\alpha^* \in \mathcal{I}(X - (B \cup \{c\}))$. Similarly, if $\alpha^* \in F_B$, then $c \in \operatorname{dom}(\alpha) - \operatorname{im}(\alpha)$, $\operatorname{span}(\alpha^*) \cap A = \emptyset$, and $\alpha^* \in \mathcal{I}(X - (A \cup \{c\}))$. If $\alpha^* \in F_0$, then clearly $\alpha^* \in \mathcal{I}(X - (A \cup B \cup \{c\}))$. Thus, $S_c^* = F_A \cup F_B \cup F_0$ and the sets F_A, F_B, and F_0 are pairwise disjoint.

By Lemma 4.15, F_A, F_B, and F_0 are sets of commuting transformations (as subsets of S_c^*). Let F be any subset of S_c^* and denote by $\langle F \rangle$ the semigroup generated by F. Then $\langle F \rangle$ is clearly commutative. Suppose to the contrary that $\langle F \rangle$ is not a nilpotent semigroup. Then it contains a nonzero idempotent, say $\varepsilon = \alpha_1^* \cdots \alpha_k^*$, where $\alpha_i^* \in F$. Let $x \in X$ be any element fixed by ε. Then $x(\alpha_1^* \cdots \alpha_k^*) = x$, and so $x(\alpha_1 \cdots \alpha_k) = x$ since each α_i^* is a restriction of α_i. But this is a contradiction since $\alpha_1 \cdots \alpha_k$ is a nilpotent as an element of S. Thus $\langle F \rangle$ is a nilpotent semigroup.

Hence, by Proposition 4.12 and the inductive hypothesis applied to $\langle F_A \cup F_0 \rangle \subseteq \mathcal{I}(X - (B \cup \{c\}))$, $\langle F_B \cup F_0 \rangle \subseteq \mathcal{I}(X - (A \cup \{c\}))$, and $\langle F_0 \rangle \subseteq \mathcal{I}(X - (A \cup B \cup \{c\}))$, we have

$$|F_A| + |F_0| \leq \lambda_{n-t-1}, \ |F_B| + |F_0| \leq \lambda_{n-p-1}, \ |F_0| \leq \lambda_{n-p-t-1}. \tag{4.6}$$

Suppose $\alpha^* \in F_A$. Then $c \in \operatorname{im}(\alpha) - \operatorname{dom}(\alpha)$, and so $a\alpha = c$ for some $a \in X$. Note that $a \in A$. Fix $a_0 \in \operatorname{span}(\alpha^*) \cap A$. Suppose to the contrary that $a_0 = a$. Then $a_0 \notin \operatorname{dom}(\alpha^*)$ since $a_0 = a = c\alpha^{-1} \in U_\alpha$ and $\alpha^* = \alpha|_{X-U_\alpha}$. Hence $a_0 \in \operatorname{im}(\alpha^*)$, that is, $x\alpha^* = a_0 = a$ for some $x \in \operatorname{dom}(\alpha^*)$. But this is a contradiction since $x = a\alpha^{-1} \in U_\alpha$, and so $x \notin \operatorname{dom}(\alpha^*)$. We have proved that $a_0 \neq a$. Suppose $\alpha^* = \beta^*$. By the foregoing argument, there is $a' \in A$ such that $a'\beta = c$ and $a' \neq a_0$. Moreover, if $a = a'$, then $\alpha = \beta$ by Lemma 4.14.

It follows that any $\alpha^* \in F_A$ has at most $p-1$ preimages under the mapping * (which correspond to the number of elements from the set $A - \{a_0\}$ that α can map to c if $\alpha^* \in F_A$). By similar arguments, any $\alpha \in F_B$ has at most $t-1$ preimages under *, and any $\alpha^* \in F_0$ has at most $p+t$ preimages under *.

These considerations about the number of preimages that an element of S_c^* can have, together with (4.6), give

$$\begin{aligned}|S_c| &\leq (p-1)|F_A| + (t-1)|F_B| + (p+t)|F_0| \\ &= (p-1)(|F_A| + |F_0|) + (t-1)(|F_B| + |F_0|) + 2|F_0| \\ &\leq (p-1)\lambda_{n-t-1} + (t-1)\lambda_{n-p-1} + 2\lambda_{n-p-t-1},\end{aligned}$$

which completes the proof. □

The following proposition will finish our inductive proof of (4.3). The proposition is stronger than what we need in this section, but we will also use it in the proof of the general case.

Proposition 4.18. *Let X be a set with $n \geq 4$. Let S be a commutative nilpotent subsemigroup of $\mathcal{I}(X)$ with $C \neq \emptyset$. Then:*

(1) *If $n \leq 7$, then $|S| < \lambda_n$.*

(2) *If $n \geq 8$, then $|S| < \lambda_n - n$.*

Proof. We have checked that (1) is true by direct calculations using GRAPE [36], which is a package for GAP [17]. For $n \in \{4, 5, 6, 7\}$, we have calculated the orders of the maximal commutative nilpotent semigroups and the number of semigroups of each order. The following table contains the maximum order of a commutative nilpotent semigroup (row 2) and the number of commutative nilpotent semigroups of the maximum order.

n	4	5	6	7
Max order	7	13	34	73
No of sgps of max order	6	20	20	70

The numbers in the second row of the table are λ_4, λ_5, λ_6, and λ_7 (see the table in Lemma 4.7). The numbers in the third row are $\binom{4}{2}$, $2\binom{5}{2}$, $\binom{6}{3}$, and $2\binom{7}{3}$. This means that the commutative nilpotent semigroups of the maximum order are the balanced null semigroups $S_{K,L}$ since, for $m = \lfloor \frac{n}{2} \rfloor$, there are $\binom{n}{m}$ such semigroups if n is even, and $2\binom{n}{m}$ such semigroups if n is odd (see the proof of Theorem 5.3). Since $C = \emptyset$ for each semigroup $S_{K,L}$ (balanced or not), (1) follows.

To prove (2), suppose $n \geq 8$. Let $c \in C$ be an element that satisfies one of the conditions (1)–(3) from Lemma 4.11. Let $p = |A_c|$ and $t = |B_c|$. By (4.5) and Lemma 4.17,

$$|S| = |S - S_c| + |S_c| \leq \lambda_{n-1} + (p-1)\lambda_{n-t-1} + (t-1)\lambda_{n-p-1} + 2\lambda_{n-p-t-1}. \tag{4.7}$$

We consider four possible cases.

Case 1. $p \geq 2$ and $t \geq 2$.

By (4.7),

$$\begin{aligned}|S| &\leq \lambda_{n-1} + (p-1)\lambda_{n-t-1} + (t-1)\lambda_{n-p-1} + 2\lambda_{n-p-t-1} \\ &\leq \lambda_{n-1} + (p-1)\lambda_{n-3} + (t-1)\lambda_{n-3} + 2\lambda_{n-5} \qquad (4.8) \\ &\leq \lambda_{n-1} + (n-3)\lambda_{n-3} + 2\lambda_{n-5}, \qquad (4.9)\end{aligned}$$

where (4.8) follows from $n-t-1, n-p-1 \leq n-3$ and $n-p-t-1 \leq n-5$, and (4.9) from $p+t \leq n-1$ (so $p+t-2 \leq n-3$). For $n = 8$ and $n = 9$, $\lambda_{n-1} + (n-3)\lambda_{n-3} + 2\lambda_{n-5} < \lambda_n - n$ by direct calculations:

n	8	9
$\lambda_n - n$	201	492
$\lambda_{n-1} + (n-3)\lambda_{n-3} + 2\lambda_{n-5}$	144	427

For $n \geq 10$, $\lambda_{n-5} > n$ (see the table in Lemma 4.7), and so

$$\begin{aligned}
|S| &\leq \lambda_{n-1} + (n-3)\lambda_{n-3} + 2\lambda_{n-5} \\
&< \lambda_{n-1} + (n-3)\lambda_{n-3} + 3\lambda_{n-5} - n \qquad (4.10)\\
&< \lambda_{n-1} + (n-3)\lambda_{n-3} + \tfrac{3}{4}\lambda_{n-3} - n \qquad (4.11)\\
&< \lambda_{n-1} + (n-2)\lambda_{n-3} - n \\
&\leq \lambda_{n-1} + \left\lfloor \tfrac{n}{2} \right\rfloor \lambda_{n-2} - n \qquad (4.12)\\
&= \lambda_n - n, \qquad (4.13)
\end{aligned}$$

where (4.10) follows from $\lambda_{n-5} > n$ when $n \geq 10$ (see Lemma 4.7 and the table in its proof), (4.11) from $\lambda_{n-3} > 4\lambda_{n-5}$ when $n \geq 10$ (see Lemma 4.7 and the table in its proof), (4.12) from $\lambda_{n-2} > 2\lambda_{n-3} > \frac{n-2}{\lfloor \frac{n}{2} \rfloor}\lambda_{n-3}$ when $n \geq 8$ (see Lemma 4.7), and (4.13) from Lemma 4.4.

Case 2. $p = 1$ and $t = 1$.

Then, by (4.7),

$$\begin{aligned}
|S| &\leq \lambda_{n-1} + 2\lambda_{n-3} \\
&< \lambda_{n-1} + 3\lambda_{n-3} - n \qquad (4.14)\\
&< \lambda_{n-1} + \tfrac{3}{2}\lambda_{n-2} - n \qquad (4.15)\\
&< \lambda_{n-1} + \left\lfloor \tfrac{n}{2} \right\rfloor \lambda_{n-2} - n \\
&= \lambda_n - n, \qquad (4.16)
\end{aligned}$$

where (4.14) follows from $\lambda_{n-3} > n$ when $n \geq 8$, (4.15) from $\lambda_{n-2} > 2\lambda_{n-3}$ when $n \geq 8$ (see Lemma 4.7), and (4.16) from Lemma 4.4.

Case 3. $p = 1$ and $2 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Again by (4.7),

$$\begin{aligned}
|S| &\leq \lambda_{n-1} + (t-1)\lambda_{n-2} + 2\lambda_{n-t-2} \\
&\leq \lambda_{n-1} + (\left\lfloor \tfrac{n}{2} \right\rfloor - 1)\lambda_{n-2} + 2\lambda_{n-4} \\
&\leq \lambda_{n-1} + (\left\lfloor \tfrac{n}{2} \right\rfloor - 1)\lambda_{n-2} + 3\lambda_{n-4} - n + 1 \qquad (4.17)\\
&< \lambda_{n-1} + (\left\lfloor \tfrac{n}{2} \right\rfloor - 1)\lambda_{n-2} + \tfrac{3}{4}\lambda_{n-2} - n + 1 \qquad (4.18)\\
&= \lambda_{n-1} + \left\lfloor \tfrac{n}{2} \right\rfloor \lambda_{n-2} - \tfrac{1}{4}\lambda_{n-2} - n + 1 \\
&< \lambda_{n-1} + \left\lfloor \tfrac{n}{2} \right\rfloor \lambda_{n-2} - n \qquad (4.19)\\
&= \lambda_n - n, \qquad (4.20)
\end{aligned}$$

where (4.17) follows from $\lambda_{n-4} \geq n-1$ when $n \geq 8$ (see Lemma 4.7 and the table in its proof), (4.18) from $\lambda_{n-2} > 4\lambda_{n-4}$ when $n \geq 8$ (see Lemma 4.7 and the table in its proof), (4.19) from $\frac{1}{4}\lambda_{n-2} > 1$ for $n \geq 6$, and (4.20) from Lemma 4.4.

Case 4. $2 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $t = 1$.

This case is symmetric to Case 3. $\square$

The inductive proof of (4.3) is complete. As a bonus, we have Proposition 4.18. We can now prove the main theorem of this section.

Theorem 4.19. *Let X be a set with $n \geq 1$ elements and let $m = \left\lfloor \frac{n}{2} \right\rfloor$. Then:*

(1) *The maximum cardinality of a commutative nilpotent subsemigroup of $\mathcal{I}(X)$ is*

$$\lambda_n = \sum_{r=0}^{m} \binom{m}{r}\binom{n-m}{r} r!.$$

(2) *If $n \notin \{1,3\}$, then the only commutative nilpotent subsemigroups of $\mathcal{I}(X)$ of order λ_n are the balanced null semigroups $S_{K,L}$.*

Proof. Let S be a commutative nilpotent subsemigroup of $\mathcal{I}(X)$. If $n=1$, then $S=\{0\}$, so $|S|=1=\lambda_1$. Let $n\geq 2$. If $S=S_{K,L}$ is a balanced null semigroup, then $|S|=\lambda_n$ by Lemma 4.2. Suppose S is not one of the balanced null semigroups $S_{K,L}$. If $C=\emptyset$, then $|S|<\lambda_n$ by Proposition 4.12. Suppose $C\neq\emptyset$. If $n=3$, then $S=\langle[i\,j\,k]\rangle=\{0,[i\,j\,k],[i\,k]\}$, where i,j,k are pairwise distinct elements of X, so $|S|=3=\lambda_3$. If $n\geq 4$, then $|S|<\lambda_n$ by Proposition 4.18. The result follows. □

5 The Largest Commutative Semigroups in $\mathcal{I}(X)$

In this section, we determine the maximum order of a commutative subsemigroup of $\mathcal{I}(X)$, and describe the commutative subsemigroups of $\mathcal{I}(X)$ of the maximum order (Theorem 5.3).

Lemma 5.1. *Let X be a set with $n<10$ elements. Suppose S is a commutative subsemigroup of $\mathcal{I}(X)$ such that $S\neq E(\mathcal{I}(X))$, where $E(\mathcal{I}(X))$ is the semilattice of idempotents of $\mathcal{I}(X)$. Then $|S|<2^n$.*

Proof. The lemma is vacuously true when $n=1$. It is also true when $n=2$ since then the only maximal commutative subsemigroups of $\mathcal{I}(X)$ other than $E(\mathcal{I}(X))$ are $\mathrm{Sym}(X)\cup\{0\}$ and $\{0,1,[i\,j]\}$, where i and j are distinct elements of X. Let $n\geq 3$ and suppose, as the inductive hypothesis, that the result is true whenever $|X|<n$. Let $G=S\cap\mathrm{Sym}(X)$ and $T=S-G$.

Suppose G is a semiregular subgroup of $\mathrm{Sym}(X)$ and T is a nilpotent semigroup. Then $|G|\leq n$ (since G is semiregular) and $|T|\leq\lambda_n$ (by Theorem 4.19). Thus $|S|\leq\lambda_n+n<2^n$, where the latter inequality follows from the table below.

n	3	4	5	6	7	8	9
λ_n+n	6	11	18	40	80	217	510
2^n	8	16	32	64	128	256	512

Suppose G is not a semiregular group or T is not a nilpotent semigroup. Then, by Lemmas 2.3 and 3.1, there is a partition $\{A,B\}$ of X such that $S\cong S_A\times S_B$, where S_A is a commutative subsemigroup of $\mathcal{I}(A)$ and S_B is a commutative subsemigroup of $\mathcal{I}(B)$. If $S\subseteq E(\mathcal{I}(X))$, then $|S|<|E(\mathcal{I}(X))|=2^n$. Suppose S is not included in $E(\mathcal{I}(X))$. Then at least one of S_A and S_B, say S_A, must contain an element that is not an idempotent. Let $k=|S_A|$. By the inductive hypothesis, $|S_A|<2^k$ and $|S_B|\leq 2^{n-k}$, and so $|S|=|S_A|\cdot|S_B|<2^k\cdot 2^{n-k}=2^n$. □

Lemma 5.2. *Let $n=|X|\geq 5$. Suppose $S=G\cup T$ is a commutative subsemigroup of $\mathcal{I}(X)$ such that G is a nontrivial semiregular subgroup of $\mathrm{Sym}(X)$ and T is a subsemigroup of $S_{A,B}$, where $\{A,B\}$ is a partition of X. Then $|S|<\lambda_n+1$.*

Proof. Let $k=|A|$, so $|B|=n-k$. We have $|G|\leq n$ (since G is semiregular) and $|T|\leq|S_{A,B}|\leq\lambda_n$ (by Proposition 4.12). If $k=1$ or $k=n-1$, then $|T|\leq|S_{A,B}|=n$, and so $|S|\leq n+n=2n<\lambda_n+1$ since $n\geq 5$ (see the table in Lemma 4.7).

Suppose $1<k<n-1$. The semigroup $S_{A,B}$ contains $|A|\cdot|B|=k(n-k)$ nilpotents $[x\,y]$. Let σ be a nontrivial element of G. Then no nilpotent $[x\,y]$ commutes with σ (by Proposition 2.2), and so such a nilpotent cannot be in T. Thus $|T|\leq|S_{A,B}|-k(n-k)\leq\lambda_n-k(n-k)$. But, since $n\geq 5$ and $1<k<n-1$, we have $k(n-k)\geq n$ by elementary algebra, and so

$$|S|=|G|+|T|\leq n+\lambda_n-k(n-k)\leq n+\lambda_n-n=\lambda_n<\lambda_n+1.$$

□

We can now prove the main theorem of the paper regarding largest commutative subsemigroups of $\mathcal{I}(X)$.

Theorem 5.3. *Let X be a set with $n\geq 1$ elements and let $m=\left\lfloor\frac{n}{2}\right\rfloor$. Then:*

(1) *If $n < 10$, then the maximum cardinality of a commutative subsemigroup of $\mathcal{I}(X)$ is 2^n, and the semilattice $E(\mathcal{I}(X))$ is the unique commutative subsemigroup of $\mathcal{I}(X)$ of order 2^n.*

(2) *Suppose $n \geq 10$. Then the maximum cardinality of a commutative subsemigroup of $\mathcal{I}(X)$ is*

$$\lambda_n + 1 = \sum_{r=0}^{m} \binom{m}{r}\binom{n-m}{r} r! + 1.$$

(a) *If n is even, then there are exactly $\binom{n}{m}$, pairwise isomorphic, commutative subsemigroups of $\mathcal{I}(X)$ of order $\lambda_n + 1$, namely the balanced null monoids $S_{K,L} \cup \{1\}$.*

(b) *If n is odd, then there are exactly $2\binom{n}{m}$, pairwise isomorphic, commutative subsemigroups of $\mathcal{I}(X)$ of order $\lambda_n + 1$, namely the balanced null monoids $S_{K,L} \cup \{1\}$.*

Proof. Statement (1) follows immediately from Lemma 5.1 and the fact that if $|X| = n$, then $|E(\mathcal{I}(X))| = 2^n$.

To prove (2), suppose $n \geq 10$. Each of the balanced null monoids $S_{K,L} \cup \{1\}$ has order $\lambda_n + 1$ by Lemma 4.2. If n is even, then $|K| = |L| = m$, and so there are $\binom{n}{m}$ balanced null semigroups $S_{K,L}$ (since we have $\binom{n}{m}$ choices for K and $L = X - K$ is determined when K has been selected). If n is odd, then the number doubles since we have $\binom{n}{m}$ such semigroups when $|K| = m$ and another $\binom{n}{m}$ when $|K| = n - m$.

Let S be a commutative subsemigroup of $\mathcal{I}(X)$ that is different from the balanced null monoids $S_{K,L} \cup \{1\}$. Our objective is to prove that

$$|S| < \lambda_n + 1 = \sum_{r=0}^{m} \binom{m}{r}\binom{n-m}{r} r! + 1. \tag{5.21}$$

Proceeding by induction on $n = |X|$, we suppose that the statement is true for every X with $10 \leq |X| < n$. Let $G = S \cap \mathrm{Sym}(X)$ and $T = S - G$.

Suppose G is a semiregular subgroup of $\mathrm{Sym}(X)$ and T is a nilpotent semigroup. If G is trivial, then T is not a balanced null semigroup, and hence $|S| < \lambda_n + 1$ by Theorem 4.19. So assume that $G \neq \{1\}$. Let

$$C = \{c \in X : c \in \mathrm{dom}(\alpha) \cap \mathrm{im}(\beta) \text{ for some } \alpha, \beta \in T\}.$$

If $C = \emptyset$, then $T \subseteq S_{A,B}$, where $\{A, B\}$ is a partition of X, and so $|S| < \lambda_n + 1$ by Lemma 5.2. Suppose $C \neq \emptyset$. Then $|G| \leq n$ (since G is semiregular) and $|T| < \lambda_n - n$ (by Proposition 4.18). Thus $|S| = |G| + |T| < n + \lambda_n - n = \lambda_n < \lambda_n + 1$.

Suppose G is not a semiregular subgroup of $\mathrm{Sym}(X)$ or T is a not a nilpotent semigroup. Then, by Lemmas 2.3 and 3.1, there is a partition $\{A, B\}$ of X such that $S \cong S_A \times S_B$, where S_A is a commutative subsemigroup of $\mathcal{I}(A)$ and S_B is a commutative subsemigroup of $\mathcal{I}(B)$. Notice that $1 \leq |A|, |B| < |X| = n$. We may assume that $|A| \leq |B|$. Let $k = |A|$. Then $1 \leq k < n$ and $|B| = n - k$. We consider three possible cases.

Case 1. $k < 10$ and $n - k < 10$.

Then, by Lemma 5.1, $|S_A| \leq 2^k$ and $|S_B| \leq 2^{n-k}$, and so

$$|S| = |S_A| \cdot |S_B| \leq 2^k \cdot 2^{n-k} = 2^n < \lambda_n + 1,$$

where the last inequality is true since $n \geq 10$ (see Lemma 4.7 and the table in its proof).

Case 2. $k < 10$ and $n - k \geq 10$.

Then, $|S_A| \leq 2^k$ (by Lemma 5.1) and $|S_B| \leq \lambda_{n-k} + 1$ (by Theorem 4.19 and the inductive hypothesis). Thus, by (1) of Lemma 4.6,

$$|S| = |S_A| \cdot |S_B| \leq 2^k(\lambda_{n-k} + 1) < \lambda_n + 1.$$

Case 3. $k \geq 10$ and $n - k \geq 10$.

Then, by Theorem 4.19 and the inductive hypothesis, $|S_A| \leq \lambda_k + 1$ and $|S_B| \leq \lambda_{n-k} + 1$. Thus, by (2) of Lemma 4.6,

$$|S| = |S_A| \cdot |S_B| \leq (\lambda_k + 1)(\lambda_{n-k} + 1) < \lambda_n + 1.$$

Hence, in all cases, $|S| < \sum_{r=0}^{m} \binom{m}{r}\binom{n-m}{r} r! + 1$, which concludes the proof of (2). $\square$

It follows from Theorem 5.3 that every symmetric inverse semigroup $\mathcal{I}(X)$ has, up to isomorphism, a unique commutative subsemigroup of maximum order. In comparison, the symmetric group $\mathrm{Sym}(X)$ has, up to isomorphism, a unique abelian subgroup of maximum order if $|X| = 3k$ or $|X| = 3k + 2$, and two abelian subgroups of maximum order if $|X| = 3k + 1$ [11, Theorem 1].

6 The Clique Number and Diameter of $\mathcal{G}(\mathcal{I}(X))$

In this section, we determine the clique number of the commuting graph of $\mathcal{I}(X)$ and the diameter of the commuting graph of every nonzero ideal of $\mathcal{I}(X)$. The exception is the case of $\mathcal{G}(\mathcal{I}(X))$ when $n = |X|$ is odd and composite, and not a prime power, where we are only able to say that the diameter is either 4 or 5.

Let Γ be a simple graph, that is, $\Gamma = (V, E)$, where V is a finite non-empty set of vertices and $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$ is a set of edges. We will write $u - v$ to mean that $\{u, v\} \in E$. (If $\mathcal{G}(S)$ is the commuting graph of a semigroup S, then for all vertices a and b of $\mathcal{G}(S)$, $a - b$ if and only if $a \neq b$ and $ab = ba$.)

A subset K of V is called a *clique* in Γ if $u - v$ for all distinct $u, v \in K$. The *clique number* of Γ is the largest integer r such that Γ has a clique K with $|K| = r$.

Let $u, w \in V$. A *path* in Γ of length $m - 1$ ($m \geq 1$) from u to w is a sequence of pairwise distinct vertices $u = v_1, v_2, \ldots, v_m = w$ such that $v_i - v_{i+1}$ for every $i \in \{1, \ldots, m-1\}$. The *distance* between vertices u and w, denoted $d(u, w)$, is the smallest integer $k \geq 0$ such that there is a path of length k from u to w. If there is no path from u to w, we say that the distance between u and w is infinity, and write $d(u, w) = \infty$. The maximum distance $\max\{d(u, w) : u, w \in V\}$ between vertices of Γ is called the *diameter* of Γ. Note that the diameter of Γ is finite if and only if Γ is connected.

It follows easily from Proposition 2.2 that the only central elements of $\mathcal{I}(X)$ are the zero and identity transformations. Therefore, the following result is an immediate corollary of Theorem 5.3. (Note that if $|X| = 1$, then $\mathcal{I}(X)$ is a commutative semigroup.)

Corollary 6.1. *Let X be a set with $n \geq 2$ elements and let $m = \lfloor \frac{n}{2} \rfloor$. Then:*

(1) *If $n < 10$, then the clique number of the commuting graph of $\mathcal{I}(X)$ is $2^n - 2$.*

(2) *If $n \geq 10$, then the clique number of the commuting graph of $\mathcal{I}(X)$ is*

$$\lambda_n - 1 = \sum_{r=0}^{m} \binom{m}{r}\binom{n-m}{r} r! - 1.$$

It is well known (see [19, Exercises 5.11.2 and 5.11.4]) that $\mathcal{I}(X)$ has exactly $n + 1$ ideals, $J_0, J_1, \ldots, J_n$, where

$$J_r = \{\alpha \in \mathcal{I}(X) : \mathrm{rank}(\alpha) \leq r\}$$

for $0 \leq r \leq n$. Each ideal J_r is principal and any $\alpha \in \mathcal{I}(X)$ of rank r generates J_r. The ideal $J_0 = \{0\}$ consists of the zero transformation. Our next objective is to find the diameter of the commuting graph of every proper nonzero ideal $\mathcal{I}(X)$.

Lemma 6.2. *Let $n \geq 2$. Suppose $\alpha \in \mathcal{I}(X) - \{0, 1\}$ is not an n-cycle or a nilpotent of index n. Then there exists an idempotent $\varepsilon \in \mathcal{I}(X) - \{0, 1\}$ such that* $\mathrm{rank}(\varepsilon) \leq \mathrm{rank}(\alpha)$ *and $\alpha\varepsilon = \varepsilon\alpha$.*

Proof. If α is a permutation that is not an n-cycle, then α commutes with the restriction of the identity to the domain of any cycle of α. If α is nilpotent, but not of index n, then α commutes with the restriction of the identity to the span of any chain of α. Finally, if α is neither invertible nor nilpotent, then α commutes with its unique idempotent power. In all cases, the selected idempotent is different from 0 and 1 and has rank less than or equal to the rank of α. $\square$

Lemma 6.3. *Let $n \geq 3$. Suppose $\alpha, \beta \in \mathcal{I}(X) - \{0,1\}$ such that neither α nor β is an n-cycle. Then in the commuting graph $\mathcal{G}(\mathcal{I}(X))$, there is a path from α to β of length at most 4 such that all vertices in the path have rank at most $\max\{\mathrm{rank}(\alpha), \mathrm{rank}(\beta)\}$.*

Proof. Suppose neither α nor β is a nilpotent of index n. Then, by Lemma 6.2, there are idempotents $\varepsilon_1, \varepsilon_2 \in \mathcal{I}(X) - \{0,1\}$ such that $\mathrm{rank}(\varepsilon_1) \leq \mathrm{rank}(\alpha)$, $\mathrm{rank}(\varepsilon_2) \leq \mathrm{rank}(\beta)$, $\alpha - \varepsilon_1$, and $\varepsilon_2 - \beta$. Since idempotents in $\mathcal{I}(X)$ commute, $\alpha - \varepsilon_1 - \varepsilon_2 - \beta$.

Suppose $\alpha = [y_1\, y_2 \dots y_n]$ is a nilpotent of index n and β is not a nilpotent of index n. Let ε_1 be the idempotent with $\mathrm{dom}(\varepsilon_1) = \{y_1, y_n\}$ (note that $\mathrm{rank}(\varepsilon_1) \leq \mathrm{rank}(\alpha)$) and ε_2 be an idempotent different from 0 and 1 such that $\mathrm{rank}(\varepsilon_2) \leq \mathrm{rank}(\beta)$ and $\varepsilon_2 - \beta$ (such an idempotent exists by Lemma 6.2). Then $\alpha - [y_1\, y_n] - \varepsilon_1 - \varepsilon_2 - \beta$.

Finally, suppose $\alpha = [y_1\, y_2 \dots y_n]$ and $\beta = [x_1\, x_2 \dots x_n]$ are nilpotents of index n. If $\{y_1, y_n\} \cap \{x_1, x_n\} = \emptyset$, then $[y_1\, y_n]$ and $[x_1\, x_n]$ commute, and so $\alpha - [y_1\, y_n] - [x_1\, x_n] - \beta$. Suppose $\{y_1, y_n\} \cap \{x_1, x_n\} \neq \emptyset$. Now, if $n \geq 4$, there is $z \in X - \{y_1, y_n, x_1, x_n\}$. Let ε be the idempotent with $\mathrm{dom}(\varepsilon) = \{z\}$. Then, by Proposition 2.2, $\alpha - [y_1\, y_n] - \varepsilon - [x_1\, x_n] - \beta$.

It remains to consider the case when $n = 3$ and $\alpha = [x\, y\, z]$ and β are distinct nilpotents of index 3. We want to show that $d(\alpha, \beta) \leq 4$. Since $\alpha - [x\, z]$, it suffices to show that $d([x\, z], \beta) \leq 3$. If $\beta = [x\, z\, y]$, then $[x\, z] - [x\, y] - [x\, z\, y]$; if $\beta = [y\, x\, z]$, then $[x\, z] - [y\, z] - [y\, x\, z]$; if $\beta = [y\, z\, x]$, then $[x\, z] - [y\, z] - [y\, x] - [y\, z\, x]$; if $\beta = [z\, x\, y]$, then $[x\, z] - [x\, y] - [z\, y] - [z\, x\, y]$; finally, if $\beta = [z\, y\, x]$, then $[x\, z] - \varepsilon - [z\, x] - [z\, y\, x]$, where ε is the idempotent with $\mathrm{dom}(\varepsilon) = \{y\}$. Thus $d(\alpha, \beta) \leq 4$. $\square$

Lemma 6.4. *Let $n \geq 2$. Suppose $\alpha, \beta \in \mathcal{I}(X) - \{0,1\}$ with $\alpha\beta = \beta\alpha$. Then*

(1) *If $\alpha = [x_1 \dots x_n]$ is a nilpotent of index n, then there is $q \in \{1, \dots, n-1\}$ such that $\beta = \alpha^q$.*

(2) *If $\alpha = (x_0\, x_1 \dots x_{n-1})$ is an n-cycle, then there is $q \in \{1, \dots, n-1\}$ such that $\beta = \alpha^q$.*

Proof. Suppose $\alpha = [x_1 \dots x_n]$. Since $\beta \notin \{0,1\}$, it follows by Proposition 2.2 that there is $t \in \{1, \dots, n-1\}$ such that $\mathrm{dom}(\beta) \cap \{x_1, \dots, x_n\} = \{x_1, \dots, x_t\}$ and

$$x_1\beta = x_{n-t+1},\ x_2\beta = x_{n-t+2}, \dots,\ x_t\beta = x_n.$$

Thus $\beta = \alpha^q$, where $q = n - t$, and $q \notin \{0, n\}$ (since $1 \leq t \leq n-1$). We have proved (1).

Suppose $\alpha = (x_0\, x_1 \dots x_{n-1})$. Since $\beta \neq 0$, $\{x_0, x_1, \dots, x_{n-1}\} \subseteq \mathrm{dom}(\beta)$ by Proposition 2.2. Let $x_q = x_0\beta$, where $q \in \{0, 1, \dots, n-1\}$, and note that $q \neq 0$ since $\alpha \neq 1$. Then, by Proposition 2.2, $x_i\beta = x_{q+i}$ for every $i \in \{0, \dots, n-1\}$ (where $x_{q+i} = x_{q+i-n}$ if $q + i \geq n$). Thus $\beta = \alpha^q$. We have proved (2). $\square$

Lemma 6.5. *Let $n \geq 3$. Then there are nilpotents $\alpha, \beta \in \mathcal{I}(X)$ of index n such that $d(\alpha, \beta) = 4$.*

Proof. Let $\alpha = [x_1\, x_2 \dots x_k\, y_1\, y_2 \dots y_m]$ and $\beta = [y_m \dots y_2\, y_1\, x_k \dots x_2\, x_1]$, where $k + m = n$ and $k = \lceil \frac{n}{2} \rceil$. If $n \geq 4$, then $d(\alpha, \beta) \leq 4$ by Lemma 6.3. If $n = 3$, then $\alpha = [x_1\, x_2\, y_1] - [x_1\, y_1] - \varepsilon - [y_1\, x_1] - [y_1\, x_2\, x_1] = \beta$, where ε is the idempotent with $\mathrm{dom}(\varepsilon) = \{x_2\}$, so $d(\alpha, \beta) \leq 4$.

Note that α and β do not commute, so $d(\alpha, \beta) \geq 2$. Suppose $\alpha - \gamma - \delta - \beta$ is a path from α to β of length 3. By Lemma 6.4, $\gamma = \alpha^p$ and $\delta = \beta^q$ for some $p, q \in \{1, \dots, n-1\}$. We may assume that $p \geq k$. (If not, then there exists an integer t such that $k \leq pt \leq n-1$, and so α^p can be replaced with $\alpha^{pt} = (\alpha^p)^t$ in the path.) Similarly, we may assume that $q \geq k$. Then

$$\alpha^p = [x_1\, y_i] \sqcup [x_2\, y_{i+1}] \sqcup \dots \sqcup [x_{m-i+1}\, y_m] \text{ and } \beta^q = [y_m\, x_j] \sqcup [y_{m-1}\, x_{j-1}] \sqcup \dots \sqcup [y_{m-j+1}\, x_1],$$

for some $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, k\}$ (with $j \in \{1, \dots, k-1\}$ when n is odd). But then α^p and β^q do not commute (since $x_{m-i+1}(\alpha^p\beta^q) = x_j$ and $x_{m-i+1} \notin \mathrm{dom}(\beta^q\alpha^p)$), which is a contradiction.

We have proved that there is no path from α to β of length 3. But then there is no path from α to β of length 2 either since any such path would have the form $\alpha - \alpha^p - \beta$ (and then $\alpha - \alpha^p - \beta^2 - \beta$ would be a path of length 3) or $\alpha - \beta^q - \beta$ (and then $\alpha - \alpha^2 - \beta^q - \beta$ would be a path of length 3). It follows that $d(\alpha, \beta) = 4$. $\square$

Lemma 6.6. *Let $n \geq 3$ and $\lfloor \frac{n-1}{2} \rfloor < r < n-1$. Then there are $\alpha, \beta \in J_r$ such that for every nonzero $\gamma \in \mathcal{I}(X)$, if $\alpha - \gamma - \beta$, then $\gamma = 1$.*

Proof. Consider a nilpotent $\alpha = [x\, z_1 \ldots z_{r-1}\, y]$ of rank r (possible since $r < n-1$). Since $r > \lfloor \frac{n-1}{2} \rfloor$, we have $r > \frac{n-1}{2}$, and so $2r \geq n$. Therefore, there are pairwise distinct elements $w_1, \ldots, w_{r-1}$ of X such that $\{x, y, x_1, \ldots, x_{r-1}, w_1, \ldots, w_{r-1}\} = X$. Let $\beta = [y\, w_1 \ldots w_{r-1}\, x] \in J_r$, and suppose $0 \neq \gamma \in \mathcal{I}(X)$ is such that $\alpha - \gamma - \beta$. We want to prove that $\gamma = 1$.

Since $\gamma \neq 0$ and $\operatorname{span}(\alpha) \cup \operatorname{span}(\beta) = X$, we have $\operatorname{dom}(\gamma) \cap \operatorname{span}(\alpha) \neq \emptyset$ or $\operatorname{dom}(\gamma) \cap \operatorname{span}(\beta) \neq \emptyset$. We may assume that $\operatorname{dom}(\gamma) \cap \operatorname{span}(\alpha) \neq \emptyset$. Then, since $\alpha\gamma = \gamma\alpha$, $x \in \operatorname{dom}(\gamma)$ by Proposition 2.2. Since $\beta\gamma = \gamma\beta$, $x \in \operatorname{dom}(\gamma)$ and Proposition 2.2 imply that $\operatorname{span}(\beta) \subseteq \operatorname{dom}(\gamma)$ and γ maps β onto a terminal segment of some chain in β. But β is a single chain, so γ must map β onto β, which is only possible if γ fixes every element of $\operatorname{span}(\beta)$. We now know that $\operatorname{dom}(\gamma) \cap \operatorname{span}(\beta) \neq \emptyset$. By the foregoing argument, with the roles of α and β reversed, we conclude that γ must also fix every element of $\operatorname{span}(\alpha)$. Hence $\gamma = 1$. $\square$

We can now determine the diameter of $\mathcal{G}(J_r)$ for every $r < n$.

Theorem 6.7. *Let $n = |X| \geq 3$ and let J_r be a proper nonzero ideal of $\mathcal{I}(X)$. Then:*

(1) *The diameter of $\mathcal{G}(J_{n-1})$ is 4.*

(2) *If $\lfloor \frac{n-1}{2} \rfloor < r < n-1$, then the diameter of $\mathcal{G}(J_r)$ is 3.*

(3) *If $1 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$, then the diameter of $\mathcal{G}(J_r)$ is 2.*

Proof. We first note that for every $r \in \{1, \ldots, n-1\}$, the only central element of J_r is 0.

To prove (1), observe that $J_{n-1} = \mathcal{I}(X) - \operatorname{Sym}(X)$. The diameter of $\mathcal{G}(J_{n-1})$ is at least 4 by Lemma 6.5, and at most 4 by Lemma 6.3.

To prove (2), suppose $\lfloor \frac{n-1}{2} \rfloor < r < n-1$. Then the diameter of $\mathcal{G}(J_r)$ is at least 3 by Lemma 6.6. Let $\alpha, \beta \in J_r$. Since $r < n-1$, neither α nor β is an n-cycle or a nilpotent of index n. Thus, by Lemma 6.2, there are idempotents $\varepsilon_1, \varepsilon_2 \in J_r - \{0\}$ such that $\alpha\varepsilon_1 = \varepsilon_1\alpha$ and $\beta\varepsilon_2 = \varepsilon_2\beta$. Since the idempotents in $\mathcal{I}(X)$ commute, we have $\alpha - \varepsilon_1 - \varepsilon_2 - \beta$, so the diameter of $\mathcal{G}(J_r)$ is at most 3.

To prove (3), suppose $1 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$. Then the diameter of $\mathcal{G}(J_r)$ is at least 2 since for any distinct $x, y \in X$, the nilpotents $[x\, y]$ and $[y\, x]$ (which are in J_r since $r \geq 1$) do not commute. Let $\alpha, \beta \in J_r - \{0\}$. We have $r \leq \lfloor \frac{n-1}{2} \rfloor \leq \frac{n-1}{2}$, and so $2r \leq n-1 < n$. Therefore,

$$|\operatorname{im}(\alpha) \cup \operatorname{im}(\beta)| \leq |\operatorname{im}(\alpha)| + |\operatorname{im}(\beta)| \leq r + r = 2r < n,$$

and so there is $x \in X$ such that $x \notin \operatorname{im}(\alpha) \cup \operatorname{im}(\beta)$. By the same argument, there is $y \in X$ such that $y \notin \operatorname{dom}(\alpha) \cup \operatorname{dom}(\beta)$. If $x = y$, then $\alpha - \varepsilon - \beta$, where ε is the idempotent with $\operatorname{dom}(\varepsilon) = \{x\}$. If $x \neq y$, then $\alpha - [x\, y] - \beta$. Thus, the diameter of $\mathcal{G}(J_r)$ is at most 2. $\square$

We now want to prove that if $n \geq 4$ is even, then the diameter of $\mathcal{G}(\mathcal{I}(X))$ is 4.

Definition 6.8. Let $\gamma, \delta \in \mathcal{I}(X)$. We say that γ and δ are *aligned* if there exists an integer $r \geq 2$ and pairwise distinct elements $a_1, \ldots, a_r, c_1, \ldots, c_{r-1}, b_1$ of X such that

$$\gamma = (a_1\, b_1) \sqcup (a_2\, c_1) \sqcup (a_3\, c_2) \sqcup \cdots \sqcup (a_{r-1}\, c_{r-2}) \sqcup (a_r\, c_{r-1}),$$
$$\delta = (a_1\, c_1) \sqcup (a_2\, c_2) \sqcup (a_3\, c_3) \sqcup \cdots \sqcup (a_{r-1}\, c_{r-1}) \sqcup (a_r\, b_1).$$

The following lemma follows immediately from Definition 6.8

Lemma 6.9. *Let $\gamma, \delta \in \mathcal{I}(X)$ be aligned. Then, with the notation from* Definition 6.8,

$$\gamma - (a_1\, a_2 \ldots\, a_r) \sqcup (b_1\, c_1 \ldots\, c_{r-1}) - \delta.$$

Lemma 6.10. *Let $n = 2k = |X| \geq 4$ be even. Suppose $\alpha, \beta \in \mathrm{Sym}(X)$ are joins of k cycles of length 2 with no cycle in common. Then $\alpha = \gamma \sqcup \alpha'$ and $\beta = \delta \sqcup \beta'$, where γ and δ are aligned.*

Proof. Select any cycle $(a_1\, b_1)$ in α. Then β has a cycle $(a_1\, c_1)$ with $c_1 \neq b_1$ (since α and β have no cycle in common). Continuing, α must have a cycle $(a_2\, c_1)$, and so β must have either a cycle $(a_2\, b_1)$ or a cycle $(a_2\, c_2)$ with $c_2 \neq b_1$. In the latter case, α must have a cycle $(a_3\, c_2)$, and so β must have a cycle $(a_3\, b_1)$ or a cycle $(a_3\, c_3)$ with $c_3 \neq b_1$. This process must terminate after at most k, say r, steps. That is, at step r, we will obtain a cycle $(a_r\, c_{r-1})$ in α and a cycle $(a_r\, b_1)$ in β. Hence

$$\alpha = (a_1\, b_1) \sqcup (a_2\, c_1) \sqcup (a_3\, c_2) \sqcup \cdots \sqcup (a_{r-1}\, c_{r-2}) \sqcup (a_r\, c_{r-1}) \sqcup \alpha',$$
$$\beta = (a_1\, c_1) \sqcup (a_2\, c_2) \sqcup (a_3\, c_3) \sqcup \cdots \sqcup (a_{r-1}\, c_{r-1}) \sqcup (a_r\, b_1) \sqcup \beta',$$

where $\alpha' = \beta' = 0$ if $r = k$. The proof is completed by the observation that $\gamma = (a_1\, b_1) \sqcup (a_2\, c_1) \sqcup (a_3\, c_2) \sqcup \cdots \sqcup (a_{r-1}\, c_{r-2}) \sqcup (a_r\, c_{r-1})$ and $\delta = (a_1\, c_1) \sqcup (a_2\, c_2) \sqcup (a_3\, c_3) \sqcup \cdots \sqcup (a_{r-1}\, c_{r-1}) \sqcup (a_r\, b_1)$ are aligned. ☐

Lemma 6.11. *Let $n \geq 6$ be composite. Suppose $\alpha, \beta \in \mathcal{I}(X) - \{0, 1\}$ such that α is an n-cycle and β is not an n-cycle. Then $d(\alpha, \beta) \leq 4$.*

Proof. Suppose β is not a nilpotent of index n. Since n is composite, there is a divisor k of n with $1 < k < n$. Then $\alpha^k \in \mathcal{I}(X) - \{0, 1\}$ is not an n-cycle. Thus, by Lemma 6.2, there are idempotents $\varepsilon_1, \varepsilon_2 \in \mathcal{I}(X) - \{0, 1\}$ such that $\alpha^k - \varepsilon_1$ and $\varepsilon_2 - \beta$. Then $\alpha - \alpha^k - \varepsilon_1 - \varepsilon_2 - \beta$, and so $d(\alpha, \beta) \leq 4$.

Suppose $\beta = [x_1\, x_2 \ldots\, x_n]$ is a nilpotent of index n. Let k be the largest proper divisor of n. Then $\alpha = \rho_1 \sqcup \cdots \sqcup \rho_k$, where each ρ_i is a cycle of length $\frac{n}{k}$. Since $n \geq 6$, we have $k > 2$. Thus, there exists $t \in \{1, \ldots, k\}$ such that $x_1, x_n \notin \mathrm{span}(\rho_t)$. Let ε be the idempotent with $\mathrm{dom}(\varepsilon) = \mathrm{span}(\rho_t)$. Then $\varepsilon \neq 0, 1$ and, by Proposition 2.2, $\alpha - \alpha^k - \varepsilon - [x_1\, x_n] - \beta$. Hence $d(\alpha, \beta) \leq 4$. ☐

Theorem 6.12. *Let $n = |X| \geq 4$ be even. Then the diameter of $\mathcal{G}(\mathcal{I}(X))$ is 4.*

Proof. Let $\alpha, \beta \in \mathcal{I}(X) - \{0, 1\}$. We will prove that $d(\alpha, \beta) \leq 4$. If neither α nor β is an n-cycle, then $d(\alpha, \beta) \leq 4$ by Lemma 6.3.

Suppose α is an n-cycle and β is not an n-cycle. If $n \geq 6$, then $d(\alpha, \beta) \leq 4$ by Lemma 6.11. If $n = 4$ and β is not a nilpotent of index 4, then $d(\alpha, \beta) \leq 4$ again by Lemma 6.11 (where the assumption $n \geq 6$ was only used when β was a nilpotent of index n). Let $n = 4$, $\alpha = (x\, y\, z\, w)$, and $\beta = [a\, b\, c\, d]$. Then $\alpha^2 = (x\, z) \sqcup (y\, w)$, $\beta^3 = [a\, d]$, and so it suffices to find a path of length 2 from $(x\, z) \sqcup (y\, w)$ to $[a\, d]$. If $\{a, d\} = \{x, z\}$ or $\{a, d\} = \{y, w\}$, then $(x\, z) \sqcup (y\, w) - \varepsilon - [a\, d]$, where ε is the idempotent with $\mathrm{dom}(\varepsilon) = \{a, d\}$. Otherwise, we may assume that $a = x$ and $d = w$, and then $(x\, z) \sqcup (y\, w) - [x\, y] \sqcup [z\, w] - [x\, w] = [a\, d]$. Hence $d(\alpha, \beta) \leq 4$.

Suppose α and β are n-cycles. Then for $k = n/2$, α^k and β^k are joins of k cycles of length 2. Therefore, it suffices to find a path of length 2 from α^k to β^k. If α^k and β^k have a cycle in common, say $(a\, b)$, then $\alpha^k - \varepsilon - \beta^k$, where ε is the idempotent with $\mathrm{dom}(\varepsilon) = \{a, b\}$.

Suppose α^k and β^k have no common cycle. Then $\alpha^k = \gamma \sqcup \alpha'$ and $\beta^k = \delta \sqcup \beta'$, where $\gamma, \alpha', \delta, \beta'$ are as in Lemma 6.10. By Lemma 6.9, there is $\eta \in \mathcal{I}(X)$ such that $\mathrm{span}(\eta) = \mathrm{span}(\gamma) = \mathrm{span}(\delta)$ and $\gamma - \eta - \delta$. It follows that

$$\alpha^k = \gamma \sqcup \alpha' - \eta - \delta \sqcup \beta' = \beta^k.$$

We have proved that $d(\alpha, \beta) \leq 4$ for all $\alpha, \beta \in \mathcal{I}(X)$, which shows that the diameter of $\mathcal{G}(\mathcal{I}(X))$ is at most 4. Since the diameter of $\mathcal{G}(\mathcal{I}(X))$ is at least 4 by Lemma 6.5, the proof is complete. ☐

Suppose $n = 2$, say $X = \{x, y\}$. Then the commuting graph $\mathcal{G}(\mathcal{I}(X))$ has one edge, $(x) - (y)$ (recall that in our notation (x) is the idempotent with domain $\{x\}$), and three isolated vertices, $(x\, y)$, $[x\, y]$, and $[y\, x]$. Hence, the diameter of $\mathcal{G}(\mathcal{I}(X))$ is ∞, $\mathcal{G}(\mathcal{I}(X)$ has three connected components of diameter 0, and one connected component of diameter 1.

The following proposition and Theorem 6.17 partially solve the problem of finding the diameter of $\mathcal{G}(\mathcal{I}(X))$ when n is odd.

Proposition 6.13. *Let $n = |X| \geq 3$ be odd. Then:*

(1) *If n is prime, then the diameter of $\mathcal{G}(\mathcal{I}(X))$ is ∞.*

(2) *If n is composite, then the diameter of $\mathcal{G}(\mathcal{I}(X))$ is either 4 or 5.*

Proof. Suppose $n = p$ is an odd prime. Consider a p-cycle $\alpha = (x_0\, x_1 \dots x_{p-1})$ and let $\beta \in \mathcal{I}(X) - \{0, 1\}$ with $\alpha\beta = \beta\alpha$. By Lemma 6.4, $\beta = \alpha^q$ for some $q \in \{1, \dots, p-1\}$. Thus, since p is prime, β is also a p-cycle. It follows that if γ is a vertex of $\mathcal{G}(\mathcal{I}(X))$ that is not a p-cycle, then there is no path in $\mathcal{G}(\mathcal{I}(X))$ from α to γ. Hence $\mathcal{G}(\mathcal{I}(X))$ is not connected, and so the diameter of $\mathcal{G}(\mathcal{I}(X))$ is ∞. We have proved (1).

Suppose n is odd and composite (so $n \geq 9$). Let $\alpha, \beta \in \mathcal{I}(X) - \{0, 1\}$. If α or β is not an n-cycle, then $d(\alpha, \beta) \leq 4$ by Lemmas 6.3 and 6.11. Suppose α and β are n-cycles. Let k be a proper divisor of n ($1 < k < n$). Then $\alpha = \rho_1 \sqcup \dots \sqcup \rho_k$ and $\beta = \sigma_1 \sqcup \dots \sqcup \sigma_k$, where each ρ_i and each σ_i is a cycle of length $\frac{n}{k}$. Let ε_1 and ε_2 be the idempotents with $\mathrm{dom}(\varepsilon_1) = \mathrm{span}(\rho_1)$ and $\mathrm{dom}(\varepsilon_2) = \mathrm{span}(\sigma_1)$. Then, $\alpha^k, \beta^k \neq 1$ (since $k < n$), $\varepsilon_1, \varepsilon_2 \neq 1$ (since $k > 1$), and $\alpha - \alpha^k - \varepsilon_1 - \varepsilon_2 - \beta^k - \beta$. Hence $d(\alpha, \beta) \leq 5$, and so the diameter of $\mathcal{G}(\mathcal{I}(X))$ is at most 5. On the other hand, the diameter of $\mathcal{G}(\mathcal{I}(X))$ is at least 4 by Lemma 6.5. We have proved (2). $\square$

In the case that n is a prime, we can specify the number and diameters of the connected components of $\mathcal{G}(\mathcal{I}(X)$.

Proposition 6.14. *Let $n = |X|$ be an odd prime. Then $\mathcal{G}(\mathcal{I}(X))$ has $(n-2)!$ connected components of diameter 1 and one connected component of diameter 4.*

Proof. As each n-cycle only commutes with its powers and 0, the non-identity powers of any n-cycle form a connected component of $\mathcal{G}(\mathcal{I}(X))$ with diameter 1. Each such component has $n - 1$ members and hence there are $(n-1)!/(n-1) = (n-2)!$ of them.

By Lemma 6.3, the non-n-cycles form a connected component of $\mathcal{G}(\mathcal{I}(X))$ with diameter at most 4, while by Lemma 6.5, this diameter is at least 4. $\square$

We will now prove that when $n = p^k$ is a power of an odd prime p, with $k \geq 2$, then the diameter of $\mathcal{G}(\mathcal{I}(X))$ is 5.

Definition 6.15. Let $\alpha = \rho_1 \sqcup \rho_2 \sqcup \dots \sqcup \rho_k \in \mathrm{Sym}(X)$ and let $\gamma \in \mathcal{I}(X)$ with $\alpha\gamma = \gamma\alpha$. We define a partial transformation h^α_γ on the set $A = \{\rho_1, \rho_2, \dots, \rho_k\}$ of cycles of α by:

$$\mathrm{dom}(h^\alpha_\gamma) = \{\rho_i \in A : \mathrm{span}(\rho_i) \cap \mathrm{dom}(\gamma) \neq \emptyset\},$$
$$\rho_i h^\alpha_\gamma = \text{the unique } \rho_j \in A \text{ such that } (\mathrm{span}(\rho_i))\gamma = \mathrm{span}(\rho_j).$$

Note that h^α_γ is well defined and injective by Proposition 2.2.

The case of $n = 3^2 = 9$ is special and we consider it in the following lemma.

Lemma 6.16. *Let $n = |X| = 9$. Then there are 9-cycles α and β in $\mathrm{Sym}(X)$ such that the distance between α and β in $\mathcal{G}(\mathcal{I}(X))$ is 5.*

Proof. Let $X = \{1, 2, \dots, 9\}$, and consider the following 9-cycles in $\mathrm{Sym}(X)$:

$$\alpha = (1\,2\,3\,4\,5\,8\,7\,6\,9) \text{ and } \beta = (1\,4\,7\,2\,5\,8\,3\,6\,9).$$

We claim that the distance between α and β in $\mathcal{G}(\mathcal{I}(X))$ is 5. We know that $d(\alpha, \beta) \leq 5$ by Proposition 6.13. Suppose to the contrary that $d(\alpha, \beta) < 5$. Then there are $\delta, \gamma, \eta \in \mathcal{I}(X) - \{0, 1\}$ such that $\alpha - \delta - \gamma - \eta - \beta$. Then, by Lemma 6.4, $\delta = \alpha^p$ and $\eta = \beta^q$ for some $p, q \in \{1, \dots, 8\}$. The exponent p is 3, 6, or relatively prime to 9. In the latter case, there is $t \in \{1, \dots, 8\}$, relatively prime to 9, such that $pt \equiv 1 \pmod 9$. Since γ commutes with $\delta = \alpha^p$, it also commutes with $(\alpha^p)^{3t} = (\alpha^{pt})^3 = (\alpha^1)^3 = \alpha^3$.

If $p = 6$, then γ commutes with $(\alpha^6)^5 = (\alpha^{10})^3 = (\alpha^1)^3 = \alpha^3$. Hence, in either case, γ commutes with α^3. By a similar argument, γ also commutes with β^3, and so

$$\alpha^3 = (1\,4\,7) \sqcup (2\,5\,6) \sqcup (3\,8\,9) - \gamma - (1\,2\,3) \sqcup (4\,5\,6) \sqcup (7\,8\,9) = \beta^3.$$

Since $\gamma \neq 0$, there is a cycle σ in β^3 such that $\mathrm{span}(\sigma) \subseteq \mathrm{dom}(\gamma)$ (by Proposition 2.2). Therefore, 1, 4, or 7 is in $\mathrm{dom}(\gamma)$, and so, since γ commutes with α^3 and $(1\,4\,7)$ is a cycle in α^3, we have $(1\,4\,7) \in \mathrm{dom}(h_\gamma^{\alpha^3})$. There are three possible cases.

Case 1. $(1\,4\,7)h_\gamma^{\alpha^3} = (1\,4\,7)$.

Then $1\gamma = 1$, 4, or 7. If $1\gamma = 1$, then $4\gamma = 4$ and $7\gamma = 7$ by Proposition 2.2. But then, since γ commutes with β^3, γ must fix every element of every cycle of β^3, that is, $\gamma = 1$. This is a contradiction. Suppose $1\gamma = 4$. Then $(1\,2\,3)h_\gamma^{\beta^3} = (4\,5\,6)$ with $2\gamma = 5$ and $3\gamma = 6$. But then $(2\,5\,6)h_\gamma^{\alpha^3} = (2\,5\,6)$ and $(3\,8\,9)h_\gamma^{\alpha^3} = (2\,5\,6)$, which is a contradiction since $h_\gamma^{\alpha^3}$ is injective. If $1\gamma = 7$, we obtain a contradiction in a similar way.

Case 2. $(1\,4\,7)h_\gamma^{\alpha^3} = (2\,5\,6)$.

Then $1\gamma = 2$, 5, or 6. If $1\gamma = 2$, then $4\gamma = 5$ and $7\gamma = 6$, and so $(4\,5\,6)h_\gamma^{\beta^3} = (4\,5\,6)$ and $(7\,8\,9)h_\gamma^{\beta^3} = (4\,5\,6)$. This is a contradiction since $h_\gamma^{\beta^3}$ is injective. If $1\gamma = 5$, then $4\gamma = 6$, and so $(1\,2\,3)h_\gamma^{\beta^3} = (4\,5\,6)$ and $(4\,5\,6)h_\gamma^{\beta^3} = (4\,5\,6)$, again a contradiction. Finally, if $1\gamma = 6$, then $7\gamma = 5$, and so $(1\,2\,3)h_\gamma^{\beta^3} = (4\,5\,6)$ and $(7\,8\,9)h_\gamma^{\beta^3} = (4\,5\,6)$, also a contradiction.

Case 3. $(1\,4\,7)h_\gamma^{\alpha^3} = (3\,8\,9)$.

In this case, we also obtain a contradiction by the argument similar to the one used in Case 2.

Therefore, the assumption $d(\alpha, \beta) < 5$ leads to a contradiction, and so $d(\alpha, \beta) \geq 5$. Since we already know that $d(\alpha, \beta) \leq 5$, we have $d(\alpha, \beta) = 5$. □

Theorem 6.17. *Let $|X| = n = p^k$, where p is an odd prime and $k \geq 2$. Then the diameter of $\mathcal{G}(\mathcal{I}(X))$ is 5.*

Proof. By Proposition 6.13, it suffices to find two n-cycles α and β in $\mathrm{Sym}(X)$ such that the distance between α and β in $\mathcal{G}(\mathcal{I}(X))$ is at least 5. If $n = 9$, then such cycles exist by Lemma 6.16.

Suppose $n > 9$ and let $X = \{1, 2, \ldots, n\}$. If $\alpha, \beta \in \mathcal{I}(X)$ are n-cycles such that $\alpha - \delta - \gamma - \eta - \beta$ for some $\delta, \gamma, \eta \in \mathcal{I}(X) - \{0, 1\}$, then, by the argument similar to the one we used in Lemma 6.16, we may assume that $\delta = \alpha^q$ and $\eta = \beta^q$, where $q = p^{k-1}$. Note that then δ and η are joins of q cycles, each cycle of length p. Consider the following $\delta, \eta \in \mathrm{Sym}(X)$:

$$\delta = (1\ \ 2\ \ \ldots\ \ p) \sqcup (p+1\ \ p+2\ \ \ldots\ \ 2p) \sqcup \cdots \sqcup (n-p+1\ \ n-p+2\ \ \ldots\ \ n),$$

$$\begin{array}{rllllll}
\eta = & (1 & q-1 & 2q-2 & \ldots & n-3q-p+3 & n-2q-p+2 & n-q-p+1) \\
& \sqcup\,(2 & q & 2q-1 & \ldots & n-3q-p+4 & n-2q-p+3 & n-q-p+2) \\
& \sqcup\,(3 & q+1 & 2q & \ldots & n-3q-p+5 & n-2q-p+4 & n-q-p+3) \\
& \sqcup\,(4 & q+2 & 2q+1 & \ldots & n-3q-p+6 & n-2q-p+5 & n-q-p+4) \\
& \vdots & & & & & & \\
& \sqcup\,(q-3 & 2q-5 & 3q-6 & \ldots & n-2q-p-1 & n-q-p-2 & n-p-3) \\
& \sqcup\,(q-2 & 2q-4 & 3q-5 & \ldots & n-2q-p & n-q-p-1 & n-p-2) \\
& \sqcup\,(n-p+1 & 2q-3 & 3q-4 & \ldots & n-2q-p+1 & n-q-p & n-p-1) \\
& \sqcup\,(n-p+2 & n-p+3 & n-p+4 & \ldots & n-1 & n & n-p)
\end{array}$$

The construction of δ is straightforward. Regarding η, the last cycle,

$$\tau = (n-p+2\ \ n-p+3\ \ n-p+4\ \ \ldots\ \ n-1\ \ n\ \ n-p),$$

is special. (Its role will become clear in the second part of the proof). If $\tau' = (x_1\, x_2 \dots x_p)$ is any other cycle in η, then $x_{i+1} - x_i = q - 1$ for every $i \in \{2, \dots, p-1\}$. Here and in the following, we assume cycles are always represented by expressions listing the elements in the fixed orders from the definitions of δ and η, so that we may speak of the position of an element in a cycle.

Let α and β be n-cycles such that $\alpha^q = \delta$ and $\beta^q = \eta$. As δ and η consist of q disjoint cycles of length p, such α and β exist. We claim that $d(\alpha, \beta) \geq 5$. Suppose to the contrary that $d(\alpha, \beta) < 5$. Then, by the foregoing argument, there exists $\gamma \in \mathcal{I}(X) - \{0, 1\}$ with $\delta - \gamma - \eta$.

Define a binary relation $\sim$ on X by: $x \sim y$ if there exists a cycle ρ in δ or in η with $\{x, y\} \subseteq \operatorname{span}(\rho)$. Let $\sim^*$ be the transitive closure of $\sim$. It follows from Proposition 2.2 that $\sim$ preserves the following two properties: "γ is defined at x" and "γ fixes x". It is then clear that $\sim^*$ preserves these properties as well. We will write $x \sim_\delta y$ if x and y are in the same cycle of δ, and $x \sim_\eta y$ if x and y are in the same cycle of η (so $\sim = \sim_\delta \cup \sim_\eta$).

We claim that $\sim^* = X \times X$. Consider the set $A = \{n-q-p+1, n-q-p+2, \dots, n-p\}$ of the rightmost elements of the cycles in η. Note that A contains $t = q/p$ multiples of p:

$$n-p,\ n-2p,\ \dots,\ n-tp. \tag{6.22}$$

Let $i \in \{1, 2, \dots, t-1\}$. We claim that $n-ip \sim^* n-(i+1)p$. First, we have $n-ip \sim_\delta n-ip-p+1$ since $(n-ip-p+1\ n-ip-p+2\ \dots\ n-ip)$ is a cycle in δ. Next, $n-ip-p+1$ is a rightmost element of a cycle in η that is different from τ (the last cycle). We have already observed that $n-ip-p+1-(q-1)$ is the preceding element in the same cycle. Thus

$$n-ip-p+1 \sim_\eta n-ip-p+1-(q-1) = n-q-ip-p+2.$$

Further, $n-q-ip-p+2 \sim_\delta n-q-ip-p+1$, and finally

$$n-q-ip-p+1 \sim_\eta n-q-ip-p+1+(q-1) = n-ip-p = n-(i+1)p.$$

To summarize,

$$n-ip \sim_\delta n-ip-p+1 \sim_\eta n-q-ip-p+2 \sim_\delta n-q-ip-p+1 \sim_\eta n-ip-p = n-(i+1)p.$$

It follows by the transitivity of $\sim^*$ that any two multiples of p from (6.22) are $\sim^*$-related. Let $x, y \in X$. Then there are $z, w \in A$ such that $x \sim_\eta z$ and $y \sim_\eta w$. Now, z must be in some cycle of δ whose rightmost element is a multiple of p. Since $z \in A$, that multiple must come from (6.22), that is, $z \sim_\delta n-jp$ for some $j \in \{1, 2, \dots, t\}$, where $t = q/p$. Similarly, $w \sim_\delta n-lp$ for some $l \in \{1, 2, \dots, t\}$. Hence

$$x \sim_\eta z \sim_\delta n-jp \sim^* n-lp \sim_\delta w \sim_\eta y.$$

Thus $x \sim^* y$, and so $\sim^* = X \times X$.

As $\gamma \neq 0$, γ must be defined on some element of X. Since $\sim^*$ preserves the statement "γ is defined at x" and $\sim^* = X \times X$, we have $\operatorname{dom}(\gamma) = X$.

Consider the cycle $\rho = (n-p+1\ \ n-p+2\ \ \dots\ \ n)$ in δ and the cycle

$$\tau = (n-p+2\ \ n-p+3\ \ n-p+4\ \ \dots\ \ n-1\ \ n\ \ n-p)$$

in η, and note that $\operatorname{span}(\rho) \cap \operatorname{span}(\tau)$ consists of $p-1$ elements. Thus, $\operatorname{span}(\rho h_\gamma^\delta) \cap \operatorname{span}(\tau h_\gamma^\eta)$ also consists of $p-1$ elements. However, for all cycles ρ' in δ and τ' in η, if $\rho' \neq \rho$, then $\operatorname{span}(\rho') \cap \operatorname{span}(\tau')$ consists of either 1 or 2 elements, where 2 is only possible when $n = p^2$. In the latter case, $p \geq 5$ (since $n > 9$), and so $p - 1 > 2$. If $n = p^k$ with $k > 2$, then $p - 1 > 1$ (even when $p = 3$). Hence $\rho h_\gamma^\delta = \rho$ since otherwise we would have $|\operatorname{span}(\rho h_\gamma^\delta) \cap \operatorname{span}(\tau h_\gamma^\eta)| < p - 1$. Applying the same argument to τ, we see that $\tau h_\gamma^\eta = \tau$.

These two conditions imply that the element $n - p$ that occurs in τ must be fixed by γ. Since $\sim^*$ preserves the statement "γ fixes x" and $\sim^* = X \times X$, it follows that γ fixes every element of X. So $\gamma = 1$, which is a contradiction. We have proved that $d(\alpha, \beta) \geq 5$.

It now follows from Proposition 6.13 that the diameter of $\mathcal{G}(\mathcal{I}(X))$ is 5. $\square$

We conclude this section with a discussion of the diameter of the commuting graph of the symmetric group $\mathrm{Sym}(X)$. Iranmanesh and Jafarzadeh have proved [20, Theorem 3.1] that if n and $n-1$ are not primes, then the diameter of $\mathcal{G}(\mathrm{Sym}(X))$ is at most 5. (If n or $n-1$ is a prime, then the diameter of $\mathcal{G}(\mathrm{Sym}(X))$ is ∞.)

Dolžan and Oblak have strengthened this result [13, Theorem 4] by showing that if n and $n-1$ are not primes, then the distance between $\alpha = (1\,2\,\ldots\,n)$ and $\beta = (1\,2\,\ldots\,n-1) \sqcup (n)$ in $\mathcal{G}(\mathrm{Sym}(X))$ is at least 5 (so the diameter of $\mathcal{G}(\mathrm{Sym}(X))$ is exactly 5). However, in our opinion, their proof contains a small point needing clarification. They state that if $\rho, \sigma, \tau \in \mathrm{Sym}(X)$ are such that $\rho - \sigma - \tau$ and the length of any cycle in ρ is relatively prime to the length of any cycle in τ, then σ must fix every point in X, and so $\sigma = 1$. However, this statement is not true, even with the additional assumptions that ρ is the power of an n-cycle and τ is the power of a disjoint join between an $(n-1)$-cycle and a 1-cycle. Let $X = \{1, 2, \ldots, 10\}$, and consider

$$\rho = (1\,2) \sqcup (3\,4) \sqcup (5\,6) \sqcup (7\,8) \sqcup (9\,10) \quad \text{and} \quad \tau = (1\,3\,5) \sqcup (2\,4\,6) \sqcup (7\,8\,9) \sqcup (10).$$

Then for $\sigma = (1\,3\,5) \sqcup (2\,4\,6) \sqcup (7) \sqcup (8) \sqcup (9) \sqcup (10)$, we have $\rho - \sigma - \tau$ but $\sigma \neq 1$.

It is possible to fix this gap by taking into account the special form of α and β in the original proof. We do this in the following lemma.

Lemma 6.18. *Let $X = \{1, 2, \ldots, n\}$, where neither n nor $n-1$ is a prime. Then, the distance between $\alpha = (1\,2\,\ldots\,n)$ and $\beta = (1\,2\,\ldots\,n-1) \sqcup (n)$ in $\mathcal{G}(\mathrm{Sym}(X))$ is at least* 5.

Proof. Suppose to the contrary that $d(\alpha, \beta) < 5$. Then $\alpha - \rho - \sigma - \tau - \beta$ for some $\rho, \sigma, \tau \in \mathrm{Sym}(X) - \{1\}$. It easily follows from the proof of Lemma 6.4 that $\rho = \alpha^m$ and $\tau = \beta^k$ for some $m \in \{1, \ldots, n-1\}$ and some $k \in \{1, \ldots, n-2\}$. We may assume that m is a proper divisor of n. (If m and n are relatively prime, then $\alpha^m = \alpha$, and so we may replace $\rho = \alpha^m$ in $\alpha - \rho - \sigma - \tau - \beta$ with $\alpha^{m'}$, where m' is any proper divisor of n. Similarly, if $m = lm'$, where l and n are relatively prime and m' is a proper divisor of n, we can replace $\rho = \alpha^m$ with $\alpha^{m'}$.) Similarly, we may assume that k is a proper divisor of $n-1$. Note that m and k are relatively prime. The permutation $\rho = \alpha^m$ is the join of m cycles, each of length $t = n/m$:

$$\rho = \alpha^m = \lambda_1 \sqcup \lambda_2 \sqcup \cdots \sqcup \lambda_m. \tag{6.23}$$

Consider the cyclic group $\mathbb{Z}_n = \{1, 2, \ldots, n\}$ of integers modulo n and the subgroup $\langle m \rangle$ of $\mathbb{Z}_n$. Then the spans of the cycles in $\rho = \alpha^m$ are precisely the cosets of the group $\langle m \rangle$. Since k and m are relatively prime, the cosets of $\langle m \rangle$ are

$$\langle m \rangle + k,\ \langle m \rangle + 2k,\ \ldots,\ \langle m \rangle + mk.$$

We may order the cycles in (6.23) in such a way that $\mathrm{span}(\lambda_i) = \langle m \rangle + ik$ for every $i \in \{1, 2, \ldots, m\}$.

Since (n) is the only 1-cycle in $\tau = \beta^k$, σ fixes n by Proposition 2.2. Recall that, by Proposition 2.2, if σ fixes some element of a cycle in ρ or in τ, then it fixes all elements of that cycle. Thus σ fixes all elements of $\mathrm{span}(\lambda_m)$ (since $\mathrm{span}(\lambda_m) = \langle m \rangle + mk = \langle m \rangle$ contains n). Since $m \le n/2$ and $k \le (n-1)/2$, there is $x \in \mathrm{span}(\lambda_m)$ such that $x + k \le n - 1$ (in standard, non-modular addition). Thus x and $x + k$ are in the same cycle of τ (since $\tau = \beta^k$ is a join of (n) and k cycles, each of length $s = (n-1)/k$, and the span of each cycle of length s is closed under addition of k modulo $n-1$). Hence, since σ fixes x, σ also fixes $x + k$. But $x + k \in \mathrm{span}(\lambda_1)$ (since $\mathrm{span}(\lambda_1) = \langle m \rangle + k = (\langle m \rangle + mk) + k$), and so σ fixes all elements of $\mathrm{span}(\lambda_1)$.

Applying the foregoing argument $m-2$ more times, to cycles $\lambda_1, \ldots, \lambda_{m-1}$, will show that σ fixes all elements of every cycle in ρ. Hence $\sigma = 1$, which is a contradiction. Thus $d(\alpha, \beta) \ge 5$. $\square$

7 Problems

In the process of proving Theorem 5.3, we came across a purely combinatorial conjecture that, if true, could simplify some of the proofs. As this combinatorial problem may be of interest regardless of the commuting graphs, we present it here.

Problem 7.1. Let $s, t > 1$ be natural numbers. Suppose A is an $s \times t$ matrix with entries from some set S such that:

(a) entries in each row of A are pairwise distinct;

(b) entries in each column of A are pairwise distinct; and

(c) there is no $a \in S$ such that a occurs in *every* row of A or a occurs in *every* column of A.

For given s and t find the smallest S that satisfies the three conditions above. In particular, it would be useful to simplify the proofs if such an A contains at least $s + t - 1$ distinct entries. However, Simon R. Blackburn (Department of Mathematics, Royal Holloway, University of London), in a private communication, provided an elegant counter-example.

For a graph $G = (V, E)$, denote by $\mathrm{Aut}(G)$ the group of automorphisms of G. Recall that $T(X)$ denotes the semigroup of full transformations on X. The automorphism groups of the commuting graphs of $T(X)$ and of $\mathcal{I}(X)$ are, comparatively to the size of the graphs themselves, very large. We list here their cardinalities for small values of $n = |X|$, which we have obtained using GAP [17] and GRAPE [36].

n	$\lvert\mathrm{Aut}(\mathcal{G}(\mathcal{I}(X)))\rvert$	$\lvert\mathrm{Aut}(\mathcal{G}(T(X)))\rvert$
2	$2^2 \cdot 3$	$2 \cdot 3$
3	$2^9 \cdot 3$	$2^5 \cdot 3^4$
4	$2^{38} \cdot 3^5$	$2^{34} \cdot 3$
5	$2^{231} \cdot 3^{44} \cdot 5^2$	$2^{410} \cdot 3^9 \cdot 5^2$

Problem 7.2. Describe the automorphism groups of the commuting graphs of $\mathcal{I}(X)$, $T(X)$, and $Sym(X)$.

The diameter of the commuting graph of $T(X)$ has been determined in [6, Theorems 2.22]. The following problem is easy to state and understand but appears to be very challenging.

Problem 7.3. Find the clique number of the commuting graph of $T(X)$.

A related problem is to determine the chromatic number of a given commuting graph.

Problem 7.4. Find the chromatic numbers of the commuting graphs of $\mathcal{I}(X)$, $T(X)$, and $\mathrm{Sym}(X)$.

The problem of finding the exact value of the diameter of $\mathcal{G}(\mathcal{I}(X))$ when n is odd and divisible by at least two primes remains open. By Lemmas 6.3 and 6.11, $d(\alpha, \beta) \leq 4$ for all $\alpha, \beta \in \mathcal{I}(X)$ such that α or β is not an n-cycle. So the exact value of the diameter (which is 4 or 5) depends on the answer to the following question.

Problem 7.5. *Let $n \geq 15$ be odd and divisible by at least two primes. Are there n-cycles $\alpha, \beta \in \mathcal{I}(X)$ such that $d(\alpha, \beta) = 5$?*

As the referee has pointed out, the idempotents of the symmetric inverse semigroup are "almost central" in that they commute with every element in their local submomoid. He or she suggests that an alternative, idempotent-free definition of the commuting graph might be of interest in the context of inverse semigroups. Such a modification would mainly affect our results on the diameter of the graph, which rely heavily on the commutativity of the idempotents.

Problem 7.6. Find the diameter of the graph that is obtained from $\mathcal{G}(\mathcal{I}(X))$ by removing all idempotent vertices.

The commuting graphs of finite groups have attracted a great deal of attention. There is a parallel concept of the non-commuting graph of a finite group, which has also been the object of intensive study [1, 12, 30, 40]. (A *non-commuting graph* of a finite nonabelian group G is a simple graph whose vertices are the non-central elements of G and two distinct vertices x, y are adjacent if $xy \neq yx$.) Once again, the concept carries over to semigroups, but nothing is known about the non-commuting graphs of semigroups.

Problem 7.7. Find the diameters, clique numbers, and chromatic numbers of the non-commuting graphs of $T(X)$ and $\mathcal{I}(X)$. Is it true that for every natural n, there exists a semigroup whose non-commuting graph has diameter n?

In the present paper and [6], the commuting graphs of $\mathcal{I}(X)$, $T(X)$, and their ideals have been investigated. However, there are many other subsemigroups of $\mathcal{I}(X)$ and $T(X)$ that have been intensively studied (see [14, 16]).

Problem 7.8. Calculate the diameters, clique numbers, and chromatic numbers of commuting and non-commuting graphs of various subsemigroups of $\mathcal{I}(X)$ and $T(X)$.

Acknowledgment The authors would like to thank the referee for his or her helpful remarks and for suggesting the interesting modification of the commuting graph mentioned in Problem 7.6.

The first author was partially supported by FCT through the following projects: PEst-OE/MAT/UI1043/2011, Strategic Project of Centro de Álgebra da Universidade de Lisboa; and PTDC/MAT/101993/2008, Project Computations in groups and semigroups .

The research of the second author leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement no. PCOFUND-GA-2009-246542 and from the Foundation for Science and Technology of Portugal.

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