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Inverse Semigroups with Idempotent-Fixing Automorphisms

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Abstract A celebrated result of J. Thompson says that if a finite group G has a fixed-point-free automorphism of prime order, then G is nilpotent. The main purpose of this note is to extend this result to finite inverse semigroups. An earlier related result of B. H. Neumann says that a uniquely 2-divisible group with a fixed-point-free automorphism of order 2 is abelian. We similarly extend this result to uniquely 2-divisible inverse semigroups.

Keywords inverse semigroup, automorphism group

1 Introduction and main results

An important result in finite group theory is the following due to J. Thompson [9].

Theorem 1 *Let G be a finite group with a fixed-point-free automorphism of prime order. Then G is nilpotent.*

The main purpose of this note is to extend this result to finite inverse semigroups. Standard references for inverse semigroups are ([1], Chap. 5), [3] [7]. We denote, as usual, the set of idempotents of a semigroup S by $E(S)$, the automorphism group by $\mathrm{Aut}(S)$, and the fixed point set of $\alpha \in \mathrm{Aut}(S)$ by $\mathrm{Fix}(\alpha) := \{x \in S \mid x\alpha = x\}$. Our first main result is the following.

Theorem 2 *Let S be a finite inverse semigroup and let $\alpha \in \mathrm{Aut}(S)$ have prime order and satisfy $\mathrm{Fix}(\alpha) = E(S)$. Then S is a nilpotent Clifford semigroup.*

Here, nilpotence of a finite Clifford semigroup is in the sense defined by Kowol and Mitsch [2].

An earlier result than Thompson's is the following of B. H. Neumann [5].

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Theorem 3 *Let G be a uniquely 2-divisible group with a fixed-point-free automorphism α of order 2. Then $x\alpha = x^{-1}$ for all $x \in G$ and hence G is abelian.*

Here uniquely 2-divisible means that the squaring map $x \mapsto x^2$ is a bijection. Neumann used this result to prove that a finite group with a fixed-point-free automorphism of order 2 must be abelian, for such a group must have odd order and then Theorem 3 applies. Neumann later outlined a different proof in the finite case in [6] by observing that an automorphism α being fixed-point-free is equivalent to the injectivity of the function $x \mapsto x^{-1} \cdot x\alpha$. By finiteness, the same function must also be surjective. This together with $\alpha^2 = 1$ easily implies the desired result. In the same paper, he showed that if one instead assumes $\alpha^3 = 1$, then G is nilpotent of class 2.

Theorem 3 is of interest on its own because the hypothesis is independent of cardinality. Our second main result is to generalize it to inverse semigroups.

Theorem 4 *Let S be a uniquely 2-divisible inverse semigroup and let $\alpha \in \mathrm{Aut}(S)$ satisfy $\alpha^2 = 1$ and $\mathrm{Fix}(\alpha) = E(S)$. Then $x\alpha = x^{-1}$ for all $x \in S$ and hence S is commutative.*

2 A Key Lemma

We will freely use standard identities in inverse semigroups, such as $(x^{-1})^{-1} = x$ and the antiautomorphic inverse property $(xy)^{-1} = y^{-1}x^{-1}$ ([1], Proposition 5.12). In the lemma below, we also use the natural partial order ([1], §5.2).

The critical tool in the proofs of Theorems 2 and 4 is the following lemma analogous to a key result in [6].

Lemma 1 *Let S be an inverse semigroup, let $\alpha \in \mathrm{Aut}(S)$, and define $\psi : S \to S$ by $x\psi = x^{-1} \cdot x\alpha$ for all $x \in S$. If $\mathrm{Fix}(\alpha) = E(S)$, then ψ is injective.*

Proof Assume $\mathrm{Fix}(\alpha) = E(S)$ and suppose $a\psi = b\psi$ for some $a, b \in S$. Then $ba^{-1} \cdot a\alpha = bb^{-1} \cdot b\alpha$. Applying α^{-1} to both sides, we get

$$(ba^{-1})\alpha^{-1}a = (bb^{-1})\alpha^{-1}b = bb^{-1}b = b\,. \tag{1}$$

Thus $(ba^{-1})\alpha^{-1}aa^{-1} = ba^{-1}$. Applying α to both sides of this, we get (starting with the right side) $(ba^{-1})\alpha = ba^{-1}(aa^{-1})\alpha = ba^{-1}aa^{-1} = ba^{-1}$. Thus $ba^{-1} \in \mathrm{Fix}(\alpha)$. By hypothesis, $ba^{-1} \in E(S)$, and so using (1), we get $ba^{-1}a = b$, that is, $b \leq a$ (see [1, Proposition 5.2.1]). By the obvious symmetry, we also have $a \leq b$, and thus $a = b$, which is what we desired to prove. □

The hypothesis of Lemma 1 cannot be weakened to $\mathrm{Fix}(\alpha) \subseteq E(S)$.

Example 1 Consider the three element chain $0 < a < 1$, and let α be the automorphism defined by $0\alpha = 0$, $a\alpha = 1$ and $1\alpha = 0$. Then $1 \cdot 1\alpha = 1 \cdot a = 1 = a \cdot 1 = a \cdot a\alpha$, so that ψ is not injective.

A weak converse of Lemma 1 holds, although we will not use it in our proofs of Theorems 2 and 4. We are grateful to the anonymous referee for suggesting the proof of the second part of the following lemma.

Lemma 2 *Let S be an inverse semigroup, let $\alpha \in \mathrm{Aut}(S)$, and define $\psi : S \to S$ by $x\psi = x^{-1} \cdot x\alpha$ for all $x \in S$. If ψ is injective, then* $\mathrm{Fix}(\alpha) \subseteq E(S)$*. If, in addition, we assume that for each $e \in E(S)$, the $\langle\alpha\rangle$-orbit of e is finite, then* $\mathrm{Fix}(\alpha) = E(S)$.

In particular, the last part of the lemma applies if α has finite order.

Proof First, suppose ψ is injective and $a\alpha = a$. Then

$$a\psi = a^{-1} \cdot a\alpha = a^{-1}a = a^{-1}aa^{-1}a = (a^{-1}a)^{-1} \cdot (a^{-1}a)\alpha = (a^{-1}a)\psi\,.$$

By injectivity of ψ, $a = a^{-1}a$, and thus $aa = aa^{-1}a = a$, as claimed.

For the remaining claim, let e be an idempotent and suppose $e\alpha^n = e$ for some $n \in \mathbb{N}$. An easy induction argument shows that for all $k \in \mathbb{N}$,

$$e\psi^k \in E(S)\,, \qquad e\psi^k = e\psi^{k-1} \cdot e\alpha^k\,, \qquad \text{and} \qquad e \cdot e\psi^k = e\psi^k\,.$$

Thus

$$e\psi^n = e\psi^{n-1} \cdot e\alpha^n = e\psi^{n-1} \cdot e = e \cdot e\psi^{n-1} = e\psi^{n-1}\,.$$

Since ψ is injective, we eventually get $e\psi = e$. This can be written as $e \cdot e\alpha = e$, that is, $e \le e\alpha$. Now since automorphisms preserve the natural partial order in an inverse semigroup, we have that $e\alpha^j \le e\alpha^{j+1}$ for all $j \in \mathbb{N}$. Thus $e \le e\alpha \le e\alpha^2 \le \cdots \le e\alpha^{n-1} \le e\alpha^n = e$. Therefore $e = e\alpha$ as desired. This completes the proof. □

We conclude this section by noting that the full converse of Lemma 1 is false.

Example 2 Let $(\mathbb{Z}, \wedge)$ be the integers with the usual meet operation. Define $x\alpha = x+1$. Then α is an automorphism: $(x\wedge y)\alpha = (x\wedge y)+1 = (x+1)\wedge(y+1) = x\alpha\wedge y\alpha$. We have $x\psi = x \wedge x\alpha = x \wedge (x+1) = x$ for all x, so that ψ is trivially injective. However, α evidently has no fixed points.

3 Proofs of the main results

We now prove Theorem 2. Recall that if S is an inverse semigroup and α is an automorphism of S, then we have $(a^{-1})\alpha = (a\alpha)^{-1}$. By Lemma 1, the map ψ is injective. Since S is finite, ψ is also surjective. For $x \in S$, let $y \in S$ satisfy $x = y\psi$. Then

$$xx^{-1} = y\psi \cdot (y\psi)^{-1} = y^{-1} \cdot y\alpha \cdot (y\alpha)^{-1} \cdot y = y^{-1} \cdot (yy^{-1})\alpha \cdot y = y^{-1} \cdot yy^{-1} \cdot y = y^{-1}y\,,$$

and

$$x^{-1}x = (y\psi)^{-1} \cdot y\psi = (y\alpha)^{-1} \cdot y \cdot y^{-1} \cdot y\alpha = (y^{-1} \cdot yy^{-1} \cdot y)\alpha = (y^{-1} \cdot y)\alpha = y^{-1}y\,.$$

We conclude that $x^{-1}x = xx^{-1}$. Therefore S is a completely regular and inverse semigroup, hence it is a Clifford semigroup (see [1, Theorem 4.2.1]). Now $S = \bigcup G_\beta$ is a (strong) semilattice of groups G_β (see [1, Theorem 4.2.1]). These groups, which are the $\mathcal{H}$-classes of S (see [8, Theorem II.1.4]), are permuted by α since automorphisms preserve Green's relations. But by assumption, α fixes the identity element of each group, and hence α restricts to an automorphism of each G_β. Now we apply Theorem 1 to conclude that each group G_β is nilpotent. Finally, we appeal to a key feature of the Kowol-Mitsch notion of nilpotence for finite Clifford semigroups: *if* $S = \bigcup G_\beta$ *is a strong semilattice of groups* G_β*, then* S *is nilpotent if and only if each* G_β *is nilpotent* ([2], Theorem 4.1, p. 442). This completes the proof of Theorem 2.

For Theorem 4, the squaring map $x \mapsto x^2$ is assumed to be bijective, and so we denote the unique square root of an element $x \in S$ by $x^{1/2}$. We have $(x\alpha)^{1/2} = (x^{1/2})\alpha$, as can be seen immediately from squaring both sides. Similarly, $(x^{1/2})^{-1} = (x^{-1})^{1/2}$, and we write $x^{-1/2}$ for this common expression.

For the function $x\psi = x^{-1} \cdot x\alpha$, we note that

$$(x\psi)\alpha = (x\alpha)^{-1} \cdot x\alpha^2 = (x\alpha)^{-1} \cdot x = (x\psi)^{-1} \tag{2}$$

since $\alpha^2 = 1$. Thus we compute

$$[(x\psi)^{-1/2}]\psi = (x\psi)^{1/2} \cdot ((x\psi)^{-1/2})\alpha = (x\psi)^{1/2}(x\psi)^{1/2} = x\psi\,,$$

using (2) in the last step. By Lemma 1, ψ is injective, and so we conclude

$$(x\psi)^{-1/2} = x \tag{3}$$

for all $x \in S$. Therefore

$$x\alpha = [(x\psi)^{-1/2}]\alpha = (x\psi)^{1/2} = [(x\psi)^{-1/2}]^{-1} = x^{-1}\,,$$

using (3) in the first and last equalities, and (2) in the second. Finally, the commutativity of S follows because the inversion mapping $x \mapsto x^{-1}$ is both an automorphism (thus $(xy)^{-1} = x^{-1}y^{-1}$) and an antiautomorphism (thus $(xy)^{-1} = y^{-1}x^{-1}$); now the identities $x^{-1}y^{-1} = y^{-1}x^{-1}$ and $(x^{-1})^{-1} = x$ imply that $xy = yx$. This completes the proof of Theorem 4.

4 Remarks and Problems

A property which was useful in our proofs is that an automorphism α of an inverse semigroup preserves the inversion map, that is, $(x^{-1})\alpha = (x\alpha)^{-1}$. This same property also holds for completely regular semigroups, and so it is natural to ask if analogs of our results hold in that setting as well. For instance, we offer the following:

Conjecture 1 Let S be a uniquely 2-divisible completely regular semigroup and let $\alpha \in \mathrm{Aut}(S)$ satisfy $\alpha^2 = 1$ and $\mathrm{Fix}(\alpha) = E(S)$. Then $x\alpha = x^{-1}$ for all $x \in S$.

Consideration of the identity mapping on a finite left zero band with at least two elements shows that one cannot strengthen the conclusion of the conjecture to commutativity.

One might also try regular involuted semigroups, that is, semigroups with a unary operation $'$ such that the identities

$$(xy)' = y'x' \qquad x'' = x \qquad x = xx'x\,.$$

hold. However, we do not get an immediate generalization of, say, Theorem 4. For instance, let S be the band with the following multiplication table:

$\cdot$	1	2	3	4
1	1	3	3	1
2	4	2	2	4
3	1	3	3	1
4	4	2	2	4

Let α be the identity mapping on S and let $'$ be the unary operation defined by $1' = 2$, $2' = 1$, $3' = 3$, $4' = 4$. Then $(S, \cdot, ')$ is a regular involuted semigroup, but we have neither $x\alpha = x'$ for all $x \in S$ nor that S is commutative.

Despite this, it is certainly reasonable to guess that other classes of regular semigroups might yield interesting results.

Problem 1 Extend Theorem 3 to other classes of regular semigroups.

Cancellative semigroups form another natural class of semigroups closely related to groups. Therefore the next problem is very natural.

Problem 2 Does an analog of Theorem 3 hold for cancellative semigroups?

Regarding nilpotence, we followed the definition of [2] for finite Clifford semigroups. This definition was motivated by the fact that nilpotence of (finite) groups can be characterized in various different ways, and the authors of [2] wished to keep these characterizations in (finite) inverse semigroups ([2], Main Theorem, p. 448). This is, of course, a rather strong requirement and suggests why this notion of nilpotence does not extend much beyond Clifford semigroups.

Problem 3 Find appropriate notions of nilpotence for other classes of semigroups, containing the class of all groups, such that the restriction of the notion to groups is equivalent to the usual one, and, in addition, a generalization of Theorem 1 holds for that class of semigroups.

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References

1. J. M. Howie, *Fundamentals of semigroup theory.* London Mathematical Society Monographs. New Series, 12. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
2. G. Kowol and H. Mitsch, *Nilpotent inverse semigroups with central idempotents*, Trans. Amer. Math. Soc. **271** (1982), 437–449.
3. M. V. Lawson, *Inverse semigroups. The theory of partial symmetries*, World Sci. Publ. Co., Inc., River Edge, NJ, 1998.
4. W. McCune, Prover9 and Mace4, version LADR–2009–11A, `http://www.cs.unm.edu/~mccune/prover9/`
5. B. H. Neumann, *On the commutativity of addition*, J. London Math. Soc. **15** (1940), 203–208.

6. B. H. Neumann, *Groups with automorphisms that leave only the neutral element fixed*, Arch. Math. (Basel) **7** (1956), 1–5.
7. M. Petrich, *Inverse semigroups*, Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1984.
8. M. Petrich and N. Reilly, *Completely regular semigroups*, Canadian Mathematical Society Series of Monographs and Advanced Texts **23**, Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999.
9. J. Thompson, *Finite groups with fixed-point-free automorphisms of prime order*, Proc. Nat. Acad. Sci. **45** (1959), 578–581.