Molaie's Generalized Groups are Completely Simple Semigroups

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Abstract

In [2] Molaei introduces generalized groups, a class of algebras of interest to physics, and proves some results about them.

The aim of this note is to prove that Generalized Groups are the Completely Simple Semigroups.

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1 The Main Theorem

Let **G** be a groupoid, that is, $\mathbf{G} = (G, f)$ where X is a non-empty set and f is a binary operation $f : G \times G \to G$. As usual, for $a, b \in G$, we denote $f(a, b)$ by ab. Suppose that **G** satisfies the following axioms:

(M1) $(\forall_{x,y,z\in G})\ (xy)z = x(yz)$;

(M2) $(\forall_{x\in G})(\exists^1_{e(x)\in G})\ xe(x) = x = e(x)x$;

(M3) $(\forall_{x\in G})(\exists_{x^{-1}\in G})\ xx^{-1} = e(x) = x^{-1}x$.

In [2] Molaei introduces these groupoids and calls them *Generalized Groups*. However, since the name Generalized Group already appears in literture defining a different algebraic structure, we are going to call the groupoids satisfying (M1)-(M3) *Molaei's Generalized groups*. It is well known that a groupoid satisfying (M1) is called a *Semigroup*.

We start by giving an example of a Molaei's generalized group. Let I, L be two nonempty sets and let G be a group. Moreover, let $P = (p_{l,i})_{l\in L, i\in I}$ be a $L \times I$ matrix with entries in G. In $I \times G \times L$ consider the following product: for all $(i, g, l), (j, h, m) \in I \times G \times L$,

$$(i, g, l)(j, h, m) = (i, gp_{m,j}h, m).$$

It is easy to verify that $I \times G \times L$ with this product is a semigroup which will be denoted by $\mathcal{M}(G; I, L; P)$.

Now let $(i, g, l) \in \mathcal{M}(G; I, L; P)$ and consider the element $(i, p_{l,i}^{-1}, l)$ of $\mathcal{M}(G; I, L; P)$. We claim that $(i, p_{l,i}^{-1}, l)$ is the unique element $e \in \mathcal{M}(G; I, L; P)$ such that $(i, g, l)e = e(i, g, l) = (i, g, l)$ and hence $\mathcal{M}(G; I, L; P)$ satisfies $(M1)$. In fact

$$(i, g, l)(i, p_{l,i}^{-1}, l) = (i, gp_{l,i}p_{l,i}^{-1}, l) = (i, g, l)$$

and

$$(i, p_{l,i}^{-1}, l)(i, g, l) = (i, p_{l,i}^{-1}p_{l,i}g, l) = (i, g, l).$$

Now suppose that $(i, g, l)(j, h, m) = (i, g, l)$. Then $(i, gp_{l,j}h, m) = (i, g, l)$ and hence $l = m$, $p_{l,j}^{-1} = h$. In the same way, $(j, h, m)(i, g, l) = (i, g, l)$ implies $j = i$ and $p_{m,i}^{-1} = h$. Thus $l = m, i = j$ and $p_{l,j}^{-1} = p_{l,i}^{-1} = p_{m,j}^{-1} = h$. Thus $(j, h, m) = (i, p_{l,i}^{-1}, l)$. It is proved that $(i, p_{l,i}^{-1}, l) = e((i, g, l))$ and hence $\mathcal{M}(G; I, L; P)$ satisfies $(M1)$.

To prove that $\mathcal{M}(G; I, L; P)$ satisfies $(M2)$, let $(i, g, l) \in \mathcal{M}(G; I, L; P)$ and consider the element $(i, p_{l,i}^{-1}g^{-1}p_{l,i}^{-1}, l)$ of $\mathcal{M}(G; I, L; P)$. It is obvious that

$$(i, p_{l,i}^{-1}g^{-1}p_{l,i}^{-1}, l)(i, g, p) = (i, g, p)(i, p_{l,i}^{-1}g^{-1}p_{l,i}^{-1}, l) = (i, p_{l,i}^{-1}, l) = e((i, g, p))$$

and hence $\mathcal{M}(G; I, L; P)$ is a generalized group. In the main theorem we prove that every generalized group is isomorphic to some semigroup $\mathcal{M}(G; I, L; P)$. But before stating the main theorem we introduce a definition.

A semigroup S is said to be *Completely Simple* if

(C1) $SaS = S$, for all $a \in S$;

(C2) If $e, f \in S$ are idempotents (that is, $e^2 = e$ and $f^2 = f$) and $e = ef = fe$, then $e = f$.

Theorem 1.1 *Let S be a semigroup. The following are equivalent:*

(1) *S is isomorphic to some $\mathcal{M}(G; I, L; P)$;*

(2) *S is completely simple;*

(3) *S is a Molaie's generalized group.*

Proof: We have proved above that every semigroup $\mathcal{M}(G; I, L; P)$ is a Molaie's generalized group. Therefore (1) implies (3).

That (2) implies (1) follows from Rees Theorem, the most famous theorem in Semigroup Theory. An account of this theorem can be found in [1] (Chapter 3).

It remains to prove that (3) implies (2). Observe first that for all idempotent $e \in S$ we have $e(e) = e$ (because $ee = ee = e$). Now suppose that we have two idempotents $e, f \in S$ such that $e = ef = fe$. Then, by $(M2)$, $e(e) = f$ and hence $e = f$ (because $e(e) = e$, for all idempotents in S). Condition $(C2)$ is proved.

Now observe that $Se(x)S = SxS$, for all $x \in S$. In fact $x = e(x)e(x)x$ (because, by $(M2)$, $e(x)e(x)x = e(x)x = x$) and hence $SxS \subseteq Se(x)S$. Conversely,

$$e(x)xx^{-1} = xx^{-1}xx^{-1} = x(x^{-1}x)x^{-1} = xe(x)x^{-1} = xx^{-1} = e(x).$$

Thus $e(x) = e(x)xx^{-1}$ and hence $Se(x)S \subseteq SxS$. Thus, for all $x \in S$, we have $Se(x)S = SxS$.

To prove $(C1)$ we want to prove that $S \subseteq SyS$, for all $y \in S$. Thus let $x, y \in S$. We claim that $x \in SyS$. In fact, let $z = e(x)ye(x)$. It is obvious that $ze(x) = e(x)z = z$ and hence, by $(M2)$, $e(z) = e(x)$. Now

$$x \in Se(x)S = Se(z)S = SzS = Se(x)ye(x)S \subseteq SyS.$$

It is proved that $S \subseteq SyS$, for all $y \in S$, and hence S is completely simple. The theorem follows. ■

Finally we observe that [2] there is a problem with the proof of Theorem 2.3. Let S be a Molaei's generalized group and $x \in S$. Then let $R(X) = \{y \in S \mid xy = e(x) = yx\}$. In the referred theorem it is proved that $|R(x)| = 1$ but the proof uses the unproved assumption that $e(x)e(x^{-1}) = e(x^{-1})e(x)$.

Lemma 1.2 *Let S be a Molaei's generalized group. Then the following are equivalent*

(a) $e(x) = e(x^{-1})$, *for every* $x \in S, x^{-1} \in R(x)$.

(b) $(x^{-1})^{-1} = x$, *for every* $x \in S, x^{-1} \in R(x)$ *and* $(x^{-1})^{-1} \in R(x^{-1})$.

(c) $x^{-1}xx^{-1} = x^{-1}$, *for every* $x \in S, x^{-1} \in R(x)$.

(d) $|R(x)| = 1$, *for every* $x \in S$.

(e) $e(x)e(x^{-1}) = e(x^{-1})e(x)$, *for every* $x \in S, x^{-1} \in R(x)$.

Proof: We start by proving that (a) implies (b). Let $x \in S$ and $x^{-1} \in R(x)$. Then

$$\begin{aligned} e(x^{-1}) = x^{-1}(x^{-1})^{-1} &\Rightarrow xe(x^{-1}) = xx^{-1}(x^{-1})^{-1} \\ &\Rightarrow xe(x^{-1}) = e(x)(x^{-1})^{-1} \\ &\Rightarrow xe(x) = e(x^{-1})(x^{-1})^{-1} (\text{by } (a)) \\ &\Rightarrow x = (x^{-1})^{-1} \end{aligned}$$

thus proving (b).

To prove that (b) implies (c), let $x \in S, x^{-1} \in R(x)$ and $(x^{-1})^{-1} \in R(x^{-1})$. Then we have

$$\begin{aligned} x^{-1}(x^{-1})^{-1} = e(x^{-1}) &\Rightarrow x^{-1}(x^{-1})^{-1}x^{-1} = e(x^{-1})x^{-1} \\ &\Rightarrow x^{-1}(x^{-1})^{-1}x^{-1} = x^{-1} \\ &\Rightarrow x^{-1}xx^{-1} = x^{-1} \text{ (by } (b)) \end{aligned}$$

and (c) follows.

To prove that (c) implies (d), let $y, z \in R(x)$. Then we have

$$y = yxy = y(xy) = ye(x) = y(xz) = (yx)z = e(x)z = (zx)z = z,$$

where the first and last equality follow from (c). It is proved that $|R(x)| = 1$.

Conversely, (d) implies (c). In fact, let $x^{-1} \in R(x)$. Then

$$x(x^{-1}xx^{-1}) = (xx^{-1})(xx^{-1}) = e(x)xx^{-1} = xx^{-1} = e(x)$$

and

$$(x^{-1}xx^{-1})x = x^{-1}xe(x) = x^{-1}x = e(x).$$

Thus $x^{-1}xx^{-1} \in R(x)$ and hence, by (d), $x^{-1}xx^{-1} = x^{-1}$.

We prove now that (c) implies (a). In fact, for all $x \in S$ and $x^{-1} \in R(x)$, we have $e(x)x^{-1} = x^{-1}xx^{-1} = x^{-1}$ and $x^{-1}e(x) = x^{-1}xx^{-1} = x^{-1}$. Thus $e(x) = e(x^{-1})$. Thus the first four conditions are all equivalent.

Finally, it is obvious that (a) implies (e). Conversely, let $x \in S$ and $x \in R(x)$. Then

$$\begin{aligned} e(x)e(x^{-1}) = e(x^{-1})e(x) &\Rightarrow e(x)e(x^{-1})x^{-1} = e(x^{-1})e(x)x^{-1} \\ &\Rightarrow e(x)x^{-1} = e(x^{-1})e(x)x^{-1} \\ &\Rightarrow e(x)x^{-1}x = e(x^{-1})e(x)x^{-1}x \\ &\Rightarrow e(x)e(x) = e(x^{-1})e(x)e(x) \\ &\Rightarrow e(x) = e(x^{-1})e(x). \end{aligned}$$

Now

$$e(x) = e(x^{-1})e(x) \Rightarrow e(x)x = e(x^{-1})e(x)x \Rightarrow x = e(x^{-1})x.$$

Similarly,

$$e(x) = e(x)e(x^{-1}) = e(x^{-1})e(x) \Rightarrow xe(x) = xe(x^{-1})e(x) \Rightarrow x = xe(x^{-1}).$$

It is proved that $e(x) = e(x^{-1})$. The lemma follows. ■

It is easy to prove, using Theorem 1.1, that each one of the equivalent conditions in the previous lemma hold in a Molaei's generalized group. However, we provide a direct proof for (a).

Lemma 1.3 *Let $x \in S$ and $x \in R(x)$. Then $e(x) = e(x^{-1})$.*

Proof: Let $x \in S$. Then

$$\begin{aligned} e(x)e(x^{-1}) &= (xx^{-1})((x^{-1})^{-1}x^{-1}) \\ &= x(x^{-1}(x^{-1})^{-1})x^{-1} \\ &= xe(x^{-1})x^{-1} \\ &= x(e(x^{-1})x^{-1}) \\ &= xx^{-1} \\ &= e(x). \end{aligned}$$

Similarly we prove that $e(x^{-1})e(x) = e(x)$. Thus it is proved that

$$e(x)e(x^{-1}) = e(x) = e(x^{-1})e(x). \tag{1}$$

On the other hand

$$e(x)e(x) = (xx^{-1})(xx^{-1}) = x(x^{-1}x)x^{-1} = xe(x)x^{-1} = (xe(x))x^{-1} = xx^{-1} = e(x). \tag{2}$$

Thus, by (M2) together with (1) and (2), it follows that $e(x) = e(x^{-1})$. The lemma follows. ∎

References

[1] J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford Science Publications, Oxford, (1995).

[2] M. R. Molaei, *Generalized Groups*, Buletinul Institutului Politechnic din Iaşi, Tom. XLV (XLIX), 21-24 (1999).

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