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Geometric Harmonic Analysis

C_p weights, John–Nirenberg estimates and Hajłasz capacity density conditions

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Thunder only happens when it's raining

Fleetwood Mac

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Resumen de la tesis

Esta tesis es la recopilación de los resultados obtenidos durante mi doctorado, que empezó en enero de 2018 y terminará a finales del 2021. La materia principal está dividida en 5 Capítulos, los Capítulos 2–6. Estos capítulos se pueden reunir en 3 partes, siendo la primera los Capítulos 2 y 3, relacionada con pesos C_p; la segunda, los Capítulos 4 y 5, relacionada con el teorema de John–Nirenberg; y la última, el capítulo 6, dedicada a la capacidad de Hajłasz. En estas páginas damos un resumen de los resultados obtenidos en cada parte, así como una breve motivación de los mismos.

La clase C_p de pesos

Uno de los principales conceptos en análisis matemático, y más precisamente en análisis armónico, es la clase A_∞ de Muckenhoupt. Esta clase, introducida por Muckenhoupt en los años 70, ha sido un eje principal del análisis armónico desde su origen. Entre las diferentes propiedades importantes de la clase A_∞, encontramos la doblantez, que grosso modo quiere decir que la medida pesada de una bola grande se puede controlar por la medida pesada de bolas más pequeñas contenidas en la misma, siempre que la razón entre los radios esté controlada

Pese a que los pesos A_∞ son muy importantes e incluso llegamos a utilizarlos en esta tesis, nuestro principal objeto de estudio es la clase C_p. Debido a la dificultad de trabajar con esta clase de pesos, no hay un tratamiento sistemático de ellos. Una de las dificultades de los pesos C_p es que no son necesariamente doblantes, lo que es una de las principales diferencias con respecto a A_∞. El primer capítulo de esta tesis se puede ver como un listado de técnicas que pueden ser útiles a la hora de trabajar con estos pesos.

La clase C_p de pesos fue introducida por Muckenhoupt en [100], y está relacionada con la desigualdad en norma pesada entre la transformada de Hilbert y la función maximal de Hardy–Littlewood. Esta desigualdad había sido probada para pesos A_∞ por Coifman y Fefferman [20], pero, como Muckenhoupt demostró, existen más pesos para los cuales se cumple. Encontró una condición necesaria para que esta desigualdad se cumpla, y la bautizó C_p. La p en el nombre C_p responde al exponente de la norma L^p pesada entre la transformada de Hilbert y la función maximal de Hardy–Littlewood. Desafortunadamente, Muckenhoupt no pudo demostrar que la condición sea también suficiente, pero sí que llegó a conjeturarlo. Esta conjetura, hoy en día conocida como la conjetura de Muckenhoupt, aún sigue sin resolverse.

Poco después de los resultados de Muckenhoupt, Sawyer estudió el problema en dimensiones superiores [109]. Usando métodos similares a los de Muckenhoupt, demostró que C_p es una condición necesaria para que la desigualdad en norma pesada entre cada una de las transformadas de Riesz y la función maximal de Hardy–Littlewood. Esto es una generalización directa de los resultados de Muckenhoput a dimensiones superiores. Pero no sólo esto, en el mismo trabajo Sawyer demostró que la condición C_q es suficiente para que la desigualdad en L^p se cumpla, siendo $1 < p < q$. Es pertinente en este momento comentar que estas clases están ordenadas en el sentido de que C_p contiene a C_q siempre que $p < q$, es decir, C_q es una condición más restrictiva que C_p. Es más, Sawyer demostró que, en este caso, la desigualdad se cumple para todo operador de Calderón–Zygmund. Claramente, los resultados de Sawyer no resuelven la conjetura de Muckenhoupt, pero es una gran respuesta parcial.

Aunque la conjetura no esté resuelta, había esperanza para una solución sencilla en términos de automejora. Es abiertamente conocido que un peso en A_p pertenece a $A_{p-\varepsilon}$ para cierto $\varepsilon > 0$ pequeño que depende del peso. Si una condición similar se cumpletra en el contexto de C_p, la conjetura se resolvería automáticamente. Pero no es así, como probaron Kahanpää y Mejlbro [69]. En dimensión uno y para cualquier $p > 1$, construyeron un peso que pertenece a C_p pero no a ningún C_q si $q > p$. En la sección 2.9 damos una generalización de este resultado a un contexto un poco más general, dando una nueva prueba del mismo resultado. Estas construcciones juegan con la geometría del soporte del peso, lo que es un ejemplo del comportamiento extraño que estos pesos pueden tener.

Unos años después del resultado de Kahanpää y Mejlbro, una desigualdad nueva fue demostrada en el contexto de C_p. En [117], Yabuta demostró que la desigualdad de Fefferman–Stein se cumple en L^p con peso si el peso está en C_q para algún $1 < p < q$. Esta desigualdad es la desigualdad en norma con peso entre el operador maximal de Hardy–Littelwood y el operador maximal agudo de Fefferman–Stein, que se cumple para funciones acotadas con soporte compacto. En este contexto, la necesidad de C_p también fue probada, lo que establece el paralelismo entre esta desigualdad y la de Coifman–Fefferman. Estas dos desigualdades estan estrechamente relacionadas, con lo que es esperable este tipo de paralelismos.

Los resultados mencionados anteriormente eran los únicos resultados conocidos durante un largo tiempo, hasta que este problema fue revisitado. Lerner dio un paso en adelante en la solución de la conjetura de Muckenhoupt. Introdujo una clase, llamada $\tilde{C}_p$ que está contenida en C_p y contiene a C_q si $1 < p < q$, y demostró que esta clase es suficiente para que se cumplan tanto la desigualdad de Coifman–Fefferman como la de Fefferman–Stein.

Varias estimaciones para distintos operadores pueden encontrarse en el trabajo de Cejas, Li, Pérez y Rivera-Ríos [17]. Entre estas estimaciones están algunas desigualdades de tipo débil, así como desigualdades con operadores multilineales de Calderón–Zygmund.

Pese a que en la última década, se han cuantificado satisfactoriamente varias desigualdades con peso en términos del peso, esto sólo ha ocurrido para pesos A_p o A_∞. En esta tesis presentamos una manera de hacer lo mismo para pesos C_p, y daremos una cuantificación de las desigualdades de Coifman–Fefferman y de Fefferman–Stein. Para ello, presentamos una constante, llamada la constante C_p del peso, que codifica el tamaño del peso en la clase C_p. La idea es que cuanto más pequeña sea la constante, mejor es el peso.

Una vez hemos definido esta constante, demostramos una desigiualdad inversa de Hölder, más débil que la estándar, que caracteriza la clase C_p. Cuantificamos el exponente de esta desigualdad en términos de la constante C_p. Esta cuantificación es en realidad un resultado paralelo a la desigualdad de Hölder inversa precisa de Hytönen, Pérez y Rela , [62, 63]. Por este motivo, afirmamos que nuestra cuantificación es también precisa.

La definición de la constante C_p, así como la desigualdad de Hölder inversa, están contenidas en el capítulo 2. Además, en ese capítulo se encuentra una discusión de la propia clase C_p, algunas de sus propiedades principales y algunos ejemplos. También expandimos el contraejemplo de automejora de Kahanpää y Mejlbro. Nuestra prueba se generaliza a dimensiones superiores.

La traducción de la cuantificación de la desigualdad de Hölder inversa a las cuantificaciones de las desigualdades en norma no es difícil una vez que sean identificadas las herramientas que son necesarias. En este caso, utilizamos una desigualdad de los buenos lambdas que tomamos prestada desde [7]. Esta desigualdad combinada con nuestra desigualdad de Hölder inversa nos permite ontener una desigualdad cuantificada de la desigualdad de Coifman–Fefferman. Aun así, debido al comportamiento no local de las clases C_p, aparece un término logarítmico en la constante, con lo que no podemos decir que esta dependencia sea precisa. El término logarítmico es totalmente inevitable mediante nuestros métodos.

Aunque por lo general los operadores sparse suelen dar resultados precisos en términos de la constante de los pesos, en este caso no es así debido a la naturaleza no local e estos pesos. Aún así, damos estimaciones de estos operadores en el contexto de C_p, que aunque no sean precisas no dejan de ser interesantes por su novedad, ya que incluso de manera cualitativa no eran conocidas hasta ahora. La cuantificación de la desigualdad de Coifman–Fefferman para operadores de Calderón–Zygmund, operadores integrales singulares rough y formas sparse está en el Capítulo 3. La cuantificación de la desigualdad de Fefferman–Stein se pospone hasta el Capítulo 4, en el que también demostramos una desigualdad de los buenos lambdas entre los operadores maximal y maximal agudo con el decaimiento correcto, que es exponencial.

Estimaciones para BMO

La segunda parte de esta tesis está dedicada a obtener estimaciones para funciones de tipo BMO y está contenida en los capítulos 4 y 5.

El espacio de funciones de oscilación media acotada, BMO, es un espacio clásico en el análisis matemático. Sirve como una alternativa adecuada a L^∞ en ciertos casos, como, por ejemplo, la integral singular de una función acotada no está acotada pero sí en BMO. Aunque es más grande que L^∞, (y por tanto es una condición más débil) este espacio es suficientemente grande como para servir de punto de interpolación.

Más allá de ser el sustituto de L^∞ en alguna situación, el espacio BMO es interesante por derecho propio. La propiedad más importante es el teorema de John–Nirenberg, que afirma que estas funciones son en realidad localmente exponencialmente integrables. Esto puede verse como una propiedad de automejora, ya que empezando con una condición de integrabilidad L^1, obtenemos una integrabilidad exponencial. Fenómenos parecidos ocurren para otros objetos, como desigualdades de Poincaré o de Poincaré–Sobolev, y también para objetos geométricos como condiciones de densidad de capacidad como en el Capítulo 6.

En relación a el espacio BMO, está la función maximal aguda de Fefferman–Stein. Que esta función esté acotada es equivalente a que la función original esté en BMO. Pero esta no es la primera vez que esta función maximal aparece en esta tesis, ya que ya apareció de manera tangencial en relación a los pesos C_p. Fue Yabuta quien demostró en [117] la relación entre la función maximal de Hardy–Littlewood y la función maximal aguda de Fefferman–Stein en el contexto de pesos C_p.

Para obtener una cuantificación de esta desigualdad en términos de la constante C_p del peso, necesitábamos una desigualdad de los buenos lambdas con decaimiento exponencial entre las funciones maximales de Hardy–Littlewood y Fefferman–Stein. Una desigualdad de estas características no estaba disponible, así que para obtenerla trazamos un nuevo camino, que nos llevó a obtener dos extensiones del teorema de John–Nirenberg. La cuantificación de la desigualdad de Fefferman–Stein para pesos C_p aparece como consecuencia natural de estas extensiones, pero también siguen resultados adicionales.

Entre las consecuencias obtenidas de las extensiones de John–Nirenberg, está una versión de una desigualdad con peso de Muckenhoupt y Wheeden. Esta desigualdad con peso para pesos A_p se puede interpretar como un resultado de automejora, porque empezando con integrabilidad de L^1 sin peso obtenemos $L^{r,\infty}$ con peso para algún $r > 1$.

Las extensiones de John–Nirenberg y sus consecuencias están en el capítulo 4.

Como hemos comentado anteriormente, el teorema de John–Nirenberg se puede interpretar como un resultado de automejora. Partiendo de cotas para oscilaciones de tipo L^1 obtenemos cotas para oscilaciones de tipo exponencial. Tenemos un punto de partida, en este caso, oscilaciones medias acotadas, y mejoramos ese punto de partida a una condición mejor, en este caso la integrabilidad exponencial. Una pregunta natural sería si se puede tomar un punto de partia más débil, es decir, si empezando con una condición más suave que BMO se puede obtener el mismo resultado. Si es así, ¿cuán débil puede ser esa condición?

Esta no es una nueva pregunta, ya que fue planteada por John [65] y por Strömberg [113]. La condicion minimal correcta para BMO es la oscilación media acotada en términos de oscilaciones L^φ para una función cóncava φ, que puede arbitrariamente lento. Recientemente, Logunov, Slavin, Stolyarov, Vasyunin y Zatitskiy [90] dieron una estimación explícita y cuantitativa de este resultado, dando una estimación de la norma de BMO de la función en términos de la escala de φ. Esta estimación tiene la desventaja de que no es homogénea en la función. En este trabajo, damos una prueba nueva y completamente transparente del mismo resultado, que resulta en una estimación homogénea. Nuestra prueba se puede extender a otras geometrías.

También estudiamos el mismo problema en contextos más generales que la geometría euclídea. Por ejemplo, extendemos nuestros resultados a espacios de tipo homogéneo, que son espacios quasi-métricos con una medida doblante. Podemos realizar esta extensión porque nuestro método en el espacio euclídeo es bastante sencillo y por tanto fácilmente generalizable. Es más, también utilizamos la geometría peculiar de $\mathbb{R}^n$ para obtener el mismo resultado para ciertas medidas no doblantes en $\mathbb{R}^n$. No podemos dar el mismo resultado en espacios métricos generales con medidas no doblantes, ya que la geometría de $\mathbb{R}^n$ es muy especial. Todos estos resultados de minimalidad de BMO están en el Capítulo 5.

Capacidades de Hajłasz

En la última parte de esta tesis, el capítulo 6, tratamos condiciones de densidad de capacidad en términos de gradientes de Hajłasz y su automejora en espacios métricos abstractos. Esta es la primera vez se obtiene la automejora de una condición de densidad de capacidad en términos de un gradiente no local. La manera en que esta última parte está conectada con el resto de la tesis no es del todo trivial. Empezamos intentando demostrar propiedades de automejora de ciertas desigualdades de Hardy fraccionarias en espacios métricos abstractos, lo que estaría más conectado con la Sección 4.5. Tal resultado no pudo ser obtenido, pero en su búsqueda dimos con los resultados que aquí presentamos.

El estudio de automejora de condiciones de densidad de capacidades fue iniciado por Lewis [88], donde se estudió la automejora de una condición de densidad de capacidad en términos de potenciales de Riesz en $\mathbb{R}^n$. A este resultado le siguió el trabajo de Mikkonen [98], donde se obtuvieron estimadas de tipo Maz'ya con peso para el p-Laplaciano, así como el trabajo de Björn, MacManus y Shanmugalingam [6], donde se obtuvieron estimaciones parecidas en espacios métricos. En este último trabajo, se utilizan gradientes superiores, que son una manera de introducir el concepto de derivada a espacios métricos. Estos gradientes superiores son objetos locales, ya que su valor solamente depende de un entorno del punto.

En este trabajo, trabajaremos con gradientes de β-Hajłasz, que fueron introducidos por Hajłasz en [49] para $\beta = 1$. Su naturaleza es altamente no local por su definición, pero asimismo su definición es bastante natural. En el caso fraccionario $0 < \beta < 1$, la misma definición aparece de manera orgánica. Es más, se puede demostrar que los gradientes superiores son en realidad gradientes 1-Hajłasz, lo que de alguna manera quiere decir que los gradientes de Hajłasz son algo más generales y una herramienta más versátil que los gradientes superiores.

Una de las ventajas principales de trabajar con estos gradientes de Hajłasz es que las desigualdades de Poincaré se cumplen para todo exponente sin ninguna hipótesis extra en la medida. Es decir, para cualquier función y cualquiera de sus posibles gradientes de Hajłasz se cumple la desigualdad de Poincaré pertinente, ver Sección 6.8. Esto no es cierto para otras derivadas, como los gradientes superiores, y normalmente al trabajar con estos gradiente se requiere la hipótesis ad hoc de que desigualdades de Poincaré se cumplan. Esta hipótesis extra excluye ciertos espacios de medida doblantes en $\mathbb{R}$, ver [5].

En este trabajo, introducimos una condición de densidad de capacidad similar a otras condiciones de densidad de capacidad, pero en términos de gradientes de Hajłasz . Esta condición depende de dos parámetros, el orden de derivación β y el parámetro de tamaño p, que se mide en términos de integrabilidad. Probamos que esta condición de densidad de capacidad se automejora en ambos parámetros β y p. Más precisamente, demostramos que un conjunto E satisface una condición de densidad de (β, p)-capacidad si y sólo si su codimensión superior de Assouad es estrictamente menor que βp. Es decir, siempre habrá un pequeño margen para bajar un poco tanto β como p de manera que su producto siga siendo mayor que la codimensión superior de Assouad del conjunto.

Esta caracterización de la condición de densidad de capacidad en términos de la codimensión superior de Assouad es bastante técnica. Es moderadamente sencillo demostar que la cota en la codimensión implica la condición de densidad de capacidad, utilizando una desigualdad de tipo Maz'ya. También es relativamente sencillo demostrar que la condición de densidad de capacidad implica una cota no estricta en la codimensión de Assouad. La parte complicada es obtener la cota estricta.

Para ello, combinamos una técnica de utilizar desigualdades de Poincaré y de Hardy en este contexto, y utilizamos técnicas conocidas de automejora para estas desigualdades. El estudio de estas propiedades de automejora fue iniciada por Keith y Zhong en el celebrado trabajo [71]. En este camino, nos unimos a una línea de investigación iniciada por Korte, Lehrbäck y Tuominen en [76], donde relacionaron una condición similar a nuestra condición de densidad de capacidad a desigualdades de Hardy. Combinamos todos estos métodos y los adaptamos a nuestro contexto para demostrar nuestros resultados.

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1

Introduction

This thesis is the compilation of the results obtained during my PhD, which started in January 2018 and is being completed in the end of 2021. The main matter is divided into five chapters, Chapters 2–6. Each of these chapters has its own introductory part, some longer some shorter. This chapter is intended to be an introduction to the whole thesis. Without going into technical details, in this Chapter we will not only motivate the results and the content of the dissertation, but we also explain how and why these results came to be studied. We also introduce the main notation and some preliminary concepts that will be used throughout the dissertation.

1.1 The C_p class

One of the main concepts in Analysis, and more in particular in Harmonic Analysis, is the class A_∞ of weights. This class was introduced my Muckenhoupt in the 70's and has since shaped the history of harmonic analysis. Among the many important properties of A_∞ weights, there is the doubling property, which grosso modo states that the weighted measure of a big ball is controlled by the weighted measure of smaller balls contained in it, provided that the ratio between the radii is controlled.

Although A_∞ weights are so important and we do use them in this dissertation, we are mainly concerned with the C_p class. Due to the difficulty of working with this class of weights, there is not a systematic treatment of it. One of the difficulties of the C_p weights is that they need not be doubling, which is one of the main differences of these classes with A_∞. The first chapter of this thesis can be seen as a collection of techniques that can be useful for working with the C_p classes.

The C_p classes of weights were introduced in [100] by Muckenhoupt, and are related to the weighted norm inequality between the Hilbert transform and the Hardy–Littlewood maximal function. This inequality had been shown to hold in any dimension for A_∞ weights by Coifman and Fefferman [20], but, as Muckenhoupt proved, there are more weights for which that inequality holds. He found a necessary condition, which he denoted C_p. The p in the name C_p refers to the exponent for the weighted L^p norm inequality between the Hilbert transform and the Hardy–Littlewood maximal operator. Alas, Muckenhoupt was not able to prove that this condition is sufficient for the norm inequality to hold, but he conjectured that it is. That conjecture, known as Muckenhoupt's conjecture, is still not solved.

Shortly after Muckenhouopt's results, Sawyer studied the problem in higher dimensions [109]. Using the methods of Muckenhoupt, he proved that C_p is a necessary condition for the norm inequality between all Riesz transforms and the Hardy–Littlewood maximal operator to hold. This is a direct generalization to higher dimensions of the result by Muckenhoupt. But not only this, Sawyer was able to prove that C_q condition is sufficient for the L^p norm inequality to hold, when $q > p > 1$. It is pertinent to comment here that these classes are nested in the sense that C_p contains C_q for $p < q$, that is, C_q is a stronger condition than C_p. Moreover, Sawyer proved that, in this case, the inequality actually holds for all Calderón–Zygmund operators. That inequality is known as the Coifman–Fefferman inequality. Clearly, the result of Sawyer does not directly solve the conjecture, but it is a great partial answer.

Even if the conjecture was not solved, there was hope for an easy solution in the terms of self-improvement. It is well known that a weight in A_p belongs to $A_{p-\varepsilon}$ for some small $\varepsilon > 0$ that depends on the weight. If a similar property were to hold for C_p weights, that is, if for a weight in C_p there would exist a positive $\varepsilon > 0$ such that the weight belonged to $C_{p+\varepsilon}$, then Sawyer's result would imply the positive answer to the conjecture. That hope was hastily dismissed by Kahanpää and Mejlbro [69], who, in dimension one and for any $p > 1$, constructed a weight that belongs to C_p but not to C_q for any $q > p$. This construction plays with the support of the weight and serves as an example of the strange behavior that C_p weights can have.

A few years later, a different type of inequality was proved to hold in the context of C_p weights. In [117], Yabuta showed that the Fefferman–Stein inequality in weighted L^p spaces holds for C_q weights if $q > p > 1$. This inequality is the weighted norm inequality between the Hardy–Littlewood maximal function and the Fefferman–Stein sharp maximal function, and it holds for bounded functions of compact support. In the same work, the necessity of the weight belonging to C_p was also proved. Therefore, the same dynamic as for the Coifman–Fefferman inequality also happens for the Fefferman–Stein inequality. These two inequalities are deeply related, so the parallelism for C_p weights between both of them is understandable.

The results mentioned above were the only results on C_p weights for a long time, until the problem was revisited once again. Lerner made a step forward in solving Muckenhoupt's conjecture in [83]. He introduced what he called the class $\widetilde{C}_p$ of weights, which is contained by C_p and contains C_q for all $q > p$. He actually showed that this new class is sufficient for the Fefferman–Stein inequality to hold and, therefore, also the Coifman–Fefferman inequality to hold.

Many estimates for C_p weights were also given by Cejas, Li, Pérez and Rivera-Ríos in [17]. In that work, a wide collection of new estimates are given for C_p weights, that include weak-type Coifman–Fefferman estimates and also Coifman–Fefferman estimates for linear operators that satisfy a condition involving the Fefferman–Stein maximal operator. Among these operators one can find multilinear Calderón–Zygmund operators, some pseudodifferential operators, and many others.

Not only that, but in [86] Lerner characterized the class of weights for which the Fefferman–Stein inequality holds in weighted weak L^p spaces. This class of weights obtained the name SC_p, standing for *strong* C_p. This class is stronger than C_p but weaker than C_q if $q > p$.

In the last decade, many quantitative inequalities have been found for weights of classes A_p or A_∞. The most important of such results is probably the solution of the A_2 conjecture. Proved by Hytönen [61], it states that the weighted L^2 norm of a Calderón–Zygmund operator is controlled by the A_2 constant of the weight with a linear dependence. This result was later improved by Hytönen and Pérez in [62] where the authors combine the A_2 constant with the A_∞ constant, which is a more precise mixed-type estimate.

Although many norm weighted inequalities have been satisfactorily quantified in terms of the weight, this has only happened in the setting of A_p or A_∞ weights. In this thesis we present a way of obtaining quantitative estimates for C_p weights, and we actually give a quantification of both Coifman–Fefferman and Fefferman–Stein inequalities. In order to do that, we introduce a constant, called the C_p constant of the weight, that encodes the size of the weight in the C_p class. The idea is that the smaller the constant is, the better the weight is.

Once the constant is defined, we obtain a weak reverse Hölder inequality for C_p weights, in which the dependence of the reverse Hölder exponent is quantified by the C_p constant. This quantification is actually the same as in the sharp Reverse Hölder inequality for A_∞ weights proved by Hytönen, Pérez and Rela in [62] and [63]. That is why we claim that our reverse Hölder inequality we present for C_p weights is also sharp.

The definition of the C_p constant and the reverse Hölder inequality are contained in Chapter 2. In this chapter, we provide a discussion on the C_p class itself, some of its main properties and a few examples. Also, we expand on the counterexample of Kahanpää and Mejlbro that disproves the self-improvement of C_p weights. We give a new proof of this fact that can be expanded to higher dimensions.

The translation of the quantification of the reverse Hölder inequality to the quantification of the Coifman–Fefferman inequality is not a difficult task once the correct tools are identified. The most important tool is a good-λ inequality with exponential decay between the Calderón–Zygmund and the maximal Hardy–Littlewood operators, that we borrow from [7]. This inequality, combined with the sharp reverse Hölder inequality allows us to obtain a quantification of the Coifman–Fefferman inequality for C_p weights. Nevertheless, due to the non-local nature of C_p weights, a logarithmic dependence on the constant is added, and not the desired linear dependence. This extra logarithmic term is unavoidable by our methods.

We are also able to prove Coifman–Fefferman inequalities for more general operators. Precisely, rough homogeneous singular integral operators. The lack of regularity of the kernel of these operators makes it impossible for them to satisfy a good-λ inequality, less one with exponential decay. Therefore, we need to use the technique of sparse domination in order to prove C_p Coifman–Fefferman inequalities for rough operators. To the best of our knowledge, no C_p estimate was known to be satisfied by rough operators until our result.

Although the sparse domination technique is known for delivering sharp estimates on weights, that is not the case for C_p weights, sadly. Here, the intricate non-local properties of C_p weights make sparse domination technique not optimal, so the estimates obtained for rough operators do not look sharp at all. Nevertheless, their novelty makes them interesting, since such weighted estimates had not been proven before, even qualitatively. The quantification of Coifman–Fefferman inequalities for Calderón–Zygmund operators, rough homogeneous singular integrals and sparse forms is given in Chapter 3. The quantification of the inequality of Fefferman–Stein is postponed until Chapter 4, where we obtain a good-λ inequality with exponential decay between the Hardy–Littlewood maximal operator and the sharp maximal function of Fefferman–Stein.

1.2 BMO estimates

The second part of the dissertation is devoted to obtaining estimates for BMO functions, which is contained in Chapters 4 and 5.

The space of functions of bounded mean oscillation, BMO, is a classical space in analysis. It serves as an adequate alternative for L^∞ in some cases, for example, the singular integral of a bounded function is not bounded, but lies in BMO. Although the space is larger than L^∞ (and therefore the BMO condition is weaker than boundedness), BMO is strong enough to be used as an interpolation end-point.

Even though it can be used to substitute L^∞ in some cases, the space BMO is interesting in its own right. The most important property of this space is the John–Nirenberg theorem, that states that these functions are actually locally exponentially integrable. This can be seen as a self-improvement result on integrability, since starting from an L^1 integrability condition we obtain an exponential integrability condition. Similar phenomena hold for Poincaré and Sobolev–Poincaré inequalities, as shown in [107], and also for capacity density conditions as in Chapter 6.

Related to the BMO space, there is the Fefferman–Stein sharp maximal function. This function being in L^∞ is equivalent to the original function being in BMO. But this is not the first time this maximal object appears in this dissertation, since we already dealt with it tangentially while talking about C_p weights. It was Yabuta [117] who showed the relation between the Hardy–Littlewood and the Fefferman–Stein maximal functions in the context of C_p weights, which is called the Fefferman–Stein inequality.

In order to get a precise quantification of the Fefferman–Stein inequality for C_p weights, we needed a good-λ type inequality between the Hardy–Littlewood and the Fefferman–Stein maximal functions. More precisely, we needed a good-λ inequality between them with an exponential decay. Such an inequality was not available to us in the literature, so in order to prove it, we took a new route. This new route led us to finding some extensions of the John–Nirenberg theorem. The quantification of the Fefferman–Stein inequality for C_p weights comes naturally following one of those extensions of the John–Nirenberg theorem, but further consequences also follow.

Among the consequences that were obtained from the extensions of the John–Nirenberg theorem, we find a version of a weighted inequality of Muckenhoupt and Wheeden. This weighted inequality for A_p weights can be seen as a self-improvement result, since starting from a local unweighted L^1 estimate we obtain a weighted $L^{r,\infty}$ estimate for some $r > 1$.

The extensions of the John–Nirenberg theorem, along with their consequences and generalizations are presented in Chapter 4.

As we commented before, John–Nirenberg can be seen as a self-improvement result. That is, a function that a priori has uniformly bounded L^1-type oscillations actually has uniformly bounded exponential-type oscillations. We have a starting point, in this case, the function belonging to BMO and we improve that starting point to a better condition, in this case, exponential integrability. A natural question to be asked is if this starting point can be weaker, that is, can we have a softer condition that self-improves to BMO? If so, how much can we weaken this condition?

This is not a new question, it was already addressed by John in [65] and Strömberg in [113]. The correct minimal condition for BMO is the uniform boundedness of L^φ-oscillations for a concave function φ, which can grow as slow as we want. More recently, an explicit quantitative estimate of the BMO norm in terms of the scale function φ was given by Logunov, Slavin, Stolyarov, Vasyunin and Zatitskiy in [90]. This estimate for the norm has the disadvantage of not being homogeneous on the function. We present a new proof of the same minimality that yields a homogeneous norm estimate, also being completely transparent. Our proof can also be extended to other geometries.

We will also study the same problem in more general contexts beyond euclidean spaces. For example, we extend our methods to spaces of homogeneous type, which are quasi-metric spaces equipped with a doubling measure. We can do this because the method for the euclidean case is quite transparent and it can easily be generalized. Moreover, we can also use the geometry of $\mathbb{R}^n$ to obtain similar results for non-doubling measures in $\mathbb{R}^n$. We are not able to obtain the same results in abstract quasi-metric (or even metric) spaces without doubling of the measure, because we use special properties of the geometry of $\mathbb{R}^n$ in this case. All these results concerning minimality for BMO are developed in Chapter 5.

1.3 Hajłasz capacity density condition

The last part of this thesis, Chapter 6 concerns capacity density conditions in terms of Hajłasz gradients and their self-improvement in abstract metric spaces. This is the first time that a capacity density condition concerning non-local gradients is proved to be self-improving in metric spaces. The way this chapter is connected to the thesis is not as straightforward as the previous chapters. We started trying to prove self-improvement properties of fractional Hardy inequalities in metric spaces, that is somewhat related to the self-improvement of Poincaré inequalities of Section 4.5. The improvement of Hardy inequalities was not obtained but, in the process, we started working with Hajłasz gradients and eventually came to our results.

The study of self-improvement of capacity density conditions can be tracked back to the seminal work of Lewis [88], in which self-improvement of a capacity density condition concerning Riesz potentials was established on $\mathbb{R}^n$. This result was followed by the work of Mikkonen [98] in which p-Laplace weighted Maz'ya estimates were obtained, and also by the work of Björn, MacManus and Shanmugalingam [6] in which similar estimates were obtained in metric spaces. This last result uses upper gradients, which are a way of defining derivatives of functions in metric spaces. Upper gradients are local objects, since their value only depends on a neighborhood of the point.

We work with β-Hajłasz gradients, which were introduced in [49] for $\beta = 1$. Their nature is non-local because of their definition, but that same definition is at the same time quite natural. In the fractional case $0 < \beta < 1$, the same definition appears naturally. Moreover, upper gradients can be shown to be 1-Hajłasz gradients, which makes Hajłasz gradients a slightly more general and versatile tool than upper gradients.

One of the main advantages of working with Hajłasz gradients is that Poincaré inequalities hold without any extra assumption on the measure. That is, for any function and any Hajłasz gradient of it, a local Poincaré inquality holds, see Section 6.3. This is not true for other derivatives, such as upper gradients, and usually those kind of settings require extra hypotheses that exclude some interesting cases such as certain doubling measures in $\mathbb{R}$, see [5].

We introduce a capacity density condition similarly as other capacity density conditions, but in terms of Hajłasz gradients. This condition thus has two parameters, the derivative order β and the size parameter p in terms of integrability. We prove that our condition is self-improving in both β and p. More precisely, we prove that a set E satisfies a (β, p)-capacity density condition if and only if its upper Assouad codimension is strictly smaller than the product βp. That is, there is always a small room for lowering both β and p in a way that their product is still strictly greater than the upper Assouad codimension.

The characterization of the capacity density condition in terms of the Assouad codimension is quite technical. It is fairly easy to prove that the strict bound on the Assouad codimension implies the capacity density condition, using an inequality of Maz'ya type. It is also not difficult to prove that the capacity density condition implies that the Assouad codimension is smaller or equal to the product βp. The difficult part is proving that this bound is strict.

In order to do that, we combine a technique of using Poincaré inequalities and Hardy-type inequalities to this setting, and we also use some techniques of self-improvement of Poincaré inequalities. The study of such self-improvement properties was initiated by Keith and Zhong in thier celebrated work [71]. In this respect, we joint the line of research initiated by Korte Lehrbäck and Tuominen in [76] in which a condition similar to our capacity density condition was related to Hardy inequalities. In [74], a maximal function approach for these methods was proposed to obtaining Poincaré inequalities. We combine all the methods above for Hardy and Poincaré inequalities, and we elaborate on those arguments to obtain our results concerning capacity density conditions.

1.4 Preliminaries and notation

This thesis has several parts that are contained in a few works that have been developed with different sets of people at different points in time. Therefore, keeping a unified notation has not been particularly easy and the notation in this document may be different to the notation in the referred works and even not-standard at some points. I apologize in advance. I did my best to unify all this expressions and notations to my liking and always trying to make everything easy to understand and keeping it correct.

1.4.1 Basic notation

The characteristic function of a set E in an ambient space X will be denoted by χ_E, ignoring the ambient space. That is,

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

For an exponent $1 \leq p \leq \infty$, we use the standard notation p' to denote the Hölder conjugate exponent, that is, $1' = \infty$, $\infty' = 1$ and $p' = p/(p-1)$ for $1 < p < \infty$.

By a weight we mean a nonnegative locally integrable function, usually denoted by w. Although generally weights are assumed to be positive almost everywhere, we will let them vanish on sets of positive measure for reasons that will become apparent shortly. Abusing ever so slightly the notation, we identify the weight function w with the measure that it defines, that is, for a measurable set F we write

$$w(F) = \int_F w(x)dx,$$

where dx denotes the Lebesgue measure, or the measure of the ambient space. Clearly, weighted measures are always absolutely continuous with respect to the ambient measure.

A cube in $\mathbb{R}^n$ is a cartesian product of n intervals of the same length, usally denoted by the letter Q. That is, a set of the form

$$Q = \prod_{j=1}^{n} [a_j, b_j] = [a_1, b_1] \times \cdots \times [a_n, b_n].$$

Since all of our measures in $\mathbb{R}^n$ are absolutely continuous with respect to the Lebesgue measures, it will not be of importance whether the intervals are open, closed or half-open, since sets of measure zero will not matter in this context.

When we work on $\mathbb{R}^n$, the Lebesgue measure will be denoted by $|\cdot|$.

For a locally integrable function f, we will use two notations for averages over a set Q, which will mostly be either a cube or a ball:

$$f_Q = \fint_Q f(x)dx = \frac{1}{|Q|}\int_Q f(x)dx.$$

In the context of a metric space (X, μ), the same notation will be used for averages, that is, for a set E with finite measure and an integrable function f, the average of f over E will be denoted by

$$f_E = \fint_E f(y)d\mu(y) = \frac{1}{\mu(E)}\int_E f(y)d\mu(y).$$

Another standard notation for averages, usually used in the context of sparse operators and forms is the following one:

$$\langle f\rangle_Q = f_Q = \fint_Q f(x)dx.$$

Moreover, with this notation we can denote also L^p-averages, that is, for a positive function f,

$$\langle f\rangle_{p,Q} = \left((f^p)_Q\right)^{\frac{1}{p}} = \left(\fint_Q f(x)^p dx\right)^{\frac{1}{p}}.$$

Also, this notation allows us to incorporate weighted averages quite naturally:

$$\langle f\rangle_{p,Q}^{w} = \left(\frac{1}{w(Q)}\int_Q f(x)^p w(x)dx\right)^{\frac{1}{p}}.$$

We use standard Lebesgue and Lorentz spaces. That is, for a measure space (X, μ), exponent $0 < p < \infty$, and a measurable function on X, we define

- $\|f\|_{L^p(X)} = \left(\int_X |f(x)|^p d\mu(x)\right)^{\frac{1}{p}}$;
- $\|f\|_{L^\infty(X)} = \operatorname{ess\,sup}_{x\in X}|f(x)| = \inf\{M > 0 : \mu(\{x \in X : |f(x)| > M\}) = 0\}$.
- $\|f\|_{L^{p,\infty}(X)} = \sup_{t>0} t\mu(\{x \in X : |f(x)| > t\})^{\frac{1}{p}}$.

1.4.2 The Hardy–Littlewood maximal operator

The Hardy–Littlewood maximal operator M is defined for a locally integrable function f defined on $\mathbb{R}^n$ as

$$Mf(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)|dy,$$

where the supremum is taken over all cubes containing the point x. Taking open or closed cubes does not change the result.

The Hardy–Littlewood maximal operator has the following boundedness properties in $\mathbb{R}^n$:

- $\|Mf\|_{L^p(\mathbb{R}^n)} \leq \kappa_n\, (p')^{\frac{1}{p}}\, \|f\|_{L^p(\mathbb{R}^n)}$, for $1 < p \leq \infty$.
- $\|Mf\|_{L^{1,\infty}(\mathbb{R}^n)} \leq 3^n \|f\|_{L^1(\mathbb{R}^n)}$.

For $0 < s < \infty$, the operator M_s is defined by the expression

$$M_s f(x) = M(|f|^s)(x)^{\frac{1}{s}} = \sup_{x\in Q} \left(\fint_Q |f(y)|^s dy \right)^{\frac{1}{s}}.$$

This operator has the following boundedness-properties:

- $\|M_s f\|_{L^p(\mathbb{R}^n)} \leq \kappa_{n,p,s}\, \|f\|_{L^p(\mathbb{R}^n)}$, for $s < p \leq \infty$.
- $\|M_s f\|_{L^{s,\infty}(\mathbb{R}^n)} \leq \kappa_{n,s} \|f\|_{L^s(\mathbb{R}^n)}$.

The maximal operator can also be defined in metric spaces using balls, and it satisfies the same boundedness properties.

1.4.3 Covering and decomposition techniques

For a fixed cube Q, the family of dyadic descendants of Q, $\mathcal{D}(Q)$ is obtained by dividing Q into the 2^n cubes that come from splitting each side of Q in two intervals of half lenght, and iterating this process.

We state and prove the Calderón–Zygmund decomposition technique in its simplest form. This useful technique was first used by Calderón and Zygmund to prove the boundednes of some singular integral operators [10]. Although it can be done in different scenarios, such as the whole space $\mathbb{R}^n$, we are going to describe how it can be applied while working locally at some cube Q. The idea is to decompose the cube Q into smaller cubes such that the function is somehow controlled in each of these smaller cubes.

Lemma 1.1 – Calderón–Zygmund decomposition

Let Q be a cube in $\mathbb{R}^n$ and $f \in L^1(Q)$ such that $|f|_Q = 1$. Choose a stopping-time $\lambda > 1$. There exists a family of cubes $\mathcal{Q} = \{Q_j\}_j \subset \mathcal{D}(Q)$ with the following properties:

- The cubes in $\mathcal{Q}$ are pairwise disjoint;
- For each $Q_j \in \mathcal{Q}$, we have $\lambda < \fint_{Q_j} |f(y)| dy \leq 2^n\, \lambda$;
- $\displaystyle\sum_{Q_j \in \mathcal{Q}} |Q_j| \leq \frac{|Q|}{\lambda}$;
- for almost every $x \in Q \setminus \bigcup_{Q_j \in \mathcal{Q}} Q_j$, we have $|f(x)| \leq \lambda$.

Proof. We use the following iteration. We divide Q into its 2^n children and test the condition

$$\fint_P |f(y)| dy > \lambda, \tag{1.1}$$

for each P dyadic child of Q. We add the ones that satisfy (1.1) to the family $\mathcal{Q}$. For the non chosen children, we divide them into their children and continue the process.

The obtained family is clearly pairwise disjoint. The second property holds because each of the chosen child satisfies (1.1) and its direct parent does not. The third property also follows from (1.1), since

$$\sum_{Q_j\in\mathcal{Q}} |Q_j| < \frac{1}{\lambda}\sum_{Q_j\in\mathcal{Q}}\int_{Q_j}|f(y)|dy \le \frac{1}{\lambda}\int_Q |f(y)|dy = \frac{|Q|}{\lambda}.$$

The last property follows from the Lebesgue differentiation theorem. □

Lemma 1.2 – Vitali covering

Let $\mathbb{X}$ be a space of homogeneous type and let $\mathcal{B}$ be a collection of balls in $\mathbb{X}$ with bounded radius. There exists a subcolection $\mathcal{B}^* \subset \mathcal{B}$ of pairwise disjoint balls such that

$$\bigcup_{B\in\mathcal{B}} B \subseteq \bigcup_{B\in\mathcal{B}^*} \kappa(4\kappa+1)B. \tag{1.2}$$

Proof. Let R denote the supremum of the radii of balls in $\mathcal{B}$. Divide the family $\mathcal{B}$ into $\mathcal{B}_n$ containing the balls in $\mathcal{B}$ with radius in $(2^{-n-1}R, 2^{-n}R]$, for $n \ge 0$. We define a sequence of families as follows. Let $H_0 = \mathcal{B}_0$ and let $\mathcal{B}_0^*$ be a maximal subcollection of H_0 of pairwise disjoint balls, which exists by Zorn's Lemma. Then, we define inductively

$$H_{n+1} = \{B \in \mathcal{B}_{n+1} : B\cap C = \emptyset, \text{ for all } C \in \bigcup_{m=0}^{n} \mathcal{B}^*\},$$

and $\mathcal{B}_{n+1}^*$ a maximal subcollection of pairwise disjoint balls of H_{n+1}. Then the family

$$\mathcal{B}^* = \bigcup_{n=0}^{\infty} \mathcal{B}^*$$

consists of pairwise disjoint balls and satisfies (1.2). □

1.4.4 Good-λ inequalities

In this section we introduce the technique of good-λ inequalities that will be used throughout the dissertation. First, we will use what we call the layer-cake formula or Cavalieri principle. [45, Proposition 1.1.4]

Lemma 1.3 – Layer cake formula

Let (X,μ) be a measure space and ψ a differentiable nonnegative function on $[0,\infty)$. Then for every positive function f on X, we have

$$\int_X \psi(f(x))d\mu(x) = \int_0^\infty \psi'(t)\mu(\{x\in X : f(x) > t\})dt.$$

The following lemma is the in the same spirit as the layer cake formula, but it constitutes a discretization on the height.

Lemma 1.4

Let $f \in L^p(X)$ for some measure space (X, μ). Then

$$\int_X f(x)^p d\mu(x) \equiv_p \sum_{k \in \mathbb{Z}} 2^{kp} \mu(\{x \in X : f(x) > 2^k\}),$$

where the implicit constants only depend on p.

Finally, we introduce the good-λ technique that is used throughout the dissertation many times.

Lemma 1.5 – Good-λ inequalities

Let (X, μ) be a measure space and let f, g be to functions satisfying

$$\mu(\{x \in X : f(x) > t, g(x) \leq \lambda t\}) \leq \varphi(\lambda),$$

for all $0 < t < \infty$ and $0 < \lambda < 1$, and some continuous function φ such that $\varphi(0) = 0$. Then $\|f\|_{L^p(X)} \leq_{\varphi,p} \|g\|_{L^p(X)}$ for all $0 < p < \infty$.

Proof. We use the Layer cake formula in Lemma 1.3. We have

$$\begin{aligned}\|f\|_{L^p(X)} &= p \int_0^\infty t^{p-1} \mu\big(\{x \in X : f(x) > t\}\big) dt \\ &\leq p \int_0^\infty t^{p-1} \mu\big(\{x \in X : g(x) > \lambda t\}\big) dt \\ &+\varphi(\lambda) p \int_0^\infty t^{p-1} \mu\big(\{x \in X : f(x) > t\}\big) dt \\ &= \lambda^{-p} \|g\|_{L^p(X)} + \varphi(\lambda) \|f\|_{L^p(X)}.\end{aligned}$$

Now, since φ is continuous and going to zero, one can choose λ small enough so that $\varphi(\lambda) \leq \frac{1}{2}$. Passing then the last term to the left hand side finishes the argument. □

2

The class C_p of weights

Some of the results in this chapter are contained in the following works:

[12] Canto, J. *Sharp Reverse Hölder inequality for C_p Weights and Applications*, The Journal of Geometric Analysis (2021) **31**: 4165–4190.

[13] Canto, J., Li, K., Roncal, L., Tapiola, O. C_p estimates for rough homogeneous singular integrals and sparse forms, Annalli della Scuola Normale Superiore di Pisa, clase di Scienze (5) Vol XXII (2021), 1131–1168.

In this chapter we will discuss and further develop the results from these two works that focus on the structure of the C_p classes of weights. We define the C_p constant, prove a quantitative sharp Reverse Hölder inequality for C_p weights and show the lack of self-improvement for these classes.

The results concerning quantitative weighted norm inequalities will be discussed in the following chapter. Nevertheless, since weighted norm inequalities are intrinsically tied to C_p weights, they will appear throughout this chapter.

First and foremost, let me make a probably silly but important comment that was noted to me by Javier Martínez-Perales. The most annoying thing about working with the C_p class is precisely its name, that is, the notation C_p. Usually in analysis, whenever we want to emphasize that a quantity A is bounded by another quantity B times a constant "that is somehow relevant but not that much" depending on a parameter, say p, we write inequalities of the form

$$A \leq C_p B.$$

The problem is therefore evident here: we should not use, in this text, the usual notation of the ever-changing constant C and use subscripts to specify the parameters in which this constant depends, because of the confusion it may cause as it is the name of one of the main object of study in this thesis. That is why we will name these kind of constants (whenever we are in the C_p context) by the letter κ.

2.1 Historical introduction

Weighted inequalities have been a core field of study in Harmonic Analysis since the 70's. It was Muckenhoupt [99] who introduced the A_p class of weights and proved its characterization in terms of the boundedness of the Hardy–Littlewood maximal function. Later, it was shown that A_p weights satisfy further properties, such as the boundedness of the Hilbert transform [60] or Calderón–Zygmund operators [20] in weighted Lebesgue spaces. Since A_p weights are not the target class of weights for this dissertation, we will not go into detail on that topic, but we refer to [29, 40] for more detailed information.

One of the most interesting properties of A_p weights is the reverse Hölder inequality they satisfy, originally proved by Muckenhoupt. Without going into much detail at the moment, this means that a weight in A_p will locally be L^q integrable, for some $1 < q < \infty$, this q being the *reverse Hölder exponent*. A weight satisfying a reverse Hölder inequality is called a *Reverse Hölder* weight, and the class of weights satisfying a reverse Hölder inequality with exponent q is usually caled RH_q, but it has sometimes been denoted by B_q [8, Chapter 3]. It is not clear where the name B_q originated, but it is reasonable to think that it comes as a continuation to A_p in the alphabet.

Let us make two remarks here about the A_p and B_q classes. First of all, a weight is in some A_p if and only if it belongs to some B_q, but no relation between p and q can exist. This was proved by Coifman and Fefferman [20] who showed that the union of all A_p classes and the union of all B_q classes coincide and equal A_∞.

The second remark is their nestedness. That is, whenever $p < q$, we have the inclusions $A_p \subset A_q$ and $B_q \subset B_p$. This nestedness property is but a consequence of the definitions of these classes and the (standard) Hölder inequality. But the important key here is that there is self-improvement in some sense. That is, if a weight belongs to A_p for some p, it also belongs to A_q for some $q < p$, this q depending on the weight. The same is true for B_p, that is, a weight belonging to B_p also belongs to B_q for some $q > p$, and this q depends on the weight. This fact was first proved by Ghering [41], and it is a key fact in the A_p theory.

Continuing with the alphabet, the class C_p of weights was introduced by Muckenhoupt in [100]. In its inception, it appeared as an attempt to characterize the weighted norm inequality between the Hilbert transform and the Hardy–Littlewood maximal function. That is, the original problem was to, for a fixed p, study which weights w satisfy the norm inequality

$$\|Hf\|_{L^p(w)} \leq \kappa \, \|Mf\|_{L^p(w)}, \tag{2.1}$$

for all bounded f with compact support and κ independent from f, where H denotes the Hilbert transform and M the Hardy–Littlewood maximal operator.

The answer to this problem has not been completely found, but partial answers have been given. The first answers were by Muckenhoupt [100] and by Sawyer [109], who established that $w \in C_p$ is necessary for (2.1) and its analogue in higher dimensions with the Riesz transforms, but the sufficient condition that was found there was C_q for some $q > p$. More precisely, Sawyer proved the following sufficient condition. If a weight is in C_q for $1 < p < q < \infty$ and T is a Calderón–Zygmund operator, see Chapter 3, there exists $\kappa > 0$ such that for all bounded functions f with compact support, the following inequality holds

$$\|T^* f\|_{L^p(w)} \leq \kappa \|Mf\|_{L^p(w)}, \tag{2.2}$$

where T^* denotes the maximally truncated singular integral operator related to T, see Chapter 3.

Sawyer also proved that if there exists $K > 0$ such that for each of the Riesz transforms R_j and bounded f with compact support, the following holds for some $1 < p < \infty$,

$$\|R_j f\|_{L^p(w)} \leq \kappa \|Mf\|_{L^p(w)},$$

then the weight w is in C_p.

Understandingly, it was conjectured by Muckenhoupt that C_p is the correct sufficient condition. This conjecture is known as the Muckenhoupt conjecture.

Conjecture 2.1 – Muckenhoupt [100, 109]

Let $1 < p < \infty$ and let w be a weight. Then $w \in C_p$ if and only if there for each Calderón–Zygmund operator T, there exists a constant κ such that for every bounded function f with compact support, the following inequality holds

$$\|T^* f\|_{L^p(w)} \leq \kappa \, \|Mf\|_{L^p(w)}.$$

The most simple way of proving this conjecture would be, using the known sufficient conditions, to prove a self-improvement property between C_p classes. That is, to prove that a weight in C_p belongs in C_q for some $q > p$. This is not true, as was shown by Kahanpaa and Mejlbro in [69], see Section 2.9.

Another important norm inequality that is satisfied by C_p weights is the Fefferman–Stein inequality. This is an inequality between a function f and the sharp maximal function $M^\sharp f$. A result by Yabuta [117] states that the $L^p(w)$-norm of Mf is bounded by that of $M^\sharp f$ for weights in C_q for $q > p$, and C_p is necessary. The parallelism between this result and that of Sawyer's is clear. A more detailed discussion on the Fefferman–Stein inequality will be given in Chapter 4, where also a quantitative estimate is given.

2.2 A brief note on A_∞ weights

In this section, we state a few well-known facts about the class A_∞. Although this class is not the target class of study on this Chapter, we have included this section in order to establish parallelisms between A_∞ and C_p, which is the main object of study in this Chapter. The results in this section will not be proven here, but the reader is welcomed to visit references on this topic such as [45, Chapter 7]

There are many equivalent definitions of A_∞ weights, see for example an extensive list on [30]. We choose to give this one here.

Definition 2.2

Let w be a weight. We say that w is an A_∞ weight, and we write $w \in A_\infty$ if there exist constants $\kappa \geq 1$ and $\varepsilon > 0$ such that for all cubes $Q \in \mathbb{R}^n$ and all measurable $E \subseteq Q$, the following inequality holds:

$$w(E) \leq \kappa \left(\frac{|E|}{|Q|}\right)^{\varepsilon} w(Q). \tag{2.3}$$

We recall the definition of the A_∞ constant of Fujii–Wilson, that was introduced by Hytönen, Pérez and Rela in the works [62, 63] in order to give mixed type A_p-A_∞ estimates.

Definition 2.3

Let w be a weight. The A_∞ constant of w is the number

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)(x)dx,$$

where the supremum is taken over all cubes Q.

It is a well known fact that A_∞ weights satisfy a Reverse Hölder inequality, for example, see Coifman and Fefferman [20]. See also the books [29, 40] a more detailed discussion on the topic.

Proposition 2.4

Let w be a weight. The following statements are equivalent

1. $w \in A_\infty$;
2. There exist $\delta > 0$ and $\kappa > 0$ such that for all cubes Q,

$$\left(\fint_Q w(x)^{1+\delta} dx \right)^{\frac{1}{1+\delta}} \leq \kappa \frac{w(Q)}{|Q|}.$$

A quantitative version of this Reverse Hölder inequality was given by Hytönen, Pérez in [62] and later by them and Rela in [63], in which the exponent is explicitly given in terms of the Fujii–Wilson constant of the weight. We state it here.

Theorem 2.5 – Sharp Reverse Hölder Inequality for A_∞ weights, [63]

Let $w \in A_\infty$ and let Q be a cube. Then

$$\left(\fint_Q w(x)^{1+\delta} dx \right)^{\frac{1}{1+\delta}} \leq 2 \fint_Q w(x) dx,$$

for any $\delta > 0$ such that $0 < \delta \leq \frac{1}{2^{n+1}[w]_{A_\infty} - 1}$.

In Section 2.5 we will give a version of this inequality for C_p weights.

2.3 Weights of class C_p

Let us give the definition for this class of weights as was given originally in [100] in $\mathbb{R}$ and then in [109] for higher dimensions.

Definition 2.6

Let $1 < p < \infty$. We say that a weight w is of class C_p, and we write $w \in C_p$ if there exist constants $\kappa > 0$ and $\varepsilon > 0$ such that for every cube $Q \subset \mathbb{R}^n$ and every measurable $E \subset Q$ we have

$$w(E) \leq \kappa \left(\frac{|E|}{|Q|}\right)^{\varepsilon} \int_{\mathbb{R}^n} (M\chi_Q(x))^p(x) w(x) dx. \tag{2.4}$$

Here M denotes the standard Hardy–Littlewood maximal operator, see Section 1.4.2.

At a first sight, the difference between C_p and A_∞ is the appearance of the quantity $\int_{\mathbb{R}^n} (M\chi_Q)^p w$ in (2.4) playing the role that $w(Q)$ plays in (2.3). This quantity recurrently appears whenever C_p weights are on the menu, and its non local behavior presents the main difficulties that arise in the study of these weights. We will call it the "C_p-tail of w at Q".

The way to interpret (2.4) is in a continuity sense, that w measures small sets in a controlled way; that is, the ratio between the weighted measure $w(E)$ and the C_p-tail at Q has to be bounded by a power of the ratio between the Lebesgue measures of E and Q.

Examples 2.7

Let us give a few examples of weights belonging to C_p.

- All A_∞ weights are in C_p, see bellow.
- In dimension one, $\chi_{(0,\infty)} \in C_p$ for all p.
- No integrable function can be C_p.

Let us make a digression and make a few comments on C_p-tails. First, note that $M\chi_Q$ is a function that is positive everywhere, that takes value 1 in the cube Q and tends to zero at infinity. This makes it clear that any weight in the A_∞ class belongs to C_p. Moreover, since $M\chi_Q \leq 1$, we have that for $(M\chi_Q)^p \geq (M\chi_Q)^q$ for $p \leq q$, which implies $C_q \subseteq C_p$ for $p \leq q$. In short, we have

$$A_\infty \subseteq C_q \subseteq C_p, \quad p \leq q.$$

Later, we will show that these inclusions are strict whenever $p < q$.

Note that the inclusions go in the opposite way as for A_p, that satisfy $A_p \subset A_q$ for $p \leq q$.

As mentioned before, C_p-tails play an important role in the analysis of C_p weights, so let us take a look at them. First, let us take a look at the maximal function of the characteristic of a cube, that is, $M\chi_Q$. We have a pointwise estimate which will be used many times throughout this dissertation.

Lemma 2.8

Let Q be a cube of side-lenght $\ell(Q)$ and center x_Q. There exist constants depending only on the dimension n such that

$$M\chi_Q(x) \simeq \frac{|Q|}{|Q| + \operatorname{dist}(x, Q)^n} \simeq \frac{|Q|}{(\ell(Q) + |x - x_Q|)^n}.$$

Lemma 2.9

Let Q be a cube and $\lambda > 1$. Then the following pointwise inequality holds almost everywhere,

$$M\chi_Q(x) \le M\chi_{\lambda Q}(x) \le \lambda^n M\chi_Q(x).$$

The proof of the preceding lemma can be generalized to not only dilates of cubes but to nested cubes.

Lemma 2.10

Let P, Q be two cubes, such that $Q \subset P$. Then, for almost all $x \in \mathbb{R}^n$ the following inequality holds,

$$M\chi_Q(x) \le M\chi_P(x) \le \left(\frac{|P|}{|Q|}\right)^n M\chi_Q(x).$$

Clearly, Lemma 2.10 says that C_p-tails of different cubes can always be compared if they are *not too far away* and their sizes are *not too different*, loosely speaking. This holds for all measures because the estimate in Lemma 2.10 is a pointwise one, and therefore independent of the measure.

If we try to keep up the analogy with the A_∞ counterpart, something similar still holds. What we mean by this is that, if a weight is in A_∞, two cubes that are *not too far away* and whose sizes are *not too different* then their weighted measures are also *not too different.* But here we do need that the weight belongs to A_∞ because the counterpart of Lemma 2.10 in the A_∞ world, which would be a pointwise inequality between characteristics of nested cubes fails drastically.

This small discussion, although somewhat loose and non-rigorous, pictures some of the difficulties that arise while working with C_p weights, that is, their non-local nature.

We also present the following interesting property, that states that in order to compute the C_p-tail of a C_p weight w at a cube Q, the values that w takes inside the cube Q are not really important, that is, we can make a hole in the cube while computing the tail and still obtain an equivalent quantity.

Lemma 2.11

Let $p > 1$ and $w \in C_p$. Then there exists a constant $\kappa = \kappa_{p,w} > 0$ such that for any cube Q we have

$$\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)dx \le \kappa \int_{\mathbb{R}^n \setminus Q} M\chi_Q(x)^p w(x)dx.$$

Proof. Let us fix a cube Q and set $\alpha = (2\kappa_w)^{\frac{1}{n\varepsilon_w}}$, where κ_w and ε_w are the constants in the definition of C_p (2.4). Notice that $\alpha \ge 1$. Now applying the C_p condition for αQ and Q gives us

$$\begin{aligned} w(Q) &\le \kappa_w \left(\frac{|Q|}{\alpha^n |Q|}\right)^{\varepsilon_w} \int_{\mathbb{R}^n} \left(M\chi_{\alpha Q}(x)\right)^p w(x)dx \\ &= \frac{1}{2}\int_{\mathbb{R}^n} \left(M\chi_{\alpha Q}(x)\right)^p w(x)dx \\ &\le \frac{1}{2} w(Q) + \frac{1}{2}\int_{\mathbb{R}^n \setminus Q} \left(M\chi_{\alpha Q}(x)\right)^p w(x)dx, \end{aligned}$$

since $M\chi_{\alpha Q} = 1$ on Q. In particular,

$$w(Q) \leq \int_{\mathbb{R}^n \setminus Q} (M\chi_{\alpha Q}(x))^p w(x)dx$$

Thus,

$$\begin{aligned}
\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)dx &= w(Q) + \int_{\mathbb{R}^n \setminus Q} (M\chi_Q(x))^p w(x)dx \\
&\leq \int_{\mathbb{R}^n \setminus Q} (M\chi_{\alpha Q}(x))^p w(x)dx + \int_{\mathbb{R}^n \setminus Q} (M\chi_Q(x))^p w(x)dx \\
&\leq \int_{\mathbb{R}^n \setminus Q} (\kappa_\alpha M\chi_Q(x))^p w(x)dx + \int_{\mathbb{R}^n \setminus Q} (M\chi_Q(x))^p w(x)dx \\
&\leq \kappa_{\alpha,p} \int_{\mathbb{R}^n \setminus Q} (M\chi_Q(x))^p w(x)dx,
\end{aligned}$$

where we used Lemma 2.35 in the second to last inequality. □

Let us now compute, or more precisely, estimate the C_p-tails of the constant weight $w = 1$.

Lemma 2.12

There exists a dimensional constant $\kappa_n > 0$ such that for all $1 < p < \infty$ and all Q cubes, the following estimates are true

$$|Q| \leq \int_{\mathbb{R}^n} (M\chi_Q(x))^p dx \leq \kappa_n \, p' \, |Q|.$$

Proof. The first inequality is trivial, since $M\chi_Q \geq 1$ almost everywhere on Q. The second inequality follows from the operator norm of the Hardy–Littlewood maximal operator on $L^p(\mathbb{R}^n)$, see for example [29]. □

Finally, let us make a comment on weights that have infinite C_p tails. Such weights exist, even in the A_∞ class: the weight $w(x) = |x|^\alpha$ has infinite C_p-tails for big enough $\alpha > 0$ and fixed p. The following lemma illustrates that a weight has either infinite C_p-tails at every cube or the tails are finite at every cube.

Lemma 2.13

Let $1 < p < \infty$ and let w be a weight. Suppose that the C_p-tail of w at Q is infinite for some cube Q, that is, there exists a cube Q such that

$$\int_{\mathbb{R}^n} (M\chi_Q(x))^p w(x)dx = \infty.$$

Then the same is true for all cubes.

Therefore, we will say that a weight w has *infinite C_p-tails* if the C_p-tail of some cube (and therefore all cubes) is infinite. Clearly, these weights are always in C_p, since the right hand side of inequality (2.4) is infinite. However, these are not very interesting weights as the following lemma tries to show.

Lemma 2.14

Let w be a weight with infinite C_p-tails. Then for every non-zero function f, the following is true:

$$\|Mf\|_{L^p(w)} = \infty.$$

Many of the quantitative norm inequalities that we study in the next chapter have the $L^p(w)$ norm of the Hardy–Littlewood maximal operator on the right-hand side and therefore hold trivially for weights with infinite C_p-tails, as a consequence of Lemma 2.13. Thus, having finite C_p-tails will be a common hypothesis in the proofs of results of that nature.

As an end to this section, let us mention that it is known that C_p weights satisfy a weaker version of the Reverse Hölder inequality, in which the C_p-tail appears. See for example [8]. In Section 2.5 we will give a more detailed discussion on this and also provide a quantitative estimate on the exponent, but let us state this equivalence here.

Even though this inequality is clearly weaker than the Reverse Hölder inequality satisfied by A_∞ weights, we will refer to it as the Reverse Hölder Inequality for C_p weights, or just Reverse Hölder inequality.

Proposition 2.15 – [8, Lemma 7.7]

Let $1 < p < \infty$ and let w be a weight. The following statements are equivalent:

1. $w \in C_p$;
2. There exist $\delta > 0$ and $\kappa > 0$ such that for all cubes Q,

$$\left(\fint_Q w(x)^{1+\delta}dx\right)^{\frac{1}{1+\delta}} \leq \frac{\kappa}{|Q|}\int_{\mathbb{R}^n}\left(M\chi_Q(x)\right)^p w(x)dx.$$

The parallelism between Proposition 2.4 and Proposition 2.15 is clear: in the C_p world, the role that $w(Q)$ plays in the A_∞ world is played by the C_p-tail.

2.4 The C_p constant

In this section, we provide a constant for the C_p class in the spirit of the Fujii–Wilson constant for A_∞ weights from Definition 2.3. This constant will be used in the following chapter for giving quantitative estimates for norm inequalities between a wide variety of operators and objects. In order to introduce this constant, we keep Definition 2.3 in mind and take the parallelism between A_∞ and C_p to the next level.

Definition 2.16 – C_p constant

For an arbitrary non-zero weight w, we define

$$[w]_{C_p} := \sup_Q \frac{\displaystyle\int_Q M(\chi_Q w)(x)dx}{\displaystyle\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)dx},$$

where the supremum is taken over all cubes Q. If $w = 0$, we set $[w]_{C_p} = 0$.

Notice that if w is not identically zero, the quantity on the denominator is always strictly greater than zero.

Remark 2.17 A non-zero weight w has infinite C_p-tails if and only if $[w]_{C_p} = 0$. Indeed, if w has infinite C_p-tails then the denominator equals infinity and we have $[w]_{C_p} = 0$. Conversely, if $[w]_{C_p} = 0$ we have that for every cube Q,

$$\frac{1}{\int_{\mathbb{R}^n} (M\chi_Q)^p w} \int_Q M(\chi_Q w) = 0.$$

Let us first check that the C_p constant is actually finite for C_p weights.

Proposition 2.18

Let $w \in C_p$. Then $[w]_{C_p} < \infty$.

Proof. We may assume that w has finite C_p-tails. Let $\delta > 0$ be as in Proposition 2.15. Then, for all cubes Q, we have

$$\begin{aligned}
⨍_Q M(w\chi_Q)(x)dx &\leq \left(⨍_Q M(w\chi_Q)(x)^{1+\delta}dx\right)^{\frac{1}{1+\delta}} \\
&\leq \kappa \left(⨍_Q w(x)^{1+\delta}dx\right)^{\frac{1}{1+\delta}} \\
&\leq \kappa \frac{1}{|Q|} \int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)dx,
\end{aligned}$$

where we have used that the maximal function is bounded in $L^{1+\delta}(\mathbb{R}^n)$ and the Reverse Hölder inequality from Proposition 2.15. Rearranging the terms and taking the supremum over all cubes Q, we obtain the result. □

Remark 2.19 For any weight w we have the following relation between the different constants for $q \leq p$, $[w]_{C_q} \leq [w]_{C_p} \leq [w]_{A_\infty}$.

Example 2.20

In dimension one, we have $[1]_{C_p} = \frac{p-1}{p+1}$, and in higher dimensions, $[1]_{C_p} \simeq_n \frac{1}{p'}$. In particular this shows that the constant C_p can be arbitrarily small. For $p > 1$ and small ε, for $w_\varepsilon(x) = |x|^{n(p-1-\varepsilon)}$ we have $[w_\varepsilon]_{C_p} \lesssim \varepsilon$.

As the previous example illustrates, for a fixed p and for any $\varepsilon > 0$ there exists a weight w satisfying $0 < [w]_{C_p} \leq \varepsilon$. This is a huge difference with the A_∞ constant, and the first moment in which the parallelism breaks.

The fact that the C_p constant can be arbitrarily small makes quantitative estimates take an awkward form in which expressions of the likes of $(1 + [w]_{C_p})$ appear. So far, we have not found a way of making this expressions less awkward.

2.5 The Sharp Reverse Hölder Inequality

In this section we will state and prove a result that is analogous to Theorem 2.5 for C_p weights. With this result, we can confirm that the definition of the C_p constant is the correct one, since it lets us have a quantitatively sharp Reverse Hölder inequality in the same sense as the one for A_∞.

Theorem 2.21
Let $1 < p < \infty$ and let w be a weight such that $0 \le [w]_{C_p} < \infty$. Then $w \in C_p$ and w satisfies, for $\delta = \frac{1}{B(1+[w]_{C_p})}$, with

$$B = \frac{2^{1+4np+3n}(20)^n}{1-2^{-n(p-1)}},$$

$$\left(\fint_Q w(x)^{1+\delta}dx\right)^{\frac{1}{1+\delta}} \le \frac{4}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)dx. \tag{2.5}$$

Remark 2.22 Notice that B depends on the dimension and on p. Moreover, we have $B \to \infty$ whenever p tends to either ∞ or 1.
Remark 2.23 The quantification in terms of the parameters ε and κ in (2.6) is $\kappa = 2$ and

$$\varepsilon = \frac{1-2^{-n(p-1)}}{2^{2np+3n}(20)^n}\,\frac{1}{1+[w]_{C_p}^{-1}}. \tag{2.6}$$

In particular, we have that both ε and δ are smaller than one.
Remark 2.24 Also, we note that, since we can show that $\kappa = 2$ for the correct ε, we may always assume that $\kappa = 2$ in the definition of C_p weights in Definition 2.6.

We may assume that w has finite C_p-tails, that is, $[w]_{C_p} > 0$. Indeed, if $[w]_{C_p} = 0$ then the right side of (2.5) equals infinity and the theorem is trivially true.

The proof is inpired by a remark from [3, Section 8.1], and by the proof given in [63] of the RHI for A_∞ weights.

We now introduce a functional over cubes that serves as a discrete analogue for the C_p-tail. Define, for a cube Q

$$T_{C_p}(Q,w) := \sum_{k=0}^{\infty} 2^{-n(p-1)k}\fint_{2^kQ} w(x)dx. \tag{2.7}$$

We note that $\alpha = \sum_{k\ge 0} 2^{-n(p-1)k} = (2^{n(p-1)})' < \infty$ only depends on n and p. In the following lemma we prove that the discrete and continuous C_p-tails are equivalent.

Lemma 2.25
Let $\beta = \sum_{l=0}^{\infty} 2^{-npl}$. Then, for every weight w and every cube Q, we have

$$\frac{1}{\beta}\,T_{C_p}(Q,w) \le \frac{1}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)dx \le \frac{4^{np}}{\beta}\,T_{C_p}(Q,w). \tag{2.8}$$

As a corollary of this, we have that $T_{C_p}(Q,w) < \infty$ for every cube Q whenever w has finite C_p-tails.

Proof. Observe that $\beta = \sum_{l=0}^{\infty} 2^{-npl} = (2^{np})'$ and hence $\beta < 2$. Note that for all $x \in 2^kQ \setminus 2^{k-1}Q$ we have $2^{-kn} \le M\chi_Q(x) \le 2^{-n(k-2)}$. Then

$$\frac{1}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)dx = \fint_Q w + \sum_{k=1}^{\infty}\frac{1}{|Q|}\int_{2^kQ\setminus 2^{k-1}Q} M\chi_Q(x)^p w(x)dx,$$

so we actually have

$$\begin{aligned}
&\fint_Q w(x)dx + \sum_{k=1}^{\infty} \frac{2^{-npk}}{|Q|} w(2^kQ \setminus 2^{k-1}Q) \\
&\qquad\qquad \le \frac{1}{|Q|} \int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)dx \\
&\qquad\qquad \le \fint_Q w(x)dx + \sum_{k=1}^{\infty} \frac{2^{-np(k-2)}}{|Q|} w(2^kQ \setminus 2^{k-1}Q) \\
&\qquad\qquad \le 4^{np} \left(\fint_Q w(x)dx + \sum_{k=1}^{\infty} \frac{2^{-npk}}{|Q|} w(2^kQ \setminus 2^{k-1}Q) \right).
\end{aligned}$$

Now we rewrite (2.7) in the following way

$$\begin{aligned}
&\sum_{k=0}^{\infty} 2^{-n(p-1)k} \fint_{2^kQ} w(x)dx \\
&\quad = \fint_Q w(x)dx + \sum_{k=1}^{\infty} \frac{2^{-npk}}{|Q|} \left(\int_Q w(x)dx + \sum_{j=1}^{k} \int_{2^jQ \setminus 2^{j-1}Q} w(x)dx \right) \\
&\quad = \beta \fint_Q w(x)dx + \frac{1}{|Q|} \sum_{j=1}^{\infty} \left(\sum_{k=j}^{\infty} 2^{-npk} \right) \int_{2^jQ \setminus 2^{j-1}Q} w(x)dx \\
&\quad = \beta \left(\fint_Q w(x)dx + \frac{1}{|Q|} \sum_{j=1}^{\infty} 2^{-pnj} \int_{2^jQ \setminus 2^{j-1}Q} w(x)dx \right).
\end{aligned}$$

This finishes the proof of (2.8). □

The next proposition is a first approach to the result in Theorem 2.21. It constitutes a bound for the $L^{1+\delta}$ norm of the maximal function of the weight, locally at a cube, in terms of the C_p-tail of the weight.

Proposition 2.26

Let w be a weight and $p > 1$. Suppose that there exists a constant $0 < \gamma < \infty$ such that for every cube Q

$$\fint_Q M(\chi_Q w)(x)dx \le \gamma T_{C_p}(Q, w). \tag{2.9}$$

Then there exists $0 < \delta \le \frac{1}{A \max(\gamma,1)}$, with

$$A = 20^n \frac{2^{1+3n}}{1 - 2^{-n(p-1)}},$$

such that for every cube Q,

$$\fint_Q M(\chi_Q w)(x)^{1+\delta} dx \le 2^{1+n(2p+3)}\, \gamma\, T_{C_p}(Q, w)^{1+\delta}.$$

Note that the infimum of the constants γ such that (2.9) holds is equivalent to the C_p constant of w, because of Lemma 2.25. In this case we will have $0 < [w]_{C_p} < \infty$.

Proof. Fix $Q = Q(x_0, R)$, that is, the cube centered at the point x_0 and with side length $2R$. Note that $Q(x, R)$ is a ball of radius R with the l^∞ distance in $\mathbb{R}^n$. The proof will be carried out following some steps.

Step 1. Let $r, \rho > 0$ and $l \in \mathbb{Z}$ be numbers that satisfy $R \le r < \rho \le 2R$ and $2^l(\rho - r) = R$. This in particular implies $l \ge 0$.

We define a new maximal operator that is a discrete centered Hardy–Littlewood maximal operator, with scales at a geometric sequence:

$$\widetilde{M}v(x) := \sup_{k\in\mathbb{Z}} \fint_{Q(x,2^k(\rho-r))} |v(x)|dx.$$

One can prove the following pointwise bounds between the different maximal functions

$$\widetilde{M}v \le Mv \le \kappa\widetilde{M}v,$$

where $\kappa = 4^n$. The first inequality is obvious, and the second one follows from the doubling property of the Lebesgue measure. For $t \ge 0$ and a function F we define the truncated function $F_t = \min(F, t)$. Now fix $m > 0$ with the intention of working with truncation at level m and letting $m \to \infty$ in the end. Call $Q_r = Q(x_0, r)$ and $Q_\rho = Q(x_0, \rho)$.

For any $\delta > 0$ that will be chosen later, we have, using the layer cake formula from Section 1.4.4,

$$\begin{aligned}\int_{Q_r} (M(\chi_{Q_r} w)(x))_m^{1+\delta} dx &\le \kappa^{1+\delta} \int_{Q_r} (\widetilde{M}(\chi_{Q_r} w)(x))_m^{\delta}\, \widetilde{M}(\chi_{Q_r} w)(x)dx \\ &\le \kappa^{1+\delta} \int_{Q_r} (\widetilde{M}(\chi_{Q_\rho} w)(x))_m^{\delta}\, \widetilde{M}(\chi_{Q_\rho} w)(x)dx \\ &\le \kappa^{1+\delta}\delta \int_0^m \lambda^{\delta-1} u(\{x \in Q_r : u(x) > \lambda\})d\lambda,\end{aligned}$$

where $u = \widetilde{M}(\chi_{Q_\rho} w)$. We have used that the maximal operator $\widetilde{M}$ is increasing on the function. To state it in a separate line, we have

$$\int_{Q_r} (M(\chi_{Q_r} w))_m(x)^{1+\delta} dx \le \kappa^{1+\delta}\delta \int_0^m \lambda^{\delta-1} u(\{x \in Q_r : u(x) > \lambda\})d\lambda. \tag{2.10}$$

Step 2. Now we pick $\lambda_0 := 2^{n(l+1)} T_{C_p}(2Q, w)$, which is finite by hypothesis. It is easy to see that for $x \in Q_r$ and $k \ge 0$, by the choice of λ_0, we have

$$\fint_{Q(x,2^k(\rho-r))} \chi_{Q_\rho}(y)w(y)dy \le \lambda_0. \tag{2.11}$$

Indeed, we have that $Q_\rho \subset 2Q$, so we can make

$$\begin{aligned}\fint_{Q(x,2^k(\rho-r))} \chi_{Q_\rho}(y)w(y)dy &\le \fint_{Q(x,2^k(\rho-r))} \chi_{2Q}(y)w(y)dy \\ &= \frac{|2Q|}{|Q(x, 2^k(\rho - r))|} \fint_{2Q} w(y)dy \\ &\le 2^{n(l+1-k)}\, T_{C_p}(2Q, w)\end{aligned}$$

$$\leq\ 2^{n(l+1)}\, T_{C_p}(2Q,w).$$

This completes the proof of (2.11) when $x \in Q_r$ and $k \geq 0$.

Let $\lambda > \lambda_0$ and $x \in Q_r \cap \{u > \lambda\}$. By the definition of u and the choice of λ_0, the fact that $Q(x, 2^k(\rho - r)) \subset Q_\rho$ when $k < 0$ together with (2.11) imply

$$u(x) = \sup_{k<0} \fint_{Q(x,2^k(\rho-r))} \chi_{Q_\rho} w \ = \sup_{k<0} \fint_{Q(x,2^k(\rho-r))} w.$$

For such an x, let $k_x = \max\{k : \fint_{Q(x,2^k(\rho-r))} w > \lambda\}$. Trivially, we have

$$Q_r \cap \{u > \lambda\} \subset \bigcup_{x \in Q_r \cap \{u>\lambda\}} Q(x, \frac{1}{5} 2^{k_x}(\rho - r)).$$

We use the Vitali covering Lemma 1.2 for infinite sets and choose a countable collection of $x_i \in Q_r \cap \{u > \lambda\}$ so that the family of cubes $Q_i = Q(x_i, 2^{k_{x_i}}(\rho - r))$ satisfy the following properties:

– $\{x \in Q_r : u(x) > \lambda\} \subset \bigcup_i Q_i$;
– the cubes $\frac{1}{5} Q_i$ are pairwise disjoint;
– $\fint_{Q_i} w(y)dy > \lambda$,
– $\fint_{2^k Q_i} w(y)dy \leq \lambda$, for any $k \geq 1$
– $Q_i \subset Q_\rho$.

We make the following claim. If we denote $Q_i^* = 2Q_i$ then for all $x \in Q_i \cap Q_r$,

$$u(x) \leq 2^n M(\chi_{Q_i^*} w)(x). \tag{2.12}$$

Indeed, fix $x \in Q_i \cap Q_r$ and $k < 0$. If $k \geq k_{x_i}$ then by the stopping time we get

$$\begin{aligned}
\fint_{Q(x,2^k(\rho-r))} w(y)dy &\leq \frac{|Q(x_i, 2^{k+1}(\rho - r))|}{|Q(x, 2^k(\rho - r))|} \fint_{Q(x_i,2^{k+1}(\rho-r))} w(y)dy \\
&\leq 2^n \lambda \\
&\leq 2^n \fint_{Q_i} w(y)dy \\
&\leq 2^n M(\chi_{Q_i^*} w)(x).
\end{aligned}$$

In the other case, namely $k < k_{x_i}$ we have $Q(x, 2^k(\rho - r)) \subset Q_i^* \cap Q_\rho$ and hence

$$\fint_{Q(x,2^k(\rho-r))} w(y)dy \leq M(\chi_{Q_i^*} w)(x),$$

and thus the claim (2.12) is proved.

Step 3. We use now this claim together with the stopping time and the hypothesis (2.9) to see

$$u(\{x \in Q_r : u(x) > \lambda\}) \leq \sum_i u(Q_i \cap Q_r)$$

$$
\begin{aligned}
&\le \sum_i \int_{Q_i\cap Q_r} u(y)dy \\
&\le 2^n \sum_i \int_{Q_i\cap Q_r} M(\chi_{Q_i^*} w)(y)dy \\
&\le 2^n \sum_i |Q_i^*| \fint_{Q_i^*} M(\chi_{Q_i^*} w)(y)dy \\
&\le 2^n \gamma \sum_i |Q_i^*| T_{C_p}(Q_i^*, w).
\end{aligned}
$$

But, using the properties of Q_i we get

$$
T_{C_p}(Q_i^*, w) = \sum_{k=0}^{\infty} 2^{-nk(p-1)} \fint_{2^{k+1}Q_i} w(y)dy \le \lambda\alpha.
$$

Therefore, we have

$$
u(\{x\in Q_r : u(x) > \lambda\}) \le 2^n \gamma \sum_i |Q_i^*| \alpha\lambda \le (20)^n \gamma\alpha \left|\bigcup_i Q_i\right| \lambda,
$$

where in the last inequality we have used that $\frac{1}{5}Q_i$ are disjoint. Since each one of the cubes Q_i satisfies the properties $Q_i \subset Q_\rho$ and $\lambda < \fint_{Q_i} w$, we have

$$
\bigcup_i Q_i \subset \{x\in Q_\rho : M(\chi_{Q_\rho} w)(x) > \lambda\}.
$$

Therefore, we have obtained for $\lambda > \lambda_0$

$$
u(\{x\in Q_r : u(x) > \lambda\}) \le (20)^n \,\alpha\,\gamma\,\lambda\, |\{x\in Q_\rho : M(\chi_{Q_\rho} w)(x) > \lambda\}|.
$$

Plugging everything on what we had in (2.10) we have

$$
\begin{aligned}
&\int_{Q_r} (M(\chi_{Q_r}))_m(x)^{1+\delta} dx \qquad (2.13)\\
&\quad\le \kappa^{1+\delta}\,\lambda_0^\delta\, u(Q_r) + \kappa^{\delta+1}(20)^n\gamma\,\alpha\,\delta \int_{\lambda_0}^m \lambda^\delta\, |\{x\in Q_\rho : M(\chi_{Q_\rho} w)(x) > \lambda\}|\, d\lambda.
\end{aligned}
$$

Step 4. For $t > 0$, we define the function

$$
\varphi(t) = \int_{Q_t} M(\chi_{Q_t} w)_m(x)^{1+\delta}\, dx.
$$

Observe that $\varphi(t) \le (2t)^n m^{1+\delta} < \infty$ for any $t > 0$. We claim that there exists some $K_1 > 0$ that depends on n, p, δ such that

$$
\varphi(r) \le K_1\,\gamma\,|Q|\,2^{nl\delta}\left(T_{C_p}(Q,w)\right)^{1+\delta} + \delta\,\kappa^{\delta+1}(20)^n\gamma\,\alpha\,\varphi(\rho). \qquad (2.14)
$$

Indeed, combining (2.13) we obtained before in the following way:

$$
\begin{aligned}
\varphi(r) &\le K_1\,\gamma\,|Q| 2^{nl\delta}\left(T_{C_p}(Q,w)\right)^{1+\delta} + \kappa^{\delta+1}(20)^n\gamma\,\alpha\,\frac{\delta}{\delta+1}\int_{Q_\rho} M(\chi_{Q_\rho} w)_m(x)^{\delta+1}dx \\
&\le K_1\,\gamma\,|Q| 2^{nl\delta}\left(T_{C_p}(Q,w)\right)^{1+\delta} + \kappa^{\delta+1}(20)^n\gamma\,\alpha\,\delta\,\varphi(\rho),
\end{aligned}
$$

where $K_1 = 2^{n(p+1)(\delta+1)}$, and where we have used

$$\begin{aligned} u(Q_r) &= \int_{Q_r} \widetilde{M}(\chi_{Q_\rho} w)(x)dx \\ &\leq |2Q| \fint_{2Q} M(\chi_{2Q} w)(x)dx \\ &\leq 2^n |Q| \gamma T_{C_p}(2Q, w) \\ &\leq 2^{np} |Q| \gamma T_{C_p}(Q, w), \end{aligned}$$

since

$$T_{C_p}(2Q, w) \leq 2^{n(p-1)} T_{C_p}(Q, w).$$

This yields the claim (2.14).

Step 5. Now we present an iteration scheme starting from claim (2.14). Remember that $l \geq 0$ was an integer such that $2^l(\rho - r) = R$. Set

$$\begin{aligned} t_0 &= R, \\ t_{i+1} &= t_i + 2^{-(i+1)} R = \sum_{j=0}^{i+1} 2^{-j} R, \quad i \geq 0. \end{aligned}$$

Clearly, $t_i \to 2R$ as $i \to \infty$. This way, $2^{i+1}(t_{i+1} - t_i) = R$ and we can substitute $\rho = t_{i+1}$, $t_i = r$, and $l = i + 1$ in (2.14). That is, we have the estimate for $\varphi(t_i)$ in terms of $\varphi(t_{i+1})$:

$$\varphi(t_i) \leq K_2 2^{n\delta i} + K_3 \varphi(t_{i+1}),$$

where $K_2 = K_1 2^{n\delta} \gamma |Q| (T_{C_p}(Q, w))^{1+\delta}$ and $K_3 = \kappa^{\delta+1} 20^n \alpha \gamma \delta$. Therefore, iterating this last inequality i_0 times we get

$$\begin{aligned} \varphi(R) &= \varphi(t_0) \\ &\leq K_2 \sum_{j=0}^{i_0-1} (K_3 2^{n\delta})^j + K_3^{i_0} \varphi(t_{i_0}) \\ &\leq K_2 \sum_{j=0}^{i_0-1} (K_3 2^{n\delta})^j + (K_3)^{i_0} \varphi(2R). \end{aligned} \tag{2.15}$$

We choose $0 < \delta < 1$ small enough so that we have the relation

$$K_3 2^{n\delta} = 20^n \kappa^{\delta+1} \gamma\, \alpha\, \delta\, 2^{n\delta} < 1/2. \tag{2.16}$$

We will postpone the choice of δ for the sake of finishing the argument. Once we have (2.16), we can take the limit $i_0 \to \infty$ in (2.15). The sum is bounded by 2 and the second term goes to zero since $\varphi(2R) < \infty$. Hence

$$\begin{aligned} \varphi(R) \leq 2K_2 &= 2^{1+n\delta+n(\delta+1)(p+1)} \gamma |Q| (a_{C_p}(Q))^{1+\delta} \\ &< 2^{1+n(2p+3)} \gamma |Q| (T_{C_p}(Q, w))^{1+\delta}, \end{aligned}$$

and then

$$\frac{1}{|Q|} \int_Q M(\chi_Q w)_m(x)^{1+\delta} dx \leq 2^{1+n(2p+3)} \gamma \left(T_{C_p}(Q, w)\right)^{1+\delta}.$$

Now, letting $m \to \infty$ and using the Fatou lemma we can conclude the proof.

To finish the proof, we make the choice of δ as follows. Coming back to (2.16) we see that, since we have δ in the exponent and γ can be arbitrarily small, we have to choose $\delta = \frac{1}{A(1+\gamma)}$ with

$$A = 2\kappa^2(20)^n 2^n \alpha = (20)^n \frac{2^{1+3n}}{1-2^{-n(p-1)}}. \qquad \square$$

Using the last Proposition, we are in shape of proving Theorem 2.21. We are going to use arguments similar to those from [63, Theorem 2.3].

Proof of Theorem 2.21. Fix a cube Q. Let M_Q denote the maximal operator with respect to the dyadic children of Q, that is

$$M_Q v(x) = \sup_{\substack{R\in\mathcal{D}(Q)\\ x\in R}} \frac{1}{|R|}\int_R |v(y)|dy, \quad x\in Q.$$

By the Lebesgue differentiation theorem, we have the estimate

$$\int_Q w(x)^{1+\delta}dx \le \int_Q M_Q w(x)^\delta w(x)dx.$$

Call now $\Omega_\lambda = \{x\in Q : M_{d,Q}w(x) > \lambda\}$. For $\lambda \ge w_Q$ we make the Calderón–Zygmund decomposition, see Section 1.4.3 for more details, of w at height λ to obtain $\Omega_\lambda = \bigcup_j Q_j$ with Q_j pairwise disjoint and

$$\lambda < \frac{1}{|Q_j|}\int_{Q_j} w(x)dx \le 2^n\lambda.$$

Multiplying by $|Q_j|$ and summing on j, this inequality chain becomes

$$\lambda|\Omega_\lambda| \le w(\Omega_\lambda) \le 2^n\lambda|\Omega_\lambda|.$$

Therefore, we can make the following computations

$$\begin{aligned}
\fint_Q \Big(M_Q w(x)\Big)^\delta w(x)dx &= \frac{1}{|Q|}\int_0^\infty \delta\lambda^{\delta-1}w(\Omega_\lambda)d\lambda \\
&\le w_Q^{\delta+1} + \frac{1}{|Q|}\int_{w_Q}^\infty \delta\lambda^{\delta-1}w(\Omega_\lambda)d\lambda \\
&\le w_Q^{\delta+1} + \delta 2^n \frac{1}{|Q|}\int_{w_Q}^\infty \lambda^\delta|\Omega_\lambda|d\lambda \\
&\le w_Q^{\delta+1} + 2^n\frac{\delta}{\delta+1}\frac{1}{|Q|}\int_Q \Big(M_Q w(x)\Big)^{1+\delta}dx.
\end{aligned}$$

Now we apply Proposition 2.26. We have $[w]_{C_p} \le \beta\gamma \le 4^{np}[w]_{C_p}$, so we need $\delta \le \beta/A(1+[w]_{C_p})$, with β as in Lemma 2.25. So we get

$$\begin{aligned}
\fint_Q \Big(M_{d,Q}w(x)\Big)^\delta w(x)dx &\le (1+2^{1+n(2p+4)}\frac{\delta}{\delta+1}\gamma)\,\Big(T_{C_p}(Q,w)\Big)^{1+\delta} \\
&\le (1+2^{1+n(2p+4)}\frac{\delta}{\delta+1}[w]_{C_p}\frac{4^{np}}{\beta}) \\
&\quad \times \left(\frac{\beta}{|Q|}\int_{\mathbb{R}^n}\Big(M\chi_Q(x)\Big)^p w(x)dx\right)^{1+\delta},
\end{aligned}$$

where we have used Lemma 2.25. Now, since we have $2^{4np}/\beta$ multiplying δ, we have to change the choice of δ slightly and make

$$\delta \leq \frac{2^{-4np}}{\beta}\frac{\beta}{A(1+[w]_{C_p})} = \frac{1}{B(1+[w]_{C_p})}.$$

This finishes the proof of the theorem. □

2.5.1 Sharpness of the exponent

In this section, we are going to discuss the sharpness of the dependence of δ on the C_p constant of the weight in the statement of Theorem 2.21.

For a cube Q, it is clear that $M\chi_Q$ equals 1 on the cube and is smaller than 1 outside the cube. Therefore $(M\chi_Q)^p$ converges to χ_Q a.e. when $p \to \infty$. Moreover, for a weight w with finite C_{p_0}-tails for some $p_0 < \infty$, by the Dominated Convergence theorem we have

$$\lim_{p\to\infty} \int_{\mathbb{R}^n} (M\chi_Q(x))^p w(x)dx = w(Q).$$

For any weight $w \in A_\infty$, we have by the definition of the constant $[w]_{A_\infty}$ that for any cube Q

$$\int_Q M(w\chi_Q)(x)dx \leq [w]_{A_\infty} w(Q) \leq [w]_{A_\infty} T_{C_p}(Q,w),$$

where $T_{C_p}(Q,w) = \sum_{k\geq 0} 2^{-n(p-1)k} \fint_{2^kQ} w$ is the discrete C_p-tail introduced in the previous section.

If we modify slightly the proof of Proposition 2.26 and Theorem 2.21 and add some extra hypothesis, we can recover the RHI for A_∞ weights. We explain how to do this in this section.

Fix a number $s > 1$. This will be the dilation parameter, which was $s = 2$ in the previous section. We plan on letting t tend to one in the end. We introduce the corresponding discrete C_p-tail with respect to s, that is,

$$T^s_{C_p}(Q,w) = \sum_{k\geq 0} s^{-n(p-1)k} \fint_{s^kQ} w(x)dx.$$

Note that for any weight w with finite C_{p_0}-tails for some $p_0 < \infty$, we have, using the dominated convergence theorem, that $\lim_{p\to\infty} T^s_{C_p}(Q) = w_Q$ for any $s > 1$. Also, for a fixed $s > 1$ we introduce the corresponding discrete C_p constant

$$[w]_{C_p,s} := \sup_Q \frac{\int_Q M(\chi_Q w)}{T^s_{C_p}(Q,w)}.$$

Remark 2.27 For a weight $w \in A_\infty$ and any $s > 1$ we have $\lim_{p\to\infty}[w]_{C_p,s} \leq [w]_{A_\infty}$. Indeed, we claim that if $w \in A_\infty$, then $T^s_{C_p}(Q,w)$ is finite for all Q for big enough p_0. Then, by the Dominated Convergence Theorem, $\lim_{p\to\infty} T^s_{C_p}(Q,w) = w(Q)$ and the result follows. In order to see that $T^s_{C_p}(Q,w)$ is finite, we use that $w \in A\infty$ is doubling, that is, there exists a constant $\kappa \geq 1$ such that

$$w(sQ) \leq \kappa w(Q)$$

for all Q. Then,

$$\begin{aligned} T^s_{C_p}(Q,w) &= \sum_{k\geq 0} s^{-n(p-1)k} \fint_{s^kQ} w(x)dx \\ &= \sum_{k\geq 0} s^{-n(p-1)k} \frac{w(s^kQ)}{|s^kQ|} \\ &\leq \sum_{k\geq 0} s^{-n(p-1)k} \frac{\kappa^k w(Q)}{s^{nk}|Q|} \\ &= w_Q \sum_{k\geq 0} \left(\frac{\kappa}{s^{np}}\right)^k, \end{aligned}$$

which is finite for big enough p.

Theorem 2.28

Fix $2 \geq s > 1$ and $1 < p < \infty$. For a weight w in C_p and $\delta = \frac{1}{A_{s,p}(1+[w]_{C_p,s})}$ and every cube Q, with

$$A_{s,p} = \frac{5^n 2^{1+5n}}{1 - s^{-n(p-1)}},$$

we have

$$\left(\frac{1}{|Q|}\int_Q w(x)^{1+\delta}dx\right)^{\frac{1}{1+\delta}} \leq (2^n+1)\, T^s_{C_p}(sQ,w). \tag{2.17}$$

Before we prove this theorem, we give a proof of Theorem 2.5 as a corollary. More precisely, we obtain a Reverse Hölder inequality for weights $w \in A_\infty$ in which the dependence of the exponent on the A_∞ constant is of the same order of the one in Theorem 2.5, with a worse dimensional constant. This will show that the dependence of the exponent δ on the C_p constant is sharp in that sense, because Theorem 2.5 is sharp.

Let $w \in A_\infty$. By Remark 2.27, we can let $p \to \infty$ in equation (2.17) and we obtain

$$\left(\frac{1}{|Q|}\int_Q w^{1+\delta_\infty}\right)^{\frac{1}{1+\delta_\infty}} \leq (2^n+1)\, w_{sQ}, \tag{2.18}$$

where

$$\delta_\infty = \lim_{p\to\infty} \frac{1-s^{-n(p-1)}}{5^n 2^{1+5n}\max(1,[w]_{C_p,s})} = \frac{1}{5^n 2^{1+5n}[w]_{A_\infty}}.$$

Now we let $s \to 1$ in (2.18) and obtain

$$\left(\frac{1}{|Q|}\int_Q w^{1+\delta_\infty}\right)^{\frac{1}{1+\delta_\infty}} \leq (2^n+1)\, w_Q,$$

which is in fact the reverse Hölder inequality for A_∞ weights.

Remark 2.29 The dimensional constants are bigger from those in Theorem 2.5, but the dependence on the weight is essentially the same. Because of this, we obtain that the dependence on w in Theorem 2.21 is sharp.

Proof of Theorem 2.28. We repeat the first three steps of the proof of Proposition 2.26, with the following modifications. This time, r, ρ, l will satisfy $s^l(\rho - r) = R$

and $R \le r < \rho \le R$. Also, now we will use the maximal operator $\widetilde{M}v(x) = \sup_{k\in\mathbb{Z}} \fint_{Q(x,s^k(\rho-r))} u$, and some other trivial changes.

For the fourth step, we leave $T^s_{C_p}(sQ)$ in the equation, so we get

$$\varphi(r) \le s^{n(\delta+1)}\gamma|Q|s^{n\delta l}\left(T^s_{C_p}(sQ)\right)^{1+\delta} + (\kappa^{1+\delta}(5s^2)^n\gamma\alpha_s)\ \delta\ \varphi(\rho),$$

where $\alpha_s = \sum_{k\ge 0} s^{-nk(p-1)} = (1-s^{-n(p-1)})^{-1}$. We make a similiar iteration scheme, namely $t_0 = R$ and $t_{i+1} = t_i + s^{-(i+1)}R \le sR$. Now the condition for δ translates to $\delta \le \frac{1}{A_{s,p}\max(1,\gamma)}$ where

$$A_{s,p} = \frac{5^n 2^{1+5n}}{1-s^{-n(p-1)}}.$$

The main difference is that now we get

$$\frac{1}{|Q|}\int_Q M(\chi_Q w)_m(x)^{1+\delta}dx \le 2^{1+5n}\gamma\big(T^s_{C_p}(sQ,w)\big)^{1+\delta},$$

where the right part stays bounded whenever $p\to\infty$. Now we use Fatou lemma and make $m\to\infty$ to get

$$\frac{1}{|Q|}\int_Q M(\chi_Q w)(x)^{1+\delta}dx \le 2^{1+5n}\gamma\big(T^s_{C_p}(sQ,w)\big)^{1+\delta}. \tag{2.19}$$

Finally we make the argument in the proof of Theorem 2.21 and combine it with (2.19). We get

$$\begin{aligned}\fint_Q w(x)^{1+\delta}dx &\le (w_Q)^{1+\delta} + 2^n\frac{\delta}{1+\delta}\frac{1}{|Q|}\int_Q M_Q w(x)^{1+\delta}dx\\ &\le (w_Q)^{1+\delta} + 2^n\frac{\delta}{1+\delta}2^{1+5n}\gamma\big(T^s_{C_p}(sQ)\big)^{1+\delta}\\ &\le (2^n+\delta 2^{1+6n}\gamma)\big(T^s_{C_p}(sQ,w)\big)^{1+\delta}\\ &\le (2^n+1)\big(T^s_{C_p}(sQ,w)\big)^{1+\delta},\end{aligned}$$

if $\delta \le \frac{1}{2^{1+6n}\gamma}$, which is true by the choice of δ. This finishes the proof. $\square$

2.6 Weak self-improvement properties of C_p

It is well-known that A_p weights are self-improving: if $w \in A_p$, then there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$ [20, Lemma 2]. Since this is a particularly convenient property in many proofs, it would be desirable if C_p weights had a similar property, i.e. for every $w \in C_p$ there existed $\varepsilon > 0$ such that $w \in C_{p+\varepsilon}$. In particular, this property together with Sawyer's results would prove Muckenhoupt's conjecture Conjecture 2.1. Unfortunately, this is not true due to an example by Kahanpää and Mejlbro [69, Theorem 11]. We discuss their counterexample and its generalizations in detail in Section 2.9.

The failure of this self-improving property raises natural questions about weaker self-improving properties of C_p weights. For example, although the well-known self-improving property of classical Reverse Hölder weights [41, Lemma 3] fails in spaces of homogeneous type [2, Section 7], the weights are still self-improving in a weak sense even in this more general setting [2, Section 6] (see also [119, Theorem 3.3]). Although we show in Section 2.7 that weakening the definition of C_p in an obvious way does not actually change the structure of the corresponding weight class, various self-improvement and Reverse Hölder questions remain open. In particular:

Open Problem 2.30

Suppose that $w \in C_p$ for some $1 < p < \infty$ and let δ be the Reverse Hölder exponent from Theorem 2.21. Do there exist $c_w > 1$ and $K_w > 1$ such that

$$\left(\fint_Q w(x)^{c(1+\delta)}dx\right)^{\frac{1}{c(1+\delta)}} \leq \frac{K_w}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)dx$$

for every cube Q and every $1 < c \leq c_w$?

In this section, we record two observations related to Problem 2.30. First, we prove the following analogue to the well-known A_∞ result that states that for any $w \in A_\infty$, there exists some small $\varepsilon > 0$ such that $w^{1+\varepsilon} \in A_\infty$ (see e.g. [64, Corollary 3.17]). This property is what we call *weak self-improvement* property of C_p.

Proposition 2.31

Let $w \in C_p$ for some $1 < p < \infty$. Then there exists $\varepsilon_0 > 0$ such that $w^{1+\varepsilon} \in C_p$ for every $0 < \varepsilon \leq \varepsilon_0$.

Proof. Let δ be the Reverse Hölder parameter from Theorem 2.21 and set $\varepsilon_0 = \frac{\delta}{2}$. Then, for $s = 1 + \frac{\delta}{2+\delta}$, we have $s(1+\varepsilon_0) = 1+\delta$. Thus, we get

$$\begin{aligned}\left(\fint_Q w(x)^{(1+\varepsilon_0)s}dx\right)^{\frac{1}{s}} &\leq \left(\frac{1}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)dx\right)^{\frac{1+\delta}{s}}\\ &= \left(\frac{1}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)dx\right)^{1+\varepsilon_0}\\ &\leq \left(\frac{1}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^p dx\right)^{\frac{1+\varepsilon_0}{1+\frac{1}{\varepsilon_0}}}\left(\frac{1}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)^{1+\varepsilon_0}dx\right)\\ &\leq (c_n\, p')^{\varepsilon_0}\cdot\frac{1}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)^{1+\varepsilon_0}dx,\end{aligned}$$

where we used first Theorem 2.21, then the standard Hölder's inequality and finally the L^p-boundedness of the Hardy–Littlewood maximal operator. Thus, the weight $w^{1+\varepsilon_0}$ satisfies a Reverse Hölder inequality in the sense of Theorem 2.21 and therefore $w^{1+\varepsilon_0} \in C_p$.

The fact that now also $w^{1+\varepsilon} \in C_p$ for every $0 < \varepsilon \leq \varepsilon_0$ follows easily from Hölder's inequality, since the $L^{1+\varepsilon}$ average on a cube is bounded by the $L^{1+\varepsilon_0}$ average. □

In the light of Proposition 2.31, we propose a new problem whose positive answer would also imply a positive answer to Problem 2.30:

Open Problem 2.32

Suppose that $w \in C_p$ for some $1 < p < \infty$. Do there exist $\varepsilon_0 > 0$ and $\kappa \geq 1$ such that

$$\left(\frac{1}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)^{1+\varepsilon}dx\right)^{\frac{1}{1+\varepsilon}} \leq \kappa \frac{1}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)dx \tag{2.20}$$

for every cube Q and every $0 < \varepsilon \leq \varepsilon_0$?

As a consequence of Proposition 2.31 we get something slightly worse than (2.20). We can bound the C_p-tail of $w^{1+\delta}$ by the $C_{\frac{p+\delta}{1+\delta}}$-tail of w for δ smaller than the Reverse Hölder exponent of w.

Corollary 2.33

Suppose $w \in C_p$ for some $1 \leq p < \infty$ and let δ_0 be the Reverse Hölder exponent from Theorem 2.21. Then for every $0 < \delta \leq \delta_0$ and every cube Q we have

$$\left(\frac{1}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)^{1+\delta}dx\right)^{\frac{1}{1+\delta}} \leq C_{n,p,\delta}\frac{1}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^{\frac{p+\delta}{1+\delta}} w(x)dx,$$

Note that when $p > 1$ the exponent $\frac{p+\delta}{1+\delta}$ is strictly greater than p.

Since the proof of Corollary 2.33 is a fairly technical computation, we formulate explicitly the following well-known embedding property of ℓ^p spaces:

Lemma 2.34

Let $0 < \alpha < \beta < \infty$. Then, for positive numbers a_n, $n \in \mathbb{N}$, we have

$$\left(\sum_n a_n^\beta\right)^{\frac{1}{\beta}} \leq \left(\sum_n a_n^\alpha\right)^{\frac{1}{\alpha}}.$$

Proof. Since $a_n > 0$, it is clear that for any n it holds

$$\frac{a_n^\alpha}{\sum_m a_m^\alpha} \leq 1.$$

Then, since $\beta > \alpha$, we have

$$\begin{aligned}
\left(\sum_n a_n^\beta\right)^{\frac{1}{\beta}} &= \left(\sum_n a_n^\beta\right)^{\frac{1}{\beta}} \frac{\left(\sum_m a_m^\alpha\right)^{\frac{1}{\alpha}}}{\left(\sum_m a_m^\alpha\right)^{\frac{1}{\alpha}}} \\
&= \left(\sum_n \left(\frac{a_n^\alpha}{\sum_m a_m^\alpha}\right)^{\frac{\beta}{\alpha}}\right)^{\frac{1}{\beta}} \left(\sum_m a_m^\alpha\right)^{\frac{1}{\alpha}} \\
&\leq \left(\sum_n \frac{a_n^\alpha}{\sum_m a_m^\alpha}\right)^{\frac{1}{\beta}} \left(\sum_m a_m^\alpha\right)^{\frac{1}{\alpha}} \\
&= \left(\sum_m a_m^\alpha\right)^{\frac{1}{\alpha}},
\end{aligned}$$

which finishes the proof. □

Proof of Corollary 2.33. We argue by discretizing the tail. By Lemma 2.25, we have

$$\frac{1}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)dx \approx_{n,p} \sum_{k=0}^{\infty} 2^{-n(p-1)k} \fint_{2^k Q} w(x)dx,$$

for $1 \le p < \infty$ and any weight w. The implicit constants do not blow up when p tends to 1, but they do blow up when $p \to \infty$. We get

$$\begin{aligned}
\frac{1}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^p w(x)^{1+\delta}(x)dx &\overset{(A)}{\approx}_{n,p} \sum_{k=0}^{\infty} 2^{-n(p-1)k} \fint_{2^k Q} w(x)^{1+\delta}dx \\
&\overset{(B)}{\lesssim} \sum_{k=0}^{\infty} 2^{-n(p-1)k} \Big(\frac{1}{|2^k Q|}\int_{\mathbb{R}^n} M\chi_{2^k Q}(x)^p w(x)dx\Big)^{1+\delta} \\
&\overset{(A)}{\lesssim} \sum_{k=0}^{\infty} 2^{-n(p-1)k} \Big(\sum_{j=0}^{\infty} 2^{-n(p-1)j} \fint_{2^{j+k}Q} w(x)dx\Big)^{1+\delta} \\
&\overset{(C)}{\le} \Big(\sum_{k,j=0}^{\infty} 2^{-n(p-1)\frac{k}{1+\delta}} 2^{-n(p-1)j} \fint_{2^{j+k}Q} w\Big)^{1+\delta} \\
&= \Big(\sum_{m=0}^{\infty} \Big(\sum_{i=0}^{m} 2^{-n(p-1)\left(\frac{i}{1+\delta}+(m-i)\right)}\Big) \fint_{2^m Q} w(x)dx\Big)^{1+\delta} \\
&\overset{(D)}{\lesssim} \Big(\sum_{m=0}^{\infty} 2^{-n(p-1)\frac{m}{1+\delta}} \fint_{2^m Q} w(x)dx\Big)^{1+\delta} \\
&= \Big(\sum_{m=0}^{\infty} 2^{-n\left(\frac{p+\delta}{1+\delta}-1\right)m} \fint_{2^m Q} w(x)dx\Big)^{1+\delta} \\
&\overset{(A)}{\approx}_{n,p,\delta} \Big(\frac{1}{|Q|}\int_{\mathbb{R}^n} M\chi_Q(x)^{\frac{p+\delta}{1+\delta}} w(x)dx\Big)^{1+\delta},
\end{aligned}$$

where we (A) used the discretization, (B) used the Reverse Hölder inequality, (C) applied Lemma 2.34 with $\alpha = \frac{1}{1+\delta}$ and $\beta = 1$, and (D) calculated the geometric sum and made obvious estimates. □

2.7 On weak C_p and dyadic C_p

When we compare the characterizations of A_∞ (2.3) and C_p (2.4), it is obvious that $A_\infty \subset C_p$ for every p. However, A_∞ weights are not good representatives of C_p weights because the C_p classes are much bigger than the A_∞ class. For example, A_∞ weights are always doubling and they cannot vanish in a set of positive measure whereas C_p weights can grow arbitrarily fast and their supports can contain holes of infinite measure. Thus, the structure of a general C_p weight can be very messy.

In this section, we introduce weak and dyadic C_p weights as an analogy to weak and dyadic A_∞ weights. Although these new classes of weights seem like they are larger than C_p, this is not the case: weak and dyadic C_p weights are just C_p weights. We also consider some examples and properties related to C_p weights.

We start by proving an elementary lemma for the Hardy–Littlewood maximal operator similar to Lemma 2.9 with a general set.

Lemma 2.35

Let $Q_0 \subset \mathbb{R}^n$ be a cube and $E_0 \subset Q_0$ a measurable subset such that $|E_0| \geq \eta |Q_0|$ for some $0 < \eta \leq 1$. Then there exists a dimensional constant κ_n such that

$$M(\chi_{Q_0})(x) \leq \frac{\kappa_n}{\eta} M(\chi_{E_0})(x)$$

for almost every $x \in \mathbb{R}^n$.

Proof. Let $Q(x,r)$ be the cube with center point x and side length r. There exists a structural constant $K_n \geq 1$ such that

$$E_0 \subset Q_0 \subset Q(x, K_n(\operatorname{dist}(x,Q_0) + \ell(Q_0))).$$

The proof now consists of two cases:

Case 1. Suppose that $\operatorname{dist}(x,Q_0) \leq \ell(Q_0)$. Then $Q_0 \subset Q(x, 2K_n\ell(Q_0)) =: Q_x$ and $|Q_0| \approx |Q_x|$. Thus,

$$M(\chi_{E_0})(x) \geq \frac{|E_0 \cap Q_x|}{|Q_x|} \approx \frac{|E_0|}{|Q_0|} \geq \eta \geq \eta M(\chi_{Q_0})(x).$$

Case 2. Suppose that $\operatorname{dist}(x,Q_0) > \ell(Q_0)$. Then

$$\begin{aligned} M(\chi_{Q_0})(x) &= \sup_{r > \operatorname{dist}(x,Q_0)} \frac{|Q_0 \cap Q(x,r)|}{|Q(x,r)|} \\ &\leq \sup_{r > \operatorname{dist}(x,Q_0)} \frac{\kappa_n' |Q_0|}{|Q(x,2K_n r)|} \\ &\leq \sup_{r > \operatorname{dist}(x,Q_0)} \frac{\kappa_n'}{\eta} \frac{|E_0|}{|Q(x,2K_n r)|} \\ &= \sup_{r > \operatorname{dist}(x,Q_0)} \frac{\kappa_n'}{\eta} \frac{|E_0 \cap Q(x,2K_n r)|}{|Q(x,2K_n r)|} \\ &\leq \frac{\kappa_n'}{\eta} M(\chi_{E_0})(x). \end{aligned}$$

□

2.7.1 Weak A_∞ weights

Let us recall the definition of the weak A_∞ classes. The Fujii–Wilson type characterization of these weights was studied in detail in [2] but earlier they have appeared in other forms in the study of e.g. weighted norm inequalities [108] and elliptic partial differential equations and quantitative rectifiability; see e.g. [58] and references therein.

Definition 2.36 – Weak A_∞

Suppose that $\gamma \geq 1$. We say that a weight w belongs to the γ*-weak* A_∞ *class* A_∞^γ if there exist positive constants $\kappa, \delta > 0$ such that

$$w(E) \leq \kappa \left(\frac{|E|}{|Q|} \right)^\delta w(\gamma Q) \tag{2.21}$$

for any cube Q and any measurable subset $E \subset Q$, where γQ is the cube of side length $\gamma \ell(Q)$ with the same center point as Q.

We denote $A_\infty^{\text{weak}} := \bigcup_{\gamma\geq 1} A_\infty^\gamma$. It was shown in [2] that this definition does not give us a continuum of different weak A_∞ classes but the dilation parameter γ is irrelevant for the structure of the class as long as $\gamma > 1$:

Theorem 2.37 – [2]
We have

i) $A_\infty \subsetneq A_\infty^\gamma$ for every $\gamma > 1$;

ii) $A_\infty^\gamma = A_\infty^{\text{weak}}$ for every $\gamma > 1$;

iii) $w \in A_\infty^{\text{weak}}$ if and only if for every $\lambda > 1$ there exists a constant $[w]_{A_\infty^\lambda}$ such that, for every cube Q,
$$\int_Q M(\chi_Q w)(x)dx \leq [w]_{A_\infty^\lambda} w(\lambda Q).$$

2.7.2 Weak C_p and dyadic C_p

Let us then consider two generalizations of the C_p class. Suppose that $\gamma \geq 1$. We write

i) $w \in C_p^{\mathscr{D}}$ if w satisfies condition (2.4) for all $Q \in \mathscr{D}$ instead of all cubes;

ii) $w \in C_p^\gamma$, if w satisfies condition (2.4) for $\chi_{\gamma Q}$ instead of χ_Q, and all cubes Q;

iii) $w \in C_p^{\text{weak}}$ if $w \in \bigcup_{\alpha\geq 1} C_p^\alpha$.

We also define $A_\infty^{\mathscr{D}}$ similarly as $C_p^{\mathscr{D}}$.

Usually, these types of generalizations genuinely weaken the objects in question. For example, in the case of A_∞, we already saw that A_∞ is a proper subset of A_∞^{weak}, and since $1_{[0,\infty)} \in A_\infty^{\mathscr{D}}$, we also have $A_\infty \subsetneq A_\infty^{\mathscr{D}}$. However, because of the non-local nature of the C_p condition, these generalizations for C_p classes just end up giving us back C_p, as the following proposition illustrates.

Proposition 2.38
We have $C_p = C_p^{\mathscr{D}} = C_p^\gamma = C_p^{\text{weak}}$ for every $\gamma \geq 1$.

Proof. The inclusions
$$C_p \subset C_p^{\mathscr{D}} \quad \text{and} \quad C_p \subset C_p^\gamma \subset C_p^{\text{weak}}$$
are obvious and
$$C_p \supset C_p^\gamma \supset C_p^{\text{weak}}$$
follow from Lemma 2.35. Thus, we only need to show that $C_p^{\mathscr{D}} \subset C_p$.

Suppose that $w \in C_p^{\mathscr{D}}$ and let $Q \subset \mathbb{R}^n$ be any cube and $E \subset Q$ a measurable set. There exists 2^n dyadic cubes $Q_i \in \mathscr{D}$ and a uniformly bounded constant $\alpha \geq 1$ such that

1) the cubes Q_i are pairwise disjoint,

2) $\ell(Q_i) \approx \ell(Q)$,

3) $Q \subset \bigcup_i Q_i \subset \alpha Q$.

Applying the $C_p^{\mathscr{D}}$ property to the sets $Q_i \cap E$ and Lemma 2.35 to $M(\chi_{\alpha Q})$ gives us

$$\begin{aligned} w(E) = \sum_i w(E \cap Q_i) &\leq \kappa \sum_i \left(\frac{|E \cap Q_i|}{|Q_i|}\right)^\varepsilon \int_{\mathbb{R}^n} M\chi_{Q_i}(x)^p w(x)dx \\ &\leq \kappa \sum_i \left(\frac{|E|}{|Q|}\right)^\varepsilon \int_{\mathbb{R}^n} M\chi_{\alpha Q}(x)^p w(x)dx \\ &\leq \kappa 2^n \left(\frac{|E|}{|Q|}\right)^\varepsilon \int_{\mathbb{R}^n} M\chi_{Q}(x)^p w(x)dx. \end{aligned}$$

□

2.7.3 Examples and some properties of C_p weights

In this section, we gather some known results from the literature and consider some other examples and properties of C_p weights, along with a few results that we have already discussed. The aim is to have a compiled list of properties and examples that are related to C_p weights.

i) From A_p theory, (2.21), Lemma 2.35, [2] and Theorem 2.43, it follows that for $1 < p < q < \infty$ we have

$$A_1 \subsetneq A_p \subsetneq A_q \subsetneq A_\infty \subsetneq A_\infty^{\text{weak}} \subsetneq C_q \subsetneq C_p \subsetneq C_1.$$

ii) If follows easily from the argument in [2, Example 3.2] that A_∞^{weak} contains all non-negative functions that are monotonic in each variable. By i), all these functions are also contained in C_p for every p. In particular, C_p weights are generally non-doubling.

iii) If $w \in C_p$ is a doubling weight such that $w(2Q) \leq 2^p w(Q)$, where $2Q$ is the concentric dilation of Q with $\ell(2Q) = 2\ell(Q)$, then $w \in A_\infty$ [8, Section 7].

iv) If $w \in A_\infty$, then $w\chi_{[0,\infty)} \in C_p$ for every $1 \leq p < \infty$ [99].

v) More generally, if $w \in A_\infty$ and g is a *convexely contoured* weight (i.e. a weight such that $\{x \in \mathbb{R}^n : g(x) < \alpha\}$ is a convex set for every $\alpha \geq 0$), then $wg \in C_p$ for every $1 \leq p < \infty$ [8, Proposition 7.3].

vi) If w is a compactly supported weight, then $w \notin C_p$ for any p. It is straightforward to prove this. Let us denote $P := \operatorname{supp} w$. For every $k \in \mathbb{N}$, let P_k be a cube such that $P \subset P_k$ and $|P_k| \geq 2^k |P|$. Now, for $E = P$, we have

$$\int_{\mathbb{R}^n} M\chi_{P_k}(x)^p w(x)dx = \int_P M\chi_{P_k}(x)^p w(x)dx = \int_P w = w(P) \in (0,\infty)$$

for every k since w is locally integrable. However,

$$\left(\frac{|E|}{|P_k|}\right)^\varepsilon \leq \left(\frac{|P|}{2^k|P|}\right)^\varepsilon \searrow 0 \quad \text{as } k \nearrow \infty$$

for every $\varepsilon > 0$. Thus, there do not exist constants C and ε such that (2.4) holds for every cube Q. This argument also proves that if $w \in C_p$, then $w \notin L^1(\mathbb{R}^n)$.

vii) Even though C_p weights cannot have compact support, their support can have arbitrarily small measure. Indeed, suppose that $w \in A_\infty$ and

$$P = \bigcup_{k=1}^{\infty} [10^k, 10^k + \tfrac{1}{2^k}].$$

Then $|P| = 1$ but P is unbounded. We set

$$v(x) := w(x) 1_P(x).$$

- If $w(x) = x^4$, then $\int_{\mathbb{R}} M(\chi_Q)^2 v = \infty$ for every cube Q and thus, $v \in C_2$.
- If $w(x) = 1$, then w is integrable and, by vi), $w \notin C_p$ for any p.

viii) Suppose that w is a weight such that $w(x) \geq \alpha > 0$ for every $x \in \mathbb{R}^n \setminus A$, where A is a bounded set. Since $M(\chi_Q) \notin L^1(dx)$ for any cube Q, we have

$$\int_{\mathbb{R}^n} M(\chi_Q)(x) w(x) dx \geq \alpha \int_{\mathbb{R}^n \setminus A} M(\chi_Q)(x) dx = \infty$$

and thus, $w \in C_1$.

2.8 The C_ψ classes of Lerner

The classes C_ψ were introduced by Lerner in [83] as intermediate classes between C_p and C_q for $q > p \geq 1$ and a new way to attack Muckenhoupt's conjecture Conjecture 2.1. If $1 < p < q < \infty$, we know that C_p is necessary and C_q is sufficient for (2.2) to hold, so it makes sense to use a intermediate scale between the L^p and L^q norms of $M\chi_Q$ in (2.4).

To be more precise, we define generalizations of C_p classes that depend on a Young function ψ instead of p. As we will see, the choice of the function ψ affects the structure of the class in a significant way.

Definition 2.39

Let ψ be a function defined on $[0, 1]$. We denote $w \in C_\psi$ if there exist constants $\kappa_w, \varepsilon_w > 0$ such that for every cube Q and measurable $E \subset Q$ we have

$$w(E) \leq \kappa_w \left(\frac{|E|}{|Q|}\right)^{\varepsilon_w} \int_{\mathbb{R}^n} \psi\big(M\chi_Q(x)\big) w(x) dx. \tag{2.22}$$

Without loss of generality, we may assume that $\kappa_w \geq 1$.

Example 2.40

If we choose the function ψ in a suitable way, we recover classes that we have considered earlier:

- Let $\psi_p(t) = t^p$, for $1 < p < \infty$. Then $C_{\psi_p} = C_p$.
- Let $\psi_\infty = \chi_{\{1\}}$. Then we have $\psi_\infty(M\chi_Q) = \chi_Q$ and thus, $C_{\psi_\infty} = A_\infty$.

- Let $0 < a < 1$ and $\psi_a = \chi_{[a,1]}$. Then we have $\psi_a(M\chi_Q) = \chi_{\kappa_a Q}$ for some constant $\kappa_a > 1$ and thus, $C_{\psi_a} = A_\infty^{\text{weak}}$.

From now on, we consider a C_ψ class with a carefully chosen ψ. Similarly chosen functions would yield similar results, but we have stick to one choice.

Definition 2.41

Let $p > 1$. We set $\widetilde{C}_p := C_{\varphi_p}$ for the function φ_p such that $\varphi_p(0) = 0$ and

$$\varphi_p(t) = \frac{t^p}{\log^2(1 + \frac{1}{t})}, \quad t \in (0, 1].$$

For notational convenience, we also set $\varphi_p(t) = \varphi_p(1)$ for every $t > 1$. It is straightforward to check that the function φ_p satisfies the following properties:

1. $\lim_{t\to 0} \varphi_p(t) = 0$ and $\varphi_p(1) = \frac{1}{\log^2 2} > 1$,
2. both φ_p and $t \mapsto t^{-1}\varphi_p(t)$ are increasing functions,
3. $\varphi_p(2t) \le \kappa\varphi_p(t)$ for some $\kappa > 0$ and all $t \ge 0$ (and thus, $\varphi_p(\lambda t) \le \kappa_\lambda \varphi_p(t)$ for any $\lambda > 1$ and $t \ge 0$),
4. $\int_0^1 \varphi_p(t)\frac{dt}{t^{p+1}} < \infty$.

The key property of $\widetilde{C}_p$ is that $\bigcup_{q>p} C_q \subset \widetilde{C}_p$ and we have

$$w \in \widetilde{C}_p \quad \Longrightarrow \quad \|Mf\|_{L^p(w)} \le \kappa \, \|M^\sharp f\|_{L^p(w)} \Longrightarrow \quad w \in C_p, \tag{2.23}$$

where $M^\sharp$ is the sharp maximal operator of Fefferman and Stein [37]. The implications (2.23) were first proven by Yabuta [117, Theorem 1, Theorem 2] for $w \in \bigcup_{q>p} C_q$ and then improved by Lerner [83, Theorem 6.1] to this form. By [83, Remark 6.2] and [17, Subsection 1.5], we know that this result also gives us (2.2) for e.g. Calderón–Zygmund operators and every $w \in \widetilde{C}_p$.

Theorem 2.42 – [83, Remark 6.2], [17, Subsection 1.5]

In any dimension, we have: If $w \in \widetilde{C}_p$ then Coifman–Fefferman inequality (2.2) holds for Calderón–Zygmund operators.

2.9 The counterexample of Kahanpää–Mejlbro

This last section is devoted to the counterexample constructed by Kahanpää and Mejlbro in [69], a counterexample that disproves the self-improvement of the C_p classes. Because of the limited availability of [69], and for convenience of the reader, we give a self-contained description of their counterexample.

We give a detailed proof of the failure of the self-improving properties of C_p classes and generalize this also to the context of $\widetilde{C}_p$. Although we use many central ideas of Kahanpää and Mejlbro, the proof we present here is different from the one given in

[69]. In particular, we avoid using the explicit Hilbert transform estimates that had a key role in [69] and our techniques allow us to consider dimensions higher than 1.

2.9.1 The Kahanpää–Mejlbro weights

As we mentioned earlier, Muckenhoupt's conjecture would be trivially true if every C_p weight was self-improving with respect to p. Unfortunately, this is not true due to a construction by Kahanpää and Mejlbro. Let us describe this construction.

For every integer $k \in \mathbb{Z}$, let us define the intervals

$$I_k := [4k-3, 4k-1] \quad \text{and} \quad J_k := \left[4k - \frac{1}{2}\ell_k, 4k + \frac{1}{2}\ell_k\right],$$

where $\ell_k \in (0,1]$ are numbers such that $\inf_{k\in\mathbb{Z}} \ell_k = 0$. For the sake of choosing, we let $\ell_k = 2^{-|k|-1}$. See Figure 2.1.

FIGURE 2.1: The distribution of the intervals I_k (red) and J_k (green)

Let also $h = (h_k)_{k\in\mathbb{Z}}$ be a sequence of heights such that $0 < h_k < 1$ for every $k \in \mathbb{Z}$. We define the weight w_h to have value 1 in each of the intervals I_k and value h_k in each of the J_k. That is, we define

$$w_h = \sum_{k\in\mathbb{Z}} \chi_{I_k} + \sum_{k\in\mathbb{Z}} h_k \chi_{J_k}. \tag{2.24}$$

Note that all weights of the form (2.24) have the same support and all agree on each of the intervals I_k. They are completely determined by the sequence of heights.

We note that in [69], the sum in the definition of w was indexed as $k \geq 0$. We have decided to index as $k \in \mathbb{Z}$ because of symmetry and because this way it is easier to generalize the construction to higher dimensions.

Theorem 2.43 – [69, Theorem 11, Proposition 12]
Let $p > 1$. For suitable choices of the sequence of heights $h = (h_k)_{k\in\mathbb{Z}}$, the weight w_h satisfies $w_h \in C_p$ and $w_h \notin C_{p+\varepsilon}$ for any $\varepsilon > 0$. In particular,

$$C_p \setminus \bigcup_{q>p} C_q \neq \emptyset. \tag{2.25}$$

The property (2.25) can also be seen as a corollary of Theorem 2.45 a).

2.9.2 The Kahanpää–Mejlbro weights and $\widetilde{C}_p$

Since $\varphi_p(t) \leq \kappa t^p$ for all $t \in [0,1]$, we have $\widetilde{C}_p \subset C_p$. On the other hand, since $t^q \leq \kappa\varphi_p(t)$ for every $q > p$, we have $C_q \subset \widetilde{C}_p$ for any $q > p$. Thus, for any $p > 1$, we have

$$\bigcup_{q>p} C_q = \bigcup_{\varepsilon>0} C_{p+\varepsilon} \subset \widetilde{C}_p \subset C_p. \tag{2.26}$$

This raises a natural question: Are these inclusions strict? If the first one is not, we get a self-improving property for $\widetilde{C}_p$ weights. If the second one is not, we have solved Muckenhoupt's conjecture. Unfortunately, we will next show that $\widetilde{C}_p \setminus \bigcup_{q>p} C_q \neq \emptyset$

and $C_p \setminus \widetilde{C}_p \neq \emptyset$. This does not prove or disprove Muckenhoupt's conjecture but it is one step closer to understanding the solution.

Our main tool for proving that the inclusions in (2.26) are strict in dimension one is the following generalization of Kahanpää–Mejlbro techniques:

Theorem 2.44

Let $1 < p < \infty$, h a sequence of heights and let w_h be a weight as in (2.24).

i) If $w_h \in C_p$, then there exists $\kappa > 0$ such that $h_k \leq \kappa(\ell_k)^{p-1}$.

ii) If $h_k = (\ell_k)^{p-1}$, then $w_h \in C_p$.

iii) If $w_h \in \widetilde{C}_p$, then there exists $\kappa > 0$ such that $h_k \leq \kappa \displaystyle\int_0^{\ell_k} \varphi_p(t)\frac{dt}{t^2}$.

iv) If $h_k = \frac{\varphi_p(\ell_k)}{\ell_k}$, then $w_h \in \widetilde{C}_p$.

One can also prove similar statements as iii) and iv) for the more general class C_ψ assuming that ψ satisfies certain conditions, but for the sake of simplicity we only consider the class $\widetilde{C}_p$.

We will postpone the proof of Theorem 2.44 until the next section. Nevertheless, let us explain how it can be used to prove the strictness of the inclusions in (2.26), or more precisely, that they are not self-improving.

Theorem 2.45

The following are true:

a) $C_p \setminus \widetilde{C}_p \neq \emptyset$,

b) $\cup_{\varepsilon>0} C_{p+\varepsilon} \subsetneq \widetilde{C}_p$.

Proof. We construct weights w of the type (2.24) and then use Theorem 2.44 to prove the claims.

Let us prove first statement a). Let us set $h_k = (\ell_k)^{p-1}$ for every $k \in \mathbb{Z}$. By part ii) of Theorem 2.44, we know that $w \in C_p$. Let us then use part iii) of Theorem 2.44 to show that $w \notin \widetilde{C}_p$. It is enough to show that

$$\inf_{0<t<1} \frac{\int_0^t \varphi_p(s)\frac{ds}{s^2}}{t^{p-1}} = 0.$$

This can be seen easily by computing the limit as $t \to 0^+$: by L'Hôpital's rule and the Fundamental theorem of calculus, we have

$$\lim_{t\to 0^+} \frac{\int_0^t \varphi_p(s)\frac{ds}{s^2}}{t^{p-1}} = \lim_{t\to 0^+} \frac{\varphi_p(t)t^{-2}}{(p-1)t^{p-2}} = \frac{1}{p-1}\lim_{t\to 0^+} \frac{\varphi_p(t)}{t^p} = 0.$$

Thus, by part iii) of Theorem 2.44, $w \notin \widetilde{C}_p$, which proves statement a).

Let us now prove statement b). Let us set

$$h_k = \frac{\varphi_p(\ell_k)}{\ell_k} = \frac{(\ell_k)^{p-1}}{\log^2(1+\frac{1}{\ell_k})}.$$

for every $k \in \mathbb{Z}$. By part iv) of Theorem 2.44, we know that $w \in \widetilde{C}_p$. We then use part i) of Theorem 2.44 to show that $w \notin C_{p+\varepsilon}$ for any $\varepsilon > 0$. To see this, we prove

$$\inf_{0<t<1} \frac{t^{p+\varepsilon-1}}{\varphi_p(t)t^{-1}} = 0.$$

As in the previous case, we show this by computing the limit as $t \to 0^+$. We get

$$\lim_{t\to 0^+} \frac{t^{p+\varepsilon-1}}{\varphi_p(t)t^{-1}} = \lim_{t\to 0^+} t^{\varepsilon} \log^2\left(\frac{1+t}{t}\right) = \lim_{t\to 0^+} t^{\varepsilon}\big(\log(1+t) - \log(t)\big)^2 = 0,$$

since $x^{\alpha}\log(x) \to 0$ as $x \to 0^+$ for any $\alpha > 0$. Thus, by part i) of Theorem 2.44, $w \notin C_{p+\varepsilon}$. for any $\varepsilon > 0$. □

From Theorem 2.42 we know that $\widetilde{C}_p$ is sufficient for (2.2), but from Theorem 2.45 b) there exists a weight $w \in \widetilde{C}_p \setminus \cup_{\varepsilon>0} C_{p+\varepsilon}$. In particular, this proves that $C_{p+\varepsilon}$ is the correct sufficient condition for the Coifman–Fefferman inequality (2.2) to hold. We state this fact in the following Corollary.

Corollary 2.46

The condition $C_{p+\varepsilon}$ is not necessary for (2.2) to hold for Calderón–Zygmund operators.

2.9.3 Proof of Theorem 2.44

The following counterpart of [69, Proposition 8] will be useful for us in the proof of Theorem 2.44. It is the analogue of Lemma 2.11 but for $\widetilde{C}_p$, in which the $\widetilde{C}_p$-tail is bounded by the tail with a hole on the integration domain.

Lemma 2.47

Let $p > 1$ and $w \in \widetilde{C}_p$. Then there exists a constant $\kappa = \kappa_{\varphi,w} > 0$ such that for any cube Q we have

$$\int_{\mathbb{R}^n} \varphi_p(M\chi_Q(x))w(x)dx \le \kappa \int_{\mathbb{R}^n\setminus Q} \varphi_p(M\chi_Q(x))w(x)dx.$$

Proof. Let us fix a cube Q and set $\alpha = (2\varphi_p(1)\kappa_w)^{\frac{1}{n\varepsilon_w}}$, where C_w and ε_w are the constants in the definition of $\widetilde{C}_p = C_{\varphi_p}$ (2.22). Notice that $\alpha \ge 1$. Now applying the $\widetilde{C}_p$ condition for αQ and Q gives us

$$\begin{aligned} w(Q) &\le \kappa_w\left(\frac{|Q|}{\alpha^n|Q|}\right)^{\varepsilon_w} \int_{\mathbb{R}^n} \varphi_p(M\chi_{\alpha Q}(x))w(x)dx \\ &= \frac{1}{2\varphi_p(1)} \int_{\mathbb{R}^n} \varphi_p(M\chi_{\alpha Q}(x))w(x)dx \\ &\le \frac{1}{2}w(Q) + \frac{1}{2\varphi_p(1)} \int_{\mathbb{R}^n\setminus Q} \varphi_p(M\chi_{\alpha Q}(x))w(x)dx, \end{aligned}$$

since $M\chi_{\alpha Q} = 1$ on Q and $\varphi_p(1) > 1$. In particular,

$$w(Q) \le \frac{1}{\varphi_p(1)} \int_{\mathbb{R}^n\setminus Q} \varphi_p(M\chi_{\alpha Q}(x))w(x)dx.$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi_p(M\chi_Q(x))w(x)dx &= \varphi_p(1)w(Q) + \int_{\mathbb{R}^n\setminus Q} \varphi_p(M\chi_Q(x))w(x)dx \\ &\le \int_{\mathbb{R}^n\setminus Q} \varphi_p(M\chi_{\alpha Q}(x))w(x)dx + \int_{\mathbb{R}^n\setminus Q} \varphi_p(M\chi_Q(x))w(x)dx \\ &\overset{(A)}{\le} \int_{\mathbb{R}^n\setminus Q} \varphi_p(c_\alpha M\chi_Q(x))w(x)dx + \int_{\mathbb{R}^n\setminus Q} \varphi_p(M\chi_Q(x))w(x)dx \\ &\overset{(B)}{\le} C_\alpha \int_{\mathbb{R}^n\setminus Q} \varphi_p(M\chi_Q(x))w(x)dx,\end{aligned}$$

where we used (A) Lemma 2.35 and the fact that φ_p is increasing, and (B) the doubling property of φ_p. □

Proof of Theorem 2.44. Let us fix an interval I and a subset $E \subset I$. We denote $A := \bigcup_k I_k$. It is straightforward to check that for almost every $x \in A$ and every $r > 0$ we have

$$|A \cap (x-r, x+r)| \ge \kappa_A\, r, \tag{2.27}$$

for a uniformly bounded constant $\kappa_A > 0$. We remark the similarity of condition (2.27) to capacity density condition and measure density conditions from Chapter 6

Let us begin proving i). Suppose that $w \in C_p$. Notice that by the definition of the weight w, we have $h_k\ell_k = w(J_k)$. To simplify the notation, we only consider the case $k = 0$ and denote $h := h_0$, $\ell := \ell_k$ and $J_0 := J$. Now applying the C_p condition for the set $J = [-\frac{1}{2}\ell, \frac{1}{2}\ell]$ gives us

$$\begin{aligned}h\ell = w(J) &\le \int_{\mathbb{R}} \big(M\chi_J(x)\big)^p w(x)dx \\ &\overset{(A)}{\le} K \int_{\mathbb{R}\setminus J} \big(M\chi_J(x)\big)^p w(x)dx \\ &= \kappa \int_{|x|>\frac{\ell}{2}} (M\chi_J(x))^p w(x)\, dx \\ &\overset{(B)}{=} \kappa \int_{|x|>1} \Big(\sup_{I'\ni x} \frac{|I\cap J|}{|I|}\Big)^p w(x)\, dx \\ &\overset{(C)}{\le} \kappa_p \int_{|x|>1} \Big(\frac{|J|}{|x|}\Big)^p\, dx \\ &\le \kappa_p\, \ell^p \int_{|x|>1} |x|^{-p}\, dx \le \kappa_p\, \ell^p,\end{aligned}$$

where we used (A) Lemma 2.11, (B) the fact that $w(x) = 0$ for every x such that $\frac{\ell}{2} < |x| < 1$, and (C) for $|x| > 1$ we have $|I'| \gtrsim |x|$ for every interval I' such that $J \cap I' \ne \emptyset$. Thus, we have $h \le \kappa \ell^{p-1}$.

Let us now prove ii). Suppose that $h_k = (\ell_k)^{p-1}$ for each $k \in \mathbb{Z}$. We want to show that there exist constants $\kappa > 0$ and $\varepsilon > 0$ that are independent of I and E and

$$w(E) \le \kappa\Big(\frac{|E|}{|I|}\Big)^{\varepsilon} \int_{\mathbb{R}} (M\chi_I(x))^p w(x)dx.$$

Naturally, we may assume that $w(I) > 0$. We split the proof into two cases, depending on the interaction between I and the support of w.

Case 1: $|I \cap A| > 0$. By (2.27), we know that there exists a point $x_0 \in I \cap A$ such that

$$\big|A \cap (x_0 - |I|, x_0 + |I|)\big| \geq K_A\,|I|.$$

See Figure 2.2 for this case.

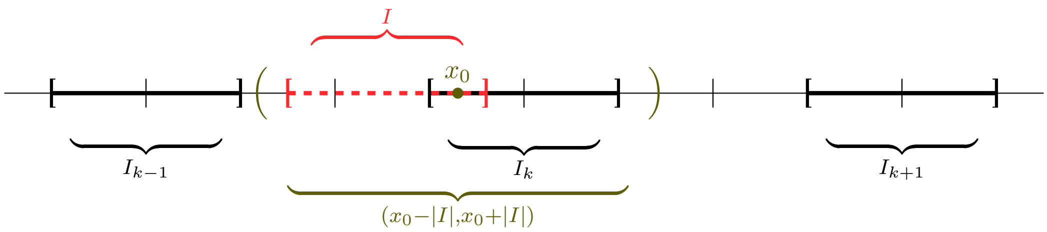

FIGURE 2.2: Case 1: $|I \cap A| > 0$.

Thus, since $1_A \leq w \leq 1$ a.e. and it holds that $(x_0 - |I|, x_0 + |I|) \subset 3I$, we have

$$\begin{aligned} w(E) \leq |E| &\leq K_A^{-1}\frac{|E|}{|I|}\big|A \cap (x_0 - |I|, x_0 + |I|)\big| \\ &\leq K_A^{-1}\frac{|E|}{|I|}w(3I) \\ &\leq K_{A,p}\frac{|E|}{|I|}\int_{\mathbb{R}}(M\chi_I)^p w, \end{aligned}$$

where we used Lemma 2.35 in the last inequality.

Case 2: $|I \cap A| = 0$. In this case, we only have exactly one $k_0 \in \mathbb{Z}$ such that $I \cap J_{k_0} \neq \emptyset$. Let us consider two subcases.

Case 2a: $|I| \leq |\Omega_{k_0}|$. In this case, we know that $w \leq (\ell_{k_0})^{p-1}$ on $E \cap J_{k_0}$. Thus, we get

$$w(E) \leq (\ell_{k_0})^{p-1}|J_{k_0} \cap E| \leq (\ell_{k_0})^{p-1}|E| = (\ell_{k_0})^{p-1}\frac{|E|}{|I|}|I|. \tag{2.28}$$

Since $I \cap J_{k_0} \neq \emptyset$ and $|I| \leq |J_{k_0}|$, there exists $\widetilde{I} \subset J_{k_0}$ such that $\widetilde{I} \subset 3I$ and $|\widetilde{I}| = |I|$. See Figure 2.3.

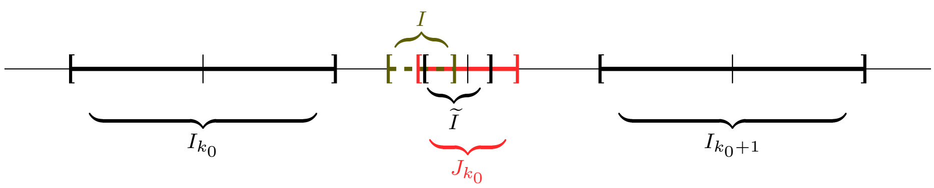

FIGURE 2.3: Case 2a: $|I \cap A| = 0$, $|I| \leq |J_{k_0}|$.

Thus, we have

$$\int_{\mathbb{R}}(M\chi_I(x))^p w(x)dx \geq \int_{\widetilde{I}}(M\chi_I(x))^p w(x)dx \geq K\,h_{k_0}|\widetilde{I}| = (\ell_{k_0})^{p-1}|I|. \tag{2.29}$$

Combining (2.28) and (2.29) then gives us

$$w(E) \leq \kappa\frac{|E|}{|I|}\int_{\mathbb{R}}(M\chi_I(x))^p w(x)dx,$$

which is what we wanted.

Case 2b: $|I| > |J_{k_0}|$. In this case, we have obviously $J_{k_0} \subset 3I$. See Figure 2.4.

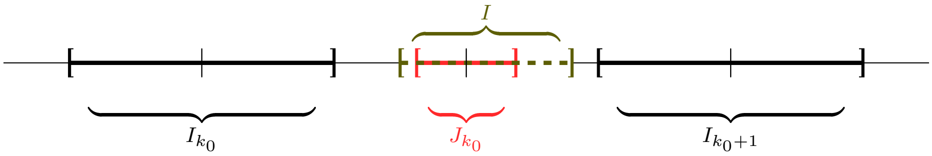

FIGURE 2.4: Case 2b: $|I \cap A| = 0$, $|I| > |J_{k_0}|$.

Let x_I be the center of I. Since $w = 1$ on I_{k_0+1} and $|I| \leq |I_{k_0+1}| = 2$, we have

$$\begin{aligned}
\int_{\mathbb{R}} (M\chi_I(x))^p w(x)dx &\geq \int_{\mathbb{R}} \Big(\frac{|I|}{|I| + |x - x_I|}\Big)^p w(x)dx \\
&\geq \int_{I_{k_0+1}} \Big(\frac{|I|}{|I| + |x - x_I|}\Big)^p w(x)dx \\
&\geq |I|^p \int_{I_{k_0+1}} \Big(\frac{1}{|I_{k_0+1}|}\Big)^p dx \\
&= 2^{1-p}|I|^p.
\end{aligned}$$

Since $\ell_{k_0} = |J_{k_0}| < |I|$, we also have

$$w(E) \;=\; (\ell_{k_0})^{p-1}|J_{k_0} \cap E| \;\leq\; |J_{k_0}|^{p-1}|E| \;\leq\; |I|^{p-1}|E| \;=\; |I|^p \frac{|E|}{|I|}.$$

Combining these two estimates gives us what we wanted. This completes the proof of part ii).

The proof of part iii) is similar to the proof of part i). We use the same notation as in the proof of part i). Using Lemma 2.47, the facts that φ_p is increasing and doubling, and that $|J| = \ell$, we get

$$\begin{aligned}
h\ell = w(J) &\leq \kappa \int_{\mathbb{R}} \varphi_p\big(M\chi_J(x)\big)w(x)dx \\
&\leq \kappa \int_{|x|>\ell} \varphi_p\big(M\chi_J(x)\big)w(x)dx \\
&\leq \kappa \int_{|x|>1} \varphi_p\Big(\frac{\ell}{|x|}\Big)\, dx \\
&\leq \kappa \int_1^{\infty} \varphi_p\Big(\frac{\ell}{x}\Big)\, dx \\
&\leq \kappa\, \ell \int_0^{\ell} \varphi_p(x)\, \frac{dx}{x^2},
\end{aligned}$$

where we used integration by substitution in the last step.

In order to prove vi), we argue as in the proof of ii). The cases 1 and 2a are essentially the same, since the value of h_k does not really play a role in these cases. Let us prove the case 2b. We get

$$\begin{aligned}
\int_{\mathbb{R}} \varphi_p\big(M\chi_J(x)\big)w(x)dx &\geq \int_{\mathbb{R}} \varphi_p\Big(\frac{|I|}{|I| + |x - x_I|}\Big)w(x)dx \\
&\geq \int_{I_{k_0+1}} \varphi_p\Big(\frac{|I|}{|I| + |x - x_I|}\Big)w(x)dx
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{I_{k_0+1}} \varphi_p\Big(\frac{|I|}{2|I_{k_0+1}|}\Big)\,dx \\
&\geq \int_{I_{k_0+1}} \varphi_p\Big(\frac{|I|}{4}\Big)\,dx \\
&\geq \kappa_\varphi\, \varphi_p(|I|),
\end{aligned}
$$

since φ_p is increasing and doubling and $w = 1$ a.e. on I_{k_0+1}. Also, we have

$$
w(E) \;=\; \frac{\varphi_p(\ell_{k_0})}{\ell_{k_0}}|J_{k_0} \cap E| \;\leq\; \frac{\varphi_p(|J_{k_0}|)}{|J_{k_0}|}|E| \;\leq\; \frac{\varphi_p(|I|)}{|I|}|E| \;=\; \varphi_p(|I|)\frac{|E|}{|I|},
$$

where we used the fact that $t \mapsto \frac{\varphi_p(t)}{t}$ is an increasing function in the last inequality. This finishes the proof. $\square$

2.9.4 Kahanpää–Mejlbro weights in higher dimensions

Although the definition of $\widetilde{C}_p$ makes sense in every dimension, the proof of Theorem 2.45 works only in dimension one since it relies on the one-dimensional construction of Kahanpää–Mejlbro weights and their properties. In this section, we explain how the construction and the the proofs of Theorem 2.44 and Theorem 2.45 can be generalized for higher dimensions.

For a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $r > 0$, we let $Q(x, r)$ be the (closed) cube centered at x with side length $2r$:

$$
Q(x, r) \coloneqq [x_1 - r, x_1 + r] \times \ldots \times [x_n - r, x_n + r].
$$

Let us construct the n-dimensional analogue of the set A from the proof of Theorem 2.44. For every $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, we set

$$
R_m \coloneqq Q(4m - 2, 1) = [4m_1 - 3, 4m_1 - 1] \times \ldots \times [4m_n - 3, 4m_n - 1]\,.
$$

We now use the cubes R_m similarly as the intervals I_k and set $A \coloneqq \bigcup_{m \in \mathbb{Z}^n} R_m$.

Lemma 2.48

There exists a constant $\kappa_A > 0$ such that, for every $x \in A$ and $r > 0$,

$$
|A \cap Q(x, r)| \geq \kappa_A r^n. \tag{2.30}
$$

As in the one dimensional case, we remark the similarity of this condition to the capacity density conditions and measure density condition from Chapter 6.

Proof. Let us fix $x \in A$ and $0 < r < \infty$. Then x lies inside exactly one of the cubes R_m. Let us denote this cube by Q_0.

Suppose that $0 < r < 2$. Let us break $Q(x, r)$ into 2^n subcubes of side length r. Since $x \in Q_0$ and $\ell(Q_0) = 2 > r$, at least one of the subcubes has to lie inside Q_0. Let us denote this subcube by P. Thus,

$$
|A \cap Q(x, r)| \geq |Q_0 \cap Q(x, r)| \geq |P| = r^n.
$$

Suppose now that $2 + 4j \leq r < 2 + 4(j + 1)$ for some $j \geq 0$. There are at least $(2j + 1)^n$ cubes R_m contained in $Q(x, r)$. See Figure 2.5. Since each of these cubes has measure 2^n, we get

$$|A \cap Q(x,r)| \geq (2j+1)^n 2^n = \frac{(4j+2)^n}{r^n} r^n \geq \frac{(4j+2)^n}{(2+4(j+1))^n} r^n = \left(\frac{1}{3}\right)^n r^n. \qquad \square$$

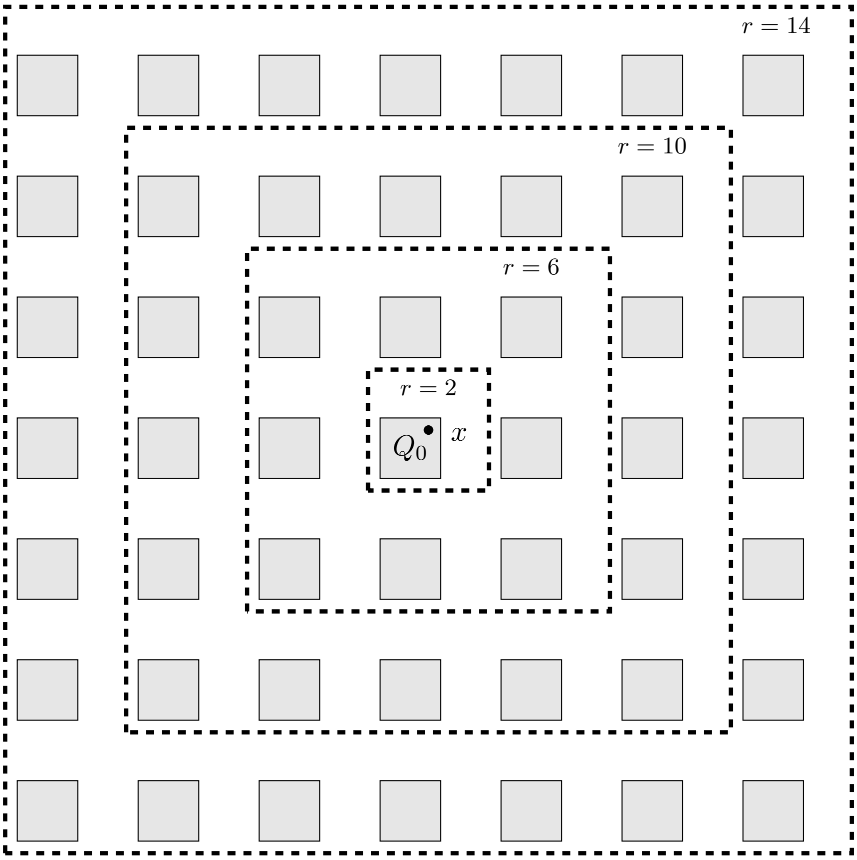

FIGURE 2.5: Scheme in dimension 2. $Q_0 = [-1,-3]^2$ and each of the grey cubes has sidelength 2, at distance 2 from each other. The dashed lines represent cubes centered at x with sidelength $2r$, for the values $r = 2, 6, 10, 14$.

Let us finally construct the n-dimensional weights. For every $m \in \mathbb{Z}^n$, let ℓ_m be a number such that $0 < \ell_m < 1$ and $\inf_m \ell_m = 0$, for example, $\ell_m = 1/(|m|+1)$. We set

$$P_m := Q\left(4m, \frac{\ell_m}{2}\right) = \left[4m_1 - \frac{\ell_m}{2}, 4m_1 + \frac{\ell_m}{2}\right] \times \ldots \times \left[4m_n - \frac{\ell_m}{2}, 4m_n + \frac{\ell_m}{2}\right],$$

for every $m \in \mathbb{Z}$. Thus, we have $\ell(P_m) = \ell_m$. See Figure 2.6 for a visual description of the sets A and P_m in dimension 2. These cubes will be the support of our weight.

Fix a sequence of heights, $h = (h_m)_{m \in \mathbb{Z}^n}$, that will be indexed by $\mathbb{Z}^n$, and that satisfy $0 < h_m < 1$ for all $m \in \mathbb{Z}^n$. We define the Kahanpää–Mejlbro weight w_h in an analogous way as in dimension one, that is,

$$w = 1_A + \sum_{m \in \mathbb{Z}^n} h_m 1_{P_m}, \tag{2.31}$$

Naturally, these weights share a lot of properties with their 1-dimensional counterparts but because of the dimension, we have to make some modifications.

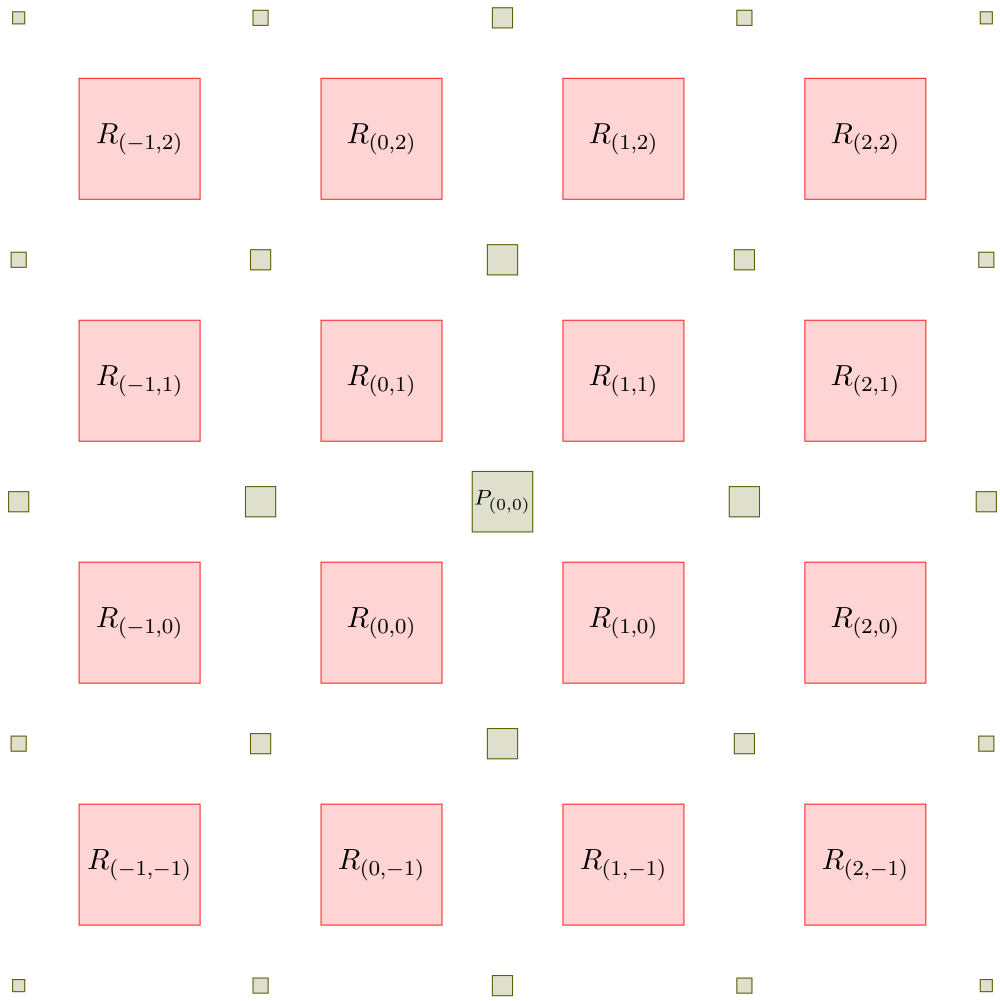

FIGURE 2.6: The cubes R_m (in red) and P_m (in green) in $\mathbb{R}^2$, with $m = (m_1, m_2)$, for $\ell_m = \frac{1}{|m|+1}$. Each R_m has side length 2 and P_m has sidelength ℓ_m.

An analogue of Theorem 2.44 holds for these n-dimensional weights in the following form.

Theorem 2.49

Let $h = (h_m)_{m\in\mathbb{Z}}$ be a sequence of heights and let w_h be the weight as in (2.31). The following statements hold.

i) If $w_h \in C_p$, then there exists $K > 0$ such that $h_m \leq K(\ell_m)^{n(p-1)}$.

ii) If $h_m = (\ell_m)^{n(p-1)}$, then $w_h \in C_p$.

iii) If $w_h \in \widetilde{C}_p$, then there exists $K > 0$ such that $h_m \leq K \int_0^{(\ell_m)^n} \varphi_p(t)\, \frac{dt}{t^2}$.

iv) If $h_m = \frac{\varphi_p(\ell_m^n)}{\ell_m^n}$, then $w_h \in \widetilde{C}_p$.

The correct exponent is now $n(p-1)$ instead of $p-1$ because $|P_m| = (\ell_m)^n$.

The proof of this theorem is essentially the same as in the 1-dimensional case. Since Lemma 2.11 and Lemma 2.47 hold in any dimension, the proofs of i) and iii) work also in any dimension. Parts ii) and iv) also hold because of (2.30) and there are no more cases than the 1-dimensional cases 1, 2a and 2b. The rest of the computations are essentially the same as before.

With the help of Theorem 2.49, it is straightforward to generalize Theorem 2.45 for higher dimensions:

Theorem 2.50

In any dimension, we have

a) $C_p \setminus \widetilde{C}_p \neq \emptyset$,

b) $\bigcup_{\varepsilon>0} C_{p+\varepsilon} \subsetneq \widetilde{C}_p$.

In particular, the condition $C_{p+\varepsilon}$ is not necessary for (2.2) to hold for Calderón–Zygmund operators.

3

Quantitative weighted C_p estimates

Some of the results in this chapter are contained in the following works:

[12] Canto, J. *Sharp Reverse Hölder inequality for C_p Weights and Applications*, The Journal of Geometric Analysis (2021) **31**: 4165–4190.

[13] Canto, J., Li, K., Roncal, L., Tapiola, O. C_p estimates for rough homogeneous singular integrals and sparse forms, Annalli della Scuola Normale Superiore di Pisa, clase di Scienze (5) Vol XXII (2021), 1131–1168.

In this section, we will give quantitative weighted norm inequalities, mostly between singular integral operators and maximal operators. This inequalities will be for C_p weights and will use the C_p constant that was defined in Section 2.4. More precisely, we will provide quantitative Coifman–Fefferman-type inequalities for Calderón–Zygmund operators and rough homogeneous singular integral operators.

Although it is a quantitative weighted norm inequality, we postpone the discussion on the Fefferman–Stein inequality until Chapter 4. This is because an exponential decayed good-λ inequality between the sharp maximal operator and the Hardy–Littlewood maximal operator is needed in order to obtain that inequality, which is obtained in that Chapter.

3.1 Definitions of the main operators

Aquí meter definiciones de los operadores de Calderón–Zygmund, operadors de tipo rough, de los sparse, meter todo tipo de relaciones de dominación y hablar un poco de qué es lo que vamos a hacer con los pesos y así. Para que las secciones sucesivas sean directamente de resultado resultado pim pam pum.

3.1.1 Calderón–Zygmund operators

Let $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a function defined away from the diagonal $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ that satisfies the following conditions for some constant $\kappa_K > 0$, the size condition

$$|K(x,y)| \leq \frac{\kappa_K}{|x-y|^n}, \tag{3.1}$$

and the regularity condition

$$|K(x,y) - K(x,y')| + |K(y,x) - K(y',x)| \leq \kappa_K \frac{|y-y'|^{\varepsilon_K}}{|x-y|^{n+\varepsilon_K}}, \tag{3.2}$$

for $2|y - y'| \leq |x - y|$ and some $\varepsilon_K > 0$. A function satisfying (3.1) and (3.2) is called a Calderón–Zygmund kernel.

Definition 3.1

Let K be a Calderón–Zygmund kernel satisfying (3.1) and (3.2). A Calderón–Zygmund operator associated to K is a linear operator $T : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ that satisfies

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)\, f(y)\, dy,$$

for $f \in C_c^\infty(\mathbb{R}^n)$ and x outside of the support of f.

Note that one Calderón–Zygmund operator has multiple operators associated to it. Nevertheless, there is a unique sublinear operator T^*, which is called the maximally truncated Calderón–Zygmund singular integral operator, which is defined by

$$T^* f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} K(x,y) f(y) dy \right|. \tag{3.3}$$

Calderón–Zygumnd kernels were introduced by Coifman and Meyer in [21], where they were named as *standard kernels*. They were eventually called Calderón–Zygmund kernels because of the work of Calderón and Zygmund [10], in which the boundedness of similar operators was proved under a slightly regularity condition stronger than (3.2). Other regularity conditions have been widely studied, such as Hörmander condition [59], or Dini condition missing ref. For more on Calderón–Zygmund operators, we refer to [45, 67, 112]

3.1.2 Rough homogeneous singular integral operators

Rough homogeneous singular integral operators are convolution operators whose kernel is homogeneous of degree $-n$ but satisfies no regularity condition. Let Ω be a bounded function defined on $\mathbb{S}^{n-1}$ that satisfies the cancellation property

$$\int_{\mathbb{S}^{n-1}} \Omega(x) d\sigma(x) = 0. \tag{3.4}$$

Definition 3.2

Let $\Omega \in L^{\infty}(\mathbb{S}^{n-1})$ satisfy the cancellation condition (3.4). The rough singular integral operator associated to Ω is defined by the expression

$$T_{\Omega} f(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{\Omega(\frac{y}{|y|})}{|y|^n} f(x-y)\, dy, \tag{3.5}$$

for $f \in C_c^{\infty}(\mathbb{R}^n)$.

Cancellation condition (3.4) is necessary for (3.5) to be well defined. These operators have been studied intensively by numerous authors both in unweighted and weighted settings, see e.g. [11, 18, 19, 28, 31, 47, 57, 64, 110, 115].

3.1.3 Sparse operators and sparse forms

Sparse operators come as a newish technique to obtain sharp dominance over several different operators. There have been different definitions for what they are.

Definition 3.3

Let $\mathcal{S}$ be a collection of cubes in $\mathbb{R}^n$, and let $0 < \gamma < 1$. We say that $\mathcal{S}$ is γ-sparse if for every $Q \in \mathcal{S}$, there exists an exceptional set $E_Q \subset Q$ such that $|E_Q| \geq \gamma |Q|$ and the sets $\{E_Q\}_{Q \in \mathcal{S}}$ are pairwise disjoint.

In most cases, we are going to assume that $\gamma = \frac{1}{2}$ and we will not explicitly mention the parameter gamma. Therefore, we will just say that the family $\mathcal{S}$ is sparse.

Example 3.4

Let $I_k = (0, 2^k)$ for $k \in \mathbb{Z}$. The family $\mathcal{S} = \{I_k\}_{k \in \mathbb{Z}}$ is sparse. Indeed, let $E_{I_k} = (2^{k-1}, 2^k) \subset I_k$. Then clearly $|E_{I_k}| = \frac{1}{2}|I_k|$ and they are pairwise disjoint.

Now, let us define the sparse operator over a sparse family $\mathcal{S}$. It is a sublinear operator $\mathcal{A}_{\mathcal{S}}$, that, applied to a locally integrable function f has the form

$$\mathcal{A}_{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \langle |f| \rangle_Q \, \chi_Q(x). \tag{3.6}$$

It is well known that Calderón–Zygmund operators are poinwisely bounded by sparse operators of form (3.6), see [23, 64, 79, 84].

Theorem 3.5

Let T be a Calderón–Zygmund operator as in Definition 3.1. There exists a constant κ_T such that for any function $f \in L^{\infty}(\mathbb{R}^n$ with compact support, there exists a sparse family $\mathcal{S} = \mathcal{S}(f)$ such that

$$|Tf(x)| \leq \kappa_T \mathcal{A}_{\mathcal{S}} f(x) = \kappa_T \sum_{Q\in\mathcal{S}} \langle |f| \rangle_Q \chi_Q(x).$$

Note that the sparse family in Theorem 3.5 is different for each function f, but the sparse parameter γ is uniform for each Calderón–Zygmund operator T.

Sadly, such pointwise domination is not available for rough operators, but there is an alternative. This is where sparse forms come into play, which we define now.

Definition 3.6

Let $\mathcal{S}$ be a sparse family and let $0 < \gamma \leq 1$ and $1 < t < \infty$. The sparse form $\Lambda = \Lambda_{\mathcal{S}}^{t,\gamma}$ is defined, for functions $f, g \in L^1_{loc}(\mathbb{R}^n)$ by the expression

$$\Lambda(f,g) = (t')^{\gamma} \sum_{Q\in\mathcal{S}} \langle |f| \rangle_Q^{\gamma} \, \langle |g| \rangle_{t,Q} \, |Q|.$$

In [24], it was proved that these sparse forms actually bound rough homogeneous singular integrals, in the duality pairing sense, as the following Theorem states. This is for $\gamma = 1$. For our purposes, we are going to need a similar domination result for $0 < \gamma < 1$ which will be proved in Section 3.5.2.

Theorem 3.7 – Theorem A, [24]

Suppose that T_Ω is a rough homogeneous singular integral as in Definition 3.2. Then, for any $1 < p < \infty$ and f, g bounded with compact support, we have

$$\left|\left\langle T_\Omega f \, , \, g \right\rangle\right| \leq \kappa_n \, p' \, \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \sup_{\mathcal{S}} \sum_{Q\in\mathcal{S}} \langle |f| \rangle_Q \, \langle |g| \rangle_{p,Q} \, |Q|.$$

Both Theorem 3.5 and Theorem 3.7 have been used to obtain sharp quantitative estimates for the norms of Calderón–Zygumnd operators and rough singular integral operators, respectively, in the weighted spaces $L^p(w)$ when $w \in A_p$. We will expand on this topic when we discuss weighted norm inequalities for C_p weights.

3.2 The Coifman–Fefferman inequality

In this section, we discuss the weighted norm inequality between Calderón–Zygmund operators and the Hardy–Littlewood maximal operator. This inequality, first proved by Coifman and Fefferman [20] for A_∞ weights, has the precise statement as follows.

Theorem 3.8 – Theorem III, [20]

Let $w \in A_\infty$ and let $1 < p < \infty$. Let T be a Calderón–Zygmund operator as in Definition 3.1. There exists a constant κ such that

$$\|T^* f\|_{L^p(w)} \leq \kappa \, \|Mf\|_{L^p(w)}, \tag{3.7}$$

for any $f \in L^\infty(\mathbb{R}^n)$ with compact support. Here T^* denotes the maximally truncated singular integral operator (3.3)

The classical proof of inequality (3.7) in [20] uses a good-λ inequality between the operators T^* and M. If the kernel of T is not regular enough, there is in general no good-λ inequality and even inequality (3.7) can be false, as is shown in [94].

There are ways of proving inequality (3.7) without using the good-λ inequality. For example, the proof given in [1] uses a pointwise estimate involving the sharp maximal function. Another proof can be found in [26], where the main tool is an extrapolation result that allows to obtain estimates like (3.7) for any A_∞ weight from the smaller class A_1 (see also [27]).

Inequality (3.7) is very important in the classical theory of Calderón–Zygmund operators, as it is used in the proof of many other weighted norm inequalities. The first, and probably most important consequence of (3.7) is the boundedness of T^* in $L^p(w)$ for any weight $w \in A_p$, $1 < p < \infty$, namely

$$\int_{\mathbb{R}^n} T^*f(x)^p w(x)dx \leq \kappa \int_{\mathbb{R}^n} |f(x)|^p w(x)dx.$$

This comes as a direct corollary of Muckenhoupt's theorem [99] on the boundedness of the Hardyll–Littlewood maximal operator in weighted norm.

Another consequence of inequality (3.7), though not as direct as the previous one, is the following inequality, obtained in [105]. For any weight w it holds

$$\|T^*f\|_{L^p(w)} \leq \kappa \|f\|_{L^p(M^{[p]+1}w)},$$

where $[p]$ denotes the integer part of p and M^k denotes the $k-$fold composition of M. This result is sharp since $[p]+1$ cannot be replaced by $[p]$. This is saying that inequality (3.7) encodes a lot of information. Very recently, this result was extended in [89] to the non-smooth case kernels, more precisely to the case case of rough singular operators T_Ω with $\Omega \in L^\infty(\mathbb{S}^{n-1})$, by proving inequality (3.7) for these operators. The proof of this result is quite different from the classical situation since there is no good-λ estimate involving these operators and it is a consequence of a sparse domination result for T_Ω obtained in [24] combined with the A_∞ extrapolation theorem mentioned above in [26].

Norm inequalities similar to (3.7) are true for other operators, for instance in [101] (fractional integrals) or [116] (square functions). Also, in the context of multilinear harmonic analysis one can find other examples, for example, it was shown in [87] an analogue for multilinear Calderón–Zygmund operators T, namely

$$\|T(f_1, ..., f_m)\|_{L^p(w)} \leq \kappa \|\mathcal{M}(f_1, ..., f_m)\|_{L^p(w)},$$

for $w \in A_\infty$ extending (3.7). We refer to [87] for the definition of the operator $\mathcal{M}$. The proof for the multilinear setting is in the spirit of the proof of inequality (3.7) given in [1]. There are also inequalities for (3.7) for more singular operators like the case of commutators of Calderón–Zygmund operators with BMO functions, as was proved in [106]. In this case, the result is, for $w \in A_\infty$,

$$\|[b,T]f\|_{L^p(w)} \leq c\big\|M^2 f\big\|_{L^p(w)},$$

where $[b,T]f = bTf - T(bf)$ and $M^2 = M \circ M$. The result is false for M, because the commutator is not of weak type (1,1) and it would then contradict the extrapolation result from [26].

All of the inequalities mentioned above are true for the class A_∞ of weights, but A_∞ is not the whole picture for some of them. The correct class of weights is, in some sense, the C_p class. Muckenhoupt showed in [100] that A_∞ is not necessary for the CFI (3.7), and that the correct necesary condition is C_p. About sufficiency, Sawyer [109] proved that $w \in C_{p+\eta}$ for some $\eta > 0$ is sufficient for (3.7) in the range $p \in (1, \infty)$. It is still an open conjecture if C_p is a sufficient condition.

Although C_p weights were introduced in the context of the CFI, other inequalities have been proved to hold for these weights. For example, the Fefferman–Stein inequality, between the maximal operators of Hardy–Littlewood and of Fefferman–Stein, as can be found in [117], [14] for a quantified version, [86] in the weak-type context. In [17], the authors extended Sawyer's result to a wider class of operators than Calderón–Zygmund operators, including some pseudo-differential operators and oscillatory integrals. Finally, in [13], Sawyer's result was extended to rough singular integrals and sparse forms.

For the rest of the Chapter, we are going to work with L^p estimates for weights in C_q for some $q > p$.

3.3 Marcinkiewicz integral estimates

In this section, we are going to introduce the most technical tools in this Chapter. They are the Marcinkiewicz-type integral operators. These operators arise quite naturally in the context of C_q weighted estimates, and their definition features a few concepts, such as C_q tails, level sets and the Whitney decomposition lemma.

Let us begin with a technical lemma, which shows how the sum of C_q-tails of pairwise disjoint cubes can be bounded when the weight is in C_q.

Lemma 3.9

Let $w \in C_q$. Fix $R \geq 2$ and $\delta > 0$. Then for every cube Q and any collection of pairwise disjoint cubes $\{Q_j\}_j$ that are all contained in Q we have

$$\int_{RQ} \sum_j (M\chi_{Q_j}(x))^q w(x)dx \leq \frac{1}{\kappa_1 \varepsilon} \log \frac{\kappa_2 R^{nq}}{\varepsilon\delta} w(RQ) + \delta \int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx, \quad (3.8)$$

where κ_1 and κ_2 are dimensional constants and ε is the parameter for w in (2.4). Hence, we have

$$\int_{\mathbb{R}^n} \sum_j (M\chi_{Q_j}(x))^q w(x)dx \leq \kappa_n \, 4^{nq} \, \frac{1}{\varepsilon} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x)dx. \quad (3.9)$$

Proof. For $\lambda > 0$, we will call $E_\lambda = \{x \in RQ : \sum_j M\chi_{Q_j}(x)^q > \lambda\}$. Since the cubes are pairwise disjoint, we have $\sum_j \chi_{Q_j} \in L^\infty$. Then by the exponential inequality from [36] we have $|E_\lambda| \leq \kappa_n e^{-a\lambda}|RQ|$, where c_n and a are positive dimensional constants. Then, applying the C_q condition (2.4) we get

$$w(E_\lambda) \leq 2 \left(\frac{|E_\lambda|}{|RQ|}\right)^\varepsilon \int_{\mathbb{R}^n} (M\chi_{RQ}(x))^q w(x)dx$$

$$\leq \kappa_n e^{-\varepsilon a\lambda} R^{nq} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x)dx.$$

Now we compute

$$\int_{RQ} \sum_j (M\chi_{Q_j}(x))^q w(x)dx = \int_0^\infty w(E_t)dt = \lambda w(E_\lambda) + \int_\lambda^\infty w(E_t)dt$$
$$\leq \lambda w(RQ) + \kappa_n R^{qn} \frac{1}{a\varepsilon} e^{-a\varepsilon\lambda} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x)dx.$$

We can choose λ big enough so that

$$\kappa_n R^{qn} \frac{1}{a\varepsilon} e^{-a\varepsilon\lambda} \leq \delta,$$

that is, $\lambda = \frac{1}{a\varepsilon} \log \frac{\kappa_n R^{qn}}{\delta a\varepsilon}$, and we get (3.8). In order to get (3.9), choose $R = 2$, $\delta = \frac{1}{\varepsilon}$ and use $\sum M\chi_{Q_j}^q \leq 2^{nq} M\chi_Q$ almost everywhere outside of $2Q$. □

We remark that there can be no pointwise equivalent of (3.9), that is, there exists no constant $\kappa > 0$ such that, for a cube Q and a family of pairwise disjoint cubes $\{Q_j\}_j$ contained in Q, the following estimate holds.

$$\sum_j (M\chi_{Q_j}(x))^q \leq \kappa \, (M\chi_Q(x))^q. \tag{3.10}$$

Many examples can be constructed so that (3.10) fails. Let us construct the simplest one, where the cubes, though disjoint, accumulate at a certain point. Let $Q_0 = [0, 1]^n$ and let

$$Q_j = [2^{-j}, 2^{-j+1}] \times \prod_{m=2}^{n} [0, 2^{-j}]$$

See Figure 3.1. Clearly, the cubes $Q_j \subset Q$ and they are pairwise disjoint. The idea is that the cubes accumulate around the origin.

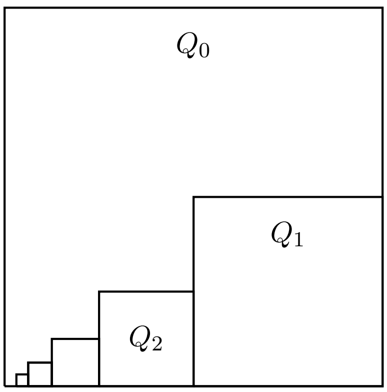

FIGURE 3.1: The cubes Q_j for $j \geq 0$ in dimension 2.

Let us define the partial sum

$$S_N(x) = \sum_{k=1}^{N} M\chi_{Q_k}(x)^q.$$

Since $\operatorname{dist}(0, Q_k) = 2^{-k} = \ell(Q_k)$ for every $k > 1$, we have that, using Lemma 2.8

$$M\chi_{Q_k}(0) \geq \kappa_n \frac{1}{2}.$$

Clearly, this means that, for N big enough, we can make S_N arbitrarily large on a neighborhood of the origin, since each of the S_N is continuous. This means that the limit $S = \lim_{N\to\infty} S_N$ is not an L^∞ function. Therefore, there is no way that (3.10) holds.

We now state the Whitney covering lemma. We are going to use this technique in order to decompose the level sets of some functions.

Lemma 3.10 – Whitney covering lemma

Given $R \geq 1$, there is $C = C(n, R)$ such that if Ω is an open subset in $\mathbb{R}^n$, then $\Omega = \cup_j Q_j$ where the Q_j are disjoint cubes satisfying

$$5R \leq \frac{\operatorname{dist}(Q_j, \mathbb{R}^n \setminus \Omega)}{\operatorname{diam} Q_j} \leq 15R,$$

$$\sum_j \chi_{RQ_j} \leq \kappa \chi_Q.$$

This decomposition technique is going to be applied to level sets of the form $\Omega = \{x \in \mathbb{R}^n : f(x) > \lambda\}$ for some function f and $\lambda > 0$. In order to ensure that these sets are open, we need the functions to be lower semicontinuous.

Definition 3.11

Let $f : \mathbb{R}^n \to \mathbb{R}$. We say that f is lower semicontinuous if for every $\lambda \in \mathbb{R}$, the set $\{x \in \mathbb{R}^n : f(x) > \lambda\}$ is open.

Clearly, continuous functions are always lower-semicontinuous. But the functions we are going to consider are also lower-semicontinuous.

Lemma 3.12

The following statements are valid.

– For any function $f \in L^1_{loc}(\mathbb{R}^n)$, its maximal function Mf is lower semicontinuous.

– Let T be a Calderón–Zygmund operator. Then T^*f is lower semicontinuous for f good enough.

We now define an auxiliary function that was used in [109]. This operator will be used to intuitively represent the integral of the function h to the power p after we apply the C_q condition.

Definition 3.13

Let h be a non-negative lower-semicontinuous function on $\mathbb{R}^n$ and k an integer. Let $\mathcal{W}(k)$ be the Whitney decomposition of the level set $\Omega_k = \{x \in \mathbb{R}^n : h(x) > 2^k\}$, that is, $\Omega_k = \bigcup_{Q\in\mathcal{W}(k)} Q$. We define the function

$$M_{p,q}h(x)^p = \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{W}(k)} 2^{kp}(M\chi_Q(x))^q. \tag{3.11}$$

We need lower-semicontinuity in this definition to ensure that we can apply Whitney's decomposition theorem. In the practice, we will apply this operator to Mf and to T^*f, which are always lower-semicontinuous by Lemma 3.12.

This expression arises naturally when estimating $L^p(w)$ norms with $w \in C_q$. Indeed, by the layer cake representation from Section 1.4.4, we have

$$\|h\|_{L^p(w)}^p = p\int_0^\infty t^{p-1}w(\{h>t\})dt \approx p\sum_{k\in\mathbb{Z}} 2^{kp} \sum_{Q\in\mathcal{Q}_k} w(Q).$$

The role that $w(Q)$ plays in the A_∞ theory is often played by $\int_{\mathbb{R}^n} M(1_Q)^q w$ in the C_q context. Therefore, the natural C_q counterpart of the above expression is

$$\sum_{k\in\mathbb{Z}} 2^{kp} \sum_{Q\in\mathcal{Q}_k} \int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx = \int_{\mathbb{R}^n} M_{p,q}h(x)^p w(x)dx.$$

Now, we prove that the Marcinkiewicz function applied to a maximal function is bounded, in the C_q weighted L^p norm by the maximal function. We prove this in a quantitative way. Note that $q > p$ is crucial in the proof.

Lemma 3.14

Let $0 < p < q < \infty$ with $1 < q$ and suppose that $w \in C_q$. Then for any f bounded with compact support, we have

$$\int_{\mathbb{R}^n} (M_{p,q}Mf(x))^p w(x)dx \le \kappa_n\, 2^{\kappa_n \frac{pq}{q-p}} \frac{1}{\varepsilon_w} \log\frac{1}{\varepsilon_w} \int_{\mathbb{R}^n} (Mf(x))^p w(x)dx, \tag{3.12}$$

where $M_{p,q}$ denotes the Marcinkiewicz integral operator as defined in (3.11) and ε_w is as in (2.6).

Proof. Let $\mathcal{W}(k)$ be the Whitney decomposition of $\Omega_k = \{x \in \mathbb{R}^n : Mf(x) > 2^k\}$, for any integer k. Let N be a positive integer to be chosen later and fix a cube P from the $k - N$ generation. We have, as in [109],

$$|\Omega_k \cap 5P| \le \kappa 2^{-N}|P|, \tag{3.13}$$

where κ depends only on the dimension n.

Now define the partial sums of the Marcinkiewicz integrals. For a fixed $k \in \mathbb{Z}$, we define the partial sum at scale k as

$$S(k) = 2^{kp} \sum_{Q\in\mathcal{W}(k)} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x)dx$$

We have called this expression partial sum because the following relation holds.

$$\int_{\mathbb{R}^n} \big(M_{p,q}Mf(x)\big)^p w(x)dx = \sum_{k\in\mathbb{Z}} S(k).$$

For a fixed $k \in \mathbb{Z}$, $N \in \mathbb{N}$ and a cube $P \in \mathcal{W}(k-N)$, we define the partial sum at scale k localized at P as

$$S(k; N, P) = 2^{kp} \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \cap P \neq \emptyset}} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x) dx.$$

Clearly, the following relation between both partial sums hold,

$$S(k) \leq \sum_{P \in \mathcal{W}(k-N)} S(k; N, P).$$

Because of the Whitney decomposition, $Q \cap R \neq \emptyset$ implies $Q \subset 5P$ for large N, so can split the integral in two parts, close to P and away from P, that is,

$$\begin{aligned} S(k; N, P) &\leq \int_{\mathbb{R}^n} 2^{kp} \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \subset 5P}} M\chi_Q(x)^q w(x) dx \\ &= \int_{10P} + \int_{(10P)^c} \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \subset 5P}} M\chi_Q(x)^q w(x) dx \\ &= I + II \quad \text{for large } \ N. \end{aligned}$$

Let us estimate I. By (3.8), for any $\eta > 0$, which will be chosen chosen later, and for $R = 10$ we get

$$I \leq 2^{kp} \frac{1}{\kappa_1 \varepsilon} \log \frac{\kappa_2 10^{nq}}{\eta \varepsilon} w(10P) + \eta 2^{kp} \int_{\mathbb{R}^n} (M\chi_P)^q w.$$

Now, let us estimate II. Standard estimates for the maximal function of characteristics of cubes show that if x_P is the center of the cube P then by Lemma 2.8 and since $1 < q$, we have

$$\begin{aligned} II &\leq \kappa_n^q \, 2^{kp} \int_{(10P)^c} \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \subset 5P}} \frac{|Q|^q}{|x - x_P|^{nq}} w(x) dx \\ &\leq \kappa_n^q \, 2^{kp} \int_{(10P)^c} \frac{1}{|x - x_P|^{nq}} \Big(\sum_{\substack{Q \in \mathcal{W}(k) \\ Q \subset 5P}} |Q| \Big)^q w(x) dx \\ &\leq \kappa_n^q \, 2^{kp} \int_{(10P)^c} \frac{|\Omega_k \cap P|^q}{|x - P|^{nq}} w(x) dx \\ &\leq \kappa_n^q \, 2^{kp} \int_{(10P)^c} \frac{2^{-qN} |P|^q}{|x - x_P|^{nq}} w(x) dx \\ &\leq \kappa_n^q \, 2^{N(p-q) + (k-N)p} \int_{\mathbb{R}^n} (M\chi_P(x))^q w(x) dx, \end{aligned}$$

where we have used (3.13) on the third inequality. Thus we have, by the Whitney decomposition theorem, for N large,

$$S(k) \leq \sum_{P \in \mathcal{W}(k-N)} S(k; N, P)$$

$$\leq \frac{1}{\kappa_1 \varepsilon} \log \frac{\kappa_2 10^{nq}}{\eta\varepsilon} 2^{kp} \sum_{P\in\mathcal{W}(k-N)} w(10P)$$
$$+ \left(\eta 2^{kp} + \kappa_n^q 2^{N(p-q)+(k-N)p}\right) \sum_{P\in\mathcal{W}(k-N)} \int_{\mathbb{R}^n} (M\chi_P(x))^q w(x)dx$$
$$\leq \frac{1}{\kappa_1 \varepsilon} \log \frac{\kappa_2 10^{nq}}{\eta\varepsilon} 2^{kp} \int_{\mathbb{R}^n} (\sum_{P\in\mathcal{W}(k-N)} \chi_{10P}(x)) w(x)dx$$
$$+ \left(\eta 2^{Np} + \kappa_n^q 2^{N(p-q)}\right) 2^{(k-N)p} \sum_{P\in\mathcal{W}(k-N)} \int_{\mathbb{R}^n} (M\chi_P(x))^q w(x)dx$$
$$\leq \kappa_n \frac{1}{\kappa_1 \varepsilon} \log \frac{\kappa_2 10^{nq}}{\eta\varepsilon} 2^{kp} w(\Omega_{k-N}) + (\eta 2^{Np} + \kappa_n^q 2^{N(p-q)}) S(k-N)$$
$$= \kappa_n 2^{Np} \frac{1}{\kappa_1 \varepsilon} \log \frac{\kappa_2 10^{nq}}{\eta\varepsilon} 2^{p(k-N)} w(\Omega_{k-N}) + (\eta 2^{Np} + \kappa_n^q 2^{N(p-q)}) S(k-N).$$

Now, since $q > p$, we can chose N so that $\kappa_n^q 2^{N(p-q)} < \frac{1}{4}$, that is, $N \geq \kappa_n \frac{q}{q-p}$; and η so that $\eta 2^{Np} < \frac{1}{4}$.

This allows us to continue the computations by

$$S(k) \leq \kappa_n 2^{\kappa_n \frac{pq}{q-p}} \frac{1}{\varepsilon}(q\kappa_n + \log\frac{1}{\varepsilon} + \kappa_n \frac{pq}{q-p}) 2^{p(k-N)} w(\Omega_{k-N}) + \frac{1}{2} S(k-N)$$
$$\leq \left(\kappa_n 2^{\kappa_n \frac{qp}{q-p}} \frac{1}{\varepsilon} \log\frac{1}{\varepsilon}\right) 2^{p(k-N)} w(\Omega_{k-N}) + \frac{1}{2} S(k-N).$$

Thus, writing $S_M = \sum_{k\leq M} S(k)$ we get

$$S_M \leq \frac{1}{2} S_{M-N} + \left(\kappa_n 2^{\kappa_n \frac{qp}{q-p}} \frac{1}{\varepsilon} \log\frac{1}{\varepsilon}\right) \sum_{k\leq M} 2^{p(k-N)} w(\Omega_{k-N})$$
$$\leq \frac{1}{2} S_M + \left(\kappa_n 2^{\kappa_n \frac{qp}{q-p}} \frac{1}{\varepsilon} \log\frac{1}{\varepsilon}\right) \sum_{k\in\mathbb{Z}} 2^{p(k-N)} w(\Omega_{k-N})$$
$$\leq \frac{1}{2} S_M + \kappa_n 2^{c_n \frac{qp}{q-p}} \frac{1}{\varepsilon} \log\frac{1}{\varepsilon} \int_{\mathbb{R}^n} (Mf(x))^p w(x)dx.$$

Now, we can argue as in [109], p. 260, to conclude that $S_M < \infty$ for each M. Then, passing it to the left hand side, we obtain

$$S_M \leq \kappa_n 2^{c_n \frac{qp}{q-p}} \frac{1}{\varepsilon} \log\frac{1}{\varepsilon} \int_{\mathbb{R}^n} (Mf(x))^p w(x)dx.$$

Now, since S_M is increasing in M, we have that

$$\sup_M S_M = \int_{\mathbb{R}^n} (M_{p,q}(Mf))^p w,$$

and therefore we conclude the proof of the lemma. □

Remark 3.15 The important part of the dependence of the constant on the exponents p and q is that the lemma will fail to be true for $p = q$, with this kind of blowup.

Remark 3.16 The correct dependence of (3.12) on the C_q constant is, after simplifications,

$$\kappa_{n,p,q}\left(1 + [w]_{C_q}\right) \log\left(e + [w]_{C_q}\right).$$

In order to have bounds for sparse operators and sparse forms, we introduce the Marcinkiewicz operator at a fixed level. These operators were introduced in [17].

Definition 3.17

Let h be a positive lower-semicontinuous function on $\mathbb{R}^n$ and k an integer. Let $\mathcal{W}(k)$ be the Whitney decomposition of the level set $\Omega_k = \{x \in \mathbb{R}^n : h(x) > 2^k\}$, that is, $\Omega_k = \bigcup_{Q\in\mathcal{W}(k)} Q$. We define the function

$$M_{k,p,q}h(x) = \Big(2^{kp} \sum_{Q\in\mathcal{Q}_k} M\chi_Q(x)^q\Big)^{\frac{1}{p}}.$$

The relation between the full and the single-scale Marcinkiewicz operators is clear and is precisely

$$\sum_{k\in\mathbb{Z}} M_{k,p,q}h(x)^p = M_{p,q}h(x)^p.$$

Let us prove an analogue of Lemma 3.9 for sparse families of cubes.

Lemma 3.18

Let Q be a cube and $\mathcal{S}$ a sparse family of cubes that are contained in Q. Suppose that $w \in C_q$ with $1 < q < \infty$. Then

$$\int_{\mathbb{R}^n} \sum_{R\in\mathcal{S}} M\chi_R(x)^q w(x)dx \le \kappa_{n,q}\big([w]_{C_q} + 1\big) \int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx.$$

Proof. We start by noticing that if $x \notin 2Q$, then we have by Lemma 2.8 and since $1 < q < \infty$,

$$\begin{aligned}\sum_{R\in\mathcal{S}} M\chi_R(x)^q &\le \kappa_n \sum_{R\in\mathcal{S}} \Big(\frac{|R|}{\operatorname{dist}(x,Q)^n}\Big)^q\\ &\le \kappa_n^q \sum_{R\in\mathcal{S}} \Big(\frac{|E_R|}{\operatorname{dist}(x,Q)^n}\Big)^q\\ &= \kappa_n^q \frac{\sum_{R\in\mathcal{S}} |E_R|^q}{\operatorname{dist}(x,Q)^{nq}}\\ &\le \kappa_n^q \Big(\frac{|Q|}{\operatorname{dist}(x,Q)^n}\Big)^q\\ &\le \kappa_n^q M\chi_Q(x)^q,\end{aligned}$$

where E_R is the exceptional set given by sparsity and we used the assumption $q > 1$ in the estimate $\sum_{R\in\mathcal{S}} |E_R|^q \le |Q|^q$. Thus, it is enough to bound $\int_{2Q} \sum_{R\in\mathcal{S}} \big(M\chi_R\big)^q w$.

Since $E_R \subset R$ and $|E_R| \ge \frac{1}{2}|R|$ for every $R \in \mathcal{S}$, we have the pointwise bound

$$\sum_{R\in\mathcal{S}} (M\chi_R(x))^q \le \kappa_n^q \sum_{R\in\mathcal{S}} (M\chi_{E_R}(x))^q,$$

almost everywhere by Lemma 2.35. Also, since the sets E_R are pairwise disjoint, we have $\sum_R (\chi_{E_R})^q \le 1 \in L^\infty$. Thus, by [36, Theorem 1 (3)] there exists $c > 0$ such that for every $\lambda > 0$ we have

$$|F_\lambda| := |\{x \in 2Q : \sum_{R\in\mathcal{S}} M\chi_R(x)^q > \lambda\}| \le \kappa e^{-\kappa\lambda}|Q|. \tag{3.14}$$

Applying the C_q condition (2.4) to $F_\lambda \subseteq 2Q$ and applying (3.14) we have

$$\begin{aligned} w(F_\lambda) &\le \kappa\Big(\frac{|F_\lambda|}{|2Q|}\Big)^\varepsilon \int_{\mathbb{R}^n} (M\chi_{2Q}(x))^q w(x)dx \\ &\le \kappa_{n,q}\, e^{-\kappa\frac{\lambda}{[w]_{C_q}+1}} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x)dx. \end{aligned} \tag{3.15}$$

Thus, for any fixed $\lambda > 0$ we have

$$\int_{2Q} \sum_{R\in\mathcal{S}} (M\chi_R(x))^q w(x)dx = \int_0^\infty w(F_t)\,dt = \int_0^\lambda w(F_t)\,dt + \int_\lambda^\infty w(F_t)\,dt = I_1 + I_2.$$

For I_1, we can use Lemma 2.35 to see that

$$\begin{aligned} I_1 &\le \lambda w(2Q) \\ &\le \lambda \int_{\mathbb{R}^n} M(\chi_{2Q}(x))^q w(x)dx \\ &\le \kappa_n^q\, \lambda \int_{\mathbb{R}^n} M(\chi_Q(x))^q w(x)dx. \end{aligned}$$

For I_2, we can use (3.15) to get

$$\begin{aligned} I_2 &\le \kappa_{n,q} \int_\lambda^\infty e^{-\kappa\frac{t}{[w]_{C_q}+1}}\,dt \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x)dt \\ &\le \kappa_{n,q}([w]_{C_q}+1)e^{-\kappa\frac{\lambda}{[w]_{C_q}+1}} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x)dx. \end{aligned}$$

Thus, we have

$$I_1 + I_2 \le \kappa\Big(\lambda + \big([w]_{C_q}+1\big)e^{-c\frac{\lambda}{[w]_{C_q}+1}}\Big) \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x)dx,$$

and choosing $\lambda = [w]_{C_q}+1$ completes the proof. □

We now relate the sum of C_p-tails of a sparse family that is contained in the level set to the Marcinkiewicz operator at the same levle.

Lemma 3.19

Let h be a non-negative lower semicontinuous function, $w \in C_q$, $1 < q < \infty$ and $0 < p < \infty$. Suppose that $k \in \mathbb{Z}$ and let $\mathcal{S} = \{R_j\}$ be a sparse collection of cubes contained in $\Omega_k = \{x\colon h(x) > 2^k\}$. Then

$$2^{kp} \sum_{R_j\in\mathcal{S}} \int_{\mathbb{R}^n} (M\chi_{R_j}(x))^q w(x)dx \le \kappa_{n,q}\big([w]_{C_q}+1\big) \int_{\mathbb{R}^n} (M_{k,p,q}h(x))^p w(x)dx.$$

Proof. Fix $k \in \mathbb{Z}$ and let $\mathcal{Q}_k = \{Q_l\}_l$ be the Whitney decomposition of Ω_k. For each $Q_l \in \mathcal{Q}_k$, let $\mathcal{S}_{k,l}$ be the family of cubes R_j whose center is contained in Q_l. Then, by the properties of the Whitney cubes and the fact that $R_j \subset \Omega_k$, we have $R_j \subset c_n Q_l$ for every $R_j \in \mathcal{S}_{k,l}$. Moreover, each $R_j \in \mathcal{S}$ is contained in exactly one of the $\mathcal{S}_{k,l}$.

The desired estimate follows now from applying Lemma 3.18 to each of the collections $\mathcal{S}_{k,l}$:

$$\begin{aligned} 2^{kp} \sum_{R_j \in \mathcal{S}} \int_{\mathbb{R}^n} (M\chi_{R_j}(x))^q w(x)dx &= 2^{kp} \sum_{Q_l \in \mathcal{Q}_k} \sum_{R_j \in \mathcal{S}_{k,l}} \int_{\mathbb{R}^n} (M\chi_{R_j}(x))^q w(x)dx \\ &\leq \kappa_{p,q}\big([w]_{C_q}+1\big) 2^{kp} \sum_{Q_l \in \mathcal{Q}_k} \int_{\mathbb{R}^n} (M\chi_{Q_l}(x))^q w(x)dx \\ &= \kappa_{p,q,n}\big([w]_{C_q}+1\big) \int_{\mathbb{R}^n} (M_{k,p,q}h(x))^p w(x)dx. \end{aligned}$$

□

Finally, as a Corollary, we give a way of bounding sparse operators in terms of C_q weights.

Corollary 3.20

Suppose that $\mathcal{S}$ is a sparse collection of cubes, f is a locally integrable function, $w \in C_q$ for $1 < q < \infty$ and $0 < p < q$. Then

$$\sum_{Q \in \mathcal{S}} \langle f \rangle_Q^p \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x)dx \leq \kappa_{p,q,n}([w]_{C_q}+1)^2 \log([w]_{C_q}+e) \|Mf\|_{L^p(w)}^p.$$

Proof. We start by making a level decomposition of the sparse family: for every $k \in \mathbb{Z}$, we set

$$\mathcal{S}_k : \{Q \in \mathcal{S} : 2^k < \langle |f| \rangle_Q \leq 2^{k+1}\}.$$

Clearly we have $\mathcal{S} = \bigcup_{k \in \mathbb{Z}} \mathcal{S}_k$. Now, for each $Q \in \mathcal{S}_k$, we have trivially $Q \subset \{Mf > 2^k\}$. Thus, Lemmas 3.19 and 3.14 give us

$$\begin{aligned} \sum_{Q \in \mathcal{S}} \langle |f| \rangle_Q^p \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x)dx &\leq 2^p \sum_{k \in \mathbb{Z}} 2^{kp} \sum_{Q \in \mathcal{S}_k} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x)dx \\ &\leq \kappa 2^p \big([w]_{C_q}+1\big) \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} (M_{k,p,q}Mf(x))^p w(x)dx. \\ &= \kappa 2^p \big([w]_{C_q}+1\big) \int_{\mathbb{R}^n} (M_{p,q}Mf(x))^p w(x)dx \\ &\leq \kappa\big([w]_{C_q}+1\big)^2 \log\big([w]_{C_q}+e\big) \|Mf\|_{L^p(w)}^p, \end{aligned}$$

where $\kappa = \kappa_{n,p,q}$. This finishes the proof. □

3.4 C_p weights and the Coifman–Fefferman inequality

We state the quantification of Theorem B from [109].

Theorem 3.21

Fix $q > p > 1$. For all Calderón–Zygmund operator T, all bounded f with compact support and all weights $w \in C_q$ we have

$$\|T^* f\|_{L^p(w)} \leq \kappa_{n,T} \left(q + \frac{qp^2}{q-p}\right) \Phi([w]_{C_q} + 1) \, \|Mf\|_{L^p(w)}, \tag{3.16}$$

where $\Phi(t) = t \log(e + t)$.

Before proving Theorem 3.21, we provide a norm estimate for the Marcinkiewicz operator from Section 3.3 in terms of the truncated maximal singular integral operator.

Lemma 3.22
Under the same assumptions of Theorem 3.21 we have

$$\int_{\mathbb{R}^n} (M_{p,q} T^* f(x))^p w(x) dx \leq \left(\kappa_n \frac{2^p}{\varepsilon} \log \frac{\kappa_n 10^{nq} 2^{p+2}}{\varepsilon} \right) \int_{\mathbb{R}^n} (T^* f(x))^p w(x) dx$$
$$+ \left(\kappa_n^q 2^{\kappa_n \frac{p^2 q}{q-p}} \frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon} \right) \int_{\mathbb{R}^n} (Mf(x))^p w(x) dx.$$

Proof. Let $\mathcal{W}(k)$ be the Whitney decomposition of the level set $\Omega_k = \{x \in \mathbb{R}^n : T^* f(x) > 2^k\}$ for integer k. One can prove as in [20] the following inequality: if $Q \in \mathcal{W}(k-1)$ and $5Q \not\subset \{Mf > 2^{k-N}\}$ for some $N \geq 1$, then

$$|\{x \in Q; T^* f > 2^k\}| \leq \kappa_T 2^{-N} |Q|. \tag{3.17}$$

Let $\mathcal{V}(k)$ be the Whitney decomposition of the set $\{x \in \mathbb{R}^n : Mf(x) > 2^k\}$. We observe that for each cube $Q \in \mathcal{W}(k-1)$ there are two cases, for a fixed N that we will chose later.

Case (a). $5Q \subset \{Mf > 2^{k-N}\}$ in which case $5Q \subset c_n I$ for some $I \in \mathcal{V}(k-N)$.

Case (b). $5Q \not\subset \{Mf > 2^{k-N}\}$ in which case (3.17) implies

$$\sum_{\substack{P \in \mathcal{W}(k) \\ P \subset 5Q}} |P| \leq c_T 2^{-N} |Q|.$$

Now define the partial sums in a similar way as in the proof of Lemma 3.9

$$S(k) = \sum_{Q \in \mathcal{W}(k)} 2^{kp} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x) dx$$

and, for a fixed $P \in \mathcal{W}(k-1)$

$$S(k; P) = \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \cap P \neq \emptyset}} 2^{kp} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x) dx \leq \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \subset 5P}} 2^{kp} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x) dx.$$

Here, the last inequality follows from the Whitney decomposition. For a fixed P, we split in two parts the integral, close from P and away from P, that is

$$S(k; P) \leq \sum_{\substack{Q \in \mathcal{W}(k) \\ Q \subset 5P}} 2^{kp} \int_{\mathbb{R}^n} (M\chi_Q(x))^q w(x) dx$$

$$
\begin{aligned}
&= \int_{10P} + \int_{(10P)^c} \sum_{\substack{Q\in\mathcal{W}(k)\\ Q\subset 5P}} 2^{kp}(M\chi_Q(x))^q w(x)dx \\
&= I + II.
\end{aligned}
$$

By (3.8) with $R = 10$ we have

$$
I \le \kappa_n \frac{1}{\varepsilon}\log\frac{\kappa_n 10^{nq}}{\varepsilon\eta} 2^{kp} w(5P) + \eta 2^{kp}\int_{\mathbb{R}^n}(M\chi_P(x))^q w(x)dx,
$$

where $\eta > 0$ is a positive number that is free and at our disposal. In a similar way as in the proof of Lemma 3.9, one can show

$$
II \le \kappa_n^q 2^{kp-Nq}\int_{\mathbb{R}^n}(M\chi_P(x))^q w(x)
$$

Combining estimates for I and II we obtain, for every case (b) cube $P \in \mathcal{W}(k-1)$,

$$
S(k;P) \le \kappa_n \frac{1}{\varepsilon}\log\frac{\kappa_n 10^{nq}}{\varepsilon\eta} 2^{kp} w(5P) + (\eta + \kappa_n^q 2^{-Nq})2^{kp}\int_{\mathbb{R}^n}(M\chi_P(x))^q w(x)dx. \quad (3.18)
$$

Thus, we can split the partial sum $S(k)$ in terms of cubes of $\mathcal{W}(k-1)$ of case (a) or (b), that is,

$$
S(k) \le \sum_{\substack{P\in\mathcal{W}(k-1)\\ P \text{ is } (a)}} S(k;P) + \sum_{\substack{P\in\mathcal{W}(k-1)\\ P \text{ is } (b)}} S(k;P) = III + IV.
$$

Now, since each of the $Q \in \mathcal{W}(k)$ of type (a) intersects at most κ of the $P \in \mathcal{W}(k-1)$, yet again due to the Whitney decomposition, we have

$$
\begin{aligned}
III &\le \kappa \sum_{I\in\mathcal{V}(k-N)} \sum_{\substack{Q\in\mathcal{W}(k)\\ Q\subset\kappa_n I}} 2^{kp}\int_{\mathbb{R}^n}(M\chi_Q(x))^q w(x)dx \\
&\le c_n^q \frac{1}{\varepsilon} \sum_{I\in\mathcal{V}(k-N)} 2^{kp}\int_{\mathbb{R}^n}(M\chi_I(x))^q w(x)dx,
\end{aligned}
$$

where we have used (3.9) and $M\chi_{\kappa_n I} \le \kappa_n M\chi_I$ using Lemma 2.8, for two different κ_n of course. For the remaining part we have by (3.18)

$$
\begin{aligned}
IV &\le \kappa_n \frac{1}{\varepsilon}\log\frac{\kappa_n 10^{nq}}{\varepsilon\eta} 2^{kp} \sum_{P\in\mathcal{W}(k-1)} w(5P) \\
&\quad + (\eta + \kappa_n^q 2^{-Nq})2^{kp} \sum_{P\in\mathcal{W}(k-1)} w(5P)\int_{\mathbb{R}^n}(M\chi_P(x))^q w(x)dx \\
&\le \kappa_n \frac{1}{\varepsilon}\log\frac{\kappa_n 10^{nq}}{\varepsilon\eta} 2^{kp}\int_{\mathbb{R}^n} w(\Omega_{k-1}) \\
&\quad + (\eta 2^p + \kappa_n^q 2^{p-Nq})2^{(k-1)p} \sum_{P\in\mathcal{W}(k-1)} \int_{\mathbb{R}^n}(M\chi_P(x))^q w(x)dx \\
&\le \kappa_n 2^p \frac{1}{a\varepsilon}\log\frac{\kappa_n 10^{nq}}{\varepsilon\eta} 2^{(k-1)p} w(\Omega_{k-1}) + \frac{1}{2}S(k-1),
\end{aligned}
$$

if we choose η small enough and N big enough. This means $\eta = 2^{-(p+2)}$ and $N \geq \kappa_n \frac{p+q}{q}$. Combining now estimates for III and IV we get

$$S(k) \leq \frac{1}{2} S(k-1) + \left(c_n 2^p \frac{1}{a\varepsilon} \log \frac{\kappa_n 10^{nq} 2^{p+2}}{\varepsilon} \right) 2^{(k-1)p} w(\Omega_{k-1})$$
$$+ \left(\kappa_n^q 2^{\kappa_n \frac{p}{q}(p+q)} \frac{1}{\varepsilon} \right) \sum_{I \in \mathcal{V}(k-N)} 2^{(k-N)p} \int_{\mathbb{R}^n} (M\chi_I)^q w.$$

Set $S_M = \sum_{k \leq M} S(k)$ and sum the previous inequality over $k \leq M$ to obtain

$$\begin{aligned} S_M &\leq \frac{1}{2} S_M + \left(\kappa_n 2^p \frac{1}{a\varepsilon} \log \frac{\kappa_n 10^{nq} 2^{p+2}}{\varepsilon} \right) \int_{\mathbb{R}^n} (T^* f(x))^p w(x) dx \\ &\quad + \left(\kappa_n^q 2^{\kappa_n \frac{p}{q}(p+q)} \frac{1}{\varepsilon} \right) \int_{\mathbb{R}^n} (M_{p,q}(Mf)(x))^p w(x) dx \\ &\leq \frac{1}{2} S_M + \left(\kappa_n 2^p \frac{1}{a\varepsilon} \log \frac{\kappa_n 10^{nq} 2^{p+2}}{\varepsilon} \right) \int_{\mathbb{R}^n} (T^* f(x))^p w(x) dx \\ &\quad + \left(\kappa_n^q 2^{\kappa_n \frac{p}{q}(p+q)} \frac{1}{\varepsilon} \right) \left(\kappa_n 2^{\kappa_n \frac{pq}{q-p}} \frac{1}{\varepsilon} \log \frac{1}{\varepsilon} \right) \int_{\mathbb{R}^n} (Mf(x))^p w(x) dx, \end{aligned}$$

by (3.12). It can be shown (cf. [109], p.262) that $S_M < \infty$, so taking it to the left and then taking the supremum over all M we obtain the desired result. $\square$

Now we are ready to prove Theorem 3.21. The prove we give is more convoluted than that in [109] and we incorporate the Reverse Hölder inequality from Theorem 2.21.

Proof of theorem 3.21. Using the exponential decay from [7], we know that if we write $\{x \in \mathbb{R}^n : T^* f(x) > 2^k\} = \bigcup_j Q_j$ as in the Whitney decomposition theorem, we have

$$|\{x \in Q_j : T^* f(x) > 2\lambda, Mf(x) \leq \gamma\lambda\}| \leq \kappa\, e^{-\frac{\varepsilon}{\gamma}} |Q_j|, \tag{3.19}$$

for any $\gamma > 0$. We call E_j to the set in the left side of (3.19). Then, if we call r to the exponent $1 + \delta$ in Theorem 2.21, we get

$$\begin{aligned} w(E_j) = |E_j| \frac{1}{|E_j|} \int_{E_j} w(x) dx &\leq |E_j| \left(\frac{1}{|E_j|} \int_{E_j} w(x)^r dx \right)^{\frac{1}{r}} \\ &\leq |E_j|^{\frac{1}{r'}} |Q_j|^{\frac{1}{r}} \left(\frac{1}{|Q_j|} \int_{Q_j} w(x)^r dx \right)^{\frac{1}{r}} \\ &\leq |E_j|^{\frac{1}{r'}} |Q_j|^{\frac{1}{r}} \frac{2}{|Q_j|} \int_{\mathbb{R}^n} (M\chi_{Q_j}(x))^q w(x) dx \\ &\leq \kappa e^{-\frac{\kappa}{\gamma r'}} \int_{\mathbb{R}^n} (M\chi_{Q_j}(x))^q w(x) dx. \end{aligned}$$

We use the standard good-λ techniques as in [109], see Section 1.4.4, combined with Lemma 3.22 to get

$$\int_{\mathbb{R}^n} T^* f(x)^p w(x) dx \leq \left(\frac{2}{\gamma} \right)^p \int_{\mathbb{R}^n} Mf(x)^p w(x) dx + \kappa e^{-\frac{\kappa}{\gamma r'}} \int_{\mathbb{R}^n} (M_{p,q} T^* f(x))^p w(x) dx$$

$$\leq \left(2^p\gamma^{-p} + e^{-\frac{\kappa}{\gamma r'}}\left(\kappa_n^q 2^{\kappa_n \frac{p^2 q}{q-p}} \frac{1}{\varepsilon^2}\log\frac{1}{\varepsilon}\right)\right)\int_{\mathbb{R}^n} Mf(x)^p w(x)dx$$
$$+ \kappa e^{-\frac{\kappa}{\gamma r'}}\left(\kappa_n 2^p \frac{1}{\varepsilon}\log\frac{\kappa_n 10^{nq}2^{p+2}}{\varepsilon}\right)\int_{\mathbb{R}^n} T^* f(x)^p w(x)dx$$

Choosing $\gamma^{-1} \sim \kappa_n(q + \frac{p^2 q}{q-p})\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ we can make

$$e^{-\frac{\kappa}{\gamma r'}}\left(\kappa_n^q 2^{\kappa_n \frac{p^2 q}{q-p}} \frac{1}{\varepsilon^2}\log\frac{1}{\varepsilon}\right) < \frac{1}{2}$$

and

$$\kappa e^{-\frac{\kappa}{\gamma r'}}\left(\kappa_n 2^p \frac{1}{a\varepsilon}\log\frac{\kappa_n 10^{nq}2^{p+2}}{\varepsilon}\right) < \frac{1}{2}.$$

and taking the term to the left side (which is possible since it is finite, see [109]) we obtain

$$\int_{\mathbb{R}^n} (T^* f(x))^p w(x)dx \leq \kappa_n^p \left(\left(q + \frac{p^2 q}{q-p}\right)\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)^p \int_{\mathbb{R}^n} Mf(x)^p w(x)dx. \qquad \square$$

Remark 3.23 We conjecture that the first q in the constant should not be there. That way $\lim_{q\to\infty} c_q < \infty$. We think this should be the case because whenever $w \in C_q$ and q is bigger, we have more information. This way we could recover a weighted inequality for the A_∞ class, though it would be a worse one than the one we mention in the introduction. For this very reason, we conjecture that the dependence on the C_q constant is not sharp in this sense.

Conjecture 3.24

Let T be a Calderón–Zygmund operator and let $1 < p < q < \infty$. There exists a constant $\kappa = \kappa_{n,p,q,T}$ such that for all $w \in C_q$ the following holds:

$$\|T^* f\|_{L^p(w)} \leq \kappa \max\left([w]_{C_q}, 1\right)\|Mf\|_{L^p(w)}.$$

Regarding Muckenhoupt's conjecture 2.1 and the C_p constant, we dare not make a quantitative conjecture in that respect, that is, how the ratio between the $L^p(w)$ norm of the singular integral and the maximal operator has to depend on the C_p constant of the weight w.

We remark that, even if usually sparse domination gives sharper quantitative bounds on the weights, this is not the case. This is because proving a bound for the sparse operator already uses the techniques of Marcinkiewicz operators. Therefore, the dependence that one obtains is essentially the same as that in (3.16).

3.5 Estimates for rough operators

In this section, we will prove Sawyer type C_p estimates for rough homogeneous singular integrals as in Section 3.1.2.

Theorem 3.25

Let T_Ω be a rough homogeneous singular integral as in Definition 3.2. The following inequalities hold:

I) if $1 < p < q < \infty$ and $w \in C_q$, then

$$\|T_\Omega f\|_{L^p(w)} \leq \kappa_{n,p,q}\big([w]_{C_q}+1\big)^3 \log\big([w]_{C_q}+e\big)\|\Omega\|_{L^\infty}\|Mf\|_{L^p(w)};$$

II) if $0 < p \leq 1 < q < \infty$ and $w \in C_q$, then

$$\|T_\Omega f\|_{L^p(w)} \leq \kappa_{n,p,q}\big([w]_{C_q}+1\big)^{1+\frac{2}{p}} \log^{\frac{1}{p}}\big([w]_{C_q}+e\big)\|\Omega\|_{L^\infty}\|Mf\|_{L^p(w)}.$$

The constant $\kappa_{n,p,q}$ satisfies $\kappa_{n,p,q} \to \infty$ as $q \to p$.

We want to emphasize that the main novelty of this result is the qualitative estimates that (to the best of our knowledge) were not known earlier. We do not know if our bounds are sharp with respect to $[w]_{C_p}$ but we strongly suspect that they are not. We also note that previous proofs for the case $0 < p < 1$ and $w \in A_\infty$ used extrapolation theory which is not available for C_p weights. Our method and quantitative bounds are new even for weights $w \in A_\infty$.

Our proof relies particularly on a sparse domination result of Conde-Alonso, Culiuc, Di Plinio and Ou:

Theorem 3.26 – [24, part of Theorem A]

Let T_Ω be a rough homogeneous singular integral as in Definition 3.2. Then, for any $1 < p < \infty$ we have

$$|\langle T_\Omega f, g\rangle| \leq \kappa_n\, p'\, \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \sup_{\mathcal{S}} \sum_{Q\in\mathcal{S}} \langle |f|\rangle_Q \langle |g|\rangle_{p,Q},$$

where the supremum is taken over all sparse collections $\mathcal{S}$, see Section 3.1.3.

An alternative approach for this result can be found in [85]. Thus, instead of working directly with rough homogeneous singular integrals, we use Theorem 3.26 to reduce the question to proving bounds for sparse forms.

Theorem 3.27

Let $\Lambda_{\mathcal{S}}^{t,\gamma}$ be the sparse form defined as

$$\Lambda_{\mathcal{S}}^{t,\gamma}(f,g) = (t')^\gamma \sum_{Q\in\mathcal{S}} \langle |f|\rangle_Q^\gamma \langle |g|\rangle_{t,Q}|Q|,$$

where $\mathcal{S}$ is a sparse collection of cubes, $t > 1$ and $0 < \gamma \leq 1$.

I) Suppose that $1 < p < q < \infty$ and $w \in C_q$. Then there exists $1 < s < 2$ such that

$$\Lambda_{\mathcal{S}}^{s,1}(f,gw) \leq \kappa_{n,p,q}\big([w]_{C_q}+1\big)^3 \log\big([w]_{C_q}+e\big)\|Mf\|_{L^p(w)}\|g\|_{L^{p'}(w)}.$$

II) Suppose that $0 < p \leq 1 < q < \infty$ and $w \in C_q$. Then there exists $1 < s < \min\{2, \frac{1}{1-p}\}$ such that

$$\Lambda_{\mathcal{S}}^{s,p}(f,w) \leq \kappa_{n,p,q}\big([w]_{C_q}+1\big)^{p+2} \log\big([w]_{C_q}+e\big)\|Mf\|_{L^p(w)}^p.$$

The constant $\kappa_{n,p,q}$ satisfies $\kappa_{n,p,q} \to \infty$ as $q \to p$.

Part I) of Theorem 3.25 follows from Theorem 3.26 and part I) of Theorem 3.27 in a very straightforward way but for part II) we need some additional considerations. In particular, we need to modify some results proven by Lerner [85] and prove a variation of the sparse domination result for the case $0 < p < 1$ (see Theorem 3.29).

We note that in [24], the authors proved similar sparse domination results also for other classes of operators, namely rough homogeneous singular integrals T_Ω with more general kernel functions Ω and Bochner–Riesz means. Their results combined with Theorem 3.27 give C_q-Coifman–Fefferman estimates also for these operators for $1 \leq p < \infty$ but for simplicity, we only consider the operators T_Ω with $\Omega \in L^\infty(\mathbb{S}^{n-1})$ satisfying $\int_{\mathbb{S}^{n-1}} \Omega \, d\sigma = 0$.

3.5.1 Proof of part I) of Theorems 3.25 and 3.27

As we stated before, part I) of Theorem 3.25 follows easily from a combination of part I) Theorem 3.27 and Theorem 3.26. Indeed, let s be the one given by Theorem 3.27. We apply Theorem 3.26 with parameter s and we get

$$\begin{aligned}
\|T_\Omega f\|_{L^p(w)} &= \sup_{\|g\|_{L^{p'}(w)}=1} |\langle T_\Omega f, gw\rangle| \\
&\leq \kappa_n \|\Omega\|_\infty \, s' \sup_{\|g\|_{L^{p'}(w)}=1} \sup_{\mathcal{S}} \sum_{Q\in\mathcal{S}} \langle f\rangle_Q \langle gw\rangle_{s,Q}|Q| \\
&\leq \kappa_{n,p,q}\|\Omega\|_\infty\big([w]_{C_q}+1\big)^3 \log\big([w]_{C_q}+e\big)\|Mf\|_{L^p(w)},
\end{aligned}$$

where we used part I) of Theorem 3.27 in the last inequality.

We now give the proof of part I) of Theorem 3.27. Let us start by recalling the dyadic Carleson embedding theorem that we need a couple of times in our proofs.

Theorem 3.28 – Carleson Embedding [62, Theorem 4.5]

Let $\mathcal{D}$ be a collection of dyadic cubes, w a weight and a_Q a non-negative number for every $Q \in \mathcal{D}$. Suppose that there exists $A \geq 0$ such that for every $R \in \mathcal{D}$ we have

$$\sum_{Q\in\mathcal{D}, Q\subset R} a_Q \leq A w(R).$$

Then, for all $1 < \alpha < \infty$ and $h \in L^\alpha(w)$, we have

$$\Big(\sum_{R\in\mathcal{D}} a_R (\langle h\rangle_R^w)^\alpha\Big)^{\frac{1}{\alpha}} \leq A^{\frac{1}{\alpha}} \alpha' \, \|h\|_{L^\alpha(w)}.$$

Let us then prove part I) of Theorem 3.27. Suppose that $1 < p < q < \infty$, and $w \in C_q$. We want to show that there exists $1 < s < 2$ such that

$$s' \sum_{Q\in\mathcal{S}} \langle |f|\rangle_Q \langle |gw|\rangle_{s,Q} |Q| \le \kappa_{w,n,p,q} \|Mf\|_{L^p(w)} \|g\|_{L^{p'}(w)}.$$

By rescaling we may assume that $\|Mf\|_{L^p(w)} = \|g\|_{L^{p'}(w)} = 1$. To simplify the notation, we also assume $f, g \ge 0$.

Let δ be the Reverse Hölder constant from Theorem 2.21 and set $s = 1 + \frac{\delta}{8p}$ and $r = 1 + \frac{1}{4p}$. It is easy to check that

$$sr < 1 + \frac{1}{2p} < p' \quad \text{and} \quad \Big(s - \frac{1}{r}\Big) r' = s + \frac{s-1}{r-1} < 1 + \delta. \tag{3.20}$$

In particular, $(s - \frac{1}{r})r'$ is an admissible exponent for the Reverse Hölder inequality in Theorem 2.21. Therefore, by Hölder's inequality and Theorem 2.21 we have

$$\begin{aligned}
\sum_{Q\in\mathcal{S}} \langle f\rangle_Q \langle gw\rangle_{s,Q} |Q| &\le \sum_{Q\in\mathcal{S}} \langle f\rangle_Q \langle g^{sr} w\rangle_Q^{\frac{1}{sr}} \langle w^{(s-\frac{1}{r})r'}\rangle_Q^{\frac{1}{sr'}} |Q| \\
&\le 2^{1-\frac{1}{sr}} \sum_{Q\in\mathcal{S}} \langle f\rangle_Q \langle g^{sr} w\rangle_Q^{\frac{1}{sr}} \Big(\frac{1}{|Q|} \int_{\mathbb{R}^n} M\chi_Q(x)^q w(x) dx \Big)^{1-\frac{1}{sr}} |Q| \\
&\le 2 \sum_{Q\in\mathcal{S}} \langle f\rangle_Q \Big(\int_{\mathbb{R}^n} M\chi_Q(x)^q w(x) dx \Big)^{1-\frac{1}{sr}} \big(\langle g^{sr}\rangle_Q^w\big)^{\frac{1}{sr}} w(Q)^{\frac{1}{sr}}.
\end{aligned}$$

Let us then split the sparse family into two parts. We set

$$\mathcal{S}_1 := \Big\{ Q \in \mathcal{S} \colon \big(\langle g^{sr}\rangle_Q^w\big)^{\frac{1}{sr}} w(Q)^{\frac{1}{sr}} \le \langle f\rangle_Q^{\frac{p}{p'}} \Big(\int_{\mathbb{R}^n} M\chi_Q(x)^q w(x) dx \Big)^{\frac{1}{sr}} \Big\}$$

and $\mathcal{S}_2 = \mathcal{S} \setminus \mathcal{S}_1$. For the collection $\mathcal{S}_1$, we use Corollary 3.20 to see that

$$\begin{aligned}
&\sum_{Q\in\mathcal{S}_1} \langle f\rangle_Q \Big(\int_{\mathbb{R}^n} M\chi_Q(x)^q w(x) dx \Big)^{1-\frac{1}{sr}} \big(\langle g^{sr}\rangle_Q^w\big)^{\frac{1}{sr}} w(Q)^{\frac{1}{sr}} \\
&\quad \le \sum_{Q\in\mathcal{S}_1} \langle f\rangle_Q \Big(\int_{\mathbb{R}^n} M\chi_Q(x)^q w(x) dx \Big)^{1-\frac{1}{sr}} \langle f\rangle_Q^{\frac{p}{p'}} \Big(\int_{\mathbb{R}^n} M\chi_Q(x)^q w(x) dx \Big)^{\frac{1}{sr}} \\
&\quad = \sum_{Q\in\mathcal{S}_1} \langle f\rangle_Q^p \int_{\mathbb{R}^n} (M\chi_Q(x)^q w(x) dx \\
&\quad \le \kappa_{n,p,q} ([w]_{C_q} + 1)^2 \log([w]_{C_q} + e) \|Mf\|_{L^p(w)}^p \\
&\quad = \kappa_{n,p,q} ([w]_{C_q} + 1)^2 \log([w]_{C_q} + e).
\end{aligned}$$

The collection $\mathcal{S}_2$ is trickier. Recall that by the discussion in Chapter 2, we may suppose that for any cube Q, the C_q-tail of w at Q is finite. Thus, we have

$$\begin{aligned}
&\sum_{Q\in\mathcal{S}_2} \langle f\rangle_Q \big(\langle g^{sr}\rangle_Q^w\big)^{\frac{1}{sr}} \Big(\int_{\mathbb{R}^n} M\chi_Q(x)^q w(x) dx \Big)^{1-\frac{1}{sr}} w(Q)^{\frac{1}{sr}} \\
&\qquad \le \sum_{Q\in\mathcal{S}_2} \big(\langle g^{sr}\rangle_Q^w\big)^{\frac{p'}{psr}} w(Q)^{\frac{p'}{psr}} \big(\langle g^{sr}\rangle_Q^w\big)^{\frac{1}{sr}}
\end{aligned}$$

$$\times \Big(\int_{\mathbb{R}^n} M\chi_Q(x))^q w(x)dx \Big)^{1-\frac{1}{sr}-\frac{p'}{psr}} w(Q)^{\frac{1}{sr}}$$

$$\leq \sum_{Q\in\mathcal{S}_2} \big(\langle g^{sr}\rangle_Q^w\big)^{\frac{p'}{sr}} w(Q)\Big(\frac{w(Q)}{\int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx}\Big)^{\frac{p'}{psr}+\frac{1}{sr}-1}$$

$$= \sum_{Q\in\mathcal{S}_2} \big(\langle g^{sr}\rangle_Q^w\big)^{\frac{p'}{sr}} w(Q)\Big(\frac{w(Q)}{\int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx}\Big)^{\frac{p'}{sr}-1}.$$

We set $\alpha = \frac{p'}{sr}$ and

$$a_Q := w(Q)\Big(\frac{w(Q)}{\int_{\mathbb{R}^n} M(\chi_Q)^q w}\Big)^{\frac{p'}{sr}-1}$$

for every cube $Q \in \mathcal{S}_2$. By (3.20), we know that $\alpha > 1$. We claim that there exists some $A > 0$ such that for any $R \in \mathcal{S}_2$ we have

$$\sum_{Q\in\mathcal{S}_2, Q\subset R} a_Q \leq A w(R). \tag{3.21}$$

Then, by the Carleson embedding (Theorem 3.28), we know that

$$\begin{aligned}\sum_{Q\in\mathcal{S}_2} \big(\langle g^{sr}\rangle_Q^w\big)^{\frac{p'}{sr}} w(Q)\Big(\frac{w(Q)}{\int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx}\Big)^{\frac{p'}{sr}-1} &= \sum_{Q\in\mathcal{S}_2} a_Q\big(\langle g^{sr}\rangle_Q^w\big)^{\alpha} \\ &\leq \big(A^{\frac{1}{\alpha}}\alpha'\|g^{sr}\|_{L^{\alpha}(w)}\big)^{\alpha} \\ &= A(\alpha')^{\alpha}\|g\|_{L^{p'}(w)}^{p'} \\ &\leq \kappa_p\, A.\end{aligned}$$

In the last inequality we have used that, by the choices of r and s, we have $1 < rs < 1 + \frac{1}{2p}$ and therefore $p' - rs > p' - 1 - \frac{1}{4p} = \frac{3p+1}{4p(p-1)}$, which gives

$$\Big(\Big(\frac{p'}{rs}\Big)'\Big)^{\frac{p'}{rs}} = \Big(\frac{p'}{p'-rs}\Big)^{p'} \leq \Big(\frac{4p^2}{3p+1}\Big)^{p'} = \kappa_p.$$

Thus, it is enough for us to prove the claim. That is, we need to show that there exists a constant $A > 0$ such that (3.21) holds. For this, fix $R \in \mathcal{S}_2$. We further split $\mathcal{S}_2$ into subcollections $\mathcal{S}_{2,j}$, $j \geq 1$, defined as

$$\mathcal{S}_{2,j} := \Big\{ Q \in \mathcal{S}_2 \colon 2^{j-1} w(Q) \leq \int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx < 2^j w(Q) \Big\}.$$

Let $\mathcal{S}_{2,j}^* = \mathcal{S}_{2,j}^*(R)$ be the collection of maximal subcubes in $\mathcal{S}_{2,j}$ which are contained in R. We now have

$$\sum_{\substack{Q\in\mathcal{S}_{2,j}\\ Q\subset R}} w(Q)\Big(\frac{w(Q)}{\int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx}\Big)^{\frac{p'}{sr}-1}$$

$$\overset{\text{(A)}}{\leq} \sum_{\substack{Q\in\mathcal{S}_{2,j}\\ Q\subset R}} 2^{1-j}\int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx \Big(\frac{2^{1-j}\int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx}{\int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx}\Big)^{\frac{p'}{sr}-1}$$

$$
\begin{aligned}
&= 2^{1-j+(1-j)\left(\frac{p'}{sr}-1\right)} \sum_{\substack{Q\in\mathcal{S}_{2,j}\\ Q\subset R}} \int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx \\
&= 2^{(1-j)\frac{p'}{sr}} \sum_{P\in\mathcal{S}_{2,j}^*} \sum_{\substack{Q\in\mathcal{S}_{2,j}\\ Q\subset P}} \int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx \\
&\overset{\text{(B)}}{\leq} 2^{(1-j)\frac{p'}{sr}}([w]_{C_q}+1) \sum_{P\in\mathcal{S}_{2,j}^*} \int_{\mathbb{R}^n} M\chi_P(x)^q w(x)dx \\
&\overset{\text{(A)}}{\leq} 2^{(1-j)\frac{p'}{sr}+j}([w]_{C_q}+1) \sum_{P\in\mathcal{S}_{2,j}^*} w(P) \\
&\overset{\text{(C)}}{\leq} 2^{(1-j)\frac{p'}{sr}+j}([w]_{C_q}+1)w(R),
\end{aligned}
$$

where we used (A) the definition of the collection $\mathcal{S}_{2,j}$, (B) Lemma 3.18 and (C) the fact that the cubes in $\mathcal{S}_{2,j}^*$ are disjoint. We now sum over j and get

$$
\sum_{\substack{Q\in\mathcal{S}_2\\ Q\subset R}} a_Q = \sum_{j\geq 1} \sum_{\substack{Q\in\mathcal{S}_{2,j}\\ Q\subset R}} a_Q \leq ([w]_{C_q}+1)2^{\frac{p'}{sr}} \sum_{j\geq 1} 2^{j\left(1-\frac{p'}{sr}\right)} w(R).
$$

Therefore (3.21) holds with

$$
A := ([w]_{C_q}+1)2^{\frac{p'}{sr}} \sum_{j\geq 1} 2^{j\left(1-\frac{p'}{sr}\right)} = 2\frac{([w]_{C_q}+1)}{1-2^{1-p'/sr}} \leq \kappa_p([w]_{C_q}+1).
$$

Putting all of the above together, we proved that for $s = 1+\frac{\delta}{8p}$ we have

$$
s' \sum_{Q\in\mathcal{S}} \langle|f|\rangle_Q \langle|gw|\rangle_{s,Q}|Q| \leq s'\left(\kappa_{n,p,q}([w]_{C_q}+1)^2 \log([w]_{C_q}+e) + \kappa_p([w]_{C_q}+1)\right).
$$

The constant $\kappa_{n,p,q}$ is the same constant as in Corollary 3.20 and thus, we have

$$
\kappa_{n,p,q} = \kappa_n 2^{\kappa_n' \frac{pq}{q-p}}.
$$

In particular, $\kappa_{n,p,q} \to \infty$ as $q \to p$. Since $\delta = \frac{1}{B([w]_{C_q}+1)}$ where $B = B(n,q)$ as in Theorem 2.21, we have

$$
s' = \frac{8p}{\delta} + 1 \approx 8pB([w]_{C_q}+1).
$$

Hence we see that

$$
s' \sum_{Q\in\mathcal{S}} \langle|f|\rangle_Q \langle|gw|\rangle_{s,Q}|Q| \leq \kappa_{n,p,q}\left([w]_{C_q}+1\right)^3 \log\left([w]_{C_q}+e\right)
$$

for a constant $\kappa_{n,p,q}$ such that $\kappa_{n,p,q} \to \infty$ as $q \to p$.

3.5.2 Sparse domination for rough singular integrals revisited

Before we prove part II) of Theorems 3.25 and 3.27, we revisit the sparse domination principle in [24] and prove a version of it that is more suitable for the case $0 < p < 1$. Let us first consider a Calderón–Zygmund operator T. It is now well-known (see e.g.

[64, 79, 84]) that T satisfies a pointwise sparse bound of the type

$$Tf(x) \leq \kappa_T \sum_{i,Q\in\mathcal{S}_i} \chi_Q(x)\langle|f|\rangle_Q.$$

Now, for $0 < p < 1$, we trivially have

$$|Tf(x)|^p \leq \kappa_T^p \sum_{i,Q\in\mathcal{S}_i} \chi_Q(x)\langle|f|\rangle_Q^p,$$

and thus, for $q = 1+\lambda$ and $w \in C_q$ for any $\lambda > 0$, Corollary 3.20 gives us

$$\begin{aligned}\int_{\mathbb{R}^n} |Tf(x)|^p w(x)dx &\leq \kappa_T^p \sum_{i,Q\in\mathcal{S}_i} w(Q)\langle|f|\rangle_Q^p \\ &\leq \kappa_T^p \sum_{i,Q\in\mathcal{S}_i} \langle|f|\rangle_Q^p \int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx \\ &\leq \kappa_{T,n,p,q}([w]_{C_q}+1)^2 \log([w]_{C_q}+e)\|Mf\|_{L^p(w)}^p.\end{aligned}$$

Qualitative version of this result was proven as a part of [17, Theorem 17] using different techniques.

To mimic this proof strategy for rough homogeneous singular integrals, we prove the following sparse domination result:

Theorem 3.29

Suppose that $0 < \theta < 1$ and $1 < s \leq \frac{1}{1-\theta}$. Then there exists a sparse collection $\mathcal{S}$ such that

$$|\langle |T_\Omega f|^\theta, g\rangle| \leq \kappa\,(s')^\theta\,\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}^\theta \sum_{Q\in\mathcal{S}} |Q|\langle|f|\rangle_Q^\theta\langle|g|\rangle_{s,Q}.$$

Our proof is strongly based on techniques used by Lerner in [85]. For a sublinear operator T and $0 < \theta < 1$, we define

$$\mathscr{M}_T^\theta(f,g)(x) := \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |T(f\chi_{\mathbb{R}^n\setminus 3Q})(y)|^\theta |g(y)|\,dy.$$

Our main tool is the following variant of [85, Theorem 3.1]:

Theorem 3.30

Let $1 \leq q \leq r$, $0 < \theta < 1$ and $s \geq 1$. Assume that T is a sublinear operator of weak type (q,q) and $\mathscr{M}_T^\theta$ satisfies the following estimate:

$$\|\mathscr{M}_T^\theta(f,g)\|_{L^{\nu,\infty}} \leq N\|f\|_{L^r}^\theta\|g\|_{L^s},$$

for exponents satisfying the relation

$$\frac{1}{\nu} = \frac{\theta}{r} + \frac{1}{s}.$$

Then for every compactly supported $f \in L^r(\mathbb{R}^n)$ and every $g \in L^s_{\text{loc}}$, there exists a sparse collection of cubes $\mathcal{S}$ such that

$$\langle |Tf|^\theta, |g|\rangle \le \kappa_{T,N} \sum_{Q\in\mathcal{S}} |Q| \langle |f|\rangle^\theta_{r,Q} \langle |g|\rangle_{s,Q},$$

where $\kappa_{T,N} = \kappa_n\big(\|T\|^\theta_{L^q\to L^{q,\infty}} + N\big)$.

Proof. The proof is essentially the same as the proof of [85, Theorem 3.1]. The only difference is the definition of the sets E_1 and E_2: the first set is the same, namely

$$E_1 = \{x \in Q_0 \colon |T(f\chi_{3Q_0})(x)| > A\langle |f|\rangle_{q,3Q_0}\},$$

and we define the second set as

$$E_2 = \{x \in Q_0 \colon \mathscr{M}^\theta_{T,Q_0}(f,g)(x) > B\langle |f|\rangle^\theta_{r,3Q_0}\langle |g|\rangle_{s,Q_0}\}.$$

The rest of the proof works as it is with the obvious changes. □

With the help of Theorem 3.30, the proof of Theorem 3.29 is fairly straightforward.

Proof of Theorem 3.29. Let T_Ω be a rough homogeneous singular integral. We want to apply Theorem 3.30 with $q = 1 = r$. Let $1 < s \le \frac{1}{1-\theta}$. Since T_Ω is of weak-type $(1,1)$ by [110], we only need to check the bound for $\mathscr{M}^\theta_{T_\Omega}$. To be more precise, we need to show that

$$\|\mathscr{M}^\theta_{T_\Omega}(f,g)\|_{L^{\nu,\infty}} \le N\|f\|^\theta_{L^1}\|g\|_{L^s}, \tag{3.22}$$

where $\frac{1}{\nu} = \theta + \frac{1}{s}$. Let us define an auxiliary operator $\mathscr{N}^\theta_{p,T_\Omega}$ by setting

$$\mathscr{N}^\theta_{p,T_\Omega} f(x) = \sup_{Q\ni x}\Big(\frac{1}{|Q|}\int_Q |T_\Omega(f\chi_{\mathbb{R}^n\setminus 3Q})(y)|^{p\theta} dy\Big)^{\frac{1}{p}}.$$

Notice that we have $\mathscr{N}^\theta_{p,T_\Omega} f(x) = \big(\mathscr{N}^1_{p\theta,T_\Omega} f(x)\big)^\theta$. By Hölder's inequality, we have the pointwise bound

$$\begin{aligned}\mathscr{M}^\theta_{T_\Omega}(f,g)(x) &\le \sup_{Q\ni x}\Big(\int_Q \big|T_\Omega(f\chi_{\mathbb{R}^n\setminus 3Q})(y)\big|^{s'\theta} dy\Big)^{\frac{1}{s'}}\Big(\int_Q |g(y)|^s dy\Big)^{\frac{1}{s}}\\ &\le \mathscr{N}^\theta_{s',T_\Omega} f(x) M_s g(x) = \big(\mathscr{N}^1_{s'\theta,T_\Omega} f(x)\big)^\theta M_s g(x).\end{aligned}$$

Now, combining this pointwise bound with Hölder's inequality for weak spaces (see e.g. [45, Ex. 1.1.15]), the straightforward estimate $\big\|\big(\mathscr{N}^1_{s'\theta,T_\Omega} f\big)^\theta\big\|_{L^{\frac{1}{\theta},\infty}} = \big\|\mathscr{N}^1_{s'\theta,T_\Omega} f\big\|^\theta_{L^{1,\infty}}$ and the weak type (s,s) of M_s, see Section 1.4.2, we get

$$\begin{aligned}\|\mathscr{M}^\theta_{T_\Omega}(f,g)\|_{L^{\nu,\infty}} &\le \nu^{-\frac{1}{\nu}}\theta^{-\theta}s^{\frac{1}{s}}\big\|\big(\mathscr{N}^1_{s'\theta,T_\Omega} f\big)^\theta\big\|_{L^{\frac{1}{\theta},\infty}}\|M_s g\|_{L^{s,\infty}}\\ &\le \kappa\nu^{-\frac{1}{\nu}}\,\theta^{-\theta}\,s^{\frac{1}{s}}\,\|\mathscr{N}^1_{s'\theta,T_\Omega} f\|^\theta_{L^{1,\infty}}\|g\|_{L^s}.\end{aligned}$$

By [85, Theorem 1.1, Lemma 3.3], we know that

$$\|\mathscr{N}^1_{s'\theta,T_\Omega} f\|_{L^{1,\infty}} \le \kappa\, s'\theta\, \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\|f\|_{L^1},$$

provided that $1 \leq s'\theta < \infty$ which is equivalent to $1 < s \leq \frac{1}{1-\theta}$. Therefore, we have

$$\begin{aligned} \|\mathscr{M}^{\theta}_{T_\Omega}(f,g)\|_{L^{\nu,\infty}} &\leq \kappa \nu^{-\frac{1}{\nu}}\,\theta^{-\theta}\,s^{\frac{1}{s}}\,(s'\theta)^{\theta}\,\|\Omega\|^{\theta}_{L^\infty(\mathbb{S}^{n-1})}\|f\|^{\theta}_{L^1}\|g\|_{L^s} \\ &\leq \kappa\,(s')^{\theta}\,\|\Omega\|^{\theta}_{L^\infty(\mathbb{S}^{n-1})}\|f\|^{\theta}_{L^1}\|g\|_{L^s}, \end{aligned}$$

since $\theta < 1 < s$, $\nu = s/(\theta s + 1)$ and

$$\nu^{-\frac{1}{\nu}}\theta^{-\theta}s^{\frac{1}{s}}(s'\theta)^{\theta} = s^{-\theta}(s')^{\theta}(s\theta+1)^{\theta+\frac{1}{s}} \leq \kappa s^{\frac{1}{s}}(s')^{\theta} \leq \kappa (s')^{\theta}.$$

Thus, (3.22) holds for $N = \kappa_n (s')^{\theta}\|\Omega\|^{\theta}_{L^\infty(\mathbb{S}^{n-1})}$. Since $\|T_\Omega\|_{L^1\to L^{1,\infty}} \leq \kappa_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}$ by [110], we can apply Theorem 3.30, which finishes the proof. □

3.5.3 Proof of part II) of Theorems 3.25 and 3.27

Firstly, we deduce part II) of Theorem 3.25 from the sparse domination presented in Theorem 3.29 and the bound for the sparse form from Theorem 3.27. Let $0 < p \leq 1$, we have

$$\|T_\Omega f\|_{L^p(w)} = \||T_\Omega f|^p\|^{\frac{1}{p}}_{L^1(w)} = |\langle |T_\Omega f|^p, w\rangle|^{\frac{1}{p}}.$$

Now, we use Theorem 3.29 to dominate the term $|\langle |T_\Omega f|^p, w\rangle|$, and apply part II) of Theorem 3.27. We get

$$\begin{aligned} |\langle |T_\Omega f|^p, w\rangle|^{\frac{1}{p}} &\leq \kappa\,\|\Omega\|_{L^\infty} s' \Big(\sum_{Q\in\mathcal{S}} |Q| \langle f\rangle^p_Q \langle w\rangle_{s,Q} \Big)^{\frac{1}{p}} \\ &\leq \kappa_{n,p,q}\|\Omega\|_{L^\infty}\big([w]_{C_q}+1\big)^{1+\frac{2}{p}} \log^{\frac{1}{p}}\big([w]_{C_q}+e\big)\|Mf\|_{L^p(w)}. \end{aligned}$$

We now turn to the proof of part II) of Theorem 3.27. Suppose that $0 < p \leq 1$, $w \in C_q$ for some $q > 1$ and $\mathcal{S}$ is a sparse collection. We want to show that there exists $1 < s < \min\{2, \frac{1}{1-p}\}$ such that

$$(s')^p \sum_{Q\in\mathcal{S}} |Q| \langle |f|\rangle^p_Q \langle w\rangle_{s,Q} \leq \kappa_{w,p,q,n}\|Mf\|^p_{L^p(w)}.$$

We choose $s = 1 + p\delta$, where δ is the Reverse Hölder exponent from Theorem 2.21. Hence $s' \leq \kappa_n([w]_{C_q}+1)/p$ and we have

$$\begin{aligned} (s')^p \sum_{Q\in\mathcal{S}} |Q| \langle |f|\rangle^p_Q \langle w\rangle_{s,Q} &\leq \kappa \Big(\frac{[w]_{C_q}+1}{p} \Big)^p \sum_{Q\in\mathcal{S}} \langle |f|\rangle^p_Q \int M\chi_Q(x)^q w(x)dx \\ &\leq \kappa p^{-p}\big([w]_{C_q}+1\big)^{p+2} \log\big([w]_{C_q}+e\big)\|Mf\|^p_{L^p(w)}, \end{aligned}$$

where we have used Corollary 3.20 in the last step. The implicit constant $\kappa_{n,p,q}$ satisfies $\kappa_{n,p,q} \to \infty$ as $q \to p$ by the same arguments as in the end of Section 3.5.1. This completes the proof of Theorem 3.27.

4

Extensions of the John–Nirenberg theorem

In this chapter, we discuss the results that were published in the following work, as well as some further results.

[14] Canto, J., Pérez, C. Extensions of the John–Nirenberg theorem, Proceedings of the American Mathematical Society **149** (2021), no. 4, 1507–1525.

We want to state that, even though C_p weights appear in this chapter, they don't play an important role here. Therefore, constants will be denoted by lowercase c.

In this chapter, we introduce two extensions of the John–Nirenberg theorem. The first of these extensions is about Muckenhoupt–Wheeden-type estimates, whereas the second is a norm estimate for the quotient of the maximal function and the sharp maximal function. Both these extensions can be used to prove the John–Nirenberg Theorem, thus the term extension.

4.1 Exponential estimates

The John–Nirenberg theorem is a result about exponential integrability. We introduce a technique of obtaining this kind of integrability in terms of L^p-integrability. That is, Proposition 4.1 states that if one can control the L^p-norm of a function as a multiple of p, then the function will actually be exponentially integrable.

Proposition 4.1

Suppose that (X, μ) is a probability space and f a non-negative function such that for every $1 \le p < \infty$ we have the L^p bound

$$\left(\int_X f(x)^p d\mu(x) \right)^{\frac{1}{p}} \le \gamma\, p,$$

for some constant γ independent from p. Then $f \in \exp(L)(X, \mu)$, meaning

$$\mu(\{x \in X : f(x) > t\}) \le e^{-\frac{t}{4\gamma}}, \quad t > 0.$$

Proof. We compute

$$\int_X \left(\exp \frac{f(x)}{4\gamma} - 1 \right) d\mu(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_X \left(\frac{f(x)}{4\gamma} \right)^n d\mu(x) \le \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{n}{4} \right)^n \le 1.$$

Therefore,

$$\begin{aligned} \mu(\{x \in X : f(x) > t\}) &= \mu(\{x \in X : \frac{f(x)}{4\gamma} - \frac{t}{4\gamma} - \log 2 > \log 2\}) \\ &\le \int_X \left(\exp \left(\frac{f(x)}{3\gamma} - \frac{t}{4\gamma} - \log 2 \right) - 1 \right) d\mu(x) \\ &= 2e^{-\frac{t}{4\gamma}} \int_X \left(\exp \frac{f(x)}{4\gamma} - 1 \right) d\mu(x). \end{aligned}$$

□

Here we present a minimization lemma that we will use in the proofs of Theorems 4.19 and 4.15, as well as Proposition 4.6.

Lemma 4.2

Let $0 < \alpha < \infty$. Then

$$\min_{1<t<\infty} t \frac{t^\alpha}{t^\alpha - 1} \le e \left(1 + \frac{1}{\alpha} \right).$$

Proof. The function $\varphi(t) = t^{\alpha+1}(t^\alpha - 1)^{-1}$ tends to infinity at 1 and infinity. So, if the derivative vanishes at a unique point, that point has to be a global minimum. The derivative has the expression

$$\varphi'(t) = \frac{(\alpha+1)t^\alpha(t^\alpha - 1) - \alpha t^{2\alpha}}{(t^\alpha - 1)^2},$$

which vanishes only at $t = (\alpha + 1)^{\frac{1}{\alpha}}$. Therefore, the global minimum is

$$\varphi\big((\alpha+1)^{\frac{1}{\alpha}}\big) = (\alpha+1)^{\frac{1}{\alpha}} \frac{\alpha+1}{\alpha} \le e \left(1 + \frac{1}{\alpha} \right).$$

□

4.2 The John–Nirenberg theorem

The classical John–Nirenberg theorem [66] states that any function of bounded mean oscillation is locally exponentially integrable, see for example [40].

But before stating the theorem, let us recall what we mean by bounded mean oscillation.

Definition 4.3

Let $f \in L^1_{loc}(\mathbb{R}^n)$ be a function. We say that f is of bounded mean oscillation, and we wwrite $f \in \mathrm{BMO}$ if

$$\|f\|_{\mathrm{BMO}} = \sup_Q \fint_Q |f(x) - f_Q| dx \ < \ \infty.$$

The quantity $\|f\|_{\mathrm{BMO}}$ is called the BMO-seminorm of f

We are not going to expand on the BMO, but let us state a few facts.

Proposition 4.4

The following properties about BMO are true.

– $\|f\|_{\mathrm{BMO}} = 0$ if and only if f is constant almost everywhere.

– The space BMO is a Banach space modulo constants.

– $\|f\|_{\mathrm{BMO}} \leq 2\|f\|_{L^\infty}$.

– $\frac{1}{2}\fint_Q |f(x) - f_Q| dx \leq \inf_{c\in\mathbb{C}} \fint_Q |f(x) - c| dx \leq \fint_Q |f(x) - f_Q| dx$

The space BMO plays quite an important part on Harmonic Analyis. Among other applications, it serves as an adequate substitute for L^∞ in some cases. For example, singular integral operators map L^∞ into BMO. Also, even if BMO is weaker then L^∞, interpolation between L^p and BMO usually works just as well as interpolation between L^p and L^∞. For more information on the BMO space, we refer to [29, 40, 46].

One of the most relevant properties of the BMO spaces is the John–Nirenberg theorem. It can be seen as a self-improvement property of integrability of BMO functions, since in its definition, these functions are locally integrable but as a consequence of this theorem, they actually have much better integrable properties than that.

Theorem 4.5 – John–Nirenberg [66]

Let $f \in \mathrm{BMO}$ and Q a cube. Then, for some dimensional constant c_n,

$$|\{x \in Q : |f(x) - f_Q| > t\}| \leq 2\, e^{-\frac{c_n\, t}{\|f\|_{\mathrm{BMO}}}} |Q|.$$

Proof. We just need to combine Proposition 4.6 with the exponential estimate from Proposition 4.1. □

We are going to give a proof of the John–Nirenberg theorem that, despite its simplicity, seems to have been overlooked throughout literature. The proof consists of using the Calderón–Zygmund decomposition technique to bound the L^p-oscillations by a power of p and using Proposition 4.1 to deduce finally the exponential integrability. A similar proof can be found in [67].

Proposition 4.6
Let $f \in \mathrm{BMO}$. Then for every cube Q and $p \geq 1$,

$$\left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx\right)^{\frac{1}{p}} \leq c_n\, p\, \|f\|_{\mathrm{BMO}}.$$

Remark 4.7 Usually this result is presented as a corollary of the John–Nirenberg theorem. Therefore, they are actually equivalent.

Proof of Proposition 4.6. We may suppose $\|f\|_{\mathrm{BMO}} = 1$ by homogeneity. Let $L > 1$ to be chosen. We make the Calderón–Zygmund decomposition of $|f - f_Q|$ in Q at height L, see Section 1.4.3 for more details. We obtain a family $\{Q_j\}$ of dyadic subcubes of Q. These cubes are pairwise disjoint with respect to the property

$$L < \frac{1}{|Q_j|}\int_{Q_j} |f(x) - f_Q| dx \leq 2^n L.$$

Moreover, if $x \notin \cup_j Q_j$, then $|f(x) - f_Q| \leq L$.

Using the disjointness, we have for almost every $x \in Q$,

$$\begin{aligned} f(x) - f_Q &= (f(x) - f_Q)\chi_{(\cup_j Q_j)^c}(x) \\ &\quad + \sum_j (f_{Q_j} - f_Q)\chi_{Q_j}(x) \\ &\quad + \sum_j (f(x) - f_{Q_j})\chi_{Q_j}(x) \\ &= A_1(x) + A_2(x) + B(x). \end{aligned}$$

By the Calderón–Zygmund decomposition, we have $|A_1| \leq L$ and $|A_2| \leq 2^n L$ almost everywhere, so $|A_1 + A_2| \leq 2^n L$ since they have disjoint support. Now, for the remaining part, we compute the norm

$$\begin{aligned} \left(\frac{1}{|Q|}\int_Q |B(x)|^p\right)^{\frac{1}{p}} &= \left(\frac{1}{|Q|}\sum_j \int_{Q_j} |f(x) - f_{Q_j}|^p dx\right)^{\frac{1}{p}} \\ &\leq \sup_R \left(\frac{1}{|R|}\int_R |f(x) - f_R|^p dx\right)^{\frac{1}{p}} \left(\sum_j \frac{|Q|}{|Q_j|}\right)^{\frac{1}{p}} \\ &\leq \frac{X}{L^{\frac{1}{p}}}, \end{aligned}$$

where X equals the corresponding supremum, which is taken over all cubes R.

Combining the estimates for A_1, A_2 and B, we have

$$\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^p dx\right)^{\frac{1}{p}} \leq 2^n L + \frac{X}{L^{\frac{1}{p}}}. \tag{4.1}$$

Since (4.1) holds for all cubes Q and $L > 1$, the right hand side being independent from Q, we can take the supremum over all cubes Q and we get

$$X \leq 2^n L + \frac{X}{L^{\frac{1}{p}}}.$$

Passing the last term to the left, we get

$$X \leq 2^n \inf_{L>1} L(L^{\frac{1}{p}})' \leq 2^{n+1}\, e\, p,$$

where in the last inequality we used Lemma 4.2. But we can only do this if $X < \infty$, which a priori might not be true. Of course, one can use the John–Nirenberg theorem to prove $X \simeq_p \|f\|_{\mathrm{BMO}}$, but since we are providing an different proof of John–Nirenberg we cannot do that.

The way of avoiding this problem is by making a truncation of f at height $m > 0$. If f is a real function, call f_m the truncation

$$f(x) = \begin{cases} -m, & f(x) < -m \\ f(x), & -m \leq f(x) \leq m \\ m, & f(x) > m. \end{cases}$$

In the case that f is a complex-valued function, one can do a similar trick, using the argument and the modulus. More precisely,

$$f(x) = \begin{cases} f(x), & |f(x)| \leq m \\ m\, e^{i\arg(f(x))}, & |f(x)| > m. \end{cases}$$

In any case, it is easy to prove

$$\frac{1}{|Q|}\int_Q |f_m(x) - (f_m)_Q| dx \leq \frac{2}{|Q|}\int_Q |f(x) - f_Q| dx$$

If we work with the functions f_m instead of f, arguing in the same way

$$X_m \leq 2^n L + \frac{X_m}{L^{\frac{1}{p}}}.$$

But now, $X_m \leq 2m < \infty$, so the rest of the proof can continue. The last step is to let $m \to \infty$ with the help of Monotone Convergence. □

The rest of the chapter is devoted to providing two extensions of the classical John–Nirenberg theorem for BMO functions. The second extension is an improvement of some classical estimates by Muckenhoupt and Wheeden [102] concerning weighted local mean oscillations. These estimates were already discussed in the work bi Omprosi, Pérez, Rela and Rivera-Ríos [103] in a more restrictive setting. The first extension constitutes an improvement of a result of Karagulyan [70], which is in turn a more precise version of the classical Fefferman–Stein inequalities relating the Hardy–Littlewood and the sharp maximal functions.

These two extensions, although different a priori, are obtained by a similar method as the proof we gave for the John–Nirenber theorem, and also using some ideas from the work by Pérez and Rela [107].

4.3 First extension: Weighted mean oscillation

The first extension of the John–Nirenberg theorem we consider in this section is motivated by a classical result of Muckenhoupt and Wheeden in [102]. In order to state said result, we introduce, in the language of [102], weighted bounded mean oscillations.

Definition 4.8

Let w be a weight on $\mathbb{R}^n$. A function f is said to be of bounded mean oscillation with respect to w if there exists some $c > 0$ such that for every cube Q, the following holds

$$\int_Q |f(x) - f_Q| dx \leq c\, w(Q). \tag{4.2}$$

The class of functions that satisfies (4.2) is called weighted BMO and is denoted by BMO_w.

There are other definitions for weighted BMO spaces, for example one where the presence of the weight comes in both sides of inequality (4.2).

This class of functions is interesting because it is connected to the theory of weighted Hardy spaces [39] and to the context of extrapolation [52] (see more details in [25]).

Theorem 4.9 – Muckenhoupt–Wheeden [102]

Let $1 \leq p < \infty$ and $w \in A_p$. Then f is of bounded mean oscillation with weight w if and only if for every $1 \leq r < \infty$ satisfying $1 \leq r \leq p'$, there exists a constant $c > 0$ such that, for all cubes Q,

$$\int_Q |f(x) - f_Q|^r w(x)^{1-r} \leq c\, w(Q). \tag{4.3}$$

As was shown in [102], the range $1 \leq r \leq p'$ is optimal, since for any given $p > 1$ there exist f, w for which $w \in A_q$ for all $q > p$ but (4.3) fails for $r = p'$.

In [103] the authors obtained a mixed-type A_p–A_∞ quantitative estimate of inequality (4.3). Here we are going to improve Theorem 1.7 from that paper, using a simplified and more transparent argument that avoids completely the use of sparse domination.

In order to do that, we are going to introduce a bumped A_p class of weights and their corresponding bumped weighted BMO space. We remark that these objects are not standard and therefore, the notation we use is also not standard.

For a weight w, exponent $r > 1$ and for any cube Q, we define the bumped measure of the cube Q as

$$w_r(Q) = |Q| \left(\fint_Q w(x)^r dx \right)^{\frac{1}{r}} = |Q|^{\frac{1}{r'}} \left(\int_Q w(x)^r (x) \right)^{\frac{1}{r}}.$$

Remark 4.10 Please note that, even if we use the word *bumped measure*, the expression w_r is definitely not a measure, since the additivity property does not hold for disjoint cubes. Nevertheless, it constitutes a *bumped* version of the measure at a cube, that is, a bumped $w(Q)$. By Hölder's inequality, it is clear that $w(Q) \leq w_r(Q)$ for any $r \geq 1$.

We also need to define a local version of the dyadic maximal operator, see 1.4.2

Definition 4.11

Let Q be a cube and let $\mathcal{D}(Q)$ be the collection of dyadic descendants of Q. The dyadic maximal operator localized to Q is defined, for a function $h \in L^1(Q)$ and $x \in Q$, by the expression

$$M_Q h(x) = \sup_{\substack{P \in \mathcal{D}(Q) \\ x \in P}} \fint_P |h(y)| dy.$$

Definition 4.12

For a weight w and $p > 1, r \geq 1$, we define the following bumped A_p constant

$$[w]_{A_p^r} = \sup_Q \left(\fint_Q w(x)^r dx \right)^{\frac{1}{r}} \left(\fint_Q w(x)^{1-p'} dx \right)^{p-1}.$$

The class of weights w such that $[w]_{A_p^r}$ is finite is called A_p^r.

Note that $[w]_{A_p} \leq [w]_{A_p^r}$ for $r \geq 1$.

Definition 4.13

Let w be a weight that is positive almost everywhere and let $r > 1$. We define the space of bumped weighted bounded mean oscillations $\mathrm{BMO}_{w,r}$ as the set of functions f such that the quantity

$$\|f\|_{\mathrm{BMO}_{w,r}} = \sup_Q \frac{1}{w_r(Q)} \int_Q |f(x) - f_Q| dx$$

is finite, where the supremum is taken over all cubes Q.

Note that when $r = 1$, we have that both weighted spaces that we define are actually the same, that is, $\mathrm{BMO}_w = \mathrm{BMO}_{w,1}$.

Let us now state the first extension of the John–Nirenberg theorem.

Theorem 4.14

Let $p, r > 1$, w such that $[w]_{A_p^r} < \infty$ and let f locally integrable such that

$$\|f\|_{\mathrm{BMO}_{w,r}} < \infty.$$

Then we have the estimate

$$\left(\frac{1}{w_r(Q)} \int_Q \left(\frac{|f(x) - f_Q|}{w(x)} \right)^{p'} w(x) dx \right)^{\frac{1}{p'}} \leq c_n \, p' \, [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}} \|f\|_{\mathrm{BMO}_{w,r}}.$$

Proof. We may suppose by homogeneity that $\|f\|_{\mathrm{BMO}_{w,r}} = 1$. We are going to use a decomposition that is similar to the Calderón–Zygmund decomposition, see Section 1.4.3. If we let $L > 1$, to be chosen later, we can choose a family of maximal subcubes $\{Q_j\}$ in Q such that

$$\frac{1}{w_r(Q_j)} \int_{Q_j} |f(x) - f_Q| dx > L. \tag{4.4}$$

Observe that if the family is empty we can see that $|f(x) - f_Q| \leq Lw(x)$ for almost every $x \in Q$ and the result is trivial. Also since $\|f\|_{\mathrm{BMO}_{w,r}} = 1$, we have that Q is not one of the selected cubes. We can check that, if Q'_j denotes the ancestor of Q_j, the following properties hold:

(i) $\displaystyle\frac{1}{w_r(Q'_j)} \int_{Q'_j} |f(x) - f_Q|\, dx \leq L;$

(ii) $\displaystyle|f_{Q_j} - f_Q| \leq 2^n\, L \left(\fint_{Q'_j} w(x)^r dx \right)^{\frac{1}{r}};$

(iii) $\displaystyle\sum_j w_r(Q_j) \leq \frac{w_r(Q)}{L}$ because of (4.4) and $\|f\|_{\mathrm{BMO}_{w,r}} = 1;$

(iv) $|f(x) - f_Q| \leq L\, w(x)$ for almost every $x \notin \cup_j Q_j$.

Using the fact that the cubes $\{Q_j\}$ are pairwise disjoint, we have for almost every $x \in Q$,

$$\begin{aligned} f(x) - f_Q &= (f(x) - f_Q)\chi_{(\cup_j Q_j)^c}(x) \\ &\quad + \sum_j (f_{Q_j} - f_Q)\chi_{Q_j}(x) \\ &\quad + \sum_j (f(x) - f_{Q_j})\chi_{Q_j}(x) \\ &= A_1(x) + A_2(x) + B(x). \end{aligned}$$

Since $p' > 1$, we can use the triangular inequality to get

$$\begin{aligned} &\left(\frac{1}{w_r(Q)} \int_Q \left(\frac{|f(x) - f_Q|}{w(x)} \right)^{p'} w(x)dx \right)^{\frac{1}{p'}} \\ &\qquad \leq \left(\frac{1}{w_r(Q)} \int_{(\cup_j Q_j)^c} \left(\frac{|f(x) - f_Q|}{w(x)} \right)^{p'} w(x)dx \right)^{\frac{1}{p'}} \\ &\qquad\quad + \left(\frac{1}{w_r(Q)} \sum_j \int_{Q_j} \left(\frac{|f_{Q_j} - f_Q|}{w(x)} \right)^{p'} w(x)dx \right)^{\frac{1}{p'}} \\ &\qquad\quad + \left(\frac{1}{w_r(Q)} \sum_j \int_{Q_j} \left(\frac{|f(x) - f_{Q_j}|}{w(x)} \right)^{p'} w(x)dx \right)^{\frac{1}{p'}} \\ &\qquad = A_1 + A_2 + B. \end{aligned}$$

Now, since $w(Q) \leq w_r(Q)$ the first term is $A_1 \leq L$, by (iv). To bound B we denote

$$X = \sup_R \left(\frac{1}{w_r(R)} \int_R \left(\frac{|f(x) - f_R|}{w(x)} \right)^{p'} w(x) \right)^{\frac{1}{p'}}.$$

and use that $\sum_j w_r(Q_j) \leq \frac{w_r(Q)}{L}$, the third property of the family of the cubes $\{Q_j\}$, to obtain:

$$B \leq X \left(\frac{1}{w_r(Q)} \sum_j w_r(Q_j) \right)^{\frac{1}{p'}} \leq X \left(\frac{1}{L} \right)^{\frac{1}{p'}}.$$

The argument for bounding A_2 is more delicate. We start the computations:

$$\begin{aligned}
A_2 &= \left(\frac{1}{w_r(Q)} \sum_j \int_{Q_j} |f_{Q_j} - f_Q|^{p'} w(x)^{p'-1} dx \right)^{\frac{1}{p'}} \\
&\leq 2^n L \left(\frac{1}{w_r(Q)} \sum_j \left(\frac{1}{|Q'_j|} \int_{Q'_j} w(x)^r dx \right)^{\frac{p'}{r}} \int_{Q_j} w(x)^{p'-1} dx \right)^{\frac{1}{p'}} \\
&\leq 2^n L \left(\frac{1}{w_r(Q)} \sum_j w_r(Q'_j) \left(\frac{1}{|Q'_j|} \int_{Q'_j} w^r \right)^{\frac{p'-1}{r}} \left(\frac{1}{|Q'_j|} \int_{Q_j} w^{p'-1} \right)^{(p'-1)(p-1)} \right)^{\frac{1}{p'}} \\
&\leq 2^n L [w]_{A_p^r}^{\frac{1}{p}} \left(\frac{1}{w_r(Q)} \sum_j |Q'_j| \left(\frac{1}{|Q'_j|} \int_{Q'_j} w(x)^r dx \right)^{\frac{1}{r}} \right)^{\frac{1}{p'}}.
\end{aligned}$$

In order to bound the term in the sum, we recall the following result by Kolmogorov. If (X, μ) is a probability space, then for $\varepsilon < 1$

$$\|g\|_{L^\varepsilon(X)} \leq \left(\frac{1}{1 - \varepsilon} \right)^{\frac{1}{\varepsilon}} \|g\|_{L^{1,\infty}(X)}.$$

We have

$$\begin{aligned}
\sum_j |Q'_j| \left(\frac{1}{|Q'_j|} \int_{Q'_j} w(x)^r dx \right)^{\frac{1}{r}} &\leq 2^n \sum_j |Q_j| \inf_{z \in Q_j} M_Q(w^r \chi_Q)(z)^{\frac{1}{r}} \\
&\leq 2^n |Q| \fint_Q M_Q(w^r \chi_Q)(x)^{\frac{1}{r}} dx \\
&\leq 2^n \frac{1}{1 - \frac{1}{r}} \|M_Q(w^r \chi_Q)\|_{L^{1,\infty}(Q, \frac{dx}{|Q|})}^{\frac{1}{r}} |Q| \\
&\leq 2^n r' |Q| \left(\fint_Q w(x)^r dx \right)^{\frac{1}{r}} \\
&= 2^n \, r' \, w_r(Q),
\end{aligned}$$

where M_Q is the local dyadic maximal operator over Q as in Definition 4.11, whose weak type $(1,1)$ bound is one. Thus, we have the bound

$$A_2 \leq 2^n \, [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}} \, L.$$

Combining the bounds for A_1, A_2 and B, we have for every cube Q and $L > 1$

$$\left(\frac{1}{w_r(Q)} \int_Q \left(\frac{|f(x) - f_Q|}{w(x)} \right)^{p'} w(x) dx \right)^{\frac{1}{p'}} \leq L + 2^n \, [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}} \, L + X \left(\frac{1}{L} \right)^{\frac{1}{p'}} \tag{4.5}$$

and thus for each L

$$X \leq 2^{n+1} \, [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}} \, L + X \left(\frac{1}{L} \right)^{\frac{1}{p'}}.$$

Hence, if we assume $X < \infty$,

$$X \leq c_n \, p' [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}}.$$

This finishes the proof in the case that $X < \infty$. In order to remove the hypothesis $X < \infty$, it is enough to replace first for each cube Q

$$\left(\frac{1}{w_r(Q)} \int_Q \left(\frac{|f(x) - f_Q|}{w(x)} \right)^{p'} w(x) dx \right)^{\frac{1}{p'}}$$

by

$$\left(\frac{1}{w_r(Q)} \int_Q \min \left\{ \frac{|f(x) - f_Q|}{w(x)}, m \right\}^{p'} w(x) dx \right)^{\frac{1}{p'}}.$$

The argument done before works exactly to get the following variant of (4.5): For every $L > 1$ and $m \geq 1$,

$$\left(\frac{1}{w_r(Q)} \int_Q \min \left\{ \frac{|f(x) - f_Q|}{w(x)}, m \right\}^{p'} w(x) dx \right)^{\frac{1}{p'}} \leq L + 2^n \, [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}} \, L + X_m \left(\frac{1}{L} \right)^{\frac{1}{p'}},$$

where now, instead of X we have X_m defined by:

$$X_m := \sup_{Q \in \mathcal{D}} \left(\frac{1}{w_r(Q)} \int_Q \min \left\{ \frac{|f(x) - f_Q|}{w(x)}, m \right\}^{p'} w(x) dx \right)^{\frac{1}{p'}} \qquad m \geq 1.$$

Then,

$$X_m \leq 2^{n+1} \, [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}} \, L + X_m \left(\frac{1}{L} \right)^{\frac{1}{p'}} \qquad L > 1, \; m \geq 1.$$

Therefore, since $X_m \leq m$ we have

$$X_m \leq c_n \, p' [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}} \qquad m \geq 1.$$

Hence for each cube Q

$$\left(\frac{1}{w_r(Q)} \int_Q \min \left\{ \frac{|f(x) - f_Q|}{w(x)}, m \right\}^{p'} dxwdx \right)^{\frac{1}{p'}} \leq c_n \, p' [w]_{A_p^r}^{\frac{1}{p}} (r')^{\frac{1}{p'}} \qquad m \geq 1.$$

Finally, let $m \to \infty$ to finish the proof. □

If the weight belongs to one of the Muckenhoupt classes A_p, we know that it satisfies a Reverse Hölder inequality, so for $r > 1$ close enough to 1 (but depending on the weight itself), we actually have

$$[w]_{A_p^r} \leq c_w\, [w]_{A_p}.$$

The following Corollary illustrates this situation with more detail.

Corollary 4.15

Let $f \in \mathrm{BMO}_{w,1}$. The following statements hold.

(i) If $w \in A_1$ we have that for every $q > 1$,

$$\left(\frac{1}{w(Q)} \int_Q \left| \frac{f(x) - f_Q}{w(x)} \right|^q w(x)dx \right)^{\frac{1}{q}} \leq c_n\, q\, [w]_{A_1}^{\frac{1}{q'}}\, [w]_{A_\infty}^{\frac{1}{q}}\, \|f\|_{\mathrm{BMO}_{w,1}},$$

and hence for any cube Q

$$\left\| \frac{f - f_Q}{w} \right\|_{\exp L\left(Q, \frac{w(x)dx}{w(Q)}\right)} \leq c_n\, [w]_{A_1}\, \|f\|_{\mathrm{BMO}_{w,1}}. \tag{4.6}$$

(ii) If $w \in A_p$ with $1 < p < \infty$ then,

$$\left(\frac{1}{w(Q)} \int_Q \left| \frac{f(x) - f_Q}{w(x)} \right|^{p'} w(x)dx \right)^{\frac{1}{p'}} \leq c_n\, p'\, [w]_{A_p}^{\frac{1}{p}}\, [w]_{A_\infty}^{\frac{1}{p'}}\, \|f\|_{\mathrm{BMO}_{w,1}}.$$

Proof. Part (1) follows from part (2) since $[w]_{A_1} \geq [w]_{A_p}$, $p > 1$. In order to prove part (2), choose $r = 1 + \delta$ with δ as in Theorem 2.5. This way $[w]_{A_p^r} \leq 2[w]_{A_p}$, $r' \leq c_n [w]_{A_\infty}$ and $w_r(Q) \leq 2w(Q)$. The result follows from Theorem 4.14. □

Remark 4.16 From (4.6), we can deduce a weighted John–Nirenberg-type estimate for BMO_w. That is, if a weight $w \in A_1$, then for any $t > 0$ and any cube Q, the following holds.

$$w(\{x \in Q : |f(x) - f_Q| > t\, w(x)\}) \leq 2e^{-\frac{t}{c_n [w]_{A_1} \|f\|_{\mathrm{BMO}_{w,1}}}}\, w(Q).$$

If the weight is actually the Lebesgue measure, this is precisely the John–Nirenberg theorem. Therefore, Theorem 4.14 can be seen as an extension of John–Nirenberg.

4.4 Second extension: an inequality of Karagulyan's

The second extension is motivated by the work of Karagulyan [70], who already provided an extension of the John–Nirenberg theorem. We improve this interesting result by providing a different more flexible proof with several different advantages. However, this first extension is also inspired by the work of Pérez and Rela [107], where a different approach to the main results from the work of Fabes Kenig and Serapioni [35] concerning degenerate Poincaré–Sobolev inequalities is found.

We obtain two different consequences of this improvement of the John–Nirenberg theorem. Firstly, we derive some degenerate Poincaré–Sobolev endpoint inequalities not available from the methods in [107]. Secondly, this improvement will be applied within the context of the C_p class of weights.

To establish this result we recall the sharp maximal function introduced by Fefferman and Stein.

Definition 4.17

Let $h \in L^1_{loc(\mathbb{R}^n)}$. The sharp maximal function of h is defined by the expression

$$M^\sharp h(x) = \sup_{x\in R} \frac{1}{|R|} \int_R |h(y) - h_R|dy,$$

where the supremum is taken over all cubes R that contain the point x.

Karagulyan proved in [70] the following interesting exponential decay. Although his work uses balls instead of cubes as a basis of differentiation, both resulting objects are equivalent.

Proposition 4.18 – Karagulyan [70]

Let $f \in L^1_{loc}$ and let B a ball in $\mathbb{R}^n$, then

$$|\{x \in B : \frac{|f(x) - f_B|}{M^\sharp f(x)} > \lambda\}| \leq \, c_n \, e^{-c_n \, \lambda}|B|.$$

The first main result of this chapter, Theorem 4.19, improves this exponential decay in several ways. On one hand, we have the decay for the local maximal function and on the other hand, we obtain weighted estimates. The method of proof is different from that in[70].

Let us now state our generalization of Proposition 4.18 and the second extension of the John–Nirenberg theorem.

Theorem 4.19

Let f be a locally integrable function. Then for any cube Q, for any $1 \leq p < \infty$ and $1 < r < \infty$, the following estimate holds

$$\left(\frac{1}{w_r(Q)} \int_Q \left(\frac{M_Q(f - f_Q)(x)}{M^\sharp f(x)}\right)^p w(x)dx\right)^{\frac{1}{p}} \leq c_n \, pr'. \tag{4.7}$$

Proof. Fix a cube Q. We make the local Calderón–Zygmund decomposition in the cube Q of the function

$$F(x) = \frac{|f(x) - f_Q|}{\operatorname{osc}(f, Q)}$$

at height $\lambda > 1$ to be precised later. We have used the notation

$$\operatorname{osc}(f, Q) = ⨍_Q |f(y) - f_Q|dy.$$

More precisely, we choose the dyadic subcubes $\{Q_j\}$ of Q, maximal for the inclusion among the cubes R that satisfy $⨍_R F(x)dx > \lambda$. The cubes $\{Q_j\}$ are pairwise disjoint and satisfy the following properties:

- $\operatorname{osc}(f, Q)\lambda < \fint_{Q_j} |f(y) - f_Q| dy \le 2^n \lambda \operatorname{osc}(f, Q)$,
- $\lambda \sum_j |Q_j| \le |Q|$,
- For $x \notin \bigcup_j Q_j$, $\quad M_Q(f - f_Q)(x) \le \lambda \operatorname{osc}(f, Q)$.

The first two properties follow from the stopping time and the maximality. To prove the third one, note that $\fint_R |f(y) - f_Q| dy / \operatorname{osc}(f, Q) \le \lambda$ for all dyadic R that contains x.

Now, by maximality of the cubes, for $x \in Q_j$ we can localize the maximal function in the following way

$$M_Q(f - f_Q)(x) = M_{Q_j}(f - f_Q)(x) \le M_{Q_j}(f - f_{Q_j})(x) + |f_Q - f_{Q_j}|. \tag{4.8}$$

Moreover, for $x \in Q_j$, we have

$$\frac{|f_Q - f_{Q_j}|}{M^\sharp f(x)} = \frac{\left|\fint_{Q_j} (f(y) - f_Q) dy\right|}{M^\sharp f(x)} \le \frac{\fint_{Q_j} |f(y) - f_Q| dy}{\operatorname{osc}(f, Q)} \le 2^n \lambda, \tag{4.9}$$

by the Calderón–Zygmund decomposition. Thus, we have found the following pointwise bound, for a.e. $x \in Q$,

$$\begin{aligned}
\frac{M_Q(f - f_Q)(x)}{M^\sharp f(x)} &= \frac{M_Q(f - f_Q)(x)}{M^\sharp f(x)} \chi_{Q \setminus \cup_j Q_j}(x) + \sum_j \frac{M_Q(f - f_Q)(x)}{M^\sharp f(x)} \chi_{Q_j}(x) \\
&\le \lambda \chi_{Q \setminus \cup_j Q_j}(x) + \sum_j \left(\frac{M_{Q_j}(f - f_{Q_j})(x)}{M^\sharp f(x)} + \frac{|f_Q - f_{Q_j}|}{M^\sharp f(x)} \right) \chi_{Q_j}(x) \\
&\le \lambda \chi_{Q \setminus \cup_j Q_j}(x) + \sum_j \left(\frac{M_{Q_j}(f - f_{Q_j})(x)}{M^\sharp f(x)} + 2^n \lambda \right) \chi_{Q_j}(x) \\
&\le 2^n \lambda + \sum_j \frac{M_{Q_j}(f - f_{Q_j})(x)}{M^\sharp f(x)} \chi_{Q_j}(x).
\end{aligned}$$

We have used (4.8) and (4.9) in the first and second inequalities respectively.

Now we compute the norm. Using the triangular inequality, Jensen's inequality and the fact that the Q_j are pairwise disjoint, we get

$$\begin{aligned}
&\left(\frac{1}{w_r(Q)} \int_Q \left(\frac{M_Q(f - f_Q)(x)}{M^\sharp f(x)} \right)^p w(x) dx \right)^{\frac{1}{p}} \\
&\qquad \le 2^n \lambda + \left(\sum_j \frac{w_r(Q_j)}{w_r(Q)} \frac{1}{w_r(Q_j)} \int_Q \left(\frac{M_{Q_j}(f - f_{Q_j})(x)}{M^\sharp f(x)} \right)^p w(x) dx \right)^{\frac{1}{p}} \\
&\qquad \le 2^n \lambda + X \left(\frac{1}{w_r(Q)} \sum_j w_r(Q_j) \right)^{\frac{1}{p}} \\
&\qquad \le 2^n \lambda + \frac{X}{\lambda^{\frac{1}{pr'}}}.
\end{aligned}$$

We have used that, by Hölder's and one of the main properties of the family Q_j,

$$\sum_j w_r(Q_j) \le w_r(Q) \left(\frac{1}{\lambda}\right)^{\frac{1}{r'}},$$

and we have set

$$X = \sup_{R\in\mathcal{D}} \left(\frac{1}{w_r(R)} \int_R \left(\frac{M_R(f-f_R)(x)}{M^\sharp f(x)}\right)^p w(x)dx\right)^{\frac{1}{p}}.$$

Now take the supremum over all dyadic cubes Q and obtain, for arbitrary $\lambda > 1$,

$$X \le 2^n\lambda + \frac{X}{\lambda^{\frac{1}{pr'}}}.$$

This in turn implies, if we assume $X < \infty$, that

$$X \le 2^n\lambda \frac{\lambda^{\frac{1}{pr'}}}{\lambda^{\frac{1}{pr'}} - 1}.$$

Applying Lemma 4.2, we see $X \le c_n\, pr'$, since $\lambda > 1$ was free. This finishes the proof if we assume that $X < \infty$.

In order to remove the hypothesis $X < \infty$, we argue as follows. Let $K > 0$ large and $\varepsilon > 0$ small. It is enough to work with

$$X_{\varepsilon,K} := \sup_{Q\in\mathcal{D}} \left(\frac{1}{w_r(Q)} \int_Q \left(\frac{M_Q(f_K-(f_K)_Q)(x)}{M^\sharp f_K(x)+\varepsilon}\right)^p w(x)dx\right)^{\frac{1}{p}} \le 2\frac{K}{\varepsilon} < \infty$$

for a suitable truncation f_K of f at height K. For example, one can take

$$f_K(x) = \begin{cases} -K, & f(x) < -K,\\ f(x), & -K \le f(x) \le K,\\ K, & K < f(x).\end{cases}$$

Making the same computations as above with some trivial changes, we can obtain the bounds for $X_{\varepsilon,K}$ independently of ε and K. Finally, monotone convergence finishes the argument, by letting $K \to \infty$ and $\varepsilon \to 0$. □

Remark 4.20 Since throughout the proof the only cubes that appear are dyadic descendants of Q, we actually obtain the stronger estimate

$$\left(\frac{1}{w_r(Q)} \int_Q \left(\frac{M_Q(f-f_Q)(x)}{M_Q^\sharp f(x)}\right)^p w(x)dx\right)^{\frac{1}{p}} \le c_n\, pr',$$

where $M_Q^\sharp$ is the sharp operator taking the supremum over dyadic descendants of Q. Since $M_Q^\sharp \le M^\sharp$, this last estimate is stronger.

Remark 4.21 We remark that the corresponding result replacing the L^p norm by the (larger) Lorentz norm $L^{p,q}$ with $1 \le q < p$ cannot be proved even in the simplest situation $w = 1$ and without M. buscar referencia

Remark 4.22 We also remark that the factor p in (4.7) (or (4.10)) it is crucial since it yields the exponential type result as follows.

4.4.1 When the weight is A_∞

Even though Theorem 4.19 holds for all positive weights, it takes a more interesting form when applied to A_∞ weights. This is because, since these weights satisfy a Reverse Hölder inequality, see Theorem 2.5, the bumped measure $w_r(Q)$ is bounded by $w(Q)$.

Corollary 4.23

Let $w \in A_\infty$ and let also $f \in L^1_{loc}(\mathbb{R}^n)$. For any cube Q and $1 < p < \infty$, the following holds

$$\left(\frac{1}{w(Q)} \int_Q \left(\frac{M_Q(f - f_Q)(x)}{M^\sharp f(x)} \right)^p w(x)dx \right)^{\frac{1}{p}} \leq c_n\, p\, [w]_{A_\infty}. \tag{4.10}$$

Proof. Since $w \in A_\infty$, we choose $r = 1 + \delta$ with δ as in Theorem 2.5. This way, $w_r(Q) \leq 2w(Q)$ and $r' \leq c_n[w]_{A_\infty}$. From this observation, applying Theorem 4.19, inequality (4.10) follows. □

Corollary 4.24

For a weight $w \in A_\infty$ and $f \in L^1_{loc}(\mathbb{R}^n)$, the following John–Nirenberg-type estimate holds.

$$\left\| \frac{M_Q(f - f_Q)}{M^\sharp f} \right\|_{\exp L(Q, \frac{w dx}{w(Q)})} \leq c_n\, [w]_{A_\infty}$$

This means that there exist dimensional constants $c_1, c_2 > 0$ such that

$$w\left(\left\{ x \in Q : \frac{M_Q(f - f_Q)(x)}{M^\sharp f(x)} > t \right\} \right) \leq c_1\, e^{-c_2\, t/[w]_{A_\infty}}\, w(Q), \quad t > 0.$$

Proof. Apply Proposition 4.1 to (4.10). □

We call this result improved John–Nirenberg estimate because if $w = 1$ and $f \in$ BMO, then $M^\sharp f(x) \leq \|f\|_{\text{BMO}}$ for a.e. x and, therefore,

$$|\{x \in Q : M_Q(f - f_Q) > t\}| \leq c_1\, e^{\frac{-c_2\, t}{\|f\|_{\text{BMO}}}}\, |Q|, \quad t > 0.$$

This implies the John–Nirenberg Theorem by Lebesgue differentiation theorem, because $M_Q(f - f_Q) \geq f - f_Q$ a.e. in Q.

Corollary 4.25

For every cube and $\lambda, \gamma > 0$ we have the following good-λ type inequality

$$w\big(\{x \in Q : M_Q(f - f_Q) > \lambda, M^\sharp f(x) \leq \gamma\lambda\}\big) \leq c_1\, e^{\frac{-c_2}{\gamma[w]_{A_\infty}}}\, w(Q).$$

4.5 Generalized Poincaré inequalities

As an application of Theorem 4.19, we improve the main result in [38], at least in the simplest situation of cubes, which at the same time provides a limiting result that could not be treated in Theorem 1.14 of [107].

Let w be an A_∞ weight and let a be a functional over cubes of $\mathbb{R}^n$. By that we mean a real-valued mapping defined over the set of cubes in $\mathbb{R}^n$. We will assume that a satisfies the $D_r(w)$ condition for some $r > 1$ as introduced in [38].

Definition 4.26

Let a be a functional over cubes and let w be a weight. We say that a satisfies the $D_r(w)$ condition, and we write $a \in D_r(w)$ if for every cube Q and every collection Λ of pairwise disjoint subcubes of Q, the following inequality holds:

$$\sum_{P\in\Lambda} w(P)\, a(P)^r \le \|a\|^r w(Q)a(Q)^r, \tag{4.11}$$

for some constant $\|a\| > 0$ that plays the role of the “norm" of a.

These kind of functionals were studied in relation with self improvement properties of generalized Poincaré inequalities in [38], further studied in [93] and more recently improved in [107]. We establish now an new endpoint result in the spirit of Theorem 1.14 in [107] which was missing since Theorem 4.19 was not available.

Theorem 4.27

Let $w \in A_\infty$ and a a functional satisfying $D_r(w)$ condition (4.11) for some $r > 1$. Let f be a locally integrable function such that for every cube Q,

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|dx \le a(Q). \tag{4.12}$$

Then, for every cube Q,

$$\|f - f_Q\|_{L^{r,\infty}\left(Q,\frac{w}{w(Q)}\right)} \le c_n\, r\, [w]_{A_\infty}\, \|a\|\, a(Q).$$

Remark 4.28 The method in [38], based on the good-λ method of Burkholder–Gundy [9], yields an exponential bound in $[w]_{A_\infty}$. We still use here the good-λ method but we use instead Corollary 4.25.

Proof of Theorem 4.27. Fix a cube Q. We have to prove that for every $t > 0$,

$$t^r\, w(\{x \in Q : |f(x) - f_Q| > t\}) \le (c_n\|a\|[w]_{A_\infty})^r\, a(Q)^r\, w(Q),$$

with c_n independent from everything but the dimension.

M_Q will denote the dyadic maximal operator localized in Q. Since $(f - f_Q) \le M_Q(f - f_Q)$ almost everywhere, we can just estimate the bigger set

$$\Omega_t = \{x \in Q : M_Q(f - f_Q)(x) > t\}.$$

Let Q_j be the maximal cubes that form Ω_t. They can be found by the Calderón–Zygmund decomposition, see Section 1.4.3. Let $q = 2^n + 1$ as in [107], and let us make the same computations that they do . We arrive to

$$w(\Omega_{qt}) \leq \sum_j w(E_{Q_j}),$$

where

$$\begin{aligned} E_{Q_j} &= \{x \in Q_j : M_Q(f - f_{Q_j})(x) > t\} \\ &= \{x \in Q_j : M_{Q_j}(f - f_{Q_j})(x) > t\}, \end{aligned}$$

by the maximality of the cubes Q_j. Now we will use the good-λ from Corollary 4.25. We use the version with the dyadic sharp maximal function in Remark 4.20. Let $\gamma > 0$ to be chosen later. Then

$$\begin{aligned} E_{Q_j} &\subseteq \{x \in Q_j : M_{Q_j}(f - f_{Q_j})(x) > t,\, M_d^\sharp f(x) \leq \gamma t\} \bigcup \{x \in Q_j : M_d^\sharp f(x) > \gamma t\} \\ &= A_j \bigcup B_j \end{aligned}$$

and therefore

$$w(E_{Q_j}) \leq w(A_j) + w(B_j).$$

For A_j sets, let $s > 1$ be the exponent for the Reverse Hölder inequality for $w \in A_\infty$ as in Theorem 2.5. Then, using Corollary 4.25, we have

$$\sum_j w(A_j) \leq c_1 e^{-\frac{c_2}{s\gamma}} \sum_j w(Q_j) = c_1 e^{-\frac{c_2}{s\gamma}} w(\Omega_t).$$

Remember that $c_1, c_2 > 0$ are dimensional constants. On the other hand, for B_j we can argue as follows. We have

$$\bigcup_j B_j \subseteq \{x \in Q : M_d^\sharp f(x) > \gamma t\} = \bigcup_i R_i,$$

where R_i are the maximal dyadic subcubes of Q such that

$$\gamma t < \frac{1}{|R_i|} \int_{R_i} |f(x) - f_{R_i}| dx.$$

Now, using the starting point (4.12), we clearly have

$$\gamma t \leq a(R_i).$$

Therefore, using that a satisfies the $D_r(w)$ condition, we have

$$\begin{aligned} \sum_j w(B_j) &\leq w\big(\{x \in Q : M_d^\sharp f(x) > \gamma t\}\big) \\ &= \sum_i w(R_i) \\ &\leq \left(\frac{1}{\gamma t}\right)^r \sum_i w(R_i) a(R_i)^r \\ &\leq \|a\|^r \left(\frac{1}{\gamma t}\right)^r w(Q) a(Q)^r. \end{aligned}$$

Now, if we put everything together, we get

$$(qt)^r w(\Omega_{qt}) \leq c_1 (tq)^r e^{-\frac{c_2}{\gamma s}} w(\Omega_t) + \left(q\frac{\|a\|}{\gamma}\right)^r w(Q)a(Q)^r.$$

Since we have qt on the left and t on the right, we define the function

$$\varphi(N) = \sup_{0<t\leq N} t^r w(\Omega_t).$$

This function is increasing, so we have

$$\varphi(N) \leq \varphi(Nq) \leq c_1 q^r e^{-\frac{c_2}{\gamma s}} \varphi(N) + \left(q\frac{\|a\|}{\gamma}\right)^r w(Q)a(Q)^r.$$

The parameter γ is free, and we make the choice so that

$$c_1 q^r e^{-\frac{c_2}{s\gamma}} = \frac{1}{2},$$

which means

$$\gamma = \frac{c_n}{r[w]_{A_\infty}}.$$

This yields the result, since $\|f - f_Q\|_{L^{r,\infty}(Q,w)} \leq \sup_N \varphi(N)$. □

4.6 Application to C_p weights: Fefferman–Stein inequality

As a second application of Theorem 4.19 we provide an improvement of theorem of Yabuta [117] concerning a classical inequality of Feffereman–Stein relating the Hardy-Littlewood maximal function M and the sharp maximal function $M^\sharp$ introduced by them in [37].

This inequality, first proved for A_∞ weights, is deeply related to the theory of C_p weights. The situation is very similar to the Coifman–Fefferman inequality that was described in Chapters 2 and 3.

Theorem 4.29 – Yabuta [117]

Let w be a weight such that the following inequality holds for all $f \in L^\infty$ with compact support,

$$\|f\|_{L^p(w)} \leq c\,\|M^\sharp f\|_{L^p(w)},$$

for a fixed constant $c > 0$. Then $w \in C_p$.

Conversely, suppose that $w \in C_q$ for some $1 < p < q < \infty$. Then, there exists a constant $c = c_{w,q,p} > 0$ such that for all $f \in L^\infty$ with compact support,

$$\|Mf\|_{L^p(w)} \leq c\,\|M^\sharp f\|_{L^p(w)}. \tag{4.13}$$

One could make a conjecture in the spirit of Muckenhoupt's conjecture 2.1, stating that $w \in C_p$ is the correct sufficient condition for (4.13) to hold.

In this line, Lerner [85] proved a characterization of weights satisfying a weak Fefferman–Stein inequality

$$\|f\|_{L^{p,\infty}(w)} \leq C\|M^\sharp f\|_{L^p(w)}.$$

The weights satisfying this inequality are of a different class of weights, called SC_p (strong C_p). This class is contained in C_p and contains $C_{p+\varepsilon}$ for every $\varepsilon > 0$.

As a consequence of Theorem 4.19, we are able to give a nice quantitative version of Yabuta's inequality, since the good-λ with exponential decay between the sharp maximal function and the Hardy–Littlewood maximal function was not available to us before.

Theorem 4.30 – Quantitative Fefferman–Stein for C_p weights

Let $1 < p < q < \infty$ and $w \in C_q$. Then for any $f \in L^\infty_c(\mathbb{R}^n)$ we have

$$\|Mf\|_{L^p(w)} \leq c_n \frac{pq}{q-p}(1+[w]_{C_q})\log(e+[w]_{C_q})\|M^\sharp f\|_{L^p(w)},$$

where the constant c_n only depends on n.

Remark 4.31 We remark that, as a consequence of Corollary 4.25, we can also obtain the following weighted inequality for A_∞ weights by standard arguments:

$$\|Mf\|_{L^p(w)} \leq c\,[w]_{A_\infty}\, \|M^\sharp f\|_{L^p(w)}, \quad 0 < p < \infty.$$

This inequality is not new, see for example [82].

Theorem 4.30 has a straight application to the wide class of operators described in [17]. Indeed, we say that an operator satisfies the (D) property if there are some constants $\delta \in (0,1)$ and $c_T > 0$ such that for all f,

$$M^\sharp_\delta(Tf)(x) \leq c_T Mf(x), \quad a.e.\ x. \tag{D}$$

Here M denotes the standard Hardy–Littlewood maximal operator and we use the notation $M^\sharp_\delta f = M^\sharp(f^\delta)^{\frac{1}{\delta}}$. This property is modeled by a result in [1] where (D) was proved for any Calderón–Zygmund operator. It also holds for some square function operators and some pseudo-differential operators. The version for multilinear Calderón–Zygmund operators was obtained in [87]. There is a more exhaustive list in [17].

Corollary 4.32

Let $1 < p < q < \infty$ and T be an operator that satisfies the property (D) with constant c_T for some $\frac{p}{q} < \delta < 1$. Then for $w \in C_q$ we have

$$\|Tf\|_{L^p(w)} \leq c_n\, c_T\, \left(\frac{pq}{\delta q - p}\ \max(1, [w]_{C_q}\log^+[w]_{C_q})\right)^{\frac{1}{\delta}} \|Mf\|_{L^p(w)}.$$

The rest of this section is devoted to proving Theorem 4.30 and Corollary 4.32. We are going to use the improved John–Nirenberg Theorem 4.19 to give a quantitative version of Theorem II in [117].

First, we need to obtain a non-dyadic unweighted version of Corollary 4.25. That is, a version of it in which the maximal operator inside actually is the

Hardy–Littlewood one and not the one that only takes into account the dyadic descendants.

Theorem 4.33

Let Q be an arbitrary cube and f a locally integrable function, non constant on Q. Then for any $\lambda > 0$ we have

$$\left|\left\{x \in Q : \frac{M\big((f - f_Q)\chi_Q\big)(x)}{M^\sharp f(x)} > \lambda\right\}\right| \leq c\, e^{-c\lambda}|Q|,$$

where $c > 0$ is a dimensional constant. Here M denotes the standard Hardy–Littlewood maximal operator.

In order to pass from the dyadic setting to the full setting, we need a bit of help. We will use a result from [22], which will allow us to obtain the general case from the dyadic setting. This is a result that says that there are $n + 1$ dyadic families such that the sum of their respective maximal operators can actually bound the Hardy–Littlewood maximal operator. We give a version of the proof by Conde-Alonso that is adjusted to our needs.

Lemma 4.34

Let $Q \subset \mathbb{R}^n$ be a cube. Then there exist $n + 1$ dyadic systems $\{\mathcal{A}_j\}_{j=0}^n$ and $n + 1$ cubes, $Q_j \in \mathcal{A}_j$ such that the following two conditions are satisfied

1. $Mf(x) \leq c_n \sum_{j=0}^n M_j f(x)$ a.e. for any function f, where M_j is the dyadic maximal function with respect to the dyadic system $\mathcal{A}_j$, $j = 0, ..., n$.
2. $Q \subset \cap_{j=0}^n Q_j$ and the $|Q| \simeq |Q_j|$ for all j.

Proof. Given the cube Q, we construct the dyadic systems as in Theorem A in [22], but with a slight change on the starting cubes.

We choose the cubes Q_{00}^j so that $Q = \cap_j Q_{00}^j$. This is possible by construction, after making a translation and dilation. Indeed, we may suppose $Q = [\frac{p_n-1}{p_n}, 1]^n$, p_n being the smallest odd integer strictly greater than n. Following the proof in [22], we have $Q_{00}^j = [0,1]^n + \frac{j}{p_n}(1, .., 1)$. These cubes satisfy property (2). Then call $Q_j = Q_{00}^j$ and apply the same procedure as in [22]. □

Proof of Theorem 4.33. Fix the cube Q and the function f, and choose $Q_0, ..., Q_n$ and $\mathcal{A}_0, ..., \mathcal{A}_n$ as in Lemma 4.34. We have

$$\begin{aligned}
&\left|\left\{x \in Q : \frac{M\big((f - f_Q)\chi_Q\big)(x)}{M^\sharp f(x)} > \lambda\right\}\right| \\
&\qquad \leq \sum_{j=0}^n \left|\left\{x \in Q : \frac{M_j\big((f - f_Q)\chi_Q\big)(x)}{M^\sharp f(x)} > \frac{\lambda}{n+1}\right\}\right| \\
&\qquad \leq \sum_{j=0}^n \left|\left\{x \in Q_j : \frac{M_j\big((f - f_Q)\chi_{Q_j}\big)(x)}{M^\sharp f(x)} > \frac{\lambda}{n+1}\right\}\right| \\
&\qquad = \sum_{j=0}^n \left|\left\{x \in Q_j : \frac{M_{Q_j}(f - f_Q)(x)}{M^\sharp f(x)} > \frac{\lambda}{n+1}\right\}\right|.
\end{aligned}$$

Now, since Q and Q_j have comparable size, we have for $x \in Q_j$,

$$\frac{|f_Q - f_{Q_j}|}{M^\sharp f(x)} \leq \frac{|Q_j|}{|Q|} \frac{\displaystyle\fint_{Q_j} |f(y) - f_{Q_j}|dy}{\displaystyle\fint_{Q_j} |f(y) - f_{Q_j}|dy} \leq c_n.$$

So, for $\frac{\lambda}{n+1} \geq c_n$ we get for each j,

$$\begin{aligned}
&\left|\left\{x \in Q_j : \frac{M_{Q_j}(f - f_Q)(x)}{M^\sharp f(x)} > \frac{\lambda}{n+1}\right\}\right| \\
&\qquad \leq \left|\left\{x \in Q_j : \frac{M_{Q_j}(f - f_{Q_j})(x)}{M^\sharp f(x)} + c_n > \frac{\lambda}{n+1}\right\}\right| \\
&\qquad \leq \left|\left\{x \in Q_j : \frac{M_{Q_j}(f - f_{Q_j})(x)}{M^\sharp f(x)} > \frac{\lambda}{n+1} - c_n\right\}\right| \\
&\qquad \leq c\, e^{-c(\lambda - c_n)}|Q_j| \\
&\qquad \leq c\, e^{-c(\frac{\lambda}{n+1} - c_n)}|Q|.
\end{aligned}$$

This finishes the proof for $\frac{\lambda}{n+1} > c_n$. The other case follows since in that case $e^{-\lambda}$ is bounded from bellow. □

We now give the key estimate, which is a good-λ estimate between M and $M^\sharp$ with exponential decay.

Proposition 4.35

Let f be a function and $\lambda > 0$. Let $\Omega_\lambda = \{x \in \mathbb{R}^n : Mf(x) > \lambda\} = \cup Q$ as in the Whitney decomposition, Proposition 3.10. Then for any Q in the decomposition and γ small enough,

$$|\{x \in Q : Mf(x) > 4^n\lambda, M^\sharp f(x) \leq \gamma\lambda\}| \leq c\, e^{-\frac{c}{\gamma}}|Q|,$$

where $c >$ only depends on the dimension.

Proof. Let $\overline{Q}$ be the multiple of Q such that $\overline{Q} \cap (\Omega_\lambda)^c \neq \emptyset$, as in the Whitney decomposition. We prove that if $x \in Q$ satisfies $Mf(x) > 4^n\lambda$ and $M^\sharp f(x) \leq \gamma\lambda$ then

$$\frac{M((f - f_{\overline{Q}})\chi_{\overline{Q}})(x)}{M^\sharp f(x)} > \frac{1}{\gamma}. \tag{4.14}$$

Then we can directly apply Theorem 4.33 and we will be done.

Let $x \in Q$. Because of the Whitney decomposition, $Mf(x) > 4^n\lambda$ implies $M(f\chi_{\overline{Q}})(x) > 4^n\lambda$. Also as a consequence of the Whitney decomposition, $|f|_{\overline{Q}} \leq \lambda$, so

$$\begin{aligned}
4^n\lambda &\leq M(f\chi_{\overline{Q}})(x) \\
&\leq M((f - f_{\overline{Q}})\chi_{\overline{Q}})(x) + |f|_{\overline{Q}} \\
&\leq M((f - f_{\overline{Q}})\chi_{\overline{Q}})(x) + \lambda,
\end{aligned}$$

which implies $M((f - f_{\overline{Q}})\chi_{\overline{Q}})(x) > \lambda$. This proves (4.14). Therefore we have

$$
\begin{aligned}
|\{x \in Q : Mf(x) > 4^n\lambda, &M^\sharp f(x) \le \gamma\lambda\}| \\
&\le |\{x \in Q : M((f - f_{\overline{Q}})\chi_{\overline{Q}})(x) > 4^n\lambda, M^\sharp f(x) \le \gamma\lambda\}| \\
&\le \left| \left\{ x \in \overline{Q} : \frac{M((f - f_{\overline{Q}})\chi_{\overline{Q}})(x)}{M^\sharp f(x)} > \frac{1}{\gamma} \right\} \right| \\
&\le c\, e^{-\frac{1}{c\gamma}} |\overline{Q}|.
\end{aligned}
$$

This ends the proof, since Q and $\overline{Q}$ have comparable size. □

Now we prove Theorem 4.30. The proof follows mainly the one in [117], but we use the good-λ inequality from Proposition 4.35. We also keep an eye for the dependence on the constant of the weight, which is in fact our main objective.

We are going use Marcinkiewicz operators that were introduced in Section 3.3 from Chapter 3.

Proof of Theorem 4.30. We may assume, arguing as in [117], that both norms are finite. Define $\Omega_k = \{x \in \mathbb{R}^n : Mf(x) > 2^k\}$ for $k \in \mathbb{Z}$. We write, following the Whitney decomposition technique that we used in Chapter 3

$$
\Omega_k = \bigcup_j Q, \quad Q \in \mathcal{W}(k) \text{ disjoint cubes.}
$$

By Proposition 4.35 we have, for each $k \in \mathbb{Z}$ and each $Q \in \mathcal{W}(k)$ the following estimate

$$
|\{x \in Q : Mf(x) > 4^n 2^k, M^\sharp f(x) \le \gamma\lambda\}| \le c\, e^{-\frac{c}{\gamma}} |Q|,
$$

which in turn yields, using Theorem 2.21,

$$
w(\{x \in Q : Mf(x) > 4^n 2^k, M^\sharp f(x) \le \gamma\lambda\}) \le c e^{-c\frac{\varepsilon}{\gamma}} \int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx,
$$

where $\varepsilon = \frac{c_n}{\max(1,[w]_{C_q})}$. These computations, together with the standard argument that uses the good-λ technique yield

$$
\begin{aligned}
\int_{\mathbb{R}^n} Mf(x)^p w(x)dx &\le 2^p \sum_{k\in\mathbb{Z}} 2^{kp} w(\Omega_k) \\
&\le (c_n)^p \sum_{k\in\mathbb{Z}} 2^{kp} w(\{x \in \mathbb{R}^n : M^\sharp f(x) > \gamma 2^k\}) \\
&\quad + c_n e^{-\frac{c\varepsilon}{\gamma}} \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{W}(k)} 2^{kp} \int_{\mathbb{R}^n} M\chi_Q(x)^q w(x)dx \\
&\le \left(\frac{c_n}{\gamma}\right)^p \int_{\mathbb{R}^n} M^\sharp f(x)^p w(x)dx + c_n e^{-\frac{c\varepsilon}{\gamma}} \int_{\mathbb{R}^n} (M_{p,q}(Mf)(x))^p w,
\end{aligned}
$$

where $M_{p,q}$ is the Marcinkiewicz operator as Definition 3.13. We now use Lemma 3.9 and obtain

$$
\int_{\mathbb{R}^n} (M_{p,q}Mf(x))^p w(x)dx \le 2^{c_n\frac{pq}{q-p}} \frac{1}{\varepsilon} \log\frac{1}{\varepsilon} \int \mathbb{R}^n Mf(x)^p w(x)dx.
$$

So, if we choose

$$\frac{1}{\gamma} = c_n \frac{pq}{q-p} \frac{1}{\varepsilon} \log \frac{1}{\varepsilon} = c_n \frac{pq}{q-p} \max(1, [w]_{C_q} \log^+ [w]_{C_q}),$$

we can absorb the last term to the left side and we obtain

$$\|Mf\|_{L^p(w)} \leq c_n \frac{pq}{q-p} \max(1, [w]_{C_q} \log^+ [w]_{C_q}) \left\| M^\sharp f \right\|_{L^p(w)}.$$

This finishes the proof. □

Remark 4.36 Note that the quantitative dependence on the constant of the weight is essentially the same as in Theorem 3.21 for the Coifman–Fefferman inequality. This is because both proofs are based on the Marcinkiewicz-integral techniques. In order to obtain better dependence for any of these inequalities, we would need other techniques.

Proof of Corollary 4.32. Since $\frac{p}{\delta} < q$, we can make the following computations:

$$\begin{aligned}
\|Tf\|_{L^p(w)} &\leq \|M_\delta(Tf)\|_{L^p(w)} = \left\| M(Tf^\delta) \right\|_{L^{\frac{p}{\delta}}(w)}^{\frac{1}{\delta}} \\
&\leq c_n \left(\frac{pq}{\delta q - p} \max(1, [w]_{C_q} \log^+ [w]_{C_q}) \right)^{\frac{1}{\delta}} \left\| M^\sharp(Tf^\delta) \right\|_{L^{\frac{p}{\delta}}(w)}^{\frac{1}{\delta}} \\
&= c_n \left(\frac{pq}{\delta q - p} \max(1, [w]_{C_q} \log^+ [w]_{C_q}) \right)^{\frac{1}{\delta}} \left\| M_\delta^\sharp(Tf) \right\|_{L^p(w)} \\
&\leq c_n \, c_T \left(\frac{pq}{\delta q - p} \max(1, [w]_{C_q} \log^+ [w]_{C_q}) \right)^{\frac{1}{\delta}} \|Mf\|_{L^p(w)}.
\end{aligned}$$

□

4.7 Further extensions: polynomial approximation

In this section we generalize Theorems 4.19 and 4.14 to the context of polynomials. More precisely, we show that the average f_Q can be replaced with an appropriate polynomial $P_Q f$ of fixed degree k. It is not clear how to obtain this polynomial approximation from the sparse techniques in [103].

Let $\mathcal{P}_k(Q)$ denote the space of polynomials of degree at most k restricted to the cube Q, and let m_k denote the dimension of $\mathcal{P}_k(Q)$, which depends only on k and n. The degree k will be frozen from now on, so we omit the subscript k if there is no room for confusion.

Proposition 4.37
The dimension of the space of polynomials in n variables of degree up to k is precisely

$$m_k = \binom{n+k+1}{n+1}.$$

Proof. For each $0 \leq j \leq k$, the □

We are going to work with the $L^2(Q)$ space with normalized measure, namely we consider the standard product $\langle f, g \rangle_Q = \fint_Q f\bar{g}$. First, we have to construct an orthonormal basis of $\mathcal{P}(Q)$. We choose any orthonormal basis of $\mathcal{P}([0,1]^n)$, namely $\{e_j\}_{j=1}^{m_k}$. For a general cube $Q = y + \ell[0,1]^n$, we choose the basis formed by

$$e_{j,Q}(x) = e_j\big(\frac{x-y}{\ell}\big).$$

Note that $\{e_{j,Q}\}_j$ is indeed an orthonormal basis because the measure in Q is normalized. Moreover, for a fixed degree k, all the basis vectors have uniformly bounded L^∞ norm for every cube Q. If we were to increase the degree k, we would just have to introduce new vectors to the basis.

We define the orthogonal projection operator, that for any integrable function gives the projection in $L^2(Q)$ to the space of polynomials.

Definition 4.38

Let Q be a cube in $\mathbb{R}^n$ and let $k \geq 1$. The projection operator P_Q is defined as

$$\begin{array}{rcl} P_Q : L^1(Q) & \longrightarrow & \mathcal{P}(Q) \\ f & \longmapsto & \sum_{j=0}^{m_k} \langle f, e_{j,Q}\rangle_Q \; e_{j,Q}, \end{array}$$

where $\{e_{j,Q}\}_{j=0}^{m_k}$ is the orthonormal basis of $P_k(Q)$ from the discussion above.

Notice then that the projection operator is indeed defined in the whole $L^1(Q)$ and not only in $L^2(Q)$ because the $e_{j,Q}$ are polynomials and therefore they belong to $L^\infty(Q)$. Using the fact that the vectors $\{e_{j,Q}\}$ are uniformly bounded, one can prove that P_Q is actually bounded from $L^1(Q)$ to $L^\infty(Q)$, as the following Proposition illustrates.

Proposition 4.39

Let Q be a cube, $k \geq 1$ and $f \in L^1(Q)$. Then the projection $P_Q f$ of f satisfies

$$|P_Q f(x)| \leq \gamma \fint_Q |f(y)| dy, \tag{4.15}$$

for any $f \in L^1(Q)$, and where γ is a constant depending only on the dimension n and on k.

Combining these properties we can show the following optimality property of the chosen polynomial projection.

Proposition 4.40

The projection $P_Q f$ is a good approximation of f in $P_k(Q)$ in the $L^p(Q)$ distance, that is,

$$\inf_{\pi \in \mathcal{P}_k} \left(\fint_Q |f(x) - \pi(x)|^p dx \right)^{\frac{1}{p}} \approx \left(\fint_Q |f(x) - P_Q f(x)|^p dx \right)^{\frac{1}{p}}.$$

Proof. The inequality in the direction "$\leq$" is trivial. To prove the opposite inequality, observe that since P_Q is a projection we have $P_Q \pi = \pi$ for any polynomial of degree at most k, and therefore by the triangle inequality

$$\left(\fint_Q |f(x) - P_Q f(x)|^p dx\right)^{\frac{1}{p}} \leq \left(\fint_Q |f(x) - \pi(x)|^p dx\right)^{\frac{1}{p}} + \left(\fint_Q |P_Q(f-\pi)(x)|^p dx\right)^{\frac{1}{p}}$$
$$\leq (1+\gamma)\left(\fint_Q |f(x) - \pi(x)|^p dx\right)^{\frac{1}{p}},$$

by (4.15). □

Before we state the main results of this section, we introduce the sharp maximal function in this polynomial context, which has form one can expect.

Definition 4.41

Let $f \in L^1_{loc}(\mathbb{R}^n)$ and let $k \geq 1$. The polynomial sharp maximal function of degree k of f is defined by the expression

$$M_k^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(x) - P_Q f(x)| dx.$$

The case $k = 0$ corresponds to the usual sharp maximal function.

We state the maximal polynomial theorem, which corresponds to Theorem 4.19 in the polynomial context.

Theorem 4.42

Let $f \in L^1_{loc}$, Q a cube, $1 < r < \infty$ and $1 \leq p < \infty$. Then

$$\left(\frac{1}{w_r(Q)}\int_Q \left(\frac{M_Q(f - P_Q f)(x)}{M_k^\sharp f(x)}\right)^p w(x)dx\right)^{\frac{1}{p}} \leq c_n\, r'\, \gamma\, p.$$

Proof of Theorem 4.42. Fix $L > 1$ and make the Calderón–Zygmund decomposition of the function

$$F(x) = \frac{|f(x) - P_Q f(x)|}{\operatorname{osc}_k(f, Q)},$$

where now

$$\operatorname{osc}_k(f, Q) = \fint_Q |f(y) - P_Q f(y)| dy.$$

We obtain cubes $\{Q_j\}$ that satisfy:

- $L < \fint_{Q_j} \frac{|f(x) - P_Q f(x)|}{\operatorname{osc}_k(f, Q)} dx \leq 2^n L$
- $\sum_{Q_j} |Q_j| \leq \frac{|Q|}{L}$
- for almost every $x \notin \bigcup_j Q_j$, it holds $\frac{|f(x) - P_Q f(x)|}{\operatorname{osc}_k(f, Q)} \leq L$

Fix one of these cubes Q_j and let $x \in Q_j$. We can localize by maximality, meaning:

$$M_Q(f - P_Q)(x) = M_{Q_j}(f - P_Q f)(x) \leq M_{Q_j}(f - P_{Q_j} f)(x) + M_{Q_j}(P_{Q_j} f - P_Q f)(x).$$

Now, the function $P_{Q_j}f - P_Q f$ is not constant, but we can bound it. Indeed, since both are polynomials of degree at most k, $Q_j \subset Q$ and both P_{Q_j} and P_Q are projection operators, we have

$$P_{Q_j}f - P_Q f = P_{Q_j}f - P_{Q_j}(P_Q f) = P_{Q_j}(f - P_Q f).$$

Therefore, using (4.15) we get

$$|P_{Q_j}f(x) - P_Q f(x)| \leq |P_{Q_j}(f - P_Q)(x)| \leq \gamma \fint_{Q_j} |f - P_Q f| \leq 2^n \ L \ \gamma \ \operatorname{osc}_k(f, Q).$$

And, since the maximal function is bounded in L^∞ with norm one, we directly have

$$M_{Q_j}(P_{Q_j}f - P_Q f)(x) \leq \gamma 2^n L \ \operatorname{osc}_k(f, Q).$$

Now, this means that for $x \in Q_j$,

$$\frac{M_Q(f - P_Q f)(x)}{M_k^\sharp f(x)} \leq 2^n L\gamma + \frac{M_{Q_j}(f - P_{Q_j}f)(x)}{M_k^\sharp f(x)}$$

and therefore

$$\begin{aligned}
&\Big(\frac{1}{w_r(Q)} \int_Q \Big(\frac{M_Q(f - P_Q f)(x)}{M_k^\sharp f(x)}\Big)^p w(x)dx\Big)^{\frac{1}{p}} \\
&\qquad \leq 2^n L\gamma + \left(\frac{1}{w_r(Q)} \sum_j \frac{w_r(Q_j)}{w_r(Q_j)} \int_{Q_j} \Big(\frac{M_{Q_j}(f - P_{Q_j}f)(x)}{M^\sharp f(x)}\Big)^p w(x)dx\right)^{\frac{1}{p}} \\
&\qquad \leq 2^n \gamma L + \frac{X}{L^{\frac{1}{p}}}.
\end{aligned}$$

From here, the result follows as in the proof of Theorem 4.19. □

Finally, we state the polynomial version of Theorem 4.14. We introduce the weighted polynomial BMO norm, that is, for a certain weight w we define

$$\|f\|_{\mathrm{BMO}_k^r(w)} := \sup_Q \frac{1}{w_r(Q)} \int_Q |f - P_Q f|.$$

Theorem 4.43

Let $1 < p < \infty$ and $r > 1$. Let w a weight and f a function satisfying $[w]_{A_p^r} < \infty$ and $\|f\|_{\mathrm{BMO}_k^r(w)} < \infty$. Then

$$\left(\frac{1}{w_r(Q)} \int_Q \left|\frac{f(x) - P_Q f}{w(x)}\right|^{p'} w(x)dx\right)^{\frac{1}{p'}} \leq c_n \, \gamma \, p' \left([w]_{A_p^r}\right)^{\frac{1}{p}} (r')^{\frac{1}{p'}} \, \|f\|_{\mathrm{BMO}_k^r(w)}.$$

Since the proofs of these theorems are very similar to the zero degree case but making only the appropriate changes that have been illustrated in the proof of, Theorem 4.42, we are just going to give a sketch of the proof of Theorem 4.43 mentioning the places in which the main changes have to be made.

Sketch of the proof of Theorem 4.43. Let us fix $L > 1$ to be chosen later and make the mixed-type Calderón–Zygmund decomposition at height L of the function

$$|f - P_Q f|.$$

That is, we select the maximal cubes $\{Q_j\}$ that satisfy

$$\frac{1}{w_r(Q_j)} \int_{Q_j} |f(y) - P_Q f(y)| dy > L.$$

As in the proof of Theorem 4.42, these cubes will satisfy

- $\displaystyle\sum_j w_r(Q_j) \leq \frac{w_r(Q)}{L}$;
- For almost every $x \in Q_j$,

$$|P_Q f(x) - P_{Q_j} f(x)| \leq 2^n\ L\ \gamma \frac{w_r(Q_j')}{|Q_j'|},$$

  where Q_j' is the parent of Q_j;
- $|f(x) - P_Q f(x)| \leq L w(x)$ almost everywhere outside of $\bigcup_j Q_j$.

Therefore, one can compute as before

$$\begin{aligned}
&\Big(\frac{1}{w_r(Q)} \int_Q \Big(\frac{f(x) - P_Q f(x)}{w(x)}\Big)^q w(x) dx\Big)^{\frac{1}{q}} \\
&\qquad \leq \left(\frac{1}{w(Q)} \int_{(\bigcup_j Q_j)^c} L\, w(x) dx\right)^{\frac{1}{q}} \\
&\qquad\qquad + \left(\frac{1}{w_r(Q)} \sum_j \int_{Q_j} |P_{Q_j} f(x) - P_Q f(x)|^q w(x)^{1-q} dx\right)^{\frac{1}{q}} \\
&\qquad\qquad + \left(\frac{1}{w_r(Q)} \sum_j \int_{Q_j} \Big(\frac{|f(x) - P_{Q_j} f(x)|}{w(x)}\Big)^q w(x) dx\right)^{\frac{1}{q}} \\
&\qquad = A_1 + A_2 + B.
\end{aligned}$$

Clearly, $A_1 \leq L$ and $B \leq \frac{X}{L^{\frac{1}{q}}}$, where

$$X = \sup_{R \in \mathcal{D}(Q)} \left(\frac{1}{w_r(R)} \int_R \Big(\frac{|f(x) - P_R f(x)|}{w(x)}\Big)^q w(x) dx\right)^{\frac{1}{q}}.$$

In order to bound A_2 we can argue as in the proof of Theorem 4.14 but using the new properties of the Calderón–Zygmund cubes to get

$$A_2 \leq 2^n\ \gamma\ L(r')^{\frac{1}{q}}\ \big([w]_{A^r_{q'}}\big)^{\frac{1}{q'}}.$$

The proof follows as in the proof of Theorem 4.14. □

Remark 4.44 One can also obtain A_∞ results analogous to Corollaries 4.25 and 4.15.

5

Minimal conditions for BMO

In this chapter, we will discuss the results that were published in the work

[15] Canto, J., Pérez, C., Rela, E. *Minimal conditions for BMO* to appear in J. Funct. Anal.

The chapter is organized as follows. In Section 5.1 we give a brief introduction to the problem that we deal with in this chapter. In Section 5.2, we introduce the Luxemburg-type expressions that will be used throughout the chapter. The main result, that is, Theorem 5.5 in which we prove the minimality conditions for the classical space BMO, comes in Section 5.3. In the last two sections, Sections 5.4 and 5.5, generalizations of the main result are give, in spaces of homogeneous type and non-doubling measures in $\mathbb{R}^n$ respectively. Finally, we also consider rectangle-based BMO, which is sometimes called bmo.

5.1 Introduction

In the previous chapter, we discussed the John–Nirenberg Theorem 4.5. In that theorem, a self-improvement property was established for functions in BMO, providing a local exponential integrability estimate. Moreover, no better self-improvement can be found, so the John–Nirenberg theorem is the maximal integrability condition for BMO.

The main concern of this chapter is precisely the opposite problem: instead of studying self-improvement properties with BMO as an starting point, we want to find how much we can weaken the initial starting point but still self-improve back to BMO. More precisely, we show that the membership of a given function to BMO can be obtained from a much weaker condition on generalized averages defined by Luxemburg type norms.

Even though this problem was already addressed in a qualitative fashion by John in [65] and later by Strömberg in [113], our point of view is more quantitative, motivated by the recent work [90] which in turn was motivated by [91]. Our results extend those in [90], giving more precise estimates that can also be applied to different contexts such as spaces of homogeneous type or non-doubling measures in $\mathbb{R}^n$.

5.2 BMO through Luxemburg

One of the main tools in this chapter concerns Orlicz-type spaces. We refer to [116] for a general discussion of the theory. These spaces provide a more precise way of studying integrability of functions, because they expand the scale of L^p-integrability.

Although the general theory of Orlicz spaces deals with convex functions, these spaces can be defined for quite general functions. Our concern in this work is with functions φ which are concave, increasing and satisfy $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$.

Let us begin our discussion in $\mathbb{R}^n$, for simplicity. Generalizing these concepts to other spaces, such as spaces of homogeneous type is fairly straightforward, see Section 5.4.

Definition 5.1

Let Q be a cube in R^n and let $\varphi : [0, \infty] \to [0, \infty]$ be an non-decreasing function. The Orlicz-type space $L_\varphi(Q, \frac{dx}{|Q|})$ with respect to φ is defined as the set of functions f for which there exists some $\lambda > 0$ such that

$$\fint_Q \varphi\left(\frac{|f(x)|}{\lambda}\right) dx < \infty. \tag{5.1}$$

Expression (5.1) is not homogeneous in f, so in order to have an expression that is actually homogeneous, we introduce the quantity

$$\|f\|_{\varphi,Q} = \inf\left\{\lambda > 0 : \fint_Q \varphi\Big(\frac{|f(x)|}{\lambda}\Big) dx \le 1\right\}. \tag{5.2}$$

Proposition 5.2

Let φ be a convex function such that $\varphi(0) = 0$. Then (5.2) is a norm.

Proof. The only property that is not trivial is the triangular inequality, which is where convexity is actually used. Let f, g be two functions defined on Q. We need to prove

$$\|f+g\|_{\varphi,Q} \leq \|f\|_{\varphi,Q} + \|g\|_{\varphi,Q}.$$

□

In our work, the function φ will not be convex but concave, so in this case, (5.2) will not satisfy the triangular inequality in general and thus it is not a norm. However, we will use sometimes the word “norm" even though (5.2) is not a norm in the usual sense. Nevertheless, using the concavity we can prove a relation with the L^1-norm in the cube

Proposition 5.3

Let φ be a concave function and Q be a cube in $\mathbb{R}^n$. Then,

$$\|f\|_{\varphi,Q} \leq \frac{1}{\varphi^{-1}(1)} \|f\|_{L^1(Q,\frac{dx}{|Q|})}.$$

Proof. Let us use $\lambda = \frac{1}{\varphi^{-1}(1)} \|f\|_{L^1(Q,\frac{dx}{|Q|})}$ as a test number for (5.2). Then ,

$$\begin{aligned}
\fint_Q \varphi\Big(\frac{|f(x)|}{\frac{1}{\varphi^{-1}(1)} \|f\|_{L^1(Q,\frac{dx}{|Q|})}}\Big)dx &\leq \varphi\left(\varphi^{-1}(1)\fint_Q \frac{|f(x)|}{\|f\|_{L^1(Q,\frac{dx}{|Q|})}}dx\right)\\
&= \varphi\big(\varphi^{-1}(1)\big)\\
&= 1,
\end{aligned}$$

where we used Jensen's inequality in the first inequality, since φ is concave. □

Finally, we define the appropriate BMO space in this context. A way of doing so might be to substitute the L^1 norm in the oscillation by means of (5.2), that is,

$$\sup_Q \|f - f_Q\|_{\varphi,Q},$$

where the supremum is taken over all cubes Q. But since f might not be a priori locally integrable, using the average f_Q is not really allowed. We note that, in the classical BMO we have, as noted in Proposition 4.4

$$\inf_{c\in\mathbb{C}} \fint_Q |f(x)-c|dx \leq \fint_Q |f(x)-f_Q|dx \leq \frac{1}{2}\inf_{c\in\mathbb{C}} \fint_Q |f(x)-c|dx. \tag{5.3}$$

Therefore, we introduce the BMO_φ space using a similar expression to (5.3)

Definition 5.4

Let φ be a function on $[0,\infty]$. The space BMO_φ is defined as the set of functions f such that the quantity

$$\begin{aligned}
\|f\|_{\text{BMO}_\varphi} &= \sup_Q \inf_c \|f-c\|_{\varphi,Q} \qquad (5.4)\\
&= \sup_Q \inf_c \inf\Big\{\lambda>0 : \fint_Q \varphi\Big(\frac{|f(x)-c|}{\lambda}\Big)dx \leq 1\Big\},
\end{aligned}$$

is finite. The supremum is taken over all cubes Q.

One easy but key observation is that, if $\|f\|_{\mathrm{BMO}_\varphi} \leq 1$, then for each Q there exits a constant c_Q such that

$$\fint_Q \varphi(|f(x) - c_Q|)dx \leq 2.$$

This definition of BMO_φ can naturally be generalized to other contexts such as SHT, $\mathbb{R}^n$ with a more general measure or even the basis of rectangles.

We will focus on the special class of increasing and concave functions φ in $[0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Such functions must be continuous and subadditive, that is,

$$\varphi(t_1 + t_2) \leq \varphi(t_1) + \varphi(t_2).$$

5.3 Minimal condition for BMO

Without further ado, let us state the main result of this chapter.

Theorem 5.5

Let φ be an increasing, concave function with $\varphi(0) = 0$ and such that $\lim_{t\to\infty} \varphi(t) = +\infty$. Then $\mathrm{BMO}_\varphi = \mathrm{BMO}$ with the following quantitative estimates:

$$\varphi^{-1}(1)\, \|f\|_{\mathrm{BMO}_\varphi} \leq \|f\|_{\mathrm{BMO}} \leq \left(2\varphi^{-1}(4) + \varphi^{-1}(2 + 2^{n+2})\right) \|f\|_{\mathrm{BMO}_\varphi}.$$

Remark 5.6 Although concavity of φ is needed for the first inequality above, just subadditivity is sufficient for the second inequality. This observation could be useful for other circumstances or functions φ.

Proof of Theorem 5.5. The first inequality follows from Proposition 5.3 in Section 5.2, so we need only to prove the second one.

Let us fix a function $f \in \mathrm{BMO}_\varphi$ with norm one, and let us fix a cube Q. Then we can find a constant c_Q such that

$$\fint_Q \varphi(|f(x) - c_Q|)dx \leq 2. \tag{5.5}$$

Recall that the goal here is to bound the oscillation of f uniformly over all cubes. To that end, we introduce the quantity

$$X = \sup_{Q \text{ cube}} \fint_Q |f(x) - c_Q|\, dx,$$

where c_Q is such that (5.5) holds for Q. Note that by last property in Proposition (4.4) it is enough to show that the bound claimed in the theorem holds for this quantity. At certain point we will need to manipulate this X, so we need to start by assuming that it is finite. In order to do that, we will work with the following truncated quantity, that is,

$$X_m = \sup_{Q \text{ cube}} \fint_Q \min\{|f(x) - c_Q|, m\}\, dx, \quad m \geq 1. \tag{5.6}$$

We consider here the usual dyadic Calderón–Zygmund decomposition of $\varphi(|f-c_Q|)$ adapted to Q at height $L > 2$, see Section 1.4.3 for more details. The result is the collection $\{Q_j\}$ of maximal dyadic subcubes of Q satisfying

- $L < \fint_{Q_j} \varphi(|f(x) - c_Q|)dx \leq 2^n L,$

- $\varphi(|f(x) - c_Q|) \leq L$, for almost every $x \in Q \setminus \bigcup_j Q_j,$

- $\frac{1}{|Q|} \sum_j |Q_j| \leq \frac{2}{L}.$

Now, let us fix a cube Q_j. For a point $x \in Q_j$, we have

$$|f(x) - c_Q| \leq |f(x) - c_{Q_j}| + |c_Q - c_{Q_j}|,$$

where c_{Q_j} is a constant so that $\fint_{Q_j} \varphi(|f - c_{Q_j}|) \leq 2$. We bound the second term as follows:

$$\begin{aligned}
|c_Q - c_{Q_j}| &= \varphi^{-1}\Big(\fint_{Q_j} \varphi(|c_Q - c_{Q_j}|)dx\Big) \\
&\leq \varphi^{-1}\Big(\fint_{Q_j} \varphi(|f(x) - c_Q|)dx + \fint_{Q_j} \varphi(|f(x) - c_{Q_j}|)dx\Big) \\
&\leq \varphi^{-1}\big(2^n L + 2\big).
\end{aligned}$$

Here we have used the definition of the norm $\|f\|_{\mathrm{BMO}_\varphi}$, the properties of the Calderón–Zygmund decomposition, and the fact that φ is subadditive and φ^{-1} increasing.

We now proceed to estimate

$$\fint_Q \min\{|f(x) - c_Q|, m\}dx,$$

for $m \in \mathbb{N}$. We split the cube into the two sets: $\bigcup_j Q_j$ and $Q \setminus \bigcup_j Q_j$. On the first one, we have a good pointwise estimate on the size of $f - c_Q$. On the second, we will use that the CZ cubes are disjoint and the previous estimate. We will use a basic but key inequality: for any choice of positive parameters a, b and m, we have that $\min\{a + b, m\} \leq \min\{a, m\} + b$. Now, we start by controlling the integral over $Q \setminus \bigcup Q_j$ as

$$\frac{1}{|Q|} \int_{Q \setminus \cup Q_j} \min\{|f(x) - c_Q|, m\}dx \leq \varphi^{-1}(L).$$

Taking this into account, we proceed to estimate the average over the cube as follows

$$\begin{aligned}
\fint_Q \min\{|f(x) - c_Q|, m\}dx &\leq \varphi^{-1}(L) + \frac{1}{|Q|} \sum_j \int_{Q_j} \min\{|f(x) - c_Q|, m\}dx \\
&= \varphi^{-1}(L) + \frac{1}{|Q|} \sum_j |Q_j| \fint_{Q_j} \min\{|f(x) - c_Q|, m\}dx.
\end{aligned}$$

The average over Q_j is controlled by using the key property about the minimum, namely

$$\begin{aligned}\fint_{Q_j} \min\{|f(x)-c_Q|, m\}dx &\leq \fint_{Q_j} \min\{|f(x)-c_{Q_j}|+|c_Q-c_{Q_j}|, m\}dx \\ &\leq \fint_{Q_j} \min\{|f(x)-c_{Q_j}|, m\}dx + |c_Q - c_{Q_j}| \\ &\leq X_m + \varphi^{-1}(2+2^n L).\end{aligned}$$

Therefore, collecting estimates we get

$$\begin{aligned}\fint_{Q} \min\{|f(x)-c_Q|, m\}dx &\leq \varphi^{-1}(L) + \frac{1}{|Q|}\sum_j |Q_j| \left(X_m + \varphi^{-1}(2+2^n L)\right) \\ &\leq \varphi^{-1}(L) + \frac{2X_m}{L} + \frac{2}{L}\varphi^{-1}(2+2^n L),\end{aligned}$$

where X_m is the quantity defined by (5.6), which is trivially bounded by m. Then, we can also take the supremum on the left hand side to obtain

$$X_m \leq \varphi^{-1}(L) + \frac{2}{L}\varphi^{-1}(2+2^n L) + \frac{2X_m}{L}.$$

Now take $L=4$ and absorb X_m into the LHS,

$$X_m \leq 2\varphi^{-1}(4) + \varphi^{-1}(2+2^{n+2}),$$

and hence for any cube Q and for any $m \in \mathbb{N}$,

$$\fint_Q \min\{|f(x)-c_Q|, m\}dx \leq 2\varphi^{-1}(4) + \varphi^{-1}(2+2^{n+2}),$$

and letting $m \to \infty$ concludes the proof of the theorem. □

Theorem 5.5 can be seen as an improvement of the main result from [90]. There, the authors deal with a quantity similar to (5.4) defined as

$$K_{\varphi,Q}(f) = \sup_{J \text{ subcube } Q} \fint_J \varphi\left(|f(x)-f_J|\, dx\right). \tag{5.7}$$

They obtain, under some conditions on φ', φ'' and φ''', that the finiteness of $K_{\varphi,Q}(f)$ implies the membership of f to $\mathrm{BMO}(Q)$. Their approach is based on the Bellman function method, and they obtain quantitative upper and lower bounds on $\|f\|_{\mathrm{BMO}}$ in terms of (5.7). However, their estimates are not homogeneous which might be a drawback for some applications.

Our proof here is based in the classical (dyadic) Calderón–Zygmund decomposition at a local level on a given cube Q. The method is transparent and allows to precisely track the involved constants to give the result in Theorem 5.5 without any regularity hypothesis on φ. Furthermore, our proof yields homogeneous estimates and it does not require a priori local integrability for f.

We can go even further in the search for minimal conditions on the function φ. We mention that in [90], the main result can be extended to almost any measurable function φ going to infinity at infinity. Our method is also able to produce a similar result.

Theorem 5.7

Let $\psi : [0,\infty) \to [0,\infty)$ be any measurable function such that $\psi(0) = 0$ and $\lim_{t\to\infty}\psi(t) = +\infty$. Then

$$\|f\|_{\mathrm{BMO}} \le c_{n,\psi}\|f\|_{\mathrm{BMO}_\psi}. \tag{5.8}$$

The main idea for the proof is to replace a general function ψ going to $+\infty$ with a related function φ for which we can apply Theorem 5.5.

Proof of Theorem 5.7. Let $\psi : [0,+\infty) \to [0,+\infty)$ be a function such that $\psi(0) = 0$ and $\lim_{t\to\infty}\psi(t) = +\infty$. Just by using the hypothesis on the behavior of ψ at infinity, we can find some non negative $t_0 \in [0,\infty)$ (depending on ψ) and a polygonal function $\varphi : [0,+\infty) \to [0,+\infty)$ which will be concave for large values of t and smaller than ψ.

More precisely, we will have that $\varphi(t) = 0$ for all $t \le t_0$ (we need to wait until ψ goes away from zero). Then, for $t \ge t_0$, φ will be constructed as a polygonal consisting of consecutive segments with endpoints (t_n, n), $(t_{n+1}, n+1)$ with $n \in \mathbb{N}$ chosen in such a way that the resulting polygonal is continuous, concave and such that $\varphi(t) \le \psi(t)$ for all $t \in [0,\infty)$. Using this auxiliary function and since we have immediately that

$$\|f\|_{\mathrm{BMO}_\varphi} \le \|f\|_{\mathrm{BMO}_\psi},$$

we will prove (5.8) for the new function φ instead of ψ. An inspection of the proof of Theorem 5.5 shows that the key step is to obtain

$$|c_Q - c_{Q_j}| \le \varphi^{-1}\big(2^n L + 2\big),$$

where the subadditivity is used. Here, we proceed as follows using the layer cake formula, see Section 1.4.3. Write $A(x) = |c_Q - f(x)|$ and $B(x) = |f(x) - c_{Q_j}|$, so

$$\begin{aligned}\int_{Q_j}\varphi(|c_Q - c_{Q_j}|)\,dx &\le \int_{Q_j}\varphi(|c_Q - f(x)| + |f(x) - c_{Q_j}|)\,dx\\ &= \int_0^\infty \varphi'(t)|\{x \in Q_j : A(x) + B(x) > t\}|\,dt\\ &= I.\end{aligned}$$

Note that φ is differentiable almost everywhere since it is a polygonal. We can split the integral at $t = 2t_0$ to obtain

$$\begin{aligned}I &= \int_0^{2t_0}\varphi'(t)|\{A + B > t\}|\,dt + \int_{2t_0}^\infty \varphi'(t)|\{A + B > t\}|\,dt\\ &\le |Q_j|\varphi(2t_0) + \int_{2t_0}^\infty \varphi'(t)|\{A > t/2\}|\,dt + \int_{2t_0}^\infty \varphi'(t)|\{B > t/2\}|\,dt\\ &= |Q_j|\varphi(2t_0) + 2\int_{t_0}^\infty \varphi'(2u)|\{A > u\}|\,du + 2\int_{t_0}^\infty \varphi'(2u)|\{B > u\}|\,du.\end{aligned}$$

Now we use that the derivative function φ' is non negative and decreasing in (t_0,∞), and so we obtain

$$\begin{aligned}I &\le |Q_j|\varphi(2t_0) + 2\int_{t_0}^\infty \varphi'(u)|\{A > u\}|\,du + 2\int_{t_0}^\infty \varphi'(u)|\{B > u\}|\,du\\ &\le |Q_j|\varphi(2t_0) + 2\int_0^\infty \varphi'(u)|\{A > u\}|\,du + 2\int_0^\infty \varphi'(u)|\{B > u\}|\,du\end{aligned}$$

$$= |Q_j|\varphi(2t_0) + 2\int_{Q_j} \varphi(|f(x) - c_Q|)dx + 2\int_{Q_j} \varphi(|f(x) - c_{Q_j}|)dx.$$

Finally, dividing by the measure of Q_j we obtain a similar estimate as in the original proof. Indeed, whenever $|c_Q - c_{Q_j}| \geq t_0$, we obtain

$$\begin{aligned}|c_Q - c_{Q_j}| &= \varphi^{-1}\Big(\fint_{Q_j} \varphi(|c_Q - c_{Q_j}|)df\Big) \\ &\leq \varphi^{-1}\Big(\varphi(2t_0) + 2\fint_{Q_j} \varphi(|f - c_Q|)dx + 2\fint_{Q_j} \varphi(|f - c_{Q_j}|)dx\Big) \\ &\leq \varphi^{-1}\big(\varphi(2t_0) + 2^{n+1}L + 4\big),\end{aligned}$$

where φ^{-1} is the inverse of φ restricted to $[t_0, +\infty)$. Otherwise, we simply bound $|c_Q - c_{Q_j}| \leq t_0$ with the obvious consequences over the final estimate. From here, the proof follows the same steps as in Theorem 5.5 to obtain

$$\|f\|_{\mathrm{BMO}} \leq c_{n,\varphi}\|f\|_{\mathrm{BMO}_\varphi} \leq c_{n,\psi}\|f\|_{\mathrm{BMO}_\psi}.$$ □

5.4 Spaces of homogeneous type

The method of proof of Theorem 5.5 is flexible enough to also solve the same problem in various different settings. We will prove the same result in the context of spaces of homogeneous type where the space $(\mathbb{X}, d, \mu)$ is endowed with a quasi metric and a doubling measure. Let us give the precise definition.

Definition 5.8 – Space of homogeneous type

A space of homogeneous type is a triplet $(\mathbb{X}, d, \mu)$ consisting on a point set $\mathbb{X}$, a quasi-metric d, and a doubling measure μ. More precisely, d is a function $d : \mathbb{X}\times\mathbb{X} \to [0, \infty)$ such that

- $d(x, y) = 0$ if and only if $x = y$;
- $d(x, y) = d(y, x)$ for all $x, y \in \mathbb{X}$;
- $d(x, z) \leq \kappa\big(d(x, y) + d(y, z)\big)$ for all $x, y, z \in \mathbb{X}$.

The constant κ is called the quasi-metric constant of $\mathbb{X}$. Moreover, we assume that the open balls with respect to d are measurable and that there exists a constant, $c_\mu > 0$, which we call the doubling constant of μ such that

$$\mu\big(B(x, 2r)\big) \leq c_\mu\, \mu\big(B(x, r)\big), \tag{5.9}$$

for all $x \in \mathbb{X}$ and all $r > 0$.

Even if d is a quasi-metric and not a metric, we can define balls of center $x \in \mathbb{X}$ and radius $r > 0$ as

$$B(x, r) = \{y \in \mathbb{X} : d(x, y) < r\}.$$

By [92], we may assume that these balls are measurable, since we can define an equivalent quasi-norm for which they actually are measurable. Thus, Definition 5.8 is coherent.

Let c is the smallest constant for which (5.9) holds. The number $D = \log_2 c$ is usually called the doubling order of μ. Then by iterating, we have

$$\frac{\mu(B)}{\mu(P)} \leq c_{\mu,\kappa} \left(\frac{r(B)}{r(P)}\right)^D, \tag{5.10}$$

for every pair P, B of balls such that $P \subset B$.

Let us now define the BMO and BMO_φ spaces in spaces of homogeneous type.

Definition 5.9

For a function φ we define the $\|f\|_{\text{BMO}_\varphi(\mathbb{X})}$ and the corresponding class as:

$$\|f\|_{\text{BMO}_\varphi(\mathbb{X})} := \sup_B \inf_{c\in\mathbb{R}} \inf \left\{\lambda > 0 : \fint_B \varphi(\frac{|f(x)-c|}{\lambda}) d\mu(x) \leq 1\right\},$$

where the supremum is taken over all balls $B \subset \mathbb{X}$. We also define $\text{BMO}(\mathbb{X})$ with the quantity

$$\|f\|_{\text{BMO}(\mathbb{X})} = \sup_B \inf_c \fint_B |f-c| d\mu.$$

Theorem 5.10

Let φ satisfy the same condition as in Theorem 5.5. Then $\text{BMO}(\mathbb{X}) = \text{BMO}_\varphi(\mathbb{X})$ and

$$\varphi^{-1}(1)\, \|f\|_{\text{BMO}_\varphi(\mathbb{X})} \leq \|f\|_{\text{BMO}(\mathbb{X})} \leq c_{\varphi,\mu}\, \|f\|_{\text{BMO}_\varphi(\mathbb{X})}.$$

The proof of this theorem requires an adapted version of the classical Calderón–Zygmund decomposition theorem and some other covering lemmas that we will develop accordingly.

In view of Lemma 1.2, we define, for a ball B, the dilation

$$B^* = \kappa(4\kappa + 1)B.$$

We also fix the following notation for dilations. Fix $\gamma > \kappa$ and we set

$$\tilde{B} := \gamma\, B.$$

This is needed because when doing the Vitali covering, dilating the balls may result in going outside the original ball B, but the following lemma guaranties that the dilated balls stay inside of $\tilde{B}$. For a ball B we denote by x_B and $r(B)$ the center and radius of B respectively.

Lemma 5.11

Let B be a ball and let $\varepsilon > 0$. There exists $L > 1$ big enough so that if P is another ball with center in B and satisfying

$$\mu(P) \leq \frac{\mu(\tilde{B})}{L},$$

then $r(P) \leq \varepsilon r(B)$. If ε is small enough, this also implies $P^* \subset \tilde{B}$.

Proof. By contradiction, suppose that there exists some $\alpha > 1$ such that $r(P) \geq \alpha r(B)$ with α independent from L. This implies $\tilde{B} \subset \kappa(\gamma+1)\frac{1}{\alpha}P$. Indeed, for $y \in \tilde{B}$,

$$\begin{aligned} d(y, x_P) &\leq \kappa\big(d(y, x_B) + d(x_B, x_P)\big) \\ &\leq \kappa\big(\gamma r(B) + r(B)\big) \\ &\leq \kappa(\gamma+1)\frac{1}{\alpha}\, r(P). \end{aligned}$$

This bound on the radii will imply a bound on the measures. Indeed, by (5.10),

$$\begin{aligned} \mu(\tilde{B}) \leq \mu\big(\frac{(\gamma+1)\kappa}{\alpha}P\big) \quad &\leq \quad c_{\mu,\kappa}\left(\frac{(\gamma+1)\kappa}{\alpha}\right)^D \mu(P) \\ &\leq \quad \frac{c_{\mu,\kappa}}{L}\left(\frac{(\gamma+1)\kappa}{\alpha}\right)^D \mu(\tilde{B}). \end{aligned}$$

This implies that $c_{\mu,\kappa}\left(\frac{(\gamma+1)\kappa}{\alpha}\right)^D \geq L$ which is not possible for L big enough.

Now we prove the last statement. We set $y \in P^*$ and we want to see $y \in \gamma B = \tilde{B}$. Indeed,

$$\begin{aligned} d(y, c_B) &\leq \kappa\big(d(y, c_P) + d(c_P, c_B)\big) \\ &\leq \kappa\big(\kappa(4\kappa+1)\varepsilon r(B) + r(B)\big) \\ &\leq \kappa\big(\kappa(4\kappa+1)\varepsilon + 1\big) r(B). \end{aligned}$$

Now, since $\gamma > \kappa$, there exists $\varepsilon > 0$ small enough such that

$$\kappa\big(\kappa(4\kappa+1)\varepsilon + 1\big) \leq \gamma.$$

Thus, $y \in \tilde{B}$ and we are done. □

Proof of Theorem 5.10. Assume that $\|f\|_{\mathrm{BMO}_\varphi(\mathbb{X})} = 1$. Set for a ball P a constant c_P such that

$$\fint_P \varphi(|f(x) - c_P|)dx \leq 2.$$

We are going to set X in a slightly different way from before, namely

$$X = \sup_P \fint_P |f(x) - c_{\tilde{P}}|dx.$$

Notice that the ball of the integral and the one inside are related but not the same. Nevertheless, it is clear that $\|f\|_{\mathrm{BMO}} \leq X$. As in the proof of Theorem 5.5, the hypothesis $X < \infty$ will be needed. This can be obtained via a truncation argument as in that proof, but we omit it for the sake of clarity. Thus, we may assume that $X < \infty$.

Now let us begin with the actual proof. Fix a ball B and $L > 1$ to be precised later. We make a decomposition in balls of the function $\varphi(|f - c_{\tilde{B}}|)$ in the spirit of Calderón–Zygmund and using the Vitali covering. By that, we mean the following process.

We are going to make a covering by balls of the set

$$\Omega_L = \{x \in B : \varphi(|f(x) - c_{\tilde{B}}|) > L\}.$$

By the Lebesge differentiation theorem, for any $x \in \Omega_L$, there exists a ball B_x centered at x and contained in $\tilde{B}$ and such that

$$\fint_{B_x} \varphi(|f(y) - c_{\tilde{B}}|) d\mu(y) > L. \tag{5.11}$$

Moreover, we can choose this B_x to be maximal with respect to the radius. That is, any other ball $B'_x \subset \tilde{B}$ satisfying (5.11) must also satisfy $r(B'_x) \leq 2r(B_x)$. This can be done since all balls contained in $\tilde{B}$ have bounded radius.

Now we have a family $\mathcal{B} = \{B_x\}_x$ and we apply the Vitali Lemma 1.2 to get a "maximal" subfamily $\mathcal{B}' = \{B_j\}$. If L is big enough, we can apply Lemma 5.11 and this ensures that $B_j^* \subset \tilde{B}$ and, by the maximality of the radius of each of the B_j, since $r(B_j^*) \geq 2r(B_j)$,

$$\fint_{B_j^*} \varphi(|f(y) - c_{\tilde{B}}|) d\mu(y) \leq L.$$

Moreover, we have the estimate

$$\begin{aligned}
\sum_j \mu(B_j) &\leq \frac{1}{L} \sum_j \int_{B_j} \varphi(|f(x) - c_{\tilde{B}}|) dx \\
&\leq \frac{1}{L} \int_{\tilde{B}} \varphi(|f(x) - c_{\tilde{B}}|) dx \\
&\leq \frac{2}{L} \mu(\tilde{B}) \leq \frac{C_{\mathbb{X}}}{L} \mu(B),
\end{aligned}$$

where $C_{\mathbb{X}}$ denotes a constant depending on the doubling property of μ.

Let us summarize all the properties of the family $\{B_j\}$:

- The balls B_j are pairwise disjoint and all contained in $\tilde{B}$.
- $\Omega_L \subset \cup_j B_j^*$
- The balls B_j^* are contained in $\tilde{B}$ and $\fint_{B_j^*} \varphi(|f(x) - c_{\tilde{B}}|) dx \leq L$.
- $\sum_j \mu(B_j^*) \leq C_{\mathbb{X}} \sum_j \mu(B_j) \leq \frac{C_{\mathbb{X}}}{L} \mu(B)$.

Now we begin to estimate $\fint |f(x) - c_{\tilde{B}}| dx$.

$$\begin{aligned}
\fint_B |f(x) - c_{\tilde{B}}| d\mu(x) &\leq \frac{1}{\mu(B)} \int_{(\Omega_L)^c} |f(x) - c_{\tilde{B}}| d\mu(x) + \frac{1}{\mu(B)} \int_{\Omega_L} |f(x) - c_{\tilde{B}}| dx \\
&\leq \varphi^{-1}(L) + \frac{1}{\mu(B)} \sum_j \int_{B_j^*} |f(y) - c_{\tilde{B}}| d\mu(y) \\
&\leq \varphi^{-1}(L) + \frac{1}{\mu(B)} \sum_j \int_{B_j^*} \Big(|f(y) - c_{\widetilde{B_j^*}}| + |c_{\tilde{B}} - c_{\widetilde{B_j^*}}| \Big) d\mu(y) \\
&= (*).
\end{aligned}$$

We now estimate $|c_{\tilde{B}} - c_{\widetilde{B_j^*}}|$:

$$|c_{\tilde{B}} - c_{\widetilde{B_j^*}}| = \varphi^{-1}\Big(\fint_{B_j^*} \varphi(|c_{\tilde{B}} - c_{\widetilde{B_j^*}}|) d\mu(y) \Big)$$

$$
\begin{aligned}
&\leq \varphi^{-1}\Big(\fint_{B_j^*} \varphi(|f(y) - c_{\tilde{B}}|) d\mu(y) + \fint_{B_j^*} \varphi(|f(y) - c_{\widetilde{B_j^*}}|) d\mu(y)\Big) \\
&\leq \varphi^{-1}\Big(L + C_{\mathbb{X}} \fint_{\widetilde{B_j^*}} \varphi(|f(y) - c_{\widetilde{B_j^*}}|) d\mu(y)\Big) \\
&\leq \varphi^{-1}\Big(L + 2C_{\mathbb{X}}\Big).
\end{aligned}
$$

And therefore,

$$
\begin{aligned}
(*) &\leq \varphi^{-1}(L) + \frac{1}{\mu(B)} \sum_j \int_{B_j^*} \Big(|f(y) - c_{\widetilde{B_j^*}}| + \varphi^{-1}\big(L + 2C_{\mathbb{X}}\big)\Big) d\mu(y) \\
&\leq \varphi^{-1}(L) + \sum_j \frac{\mu(B_j^*)}{\mu(B)} \fint_{B_j^*} \Big(|f(y) - c_{\widetilde{B_j^*}}| + \varphi^{-1}\big(L + 2C_{\mathbb{X}}\big)\Big) d\mu(y) \\
&\leq \varphi^{-1}(L) + C_{\mathbb{X}} \sum_j \frac{\mu(B_j)}{\mu(B)} \Big(\fint_{B_j^*} |f(y) - c_{\widetilde{B_j^*}}| d\mu(y) + \varphi^{-1}\big(L + 2C_{\mathbb{X}}\big)\Big) \\
&\leq \varphi^{-1}(L) + C_{\mathbb{X}} \frac{X}{L} + \frac{C_{\mathbb{X}}}{L} \varphi^{-1}(L + 2C_{\mathbb{X}}) \\
&\leq C_{L,\mathbb{X},\varphi} + C_{\mathbb{X}} \frac{X}{L}.
\end{aligned}
$$

In order to finish, we take the supremum on the left, choose L big enough and argue as in the euclidean case. □

5.5 Non-doubling measures in $\mathbb{R}^n$

We will also study the problem in $\mathbb{R}^n$ endowed with a quite general non doubling measure μ. The usual requirement is to ask for the measure to be non atomic. In that case, it is known that there is an orthogonal system of coordinates such that $\mu(\partial(Q)) = 0$ for any cube Q with sides parallel to the axes from that coordinate system, which is assumed to be the canonical one (see [95]). We mention, as an example of such measures, that a very natural choice satisfying these conditions is the class of measures with *polynomial growth*, meaning that there exists a constant $C > 0$ and a positive number α such that

$$
\mu(B(x,r)) \leq Cr^{\alpha} \qquad x \in \operatorname{supp}(\mu). \tag{5.12}
$$

The natural definitions of BMO and BMO_φ in this context are the following. We will say that $f \in \mathrm{BMO}(\mu)$ if

$$
\|f\|_{\mathrm{BMO}(\mu)} := \sup_Q \fint_Q |f - f_Q| \, d\mu < \infty,
$$

and $f \in \mathrm{BMO}_\varphi(\mu)$ if

$$
\|f\|_{\mathrm{BMO}_\varphi(\mu)} := \sup_Q \inf_{c \in \mathbb{R}} \inf \left\{ \lambda > 0 : \frac{1}{\mu(Q)} \int_Q \varphi\left(\frac{|f - c|}{\lambda}\right) \, d\mu \leq 1 \right\} < \infty.
$$

Theorem 5.12

Let φ be as in Theorem 5.5. Then, for any non-atomic measure μ, we have that $\mathrm{BMO}(\mu) = \mathrm{BMO}_\varphi(\mu)$ and

$$\varphi^{-1}(1)\, \|f\|_{\mathrm{BMO}_\varphi(\mu)} \le \|f\|_{\mathrm{BMO}(\mu)} \le c_{\varphi,n}\, \|f\|_{\mathrm{BMO}_\varphi(\mu)}.$$

The proof of the above theorem relies on a variation of the standard Calderón–Zygmund decomposition and Besicovitch's covering theorem that we borrow from [104]. The precise statement is in Lemma 5.13.

So far, we can see (and it will become clear in the actual proof) that the heart of the matter is to have the correct version of a Calderón–Zygmund decomposition adapted to the problem that we need to solve, taking into account the geometric features of the space (like in the case of spaces of homogeneous type) or the nondoubling nature of the measure (like in Theorem 5.12).

The rest of this section is dedicated to proving Theorem 5.12. Let's consider a nondoubling measure satisfying the growth condition (5.12). Therefore, it is non atomic and by [95, Theorem 2] we can choose a coordinate system such that $\mu(\partial Q) = 0$ for every cube Q defined over that system.

We will present the proof for $n = 1$ separately, since the situation there is much easier than in higher dimensions. The heart of our main argument is the Calderón–Zygmund decomposition. Here, in the nondoubling setting, we will abandon the metric to split the cubes and use the measure instead. We will construct a μ-dyadic grid of subintervals such that every interval I is divided into two subintervals each one of half of the measure of I. We sketch here the construction.

The first generation $G_1(I)$ of the dyadic grid consists of the two disjoint subintervals I_+, I_- of I satisfying $\mu(I_+) = \mu(I_-) = \mu(I)/2$ (note that this partition may be non unique, in that case we choose the one that maximizes the length of I_-, just to fix a criterion) The next generation is $G_2(I)$ is $G_1(I_+) \cup G_1(I_-)$ and then the construction procedure continues recursively. Recall that the measure is non atomic, so we can take closed intervals sharing the endpoints. We denote by $\mathcal{D}_I^\mu$ the family of all the dyadic intervals resulting from this procedure. A sequence of nested intervals in this grid will be called a *chain*. That is, a chain $\mathcal{C}$ will be of the form $\mathcal{C} = \{J_i\}_{i\in\mathbb{N}}$ such that $J_i \in G_i(I)$, and $J_{i+1} \subset J_i$ for all $i \ge 1$.

We can define $\mathcal{C}_\infty := \bigcap_{J\in\mathcal{C}} J$ as the *limit set* of the chain $\mathcal{C}$. Then, we have that $\mathcal{C}_\infty$ could be either a single point or a closed interval of positive length. In any case, we clearly have that $\mu(\mathcal{C}_\infty) = 0$. We need to get rid of those limit sets $\mathcal{C}_\infty$ of positive length, so we call them *removable*. The argument here is that in the real line there are at most countable many of them and the whole union is also a μ-null set. We denote by $\mathcal{R}$ the set of all chains with removable limits. If we define

$$E := I \setminus \bigcup_{\mathcal{C}\in\mathcal{R}} \mathcal{C}_\infty,$$

we conclude that $\mu(I) = \mu(E)$ and, in addition, for any $x \in E$, there exists a chain of nested intervals shrinking to x. Therefore the grid $\mathcal{D}_I^\mu$ forms a differential basis on E. Also, the dyadic structure of the basis guarantees the Vitali covering property (see [48, Ch.1]) and therefore this basis differentiates $L^1(E)$.

Associated to this grid we define a *dyadic* maximal operator as follows. For any $x \in E$,

$$M^{\mathcal{D}_I^\mu} f(x) = \sup_{J\in\mathcal{D}_I^\mu} \fint_J |f(y)|\, d\mu(y),$$

By a standard differentiation argument, we have that this maximal function satisfies that $f \leq M^{\mathcal{D}_I^\mu} f$, almost everywhere on E.

Now we can proceed with the proof of the 1 dimensional case of Theorem 5.12. Let us fix a function $f \in \mathrm{BMO}_\varphi(\mu)$ with norm one, and let us fix an interval I. As before, we can find a constant c_I such that

$$\fint_I \varphi(|f(y) - c_I|)\, d\mu(y) \leq 2.$$

We define again the corresponding X as

$$X = \sup_{I \text{ interval}} \fint_I |f(x) - c_I|\; d\mu(x).$$

As in the euclidean setting, a truncation argument allows us to assume $X < \infty$. Using our μ-dyadic construction, we can perform a Calderón–Zygmund decomposition of $\varphi(|f - c_I|)$ adapted to I at height $L > 2$. We then obtain a family $\{I_j\}$ of dyadic subintervals of I satisfying

- $L < \displaystyle\fint_{I_j} \varphi(|f(x) - c_I|) d\mu(x) \leq 2L,$
- $\varphi(|f(x) - c_I|) \leq L \quad$ for μ-almost every $x \in I \setminus \bigcup_j I_j$.
- $\displaystyle\frac{1}{\mu(I)} \sum_j \mu(I_j) \leq \frac{2}{L}.$

Once we have this crucial decomposition, we can develop the same proof as in the case of the Lebesgue measure. On a fixed maximal interval I_j, we write again

$$|f(x) - c_I| \leq |f(x) - c_{I_j}| + |c_I - c_{I_j}|,$$

where c_{I_j} is a constant so that $\fint_{I_j} \varphi(|f - c_{I_j}|) \leq 2$. We obtain

$$|c_I - c_{I_j}| \leq \varphi^{-1}\big(2L + 2\big)$$

Following the same line of ideas, we can control the averges to estimate the BMO μ norm

$$\begin{aligned}
\fint_I |f(x) - c_I| d\mu(x) &\leq \varphi^{-1}\big(L\big) + \frac{1}{\mu(I)} \sum_j \mu(I_j) \Big(\fint_{I_j} |f(x) - c_{I_j}| d\mu(x) + |c_I - c_{I_j}| \Big) \\
&\leq \varphi^{-1}\big(L\big) + \frac{1}{\mu(I)} \sum_j \mu(I_j) \Big(X + \varphi^{-1}\big(2L + 2\big) \Big) \\
&\leq \varphi^{-1}\big(L\big) + \frac{2}{L} X + \frac{2}{L} \varphi^{-1}\big(2L + 2\big)
\end{aligned}$$

Finally, taking the supremum over all intervals on the left hand side and choosing $L = 4$ we obtain

$$X \leq 2\varphi^{-1}\big(4\big) + \varphi^{-1}\big(10\big),$$

which finishes the proof.

Now we present the proof for $n > 1$. Again, the key step is to construct an adequate Calderón–Zygmund decomposition with dyadic structure. The ideal tool can be found in the work from [104] and consists in the following combination of the Calderón–Zygmund decomposition and Besicovitch's covering theorem. We include here the statement of that lemma.

Lemma 5.13 – Besicovitch–Calderón–Zygmund decomposition

Let Q be a cube and let $g \in L^1_\mu(Q)$ be a nonnegative function. Also let L be a positive number such that $\fint_Q g(x)\, d\mu(x) < L$. Then there is a family of quasidisjoint cubes $\{Q_j\}$ contained in Q satisfying

$$\frac{1}{\mu\left(Q_j\right)} \int_{Q_j} g d\mu = L$$

for each j and such that

$$g(x) \leq L \quad \text{ for } \mu\text{-almost every } x \in Q \setminus \bigcup_j Q_j.$$

More precisely, we can write

$$\bigcup Q_j = \bigcup_{k=1}^{B(n)} \bigcup_{Q_j \in \mathcal{F}_k} Q_j,$$

where each $\mathcal{F}_k$ is a family of disjoint cubes selected from the original collection. The number $B(n)$ is a geometric constant depending only on the dimension n known as the *Besicovitch constant*.

We can now provide the proof for Theorem 5.12 in the remaining cases $n > 1$. Let's start again with a function f such that $\|f\|_{\mathrm{BMO}_\varphi(\mu)} = 1$ and fix a cube Q and the corresponding $c_Q \in \mathbb{R}$ giving us the initial estimate

$$\fint_Q \varphi(|f(x) - c_Q|)\, d\mu(x) \leq 2.$$

We define again the corresponding X as

$$X = \sup_{Q \text{ cube}} \fint_Q |f(x) - c_Q|\, d\mu(x).$$

Applying Lemma 5.13 with $L > 2$, we obtain a quite similar collection of cubes as in the previous case. Precisely, we obtain the family of cuasidisjoint cubes $\{Q_j\}$ satisfying

- $\displaystyle\fint_{Q_j} \varphi(|f(x) - c_Q|) dx = L,$

- $\displaystyle\varphi(|f(x) - c_Q|) \leq L \quad \text{a.e. } x \in Q \setminus \bigcup_j Q_j,$

  and a minor difference in the next property:

- $\displaystyle\frac{1}{\mu(Q)} \sum_j \mu(Q_j) \leq \frac{B(n)}{L}.$

Once we have this crucial decomposition, we can develop the same proof as in the standard situation (choosing the number c_{Q_j} according to the same criterion) to obtain

$$\begin{aligned}
\fint_I |f(x) - c_Q| dx &\leq \varphi^{-1}(L) + \frac{1}{\mu(Q)} \sum_j \mu(Q_j) \Big(\fint_{Q_j} |f(x) - c_{Q_j}| dx + |c_Q - c_{Q_j}| \Big) \\
&\leq \varphi^{-1}(L) + \frac{1}{\mu(Q)} \sum_j \mu(Q_j) \Big(X + \varphi^{-1}(2L+2) \Big) \\
&\leq \varphi^{-1}(L) + \frac{B(n)}{L} X + \frac{B(n)}{L} \varphi^{-1}(2L+2)
\end{aligned}$$

Finally, taking the supremum over all cubes on the left hand side and choosing $L = 2B(n)$ we obtain

$$X \leq 2\varphi^{-1}\big(2B(n)\big) + \varphi^{-1}\big(4B(n)+2\big),$$

which finishes the proof. The assumption that $X < \infty$ can be done using the same truncation argument as before.

5.6 Rectangles and non-doubling measures

For the basis of rectangles in $\mathbb{R}^n$, the appropriate decomposition lemma is a very clever argument proven by Korenovskyy, Lerner and Stokolos in [75] known as a generalized version of Riesz's Rising sun lemma. Using that lemma we can provide a proof that extends, in some sense, Theorem 5.5 and Theorem 5.12 at the same time: we can prove the analogue result for basis of rectangles and with non doubling measures.

To present the result, we need to define here the "little" $\mathrm{bmo}(\mu)$ space in the same way of the usual $\mathrm{BMO}(\mu)$ but with rectangles instead of cubes. We refer the reader to the recent article [53] for several results on this space.

Definition 5.14

Let φ be an increasing function with $\varphi(0) = 0$ and let μ be a Radon measure. We denote by $\mathrm{bmo}_\varphi(\mu)$, little $\mathrm{BMO}_\varphi(\mu)$ the class of functions f satisfying

$$\|f\|_{\mathrm{bmo}_\varphi(\mu)} := \sup_R \inf_c \|f - c\|_{\varphi,R,\mu} < \infty,$$

where the supremum is taken over all rectangles with sides parallel to the coordinate axes and the local averages are defined as in (5.2) but with respect to the measure μ, that is,

$$\|f\|_{\varphi,R,\mu} := \inf \left\{ \lambda > 0 : \frac{1}{|R|} \int_R \varphi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1 \right\}.$$

Theorem 5.15

Let φ be as in Theorem 5.5. Then, for any non-atomic measure μ, we have that $\mathrm{bmo}(\mu) = \mathrm{bmo}_\varphi(\mu)$ and

$$\varphi^{-1}(1)\, \|f\|_{\mathrm{bmo}_\varphi(\mu)} \leq \|f\|_{\mathrm{bmo}(\mu)} \leq c_{\varphi,n}\, \|f\|_{\mathrm{bmo}_\varphi(\mu)}.$$

At this point, the only important thing is to show that we do have an appropriate decomposition lemma. We include here the statement of the aforementioned Rising sun lemma.

Lemma 5.16 – Riesz's Rising Sun

Let R be a rectangle in $\mathbb{R}^n$ and let μ be a measure such that $\mu(\partial P) = 0$ for any rectangle P (for example, a measure satisfying (5.12) or, more generally, any non-atomic measure). Let h be a real function in $L^1_\mu(R)$ and let $\lambda > h_R$. There exist an at most countable family of pairwise disjoint rectangles $R_j \subset R$ such that $h_{R_j} = \lambda$ and $h(x) \leq \lambda$ for almost every $x \in R \setminus \bigcup_j R_j$.

Moreover, the total mass of the selected cubes cannot be too big, meaning that if $h \geq 0$,

$$\sum_j \mu(R_j) = \sum_j \frac{1}{\lambda} \int_{R_j} h(x)\, d\mu(x) \leq \frac{\mu(R)}{\lambda} \fint_R h(x)\, d\mu(x).$$

Equipped with this lemma, the rest of the proof of Theorem 5.15 follows the exact same steps as in Theorem 5.5. The relevant quantity is of course

$$X = \sup_{R \text{ rectangle}} \fint_R |f(x) - c_R|\, d\mu(x), \tag{5.13}$$

where c_R is a constant such that

$$\fint_R \varphi(f(x) - c_R) d\mu(x) \leq 2.$$

Note that Lemma 5.16 is particularly well adapted to the setting of rectangles (and not useful for cubes) since the decomposition is always into rectangles, even if we start with a cube (see the discussion in [75]). Then, when intercalating the average of the form

$$\fint_{R_j} |f(x) - c_{R_j}|\, d\mu(x).$$

from the decomposition, we can control it by using our X defined in (5.13), so the proof of Theorem 5.15 can be obtained following the same line of ideas.

6

Hajłasz capacity density condition

In this chapter, we discuss and expand on the results that were obtained in

[16] Canto, J., Vähäkangas, A.V., *Hajłasz capacity density condition is self-improving,* arXiv:2108.09077v1.

That work was done during the short-stay of my PhD in University of Jyväskylä that, due to the COVID-19 pandemic had to be done online. I want to explicitly thank Antti Vähäkangas again for all the hard work that he did in order to make the online stay as welcoming and productive as it turned out to be.

Since the work that is treated in this chapter was done in the framework of the research visit, its relation to the rest of the thesis is not straightforward. The collaboration started by trying to prove self-improvement of some fractional Hardy inequalities in metric spaces, and due to the unpredictable nature of math, we ended up with the concepts that eventually find their place in this Chapter.

The results in this chapter are based on quantitative estimates and absorption arguments, where it is often crucial to track the dependencies of constants quantitatively. For this purpose, we will use the following notational convention: $C(*, \cdots, *)$ denotes a positive constant which quantitatively depends on the quantities indicated by the $*$'s but whose actual value can change from one occurrence to another, even within a single line. We remark that, in this chapter, there is no possible confusion with the weights of class C_p as there might have been in Chapters 2 and 3.

6.1 Introduction

In this chapter, we introduce a Hajłasz (β, p)-capacity density condition in terms of Hajłasz gradients of order $0 < \beta \leq 1$, see Sections 6.3 and 6.4. The main result, Theorem 6.39, states that this condition is doubly open-ended, that is, a Hajłasz (β, p)-capacity density condition is self-improving both in p and in β if X is a complete geodesic space. This result is new, since previously there was no self-improvement of similar non-local capacity density conditions in metric spaces.

The study of such conditions can be traced back to the seminal work by Lewis [88], who established self-improvement of Riesz (β, p)-capacity density conditions in $\mathbb{R}^n$. His result has been followed by other works incorporating different techniques often in metric spaces, like nonlinear potential theory [6, 98], and local Hardy inequalities [81].

A distinctive feature of our result is that we prove the self-improvement of a capacity density condition for a non-local gradient for the first time in metric spaces. We make use of a recent advance [74] in Poincaré inequalities, whose self-improvement properties were originally shown by Keith–Zhong in their celebrated work [71]. In this respect, we join the line of research initiated in [76], and continued in [33, 34], for bringing together the seemingly distinct self-improvement properties of capacity density conditions and Poincaré inequalities.

We use various techniques and concepts in the proof of Theorem 6.39. The fundamental idea is to use a geometric concept, more precisely the upper Assouad codimension, and characterize the capacity density with a strict upper bound on this codimension. Here we are motivated by the recent approach from [32], where the Assouad codimension bound is used to give necessary and sufficient conditions for certain fractional Hardy inequalities; we also refer to [80]. The principal difficulty is to prove a strict bound on the codimension. To this end we relate the capacity density condition to boundary Poincaré inequalities, and we show their self-improvement roughly speaking in two steps: (1) Keith–Zhong estimates on maximal functions and (2) Koskela–Zhong estimates on Hardy inequalities. For these purposes, respectively, we adapt the maximal function methods from [74] and the local Hardy arguments from [81].

One of the main challenges our method is able to overcome is the nonlocal nature of Hajłasz gradients [44]. More specifically, if a function u is constant in a set $A \subset X$ and g is a Hajłasz gradient of u, then $g\chi_{X \setminus A}$ is not necessarily a Hajłasz gradient of u. This fact makes it impossible to directly use the standard localization techniques. More specifically, there is no access to neither pointwise glueing lemma nor pointwise Leibniz rule, both of which are used while proving similar self-improvement properties for capacity density conditions involving p-weak upper gradients, for example, by the Wannebo approach [114] that was used in [81] to show corresponding local Hardy inequalities. The Hajłasz gradients satisfy nonlocal versions of these basic tools, both of which we employ in our method.

There is a clear advantage to working with Hajłasz gradients: Poincaré inequalities hold for all measures, see Section 6.3. Other types of gradients, such as p-weak upper gradients [4], do not have this property and therefore corresponding Poincaré inequalities need to be assumed a priori, as was the case in previous works such as [6, 33, 34, 81, 98]. We remark that this requirement already excludes many doubling measures in $\mathbb{R}$ equipped with Euclidean distance [5]. Our method has also a disadvantage. We need to assume that X is a complete geodesic space. We do not know how far this condition could be relaxed.

The outline of this paper is as follows. After a brief discussion on notation and preliminary concepts in Section 6.2, Hajłasz gradients are introduced in Section 6.3

along with their calculus and various Poincaré inequalities. Capacity density condition is discussed in Section 6.4, and some preliminary sufficient and necessary bounds on the Assouad codimension are given in Section 6.5. The most technical part of the work is contained in Sections 6.6, 6.7 and 6.8, in which the analytic framework of the self-improvement is gradually developed. Finally, the main result is given in Section 6.9, in which we show that various geometrical and analytical conditions are equivalent to the capacity density condition. The geometrical conditions are open-ended by definition, and hence all analytical conditions are seen to be self-improving or doubly open-ended.

6.2 Notation

Let us introduce some concepts that we will need in order to state the main concepts

6.2.1 Metric spaces

Throughout this chapter, unless otherwise specified, we are going to work with a metric measure space $X = (X, d, \mu)$, that is, a point-set X equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$, each of which is always an open set of the form

$$B = B(x, r) = \{y \in X \,:\, d(y, x) < r\}$$

with $x \in X$ and $r > 0$. As in [4, p. 2], we extend μ as a Borel regular (outer) measure on X. We remark that the space X is separable under these assumptions, see [4, Proposition 1.6]. We also assume that $\#X \geq 2$ and that the measure μ is *doubling*, that is, there is a constant $c_\mu > 1$, called the *doubling constant of* μ, such that

$$\mu(2B) \leq c_\mu \, \mu(B) \tag{6.1}$$

for all balls $B = B(x, r)$ in X. Here we use for $0 < t < \infty$ the notation $tB = B(x, tr)$. In particular, for all balls $B = B(x, r)$ that are centered at $x \in A \subset X$ with radius $r \leq \operatorname{diam}(A)$, we have that

$$\frac{\mu(B)}{\mu(A)} \geq 2^{-s} \left(\frac{r}{\operatorname{diam}(A)} \right)^s, \tag{6.2}$$

where $s = \log_2 c_\mu > 0$. We refer to [54, p. 31].

The closure of a set $A \subset X$ is denoted by $\overline{A}$. In particular, if $B \subset X$ is a ball, then the notation $\overline{B}$ refers to the closure of the ball B. We remark that the closure of an open ball may not be the same set as the closed ball with the same center and radius.

Remark 6.1 Although the setting is similar to that of spaces of homogeneous type that we discussed in Section 5.4, we remark that it is not quite the same, since in this case, the result of Macías–Segovia [92] that ensures the measurability of balls is no longer available.

6.2.2 Geodesic spaces

For some of the results we actually need more structure in the ambient space than just metric structure. That is why we introduce geodesic spaces.

Let X be a metric space satisfying the conditions stated in Section 6.2.1. By a *curve* we mean a nonconstant, rectifiable, continuous mapping from a compact interval of $\mathbb{R}$ to X; we tacitly assume that all curves are parametrized by their arc-length. We say that X is a *geodesic space*, if every pair of points in X can be joined by a curve whose length is equal to the distance between the two points. In particular, it easily follows that

$$0 < \operatorname{diam}(2B) \le 4 \operatorname{diam}(B) \tag{6.3}$$

for all balls $B = B(x, r)$ in a geodesic space X. The measure μ is reverse doubling in a geodesic space X, in the sense that there is a constant $0 < c_R = C(c_\mu) < 1$ such that

$$\mu(B(x, r/2)) \le c_R\, \mu(B(x, r)) \tag{6.4}$$

for every $x \in X$ and $0 < r < \operatorname{diam}(X)/2$. See for instance [4, Lemma 3.7].

We now borrow two lemmas that will be useful to us in the sequel. The first lemma concerns continuity on the radius of the measure of balls for a fixed measurable set, whereas the second lemma ensures that the restriction of the measure (and therefore the corresponding σ-algebra) to a ball still gives a doubling measure, in geodesic spaces.

Lemma 6.2 – Lemma 12.1.2, [56]

Suppose that X is a geodesic space and $A \subset X$ is a measurable set. Then the function

$$r \mapsto \frac{\mu(B(x, r) \cap A)}{\mu(B(x, r))} : (0, \infty) \to \mathbb{R}$$

is continuous whenever $x \in X$.

Lemma 6.3 – Lemma 2.5, [74]

Suppose that $B = B(x, r)$ and $B' = B(x', r')$ are two balls in a geodesic space X such that $x' \in B$ and $0 < r' \le \operatorname{diam}(B)$. Then $\mu(B') \le c_\mu^3 \mu(B' \cap B)$.

Proof. It suffices to find $y \in X$ such that $B(y, r'/4) \subset B' \cap B$. Inequality $\mu(B') \le c_\mu^3 \mu(B' \cap B)$ then follows from the doubling condition (6.1) and the fact that $B' \subset B(y, 2r')$.

Assume first that $x \in B(x', r'/4)$. In this case we may choose $y = x'$, since we have for all $z \in B(x', r'/4)$ that

$$d(z, x) \le d(z, x') + d(x', x) < r'/4 + r'/4 = r'/2 \le \operatorname{diam}(B)/2 \le r\,,$$

and hence $B(x', r'/4) \subset B' \cap B(x, r) = B' \cap B$.

Let us then consider the case $x \notin B(x', r'/4)$. Since X is a geodesic space, there exists an arc-length parametrized curve $\gamma\colon [0, \ell] \to X$ with $\gamma(0) = x'$, $\gamma(\ell) = x$ and $\ell = d(x, x')$. We claim that $y = \gamma(r'/4)$ satisfies the required condition $B(y, r'/4) \subset B' \cap B$. In order to prove the claim, let us fix a point $z \in B(y, r'/4)$. Then

$$d(z, x') \le d(z, y) + d(y, x') < r'/4 + d(\gamma(r'/4), \gamma(0)) \le r'/2 < r'\,.$$

Hence $z \in B(x', r')$ and therefore $B(y, r'/4) \subset B(x', r') = B'$. Moreover, since $\ell = d(x, x')$,

$$\begin{aligned} d(z,x) &\le d(z,y) + d(y,x) < r'/4 + d(\gamma(r'/4), \gamma(\ell)) \\ &\le r'/4 + (\ell - r'/4) = \ell = d(x,x') < r\,. \end{aligned}$$

It follows that $z \in B(x, r) = B$ and therefore $B(y, r'/4) \subset B' \cap B$. □

6.2.3 Hölder and Lipschitz functions

Let $A \subset X$. We say that $u\colon A \to \mathbb{R}$ is a β-*Hölder function,* with an exponent $0 < \beta \le 1$ and a constant $0 \le \kappa < \infty$, if

$$|u(x) - u(y)| \le \kappa\, d(x,y)^\beta \qquad \text{for all } x, y \in A\,.$$

If $u\colon A \to \mathbb{R}$ is a β-Hölder function, with a constant κ, then the classical McShane extension

$$v(x) = \inf\{u(y) + \kappa\, d(x,y)^\beta \,:\, y \in A\}\,, \qquad x \in X\,, \tag{6.5}$$

defines a β-Hölder function $v\colon X \to \mathbb{R}$, with the constant κ, which satisfies $v|_A = u$; we refer to [54, pp. 43–44]. The set of all β-Hölder functions $u\colon A \to \mathbb{R}$ is denoted by $\operatorname{Lip}_\beta(A)$. The 1-Hölder functions are also called *Lipschitz functions.*

6.3 Hajłasz gradients

We work with Hajłasz β-gradients of order $0 < \beta \le 1$ in a metric space X.

Definition 6.4

For each function $u\colon X \to \mathbb{R}$, we let $\mathcal{D}^\beta_H(u)$ be the (possibly empty) family of all measurable functions $g\colon X \to [0, \infty]$ such that

$$|u(x) - u(y)| \le d(x,y)^\beta \big(g(x) + g(y)\big) \tag{6.6}$$

almost everywhere, i.e., there exists an exceptional set $N = N(g) \subset X$ for which $\mu(N) = 0$ and inequality (6.6) holds for every $x, y \in X \setminus N$. A function $g \in \mathcal{D}^\beta_H(u)$ is called a Hajłasz β-gradient of the function u.

The Hajłasz 1-gradients in metric spaces are introduced in [49]. More details on these gradients and their applications can be found, for instance, from [43, 44, 50, 111, 118].

Proposition 6.5

The following basic properties are easy to verify for all β-Hölder functions $u, v\colon X \to \mathbb{R}$

(D1) $|a|g \in \mathcal{D}^\beta_H(au)$ if $a \in \mathbb{R}$ and $g \in \mathcal{D}^\beta_H(u)$;

(D2) $g_u + g_v \in \mathcal{D}^\beta_H(u+v)$ if $g_u \in \mathcal{D}^\beta_H(u)$ and $g_v \in \mathcal{D}^\beta_H(v)$;

(D3) If $f\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant κ, then $\kappa g \in \mathcal{D}^\beta_H(f \circ u)$ if $g \in \mathcal{D}^\beta_H(u)$.

There are both disadvantages and advantages to working with Hajłasz gradients. A technical disadvantage is their nonlocality [44]. For instance, if u is constant on some set $A \subset X$ and $g \in \mathcal{D}_H^\beta(u)$, then $g\chi_{X\setminus A}$ need not belong to $\mathcal{D}_H^\beta(u)$. This is a technical disadvantage while comparing Hajłasz gradients to weak upper gradients, since by glueing lemma, see for instance [4, Lemma 2.19], the corresponding localization property holds. This makes the application of p-weak upper gradients more flexible. However, the following nonlocal glueing lemma from [74, Lemma 6.6] holds in the setting of Hajłasz gradients. We recall the proof for convenience.

Lemma 6.6

Let $0 < \beta \le 1$ and let $A \subset X$ be a Borel set. Let $u\colon X \to \mathbb{R}$ be a β-Hölder function and suppose that $v\colon X \to \mathbb{R}$ is such that $v|_{X\setminus A} = u|_{X\setminus A}$ and there exists a constant $\kappa \ge 0$ such that $|v(x) - v(y)| \le \kappa\, d(x,y)^\beta$ for all $x, y \in X$. Then

$$g_v = \kappa\,\chi_A + g_u\chi_{X\setminus A} \in \mathcal{D}_H^\beta(v)$$

whenever $g_u \in \mathcal{D}_H^\beta(u)$.

Proof. Fix a function $g_u \in \mathcal{D}_H^\beta(u)$ and let $N \subset X$ be the exceptional set such that $\mu(N) = 0$ and inequality (6.6) holds for every $x, y \in X \setminus N$ and with $g = g_u$.

Fix $x, y \in X \setminus N$. If $x, y \in X \setminus A$, then

$$|v(x) - v(y)| = |u(x) - u(y)| \le d(x,y)^\beta \big(g_u(x) + g_u(y)\big) = d(x,y)^\beta \big(g_v(x) + g_v(y)\big)\,.$$

If $x \in A$ or $y \in A$, then

$$|v(x) - v(y)| \le \kappa\, d(x,y)^\beta \le d(x,y)^\beta \big(g_v(x) + g_v(y)\big)\,.$$

By combining the estimates above, we find that

$$|v(x) - v(y)| \le d(x,y)^\beta \big(g_v(x) + g_v(y)\big)$$

whenever $x, y \in X \setminus N$. The desired conclusion $g_v \in \mathcal{D}_H^\beta(v)$ follows. □

The following nonlocal generalization of the Leibniz rule is from [50]. The proof is recalled for the convenience of the reader. The nonlocality is reflected by the appearence of the two global terms $\|\psi\|_\infty$ and κ in the statement below.

Lemma 6.7

Let $0 < \beta \le 1$, let $u\colon X \to \mathbb{R}$ be a bounded β-Hölder function, and let $\psi\colon X \to \mathbb{R}$ be a bounded β-Hölder function with a constant $\kappa \ge 0$. Then $u\psi\colon X \to \mathbb{R}$ is a β-Hölder function and

$$(g_u\|\psi\|_\infty + \kappa|u|)\chi_{\{\psi\neq 0\}} \in \mathcal{D}_H^\beta(u\psi)$$

for all $g_u \in \mathcal{D}_H^\beta(u)$. Here $\{\psi \neq 0\} = \{y \in X : \psi(y) \neq 0\}$.

Proof. Fix $x, y \in X$. Then

$$\begin{aligned} |u(x)\psi(x) - u(y)\psi(y)| &= |u(x)\psi(x) - u(y)\psi(x) + u(y)\psi(x) - u(y)\psi(y)| \\ &\le |\psi(x)||u(x) - u(y)| + |u(y)||\psi(x) - \psi(y)|\,. \end{aligned} \tag{6.7}$$

Since u and ψ are both bounded β-Hölder functions in X, it follows that $u\psi$ is β-Hölder in X.

Fix a function $g_u \in \mathcal{D}_H^\beta(u)$ and let $N \subset X$ be the exceptional set such that $\mu(N) = 0$ and inequality (6.6) holds for every $x, y \in X \setminus N$ and with $g = g_u$. Denote $h = (g_u\|\psi\|_\infty + \kappa|u|)\chi_{\{\psi\neq 0\}}$. Let $x, y \in X \setminus N$. It suffices to show that

$$|u(x)\psi(x) - u(y)\psi(y)| \le d(x,y)^\beta (h(x) + h(y)) .$$

By (6.7), we get

$$\begin{aligned} |u(x)\psi(x) - u(y)\psi(y)| &\le |\psi(x)|d(x,y)^\beta (g_u(x) + g_u(y)) + |u(y)|\kappa d(x,y)^\beta \\ &= d(x,y)^\beta \left(|\psi(x)|(g_u(x) + g_u(y)) + \kappa|u(y)|\right) . \end{aligned} \tag{6.8}$$

Next we do a case study. If $x, y \in \{\psi \neq 0\}$, then by (6.8) we have

$$\begin{aligned} |u(x)\psi(x) - u(y)\psi(y)| &\le d(x,y)^\beta \big(g_u(x)\|\psi\|_\infty \chi_{\{\psi\neq 0\}}(x) \\ &\qquad + (g_u(y)\|\psi\|_\infty + \kappa|u(y)|)\chi_{\{\psi\neq 0\}}(y)\big) \\ &\le d(x,y)^\beta (h(x) + h(y)) . \end{aligned}$$

If $x \in X \setminus \{\psi \neq 0\}$ and $y \in \{\psi \neq 0\}$, then

$$\begin{aligned} |u(x)\psi(x) - u(y)\psi(y)| &\le d(x,y)^\beta \left(\kappa|u(y)|\chi_{\{\psi\neq 0\}}(y)\right) \\ &= d(x,y)^\beta h(y) \le d(x,y)^\beta (h(x) + h(y)) . \end{aligned}$$

The case $x \in \{\psi \neq 0\}$ and $y \in X\setminus\{\psi \neq 0\}$ is symmetric and the last case is trivial. □

A significant advantage of working with Hajłasz gradients is that Poincaré inequalities are always valid [43, 118]. The same is not true for the usual p-weak upper gradients, in which case a Poincaré inequality often has to be assumed.

The following theorem gives a (β, p, p)-Poincaré inequality for any $1 \le p < \infty$. This inequality relates the Hajłasz gradient to the given measure.

Theorem 6.8

Suppose that X is a metric space. Fix exponents $1 \le p < \infty$ and $0 < \beta \le 1$. Suppose that $u \in \operatorname{Lip}_\beta(X)$ and that $g \in \mathcal{D}_H^\beta(u)$. Then

$$\left(\fint_B |u(x) - u_B|^p \, d\mu(x)\right)^{1/p} \le 2 \operatorname{diam}(B)^\beta \left(\fint_B g(x)^p \, d\mu(x)\right)^{1/p}$$

holds whenever $B \subset X$ is a ball.

Proof. We follow the proof of [54, Theorem 5.15]. Let $N = N(g) \subset X$ be the exceptional set such that $\mu(N) = 0$ and (6.6) holds for every $x, y \in X \setminus N$. By Hölder's inequality

$$\fint_B |u(x) - u_B|^p \, d\mu(x) \le \fint_{B\setminus N} \fint_{B\setminus N} |u(y) - u(x)|^p \, d\mu(y) \, d\mu(x) .$$

Applying (6.6), we obtain

$$\begin{aligned}
&\fint_{B\setminus N}\fint_{B\setminus N}|u(y)-u(x)|^p\,d\mu(y)\,d\mu(x)\\
&\qquad\le \fint_{B\setminus N}\fint_{B\setminus N} d(x,y)^{\beta p}\,(g(x)+g(y))^p\,d\mu(y)\,d\mu(x)\\
&\qquad\le 2^{p-1}\operatorname{diam}(B)^{\beta p}\fint_{B\setminus N}\fint_{B\setminus N}(g(x)^p+g(y)^p)\,d\mu(y)\,d\mu(x)\\
&\qquad\le 2^{p}\operatorname{diam}(B)^{\beta p}\fint_{B} g(x)^p\,d\mu(x)\,.
\end{aligned}$$

The claimed inequality follows by combining the above estimates. □

In a geodesic space, even a stronger (β, p, q)-Poincaré inequality holds for some $q < p$. In the context of p-weak upper gradients, this result corresponds to the deep theorem of Keith and Zhong [71]. In our context the proof is simpler, since we have (β, q, q)-Poincaré inequalities for all exponents $1 < q < p$ by Theorem 6.8. It remains to argue that one of these inequalities self-improves to a (β, p, q)-Poincaré inequality when $q < p$ is sufficiently close to p.

Theorem 6.9

Suppose that X is a geodesic space. Fix exponents $1 < p < \infty$ and $0 < \beta \le 1$. Suppose that $u \in \operatorname{Lip}_\beta(X)$ and that $g \in \mathcal{D}_H^\beta(u)$. Then there exists an exponent $1 < q < p$ and a constant C, both depending on c_μ, p and β, such that

$$\left(\fint_B |u(x)-u_B|^p\,d\mu(x)\right)^{1/p} \le C\operatorname{diam}(B)^\beta\left(\fint_B g(x)^q\,d\mu(x)\right)^{1/q}$$

holds whenever $B \subset X$ is a ball.

Proof. Fix $Q = Q(\beta, p, c_\mu)$ such that $Q > \max\{\log_2 c_\mu, \beta p\}$. Since

$$\lim_{q\to p} Qq/(Q-\beta q) = Qp/(Q-\beta p) > p\,,$$

there exists $1 < q = q(\beta, p, c_\mu) < p$ such that $p < Qq/(Q-\beta q)$ and $\beta q < Q$. Theorem 6.8 and Hölder's inequality implies that

$$\fint_B |u(x)-u_B|\,d\mu(x) \le \left(\fint_B |u(x)-u_B|^q\,d\mu(x)\right)^{1/q} \le 2\operatorname{diam}(B)^\beta\left(\fint_B g(x)^q\,d\mu(x)\right)^{1/q}$$

whenever $B \subset X$ is a ball. Now the claim follows from [74, Theorem 3.6], which is based on the covering argument from [51]. We also refer to [42, Lemma 2.2]. □

6.4 Capacity density condition

In this section we define the capacity density condition. This condition is based on the following notion of variational capacity, and it is weaker than the well known measure density condition. We also prove boundary Poincaré inequalities for sets satisfying a capacity density condition. This is done with the aid of so-called Maz′ya's inequality, which provides an important link between Poincaré inequalities and capacities.

Definition 6.10
Let $1 \le p < \infty$, $0 < \beta \le 1$, and let $\Omega \subset X$ be an open set. The variational (β, p)-capacity of a subset $F \subset \Omega$ with $\operatorname{dist}(F, X \setminus \Omega) > 0$ is

$$\operatorname{cap}_{\beta,p}(F,\Omega) = \inf_u \inf_g \int_X g(x)^p \, d\mu(x)\,,$$

where the infimums are taken over all β-Hölder functions u in X, with $u \ge 1$ in F and $u = 0$ in $X \setminus \Omega$, and over all $g \in \mathcal{D}_H^\beta(u)$.

Remark 6.11 We may take the infimum in Definition 6.10 among all u satisfying additionally $0 \le u \le 1$. This follows by considering the β-Hölder function function $v = \max\{0, \min\{u, 1\}\}$ since $g \in \mathcal{D}_H^\beta(v)$ by Property (D3) of Proposition 6.5.

Definition 6.12
A closed set $E \subset X$ satisfies a (β, p)-capacity density condition, for $1 \le p < \infty$ and $0 < \beta \le 1$, if there exists a constant $c_0 > 0$ such that

$$\operatorname{cap}_{\beta,p}(E \cap \overline{B(x,r)}, B(x,2r)) \ge c_0 r^{-\beta p} \mu(B(x,r)) \tag{6.9}$$

for all $x \in E$ and all $0 < r < (1/8)\operatorname{diam}(E)$.

Remark 6.13 The upper bound of r by a multiple of the diameter of the set responds to the fact that we are somehow more concerned with what happens locally, that is, close to the set. If such bound was not imposed, and if $\mu(X) = \infty$, no bounded set could ever satisfy (6.9).

Example 6.14
We say that a closed set $E \subset X$ satisfies a measure density condition, if there exists a constant c_1 such that

$$\mu(E \cap \overline{B(x,r)}) \ge c_1 \, \mu(B(x,r)) \tag{6.10}$$

for all $x \in E$ and all $0 < r < (1/8)\operatorname{diam}(E)$. Assume that $1 \le p < \infty$ and $0 < \beta \le 1$, and that a set E satisfies a measure density condition. Then it is easy to show that E satisfies a (β, p)-capacity density condition, see below. We remark that the measure density condition has been applied in [72] to study Hajłasz Sobolev spaces with zero boundary values on E.

Fix $x \in E$ and $0 < r < (1/8)\operatorname{diam}(E)$. We aim to show that (6.9) holds. For this purpose, we write $F = E \cap \overline{B(x,r)}$ and $B = B(x,r)$. Let $u \in \operatorname{Lip}_\beta(X)$ be such that $0 \le u \le 1$, $u = 1$ in F and $u = 0$ in $X \setminus 2B$. Let also $g \in \mathcal{D}_H^\beta(u)$. By the properties of u and the reverse doubling inequality (6.4), we obtain

$$0 \le u_{4B} = \fint_{4B} u(y)\, d\mu(y) \le \frac{\mu(2B)}{\mu(4B)} \le c_R < 1.$$

If $y \in F$, we have $v(y) = 1$ and therefore

$$|u(y) - v_{4B}| \geq 1 - u_{4B} \geq 1 - c_R = C(c_\mu) > 0.$$

Applying the measure density condition (6.10) and the (β, p, p)-Poincaré inequality, see Theorem 6.8, we obtain

$$\begin{aligned} c_1 \mu(B) \leq \mu(F) &\leq C(c_\mu, p) \int_F |u(y) - u_{4B}|^p \, d\mu(y) \\ &\leq C(c_\mu, p) \int_{4B} |u(y) - u_{4B}|^p \, d\mu(y) \\ &\leq C(c_\mu, p) r^{\beta p} \int_{4B} g(y)^p \, d\mu(y) \leq C(c_\mu, p) r^{\beta p} \int_X g(y)^p \, d\mu(y) \,. \end{aligned}$$

By taking infimum over functions u and g as above, we see that

$$\operatorname{cap}_{\beta,p}(E \cap \overline{B(x,r)}, 2B) = \operatorname{cap}_{\beta,p}(F, 2B) \geq C(c_1, c_\mu, p) r^{-\beta p} \mu(B) \,.$$

This shows that E satisfies a (β, p)-capacity density condition (6.9).

Next, we introduce a Maz′ya-type inequality. This inequality establishes a link between capacities and Poincaré inequalities. More precisely, we can bound the size of u in a ball by the ratio of the size if a Hajłasz gradient and the capacity of the zero set of u in said ball. We refer to [96, Chapter 10] and [97, Chapter 14] for further details on inequalities of this type.

Theorem 6.15

Let $1 \leq p < \infty$, $0 < \beta \leq 1$, and let $B(z,r) \subset X$ be a ball. Assume that u is a β-Hölder function in X and $g \in \mathcal{D}^\beta_H(u)$. Then there exists a constant $C = C(p)$ such that

$$\fint_{B(z,r)} |u(x)|^p \, d\mu(x) \leq \frac{C}{\operatorname{cap}_{\beta,p}\big(\{u=0\} \cap \overline{B(z,\frac{r}{2})}, B(z,r)\big)} \int_{B(z,r)} g(x)^p \, d\mu(x) \,.$$

Here $\{u = 0\} = \{y \in X : u(y) = 0\}$.

Proof. We adapt the proof of [73, Theorem 5.47], which in turn is based on [4, Theorem 5.53]. Let $M = \sup\{|u(x)| : x \in B(z,r)\} < \infty$. By considering $\min\{M, |u|\}$ instead of u and using (D3), we may assume that u is a bounded β-Hölder function in X and that $u \geq 0$ in $B(z,r)$. Write $B = B(z,r)$ and

$$u_{B,p} = \left(\fint_B u(x)^p \, d\mu(x)\right)^{\frac{1}{p}} = \mu(B)^{-\frac{1}{p}} \|u\|_{L^p(B)} < \infty \,.$$

If $u_{B,p} = 0$ the claim is true, and thus we may assume that $u_{B,p} > 0$. We want to choose a test function for the capacity. In order to do that, we define the cut-off function ψ by the expression

$$\psi(x) = \max\Big\{0, 1 - (2r^{-1})^\beta d\big(x, B(z, \tfrac{r}{2})\big)^\beta\Big\}$$

for every $x \in X$. Then $0 \le \psi \le 1$, $\psi = 0$ in $X \setminus B(z,r)$, $\psi = 1$ in $\overline{B(z,\frac{r}{2})}$, and ψ is a β-Hölder function in X with a constant $(2r^{-1})^\beta$. Let

$$v(x) = \psi(x)\left(1 - \frac{u(x)}{u_{B,p}}\right), \qquad x \in X\,.$$

Then $v = 1$ in $\{u = 0\} \cap \overline{B(z,\frac{r}{2})}$ and $v = 0$ in $X \setminus B(z,r)$. By Lemma 6.7, and properties (D1) and (D2), the function v is β-Hölder in X and

$$g_v = \left(\frac{g}{u_{B,p}}\|\psi\|_\infty + (2r^{-1})^\beta\left|1 - \frac{u}{u_{B,p}}\right|\right)\chi_{\{\psi\neq 0\}} \in \mathcal{D}_H^\beta(v)\,.$$

Here we used the fact that $g \in \mathcal{D}_H^\beta(u)$ by assumptions. Now, the pair v and g_v is admissible for testing the capacity. Thus, we obtain

$$\begin{aligned}\operatorname{cap}_{\beta,p}\big(\{u=0\}\cap\overline{B(z,\tfrac{r}{2})}, B(z,r)\big) &\le \int_X g_v(x)^p\,d\mu(x)\\ &\le \frac{C(p)}{(u_{B,p})^p}\int_B g(x)^p\,d\mu(x) + \frac{C(p)}{r^{\beta p}(u_{B,p})^p}\int_B |u(x)-u_{B,p}|^p\,d\mu(x)\,.\end{aligned} \tag{6.11}$$

We use Minkowski's inequality and the (β,p,p)-Poincaré inequality in Theorem 6.8 to estimate the second term on the right-hand side of (6.11), and obtain

$$\begin{aligned}\left(\fint_B |u(x)-u_{B,p}|^p\,d\mu(x)\right)^{\frac{1}{p}} &\le \left(\fint_B |u(x)-u_B|^p\,d\mu(x)\right)^{\frac{1}{p}} + |u_{B,p}-u_B|\\ &\le Cr^\beta\left(\fint_B g(x)^p\,d\mu(x)\right)^{\frac{1}{p}} + \mu(B)^{-\frac{1}{p}}\big|\|u\|_{L^p(B)} - \|u_B\|_{L^p(B)}\big|.\end{aligned} \tag{6.12}$$

By the triangle inequality and the above Poincaré inequality, we have

$$\begin{aligned}\mu(B)^{-\frac{1}{p}}\big|\|u\|_{L^p(B)} - \|u_B\|_{L^p(B)}\big| &\le \mu(B)^{-\frac{1}{p}}\|u-u_B\|_{L^p(B)}\\ &= \left(\fint_B |u(x)-u_B|^p\,d\mu(x)\right)^{\frac{1}{p}}\\ &\le Cr^\beta\left(\fint_B g(x)^p\,d\mu(x)\right)^{\frac{1}{p}}.\end{aligned}$$

Together with (6.12) this gives

$$\left(\fint_B |u(x)-u_{B,p}|^p\,d\mu(x)\right)^{\frac{1}{p}} \le Cr^\beta\left(\fint_B g(x)^p\,d\mu(x)\right)^{\frac{1}{p}},$$

and thus

$$\int_B |u(x)-u_{B,p}|^p\,d\mu(x) \le C(p)r^{\beta p}\int_B g(x)^p\,d\mu(x).$$

Substituting this to (6.11) and recalling that $B = B(z,r)$, we arrive at

$$\operatorname{cap}_{\beta,p}\big(\{u=0\}\cap\overline{B(z,\tfrac{r}{2})}, B(z,r)\big) \le \frac{C(p)}{(u_{B,p})^p}\int_{B(z,r)} g(x)^p\,d\mu(x).$$

The claim follows by reorganizing the terms. □

We now establish a boundary Poincaré inequality for a set E satisfying a capacity density condition. More precisely, we prove that a function u that vanishes on the set E satisfies Poincaré inequalities at balls centered in E. The Maz′ya-type inequality in Theorem 6.15 is a key tool in the proof.

Theorem 6.16

Let $1 \le p < \infty$ and $0 < \beta \le 1$. Assume that $E \subset X$ satisfies a (β, p)-capacity density condition with a constant c_0. Then there is a constant $C = C(p, c_0, c_\mu)$ such that

$$\fint_{B(x,R)} |u(x)|^p \, d\mu(x) \le C R^{\beta p} \fint_{B(x,R)} g(x)^p \, d\mu(x)$$

whenever $u\colon X \to \mathbb{R}$ is a β-Hölder function in X such that $u = 0$ in E, $g \in \mathcal{D}_H^\beta(u)$, and $B(x, R)$ is a ball with $x \in E$ and $0 < R < \operatorname{diam}(E)/4$.

Proof. Let $x \in E$ and $0 < R < \operatorname{diam}(E)/4$. We denote $r = R/2 < \operatorname{diam}(E)/8$. Applying the capacity density condition in the ball $B = B(x, r)$ gives

$$\operatorname{cap}_{\beta,p}(E \cap \overline{B}, 2B) \ge c_0 r^{-\beta p} \mu(B) \,.$$

Write $\{u = 0\} = \{y \in X : u(y) = 0\} \supset E$. By the monotonicity of capacity and the doubling condition we have

$$\frac{1}{\operatorname{cap}_{\beta,p}(\{u = 0\} \cap \overline{B}, 2B)} \le \frac{1}{\operatorname{cap}_{\beta,p}(E \cap \overline{B}, 2B)} \le \frac{C(c_0) r^{\beta p}}{\mu(B)} \le \frac{C(c_0, c_\mu) R^{\beta p}}{\mu(2B)} .$$

The desired inequality, for the ball $B(x, R) = B(x, 2r) = 2B$, follows from Theorem 6.15. □

6.5 Necessary and sufficient geometrical conditions

In this section we adapt the approach in [32] by giving necessary and sufficient geometrical conditions for a set E to satisfy the (β, p)-capacity density condition. These conditions are given in terms of some bounds for the upper Assouad codimension [68], which we introduce now.

Definition 6.17

When $E \subset X$ and $r > 0$, the open r-neighbourhood of E is the set

$$E_r = \{x \in X : d(x, E) < r\}.$$

The upper Assouad codimension of $E \subset X$, denoted by $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E)$, is the infimum of all $Q \ge 0$ for which there is $c > 0$ such that

$$\frac{\mu(E_r \cap B(x, R))}{\mu(B(x, R))} \ge c\Big(\frac{r}{R}\Big)^Q$$

for every $x \in E$ and all $0 < r < R < \operatorname{diam}(E)$. If $\operatorname{diam}(E) = 0$, then the restriction $R < \operatorname{diam}(E)$ is removed.

Observe that a larger set has a smaller Assouad codimension. In order to develop our methods, we are going to use suitable versions of Hausdorff contents that we borrow from [80].

Definition 6.18

The (ρ-restricted) Hausdorff content of codimension $q \geq 0$ of a set $F \subset X$ is defined by

$$\mathcal{H}_\rho^{\mu,q}(F) = \inf\left\{ \sum_k \mu(B(x_k, r_k))\, r_k^{-q} : F \subset \bigcup_k B(x_k, r_k) \text{ and } 0 < r_k \leq \rho \right\}.$$

We are going to give bounds for this Hausdorff content in terms of the measure and the capacity. On the one hand, we state a lower bound for the Hausdorff content of a set truncated in a fixed ball in terms of the measure and radius of the truncating ball. The proof uses completeness via construction of a compact Cantor-type set inside E, to which mass is uniformly distributed by a Carathéodory construction.

Lemma 6.19 – [80, Lemma 5.1]

Assume that X is a complete metric space. Let $E \subset X$ be a closed set, and assume that $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) < q$. Then there exists a constant $C > 0$ such that

$$\mathcal{H}_r^{\mu,q}(E \cap \overline{B(x,r)}) \geq C r^{-q} \mu(B(x,r)) \tag{6.13}$$

for every $x \in E$ and all $0 < r < \operatorname{diam}(E)$.

On the other hand, Hausdorff contents gives a lower bound for capacity by following lemma. The proof is based on a covering argument, where the covering balls are chosen by chaining. The proof is a more sophisticated variant of the argument given in Example 6.14. Similar covering arguments via chaining have been widely used; see for instance [55].

Lemma 6.20

Let $0 < \beta \leq 1$, $1 \leq p < \infty$, and $0 < \eta < p$. Assume that $B = B(x_0, r) \subset X$ is a ball with $r < \operatorname{diam}(X)/8$, and assume that $F \subset \overline{B}$ is a closed set. Then there is a constant $C = C(\beta, p, \eta, c_\mu) > 0$ such that

$$r^{\beta(p-\eta)} \operatorname{cap}_{\beta,p}(F, 2B) \geq C \mathcal{H}_{20r}^{\mu,\beta\eta}(F)\,.$$

Proof. We adapt the proof of [32, Lemma 4.6] for our purposes. Let $u \in \operatorname{Lip}_\beta(X)$ be such that $0 \leq u \leq 1$ in X, $u = 1$ in F and $u = 0$ in $X \setminus 2B$. Let also $g \in \mathcal{D}_H^\beta(u)$. We aim to cover the set F by balls that are chosen by chaining. In order to do so, we fix $x \in F$ and write $B_0 = 4B = B(x_0, 4r)$, $r_0 = 4r$, $r_j = 2^{-j+1} r$ and $B_j = B(x, r_j)$, $j = 1, 2, \ldots$. Observe that $B_{j+1} \subset B_j$ and $\mu(B_j) \leq c_\mu^3 \mu(B_{j+1})$ if $j = 0, 1, 2, \ldots$.

By the above properties of u and the reverse doubling inequality (6.4), we obtain

$$0 \leq u_{B_0} = \fint_{B_0} u(y)\, d\mu(y) \leq \frac{\mu(2B)}{\mu(4B)} \leq c_R < 1.$$

Since $x \in F$, we find that $u(x) = 1$ and therefore

$$|u(x) - u_{B_0}| \geq 1 - u_{B_0} \geq 1 - c_R = C(c_\mu) > 0.$$

We write $\delta = \beta(p - \eta)/p > 0$. Using the Poincaré inequality in Theorem 6.8 and abbreviating $C = C(\beta, p, \eta, c_\mu)$, we obtain

$$\begin{aligned}
\sum_{j=0}^{\infty} 2^{-j\delta} &= C(1 - c_R) \leq C|u(x) - u_{B_0}| \\
&\leq C \sum_{j=0}^{\infty} |u_{B_{j+1}} - u_{B_j}| \leq C \sum_{j=0}^{\infty} \frac{\mu(B_j)}{\mu(B_{j+1})} \fint_{B_j} |u(y) - u_{B_j}| \, d\mu(y) \\
&\leq C \sum_{j=0}^{\infty} \left(\fint_{B_j} |u(y) - u_{B_j}|^p \, d\mu(y) \right)^{\frac{1}{p}} \leq C \sum_{j=0}^{\infty} r_j^{\beta} \left(\fint_{B_j} g(y)^p \, d\mu(y) \right)^{\frac{1}{p}}.
\end{aligned}$$

By comparing the series in the left- and right-hand side of these inequalities, we see that there exists $j \in \{0, 1, 2, \ldots\}$ depending on x such that

$$2^{-j\delta p} \leq C(\beta, p, \eta, c_\mu) r_j^{\beta p} \fint_{B_j} g(y)^p \, d\mu(y). \tag{6.14}$$

Write $r_x = r_j$ and $B_x = B_j$. Then $x \in B_x$ and straightforward estimates based on (6.14) give

$$\mu(B_x) r_x^{-\beta\eta} \leq C(\beta, p, \eta, c_\mu) r^{\beta(p-\eta)} \int_{B_x} g(y)^p \, d\mu(y)\,.$$

By the Vitali covering lemma 1.2, see also [4, Lemma 1.7], we obtain points $x_k \in F$, $k = 1, 2, \ldots$, such that the balls $B_{x_k} \subset B_0 = 4B$ with radii $r_{x_k} \leq 4r$ are pairwise disjoint and $F \subset \bigcup_{k=1}^{\infty} 5B_{x_k}$. Hence,

$$\begin{aligned}
\mathcal{H}^{\mu,\beta\eta}_{20r}(F) &\leq \sum_{k=1}^{\infty} \mu(5B_{x_k})(5r_{x_k})^{-\beta\eta} \leq C \sum_{k=1}^{\infty} r^{\beta(p-\eta)} \int_{B_{x_k}} g(x)^p \, d\mu(x) \\
&\leq C r^{\beta(p-\eta)} \int_{4B} g(x)^p \, d\mu(x) \leq C r^{\beta(p-\eta)} \int_X g(x)^p \, d\mu(x)\,,
\end{aligned}$$

where $C = C(\beta, p, \eta, c_\mu)$. We remark that the scale $20r$ of the Hausdorff content in the left-hand side comes from the fact that radii of the covering balls $5B_{x_k}$ for F are bounded by $20r$. The desired inequality follows by taking infimum over all functions $g \in \mathcal{D}^{\beta}_{H}(u)$ and then over all functions u as above. □

The following theorem gives an upper bound for the upper Assouad codimension for sets satisfying a capacity density condition. We emphasize the strict inequality $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) < \beta p$ and completeness in the assumptions below.

Theorem 6.21

Assume that X is a complete metric space. Let $1 \leq p < \infty$ and $0 < \beta \leq 1$. Let E be a closed set with $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) < \beta p$. Then E satisfies a (β, p)-capacity density condition.

Proof. Fix $0 < \eta < p$ such that $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) < \beta\eta$. Let $x \in E$ and $0 < r < \operatorname{diam}(E)/8$, and write $B = B(x, r)$. By a simple covering argument using the doubling condition, it follows that $\mathcal{H}^{\mu,\beta\eta}_{20r}(E\cap\overline{B}) \ge C\mathcal{H}^{\mu,\beta\eta}_{r}(E\cap\overline{B})$ with a constant C independent of B. Applying also Lemma 6.20 and then Lemma 6.19 gives

$$r^{\beta(p-\eta)}\operatorname{cap}_{\beta,p}(E\cap\overline{B},2B) \ge C\mathcal{H}^{\mu,\beta\eta}_{20r}(E\cap\overline{B}) \ge C\mathcal{H}^{\mu,\beta\eta}_{r}(E\cap\overline{B}) \ge r^{-\beta\eta}\mu(B)\,.$$

After simplification, we obtain

$$\operatorname{cap}_{\beta,p}(E\cap\overline{B},2B) \ge Cr^{-\beta p}\mu(B)\,,$$

and the claim follows. □

As a partial converse to this result, we prove by using boundary Poincaré inequalities, that a capacity density condition implies an upper bound for the upper Assouad codimension. This upper bound is not strict.

Theorem 6.22

Let $1\le p<\infty$ and $0<\beta\le 1$. Assume that $E\subset X$ satisfies a (β,p)-capacity density condition. Then $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E)\le\beta p$.

Proof. The proof of this Theorem is an adaptation of the proof of [32, Theorem 5.3] to our setting. By using the doubling condition, it suffices to show that

$$\frac{\mu(E_r\cap B(w,R))}{\mu(B(w,R))} \ge c\Big(\frac{r}{R}\Big)^{\beta p}, \tag{6.15}$$

for all $w\in E$ and $0<r<R<\operatorname{diam}(E)/4$, where the constant c is independent of w, r and R.

If $\mu(E_r\cap B(w,R))\ge\frac{1}{2}\mu(B(w,R))$, the claim is clear since $\big(\frac{r}{R}\big)^{\beta p}\le 1$. Thus we may assume in the sequel that $\mu(E_r\cap B(w,R))<\frac{1}{2}\mu(B(w,R))$, whence

$$\mu(B(w,R)\setminus E_r)\ge\tfrac{1}{2}\mu(B(w,R))>0. \tag{6.16}$$

We define a β-Hölder function $u\colon X\to\mathbb{R}$ by

$$u(x)=\min\{1,r^{-\beta}d(x,E)^{\beta}\}\,,\qquad x\in X.$$

Then $u=0$ in E, $u=1$ in $X\setminus E_r$, and

$$|u(x)-u(y)|\le r^{-\beta}d(x,y)^{\beta}\quad\text{ for all } x,y\in X.$$

We obtain

$$\begin{aligned} R^{-\beta p}\int_{B(w,R)}|u(x)|^p\,d\mu(x) &\ge R^{-\beta p}\int_{B(w,R)\setminus E_r}|u(x)|^p\,d\mu(x)\\ &= R^{-\beta p}\mu(B(w,R)\setminus E_r)\\ &\ge \tfrac{1}{2}R^{-\beta p}\mu(B(w,R)), \end{aligned} \tag{6.17}$$

where the last step follows from (6.16).

Since $u=1$ in $X\setminus E_r$ and u is a β-Hölder function with a constant $r^{-\beta}$, Lemma 6.6 implies that $g=r^{-\beta}\chi_{E_r}\in\mathcal{D}^{\beta}_{H}(u)$. We observe from (6.17) and Theorem 6.16

that

$$
\begin{aligned}
Cr^{-\beta p}\mu(E_r\cap B(w,R)) &= C\int_{B(w,R)} g(x)^p\,d\mu(x)\\
&\ge 2R^{-\beta p}\int_{B(w,R)} |u(x)|^p\,d\mu(x)\\
&\ge R^{-\beta p}\mu(B(w,R))\,,
\end{aligned}
\tag{6.18}
$$

where the constant C is independent of w, r and R. The claim (6.15) follows from (6.18). □

Observe that the upper bound $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) \le \beta p$ appears in the conclusion of Theorem 6.22. The rest of the chapter is devoted to showing the strict inequality $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) < \beta p$ for $1 < p < \infty$, which leads to a characterization of the (β, p)-capacity density condition in terms of this strict dimensional inequality.

The strategy is to combine the methods in [74] and [81] to prove a significantly stronger variant of the boundary Poincaré inequality, which involves maximal operators, see Theorem 6.28. We use this maximal inequality to prove a Hardy inequality, Theorem 6.36. This variant leads to the characterization in Theorem 6.38 of the (β, p)-capacity density condition in terms of the strict inequality $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) < \beta p$, among other geometric and analytic conditions. Certain additional geometric assumptions are needed for the proof of Theorem 6.38, namely geodesic property of X. We are not aware, to which extent this geometric assumption can be relaxed.

6.6 Local boundary Poincaré inequality

Our next aim is to show Theorem 6.28, which concerns inequalities localized to a fixed ball B_0 centered at E. The proof of this theorem requires that we first truncate the closed set E to a smaller set E_Q contained in a Whitney-type ball $\overline{Q} \subset B_0$ such that a local variant of the boundary Poincaré inequality remains valid. The choice of the Whitney-type ball Q and the construction of the set E_Q are given in this section.

This truncation construction, that we borrow from [81], is done in such a way that a local Poincaré inequality holds, see Lemma 6.26. This inequality is local in two senses: on one hand, the inequality holds only for balls $B \subset Q^*$; on the other hand, it holds for functions vanishing on the truncated set E_Q. Due to the subtlety of its consequences, the truncation in this section may seem arbitrary, but it is actually needed for our purposes.

Assume that E is a closed set in a geodesic space X. Fix a ball $B_0 = B(w, R) \subset X$ with $w \in E$ and $R < \operatorname{diam}(E)$. Define a family of balls

$$
\mathcal{B}_0 = \{B \subset X \,:\, B \text{ is a ball such that } B \subset B_0\}\,. \tag{6.19}
$$

We also need a single Whitney-type ball $Q = B(w, r_Q) \subset B_0$, where

$$
r_Q = \frac{R}{128}\,. \tag{6.20}
$$

The 4-dilation of the Whitney-type ball is denoted by $Q^* = 4Q = B(w, 4r_Q)$. Now it holds that $Q^* \subsetneq X$, since otherwise

$$\operatorname{diam}(X) = \operatorname{diam}(Q^*) \le R/16 < \operatorname{diam}(E) \le \operatorname{diam}(X)\,.$$

The following proposition illustrates a few properties that are straightforward to verify. For instance, property (W1) follows from inequality (6.3); we omit the simple proofs.

Proposition 6.23

The following properties hold:

(W1) If $B \subset X$ is a ball such that $B \cap \overline{Q} \ne \emptyset \ne 2B \cap (X \setminus Q^*)$, then $\operatorname{diam}(B) \ge 3r_Q/4$;

(W2) If $B \subset Q^*$ is a ball, then $B \in \mathcal{B}_0$;

(W3) If $B \subset Q^*$ is a ball, $x \in B$ and $0 < r \le \operatorname{diam}(B)$, then $B(x, 5r) \in \mathcal{B}_0$;

(W4) If $x \in Q^*$ and $0 < r \le 2\operatorname{diam}(Q^*)$, then $B(x, r) \in \mathcal{B}_0$.

Observe that there is some overlap between the properties (W2)–(W4). The slightly different formulations will conveniently guide the reader in the sequel.

The following Lemma 6.24 gives us the truncated set $E_Q \subset \overline{Q}$ that contains big pieces of the original set E at small scales. This big pieces property is not always satisfied by $E \cap Q$, so it cannot be used instead.

Lemma 6.24

Assume that $E \subset X$ is a closed set in a geodesic space X and that $Q = B(w, r_Q)$ for $w \in E$ and $r_Q > 0$. Let $E_Q^0 = E \cap \overline{\tfrac{1}{2}Q}$, define inductively, for every $j \in \mathbb{N}$, that

$$E_Q^j = \bigcup_{x \in E_Q^{j-1}} E \cap \overline{B(x, 2^{-j-1}r)}\,, \qquad \text{and set} \qquad E_Q = \overline{\bigcup_{j \in \mathbb{N}_0} E_Q^j}.$$

Then the following statements hold:

(a) $w \in E_Q$;

(b) $E_Q \subset E$;

(c) $E_Q \subset \overline{Q}$;

(d) $E_Q^{j-1} \subset E_Q^j \subset E_Q$ for every $j \in \mathbb{N}$.

Proof. Part (a) is is true since $w \in E_B^0$. Part (b) follows from the facts that E is closed and $\cup_j E_B^j \subset E$ by definition. To verify (c), we fix $x \in E_B^j$. If $j = 0$, then $x \in \overline{B}$. If $j > 0$, then by induction we find a sequence $x_j, \ldots, x_0$ such $x_j = x$ and, for each $k = 0, \ldots, j$, $x_k \in E_B^k$ and $x_k \in E \cap \overline{B(x_{k-1}, 2^{-k-1}r)}$ if $k > 0$. It follows that

$$d(x, w) \le \sum_{k=1}^{j} d(x_k, x_{k-1}) + d(x_0, w) \le \sum_{k=1}^{j} 2^{-k-1}r + 2^{-1}r < r\,.$$

Hence, $x \in B(w,r) \subset \overline{B}$. We have shown that $E_B^j \subset \overline{B}$ whenever $j \in \mathbb{N}_0$, whence it follows that also $E_B \subset \overline{B}$. To prove (d) we fix $j \in \mathbb{N}$ and $x \in E_B^{j-1}$. By definition we have $x \in E$ and, hence, $x \in E \cap B(x, 2^{-j-1}r) \subset E_B^j$. □

The next lemma shows that the truncated set E_Q in Lemma 6.24 really contains big pieces of the original set E at all small scales. By using these balls we later employ the capacity density condition of E, see the proof of Lemma 6.26 for details.

Lemma 6.25

Let E, Q, and E_Q be as in Lemma 6.24. Suppose that $m \in \mathbb{N}_0$ and $x \in X$ is such that $d(x, E_Q) < 2^{-m+1}r_Q$. Then there exists a ball $\widehat{B} = B(y_{x,m}, 2^{-m-1}r_Q)$ such that $y_{x,m} \in E$,

$$E \cap \overline{2^{-1}\widehat{B}} = E_Q \cap \overline{2^{-1}\widehat{B}}\,,$$

and $\widehat{B} \subset B(x, 2^{-m+2}r_Q)$.

Proof. In this proof we will apply Lemma 6.24 several times without further notice. Since $d(x, E_B) < 2^{-m+1}r$ there exists $y \in \cup_{j\in\mathbb{N}_0} E_B^j \subset E$ such that $d(y,x) < 2^{-m+1}r$. Let us fix $j \in \mathbb{N}_0$ such that $y \in E_B^j$. There are two cases to be treated.

First, let us consider the case when $j > m \geq 0$. By induction, there are points $y_k \in E_B^k$ with $k = m, \dots, j$ such that $y_j = y$ and $y_k \in E \cap \overline{B(y_{k-1}, 2^{-k-1}r)}$ for every $k = m+1, \dots, j$. It follows that

$$\begin{aligned} d(y_m, y) &= d(y_j, y_m) \\ &\leq \sum_{k=m+1}^{j} d(y_k, y_{k-1}) \\ &\leq \sum_{k=m+1}^{j} 2^{-k-1}r \\ &< 2^{-m-1}r\,. \end{aligned}$$

Take $y_{x,m} = y_m \in E_B^m \subset E$ and $\widehat{B} = B(y_m, 2^{-m-1}r)$. If $\sigma \geq 1$ and $z \in \sigma\widehat{B}$, then

$$\begin{aligned} d(z,x) &\leq d(z, y_m) + d(y_m, y) + d(y,x) \\ &\leq \sigma 2^{-m-1}r + 2^{-m-1}r + 2^{-m+1}r \\ &< \sigma 2^{-m+2}r\,, \end{aligned}$$

and thus $\sigma\widehat{B} \subset B(x, \sigma 2^{-m+2}r)$. Moreover, since $y_m \in E_B^m$, we have

$$\begin{aligned} \overline{2^{-1}\widehat{B}} \cap E &= E \cap \overline{B(y_m, 2^{-m-2}r)} \\ &\subset \bigcup_{z \in E_B^m} E \cap \overline{B(z, 2^{-m-2}r)} \\ &= E_B^{m+1} \subset E_B\,. \end{aligned}$$

On the other hand $E_B \subset E$, and thus $\overline{2^{-1}\widehat{B}} \cap E = \overline{2^{-1}\widehat{B}} \cap E_B$.

Let us then consider the case $m \geq j \geq 0$. We take $y_{x,m} = y \in E$ and $\widehat{B} = B(y, 2^{-m-1}r)$. Then, for every $\sigma \geq 1$ and each $z \in \sigma\widehat{B}$,

$$\begin{aligned} d(z,x) &\leq d(z,y) + d(y,x) \\ &< \sigma 2^{-m-1}r + 2^{-m+1}r \\ &< \sigma 2^{-m+2}r\,, \end{aligned}$$

and so $\sigma\widehat{B} \subset B(x, \sigma 2^{-m+2}r)$. Since $y \in E_B^j \subset E_B^m \subset E_B$ we can repeat the argument above, with y_m replaced by y, and it follows as above that $\overline{2^{-1}\widehat{B}} \cap E = \overline{2^{-1}\widehat{B}} \cap E_B$. □

A similar truncation procedure is a standard technique when proving the self-improvement of different capacity density conditions. It originally appears in [88, p. 180] for Riesz capacities in $\mathbb{R}^n$, and later also in [98] for $\mathbb{R}^n$ and in [6] for general metric spaces.

With the aid of big pieces inside the truncated set E_Q, we can show that a localized variant of the boundary Poincaré inequality in Theorem 6.16 holds for the truncated set E_Q, if E satisfies a capacity density condition.

Lemma 6.26

Let X be a geodesic space. Assume that $1 \leq p < \infty$ and $0 < \beta \leq 1$. Suppose that a closed set $E \subset X$ satisfies the (β, p)-capacity density condition with a constant c_0. Let $B_0 = B(w, R) \subset X$ be a ball with $w \in E$ and $R < \operatorname{diam}(E)$, and let $Q = B(w, r_Q) \subset B_0$ be the corresponding Whitney-type ball. Assume that $B \subset Q^*$ is a ball with a center $x_B \in E_Q$. Then there is a constant $K = K(p, c_\mu, c_0)$ such that

$$\fint_B |u(x)|^p \, d\mu(x) \leq K \operatorname{diam}(B)^{\beta p} \fint_B g(x)^p \, d\mu(x) \tag{6.21}$$

for all β-Hölder functions u in X with $u = 0$ in E_Q, and for all $g \in \mathcal{D}_H^\beta(u)$.

Proof. Fix a ball $B = B(x_B, r_B) \subset Q^*$ with $x_B \in E_Q$. Recall that $r_Q = R/128$ as in (6.20). Since $B \subset Q^* \subsetneq X$, we have

$$0 < r_B \leq \operatorname{diam}(B) \leq \operatorname{diam}(Q^*) \leq 8r_Q\,.$$

Hence, we can choose $m \in \mathbb{N}_0$ such that $2^{-m+2}r_Q < r_B \leq 2^{-m+3}r_Q$. Then

$$d(x_B, E_Q) = 0 < 2^{-m+1}r_Q\,.$$

By Lemma 6.25 with $x = x_B$ there exists a ball $\widehat{B} = B(y, 2^{-m-1}r_Q)$ such that $y \in E$,

$$E \cap \overline{2^{-1}\widehat{B}} = E_Q \cap \overline{2^{-1}\widehat{B}} \tag{6.22}$$

and $\widehat{B} \subset B(x_B, 2^{-m+2}r_Q) \subset B(x_B, r_B) = B$. Observe also that $B \subset 32\widehat{B}$.

Fix a β-Hölder function u in X with $u = 0$ in E_Q, and let $g \in \mathcal{D}_H^\beta(u)$. We estimate

$$\fint_B |u(x)|^p \, d\mu(x) \leq C(p) \fint_B |u(x) - u_B|^p \, d\mu(x) + C(p)|u_B - u_{\widehat{B}}|^p + C(p)|u_{\widehat{B}}|^p\,.$$

By the (β,p,p)-Poincaré inequality in Theorem 6.8, we obtain

$$\fint_B |u(x)-u_B|^p\,d\mu(x) \le C(p)\,\mathrm{diam}(B)^{\beta p}\fint_B g(x)^p\,d\mu(x)\,.$$

Using also Hölder's inequality and the doubling condition, we get

$$\begin{aligned}|u_B-u_{\widehat{B}}|^p &\le \fint_{\widehat{B}} |u(x)-u_B|^p\,d\mu(x)\\ &\le C(c_\mu)\fint_{\widehat{B}} |u(x)-u_B|^p\,d\mu(x)\\ &\le C(p,c_\mu)\,\mathrm{diam}(B)^{\beta p}\fint_{\widehat{B}} g(x)^p\,d\mu(x)\,.\end{aligned}$$

In order to estimate the remaining term $|u_{\widehat{B}}|^p$, we write

$$\{u=0\}=\{y\in X : u(y)=0\}\supset E_Q\,.$$

By using the monotonicity of capacity, identity (6.22), the assumed capacity density condition, and the doubling condition, we obtain

$$\begin{aligned}\mathrm{cap}_{\beta,p}(\{u=0\}\cap\overline{2^{-1}\widehat{B}},\widehat{B}) &\ge \mathrm{cap}_{\beta,p}(E_Q\cap\overline{2^{-1}\widehat{B}},\widehat{B})\\ &= \mathrm{cap}_{\beta,p}(E\cap\overline{2^{-1}\widehat{B}},\widehat{B})\\ &\ge c_0(2^{-m-2}r_Q)^{-\beta p}\mu(2^{-1}\widehat{B})\\ &\ge C(c_\mu,c_0)r_B^{-\beta p}\mu(B)\,.\end{aligned}$$

By Theorem 6.15, we obtain

$$\begin{aligned}|u_{\widehat{B}}|^p &\le \fint_{\widehat{B}} |u(x)|^p\,d\mu(x)\\ &\le C(p)\left(\mathrm{cap}_{\beta,p}(\{u=0\}\cap\overline{2^{-1}\widehat{B}},\widehat{B})\right)^{-1}\int_{\widehat{B}} g(x)^p\,d\mu(x)\\ &\le C(p,c_\mu,c_0)\frac{r_B^{\beta p}}{\mu(B)}\int_{\widehat{B}} g(x)^p\,d\mu(x)\\ &\le C(p,c_\mu,c_0)\,\mathrm{diam}(B)^{\beta p}\fint_{\widehat{B}} g(x)^p\,d\mu(x)\,.\end{aligned}$$

The proof is completed by combining the above estimates for the three terms. □

6.7 Maximal boundary Poincaré inequalities

This section is the most technical section of this Chapter. Here, we formulate and prove our key results, Theorem 6.28 and Theorem 6.29. These theorems give improved variants of the local boundary Poincaré inequality (6.21). The improved variants are norm inequalities for a combination of two maximal functions. Hence, we can view Theorem 6.28 and Theorem 6.29 as maximal boundary Poincaré inequalities.

The method adapts [74] to the setting of boundary Poincaré inequalities. Nevertheless, the adaptation of the argument there to our setting is nontrivial. That is why this lengthy and technical section is developed in full detail.

Let us begin by introducing the maximal operators that we are going to use throughout this section.

Definition 6.27

Let X be a geodesic space, $1 < p < \infty$ and $0 < \beta \le 1$. If $\mathcal{B} \neq \emptyset$ is a given family of balls in X, then we define a fractional sharp maximal function

$$M^{\sharp,p}_{\beta,\mathcal{B}}u(x) = \sup_{x\in B\in\mathcal{B}} \left(\frac{1}{\operatorname{diam}(B)^{\beta p}} \fint_B |u(y)-u_B|^p \, d\mu(y)\right)^{1/p}, \qquad x \in X\,, \tag{6.23}$$

whenever $u\colon X \to \mathbb{R}$ is a β-Hölder function. We also define the maximal function adapted to a given set $E_Q \subset X$ by

$$M^{E_Q,p}_{\beta,\mathcal{B}}u(x) = \sup_{x\in B\in\mathcal{B}} \left(\frac{\chi_{E_Q}(x_B)}{\operatorname{diam}(B)^{\beta p}} \fint_B |u(y)|^p \, d\mu(y)\right)^{1/p}, \qquad x \in X\,, \tag{6.24}$$

whenever $u\colon X \to \mathbb{R}$ is a β-Hölder function such that $u = 0$ in E_Q. Here x_B is the center of the ball $B \in \mathcal{B}$. The supremums in (6.23) and (6.24) are defined to be zero, if there is no ball B in $\mathcal{B}$ that contains the point x.

We are mostly interested in maximal functions for the ball family (6.19). The following is the main result in this section.

Theorem 6.28

Let X be a geodesic space. Let $1 < p < \infty$ and $0 < \beta \le 1$. Let $E \subset X$ be a closed set which satisfies the (β,p)-capacity density condition with a constant c_0. Let $B_0 = B(w,R)$ be a ball with $w \in E$ and $R < \operatorname{diam}(E)$. Let E_Q be the truncation of E to the Whitney-type ball Q as in Section 6.6. Then there exists a constant $C = C(\beta,p,c_\mu,c_0) > 0$ such that inequality

$$\int_{B_0} \left(M^{\sharp,p}_{\beta,\mathcal{B}_0}u(x) + M^{E_Q,p}_{\beta,\mathcal{B}_0}u(x)\right)^p d\mu(x) \le C \int_{B_0} g(x)^p \, d\mu(x) \tag{6.25}$$

holds whenever $u \in \operatorname{Lip}_\beta(X)$ is such that $u = 0$ in E_Q and $g \in \mathcal{D}^\beta_H(u)$.

Proof. We use the following Theorem 6.29 with $\varepsilon = 0$. Observe that the first term on the right-hand side of (6.26) is finite, since u is a β-Hölder function in X such that $u = 0$ in E_Q. Inequality (6.25) is obtained when this term is absorbed to the left-hand side after choosing the number k large enough, depending only on β, p, c_μ and c_0. □

Theorem 6.29

Let X be a geodesic space. Let $1 < q < p < \infty$ and $0 < \beta \le 1$ be such that the (β,p,q)-Poincaré inequality in Theorem 6.9 holds. Let $E \subset X$ be a closed set satisfying the (β,p)-capacity density condition with a constant c_0. Let $B_0 = B(w,R)$ be a ball with $w \in E$ and $R < \operatorname{diam}(E)$. Let E_Q be the truncation of E to the Whitney-type ball $Q = B(w,r_Q) \subset B_0$ as in Section 6.6. Let $K = K(p,c_\mu,c_0) > 0$ be the constant for the local boundary Poincaré inequality in Lemma 6.26. Assume that $k \in \mathbb{N}$, $0 \le \varepsilon < (p-q)/2$, and $\alpha = \beta p^2/(2(s+\beta p)) > 0$ with $s = \log_2 c_\mu$. Then inequality

$$
\begin{aligned}
&\int_{B_0} \left(M^{\sharp,p}_{\beta,\mathcal{B}_0} u + M^{E_Q,p}_{\beta,\mathcal{B}_0} u\right)^{p-\varepsilon} d\mu \\
&\quad \le C_1\left(2^{k(\varepsilon-\alpha)} + \frac{K 4^{k\varepsilon}}{k^{p-1}}\right) \int_{B_0} \left(M^{\sharp,p}_{\beta,\mathcal{B}_0} u + M^{E_Q,p}_{\beta,\mathcal{B}_0} u\right)^{p-\varepsilon} d\mu \\
&\qquad + C_1 C(k,\varepsilon) K \int_{B_0 \setminus \{M^{\sharp,p}_{\beta,\mathcal{B}_0} u + M^{E_Q,p}_{\beta,\mathcal{B}_0} u = 0\}} g^p \left(M^{\sharp,p}_{\beta,\mathcal{B}_0} u + M^{E_Q,p}_{\beta,\mathcal{B}_0} u\right)^{-\varepsilon} d\mu \\
&\qquad + C_3 \int_{B_0} g^{p-\varepsilon}\, d\mu \,.
\end{aligned} \tag{6.26}
$$

holds for each $u \in \operatorname{Lip}_\beta(X)$ with $u = 0$ in E_Q and every $g \in \mathcal{D}^\beta_H(u)$. Here $C_1 = C_1(\beta, p, c_\mu)$, $C_1 = C_1(\beta, p, c_\mu)$, $C_3 = C(\beta, p, c_\mu)$, $C(k,\varepsilon) = (4^{k\varepsilon} - 1)/\varepsilon$ if $\varepsilon > 0$ and $C(k, 0) = k$.

Remark 6.30 Observe that Theorem 6.29 implies a variant of Theorem 6.28 when we choose $\varepsilon > 0$ to be sufficiently small. We omit the formulation of this variant, since we do not use it. This is because of the following defect: one of the terms is the integral of $g^p\left(M^{\sharp,p}_{\beta,\mathcal{B}_0} u + M^{E_Q,p}_{\beta,\mathcal{B}_0} u\right)^{-\varepsilon}$ instead of $g^{p-\varepsilon}$. Because of its independent interest, we have however chosen to formulate Theorem 6.29 such that it incorporates the parameter ε.

The proof of Theorem 6.29 is completed in Section 6.7.4. For the proof, we need preparations that are treated in Sections 6.7.1 – 6.7.3. At this stage, we already fix X, E, B_0, Q, E_Q, K, $\mathcal{B}_0$, p, β, q, ε, k and u as in the statement of Theorem 6.29. Notice, however, that the β-Hajłasz gradient g of u is not yet fixed. We abbreviate $M^\sharp u = M^{\sharp,p}_{\beta,\mathcal{B}_0} u$ and $M^{E_Q} u = M^{E_Q,p}_{\beta,\mathcal{B}_0} u$, and denote

$$
U^\lambda = \{x \in B_0 \,:\, M^\sharp u(x) + M^{E_Q} u(x) > \lambda\}\,, \qquad \lambda > 0\,.
$$

The sets U^λ are open in X. If $F \subset X$ is a Borel set and $\lambda > 0$, we write $U^\lambda_F = U^\lambda \cap F$. We refer to these objects throughout Section 6.7 without further notice.

6.7.1 Localization to Whitney-type ball

We need a smaller maximal function that is localized to the Whitney-type ball Q. Consider the ball family

$$
\mathcal{B}_Q = \{B \subset X \,:\, B \text{ is a ball such that } B \subset Q^*\}
$$

and define

$$
M^{E_Q}_Q u = \chi_{Q^*} M^{E_Q,p}_{\beta,\mathcal{B}_Q} u\,. \tag{6.27}
$$

If $\lambda > 0$, we write

$$
Q^\lambda = \{x \in Q^* \,:\, M^{E_Q}_Q u(x) > \lambda\}\,. \tag{6.28}
$$

We estimate the left-hand side of (6.25) in terms of (6.27) with the aid of the following norm estimate. We will later be able to estimate the smaller maximal function (6.27).

Lemma 6.31

There are constants $C_1 = C(p, c_\mu)$ and $C_2 = C(\beta, p, c_\mu)$ such that

$$\int_{B_0} \big(M^\sharp u(x) + M^{E_Q} u(x)\big)^{p-\varepsilon}\, d\mu(x) \\ \leq C_1 \int_{B_0} \big(M_Q^{E_Q} u(x)\big)^{p-\varepsilon}\, d\mu(x) + C_2 \int_{B_0} g(x)^{p-\varepsilon}\, d\mu(x)$$

for all $g \in \mathcal{D}_H^\beta(u)$.

Proof. Fix $g \in \mathcal{D}_H^\beta(u)$. We have

$$\begin{aligned} &\int_{B_0} \big(M^\sharp u(x) + M^{E_Q} u(x)\big)^{p-\varepsilon}\, d\mu(x) \\ &\quad \leq C(p) \int_{B_0} \Big(M^\sharp u(x)\Big)^{p-\varepsilon}\, d\mu(x) + C(p) \int_{B_0} \big(M^{E_Q} u(x)\big)^{p-\varepsilon}\, d\mu(x)\,. \end{aligned} \tag{6.29}$$

Let $x \in B_0$ and let $B \in \mathcal{B}_0$ be such that $x \in B$. By (6.19) and the (β, p, q)-Poincaré inequality, see Theorem 6.9, we obtain

$$\begin{aligned} &\left(\frac{1}{\operatorname{diam}(B)^{\beta p}} \fint_B |u(y) - u_B|^p\, d\mu(y)\right)^{1/p} \\ &\qquad \leq C(c_\mu, p, \beta) \left(\fint_B g(y)^q\, d\mu(y)\right)^{1/q} \\ &\quad \leq C(c_\mu, p, \beta) (M(g^q \chi_{B_0})(x))^{\frac{1}{q}}\,. \end{aligned}$$

Here M is the non-centered Hardy–Littlewood maximal function operator. By taking supremum over balls B as above, we obtain

$$M^\sharp u(x) = M^{\sharp,p}_{\beta,\mathcal{B}_0} u(x) \leq C(\beta, p, c_\mu) (M(g^q \chi_{B_0})(x))^{\frac{1}{q}}\,.$$

Since $p - \varepsilon > q$, the Hardy–Littlewood maximal function theorem [4, Theorem 3.13] implies that

$$\begin{aligned} \int_{B_0} \Big(M^\sharp u(x)\Big)^{p-\varepsilon}\, d\mu(x) &\leq C(\beta, p, c_\mu) \int_{B_0} (M(g^q \chi_{B_0})(x))^{\frac{p-\varepsilon}{q}}\, d\mu(x) \\ &\leq \frac{C(\beta, p, c_\mu)}{p - q - \varepsilon} \int_{B_0} g(x)^{p-\varepsilon}\, d\mu(x)\,. \end{aligned}$$

Since $\varepsilon < (p-q)/2$, this provides an estimate for the first term in the right-hand side of (6.29).

In order to estimate the second term in the right-hand side of (6.29), we let $x \in B_0 \setminus Q^*$ and let $B \in \mathcal{B}_0$ be such that $x \in B$. We will estimate the term

$$\left(\frac{\chi_{E_Q}(x_B)}{\operatorname{diam}(B)^{\beta p}} \fint_B |u(y)|^p\, d\mu(y)\right)^{1/p},$$

where x_B is the center of B. Clearly we may assume that $x_B \in E_Q \subset \overline{Q}$. By condition (W1), we see that $\operatorname{diam}(B) \geq C \operatorname{diam}(B_0)$ and $\mu(B) \geq C(c_\mu)\mu(B_0)$. Since $B \in \mathcal{B}_0$, we have $B \subset B_0$. Thus,

$$\left(\frac{\chi_{E_Q}(x_B)}{\operatorname{diam}(B)^{\beta p}} \fint_B |u(y)|^p \, d\mu(y)\right)^{1/p} \le C(p, c_\mu) \left(\frac{1}{\operatorname{diam}(B_0)^{\beta p}} \fint_{B_0} |u(y)|^p \, d\mu(y)\right)^{1/p}.$$

By taking supremum over balls B as above, we obtain

$$M^{E_Q} u(x) = M^{E_Q,p}_{\beta,\mathcal{B}_0} u(x) \le C(p, c_\mu) \left(\frac{1}{\operatorname{diam}(B_0)^{\beta p}} \fint_{B_0} |u(y)|^p \, d\mu(y)\right)^{1/p}$$

for all $x \in B_0 \setminus Q^*$. By integrating, we obtain

$$\begin{aligned}
&\int_{B_0 \setminus Q^*} \left(M^{E_Q} u(x)\right)^{p-\varepsilon} d\mu(x) \\
&\quad\le C(p, c_\mu)\mu(B_0) \left(\frac{1}{\operatorname{diam}(B_0)^{\beta p}} \fint_{B_0} |u(y)|^p \, d\mu(y)\right)^{\frac{p-\varepsilon}{p}} \\
&\quad\le \frac{C(p, c_\mu)\mu(B_0)}{\operatorname{diam}(B_0)^{\beta(p-\varepsilon)}} \left[\left(\fint_{B_0} |u(y) - u_{Q^*}|^p \, d\mu(y)\right)^{\frac{p-\varepsilon}{p}} + |u_{Q^*}|^{p-\varepsilon}\right].
\end{aligned} \tag{6.30}$$

By the (β, p, q)-Poincaré inequality and Hölder's inequality with $q < p - \varepsilon$, we obtain

$$\begin{aligned}
&\frac{C(p, c_\mu)\mu(B_0)}{\operatorname{diam}(B_0)^{\beta(p-\varepsilon)}} \left(\fint_{B_0} |u(y) - u_{Q^*}|^p \, d\mu(y)\right)^{\frac{p-\varepsilon}{p}} \\
&\quad\le \frac{C(p, c_\mu)\mu(B_0)}{\operatorname{diam}(B_0)^{\beta(p-\varepsilon)}} \left[\left(\fint_{B_0} |u(y) - u_{B_0}|^p \, d\mu(y)\right)^{\frac{p-\varepsilon}{p}} + |u_{B_0} - u_{Q^*}|^{p-\varepsilon}\right] \\
&\quad\le \frac{C(p, c_\mu)\mu(B_0)}{\operatorname{diam}(B_0)^{\beta(p-\varepsilon)}} \left(\fint_{B_0} |u(y) - u_{B_0}|^p \, d\mu(y)\right)^{\frac{p-\varepsilon}{p}} \\
&\quad\le C(\beta, p, c_\mu)\mu(B_0) \left(\fint_{B_0} g(x)^q \, d\mu(x)\right)^{\frac{p-\varepsilon}{q}} \\
&\quad\le C(\beta, p, c_\mu) \int_{B_0} g(x)^{p-\varepsilon} \, d\mu(x).
\end{aligned}$$

On the other hand, since $Q^* = B(w, 4r_Q)$ with $w \in E_Q$ and $r_Q = R/128$, we have

$$\begin{aligned}
\frac{C(p, c_\mu)\mu(B_0)}{\operatorname{diam}(B_0)^{\beta(p-\varepsilon)}} |u_{Q^*}|^{p-\varepsilon} &\le C(p, c_\mu) \frac{\mu(Q^*)}{\operatorname{diam}(Q^*)^{\beta(p-\varepsilon)}} |u_{Q^*}|^{p-\varepsilon} \\
&\le C(p, c_\mu) \int_{Q^*} \left(\frac{\chi_{E_Q}(w)}{\operatorname{diam}(Q^*)^{\beta p}} \fint_{Q^*} |u(y)|^p \, d\mu(y)\right)^{\frac{p-\varepsilon}{p}} d\mu(x) \\
&\le C(p, c_\mu) \int_{Q^*} \left(\chi_{Q^*}(x) M^{E_Q,p}_{\beta,\mathcal{B}_Q} u(x)\right)^{p-\varepsilon} d\mu(x) \\
&= C(p, c_\mu) \int_{B_0} \left(M^{E_Q}_Q u(x)\right)^{p-\varepsilon} d\mu(x).
\end{aligned}$$

This concludes the estimate for the integral in (6.30) over $B_0 \setminus Q^*$.

To estimate the integral over the set Q^*, we fix $x \in Q^*$. Let $B \in \mathcal{B}_0$ be such that $x \in B$. If $B \subset Q^*$, then

$$\left(\frac{\chi_{E_Q}(x_B)}{\operatorname{diam}(B)^{\beta p}} \fint_B |u(y)|^p \, d\mu(y)\right)^{1/p} \le \chi_{Q^*}(x) M^{E_Q,p}_{\beta,\mathcal{B}_Q} u(x) = M^{E_Q}_Q u(x)\,.$$

Next we consider the case $B \not\subset Q^*$, and again we need to estimate the quantity

$$\left(\frac{\chi_{E_Q}(x_B)}{\operatorname{diam}(B)^{\beta p}} \fint_B |u(y)|^p \, d\mu(y)\right)^{1/p}.$$

We may assume that $x_B \in E_Q \subset \overline{Q}$. By condition (W1), we obtain $\operatorname{diam}(B) \ge C \operatorname{diam}(B_0)$ and $\mu(B) \ge C(c_\mu)\mu(B_0)$. Hence,

$$\left(\frac{\chi_{E_Q}(x_B)}{\operatorname{diam}(B)^{\beta p}} \fint_B |u(y)|^p \, d\mu(y)\right)^{1/p} \le C(p, c_\mu)\left(\frac{1}{\operatorname{diam}(B_0)^{\beta p}} \fint_{B_0} |u(y)|^p \, d\mu(y)\right)^{1/p}.$$

By taking supremum over balls B as above, we obtain

$$M^{E_Q} u(x) \le M^{E_Q}_Q u(x) + C(p, c_\mu)\left(\frac{1}{\operatorname{diam}(B_0)^{\beta p}} \fint_{B_0} |u(y)|^p \, d\mu(y)\right)^{1/p}$$

for all $x \in Q^*$. It follows that

$$\begin{aligned}\int_{Q^*} \big(M^{E_Q} u(x)\big)^{p-\varepsilon} \, d\mu(x) &\le C(p, c_\mu) \int_{B_0} \big(M^{E_Q}_Q u(x)\big)^{p-\varepsilon} \, d\mu(x) \\ &\quad + C(p, c_\mu)\mu(B_0)\left(\frac{1}{\operatorname{diam}(B_0)^{\beta p}} \fint_{B_0} |u(y)|^p \, d\mu(y)\right)^{\frac{p-\varepsilon}{p}}.\end{aligned}$$

We can now estimate as above, and complete the proof. □

The following lemma is variant of [74, Lemma 4.12]. We also refer to [50, Lemma 3.6].

Lemma 6.32
Fix $x, y \in Q^*$. Then

$$|u(x) - u(y)| \le C(\beta, c_\mu)\, d(x, y)^\beta \big(M^\sharp u(x) + M^\sharp u(y)\big) \tag{6.31}$$

and

$$|u(x)| \le C(\beta, c_\mu)\, d(x, E_Q)^\beta \left(M^\sharp u(x) + M^{E_Q} u(x)\right). \tag{6.32}$$

Furthermore, assuming that $\lambda > 0$, then the restriction $u|_{E_Q \cup (Q^* \setminus U^\lambda)}$ is a β-Hölder function in the set $E_Q \cup (Q^* \setminus U^\lambda)$ with constant $\kappa = C(\beta, c_\mu)\lambda$.

Proof. The property (W4) is used below several times without further notice. Let $z \in Q^*$ and $0 < r \le 2 \operatorname{diam}(Q^*)$. Write $B_i = B(z, 2^{-i} r) \in \mathcal{B}_0$ for each $i \in \{0, 1, \ldots\}$. Then, with the standard 'telescoping' argument, see for instance the proof of [50, Lemma 3.6], we obtain

$$|u(z) - u_{B(z,r)}| \le c_\mu \sum_{i=0}^{\infty} \fint_{B_i} |u - u_{B_i}| \, d\mu$$

$$\le c_\mu \sum_{i=0}^{\infty} 2^{\beta(1-i)} r^\beta \left(\frac{1}{\operatorname{diam}(B_i)^{\beta p}} \fint_{B_i} |u - u_{B_i}|^p \, d\mu \right)^{1/p}$$
$$\le c_\mu M^\sharp u(z) \sum_{i=0}^{\infty} 2^{\beta(1-i)} r^\beta \le C(\beta, c_\mu)\, r^\beta M^\sharp u(z) \,.$$

Fix $x, y \in Q^*$. Since $0 < d = d(x,y) \le \operatorname{diam}(Q^*)$, we obtain

$$\begin{aligned}
|u(y) - u_{B(x,d)}| &\le |u(y) - u_{B(y,2d)}| + |u_{B(y,2d)} - u_{B(x,d)}| \\
&\le C(\beta, c_\mu)\, d^\beta M^\sharp u(y) + \frac{\mu(B(y,2d))}{\mu(B(x,d))} \fint_{B(y,2d)} |u - u_{B(y,2d)}| \, d\mu \\
&\le C(\beta, c_\mu)\, d^\beta \Bigg[M^\sharp u(y) \\
&\qquad + \left(\frac{1}{\operatorname{diam}(B(y,2d))^{\beta p}} \fint_{B(y,2d)} |u - u_{B(y,2d)}|^p \, d\mu \right)^{1/p} \Bigg] \\
&\le C(\beta, c_\mu)\, d^\beta M^\sharp u(y) \,.
\end{aligned}$$

It follows that

$$\begin{aligned}
|u(x) - u(y)| &\le |u(x) - u_{B(x,d)}| + |u_{B(x,d)} - u(y)| \\
&\le C(\beta, c_\mu)\, d(x,y)^\beta \big(M^\sharp u(x) + M^\sharp u(y) \big) \,,
\end{aligned}$$

which is the desired inequality (6.31).

To prove inequality (6.32), we let $x \in Q^*$. If $d(x, E_Q) = 0$, then $x \in E_Q$ and we are done since $u = 0$ in E_Q. Therefore we may assume that $d(x, E_Q) > 0$. Then there exists $y \in E_Q \subset \overline{Q} \subset Q^*$ such that $d = d(x,y) < \min\{2d(x, E_Q), \operatorname{diam}(Q^*)\}$ and we have

$$\begin{aligned}
|u(x)| &\le |u(x) - u_{B(y,d)}| + |u_{B(y,d)}| \\
&\le C(\beta, c_\mu) d^\beta M^\sharp u(x) + c_\mu \fint_{B(y,2d)} |u| \, d\mu \\
&\le C(\beta, c_\mu) d^\beta M^\sharp u(x) + c_\mu (4d)^\beta \left(\frac{\chi_{E_Q}(y)}{\operatorname{diam}(B(y,2d))^{\beta p}} \fint_{B(y,2d)} |u|^p \, d\mu \right)^{\frac{1}{p}} \\
&\le C(\beta, c_\mu) d^\beta \left(M^\sharp u(x) + M^{E_Q} u(x) \right) \\
&\le C(\beta, c_\mu) d(x, E_Q)^\beta \left(M^\sharp u(x) + M^{E_Q} u(x) \right) \,.
\end{aligned}$$

Inequality (6.32) follows.

Fix $\lambda > 0$. Next we show that $u|(E_Q \cup (Q^* \setminus U^\lambda))$ is β-Hölder with constant $\kappa = C(\beta, c_\mu)\lambda$. Let $x, y \in E_Q \cup (Q^* \setminus U^\lambda)$. There are four cases to be considered. First, if $x, y \in E_Q$, then

$$|u(x) - u(y)| = 0 \le \kappa d(x,y)^\beta,$$

since $u = 0$ in E_Q. If $x, y \in Q^* \setminus U^\lambda$, then we apply (6.31) and obtain

$$|u(x) - u(y)| \le C(\beta, c_\mu) d(x,y)^\beta \big(M^\sharp u(x) + M^\sharp u(y) \big) \le C(\beta, c_\mu) \lambda d(x,y)^\beta \,.$$

Here we also used the fact that $Q^* \subset B_0$. If $x \in E_Q$ and $y \in Q^* \setminus U^\lambda$, we apply (6.32) and get

$$\begin{aligned}|u(x) - u(y)| = |u(y)| &\le C(\beta, c_\mu) d(y, E_Q)^\beta \left(M^\sharp u(y) + M^{E_Q} u(y) \right) \\ &\le C(\beta, c_\mu) \lambda d(x, y)^\beta .\end{aligned}$$

The last case $x \in Q^* \setminus U^\lambda$ and $y \in E_Q$ is treated in similar way. □

6.7.2 Stopping construction

We continue as in [74] and construct a stopping family $\mathcal{S}_\lambda(Q)$ of pairwise disjoint balls whose 5-dilations cover the set $Q^\lambda \subset Q^* = B(w, 4r_Q)$; recall (6.28). Let $B \in \mathcal{B}_Q$ be a ball centered at $x_B \in E_Q \subset \overline{Q}$. The *parent ball* of B is then defined to be $\pi(B) = 2B$ if $2B \subset Q^*$ and $\pi(B) = Q^*$ otherwise. Observe that $B \subset \pi(B) \in \mathcal{B}_Q$ and the center of $\pi(B)$ satisfies $x_{\pi(B)} \in \{x_B, w\} \subset E_Q$. It follows that all the balls $B \subset \pi(B) \subset \pi(\pi(B)) \subset \cdots$ are well-defined, belong to $\mathcal{B}_Q$ and are centered at E_Q. By inequalities (6.1) and (6.3), and property (W1) if needed, we have $\mu(\pi(B)) \le c_\mu^5 \mu(B)$ and $\operatorname{diam}(\pi(B)) \le 16 \operatorname{diam}(B)$.

Then we come to the stopping time argument. We will use as a threshold value the number

$$\lambda_Q = \left(\frac{1}{\operatorname{diam}(Q^*)^{\beta p}} \fint_{Q^*} |u(y)|^p \, d\mu(y) \right)^{1/p} = \left(\frac{\chi_{E_Q}(w)}{\operatorname{diam}(Q^*)^{\beta p}} \fint_{Q^*} |u(y)|^p \, d\mu(y) \right)^{1/p} .$$

Fix a level $\lambda > \lambda_Q/2$. Fix a point $x \in Q^\lambda \subset Q^*$. If $\lambda_Q/2 < \lambda < \lambda_Q$, then we choose $B_x = Q^* \in \mathcal{B}_Q$. If $\lambda \ge \lambda_Q$, then by using the condition $x \in Q^\lambda$ we first choose a starting ball B, with $x \in B \in \mathcal{B}_Q$, such that

$$\lambda < \left(\frac{\chi_{E_Q}(x_B)}{\operatorname{diam}(B)^{\beta p}} \fint_B |u(y)|^p \, d\mu(y) \right)^{1/p} .$$

Observe that $x_B \in E_Q \subset \overline{Q}$. We continue by looking at the balls $B \subset \pi(B) \subset \pi(\pi(B)) \subset \cdots$ and we stop at the first among them, denoted by $B_x \in \mathcal{B}_Q$, that satisfies the following two stopping conditions:

$$\begin{cases} \lambda < \left(\dfrac{\chi_{E_Q}(x_{B_x})}{\operatorname{diam}(B_x)^{\beta p}} \displaystyle\fint_{B_x} |u(y)|^p \, d\mu(y) \right)^{1/p} , \\ \left(\dfrac{\chi_{E_Q}(x_{\pi(B_x)})}{\operatorname{diam}(\pi(B_x))^{\beta p}} \displaystyle\fint_{\pi(B_x)} |u(y)|^p \, d\mu(y) \right)^{1/p} \le \lambda . \end{cases}$$

The inequality $\lambda \ge \lambda_Q$ in combination with the fact that $Q^* \subsetneq X$ ensures the existence of such a stopping ball.

In any case, the chosen ball $B_x \in \mathcal{B}_Q$ contains the point x, is centered at $x_{B_x} \in E_Q$, and satisfies inequalities

$$\lambda < \left(\frac{\chi_{E_Q}(x_{B_x})}{\operatorname{diam}(B_x)^{\beta p}} \fint_{B_x} |u(y)|^p \, d\mu(y) \right)^{1/p} \le 16 c_\mu^{5/p} \lambda . \tag{6.33}$$

By the $5r$-covering lemma [4, Lemma 1.7], we obtain a countable disjoint family

$$\mathcal{S}_\lambda(Q) \subset \{B_x : x \in Q^\lambda\}\,, \qquad \lambda > \lambda_Q/2\,,$$

of *stopping balls* such that $Q^\lambda \subset \bigcup_{B\in\mathcal{S}_\lambda(Q)} 5B$. Let us remark that, by the condition (W2) and stopping inequality (6.33), we have $B \subset U^\lambda$ if $B \in \mathcal{S}_\lambda(Q)$ and $\lambda > \lambda_Q/2$.

6.7.3 Level set estimates

Next we prove two technical results: Lemma 6.33 and Lemma 6.34. We follow the approach in [74] quite closely, but we give details since technical modifications are required. A counterpart of the following lemma can be found also in [71, Lemma 3.1.2]. Recall that $k \in \mathbb{N}$ is a fixed number and $\alpha = \beta p^2/(2(s+\beta p)) > 0$ with $s = \log_2 c_\mu > 0$.

Lemma 6.33

Suppose that $\lambda > \lambda_Q/2$ and let $B \in \mathcal{S}_\lambda(Q)$ be such that $\mu(U_B^{2^k\lambda}) < \mu(B)/2$. Then

$$\begin{aligned} &\frac{1}{\operatorname{diam}(B)^{\beta p}} \int_{U_B^{2^k\lambda}} |u(x)|^p \, d\mu(x) \\ &\quad \le C(p,c_\mu)2^{-k\alpha}(2^k\lambda)^p \mu(U_B^{2^k\lambda}) + \frac{C(p,c_\mu)}{\operatorname{diam}(B)^{\beta p}} \int_{B\setminus U^{2^k\lambda}} |u(x)|^p \, d\mu(x)\,. \end{aligned} \tag{6.34}$$

Proof. Fix $x \in U_B^{2^k\lambda} \subset B$ and consider the function $h\colon (0,\infty) \to \mathbb{R}$,

$$r \mapsto h(r) = \frac{\mu(U_B^{2^k\lambda} \cap B(x,r))}{\mu(B \cap B(x,r))} = \frac{\mu(U_B^{2^k\lambda} \cap B(x,r))}{\mu(B(x,r))} \cdot \left(\frac{\mu(B\cap B(x,r))}{\mu(B(x,r))}\right)^{-1}.$$

By Lemma 6.2 and the fact that B is open, we find that $h\colon (0,\infty) \to \mathbb{R}$ is continuous. Observe that $U_B^{2^k\lambda} = U^{2^k\lambda} \cap B$ is also open. Since $h(r) = 1$ for small values of $r > 0$ and $h(r) < 1/2$ for $r > \operatorname{diam}(B)$, we have $h(r_x) = 1/2$ for some $0 < r_x \le \operatorname{diam}(B)$. Write $B'_x = B(x, r_x)$. Then

$$\frac{\mu(U_B^{2^k\lambda} \cap B'_x)}{\mu(B \cap B'_x)} = h(r_x) = \frac{1}{2} \tag{6.35}$$

and

$$\frac{\mu((B \setminus U^{2^k\lambda}) \cap B'_x)}{\mu(B \cap B'_x)} = 1 - \frac{\mu(U_B^{2^k\lambda} \cap B'_x)}{\mu(B\cap B'_x)} = 1 - h(r_x) = \frac{1}{2}\,. \tag{6.36}$$

The $5r$-covering lemma [4, Lemma 1.7] gives us a countable disjoint family $\mathcal{G}_\lambda \subset \{B'_x : x \in U_B^{2^k\lambda}\}$ such that $U_B^{2^k\lambda} \subset \bigcup_{B'\in\mathcal{G}_\lambda} 5B'$. Then (6.35) and (6.36) hold for every ball $B' \in \mathcal{G}_\lambda$; namely, by denoting $B'_I = U_B^{2^k\lambda} \cap B'$ and $B'_O = (B \setminus U^{2^k\lambda}) \cap B'$, we have the following comparison identities:

$$\mu(B'_I) = \frac{\mu(B\cap B')}{2} = \mu(B'_O)\,, \tag{6.37}$$

where all the measures are strictly positive. These identities are important and they are used several times throughout the remainder of this proof.

We multiply the left-hand side of (6.34) by $\operatorname{diam}(B)^{\beta p}$ and then estimate as follows:

$$\begin{aligned}\int_{U_B^{2^k\lambda}} |u|^p\,d\mu &\le \sum_{B'\in\mathcal{G}_\lambda}\int_{5B'\cap B}|u|^p\,d\mu\\ &\le 2^{p-1}\sum_{B'\in\mathcal{G}_\lambda}\mu(5B'\cap B)|u_{B'_O}|^p + 2^{p-1}\sum_{B'\in\mathcal{G}_\lambda}\int_{5B'\cap B}|u-u_{B'_O}|^p\,d\mu\,.\end{aligned}\tag{6.38}$$

By (6.1) and Lemma 6.3, we find that

$$\mu(5B'\cap B)\le\mu(8B')\le c_\mu^3\mu(B')\le c_\mu^6\mu(B\cap B')\tag{6.39}$$

for all $B'\in\mathcal{G}_\lambda$. Hence, by the comparison identities (6.37),

$$\begin{aligned}2^{p-1}\sum_{B'\in\mathcal{G}_\lambda}\mu(5B'\cap B)|u_{B'_O}|^p &\le C(p,c_\mu)\sum_{B'\in\mathcal{G}_\lambda}\mu(B'_O)\fint_{B'_O}|u(x)|^p\,d\mu(x)\\ &= C(p,c_\mu)\sum_{B'\in\mathcal{G}_\lambda}\int_{B'_O}|u(x)|^p\,d\mu(x)\\ &\le C(p,c_\mu)\int_{B\setminus U^{2^k\lambda}}|u(x)|^p\,d\mu(x)\,.\end{aligned}$$

This concludes our analysis of the 'easy term' in (6.38). In order to treat the remaining term therein, we do need some preparations.

Let us fix a ball $B'\in\mathcal{G}_\lambda$ that satisfies $\int_{5B'\cap B}|u-u_{B'_O}|^p\,d\mu\neq 0$. We claim that

$$\fint_{5B'\cap B}|u(x)-u_{B'_O}|^p\,d\mu(x)\le C(p,c_\mu)2^{-k\alpha}(2^k\lambda)^p\operatorname{diam}(B)^{\beta p}\,.\tag{6.40}$$

In order to prove this inequality, we fix a number $m\in\mathbb{R}$ such that

$$(2^m\lambda)^p\operatorname{diam}(5B')^{\beta p}=\fint_{5B'\cap B}|u(x)-u_{B'_O}|^p\,d\mu(x)\,.\tag{6.41}$$

Let us first consider the case $m<k/2$. Then $m-k<-k/2$, and since always $\alpha<p/2$, the desired inequality (6.40) is obtained case as follows:

$$\begin{aligned}\fint_{5B'\cap B}|u-u_{B'_O}|^p\,d\mu &= 2^{(m-k)p}(2^k\lambda)^p\operatorname{diam}(5B')^{\beta p}\\ &\le 10^p\,2^{-kp/2}(2^k\lambda)^p\operatorname{diam}(B)^{\beta p}\\ &\le C(p)2^{-k\alpha}(2^k\lambda)^p\operatorname{diam}(B)^{\beta p}\,.\end{aligned}$$

Next we consider the case $k/2\le m$. Observe from (6.39) and the comparison identities (6.37) that

$$\begin{aligned}\fint_{5B'\cap B}|u(x)-u_{B'_O}|^p\,d\mu(x) &\le 2^{p-1}\fint_{5B'\cap B}|u(x)-u_{5B'}|^p\,d\mu(x)+2^{p-1}|u_{5B'}-u_{B'_O}|^p\\ &\le 2^{p+1}c_\mu^6\fint_{5B'}|u(x)-u_{5B'}|^p\,d\mu(x)\\ &\le 2^{p+1}c_\mu^6(2^k\lambda)^p\operatorname{diam}(5B')^{\beta p}\,,\end{aligned}$$

where the last step follows from condition (W3) and the fact that $5B'\supset B'_O\neq\emptyset$. By taking also (6.41) into account, we see that $2^{mp}\le 2^{p+1}c_\mu^6 2^{kp}$. On the other hand, we have

$$
\begin{aligned}
(2^m\lambda)^p \operatorname{diam}(5B')^{\beta p}\mu(B'\cap B) &\le \int_{5B'\cap B} |u(x)-u_{B'_O}|^p\,d\mu(x)\\
&\le 2^{p-1}\int_{5B'\cap B}|u(x)|^p\,d\mu(x) + 2^{p-1}\mu(5B'\cap B)|u_{B'_O}|^p\\
&\le 2^{p+1}c_\mu^6\int_B |u(x)|^p\,d\mu(x)\\
&\le 2\cdot 32^p c_\mu^{11}\lambda^p \operatorname{diam}(B)^{\beta p}\mu(B)\,,
\end{aligned}
$$

where the last step follows from the fact that $B\in\mathcal{S}_\lambda(Q)$ in combination with inequality (6.33). In particular, if $s=\log_2 c_\mu$ then by inequality (6.2) and Lemma 6.3, we obtain that

$$
\begin{aligned}
\left(\frac{\operatorname{diam}(5B')}{\operatorname{diam}(B)}\right)^{s+\beta p} &\le 20^s\frac{\operatorname{diam}(5B')^{\beta p}\mu(B')}{\operatorname{diam}(B)^{\beta p}\mu(B)}\\
&\le 20^s\cdot c_\mu^3\frac{\operatorname{diam}(5B')^{\beta p}\mu(B'\cap B)}{\operatorname{diam}(B)^{\beta p}\mu(B)}\\
&\le 2\cdot 20^s\cdot 32^p\cdot c_\mu^{14}\cdot 2^{-mp}\\
&\le 2\cdot 20^s\cdot 32^p\cdot c_\mu^{14}\cdot 2^{-kp/2}\,.
\end{aligned}
$$

This, in turn, implies that

$$
\left(\frac{\operatorname{diam}(5B')}{\operatorname{diam}(B)}\right)^{\beta p} \le 2\cdot 20^s\cdot 32^p\cdot c_\mu^{14}\cdot 2^{\frac{-k\beta p^2}{2(s+\beta p)}} = C(p,c_\mu)2^{-k\alpha}\,.
$$

Combining the above estimates, we see that

$$
\fint_{5B'\cap B}|u-u_{B'_O}|^p\,d\mu = (2^m\lambda)^p\operatorname{diam}(5B')^{\beta p} \le C(p,c_\mu)2^{-k\alpha}(2^k\lambda)^p\operatorname{diam}(B)^{\beta p}\,.
$$

That is, inequality (6.40) holds also in the present case $k/2\le m$. This concludes the proof of inequality (6.40).

By using (6.39) and (6.37) and inequality (6.40), we estimate the second term in (6.38) as follows:

$$
\begin{aligned}
2^{p-1}\sum_{B'\in\mathcal{G}_\lambda}\int_{5B'\cap B}|u(x)-u_{B'_O}|^p\,d\mu(x) &\le 2^p c_\mu^6 \sum_{B'\in\mathcal{G}_\lambda}\mu(B'_I)\fint_{5B'\cap B}|u(x)-u_{B'_O}|^p\,d\mu(x)\\
&\le C(p,c_\mu)2^{-k\alpha}(2^k\lambda)^p\operatorname{diam}(B)^{\beta p}\sum_{B'\in\mathcal{G}_\lambda}\mu(B'_I)\\
&\le C(p,c_\mu)2^{-k\alpha}(2^k\lambda)^p\operatorname{diam}(B)^{\beta p}\mu(U_B^{2^k\lambda})\,.
\end{aligned}
$$

Inequality (6.34) follows by collecting the above estimates. □

The following lemma is essential for the proof of Theorem 6.29, and it is the only place in the proof where the capacity density condition is needed. Recall from Lemma 6.26 that this condition implies a local boundary Poincaré inequality, which is used here one single time.

Lemma 6.34
Let $\lambda > \lambda_Q/2$ and $g \in \mathcal{D}_H^\beta(u)$. Then

$$\lambda^p \mu(Q^\lambda) \le C(\beta,p,c_\mu)\left[\frac{(\lambda 2^k)^p}{2^{k\alpha}}\mu(U^{2^k\lambda}) + \frac{K}{k^p}\sum_{j=k}^{2k-1}(\lambda 2^j)^p \mu(U^{2^j\lambda}) + K\int_{U^\lambda\setminus U^{4^k\lambda}} g^p\,d\mu\right].$$

Proof. By the covering property $Q^\lambda \subset \bigcup_{B\in\mathcal{S}_\lambda(Q)} 5B$ and doubling condition (6.1),

$$\lambda^p\mu(Q^\lambda) \le \lambda^p \sum_{B\in\mathcal{S}_\lambda(Q)} \mu(5B) \le c_\mu^3 \sum_{B\in\mathcal{S}_\lambda(Q)} \lambda^p \mu(B)\,.$$

Recall also that $B \subset U^\lambda$ if $B \in \mathcal{S}_\lambda(Q)$. Therefore, and using the fact that $\mathcal{S}_\lambda(Q)$ is a disjoint family, it suffices to prove that inequality

$$\lambda^p\mu(B) \le C(\beta,p,c_\mu)\left[\frac{(\lambda 2^k)^p}{2^{k\alpha}}\mu(U_B^{2^k\lambda}) + \frac{K}{k^p}\sum_{j=k}^{2k-1}(\lambda 2^j)^p\mu(U_B^{2^j\lambda}) + K\int_{B\setminus U^{4^k\lambda}} g^p\,d\mu\right] \tag{6.42}$$

holds for every $B \in \mathcal{S}_\lambda(Q)$. To this end, let us fix a ball $B \in \mathcal{S}_\lambda(Q)$.

If $\mu(U_B^{2^k\lambda}) \ge \mu(B)/2$, then

$$\lambda^p\mu(B) \le 2\lambda^p\mu(U_B^{2^k\lambda}) = 2\frac{(\lambda 2^k)^p}{2^{kp}}\mu(U_B^{2^k\lambda}) \le 2\frac{(\lambda 2^k)^p}{2^{k\alpha}}\mu(U_B^{2^k\lambda})\,,$$

which suffices for the required local estimate (6.42). Let us then consider the more difficult case $\mu(U_B^{2^k\lambda}) < \mu(B)/2$. In this case, by the stopping inequality (6.33),

$$\begin{aligned}\lambda^p\mu(B) &\le \frac{\chi_{E_Q}(x_B)}{\operatorname{diam}(B)^{\beta p}}\int_B |u(x)|^p\,d\mu(x)\\ &= \frac{\chi_{E_Q}(x_B)}{\operatorname{diam}(B)^{\beta p}}\int_X \Big(\chi_{B\setminus U^{2^k\lambda}}(x) + \chi_{U_B^{2^k\lambda}}(x)\Big)|u(x)|^p\,d\mu(x)\,.\end{aligned}$$

By Lemma 6.33 it suffices to estimate the integral over the set $B\setminus U^{2^k\lambda} = B\setminus U_B^{2^k\lambda}$; observe that the measure of this set is strictly positive. We remark that the local boundary Poincaré inequality in Lemma 6.26 will be used to estimate this integral.

Fix a number $i \in \mathbb{N}$. Since $B \subset Q^*$, it follows from Lemma 6.32 that the restriction $u|_{E_Q\cup(B\setminus U^{2^i\lambda})}$ is a β-Hölder function with a constant $\kappa_i = C(\beta,c_\mu)2^i\lambda$. We can now use the McShane extension (6.5) and extend $u|_{E_Q\cup(B\setminus U^{2^i\lambda})}$ to a function $u_{2^i\lambda}\colon X\to\mathbb{R}$ that is β-Hölder with the constant κ_i and satisfies the restriction identity

$$u_{2^i\lambda}(x) = u(x)$$

for all $x \in E_Q \cup (B\setminus U^{2^i\lambda})$. Observe that $u_{2^i\lambda} = 0$ in E_Q, since $u = 0$ in E_Q.

The crucial idea that was originally used by Keith–Zhong in [71] is to consider the function

$$h(x) = \frac{1}{k}\sum_{i=k}^{2k-1} u_{2^i\lambda}(x)\,, \qquad x \in X\,.$$

We want to apply Lemma 6.6. In order to do so, observe that $u_{2^i\lambda}|_{X\setminus A} = u|_{X\setminus A}$, where

$$A = X \setminus (B \setminus U^{2^i\lambda}) = X \setminus (B \setminus U_B^{2^i\lambda}) = (X \setminus B) \cup U_B^{2^i\lambda}\,.$$

Therefore, by Lemma 6.6 and properties (D1)–(D2), we obtain that

$$g_h = \frac{1}{k}\sum_{i=k}^{2k-1}\Big(\kappa_i \chi_{(X\setminus B)\cup U_B^{2^i\lambda}} + g\chi_{B\setminus U^{2^i\lambda}}\Big) \in \mathcal{D}_H^\beta(h)\,.$$

Observe that $U_B^{2^k\lambda} \supset U_B^{2^{(k+1)}\lambda} \supset \cdots \supset U_B^{2^{(2k-1)}\lambda} \supset U_B^{4^k\lambda}$. By using these inclusions it is straightforward to show that the following pointwise estimates are valid in X,

$$\begin{aligned}
\chi_B g_h^p &\le \left(\frac{1}{k}\sum_{i=k}^{2k-1}\Big(\kappa_i\, \chi_{U_B^{2^i\lambda}} + g\chi_{B\setminus U^{2^i\lambda}}\Big)\right)^p \\
&\le 2^p\left(\frac{1}{k}\sum_{i=k}^{2k-1}\kappa_i\, \chi_{U_B^{2^i\lambda}}\right)^p + 2^p g^p \chi_{B\setminus U^{4^k\lambda}} \\
&\le \frac{C(\beta,p,c_\mu)}{k^p}\sum_{j=k}^{2k-1}\left(\sum_{i=k}^{j} 2^i\lambda\right)^p \chi_{U_B^{2^j\lambda}} + 2^p g^p \chi_{B\setminus U^{4^k\lambda}} \\
&\le \frac{C(\beta,p,c_\mu)}{k^p}\sum_{j=k}^{2k-1}(\lambda 2^j)^p \chi_{U_B^{2^j\lambda}} + 2^p g^p \chi_{B\setminus U^{4^k\lambda}}\,.
\end{aligned}$$

Observe that $h \in \operatorname{Lip}_\beta(X)$ is zero in E_Q and h coincides with u on $B \setminus U^{2^k\lambda}$, and recall that $g_h \in \mathcal{D}_H^\beta(h)$. Notice also that $B \subset Q^*$ and $x_B \in E_Q$. The local boundary Poincaré inequality in Lemma 6.26 implies that

$$\begin{aligned}
&\frac{\chi_{E_Q}(x_B)}{\operatorname{diam}(B)^{\beta p}}\int_{B\setminus U^{2^k\lambda}} |u(x)|^p\, d\mu(x) \\
&\qquad = \frac{\chi_{E_Q}(x_B)}{\operatorname{diam}(B)^{\beta p}}\int_B |h(x)|^p\, d\mu(x) \\
&\qquad \le K\int_B g_h(x)^p\, d\mu(x) \\
&\qquad \le \frac{C(\beta,p,c_\mu)K}{k^p}\sum_{j=k}^{2k-1}(\lambda 2^j)^p \mu(U_B^{2^j\lambda}) + 2^p K\int_{B\setminus U^{4^k\lambda}} g(x)^p\, d\mu(x)\,.
\end{aligned}$$

The desired local inequality (6.42) follows by combining the estimates above. □

6.7.4 Completing proof of Theorem 6.29

We complete the proof as in [74]. Recall that $u\colon X \to \mathbb{R}$ is a β-Hölder function with $u = 0$ in E_Q and that

$$M^\sharp u + M^{E_Q} u = M^{\sharp,p}_{\beta,\mathcal{B}_0} u + M^{E_Q,p}_{\beta,\mathcal{B}_0} u\,.$$

Let us fix a function $g \in \mathcal{D}_H^\beta(u)$. Observe that the left-hand side of inequality (6.26) is finite. Without loss of generality, we may further assume that it is nonzero. By

Lemma 6.31,

$$\int_{B_0} \left(M^\sharp u(x) + M^{E_Q} u(x)\right)^{p-\varepsilon} d\mu(x)$$
$$\leq C(p, c_\mu) \int_{B_0} \left(M_Q^{E_Q} u(x)\right)^{p-\varepsilon} d\mu(x) + C(\beta, p, c_\mu) \int_{B_0} g(x)^{p-\varepsilon}(x)\, d\mu(x)\,.$$

We have

$$\int_{B_0} \left(M_Q^{E_Q} u(x)\right)^{p-\varepsilon} d\mu(x) = \int_{Q^*} \left(M_Q^{E_Q} u(x)\right)^{p-\varepsilon} d\mu(x) = (p-\varepsilon) \int_0^\infty \lambda^{p-\varepsilon} \mu(Q^\lambda)\, \frac{d\lambda}{\lambda}\,.$$

Since $Q^\lambda = Q^* = Q^{2\lambda}$ for every $\lambda \in (0, \lambda_Q/2)$, we find that

$$\begin{aligned}
(p-\varepsilon) \int_0^{\lambda_Q/2} \lambda^{p-\varepsilon} \mu(Q^\lambda)\, \frac{d\lambda}{\lambda} &= \frac{(p-\varepsilon)}{2^{p-\varepsilon}} \int_0^{\lambda_Q/2} (2\lambda)^{p-\varepsilon} \mu(Q^{2\lambda})\, \frac{d\lambda}{\lambda} \\
&\leq \frac{(p-\varepsilon)}{2^{p-\varepsilon}} \int_0^\infty \sigma^{p-\varepsilon} \mu(Q^\sigma)\, \frac{d\sigma}{\sigma} \\
&= \frac{1}{2^{p-\varepsilon}} \int_{Q^*} \left(M_Q^{E_Q} u(x)\right)^{p-\varepsilon} d\mu(x)\,.
\end{aligned}$$

On the other hand, by Lemma 6.34, for each $\lambda > \lambda_Q/2$,

$$\lambda^{p-\varepsilon} \mu(Q^\lambda) \leq C(\beta, p, c_\mu) \lambda^{-\varepsilon} \Bigg[\frac{(\lambda 2^k)^p}{2^{k\alpha}} \mu(U^{2^k\lambda}) + \frac{K}{k^p} \sum_{j=k}^{2k-1} (\lambda 2^j)^p \mu(U^{2^j\lambda}) + K \int_{U^\lambda \setminus U^{4^k\lambda}} g^p\, d\mu \Bigg].$$

Since $p - \varepsilon > 1$, it follows that

$$\begin{aligned}
\int_{Q^*} \left(M_Q^{E_Q} u(x)\right)^{p-\varepsilon} d\mu(x) &\leq 2(p-\varepsilon) \int_{\lambda_Q/2}^\infty \lambda^{p-\varepsilon} \mu(Q^\lambda)\, \frac{d\lambda}{\lambda} \\
&\leq C(\beta, p, c_\mu)(I_1(Q) + I_2(Q) + I_3(Q))\,,
\end{aligned}$$

where

$$\begin{aligned}
I_1(Q) &= \frac{2^{k\varepsilon}}{2^{k\alpha}} \int_0^\infty (\lambda 2^k)^{p-\varepsilon} \mu(U^{2^k\lambda})\, \frac{d\lambda}{\lambda}\,, \\
I_2(Q) &= \frac{K}{k^p} \sum_{j=k}^{2k-1} 2^{j\varepsilon} \int_0^\infty (2^j\lambda)^{p-\varepsilon} \mu(U^{2^j\lambda})\, \frac{d\lambda}{\lambda}\,, \\
I_3(Q) &= K \int_0^\infty \lambda^{-\varepsilon} \int_{U^\lambda \setminus U^{4^k\lambda}} g(x)^p\, d\mu(x)\, \frac{d\lambda}{\lambda}\,.
\end{aligned}$$

We estimate these three terms separately. First,

$$\begin{aligned}
I_1(Q) &\leq \frac{2^{k(\varepsilon-\alpha)}}{p-\varepsilon} \int_{B_0} \left(M^\sharp u(x) + M^{E_Q} u(x)\right)^{p-\varepsilon} d\mu(x) \\
&\leq 2^{k(\varepsilon-\alpha)} \int_{B_0} \left(M^\sharp u(x) + M^{E_Q} u(x)\right)^{p-\varepsilon} d\mu(x)\,.
\end{aligned}$$

Second,

$$\begin{aligned} I_2(Q) &\le \frac{K}{k^p}\sum_{j=k}^{2k-1} 2^{j\varepsilon}\int_0^\infty (2^j\lambda)^{p-\varepsilon}\mu(U^{2^j\lambda})\,\frac{d\lambda}{\lambda} \\ &\le \frac{K}{k^p(p-\varepsilon)}\left(\sum_{j=k}^{2k-1} 2^{j\varepsilon}\right)\int_{B_0}\left(M^\sharp u(x)+M^{E_Q}u(x)\right)^{p-\varepsilon} d\mu \\ &\le \frac{K4^{k\varepsilon}}{k^{p-1}}\int_{B_0}\left(M^\sharp u(x)+M^{E_Q}u(x)\right)^{p-\varepsilon} d\mu\,. \end{aligned}$$

Third, by Fubini's theorem,

$$\begin{aligned} I_3(Q) &\le K\int_{B_0\setminus\{M^\sharp u+M^{E_Q}u=0\}}\left(\int_0^\infty \lambda^{-\varepsilon}\chi_{U^\lambda\setminus U^{4^k\lambda}}(x)\frac{d\lambda}{\lambda}\right)g(x)^p\,d\mu(x) \\ &\le C(k,\varepsilon)K\int_{B_0\setminus\{M^\sharp u+M^{E_Q}u=0\}} g(x)^p(M^\sharp u(x)+M^{E_Q}u(x))^{-\varepsilon}\,d\mu(x)\,. \end{aligned}$$

Combining the estimates above, we arrive at the desired conclusion. □

6.8 Local Hardy inequalities

We apply Theorem 6.28 in order to obtain a local Hardy inequality, see (6.43) in Theorem 6.35. This inequality is then shown to be self-improving, see Theorem 6.36, and in this respect we follow the strategy in [81]. However, we remark that the easier Wannebo approach [114] for establishing local Hardy inequalities as in [81] is not available to us, due to absence of pointwise Leibniz and chain rules in the setting of Hajłasz gradients.

Theorem 6.35

Let X be a geodesic space. Let $1 < p < \infty$ and $0 < \beta \le 1$. Let $E \subset X$ be a closed set which satisfies the (β, p)-capacity density condition with a constant c_0. Let $B_0 = B(w, R)$ be a ball with $w \in E$ and $R < \operatorname{diam}(E)$. Let E_Q be the truncation of E to the Whitney-type ball Q as in Section 6.6. Then there exists a constant $C = C(\beta, p, c_\mu, c_0)$ such that

$$\int_{B(w,R)\setminus E_Q}\frac{|u(x)|^p}{d(x,E_Q)^{\beta p}}\,d\mu(x) \le C\int_{B(w,R)} g(x)^p\,d\mu(x) \qquad (6.43)$$

holds whenever $u \in \operatorname{Lip}_\beta(X)$ is such that $u = 0$ in E_Q and $g \in \mathcal{D}_H^\beta(u)$.

Proof. Let $u \in \operatorname{Lip}_\beta(X)$ be such that $u = 0$ in E_Q and let $g \in \mathcal{D}_H^\beta(u)$. Lemma 6.32 implies that

$$|u(x)| \le C(\beta, c_\mu)d(x,E_Q)^\beta\left(M^{\sharp,p}_{\beta,\mathcal{B}_0}u(x) + M^{E_Q,p}_{\beta,\mathcal{B}_0}u(x)\right)$$

for all $x \in Q^*$. Therefore

$$\int_{Q^*\setminus E_Q} \frac{|u(x)|^p}{d(x,E_Q)^{\beta p}}\,d\mu(x) \le C(\beta,p,c_\mu)\int_{B(w,R)} \Big(M^{\sharp,p}_{\beta,\mathcal{B}_0}u(x) + M^{E_Q,p}_{\beta,\mathcal{B}_0}u(x)\Big)^p\,d\mu(x)\,.$$

By Theorem 6.28, we obtain

$$\int_{Q^*\setminus E_Q} \frac{|u(x)|^p}{d(x,E_Q)^{\beta p}}\,d\mu(x) \le C(\beta,p,c_\mu,c_0)\int_{B(w,R)} g(x)^p\,d\mu(x)\,. \tag{6.44}$$

It remains to bound the integral over $B(w,R)\setminus Q^*$. Since $E_Q \subset \overline{Q}$ and $Q^* = 4Q$, we have $d(x,E_Q) \ge 3r_Q > R/64$ for all $x \in B(w,R)\setminus Q^*$. Thus, we obtain

$$\begin{aligned}
\int_{B(w,R)\setminus Q^*} \frac{|u(x)|^p}{d(x,E_Q)^{\beta p}}\,d\mu(x) &\le \frac{64^{\beta p}}{R^{\beta p}}\int_{B(w,R)} |u(x)|^p\,d\mu(x)\\
&\le \frac{3^p 64^{\beta p}}{R^{\beta p}}\Big(\int_{B(w,R)} |u(x)-u_{B(w,R)}|^p\,d\mu(x)\\
&\qquad + \mu(B(w,R))|u_{B(w,R)}-u_{Q^*}|^p\\
&\qquad + \mu(B(w,R))|u_{Q^*}|^p\Big)\,.
\end{aligned}$$

By the (β,p,p)-Poincaré inequality in Lemma 6.8,

$$\begin{aligned}
\int_{B(w,R)} |u(x)-u_{B(w,R)}|^p\,d\mu(x) &\le 2^p\operatorname{diam}(B(w,R))^{\beta p}\int_{B(w,R)} g(x)^p\,d\mu(x)\\
&\le C(p)R^{\beta p}\int_{B(w,R)} g(x)^p\,d\mu(x)\,.
\end{aligned}$$

For the second term, we have

$$\begin{aligned}
\mu(B(w,R))|u_{B(w,R)}-u_{Q^*}|^p &\le \mu(B(w,R)) \fint_{Q^*} |u(x)-u_{B(w,R)}|^p\,d\mu(x)\\
&\le C(c_\mu)\int_{B(w,R)} |u(x)-u_{B(w,R)}|^p\,d\mu(x)\\
&\le C(p,c_\mu)R^{\beta p}\int_{B(w,R)} g(x)^p\,d\mu(x)\,.
\end{aligned}$$

For the third term, we have $d(x,E_Q) \le d(x,w) < 4r_Q < R$ for every $x \in Q^*$. Thus,

$$\begin{aligned}
\mu(B(w,R))|u_{Q^*}|^p &\le C(c_\mu)\int_{Q^*\setminus E_Q} |u(x)|^p\,d\mu(x)\\
&\le R^{\beta p}\int_{Q^*\setminus E_Q} \frac{|u(x)|^p}{d(x,E_Q)^{\beta p}}\,d\mu(x)\,.
\end{aligned}$$

Applying inequality (6.44), we get

$$\mu(B(w,R))|u_{Q^*}|^p \le C(\beta,p,c_\mu,c_0)R^{\beta p}\int_{B(w,R)} g(x)^p\,d\mu(x)\,.$$

The desired inequality follows by combining the estimates above. □

Next we improve the local Hardy inequality in Theorem 6.35. This is done by adapting the Koskela–Zhong truncation argument from [77] to the setting of Hajłasz gradients; see also [81] and [73, Theorem 7.32] whose proof we modify to our purposes.

Theorem 6.36

Let X be a geodesic space. Let $1 < p < \infty$ and $0 < \beta \le 1$. Let $E \subset X$ be a closed set which satisfies the (β, p)-capacity density condition with a constant c_0. Let $B_0 = B(w, R)$ be a ball with $w \in E$ and $R < \operatorname{diam}(E)$. Let E_Q be the truncation of E to the Whitney-type ball Q as in Section 6.6, and let $C_1 = C_1(\beta, p, c_\mu, c_0)$ be the constant in (6.43), see Theorem 6.35. Then there exist $0 < \varepsilon = \varepsilon(p, C_1) < p - 1$ and $C = C(p, C_1)$ such that inequality

$$\int_{B(w,R)\setminus E_Q} \frac{|u(x)|^{p-\varepsilon}}{d(x, E_Q)^{\beta(p-\varepsilon)}}\, d\mu(x) \le C \int_{B(w,R)} g(x)^{p-\varepsilon}\, d\mu(x) \tag{6.45}$$

holds whenever $u \in \operatorname{Lip}_\beta(X)$ is such that $u = 0$ in E_Q and $g \in \mathcal{D}_H^\beta(u)$.

Proof. Without loss of generality, we may assume that $C_1 \ge 1$ in (6.43). Let $u \in \operatorname{Lip}_\beta(X)$ be such that $u = 0$ in E_Q and let $g \in \mathcal{D}_H^\beta(u)$. Let $\kappa \ge 0$ be the β-Hölder constant of u in X. By redefining $g = \kappa$ in the exceptional set $N = N(g)$ of measure zero, we may assume that (6.6) holds for all $x, y \in X$. Let $\lambda > 0$ and define $F_\lambda = G_\lambda \cap H_\lambda$, where

$$G_\lambda = \{x \in B(w, R) : g(x) \le \lambda\}$$

and

$$H_\lambda = \{x \in B(w, R) : |u(x)| \le \lambda d(x, E_Q)^\beta\}.$$

We show that the restriction of u to $F_\lambda \cup E_Q$ is a β-Hölder function with a constant 2λ. Assume that $x, y \in F_\lambda$. Then (6.6) implies

$$|u(x) - u(y)| \le d(x, y)^\beta \, (g(x) + g(y)) \le 2\lambda d(x, y)^\beta \, .$$

On the other hand, if $x \in F_\lambda$ and $y \in E_Q$, then

$$|u(x) - u(y)| = |u(x)| \le \lambda d(x, E_Q)^\beta \le 2\lambda d(x, y)^\beta \, .$$

The case $x \in E_Q$ and $y \in F_\lambda$ is treated in the same way. If $x, y \in E_Q$, then $|u(x) - u(y)| = 0$. All in all, we see that u is a β-Hölder function in $F_\lambda \cup E_Q$ with a constant 2λ.

We apply the McShane extension 6.5 and extend the restriction $u|_{F_\lambda \cup E_Q}$ to a β-Hölder function function v in X with constant 2λ. Then $v = u = 0$ in E_Q and $v = u$ in F_λ, thus

$$g_v = g\chi_{F_\lambda} + 2\lambda\chi_{X\setminus F_\lambda} \in \mathcal{D}_H^\beta(v)$$

by Lemma 6.6.

By applying Theorem 6.35 to the function v and its Hajłasz β-gradient g_v, we obtain

$$\begin{aligned}\int_{(B(w,R)\setminus E_Q)\cap F_\lambda} \frac{|u(x)|^p}{d(x, E_Q)^{\beta p}}\, d\mu(x) &\le \int_{B(w,R)\setminus E_Q} \frac{|v(x)|^p}{d(x, E_Q)^{\beta p}}\, d\mu(x) \\ &\le C_1 \int_{F_\lambda} g(x)^p\, d\mu(x) + C_1 2^p \lambda^p \mu(B(w, R) \setminus F_\lambda) \, .\end{aligned}$$

Since $H_\lambda = F_\lambda \cup (H_\lambda \setminus G_\lambda)$ and $C_1 \ge 1$, it follows that

$$\begin{aligned}
&\int_{(B(w,R)\setminus E_Q)\cap H_\lambda} \frac{|u(x)|^p}{d(x,E_Q)^{\beta p}}\,d\mu(x) \\
&\quad \le C_1 \int_{F_\lambda} g(x)^p\,d\mu(x) + C_1 2^p \lambda^p \mu(B(w,R)\setminus F_\lambda) \\
&\qquad + \int_{(H_\lambda\setminus E_Q)\setminus G_\lambda} \frac{|u(x)|^p}{d(x,E_Q)^{\beta p}}\,d\mu(x) \\
&\quad \le C_1 \int_{G_\lambda} g(x)^p\,d\mu(x) + C_1 2^p \lambda^p \big(\mu(B(w,R)\setminus F_\lambda) + \mu(H_\lambda \setminus G_\lambda)\big) \\
&\quad \le C_1 \int_{G_\lambda} g(x)^p\,d\mu(x) + C_1 2^{p+1} \lambda^p \big(\mu(B(w,R)\setminus H_\lambda) + \mu(B(w,R) \setminus G_\lambda)\big).
\end{aligned} \tag{6.46}$$

Here $\lambda > 0$ was arbitrary, and thus we conclude that (6.46) holds for every $\lambda > 0$.

Next we multiply (6.46) by $\lambda^{-1-\varepsilon}$, where $0 < \varepsilon < p-1$, and integrate with respect to λ over the set $(0,\infty)$. With a change of the order of integration on the left-hand side, this gives

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{B(w,R)\setminus E_Q} \left(\frac{|u(x)|}{d(x,E_Q)^{\beta}}\right)^{p-\varepsilon} d\mu(x) &\le C_1 \int_0^\infty \lambda^{-1-\varepsilon} \int_{G_\lambda} g(x)^p\,d\mu(x)\,d\lambda \\
&+ C_1 2^{p+1} \int_0^\infty \lambda^{p-1-\varepsilon} \big(\mu(B(w,R)\setminus H_\lambda) + \mu(B(w,R)\setminus G_\lambda)\big)\,d\lambda\,.
\end{aligned}$$

By the definition of G_λ, we find that the first term on the right-hand side is dominated by

$$\frac{C_1}{\varepsilon}\int_{B(w,R)} g(x)^{p-\varepsilon}\,d\mu(x)\,.$$

Using the definitions of H_λ and G_λ, the second term on the right-hand side can be estimated from above by

$$\frac{C_1 2^{p+1}}{p-\varepsilon}\left(\int_{B(w,R)\setminus E_Q}\left(\frac{|u(x)|}{d(x,E_Q)^{\beta}}\right)^{p-\varepsilon} d\mu(x) + \int_{B(w,R)} g(x)^{p-\varepsilon}\,d\mu(x)\right).$$

By combining the estimates above, we obtain

$$\begin{aligned}
&\int_{B(w,R)\setminus E_Q}\left(\frac{|u(x)|}{d(x,E_Q)^{\beta}}\right)^{p-\varepsilon} d\mu(x) \\
&\quad \le C_2 \int_{B(w,R)\setminus E_Q}\left(\frac{|u(x)|}{d(x,E_Q)^{\beta}}\right)^{p-\varepsilon} d\mu(x) + C_3 \int_{B(w,R)} g(x)^{p-\varepsilon}\,d\mu(x)\,,
\end{aligned} \tag{6.47}$$

where $C_2 = C_1 2^{p+1}\frac{\varepsilon}{p-\varepsilon}$ and $C_3 = C_1\big(1+2^{p+1}\frac{\varepsilon}{p-\varepsilon}\big)$. We choose $0 < \varepsilon = \varepsilon(C_1,p) < p-1$ so small that

$$C_2 = C_1 2^{p+1}\frac{\varepsilon}{p-\varepsilon} < \frac{1}{2}\,.$$

This allows us to absorb the first term in the right-hand side of (6.47) to the left-hand side. Observe that this term is finite, since u is β-Hölder in X and $u = 0$ in E_Q. □

6.9 Self-improvement of the capacity density condition

As an application of Theorem 6.36, we strengthen Theorem 6.22 in complete geodesic spaces. This leads to the conclusion that the Hajłasz capacity density condition is self-improving or doubly open-ended in such spaces. In fact, we characterize the Hajłasz capacity density condition in various geometrical and analytical quantities, the latter of which are all shown to be doubly open-ended.

Theorem 6.37

Let X be a geodesic space. Let $1 < p < \infty$ and $0 < \beta \le 1$. Let $E \subset X$ be a closed set which satisfies the (β, p)-capacity density condition with a constant c_0. Then there exists $\varepsilon > 0$, depending on β, p, c_μ and c_0, such that $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) \le \beta(p-\varepsilon)$.

Proof. Let $w \in E$ and $0 < r < R < \operatorname{diam}(E)$. Let E_Q be the truncation of E to the ball $Q \subset B_0 = B(w,R)$ as in Section 6.6. Let $\varepsilon > 0$ be as in Theorem 6.36. Observe that

$$E_{Q,r} = \{x \in X \,:\, d(x, E_Q) < r\} \subset \{x \in X \,:\, d(x,E) < r\} = E_r\,.$$

Hence, it suffices to show that

$$\frac{\mu(E_{Q,r} \cap B(w,R))}{\mu(B(w,R))} \ge c\Big(\frac{r}{R}\Big)^{\beta(p-\varepsilon)}, \tag{6.48}$$

where the constant c is independent of w, r and R.

If $r \ge R/4$, then the claim is clear since $\big(\frac{r}{R}\big)^{\beta(p-\varepsilon)} \le 1$ and

$$\mu(E_{Q,r} \cap B(w,R)) \ge \mu(B(w,R/4)) \ge C(\mu)\mu(B(w,R))\,.$$

The claim is clear also if $\mu(E_{Q,r} \cap B(w,R)) \ge \frac{1}{2}\mu(B(w,R))$. Thus we may assume that $r < R/4$ and that $\mu(E_{Q,r} \cap B(w,R)) < \frac{1}{2}\mu(B(w,R))$, whence

$$\mu(B(w,R) \setminus E_{Q,r}) \ge \tfrac{1}{2}\mu(B(w,R)) > 0. \tag{6.49}$$

Let us now consider the β-Hölder function $u\colon X \to \mathbb{R}$,

$$u(x) = \min\{1, r^{-\beta} d(x,E_Q)^\beta\}\,, \qquad x \in X.$$

Then $u = 0$ in E_Q, $u = 1$ in $X \setminus E_{Q,r}$, and

$$|u(x) - u(y)| \le r^{-\beta} d(x,y)^\beta \quad \text{ for all } x, y \in X.$$

We aim to apply Theorem 6.36. Recall also that $w \in E_Q$. Thus we obtain

$$\begin{aligned} \int_{B_0 \setminus E_Q} \frac{|u(x)|^{p-\varepsilon}}{d(x,E_Q)^{\beta(p-\varepsilon)}}\, d\mu(x) &\ge R^{-\beta(p-\varepsilon)} \int_{B_0 \setminus E_Q} |u(x)|^{p-\varepsilon}\, d\mu(x) \\ &\ge R^{-\beta(p-\varepsilon)} \int_{B_0 \setminus E_{Q,r}} |u(x)|^{p-\varepsilon}\, d\mu(x) \\ &\ge R^{-\beta(p-\varepsilon)} \mu(B(w,R) \setminus E_{Q,r}) \\ &\ge 2^{-1} R^{-\beta(p-\varepsilon)} \mu(B(w,R))\,, \end{aligned} \tag{6.50}$$

where the last step follows from (6.49).

Since $u = 1$ in $X \setminus E_{Q,r}$ and u is a β-Hölder function with a constant $r^{-\beta}$, Lemma 6.6 implies that $g = r^{-\beta}\chi_{E_{Q,r}} \in \mathcal{D}_H^\beta(u)$. Observe that

$$\int_{B_0} g^{p-\varepsilon}\,d\mu \le r^{-\beta(p-\varepsilon)}\mu(E_{Q,r} \cap B_0) = r^{-\beta(p-\varepsilon)}\mu(E_{Q,r} \cap B(w,R))\,.$$

Hence, the claim (6.48) follows from (6.50) and Theorem 6.36. □

The following theorem is a compilation of the results in this chapter. It states the equivalence of some geometrical conditions (1)–(2) and analytical conditions (3)–(6), one of which is the capacity density condition. We emphasize that the capacity density condition (3) is characterized in terms of the upper Assouad codimension (1); in fact, this characterization follows immediately from Theorem 6.21 and Theorem 6.37.

Theorem 6.38

Let X be a complete geodesic space. Let $1 < p < \infty$ and $0 < \beta \le 1$. Let $E \subset X$ be a closed set. Then the following conditions are equivalent:

(1) $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) < \beta p$.

(2) E satisfies the Hausdorff content density condition (6.13) for some $0 < q < \beta p$.

(3) E satisfies the (β, p)-capacity density condition.

(4) E satisfies the local (β, p, p)-boundary Poincaré inequality (6.21).

(5) E satisfies the maximal (β, p, p)-boundary Poincaré inequality (6.25).

(6) E satisfies the local (β, p, p)-Hardy inequality (6.43).

Proof. The implication from (1) to (2) is a consequence of Lemma 6.19 with $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) < q < \beta p$. The implication from (2) to (3) follows by adapting the proof of Theorem 6.21 with $\eta = q/\beta$. The implication from (3) to (4) follows from Theorem 6.26. The implication from (4) to (5) follows from the proof of Theorem 6.28, which remains valid if we assume (4) instead of the (β, p)-capacity density condition. The implication from (5) to (6) follows from the proof of Theorem 6.35. Finally, condition (6) implies the improved local Hardy inequality (6.45) and the proof of Theorem 6.37 then shows the remaining implication from (6) to (1). □

Finally, we state the main result of this paper, Theorem 6.39. It is the self-improvement or double open-endedness property of the (β, p)-capacity density condition. Namely, in addition to integrability exponent p, also the order β of fractional differentiability can be lowered. A similar phenomenon is observed in [88] for Riesz capacities in $\mathbb{R}^n$. See also [78], where solutions to nonlocal equations with measurable coefficients are shown to be both higher integrable and higher differentiable.

Theorem 6.39

Let X be a complete geodesic space, and let $1 < p < \infty$ and $0 < \beta \le 1$. Assume that a closed set $E \subset X$ satisfies the (β, p)-capacity density condition. Then there exists $0 < \delta < \min\{\beta, p-1\}$ such that E satisfies the (γ, q)-capacity density condition for all $\beta - \delta < \gamma \le 1$ and $p - \delta < q < \infty$.

Proof. We have $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) < \beta p$ by Theorem 6.38. Since $\lim_{\delta\to 0}(\beta-\delta)(p-\delta) = \beta p$, there exists $0 < \delta < \min\{\beta, p-1\}$ such that $\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) < (\beta-\delta)(p-\delta)$. Now if $\beta - \delta < \gamma \le 1$ and $p - \delta < q < \infty$, then

$$\overline{\operatorname{co\,dim}}_{\mathrm{A}}(E) < (\beta-\delta)(p-\delta) < \gamma q\,.$$

The claim follows from Theorem 6.38. □

A similar argument shows that the analytical conditions (4)–(6) in Theorem 6.38 are also doubly open ended. The geometrical conditions (1)–(2) are open-ended by definition.

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List of publications

Most of the contents of this thesis have been developed in the following works. The first three have already been published, the fourth one has been recently accepted and the last one was submitted by the time of the publication of this dissertation.

[12] Canto, J. *Sharp Reverse Hölder inequality for C_p Weights and Applications*, The Journal of Geometric Analysis (2021) **31**: 4165–4190.

[13] Canto, J., Li, K., Roncal, L., Tapiola, O. *C_p estimates for rough homogeneous singular integrals and sparse forms*, Annalli della Scuola Normale Superiore di Pisa, clase di Scienze (5) Vol XXII (2021), 1131–1168.

[14] Canto, J., Pérez, C. *Extensions of the John–Nirenberg theorem*, Proceedings of the American Mathematical Society **149** (2021), no. 4, 1507–1525.

[15] Canto, J., Pérez, C., Rela, E. *Minimal conditions for BMO* to appear in Journal of Functional Analysis.

[16] Canto, J., Vähäkangas, A.V. *The Hajłasz capacity density condition is self-improving.* arXiv:2108.09077v1