THE WALDSCHMIDT CONSTANT OF A STANDARD $\mathbb{k}$-CONFIGURATION IN $\mathbb{P}^2$

MARIA VIRGINIA CATALISANO, GIUSEPPE FAVACCHIO, ELENA GUARDO, AND YONG-SU SHIN*

ABSTRACT. A $\mathbb{k}$-configuration of type $(d_1, \dots, d_s)$, where $1 \leqslant d_1 < \dots < d_s$ are integers, is a set of points in $\mathbb{P}^2$ that has a number of algebraic and geometric properties. For example, the graded Betti numbers and Hilbert functions of all $\mathbb{k}$-configurations in $\mathbb{P}^2$ are determined by the type $(d_1, \dots, d_s)$. However the Waldschmidt constant of a $\mathbb{k}$-configuration in $\mathbb{P}^2$ of the same type may vary. In this paper, we find that the Waldschmidt constant of a $\mathbb{k}$-configuration in $\mathbb{P}^2$ of type $(d_1, \dots, d_s)$ with $d_1 \geq s \geq 1$ is s. Then we deal with the Waldschmidt constants of standard $\mathbb{k}$-configurations in $\mathbb{P}^2$ of type (a), (a, b), and (a, b, c) with $a \geq 1$. In particular, we prove that the Waldschmidt constant of a standard $\mathbb{k}$-configuration in $\mathbb{P}^2$ of type $(1, b, c)$ with $c \geq 2b + 2$ does not depend on c.

CONTENTS

1. INTRODUCTION

A set of points $\mathbb{X}$ in $\mathbb{P}^2$ is called a $\mathbb{k}$-*configuration* of type $(d_1, \dots, d_s)$, where $1 \leqslant d_1 < \dots < d_s$ are integers, when there exists a partition of $\mathbb{X} = \mathbb{X}_1 \cup \dots \cup \mathbb{X}_s$ and s distinct lines $L_1, \dots, L_s \subseteq \mathbb{P}^2$ such that, for each $i = 1, \dots, s$ we have $|\mathbb{X}_i| = d_i$, $\mathbb{X}_i \subseteq L_i$ and, for $i > 1$, $L_i \cap (\mathbb{X}_1 \cup \dots \cup \mathbb{X}_{i-1}) = \emptyset$. The last condition forces a point in $\mathbb{X}$ to belong to the set $\mathbb{X}_i$ corresponding to the largest index of a line containing it.

The $\mathbb{k}$-configurations were introduced in the 1980s by Roberts and Roitman in [26] and extensively studied in the literature for their several interesting properties, see for instance [5, 12, 14, 15, 17, 18].

In 1995, Harima [23] extended this definition to $\mathbb{P}^3$, and then in 2001 Geramita, Harima, and Shin [14, 16] generalized the definition to $\mathbb{P}^n$. Moreover, Roberts and Roitman showed that all $\mathbb{k}$-configurations in $\mathbb{P}^2$ of type $(d_1, \dots, d_s)$ have the same Hilbert function, which can be encoded from the type. This result was generalized again by Geramita, Harima, and Shin [16, Corollary 3.7] to show that all graded Betti numbers of the associated ideal of a $\mathbb{k}$-configuration in $\mathbb{P}^n$ depend on the type only. However, it should be noted that $\mathbb{k}$-configurations in $\mathbb{P}^n$ of the same type can have very different algebraic and geometric properties [6, 7].

In this paper we are interested in the study of the Waldschmidt constant.

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*Corresponding author.

The *Waldschmidt constant* of a homogeneous ideal I in $R = \Bbbk[x_0, x_1, \ldots, x_n]$ was introduced in [28] as

$$\widehat{\alpha}(I) = \lim_{t \to \infty} \frac{\alpha(I^{(t)})}{t},$$

where $I^{(t)}$ is the t-th symbolic power of the ideal I, defined by $I^{(t)} = \bigcap_{P \in \mathrm{Ass}(I)} (I^t R_P \cap R)$, and $\alpha(I^{(t)})$ is the least degree among all minimal homogeneous generators of $I^{(t)}$. In [3, Lemma 2.3.1] it was proved that this limit exists.

A prolific line of research involves the study of the Waldschmidt constant of zero dimensional schemes in $\mathbb{P}^n$, see [2, 4, 8, 9, 10, 11, 20, 21, 24, 27] just to cite some papers. In particular, in [5] and in [25], the authors give some results about the Waldschmidt constant of star configurations.

Note that if $I_{\mathbb{X}}$ is the ideal defining a set of distinct points $\mathbb{X} = \{P_1, \ldots, P_s\}$ in $\mathbb{P}^n$ and I_{P_i} is the ideal of the point P_i, then the t-th symbolic power of $I_{\mathbb{X}}$ is $I_{\mathbb{X}}^{(t)} = I_{P_1}^t \cap \cdots \cap I_{P_s}^t$, that is, $I_{\mathbb{X}}^{(t)}$ defines a *homogeneous set of fat points supported at* $\mathbb{X}$, denoted by $t\mathbb{X}$. If $I_{\mathbb{X}}$ is the ideal of a set of points $\mathbb{X}$, instead of "Waldschmidt constant of $I_{\mathbb{X}}$", we simply write "Waldschmidt constant of $\mathbb{X}$".

In [5, Section 3.3] the authors showed that two different $\Bbbk$-configurations of the same type may have different Waldschmidt constants. For an easy example, consider the following two $\Bbbk$-configurations $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{P}^2$ of type $(1, 2, 3)$.

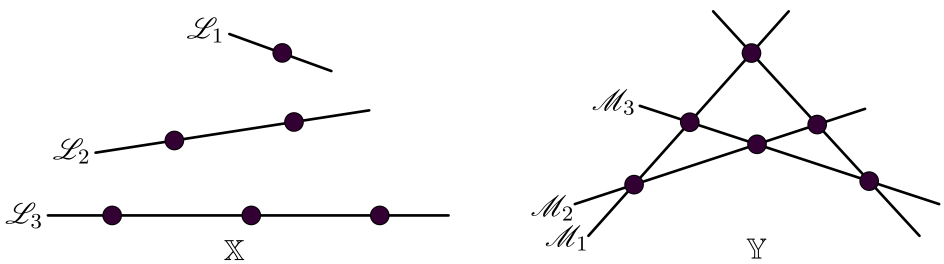

FIGURE 1. $\Bbbk$-configurations $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{P}^2$ of type $(1, 2, 3)$

Then the Waldschmidt constants of $\mathbb{X}$ and $\mathbb{Y}$ are different, i.e.,

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{7}{3} \quad \text{and} \quad \widehat{\alpha}(I_{\mathbb{Y}}) = 2,$$

respectively (see [5, 13]).

As we have seen above, $\Bbbk$-configurations in $\mathbb{P}^2$ of the same type may have different Waldschmidt constants. Here we extend some results in [5]. In particular we focus on the so called *standard* $\Bbbk$*-configurations* in $\mathbb{P}^2$, see Definition 2.4, and we find the Waldschmidt constants of all standard $\Bbbk$-configurations of type (a), (a, b) and (a, b, c), except for type $(2, 3, 5)$, as summarized in Table 1.

The paper is structured as follows.

In Section 2 we recall some definitions and useful tools; in particular we prove, in a more general context, the existence of irreducible curves in a certain linear system (see Lemma 2.7). In Section 3 we describe a method to find the Waldschmidt constant of a set $\mathbb{X}$ of points, that works in particular when $\mathbb{X}$ is supported on some lines in a specific way, e.g., when $\mathbb{X}$ is a $\Bbbk$-configuration. In Section 4 we consider particular schemes with support on lines, when the number of points on each line is bigger than the number of lines. As an application, we find the Waldschmidt constants of standard $\Bbbk$-configurations of type (a) and, for $a > 1$, of type (a, b). To complete the case (a, b), we recall the result in [11, Proposition 3.3]. In Section 5, we find the Waldschmidt constants of standard $\Bbbk$-configurations of type $(1, b, c)$. In Section 6, we find the Waldschmidt constants of standard $\Bbbk$-configurations of type (a, b, c), with $a > 1$, except the type $(2, 3, 5)$.

To lighten the reading load, the proofs of some theorems of Section 5, that are very similar to the proofs of other theorems in the same section, can be found in the Appendix, where an interested reader will find all the details.

The type of $\mathbb{X}$	Note	$\widehat{\alpha}(I_{\mathbb{X}})$	From
(a)		1	Corollary 4.2
$(1, b)$		$\frac{2b-1}{b}$	Remark 4.4
(a, b)	$a \geq 2$	2	Corollary 4.2
$(1, b, b+1)$	b even, $b \geq 4$	$\frac{9b-4}{3b}$	Theorem 5.4
$(1, b, c)$	c even, $c \leq 2b-4$	$\frac{6b+3c-4}{2b+c}$	Theorem 5.1
$(1, b, c)$	c odd, $b+1 < c \leq 2b-3$	$\frac{6b+3c-7}{2b+c-1}$	Theorem 5.2
$(1, b, 2b-2)$		$\frac{6b^2-14b+6}{2b^2-4b+1}$	Theorem 5.5
$(1, b, 2b-1)$		$\frac{6b^2-8b+1}{2b^2-2b}$	Theorem 5.6
$(1, b, 2b)$		$\frac{6b-5}{2b-1}$	Theorem 5.7
$(1, b, 2b+1)$		$\frac{6b^2-2b-3}{2b^2-1}$	Theorem 5.8
$(1, b, c)$	$c \geq 2b+2$	$\frac{3b-1}{b}$	Theorem 5.10
$(2, 3, 4)$		$\frac{17}{6}$	Theorem 6.1
$(2, 3, 5)$		$\frac{17}{6} \leq \widehat{\alpha}(I_{\mathbb{X}}) \leq \frac{71}{24}$	Remark 6.6
$(2, 3, c)$	$c \geq 6$	3	Theorem 6.5
$(2, b, c)$	$b \geq 4$	3	Theorem 6.5
(a, b, c)	$a \geq 3$	3	Theorem 6.7

TABLE 1. The Waldschmidt constant of standard $\mathbb{k}$-configurations of type (a), (a, b), (a, b, c)

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2. PRELIMINARIES

We will work with an algebraic closed field $\Bbbk$ of characteristic zero. We recall the definition of the Waldschmidt constant for an ideal (see [3, Lemma 2.3.1] for the existence of the limit, and [10] where the authors refer to that limit as "Waldschmidt constant").

Definition 2.1. For a homogeneous ideal $J \subseteq \Bbbk[\mathbb{P}^n]$ we denote by $\alpha(J)$ the initial degree of J, i.e., the least degree of nonzero elements in J. The Waldschmidt constant of J is the following limit

$$\widehat{\alpha}(J) = \lim_{t\to\infty} \frac{\alpha(J^{(t)})}{t},$$

where $J^{(t)}$ is the t-th symbolic power of J.

Note that (see the proof of [3, Lemma 2.3.1])

$$\widehat{\alpha}(J) \leq \frac{\alpha(J^{(t)})}{t}$$

for every $t > 0$.

If $I_{\mathbb{X}}$ is the ideal defining a set of distinct points $\mathbb{X} = \{P_1, \dots, P_s\}$ in $\mathbb{P}^n$ and I_{P_i} is the ideal of the point P_i, then the t-th symbolic power of $I_{\mathbb{X}}$ is $I_{\mathbb{X}}^{(t)} = I_{P_1}^t \cap \cdots \cap I_{P_s}^t$, that is, $I_{\mathbb{X}}^{(t)}$ defines a homogeneous set of fat points supported at $\mathbb{X}$, which we will denote by $t\mathbb{X}$.

In this paper we will work with special sets of simple distinct points in $\mathbb{P}^2$. By abuse of notation, we will refer to $[I_{\mathbb{X}}]_d$ as the linear system of all the plane curves of degree d containing $\mathbb{X}$, since this is, from a geometrical point of view, what the forms in $[I_{\mathbb{X}}]_d$ correspond to, and we simply write $\dim[I_{\mathbb{X}}]_d$ instead of $\dim_{\Bbbk}[I_{\mathbb{X}}]_d$.

We have the following useful lemma.

Lemma 2.2. *Let $\mathbb{X}$ be a set of simple distinct points in $\mathbb{P}^2$, and let $I_{\mathbb{X}}$ be its ideal. Let μ and d be positive integers such that the initial degree of the scheme of fat points $m\mu\mathbb{X}$ is md for each integer $m > 0$. Then the Waldschmidt constant of $I_{\mathbb{X}}$ is*

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{d}{\mu}.$$

Proof. Since, by definition, $\widehat{\alpha}(I_{\mathbb{X}}) = \lim_{t\to\infty} \frac{\alpha(I_{\mathbb{X}}^{(t)})}{t}$, if we let $t = m\mu$, we have $\alpha(I_{\mathbb{X}}^{(t)}) = \alpha(I_{m\mu\mathbb{X}}) = md$, and so

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{md}{m\mu} = \frac{d}{\mu}.$$

□

We now recall the definitions of $\Bbbk$-configurations and standard $\Bbbk$-configurations.

Definition 2.3 ([14, 15, 26])**.** Let $1 \leqslant d_1 < \ldots < d_s$ be integers and let $L_1, \dots, L_s \subseteq \mathbb{P}^2$ be distinct lines. A *$\Bbbk$-configuration of points in $\mathbb{P}^2$ of type* $(d_1, \dots, d_s)$ is a finite set $\mathbb{X}$ of points in $\mathbb{P}^2$ such that:

(1) $\mathbb{X} = \bigcup_{i=1}^s \mathbb{X}_i$, where the $\mathbb{X}_i$ are subsets of $\mathbb{X}$;
(2) $|\mathbb{X}_i| = d_i$ and $\mathbb{X}_i \subseteq L_i$ for each $i = 1, \dots, s$;
(3) L_i $(1 < i \leqslant s)$ does not contain any points of $\mathbb{X}_j$ for all $j < i$.

In analogy with [14, Section 4] in $\mathbb{P}^3$ and [15, Section 4] in $\mathbb{P}^n$, here we give an explicit definition of standard $\Bbbk$-configurations in $\mathbb{P}^2$, which are special $\Bbbk$-configurations of points in $\mathbb{P}^2$ whose coordinates are integer values.

Definition 2.4. Let $\Bbbk[x_0, x_1, x_2]$ be the homogeneous ring for $\mathbb{P}^2$, and let $(d_1, \ldots, d_s)$ be the type of a $\Bbbk$-configuration in $\mathbb{P}^2$. We construct a set of points which realizes this type, and whose points are located in the following lines L_i, where

$$L_1 = \{x_2 = (s-1)x_0\}, L_2 = \{x_2 = (s-2)x_0\}, \quad \ldots \quad , L_s = \{x_2 = 0\}.$$

On each of these lines L_i we place d_i points as follows

$$\begin{array}{ll} d_1 \text{ points on } L_1 \text{ with coordinates } [1 : j : s`1] & 0 \leq j \leq d_1`1, \\ d_2 \text{ points on } L_2 \text{ with coordinates } [1 : j : s-2] & 0 \leq j \leq d_2 - 1, \\ \vdots & \\ d_s \text{ points on } L_s \text{ with coordinates } [1 : j : 0] & 0 \leq j \leq d_s - 1. \end{array}$$

If $1 \leq d_1 < \cdots < d_s$, we call the $\Bbbk$-configuration of points in $\mathbb{P}^2$ constructed as above a *standard* $\Bbbk$-*configuration* of type $(d_1, \ldots, d_s)$.

We conclude this section with two lemmas, that are key tools for the proofs in this paper.

The first one is a technical lemma from our previous paper [5], and it is an application of Bezout's Theorem.

The second lemma is useful to compute the Waldschmidt constants of all the standard $\Bbbk$-configurations from type $(1, b, 2b-2)$ to $(1, b, 2b+1)$, since for those cases we need the existence of irreducible curves.

Lemma 2.5. *Let $m_1, \ldots, m_s$ and d be positive integers and let $P_1, \ldots, P_s$ be s points lying on a line $\mathscr{L}$ with $s > 1$. Let $\mathbb{X}$ be the scheme $m_1P_1 + \cdots + m_sP_s$. Set*

$$\mu = \left\lceil \frac{m_1 + \cdots + m_s - d}{s-1} \right\rceil, \tag{2.1}$$

and assume $[I_{\mathbb{X}}]_d \neq \{0\}$. Then

(i) *$\mu \leq d$;*
(ii) *the line $\mathscr{L}$ is a fixed component of multiplicity at least μ for the plane curves of degree d defined by the forms of the ideal $[I_{\mathbb{X}}]_d$.*

Proof. (i) Since $[I_{\mathbb{X}}]_d \neq \{0\}$, then $d \geq m_i$ for any i, hence

$$\mu = \left\lceil \frac{m_1 + \cdots + m_s - d}{s-1} \right\rceil \leq \left\lceil \frac{sd - d}{s-1} \right\rceil = d;$$

(ii) follows from [5, Lemma 2.5]. □

Remark 2.6. Note that, as we proved in (i), the condition $\mu \leq d$ follows from the hypothesis $[I_{\mathbb{X}}]_d \neq \{0\}$. (Hence the condition $\mu \leq d$ among the hypotheses of [5, Lemma 2.5] was redundant).

Lemma 2.7. *Let L, M be two distinct lines, and let b be a positive integer. Let $P_1, \ldots, P_b$, $Q_1, \ldots, Q_b$, R be distinct points such that $R \notin L \cup M$, and, for any $1 \leq i \leq b$, $P_i \in L$, $Q_i \in M$, and the point $L \cap M \notin \{P_1, \ldots, P_b, Q_1, \ldots, Q_b\}$. Moreover R, P_i, Q_j do not lie on a line, for any i and j. Then*

(i) *the scheme $\mathbb{X} = P_1 + \cdots + P_b + Q_1 + \cdots + Q_b + (b-1)R$ gives independent conditions to the curves of degree b (see Figure 2);*
(ii) *the only curve of degree b in $[I_{\mathbb{X}}]_b$ is irreducible.*

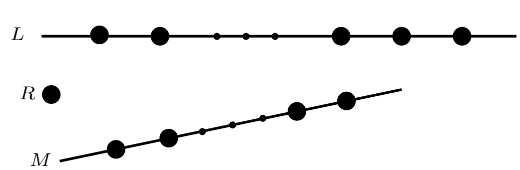

FIGURE 2. The scheme $\mathbb{X}$

Proof. (i) It is well known that the fat point $(b-1)R$ gives independent conditions to the curve of degree b. Consider the following curve $\mathscr{G}_i$ of degree b

$$\mathscr{G}_i = L + N_1 + \cdots + N_{i-1} + N_{i+1} \cdots + N_b,$$

where N_j is the line RQ_j, $j \neq i$, so that $\mathscr{G}_i$ contains the scheme $\mathbb{X} - Q_i$, but it does not contain Q_i. Analogously we can construct a curve of degree b passing through $\mathbb{X} - P_i$, that does not contain P_i. Hence $\{P_1, \ldots, P_b, Q_1, \ldots, Q_b\}$ gives independent conditions to the curves defined by the linear system $[I_{(b-1)R}]_b$, and thus (i) follows.

(ii) Note that since

$$\binom{b+2}{2} - \left(b + b + \binom{b}{2}\right) = 1,$$

then from (i) there exists only one curve of degree b through $\mathbb{X}$, say $\mathscr{C}$. Now we prove by induction on b that the curve $\mathscr{C}$ is irreducible. Obvious for $b = 1$, assume $b > 1$. Assume that

$$\mathscr{C} = \mathscr{C}_1 + \cdots + \mathscr{C}_r,$$

where $r > 1$ and the $\mathscr{C}_i$ are the irreducible components of $\mathscr{C}$. Let $b_i = \deg \mathscr{C}_i$, and let m_i be the multiplicity of $\mathscr{C}_i$ at R.

Note that if $b_i = 1$, i.e., $\mathscr{C}_i$ is a line, then $m_i \leq 1$; if $b_i > 1$, since $\mathscr{C}_i$ is irreducible, then $m_i \leq b_i - 1$.

If for each i we have $b_i > 1$, then

$$b - 1 \leq m_1 + \cdots + m_r \leq (b_1 - 1) + \cdots + (b_r - 1) = b - r,$$

hence $r \leq 1$, and we get a contradiction.

Otherwise, without loss of generality, we can assume that $b_1 = 1$, that is, $\mathscr{C}_1$ is a line.

If $R \notin \mathscr{C}_1$, then $\mathscr{C}_1$ contains at most b simple points of $\mathbb{X}$. So since the curve $\mathscr{H} = \mathscr{C}_2 + \cdots + \mathscr{C}_r$ has degree $b-1$, and contains the fat point $(b-1)R$, then it is union of $b-1$ lines through R. Moreover, recalling that R, P_i, Q_j are not collinear for any i and j, and so each line through R contains at most one point of $\mathbb{X} - (b-1)R$, then $\mathscr{H}$ cannot contains $\mathbb{X} - (b-1)R - \mathscr{C}_1$. Hence $R \in \mathscr{C}_1$ and so $\mathscr{C}_1$ contains at most one other point of $\mathbb{X}$. Hence $\mathscr{H} = \mathscr{C}_2 + \cdots + \mathscr{C}_r$ is a curve of degree $b-1$ through $\mathbb{X} - \mathscr{C}_1$, that is, through $(b-2)R$ and at least $2b-1$ points in the set $\{P_1, \ldots, P_b, Q_1, \ldots, Q_b\}$. We may assume that $\mathscr{H}$ contains $P_1 + \cdots + P_b + Q_1 + \cdots + Q_{b-1}$. By the inductive hypothesis, the only curve of degree $b-1$ through $(b-2)R + P_1 + \cdots + P_{b-1} + Q_1 + \cdots + Q_{b-1}$ is irreducible. Hence $\mathscr{H}$ has to be that curve. But $P_b \in \mathscr{H}$, so, by Bezout's Theorem, L is a component of $\mathscr{H}$, hence, since $\mathscr{H}$ is irreducible, we get $L = \mathscr{H}$. It follows that $Q_1 \in L$, a contradiction. □

3. METHOD

In this section we describe the main method that we will use to find the Waldschmidt constant of a $\mathbb{k}$-configuration $\mathbb{X}$ in $\mathbb{P}^2$. Our computation is structured as follows.

Step 1. We look for a curve $\mathscr{F}$ of degree d, which contains each point of $\mathbb{X}$ with multiplicity exactly μ, so that, for each $m > 0$, $m\mathscr{F}$ is a curve in the linear system $\left[I_{m\mu\mathbb{X}}\right]_{md}$ and so $\left[I_{m\mu\mathbb{X}}\right]_{md} \neq \{0\}$.

Step 2. We show that $\left[I_{m\mu\mathbb{X}}\right]_{md-1} = \{0\}$, for each $m \geq 1$ and we prove it by contradiction. For this purpose we define

$$\bar{m} = \min\{m | [I_{m\mu\mathbb{X}}]_{md-1} \neq \{0\}\}.$$

We prove, mostly directly, that $\bar{m} \neq 1$. For $\bar{m} > 1$, applying Lemma 2.5 several times, we show that $\mathscr{F}$ is a fixed component for the linear system $\left[I_{\bar{m}\mu\mathbb{X}}\right]_{\bar{m}d-1}$. Thus, by removing $\mathscr{F}$, we get

$$\dim \left[I_{\bar{m}\mu\mathbb{X}}\right]_{\bar{m}d-1} = \dim \left[I_{\bar{m}\mu\mathbb{X}-\mathscr{F}}\right]_{\bar{m}d-1-d}$$

and, since $\mathscr{F}$ contains each point of $\mathbb{X}$ with multiplicity exactly μ, we have

$$\left[I_{\bar{m}\mu\mathbb{X}-\mathscr{F}}\right]_{\bar{m}d-1-d} = \left[I_{(\bar{m}-1)\mu\mathbb{X}}\right]_{(\bar{m}-1)d-1}$$

and the contradiction comes from the minimality of $\bar{m}$.

Step 3. Since the initial degree of $\left[I_{m\mu\mathbb{X}}\right]$ is md, then, by Lemma 2.2 we have

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{d}{\mu}.$$

Note that if $\mathbb{X}$ is a standard $\Bbbk$-configuration, then the curve $\mathscr{F}$ strictly depends on the type of $\mathbb{X}$. In certain cases $\mathscr{F}$ is a union of lines, and in other cases it has irreducible components of higher degrees.

4. Waldschmidt constants of $\Bbbk$-configurations of type $(d_1, \dots, d_s)$ with $d_1 \geq s$

In the next lemma we compute the Waldschmidt constant of a set of points $\mathbb{X}$ contained in s lines, where each line contains at least s points of $\mathbb{X}$ and no two lines meet in a point of $\mathbb{X}$.

The following lemma will be useful for computing the Waldschmidt constants of both a $\Bbbk$-configuration of type $(d_1, \dots, d_s)$ and a standard $\Bbbk$-configuration of the same type $(d_1, \dots, d_s)$, when $d_1 \geq s$.

Lemma 4.1. *Let s be a positive integer, and let $L_1, \dots, L_s$ be distinct lines. Let $\mathbb{X}_i$ be a finite set of d_i points on the line L_i ($1 \leq i \leq s$), and let $\mathbb{X} = \bigcup_{i=1}^{s} \mathbb{X}_i$. If $d_i \geq s$, for each $1 \leq i \leq s$, and any intersection point of two lines L_i and L_j, for $i \neq j$, is not contained in $\mathbb{X}$, then the Waldschmidt constant of $\mathbb{X}$ is*

$$\widehat{\alpha}(I_{\mathbb{X}}) = s.$$

Proof. For $s = 1$, it is immediate. So we assume $s > 1$.

Let m be a positive integer. The curve $\mathscr{F} = L_1 + \cdots + L_s$ has degree s and passes through the points of $\mathbb{X}$ with multiplicity 1, hence

$$m\mathscr{F} \in [I_{m\mathbb{X}}]_{ms}.$$

Now we prove that for each $m > 0$,

$$[I_{m\mathbb{X}}]_{ms-1} = \{0\},$$

so the initial degree of $I_{m\mathbb{X}}$ will be ms and the conclusion will follow from Lemma 2.2.

Assume that for some m, $[I_{m\mathbb{X}}]_{ms-1} \neq \{0\}$. Note that if $[I_{m\mathbb{X}}]_{ms-1} \neq \{0\}$, then since each L_i contains d_i points, and each point has multiplicity m, and the degree we are considering is $ms - 1$, then by Lemma 2.5, each L_i is a fixed component of multiplicity at least $\left\lceil \frac{md_i - (ms-1)}{d_i - 1} \right\rceil$ for the plane curves of the linear system $[I_{m\mathbb{X}}]_{ms-1}$.

Now, since $d_i \geq s \geq 2$, then

$$\left\lceil \frac{md_i - (ms-1)}{d_i - 1} \right\rceil \geq 1, \tag{4.1}$$

hence $\mathscr{F}$ is a fixed component for the curves defined by this linear system.

Set

$$\bar{m} = \min\{m | [I_{m\mathbb{X}}]_{ms-1} \neq \{0\}\}. \tag{4.2}$$

First observe that $\bar{m} \neq 1$. In fact, for $m = 1$, since $\deg \mathscr{F} = s$, then $[I_{\mathbb{X}}]_{s-1} = \{0\}$. By removing $\mathscr{F}$ from the curves of the linear system $[I_{\bar{m}\mathbb{X}}]_{\bar{m}s-1}$, since any intersection point of two lines L_i and L_j is not contained in $\mathbb{X}$, we get

$$\dim[I_{\bar{m}\mathbb{X}}]_{\bar{m}s-1} = \dim[I_{\bar{m}\mathbb{X}-\mathscr{F}}]_{(\bar{m}s-1)-s} = \dim[I_{(\bar{m}-1)\mathbb{X}}]_{(\bar{m}-1)s-1},$$

and by (4.2) this is zero, a contradiction. □

Corollary 4.2. *Let $\mathbb{X}$ be a standard $\Bbbk$-configuration of type $(d_1, \dots, d_s)$ with $d_1 \geq s$. Then the Waldschmidt constant of $\mathbb{X}$ is*

$$\widehat{\alpha}(I_{\mathbb{X}}) = s.$$

Proof. It follows from the previous lemma. □

Corollary 4.3. *With notation as in Definition 2.3, if $\mathbb{X}$ is a $\Bbbk$-configuration of type $(d_1, \dots, d_s)$ with $d_1 \geq s$, then the Waldschmidt constant of $\mathbb{X}$ is*

$$\widehat{\alpha}(I_{\mathbb{X}}) = s.$$

Proof. Let $\mathscr{F} = L_1 + \cdots + L_s$, thus $m\mathscr{F} \in [I_{m\mathbb{X}}]_{ms}$. Hence

$$\widehat{\alpha}(I_{\mathbb{X}}) \leq s.$$

Now let $\mathbb{X}'$ be the subset of $\mathbb{X}$ that we get after we remove the possible points of $\mathbb{X}$ in the intersections $L_i \cap L_j$, for $i \neq j$. Let $\mathbb{X}'_i = \mathbb{X}' \cap L_i$. Recalling Definition 2.3 it is easy to show that by Lemma 4.1 we have

$$\widehat{\alpha}(I_{\mathbb{X}'}) = s.$$

Since $\mathbb{X}' \subseteq \mathbb{X}$, we have $\widehat{\alpha}(I_{\mathbb{X}'}) \leq \widehat{\alpha}(I_{\mathbb{X}})$. Thus, the conclusion follows from

$$s = \widehat{\alpha}(I_{\mathbb{X}'}) \leq \widehat{\alpha}(I_{\mathbb{X}}) \leq s.$$

□

Remark 4.4. From Corollary 4.2, we immediately get that the Waldschmidt constant of a standard $\Bbbk$-configuration of type (d_1) is 1, and of type (d_1, d_2) with $d_1 \geq 2$ is 2. For the case $(1, d_2)$ see [11, Proposition 3.3], where it is proved that if $\mathbb{X}$ is a standard $\Bbbk$-configuration of type $(1, d_2)$, then $\widehat{\alpha}(I_{\mathbb{X}}) = \frac{2d_2-1}{d_2}$.

5. Waldschmidt constants of standard $\Bbbk$-configurations of type $(1, b, c)$

In this section we compute the Waldschmidt constant of a standard $\Bbbk$-configuration $\mathbb{X}$ of type $(1, b, c)$ as in Definition 2.4, for any values of b and c.

It is interesting to note that the Waldschmidt constant stabilizes at $c = 2b + 2$, that is,

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{3b-1}{b} \quad \text{for} \quad c \geq 2b + 2$$

(see Theorem 5.10). One could expect that, for each fixed b, the Waldschmidt constant strictly increases with c until $c = 2b + 2$. But this is not always the case, as shown in Corollary 5.3, since for $c \leq 2b - 3$ it behaves in a similar way as a step function.

We fix the notation of this section, summarized in Figure 3, that will be used in the proofs.

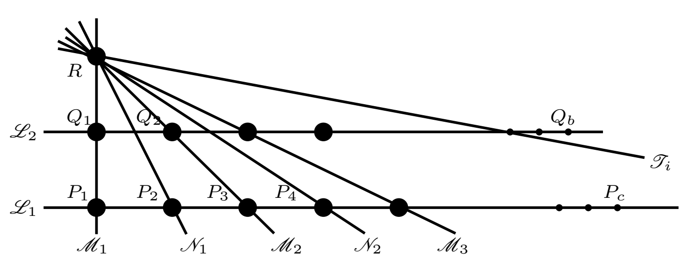

FIGURE 3. A standard $\Bbbk$-configuration of type $(1, b, c)$

Let $P_i = [1 : i-1 : 0]$, for $1 \leq i \leq c$, $Q_i = [1 : i-1 : 1]$, for $1 \leq i \leq b$, and $R = [1 : 0 : 2]$ be the points of $\mathbb{X}$ (see Definition 2.4).

Let

$$\begin{array}{ll}
\mathscr{L}_1 & \text{be the line through } P_1, P_2, \dots, P_c;\\
\mathscr{L}_2 & \text{be the line through } Q_1, Q_2, \dots, Q_b;\\
\mathscr{M}_1 & \text{be the line through } P_1, Q_1, R;\\
\mathscr{M}_2 & \text{be the line through } P_3, Q_2, R;\\
 & \vdots\\
\mathscr{M}_i & \text{be the line through } P_{2i-1}, Q_i, R, \text{ for } i \le b \text{ and } 2i \le c+1;\\
\mathscr{N}_1 & \text{be the line through } P_2, R;\\
\mathscr{N}_2 & \text{be the line through } P_4, R;\\
 & \vdots\\
\mathscr{N}_i & \text{be the line through } P_{2i}, R, \ \text{ for } 2i \le c;\\
\mathscr{T}_i & \text{be the line through } Q_i, R, \text{ for } i \le b \text{ and } 2i \ge c+2.
\end{array}$$

Note that each line $\mathscr{M}_i$ contains three points of $\mathbb{X}$, whereas the lines $\mathscr{N}_i$ and $\mathscr{T}_i$ contain two points of $\mathbb{X}$.

Theorem 5.1. *Let $\mathbb{X}$ be a standard $\mathbb{k}$-configuration of type $(1, b, c)$. If c is even and $c \le 2b - 4$, then*

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{6b + 3c - 4}{2b + c}.$$

Proof. Define

$$\mathscr{F} = \frac{2b + c - 2}{2}\mathscr{L}_1 + \frac{2b + c - 2}{2}\mathscr{L}_2 + \mathscr{M}_1 + \cdots + \mathscr{M}_{\frac{c}{2}} + \mathscr{N}_1 + \cdots + \mathscr{N}_{\frac{c}{2}} + \mathscr{T}_{\frac{c+2}{2}} + \cdots + \mathscr{T}_b.$$

$\mathscr{F}$ is the union of $\frac{6b+3c-4}{2}$ lines, and $\mathscr{F}$ contains each point of $\mathbb{X}$ with multiplicity exactly $\frac{2b+c}{2}$. Hence, for $m > 0$,

$$m\mathscr{F} \in \left[I_{\frac{2b+c}{2}m\mathbb{X}}\right]_{\frac{6b+3c-4}{2}m}.$$

Now we prove by contradiction that for each $m > 0$,

$$\dim \left[I_{\frac{2b+c}{2}m\mathbb{X}}\right]_{\frac{6b+3c-4}{2}m-1} = 0,$$

and the conclusion will follow from Lemma 2.2.

To this aim, we will use Lemma 2.5 many times in order to get a fixed component for the curves defined by the forms of $\left[I_{\frac{2b+c}{2}m\mathbb{X}}\right]_{\frac{6b+3c-4}{2}m}$.

So, assume that for some m, $\left[I_{\frac{2b+c}{2}m\mathbb{X}}\right]_{\frac{6b+3c-4}{2}m-1} \ne \{0\}$, thus by Lemma 2.5, by recalling that $c > b$, we get that $\mathscr{L}_1$ is a fixed component of multiplicity at least

$$\left\lceil \frac{\frac{2b+c}{2}cm - \frac{6b+3c-4}{2}m + 1}{c - 1} \right\rceil = \left\lceil \frac{((2b + c - 6)(c - 1) + 4c - 4b - 2)m + 2}{2(c - 1)} \right\rceil \ge \frac{2b + c - 6}{2}m \tag{5.1}$$

for the plane curves of the linear system $\left[I_{\frac{2b+c}{2}m\mathbb{X}}\right]_{\frac{6b+3c-4}{2}m-1}$.

By removing $\frac{2b+c-6}{2}m\mathscr{L}_1$ from those curves, we get

$$\dim[I_{\frac{2b+c}{2}m\mathbb{X}}]_{\frac{6b+3c-4}{2}m-1} = \dim[I_{\frac{2b+c}{2}m\mathbb{X}-\frac{2b+c-6}{2}m\mathscr{L}_1}]_{\frac{6b+3c-4}{2}m-1-\frac{2b+c-6}{2}m}.$$

If the dimension above is zero, we get a contradiction and we are done. If it is different from zero, by Lemma 2.5, by observing that $\frac{6b+3c-4}{2}m - 1 - \frac{2b+c-6}{2}m = (2b + c + 1)m - 1$, we get that $\mathscr{L}_2$ is a fixed component of multiplicity at least

(5.2)

$$\left\lceil \frac{\frac{2b+c}{2}bm - (2b + c + 1)m + 1}{b - 1} \right\rceil = \left\lceil \frac{((2b + c - 6)(b - 1) + 4b - c - 8)m + 2}{2(b - 1)} \right\rceil \ge \frac{2b + c - 6}{2}m,$$

for the plane curves of the linear system $[I_{\frac{2b+c}{2}m\mathbb{X}-\frac{2b+c-6}{2}m\mathscr{L}_1}]_{(2b+c+1)m-1}$. By removing $\frac{2b+c-6}{2}m\mathscr{L}_2$ from those curves, we get

(5.3)
$$\dim[I_{\frac{2b+c}{2}m\mathbb{X}-\frac{2b+c-6}{2}m\mathscr{L}_1}]_{(2b+c+1)m-1} = \dim[I_{\frac{2b+c}{2}m\mathbb{X}-\frac{2b+c-6}{2}m\mathscr{L}_1-\frac{2b+c-6}{2}m\mathscr{L}_2}]_{(2b+c+1)m-1-\frac{2b+c-6}{2}m},$$

where

$$\frac{2b+c}{2}m\mathbb{X} - \frac{2b+c-6}{2}m\mathscr{L}_1 - \frac{2b+c-6}{2}m\mathscr{L}_2 = \frac{2b+c}{2}mR + \sum_{P_i\in\mathscr{L}_1} 3mP_i + \sum_{Q_i\in\mathscr{L}_2} 3mQ_i.$$

If the dimension in (5.3) is zero, we get a contradiction and we are done. If it is different from zero, by Lemma 2.5, by observing that $(2b+c+1)m-1-\frac{2b+c-6}{2}m = \frac{2b+c+8}{2}m-1$, and

$$-2b+5c-8 = -2b+6c-c-8 \geq -2b+6c-(2b-4)-8 = 4(c-b)+2c-4 \geq 2c,$$

we have that $\mathscr{L}_1$ is a fixed component of multiplicity at least

$$\left\lceil \frac{3cm - \frac{2b+c+8}{2}m + 1}{c-1} \right\rceil = \left\lceil \frac{(-2b+5c-8)m+2}{2(c-1)} \right\rceil \geq m, \tag{5.4}$$

for the curves of the linear system $\left[I_{\frac{2b+c}{2}mR+\sum_{P_i\in\mathscr{L}_1}3mP_i+\sum_{Q_i\in\mathscr{L}_2}3mQ_i}\right]_{\frac{2b+c+8}{2}m-1}$. We now remove $m\mathscr{L}_1$ and we get

$$\begin{aligned} & \dim[I_{\frac{2b+c}{2}mR+\sum_{P_i\in\mathscr{L}_1}3mP_i+\sum_{Q_i\in\mathscr{L}_2}3mQ_i}]_{\frac{2b+c+8}{2}m-1} \\ = \; & \dim[I_{\frac{2b+c}{2}mR+\sum_{P_i\in\mathscr{L}_1}2mP_i+\sum_{Q_i\in\mathscr{L}_2}3mQ_i}]_{\frac{2b+c+8}{2}m-1-m}. \end{aligned}$$

So, if the dimension above is zero, we get a contradiction and we are done. If it is different from zero, then, by Lemma 2.5, by recalling that we have the hypothesis $2b \geq c+4$, and so $4b \geq 2b+c+4$, we get that $\mathscr{L}_2$ is a fixed component of multiplicity at least

(5.5)
$$\left\lceil \frac{3mb - \frac{2b+c+6}{2}m + 1}{b-1} \right\rceil = \left\lceil \frac{(4b-c-6)m+2}{2(b-1)} \right\rceil \geq \left\lceil \frac{(2b+c+4-c-6)m+2}{2(b-1)} \right\rceil = \left\lceil m + \frac{1}{(b-1)} \right\rceil \geq m,$$

for the curves of the linear system $\left[I_{\frac{2b+c}{2}mR+\sum_{P_i\in\mathscr{L}_1}2mP_i+\sum_{Q_i\in\mathscr{L}_2}3mQ_i}\right]_{\frac{2b+c+6}{2}m-1}$.

Hence

$$\begin{aligned} & \dim\left[I_{\frac{2b+c}{2}mR+\sum_{P_i\in\mathscr{L}_1}2mP_i+\sum_{Q_i\in\mathscr{L}_2}3mQ_i}\right]_{\frac{2b+c+6}{2}m-1} \\ = \; & \dim\left[I_{\frac{2b+c}{2}mR+\sum_{P_i\in\mathscr{L}_1}2mP_i+\sum_{Q_i\in\mathscr{L}_2}2mQ_i}\right]_{\frac{2b+c+4}{2}m-1}. \end{aligned}$$

If this dimension is different from zero, then we go on and we apply Lemma 2.5 to the lines $\mathscr{M}_i$, $\mathscr{N}_i$, and $\mathscr{T}_i$. Since

(5.6)
$$\left\lceil \frac{\frac{2b+c}{2}m + 2m + 2m - \frac{2b+c+4}{2}m + 1}{2} \right\rceil = \left\lceil \frac{2m+1}{2} \right\rceil > 1, \text{ and } \left\lceil \frac{\frac{2b+c}{2}m + 2m - \frac{2b+c+4}{2}m + 1}{1} \right\rceil = 1,$$

the lines $\mathscr{M}_i$, $\mathscr{N}_i$, and $\mathscr{T}_i$ are fixed components for the curves of the linear system

$$\left[I_{\frac{2b+c}{2}mR+\sum_{P_i\in\mathscr{L}_1}2mP_i+\sum_{Q_i\in\mathscr{L}_2}2mQ_i}\right]_{\frac{2b+c+4}{2}m-1}.$$

Hence, from the computations in (5.1), (5.2), (5.4), (5.5), and (5.6), we get that the following curve

$$\frac{2b+c-4}{2}m\mathscr{L}_1 + \frac{2b+c-4}{2}m\mathscr{L}_2 + \mathscr{M}_1 + \cdots + \mathscr{M}_{\frac{c}{2}} + \mathscr{N}_1 + \cdots + \mathscr{N}_{\frac{c}{2}} + \mathscr{T}_{\frac{c+2}{2}} + \cdots + \mathscr{T}_b \tag{5.7}$$

is a fixed component for the curves defined by the linear system $\left[I_{\frac{2b+c}{2}m\mathbb{X}}\right]_{\frac{6b+3c-4}{2}m-1}$.

Now set

$$\bar{m} = \min\{m \mid [I_{\frac{2b+c}{2}m\mathbb{X}}]_{\frac{6b+3c-4}{2}m-1} \neq \{0\}\}. \tag{5.8}$$

First observe that $\bar{m} \neq 1$. In fact for $m = 1$, the curve $\mathscr{F}'$ of degree $\frac{6b+3c-8}{2}$

$$\mathscr{F}' = \frac{2b+c-4}{2}\mathscr{L}_1 + \frac{2b+c-4}{2}\mathscr{L}_2 + \mathscr{M}_1 + \cdots + \mathscr{M}_{\frac{c}{2}} + \mathscr{N}_1 + \cdots + \mathscr{N}_{\frac{c}{2}} + \mathscr{T}_{\frac{c+2}{2}} + \cdots + \mathscr{T}_b$$

should be a fixed component for the linear system $[I_{\frac{2b+c}{2}\mathbb{X}}]_{\frac{6b+3c-4}{2}-1}$, so

$$\dim[I_{\frac{2b+c}{2}\mathbb{X}}]_{\frac{6b+3c-4}{2}-1} = \dim[I_{\frac{2b+c}{2}\mathbb{X}-\mathscr{F}'}]_{\frac{6b+3c-4}{2}-1-\frac{6b+3c-8}{2}} = \dim[I_{P_1+\cdots+P_c+Q_1+\cdots+Q_b}]_1 = 0,$$

a contradiction.

So $\bar{m} > 1$. By (5.7), since $\frac{2b+c-4}{2}\bar{m} \geq \frac{2b+c-2}{2}$, we get that $\mathscr{F}$ is a fixed component for the linear system $[I_{\frac{2b+c}{2}\bar{m}\mathbb{X}}]_{\frac{6b+3c-4}{2}\bar{m}-1}$, hence, by recalling that $\deg \mathscr{F} = \frac{6b+3c-4}{2}$ and $\mathscr{F}$ contains each point of $\mathbb{X}$ with multiplicity $\frac{2b+c}{2}$, we get

$$\dim[I_{\frac{2b+c}{2}\bar{m}\mathbb{X}}]_{\frac{6b+3c-4}{2}\bar{m}-1} = \dim[I_{\frac{2b+c}{2}\bar{m}\mathbb{X}-\mathscr{F}}]_{\frac{6b+3c-4}{2}\bar{m}-1-\frac{6b+3c-4}{2}} = \dim[I_{\frac{2b+c}{2}(\bar{m}-1)\mathbb{X}}]_{\frac{6b+3c-4}{2}(\bar{m}-1)-1},$$

which is zero by (5.8), a contradiction. □

Theorem 5.2. *Let $\mathbb{X}$ be a standard $\mathbb{k}$-configuration of type $(1, b, c)$. If c is odd, and $b + 1 < c \leq 2b - 3$, then*

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{6b+3c-7}{2b+c-1}.$$

Proof. Let

$$\mathscr{F} = \frac{2b+c-3}{2}\mathscr{L}_1 + \frac{2b+c-3}{2}\mathscr{L}_2 + \mathscr{M}_1 + \cdots + \mathscr{M}_{\frac{c+1}{2}} + \mathscr{N}_1 + \cdots + \mathscr{N}_{\frac{c-1}{2}} + \mathscr{T}_{\frac{c+3}{2}} + \cdots + \mathscr{T}_b.$$

$\mathscr{F}$ is the union of $\frac{6b+3c-7}{2}$ lines, and $\mathscr{F}$ contains each point of $\mathbb{X}$ with multiplicity exactly $\frac{2b+c-1}{2}$. Hence, for $m > 0$,

$$m\mathscr{F} \in \left[I_{\frac{2b+c-1}{2}m\mathbb{X}}\right]_{\frac{6b+3c-7}{2}m}.$$

By Lemma 2.2 it follows that $\widehat{\alpha}(I_{\mathbb{X}}) \leq \frac{6b+3c-7}{2b+c-1}$.

Now, by recalling that $c - 1 > b$, we can consider the standard $\mathbb{k}$-configuration $\mathbb{X}'$ of type $(1, b, c-1)$, which is contained in the standard $\mathbb{k}$-configuration $\mathbb{X}$. Hence $\widehat{\alpha}(I_{\mathbb{X}}) \geq \widehat{\alpha}(I_{\mathbb{X}'})$. Since $c-1 \leq 2b-4$ and $c-1$ is even, by Theorem 5.1 we have that $\widehat{\alpha}(I_{\mathbb{X}'}) = \frac{6b+3(c-1)-4}{2b+(c-1)} = \frac{6b+3c-7}{2b+c-1}$, and the conclusion follows. □

Corollary 5.3. *Let $\mathbb{X}$ and $\mathbb{Y}$ be standard $\mathbb{k}$-configurations of type $(1, b, c)$ and $(1, b, c+1)$, respectively. If c is even, and $c \leq 2b - 4$, then $\widehat{\alpha}(I_{\mathbb{X}}) = \widehat{\alpha}(I_{\mathbb{Y}})$.*

Proof. By Theorem 5.1 we have that $\widehat{\alpha}(I_{\mathbb{X}}) = \frac{6b+3c-4}{2b+c}$. Now by applying Theorem 5.2 to $\mathbb{Y}$ we get $\widehat{\alpha}(I_{\mathbb{Y}}) = \frac{6b+3(c+1)-7}{2b+(c+1)-1} = \frac{6b+3c-4}{2b+c} = \widehat{\alpha}(I_{\mathbb{X}})$. □

From Theorems 5.1 and 5.2, we can compute the Waldschmidt constants of any standard $\mathbb{k}$-configurations of type $(1, b, c)$, when $c \leq 2b - 3$, except for the configuration $\mathbb{X}$ of type $(1, b, b+1)$ with b even. In the following theorem we will compute the Waldschmidt constant of this type of configuration, and we will find that $\widehat{\alpha}(I_{\mathbb{X}}) = \frac{9b-4}{3b}$.

Alternatively we could have considered the subscheme $\mathbb{Y} = \mathbb{X} - P_{b+1}$, and computed the Waldschmidt constant of $\mathbb{Y}$, and found that $\widehat{\alpha}(I_{\mathbb{Y}}) = \frac{9b-4}{3b}$. With this method the conclusion would be followed from a theorem analogous to Theorem 5.2.

In the next theorem we study the case $(1, b, b+1)$, when $b \geq 4$ is even. Note that when $b = 2$, the formula in Theorem 5.4 gives $7/3$, but the correct answer is $\widehat{\alpha}(I_{\mathbb{X}}) = 9/4$ (see Theorem 5.6).

Theorem 5.4. *Let $\mathbb{X}$ be a standard $\mathbb{k}$-configuration of type $(1, b, b+1)$. If $b \geq 4$ is an even integer, then*

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{9b-4}{3b}.$$

Proof. The proof proceeds as in Theorem 5.1. See [Appendix 7, Proof of Theorem 5.4] for more details. $\square$

Now we study the standard $\Bbbk$-configurations from type $(1, b, 2b-2)$ to $(1, b, 2b+1)$. From our computations it will emerge that in this range the Waldschmidt constant is strictly increasing. A useful tool for the proofs is Lemma 2.7. Also even if the method is always the same, we prefer to give some details since the proof is more tricky than the previous cases.

Theorem 5.5. *Let $\mathbb{X}$ be a standard $\Bbbk$-configuration of type $(1, b, 2b-2)$. Then*

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{6b^2 - 14b + 6}{2b^2 - 4b + 1}.$$

Proof. Note that from the definition of a standard $\Bbbk$-configuration, we have $b > 2$. Let

- $\mathscr{C}_i$ be the irreducible curve of degree $(b-1)$ through $P_2, P_4, \ldots, P_{2b-2}, Q_1, \ldots, \widehat{Q}_i, \ldots, Q_b, (b-2)R$ for $1 \leq i \leq b-1$ (see Lemma 2.7),

and let

$$\mathscr{F} = (2b^2-5b+2)\mathscr{L}_1 + (2b^2-6b+4)\mathscr{L}_2 + (b-1)\mathscr{M}_1 + \cdots + (b-1)\mathscr{M}_{b-1} + (b-2)\mathscr{T}_b + \mathscr{C}_1 + \cdots + \mathscr{C}_{b-1}.$$

So $\mathscr{F}$ is a curve of degree $6b^2 - 14b + 6$ with multiplicity $2b^2 - 4b + 1$ at each point of $\mathbb{X}$. Hence for $m > 0$

$$m\mathscr{F} \in [I_{(2b^2-4b+1)m\mathbb{X}}]_{(6b^2-14b+6)m}.$$

We now prove that for $m > 0$,

$$[I_{(2b^2-4b+1)m\mathbb{X}}]_{(6b^2-14b+6)m-1} = \{0\}.$$

Then the result follows from Lemma 2.2.

Assume that for some m, $[I_{(2b^2-4b+1)m\mathbb{X}}]_{(6b^2-14b+6)m-1} \neq \{0\}$. Thus by Lemma 2.5, $\mathscr{L}_1$ is a fixed component of multiplicity at least

$$\left\lceil \frac{(2b^2-4b+1)(2b-2)m - (6b^2-14b+6)m + 1}{2b-3} \right\rceil \geq (2b^2-6b+3)m \tag{5.9}$$

for the plane curves of the linear system $[I_{(2b^2-4b+1)m\mathbb{X}}]_{(6b^2-14b+6)m-1}$. We remove $(2b^2-6b+3)m\mathscr{L}_1$, and we get that $\mathscr{L}_2$ is a fixed component of multiplicity at least

$$\left\lceil \frac{(2b^2-4b+1)bm - (6b^2-14b+6-2b^2+6b-3)m + 1}{b-1} \right\rceil \geq (2b^2-6b+3)m. \tag{5.10}$$

Remove $(2b^2-6b+3)m\mathscr{L}_2$. Recalling that now we are in degree $(6b^2-14b+6)m - 2(2b^2-6b+3)m - 1 = (2b^2-2b)m - 1$, and the points on $\mathscr{L}_1$ have multiplicity $(2b-2)m$, we get that $\mathscr{L}_1$ is a fixed component of multiplicity at least

$$\left\lceil \frac{(2b-2)(2b-2)m - (2b^2-2b)m + 1}{2b-3} \right\rceil = (b-2)m + \left\lceil \frac{(b-2)m+1}{2b-3} \right\rceil. \tag{5.11}$$

Hence $\mathscr{L}_1$ is a fixed component of multiplicity at least $(2b^2-6b+3)m + (b-2)m = (2b^2-5b+1)m$.

By removing $(b-2)m\mathscr{L}_1$ we get

$$\dim[I_{(2b^2-4b+1)m\mathbb{X}}]_{(6b^2-14b+6)m-1} = \dim[I_{(2b^2-4b+1)mR + \sum_{P_i\in\mathscr{L}_1} bmP_i + \sum_{Q_i\in\mathscr{L}_2}(2b-2)mQ_i}]_{(2b^2-3b+2)m-1}.$$

If the above dimension is different from zero, then each $\mathscr{M}_i$ is a fixed component of multiplicity at least

$$\left\lceil \frac{(2b^2-4b+1+b+2b-2)m - (2b^2-3b+2)m + 1}{2} \right\rceil = (b-2)m + \left\lceil \frac{m+1}{2} \right\rceil. \tag{5.12}$$

By removing the $b-1$ multiple lines $(b-2)m\mathscr{M}_i$, the residual scheme is

$$\mathbb{Y} = (b^2-b-1)mR + \sum_{P_i\in\mathscr{L}_1,\text{ with } i \text{ odd}} 2mP_i + \sum_{P_i\in\mathscr{L}_1,\text{ with } i \text{ even}} bmP_i + \sum_{i=1}^{b-1} bmQ_i + (2b-2)mQ_b,$$

and we are left in degree $(2b^2-3b+2)m-1-(b-1)(b-2)m = b^2m-1$.

Hence

$$\dim[I_{(2b^2-4b+1)m\mathbb{X}}]_{(6b^2-14b+6)m-1} = \dim[I_{\mathbb{Y}}]_{b^2m-1}.$$

If this dimension is still different from zero, then $\mathscr{T}_b$ is a fixed component of multiplicity at least

$$(5.13)\qquad (b^2-b-1+2b-2)m - b^2m + 1 = (b-3)m+1.$$

By removing $(b-3)m\mathscr{T}_b$ we get

$$\dim[I_{(2b^2-4b+1)m\mathbb{X}}]_{(6b^2-14b+6)m-1} = \dim[I_{\mathbb{Y}-(b-3)m\mathscr{T}_b}]_{b^2m-1-(b-3)m} = \dim[I_{\mathbb{Y}'}]_{(b^2-b+3)m-1},$$

where

$$\mathbb{Y}' = (b^2-2b+2)mR + \sum_{P_i\in\mathscr{L}_1,\text{ with } i \text{ odd}} 2mP_i + \sum_{P_i\in\mathscr{L}_1,\text{ with } i \text{ even}} bmP_i + \sum_{i=1}^{b-1} bmQ_i + (b+1)mQ_b.$$

If $\mathscr{H}$ is a curve of the linear system $[I_{\mathbb{Y}'}]_{(b^2-b+3)m-1}$, the multiplicity of intersection between each $\mathscr{C}_i$ and $\mathscr{H}$ is at least

$$|\mathscr{C}_i\cdot\mathscr{H}| \geq (b-2)(b^2-2b+2)m + (b-1)bm + (b-2)bm + (b+1)m = (b^3-2b^2+4b-3)m,$$

and this number is bigger than the product of the degree of $\mathscr{C}_i$ and $\mathscr{H}$, which is $(b-1)((b^2-b+3)m-1) = (b^3-2b^2+4b-3)m-b+1$. Hence, by Bézout's Theorem, each curve $\mathscr{C}_i$ is a fixed component for the curves of $[I_{\mathbb{Y}-(b-3)m\mathscr{T}_b}]_{b^2m-1-(b-3)m}$.

Now let

$$(5.14)\qquad \bar{m} = \min\{m \mid [I_{(2b^2-4b+1)m\mathbb{X}}]_{(6b^2-14b+6)m-1} \neq \{0\}\}.$$

We have $\bar{m} > 1$. In fact for $m=1$, from (5.10), (5.11), (5.12), (5.13), using also the ceiling parts, by an easy computation we get that $\mathscr{F}$ is a curve of the linear system $[I_{(2b^2-4b+1)\mathbb{X}}]_{6b^2-14b+5}$. But $\deg\mathscr{F} = 6b^2-14b+6$, a contradiction.

Hence $\bar{m} > 1$.

By the computation above $\mathscr{F}$ is a fixed component for the linear system $[I_{(2b^2-4b+1)m\mathbb{X}}]_{(6b^2-14b+6)m-1}$, hence we have

$$\begin{aligned}\dim[I_{(2b^2-4b+1)\bar{m}\mathbb{X}}]_{(6b^2-14b+6)\bar{m}-1} &= \dim[I_{(2b^2-4b+1)\bar{m}\mathbb{X}-\mathscr{F}}]_{(6b^2-14b+6)\bar{m}-1-(6b^2-14b+6)}\\ &= \dim[I_{(2b^2-4b+1)(\bar{m}-1)\mathbb{X}}]_{(6b^2-14b+6)(\bar{m}-1)-1},\end{aligned}$$

which is zero by (5.14), a contradiction. □

Theorem 5.6. *Let $\mathbb{X}$ be a standard $\Bbbk$-configuration of type $(1,b,2b-1)$. Then*

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{6b^2-8b+1}{2b^2-2b}.$$

Proof. See [Appendix 7, Proof of Theorem 5.6]. □

Theorem 5.7. *Let $\mathbb{X}$ be a standard $\Bbbk$-configuration in of type $(1,b,2b)$. Then*

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{6b-5}{2b-1}.$$

Proof. See [Appendix 7, Proof of Theorem 5.7]. □

Theorem 5.8. *Let $\mathbb{X}$ be a standard $\Bbbk$-configuration in of type $(1, b, 2b+1)$. Then*

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{6b^2 - 2b - 3}{2b^2 - 1}.$$

Proof. See [Appendix 7, Proof of Theorem 5.8]. □

Now we will prove that the Waldschmidt constant of a standard $\Bbbk$-configuration of type $(1, b, c)$ only depends on b when $c \geq 2b+2$. In order to do that, we need the following lemma.

Lemma 5.9. *Let L_1, L_2 be two distinct lines, and let b, c be positive integers, with $c \geq b+2$. Let $P_1, \ldots, P_c \in L_1$, $Q_1, \ldots, Q_b \in L_2$, and R, be distinct points such that $R \notin L_1 \cup L_2$, and the point $L_1 \cap L_2 \notin \{P_1, \ldots, P_c, Q_1, \ldots, Q_b\}$. Moreover, assume that R, P_i, Q_j do not lie on a line, for any i and j. Let $\mathbb{Y}_c$ be the scheme (see Figure 4)*

$$\mathbb{Y}_c = P_1 + \cdots + P_c + Q_1 + \cdots + Q_b + R.$$

Then

$$\widehat{\alpha}(\mathbb{Y}_c) = \frac{3b-1}{b}.$$

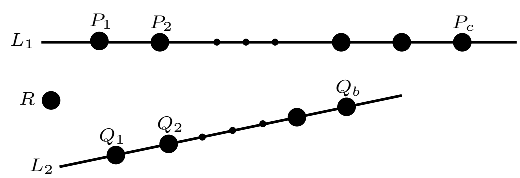

FIGURE 4. The scheme $\mathbb{Y}_c$

Proof. If $b = 1$, $\mathbb{Y}_c$ is a $\Bbbk$-configuration of type $(2, c)$, hence $\widehat{\alpha}(\mathbb{Y}_c) = 2$ follows from Corollary 4.3. The proof for $b = 2$ is analogous to the proof for $b > 2$, and it is left to the reader, so assume $b > 2$.

First we prove the lemma for $c = b+2$. For this case, we denote $\mathbb{Y}_{b+2}$ simply by $\mathbb{Y}$. Let M_i be the line through Q_i and R, ($1 \leq i \leq b$), and let

$$\mathscr{F} = bL_1 + (b-1)L_2 + M_1 + \cdots + M_b.$$

Note that $\deg \mathscr{F} = 3b - 1$, and $\mathscr{F}$ has multiplicity exactly b at all points of $\mathbb{Y}$. Hence for $m > 0$

$$m\mathscr{F} \in [I_{bm\mathbb{Y}}]_{(3b-1)m}.$$

Now we will show that for $m > 0$,

$$[I_{bm\mathbb{Y}}]_{(3b-1)m-1} = \{0\},$$

and the conclusion will follow from Lemma 2.2.

Assume that for some $m > 0$, $[I_{bm\mathbb{Y}}]_{(3b-1)m-1} \neq \{0\}$.

By Lemma 2.5, L_1 is a fixed component of multiplicity at least

$$\left\lceil \frac{b(b+2)m - (3b-1)m + 1}{b+1} \right\rceil \geq (b-2)m.$$

So we can remove $(b-2)mL_1$, and we get that

$$\dim[I_{bm\mathbb{Y}}]_{(3b-1)m-1} = \dim[I_{bm\mathbb{Y}-(b-2)mL_1}]_{(2b+1)m-1}.$$

If this dimension is different from zero, we get that L_2 is a fixed component of multiplicity at least

$$\left\lceil \frac{b^2m - (2b+1)m + 1}{b-1} \right\rceil = (b-2)m + \left\lceil \frac{(b-3)m + 1}{b-1} \right\rceil,$$

and then that L_1 is a fixed component of multiplicity at least

$$\left\lceil \frac{2(b+2)m-(b+3)m+1}{b+1} \right\rceil = m + \left\lceil \frac{1}{b+1} \right\rceil.$$

Hence

$$\begin{aligned}\dim[I_{bm\mathbb{Y}}]_{(3b-1)m-1} &= \dim[I_{bm\mathbb{Y}-(b-1)m\mathscr{L}_1-(b-2)m\mathscr{L}_2}]_{(b+2)m-1}\\ &= \dim[I_{\sum_{i=1}^{b+2} mP_i+\sum_{i=1}^{b} 2mQ_i+bmR}]_{(b+2)m-1}.\end{aligned}$$

Now, by Bezout's Theorem, each M_i is a fixed component ($1 \le i \le b$) for $[I_{bm\mathbb{Y}}]_{(3b-1)m-1}$.

Now let

$$\bar{m} = \min\{m | [I_{mb\mathbb{Y}}]_{m(3b-1)-1} \neq \{0\}\}. \tag{5.15}$$

We have $\bar{m} > 1$, in fact for $m = 1$ from the computation above, we have that $\mathscr{F}$ is a curve of degree $3b-1$ of the linear system $[I_{b\mathbb{Y}}]_{3b-2}$, a contradiction.

Hence $\bar{m} > 1$. Now from the equalities above, $\mathscr{F}$ is a fixed component for the linear system $[I_{\bar{m}b\mathbb{Y}}]_{\bar{m}(3b-1)-1}$, hence

$$\dim[I_{\bar{m}b\mathbb{Y}}]_{\bar{m}(3b-1)-1} = \dim[I_{\bar{m}b\mathbb{Y}-\mathscr{F}}]_{\bar{m}(3b-1)-1-(3b-1)} = \dim[I_{(\bar{m}-1)b\mathbb{Y}}]_{(\bar{m}-1)(3b-1)-1},$$

which is zero by (5.15), a contradiction.

Now consider the case $c > b+2$. Since also in this case $m\mathscr{F} \in [I_{bm\mathbb{Y}}]_{(3b-1)m}$, then $\widehat{\alpha}(\mathbb{Y}_c) \le \frac{3b-1}{b}$. Moreover, since $\mathbb{Y}_{b+2} \subset \mathbb{Y}_c$, then $\widehat{\alpha}(\mathbb{Y}_{b+2}) \le \widehat{\alpha}(\mathbb{Y}_c)$, and the conclusion follows. □

Theorem 5.10. *Let $\mathbb{X}$ be a standard $\mathbb{k}$-configuration of type $(1, b, c)$ with $c \ge 2b+2$. Then*

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{3b-1}{b}.$$

Proof. Let us consider the following curve $\mathscr{F}$ of degree $(3b-1)$ with multiplicities at least b at the points in $\mathbb{X}$

$$\mathscr{F} = b\mathscr{L}_1 + (b-1)\mathscr{L}_2 + \mathscr{M}_1 + \cdots + \mathscr{M}_b.$$

Then, for $m > 0$, we have $m\mathscr{F} \in [I_{mb\mathbb{X}}]_{(3b-1)m}$. By Lemma 2.2 it follows that

$$\widehat{\alpha}(I_{\mathbb{X}}) \le \frac{3b-1}{b}.$$

To conclude the proof set $\mathbb{Y} = \mathbb{X} - \{P_1, P_3, \ldots, P_{2b-1}\}$. Then, by Lemma 5.9 and since $\mathbb{Y} \subseteq \mathbb{X}$, we get

$$\frac{3b-1}{b} = \widehat{\alpha}(I_{\mathbb{Y}}) \le \widehat{\alpha}(I_{\mathbb{X}}) \le \frac{3b-1}{b}.$$

This completes the proof. □

6. Waldschmidt constants of standard $\mathbb{k}$-configurations of type (a, b, c), with $a \ge 2$.

In this section we study the Waldschmidt constant of a standard $\mathbb{k}$-configuration of type (a, b, c), with $a \ge 2$. We prove that, except for the type $(2, 3, 4)$, and for the type $(2, 3, 5)$ (see Theorem 6.1 and Remark 6.6), then the Waldschmidt constant is 3. For this section we fix the following notation (see Figure 6).

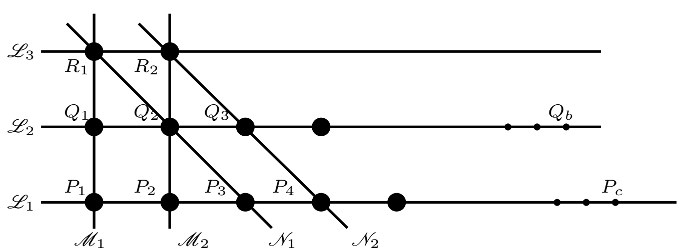

FIGURE 5. A standard $\mathbb{k}$-configuration of type $(2, b, c)$

Let $P_i = [1 : i-1 : 0]$, for $1 \le i \le c$, let $Q_i = [1 : i-1 : 1]$, for $1 \le i \le b$, let $R_1 = [1 : 0 : 2]$ and $R_2 = [1 : 1 : 2]$ be the points of $\mathbb{X}$, and let

$$\begin{array}{ll} \mathscr{L}_1 & \text{be the line through } P_1, P_2, \ldots, P_c; \\ \mathscr{L}_2 & \text{be the line through } Q_1, Q_2, \ldots, Q_b; \\ \mathscr{L}_3 & \text{be the line through } R_1, R_2; \\ \mathscr{M}_1 & \text{be the line through } P_1, Q_1, R_1; \\ \mathscr{M}_2 & \text{be the line through } P_2, Q_2, R_2; \\ \mathscr{N}_1 & \text{be the line through } P_3, Q_2, R_1; \\ \mathscr{N}_2 & \text{be the line through } P_4, Q_3, R_2. \end{array}$$

First we compute the Waldschmidt constant of a $\Bbbk$-configuration of type $(2, b, c) \neq (2, 3, 5)$.

Theorem 6.1. *Let $\mathbb{X}$ be a standard $\Bbbk$-configuration of type $(2, 3, 4)$. Then the Waldschmidt constant of $\mathbb{X}$ is*

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{17}{6}.$$

Proof. Let

$$\mathscr{C} \text{ be the conic through } P_2, P_3, Q_1, Q_3, R_1, R_2,$$

and let $\mathscr{F}$ be the following curve of degree 17, which contains each point of $\mathbb{X}$ with multiplicity 6

$$\mathscr{F} = 3\mathscr{L}_1 + 2\mathscr{L}_2 + 3\mathscr{M}_1 + 2\mathscr{M}_2 + 2\mathscr{N}_1 + 3\mathscr{N}_2 + \mathscr{C}.$$

Hence, for $m > 0$,

$$m\mathscr{F} \in [I_{6m\mathbb{X}}]_{17m}.$$

The conclusion will follows from Lemma 2.2, if we prove that for each $m > 0$,

$$\dim[I_{6m\mathbb{X}}]_{17m-1} = 0.$$

As usual, assume that for some m, $[I_{6m\mathbb{X}}]_{17m-1} \neq \{0\}$. By Lemma 2.5, $\mathscr{L}_1$ is a fixed component of multiplicity at least $\left\lceil \frac{24m-17m+1}{3} \right\rceil = \left\lceil \frac{7m+1}{3} \right\rceil \ge 2m$ for the plane curves of the linear system $[I_{6m\mathbb{X}}]_{17m-1}$. By removing $2m\mathscr{L}_1$ and assuming that the residual linear system is not empty, by Lemma 2.5, we get that $\mathscr{L}_2$ is a fixed component of multiplicity at least $\left\lceil \frac{3m+1}{2} \right\rceil$, and $\mathscr{M}_1$, $\mathscr{M}_2$, $\mathscr{N}_1$, $\mathscr{N}_2$ are fixed component of multiplicity at least $\left\lceil \frac{m+1}{2} \right\rceil$. Let

$$\bar{m} = \min\{m | [I_{6m\mathbb{X}}]_{17m-1} \neq \{0\}\}. \tag{6.1}$$

Now we claim that for $m = 1, 2, 3$, $[I_{6m\mathbb{X}}]_{17m-1} = \{0\}$. This claim can be proved directly, with the usual method. It follows that $\bar{m} \ge 4$.

From the computation above, and recalling that $\mathscr{M}_1$, $\mathscr{M}_2$, $\mathscr{N}_1$, $\mathscr{N}_2$ are fixed components of multiplicity at least $\left\lceil \frac{\bar{m}+1}{2} \right\rceil \ge 3$, then $\mathscr{F}$ is a fixed component for the linear system $[I_{6\bar{m}\mathbb{X}}]_{17\bar{m}-1}$, hence

$$\dim[I_{6\bar{m}\mathbb{X}}]_{17\bar{m}-1} = \dim[I_{6\bar{m}\mathbb{X}-\mathscr{F}}]_{17\bar{m}-1-17} = \dim[I_{6(\bar{m}-1)\mathbb{X}}]_{17(\bar{m}-1)-1},$$

which is zero by (6.1), a contradiction. □

We need the following lemma to find out the Waldschmidt constant of a standard $\Bbbk$-configuration of type $(2, 3, 6)$.

Lemma 6.2. *Let L_1, L_2 be two distinct lines, and let $P_1, \ldots, P_6 \in L_1$, and $Q_1, Q_2, Q_3 \in L_2$ be distinct points such that $L_1 \cap L_2 \notin \mathbb{Y}$, where*

$$\mathbb{Y} = P_1 + \cdots + P_6 + Q_1 + \cdots + Q_3.$$

Let m be a positive integer. Then the curve $2mL_1 + mL_2$ is a fixed component for the linear system $[I_{3m\mathbb{Y}}]_{9m-1}$.

Proof. Set

$$M = \{m' \mid 2m'L_1 + m'L_2 \text{ is a fixed component for the linear system } [I_{3m\mathbb{Y}}]_{9m-1}\}.$$

Since by Lemma 2.5, L_1 and L_2 are fixed components of multiplicity at least $\left\lceil \frac{18m-9m+1}{5} \right\rceil \geq 2$, and $\left\lceil \frac{9m-9m+1}{2} \right\rceil = 1$, respectively, then $2L_1 + L_2$ is a fixed component for $[I_{3m\mathbb{Y}}]_{9m-1}$, and so $1 \in M$. Let

$$\bar{m} = \max M.$$

If $\bar{m} \geq m$ we are done, so assume that $\bar{m} < m$. By the definition of $\bar{m}$, we have that $2\bar{m}L_1 + \bar{m}L_2$ is a fixed component for the linear system $[I_{3m\mathbb{Y}}]_{9m-1}$. Hence

$$[I_{3m\mathbb{Y}}]_{9m-1} = H \cdot [I_{3m\mathbb{Y}-2\bar{m}\mathscr{L}_1-\bar{m}\mathscr{L}_2}]_{9m-1-3\bar{m}} = H \cdot [I_{\sum_{i=1}^{6} P_i(3m-2\bar{m})+\sum_{i=1}^{3}(3m-\bar{m})Q_i}]_{9m-1-3\bar{m}},$$

where H is a form representing the curve $2\bar{m}L_1 + \bar{m}L_2$. Now, by Lemma 2.5, we get that, for the curve of the linear system $[I_{3m\mathbb{Y}-2\bar{m}L_1-\bar{m}L_2}]_{9m-1-3\bar{m}}$, L_1 is a fixed component of multiplicity at least

$$\left\lceil \frac{6(3m-2\bar{m}) - (9m-1-3\bar{m})}{5} \right\rceil = \left\lceil \frac{9m-9\bar{m}+1}{5} \right\rceil \geq 2,$$

and L_2 is a fixed component of multiplicity at least

$$\left\lceil \frac{3(3m-\bar{m}) - (9m-1-3\bar{m})}{2} \right\rceil = 1.$$

Recalling that $[I_{3m\mathbb{Y}}]_{9m-1} = H \cdot [I_{3m\mathbb{Y}-2\bar{m}\mathscr{L}_1-\bar{m}\mathscr{L}_2}]_{9m-1-3\bar{m}}$, it follows that $2(\bar{m}+1)L_1 + (\bar{m}+1)L_2$ is a fixed component for the curves of the linear system $[I_{3m\mathbb{Y}}]_{9m-1}$. A contradiction, since $\bar{m} = \max M$. □

Theorem 6.3. *Let $\mathbb{X}$ be a standard $\mathbb{k}$-configuration of type $(2, 3, 6)$. Then*

$$\widehat{\alpha}(I_{\mathbb{X}}) = 3.$$

Proof. Let $\mathscr{F}$ be the following curve of degree 9, which contains each point of $\mathbb{X}$ with multiplicity 3,

$$\mathscr{F} = 3\mathscr{L}_1 + 3\mathscr{L}_2 + 3\mathscr{L}_3.$$

Hence, for $m > 0$,

$$m\mathscr{F} \in [I_{3m\mathbb{X}}]_{9m}.$$

The conclusion will follow from Lemma 2.2, if we prove that for each $m > 0$,

$$\dim[I_{3m\mathbb{X}}]_{9m-1} = 0.$$

Assume that for some m, $[I_{3m\mathbb{X}}]_{9m-1} \neq \{0\}$. By Lemma 6.2, $2m\mathscr{L}_1 + m\mathscr{L}_2$ is a fixed component for $[I_{3m\mathbb{X}}]_{9m-1}$, hence

$$\dim[I_{3m\mathbb{X}}]_{9m-1} = \dim[I_{3m\mathbb{X}-2m\mathscr{L}_1-m\mathscr{L}_2}]_{9m-1-3m} = \dim[I_{\sum_{i=1}^{6} mP_i+\sum_{i=1}^{3} 2mQ_i+\sum_{i=1}^{2} 3mR_i}]_{6m-1}.$$

Now if we prove that this last dimension is zero, we get a contradiction.

Claim.

$$\dim[I_{\sum_{i=1}^{6} mP_i+\sum_{i=1}^{3} 2mQ_i+\sum_{i=1}^{2} 3mR_i}]_{6m-1} = 0, \quad \text{for each } m \geq 1.$$

We prove the claim by induction on m. It is easy to verify that it is true for $m = 1$, so assume $m > 1$. If this dimension is not zero, by Bezout's Theorem, $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$ are fixed components, hence

$$\dim[I_{\sum_{i=1}^{6} mP_i+\sum_{i=1}^{3} 2mQ_i+\sum_{i=1}^{2} 3mR_i}]_{6m-1} = \dim[I_{\sum_{i=1}^{6}(m-1)P_i+\sum_{i=1}^{3}(2m-1)Q_i+\sum_{i=1}^{2}(3m-1)R_i}]_{6m-4}.$$

If this dimension is still not zero, by Lemma 2.5, $\mathscr{L}_2$ and $\mathscr{L}_3$ are fixed components of multiplicity at least $\left\lceil \frac{6m-3-(6m-4)}{2} \right\rceil = 1$, and $\left\lceil \frac{6m-2-(6m-4)}{1} \right\rceil = 2$, respectively. Hence

$$\begin{aligned} & \dim[I_{\sum_{i=1}^{6}(m-1)P_i+\sum_{i=1}^{3}(2m-1)Q_i+\sum_{i=1}^{2}(3m-1)R_i}]_{6m-4} \\ = \; & \dim[I_{\sum_{i=1}^{6}(m-1)P_i+\sum_{i=1}^{3} 2(m-1)Q_i+\sum_{i=1}^{2} 3(m-1)R_i}]_{6(m-1)-1}, \end{aligned}$$

and this is zero by the inductive hypothesis. □

Theorem 6.4. *Let $\mathbb{X}$ be a standard $\Bbbk$-configuration of type $(2,4,5)$. Then the Waldschmidt constant of $\mathbb{X}$ is*

$$\widehat{\alpha}(I_{\mathbb{X}}) = 3.$$

Proof. Let $\mathscr{F}$ be the following curve of degree 6, which contains each point of $\mathbb{X}$ with multiplicity 2,

$$\mathscr{F} = 2\mathscr{L}_1 + 2\mathscr{L}_2 + 2\mathscr{L}_3.$$

Hence, for $m > 0$,

$$m\mathscr{F} \in [I_{2m\mathbb{X}}]_{6m}.$$

Now, as usual, we have to prove that for each $m > 0$, $\dim[I_{2m\mathbb{X}}]_{6m-1} = 0$. It is true for $m = 1$, so assume $m > 1$. Assume that for some m, $[I_{2m\mathbb{X}}]_{6m-1} \neq \{0\}$, and let

$$\bar{m} = \min\{m | \dim[I_{2\bar{m}\mathbb{X}}]_{6\bar{m}-1}\} \neq 0.$$

By Lemma 2.5, $\mathscr{L}_1$ is a fixed component of multiplicity at least $\left\lceil \frac{10\bar{m}-6\bar{m}+1}{4} \right\rceil \geq \bar{m}+1$. Hence

$$\dim[I_{2\bar{m}\mathbb{X}}]_{6\bar{m}-1} = \dim[I_{2\bar{m}\mathbb{X}-(\bar{m}+1)\mathscr{L}_1}]_{5\bar{m}-2}.$$

If this dimension is not zero, we get that $\mathscr{L}_2$ is a fixed component of multiplicity at least $\left\lceil \frac{8\bar{m}-5\bar{m}+2}{3} \right\rceil \geq \bar{m}+1$. Hence

$$\dim[I_{2\bar{m}\mathbb{X}}]_{6\bar{m}-1} = \dim[I_{2\bar{m}\mathbb{X}-(\bar{m}+1)\mathscr{L}_1-(\bar{m}+1)\mathscr{L}_2}]_{4\bar{m}-3}.$$

If this dimension is not zero, we get that $\mathscr{L}_3$ is a fixed component of multiplicity at least $\left\lceil \frac{4\bar{m}-4\bar{m}+3}{1} \right\rceil = 3$. It follows that $\mathscr{F}$ is a fixed component. Hence, we get a contradiction since

$$\dim[I_{2\bar{m}\mathbb{X}}]_{6\bar{m}-1} = \dim[I_{2\bar{m}\mathbb{X}-\mathscr{F}}]_{6\bar{m}-1-6} = \dim[I_{2(\bar{m}-1)\mathbb{X}}]_{6(\bar{m}-1)-1},$$

which is zero by the definition of $\bar{m}$. □

Theorem 6.5. *Let $\mathbb{X}$ be a standard $\Bbbk$-configuration of type $(2,b,c)$.*

(i) *If $b = 3$ and $c \geq 6$, then $\widehat{\alpha}(I_{\mathbb{X}}) = 3$;*
(ii) *if $b \geq 4$, then $\widehat{\alpha}(I_{\mathbb{X}}) = 3$.*

Proof. Let $\mathscr{F} = \mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_3$. Since $m\mathscr{F} \in [I_{m\mathbb{X}}]_{3m}$, then in both cases, $\widehat{\alpha}(I_{\mathbb{X}}) \leq 3$.

Now let $\mathbb{X}$ be a standard $\Bbbk$-configuration of type $(2,3,c)$, with $c \geq 6$. Then there exists a standard $\Bbbk$-configuration $\mathbb{X}'$ of type $(2,3,6)$, with $\mathbb{X}' \subseteq \mathbb{X}$. Since, by Theorem 6.3, the Waldschmidt constant of $\mathbb{X}'$ is 3, then $\widehat{\alpha}(I_{\mathbb{X}}) \geq 3$, and (i) is proved.

For (ii), since $b \geq 4$, then there exists a standard $\Bbbk$-configuration $\mathbb{X}'$ of type $(2,4,5)$, with $\mathbb{X}' \subseteq \mathbb{X}$. Since, by Theorem 6.4, the Waldschmidt constant of $\mathbb{X}'$ is 3, hence $\widehat{\alpha}(I_{\mathbb{X}}) \geq 3$, and (ii) is proved. □

Remark 6.6. From the previous results we know the Waldschmidt constant of any standard $\Bbbk$-configuration of type $(2,b,c)$, except for $\mathbb{X}$ of type $(2,3,5)$. For the case $(2,3,5)$, we found by Macaulay 2 [19] a curve $\mathscr{F}$ of degree 71 with multiplicity exactly 24 at each point of $\mathbb{X}$. The components of $\mathscr{F}$ are lines, one irreducible conic and an irreducible rational septic. This implies $\widehat{\alpha}(I_{\mathbb{X}}) \leq \frac{71}{24} < 3$. Moreover, since a $\Bbbk$-configuration of type $(2,3,4)$ is a subset of $\mathbb{X}$, this give $\frac{17}{6}$ as a lower bound (see Theorem 6.1). Hence $\frac{17}{6} \leq \widehat{\alpha}(I_{\mathbb{X}}) \leq \frac{71}{24}$.

Finally, we deal with the $\Bbbk$-configurations of type (a,b,c) when $a \geq 3$.

Theorem 6.7. *Let $\mathbb{X}$ be a standard $\Bbbk$-configuration of type (a,b,c), whith $a \geq 3$. Then the Waldschmidt constant of $\mathbb{X}$ is*

$$\widehat{\alpha}(I_{\mathbb{X}}) = 3.$$

Proof. It follows immediately from Corollary 4.2. □

Remark 6.8. We recall Chudnovsky's conjecture:
Let $\mathbb{X}$ be a finite set of distinct points in $\mathbb{P}^n$. Then, for all $m > 0$,

$$\frac{\alpha(I_{\mathbb{X}}^{(m)})}{m} \geq \frac{\alpha(I_{\mathbb{X}}) + n - 1}{n}.$$

This conjecture was proved in $\mathbb{P}^2$ by Chudnovsky (see, for instance [22, Proposition 3.1]). As an application, we wish to show that Chudnovsky's conjecture is verified by standard $\mathbb{k}$-configurations in $\mathbb{P}^2$ of type (a, b, c).

Let $\mathbb{X}$ and $\mathbb{Y}$ be standard $\mathbb{k}$-configurations in $\mathbb{P}^2$ of type (a, b, c), and (b, c), respectively. We know that $\alpha(I_{\mathbb{X}}) = 3$, and from the proof of Lemma 4.1, recalling that $b > 1$, we get that $\alpha(I_{\mathbb{Y}}^{(m)}) = 2m$. Moreover, since the scheme $m\mathbb{X} \supset m\mathbb{Y}$, then $\alpha(I_{\mathbb{X}}^{(m)}) \geq \alpha(I_{\mathbb{Y}}^{(m)})$. It follows that, for all $m > 0$,

$$\frac{\alpha(I_{\mathbb{X}}^{(m)})}{m} \geq \frac{\alpha(I_{\mathbb{Y}}^{(m)})}{m} = 2 = \frac{3 + 2 - 1}{2} = \frac{\alpha(I_{\mathbb{X}}) + n - 1}{n}.$$

7. Appendix.

We recall the notation for the proofs of theorems about standard $\mathbb{k}$-configurations of type $(1, b, c)$, summarized in Figure 6.

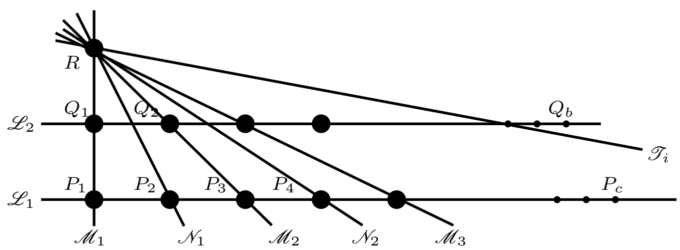

FIGURE 6. A standard $\mathbb{k}$-configuration of type $(1, b, c)$

We denote by

$\mathscr{L}_1$ be the line through $P_1, P_2, \dots, P_c$;
$\mathscr{L}_2$ be the line through $Q_1, Q_2, \dots, Q_b$;
$\mathscr{M}_1$ be the line through P_1, Q_1, R;
$\mathscr{M}_2$ be the line through P_3, Q_2, R;
$\vdots$
$\mathscr{M}_i$ be the line through P_{2i-1}, Q_i, R, for $i \leq b$ and $2i \leq c + 1$;
$\mathscr{N}_1$ be the line through P_2, R;
$\mathscr{N}_2$ be the line through P_4, R;
$\vdots$
$\mathscr{N}_i$ be the line through P_{2i}, R, for $2i \leq c$;
$\mathscr{T}_i$ be the line through Q_i, R, for $i \leq b$ and $2i \geq c + 2$.

Proof of Theorem 5.4. Let $\mathbb{X}$ be a standard $\mathbb{k}$-configuration of type $(1, b, b + 1)$. If $b \geq 4$ is an even integer, we show that

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{9b - 4}{3b}.$$

Let

$$\mathscr{F} = \frac{3b-2}{2}\mathscr{L}_1 + \frac{3b-2}{2}\mathscr{L}_2 + \mathscr{M}_1 + \cdots + \mathscr{M}_{\frac{b}{2}+1} + \mathscr{N}_1 + \cdots + \mathscr{N}_{\frac{b}{2}} + \mathscr{T}_{\frac{b}{2}+2} + \cdots + \mathscr{T}_b,$$

so $m\mathscr{F}$ is a curve in the linear system $[I_{\frac{3b}{2}m\mathbb{X}}]_{\frac{9b-4}{2}m}$. Now we need to prove that, for each $m > 0$, $\dim[I_{\frac{3b}{2}m\mathbb{X}}]_{\frac{9b-4}{2}m-1} = 0$.

By Lemma 2.5, if $\dim[I_{\frac{3b}{2}m\mathbb{X}}]_{\frac{9b-4}{2}m-1} \neq \{0\}$, then $\mathscr{L}_1$ is a fixed component of multiplicity at least

$$\left\lceil \frac{\frac{3b}{2}m(b+1) - \frac{9b-4}{2}m + 1}{b} \right\rceil = \left\lceil \frac{(3b^2-6b+4)m+2}{2b} \right\rceil \geq \frac{3b-6}{2}m, \tag{7.1}$$

for the plane curves of the linear system $[I_{\frac{3b}{2}m\mathbb{X}}]_{\frac{9b-4}{2}m-1}$.

If we remove $\frac{3b-6}{2}m\mathscr{L}_1$, we get that $\mathscr{L}_2$ is a fixed component of multiplicity at least

$$\left\lceil \frac{\frac{3b}{2}mb - (3b+1)m + 1}{b-1} \right\rceil = \left\lceil \frac{(3b^2-6b-2)m+2}{2(b-1)} \right\rceil \geq \frac{3b-6}{2}m. \tag{7.2}$$

By removing $\frac{3b-6}{2}m\mathscr{L}_2$, we have that $\mathscr{L}_1$ is a fixed component of multiplicity at least

$$\left\lceil \frac{3m(b+1) - \frac{3b+8}{2}m + 1}{b} \right\rceil = m + \left\lceil \frac{(b-2)m+2}{2b} \right\rceil. \tag{7.3}$$

After removing $m\mathscr{L}_1$, then $\mathscr{L}_2$ is a fixed component of multiplicity at least

$$\left\lceil \frac{3bm - \frac{3b+6}{2}m + 1}{b-1} \right\rceil = m + \left\lceil \frac{(b-4)m+2}{2b-2} \right\rceil. \tag{7.4}$$

Remove $m\mathscr{L}_2$. The residual scheme is

$$\mathbb{Y} = \mathbb{X} - \Big(\frac{3b-6}{2}m + m\Big)\mathscr{L}_1 - \Big(\frac{3b-6}{2}m + m\Big)\mathscr{L}_2 = \frac{3b}{2}mR + \sum_i 2mP_i + \sum_i 2mQ_i,$$

and

$$\dim[I_{\frac{3b}{2}m\mathbb{X}}]_{\frac{9b-4}{2}m-1} = \dim[I_{\mathbb{Y}}]_{\frac{3b+4}{2}m-1}.$$

Now, by Bezout's Theorem, the lines $\mathscr{M}_i$, $\mathscr{N}_i$, and $\mathscr{T}_i$ are fixed components.

Set

$$\bar{m} = \min\{m \mid [I_{\frac{3b}{2}m\mathbb{X}}]_{\frac{9b-4}{2}m-1} \neq \{0\}\}. \tag{7.5}$$

First observe that $\bar{m} \neq 1$, in fact for $m = 1$, by (7.1), (7.2), (7.3), (7.4), and using also the ceiling parts, we get that $\mathscr{F}$ is a curve of the linear system $[I_{\frac{3b}{2}\mathbb{X}}]_{\frac{9b-4}{2}-1}$, but $\mathscr{F}$ has degree $\frac{9b-4}{2}$, a contradiction.

So $\bar{m} > 1$. By by (7.1), (7.2), (7.3), (7.4), we get that $\mathscr{F}$ is a fixed component for the linear system $\left[I_{\frac{3b}{2}\bar{m}\mathbb{X}}\right]_{\frac{9b-4}{2}\bar{m}-1}$, hence, by recalling that $\deg\mathscr{F} = \frac{9b-4}{2}$ and $\mathscr{F}$ contains each point of $\mathbb{X}$ with multiplicity $\frac{3b}{2}$, we get

$$\dim\left[I_{\frac{3b}{2}\bar{m}\mathbb{X}}\right]_{\frac{9b-4}{2}\bar{m}-1} = \dim\left[I_{\frac{3b}{2}\bar{m}\mathbb{X}-\mathscr{F}}\right]_{\frac{9b-4}{2}\bar{m}-1-\frac{9b-4}{2}} = \dim\left[I_{\frac{3b}{2}(\bar{m}-1)\mathbb{X}}\right]_{\frac{9b-4}{2}(\bar{m}-1)-1},$$

which is zero by (7.5), a contradiction. □

Proof of Theorem 5.6. Let $\mathbb{X}$ be a standard $\Bbbk$-configuration of type $(1, b, 2b-1)$. We show that

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{6b^2-8b+1}{2b^2-2b}.$$

Let

- $\mathscr{C}_i$ be the irreducible curve of degree $(b-1)$ through $P_2, P_4, \ldots, P_{2b-2}, Q_1, \ldots, \widehat{Q}_i, \ldots, Q_b, (b-2)R$ for $1 \leq i \leq b$ (see Lemma 2.7), and let $\mathscr{F}$ be the following curve of degree $6b^2-8b+1$ with multiplicity $2b^2-2b$ at each point of $\mathbb{X}$.

$$\mathscr{F} = (2b^2-3b)\mathscr{L}_1 + (2b^2-4b+1)\mathscr{L}_2 + b\mathscr{M}_1 + \cdots + b\mathscr{M}_b + \mathscr{C}_1 + \cdots + \mathscr{C}_b.$$

Hence for $m>0$

$$m\mathscr{F} \in [I_{(2b^2-2b)m\mathbb{X}}]_{(6b^2-8b+1)m}.$$

We should now prove that for $m>0$,

$$[I_{(2b^2-2b)m\mathbb{X}}]_{(6b^2-8b+1)m-1} = \{0\}.$$

Since the proof is analogous to the one of Theorem 5.5, assuming that the ideals which we will consider are different from zero, we just show the computation that, from Lemma 2.5, gives how many times each component of $\mathscr{F}$ is a fixed component for the curves of the linear system $[I_{(2b^2-2b)m\mathbb{X}}]_{(6b^2-8b+1)m-1}$.

We get that $\mathscr{L}_1$ is fixed component of multiplicity at least

$$\left\lceil \frac{(2b-1)(2b^2-2b)m-(6b^2-8b+1)m+1}{2b-2} \right\rceil \geq (2b^2-4b+1)m. \tag{7.6}$$

By removing $(2b^2-4b+1)m\mathscr{L}_1$, we get that $\mathscr{L}_2$ is fixed component of multiplicity at least

$$\left\lceil \frac{b(2b^2-2b)m-(4b^2-4b)m+1}{b-1} \right\rceil = (2b^2-4b)m + \left\lceil \frac{1}{b-1} \right\rceil. \tag{7.7}$$

By removing $(2b^2-4b)m\mathscr{L}_2$, we find that $\mathscr{L}_1$ is fixed component of multiplicity at least

$$\left\lceil \frac{(2b-1)^2m-2b^2m+1}{2b-2} \right\rceil = (b-2)m + \left\lceil \frac{(2b-3)m+1}{2b-2} \right\rceil. \tag{7.8}$$

Now we remove $(b-2)m\mathscr{L}_1$ and we find that each $\mathscr{M}_i$ is fixed component of multiplicity at least

$$\left\lceil \frac{(2b^2-2b)m+2bm+(b+1)m-(2b^2-b+2)m+1}{2} \right\rceil = (b-1)m + \left\lceil \frac{m+1}{2} \right\rceil. \tag{7.9}$$

So, after we remove $((2b^2-4b+1)+(b-2))m\mathscr{L}_1 + (2b^2-4b)m\mathscr{L}_2 + \sum_{i=1}^{b}(b-1)m\mathscr{M}_i$, the residual scheme is

$$\mathbb{Y} = (b^2-b)R + \sum_{i=1}^{b}(b+1)Q_i + \sum_{\text{for } i \text{ odd}} 2mP_i + \sum_{\text{for } i \text{ even}} (b+1)mP_i,$$

and the degree we have to consider is $((6b^2-8b+1)-(2b^2-4b+1)-(b-2)-(2b^2-4b)-b(b-1))m-1 = (b^2+2)m-1$, thus

$$\dim[I_{(2b^2-2b)m\mathbb{X}}]_{(6b^2-8b+1)m-1} = \dim[I_{\mathbb{Y}}]_{(b^2+2)m-1}.$$

Now if $\mathscr{H}$ is a curve of the linear system $[I_{\mathbb{Y}}]_{(b^2+2)m-1}$, the multiplicity of intersection between each $\mathscr{C}_i$ and $\mathscr{H}$ is at least

$$|\mathscr{C}_i \cdot \mathscr{H}| \geq (b-2)(b^2-b)m + (b+1)(b-1)m + (b+1)(b-1)m = (b^3-b^2+2b-2)m,$$

and this number is bigger than the product of the degree of $\mathscr{C}_i$ and $\mathscr{H}$, which is

$$\deg \mathscr{C}_i \cdot \deg \mathscr{H} = (b-1)((b^2+2)m-1) = (b^3-b^2+2b-2)m-(b-1).$$

Hence, by Bézout's Theorem, each curve $\mathscr{C}_i$ is a fixed component for the curves of $[I_{\mathbb{Y}}]_{(b^2+2)m-1}$.

Now let

$$\bar{m} = \min\{m \mid [I_{(2b^2-2b)m\mathbb{X}}]_{(6b^2-8b+1)m-1} \neq \{0\}\}. \tag{7.10}$$

We have $\bar{m}>1$, in fact for $m=1$, by (7.6), (7.7), (7.8), (7.9), and using also the ceiling parts, we get that $\mathscr{F}$ should be a curve in the linear system $[I_{(2b^2-2b)\mathbb{X}}]_{6b^2-8b}$, but $\mathscr{F}$ has degree $6b^2-8b+1$, a contradiction.

Hence $\bar{m}>1$.

By the above computation, then $\mathscr{F}$ is a fixed component for the linear system $[I_{(2b^2-2b)m\mathbb{X}}]_{(6b^2-8b+1)m-1}$. We have

$$\begin{aligned}\dim[I_{(2b^2-2b)\bar{m}\mathbb{X}}]_{(6b^2-8b+1)\bar{m}-1} &= \dim[I_{(2b^2-2b)\bar{m}\mathbb{X}-\mathscr{F}}]_{(6b^2-8b+1)\bar{m}-1-(6b^2-8b+1)}\\ &= \dim[I_{(2b^2-2b)(\bar{m}-1)\mathbb{X}}]_{(6b^2-8b+1)(\bar{m}-1)-1}\end{aligned}$$

which is zero by (7.10), a contradiction. □

Proof of Theorem 5.7. Let $\mathbb{X}$ be a standard $\Bbbk$-configuration of type $(1, b, 2b)$. We show that

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{6b-5}{2b-1}.$$

Let

$\mathscr{C}$ be the irreducible curve of degree b through $P_2, P_4, \ldots, P_{2b}, Q_1, \ldots, Q_b, (b-1)R$ (see Lemma 2.7),

and let $\mathscr{F}$ be the following curve of degree $(6b-5)$ with multiplicity exactly $(2b-1)$ at the points of $\mathbb{X}$,

$$\mathscr{F} = (2b-2)\mathscr{L}_1 + (2b-3)\mathscr{L}_2 + \mathscr{M}_1 + \cdots + \mathscr{M}_b + \mathscr{C}.$$

Hence, for $m > 0$, $m\mathscr{F} \in [I_{m(2b-1)\mathbb{X}}]_{m(6b-5)}$. Now we will show that for each $m > 0$ we have

$$[I_{m(2b-1)\mathbb{X}}]_{m(6b-5)-1} = \{0\},$$

and the conclusion will follow from Lemma 2.2.

Assume that $[I_{m(2b-1)\mathbb{X}}]_{m(6b-5)-1} \neq \{0\}$ for some $m > 0$.

Let $\mathscr{H}$ be a curve of the linear system $[I_{m(2b-1)\mathbb{X}}]_{m(6b-5)-1}$. Then the multiplicity of the intersection between $\mathscr{C}$ and $\mathscr{H}$ is at least $(2b-1)m$ in each of the points P_i and Q_i and at least $(b-1)(2b-1)m$ in R. Since we have $2b$ points P_i and Q_i,

$$|\mathscr{C} \cdot \mathscr{H}| \geq 2b(2b-1)m + (b-1)(2b-1)m,$$

and this number is bigger than the product of the degree of $\mathscr{C}$ and $\mathscr{H}$, which is $b(m(6b-5)-1)$. In fact

$$2b(2b-1)m + (b-1)(2b-1)m - b(m(6b-5)-1) = m + b > 0.$$

Hence, by Bézout's Theorem, the curve $\mathscr{C}$ is a fixed component for the curves of $[I_{m(2b-1)\mathbb{X}}]_{m(6b-5)-1}$.

Moreover, for the curves of this linear system, by Lemma 2.5, $\mathscr{M}_i$, $(1 \leq i \leq b)$, is a fixed component of multiplicity at least

$$\left\lceil \frac{3(2b-1)m-(6b-5)m+1}{2} \right\rceil = \left\lceil \frac{2m+1}{2} \right\rceil = m+1,$$

and $\mathscr{L}_1$ is a fixed component of multiplicity at least

$$\left\lceil \frac{2b(2b-1)m-(6b-5)m+1}{2b-1} \right\rceil = (2b-3)m + \left\lceil \frac{2m+1}{2b-1} \right\rceil.$$

If we remove the curve $(2b-3)m\mathscr{L}_1$ we get

$$\dim[I_{m(2b-1)\mathbb{X}}]_{m(6b-5)-1} = \dim[I_{m(2b-1)\mathbb{X}-(2b-3)m\mathscr{L}_1})]_{(4b-2)m-1}.$$

If this dimension is different from zero, by Lemma 2.5, we get that $\mathscr{L}_2$ is a fixed component of multiplicity at least

$$\left\lceil \frac{(2b-1)m \cdot b-(4b-2)m+1}{b-1} \right\rceil = (2b-4)m + \left\lceil \frac{(b-2)m+1}{b-1} \right\rceil$$

for the curves of $[I_{m(2b-1)\mathbb{X}}]_{m(6b-5)-1}$.

Now let

(7.11) $$\bar{m} = \min\{m | [I_{m(2b-1)\mathbb{X}}]_{m(6b-5)-1} \neq \{0\}\}.$$

We have $\bar{m} > 1$, in fact for $m = 1$, by the computation above, the curve $\mathscr{F}$ of degree $6b-5$ should be a fixed component for the linear system, $[I_{(2b-1)\mathbb{X}}]_{6b-4}$, a contradiction.

Hence $\bar{m} > 1$. Since $\mathscr{F}$ is a fixed component for the linear system $[I_{m(2b-1)\mathbb{X}}]_{m(6b-5)-1}$ we have

$$\dim[I_{\bar{m}(2b-1)\mathbb{X}}]_{\bar{m}(6b-5)-1} = \dim[I_{\bar{m}(2b-1)\mathbb{X}-\mathscr{F}}]_{\bar{m}(6b-5)-1-(6b-5)} = \dim[I_{(\bar{m}-1)(2b-1)\mathbb{X}}]_{(\bar{m}-1)(6b-5)-1},$$

which is zero by (7.11), a contradiction. □

Proof of Theorem 5.8. Let $\mathbb{X}$ be a standard $\mathbb{k}$-configuration of type $(1, b, 2b+1)$. We show that

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{6b^2 - 2b - 3}{2b^2 - 1}.$$

Let

- $\mathscr{C}_i$ be the irreducible curve of degree b through $P_2, P_4, \ldots, \widehat{P}_{2i}, \ldots, P_{2b}, P_{2b+1}$, $Q_1, \ldots, Q_b, (b-1)R$ for $1 \le i \le b$,
- $\mathscr{C}_{b+1}$ be the irreducible curve of degree b through $P_2, P_4, \ldots, P_{2b}, Q_1, \ldots, Q_b, (b-1)R$;

(see Lemma 2.7 for the $b+1$ curves $\mathscr{C}_i$). Note that the curve $\mathscr{C}_1 + \cdots + \mathscr{C}_{b+1}$ has degree $b(b+1)$, multiplicity $b+1$ at each of the points $Q_1, \ldots, Q_b$, multiplicity b at each of the points $P_2, P_4, \ldots, P_{2b}, P_{2b+1}$, and multiplicity $b^2 - 1$ at R. Let

$$\mathscr{F} = (2b^2 - b - 1)\mathscr{L}_1 + (2b^2 - 2b - 2))\mathscr{L}_2 + b\mathscr{M}_1 + \cdots + b\mathscr{M}_b + \mathscr{C}_1 + \cdots + \mathscr{C}_{b+1}.$$

Then $\mathscr{F}$ is a curve of degree $(6b^2 - 2b - 3)$ with multiplicity $(2b^2 - 1)$ at each point of $\mathbb{X}$. Hence for $m > 0$

$$m\mathscr{F} \in [I_{(2b^2-1)m\mathbb{X}}]_{(6b^2-2b-3)m}.$$

We now have to prove that

$$[I_{(2b^2-1)m\mathbb{X}}]_{(6b^2-2b-3)m-1} = 0.$$

Assume that for some $m > 0$, $[I_{(2b^2-1)m\mathbb{X}}]_{(6b^2-2b-3)m-1} \ne \{0\}$.

Analogously to the proof of Theorem 5.7, let $\mathscr{H}$ be a curve of the linear system $[I_{(2b^2-1)m\mathbb{X}}]_{(6b^2-2b-3)m-1}$. Then the multiplicity of intersection between each $\mathscr{C}_i$ and $\mathscr{H}$ is at least $(2b^2-1)m$ in each of the $2b$ points P_i and Q_i and at least $(b-1)(2b^2-1)m$ in R, so,

$$|\mathscr{C}_i \cdot \mathscr{H}| \ge 2b(2b^2 - 1)m + (b-1)(2b^2-1)m,$$

and this number is bigger than the product of the degree of $\mathscr{C}_i$ and $\mathscr{H}$, which is $b((6b^2-2b-3)m-1)$. Hence, by Bézout's Theorem, each curve $\mathscr{C}_i$ is a fixed component for the curves of $[I_{(2b^2-1)m\mathbb{X}}]_{(6b^2-2b-3)m-1}$.

Moreover, for the curves of this linear system, by Lemma 2.5, each $\mathscr{M}_i$ is a fixed component of multiplicity at least

$$\left\lceil \frac{3(2b^2-1)m - (6b^2-2b-3)m + 1}{2} \right\rceil = bm + 1,$$

$\mathscr{L}_1$ is a fixed component of multiplicity at least

$$\left\lceil \frac{(2b^2-1)(2b+1)m - (6b^2-2b-3)m + 1}{2b} \right\rceil = \left\lceil \frac{(4b^3-4b^2+2)m+1}{2b} \right\rceil = (2b^2-2b)m + \left\lceil \frac{2m+1}{2b} \right\rceil,$$

and, by removing $(2b^2-2b)m\mathscr{L}_1$, we get that $\mathscr{L}_2$ is a fixed component of multiplicity at least

$$\left\lceil \frac{(2b^2-1)m \cdot b - (4b^2-3)m + 1}{b-1} \right\rceil = (2b^2 - 2b - 3)m + \left\lceil \frac{1}{b-1} \right\rceil.$$

Now let

$$\bar{m} = \min\{m | [I_{m(2b^2-1)\mathbb{X}}]_{m(6b^2-2b-3)-1} \ne \{0\}\}. \tag{7.12}$$

We have $\bar{m} > 1$, in fact for $m = 1$, by the computation above, the curve $\mathscr{F}'$of degree $6b^2 - 3b - 1$,

$$\mathscr{F}' = (2b^2 - 2b + 1)\mathscr{L}_1 + (2b^2 - 2b - 2)\mathscr{L}_2 + b\mathscr{M}_1 + \cdots + b\mathscr{M}_b + \mathscr{C}_1 + \cdots + \mathscr{C}_{b+1},$$

should be a fixed component for the linear system, so

$$\begin{aligned}\dim[I_{(2b^2-1)\mathbb{X}}]_{(6b^2-2b-3)-1} &= \dim[I_{(2b^2-1)\mathbb{X}-\mathscr{F}'}]_{(6b^2-2b-4)-(6b^2-3b-1)}\\ &= \dim[I_{(b-2)P_1+\cdots+(b-2)P_{2b+1}}]_{(b-3)}\\ &= 0,\end{aligned}$$

a contradiction.

Hence $\bar{m} > 1$. By the computation above $\mathscr{F}$ is a fixed component for $[I_{(2b^2-1)\bar{m}\mathbb{X}}]_{(6b^2-2b-3)\bar{m}-1}$, hence we have

$$\begin{aligned}\dim[I_{(2b^2-1)\bar{m}\mathbb{X}}]_{(6b^2-2b-3)\bar{m}-1} &= \dim[I_{(2b^2-1)\bar{m}\mathbb{X}-\mathscr{F}}]_{(6b^2-2b-3)\bar{m}-1-(6b^2-2b-3)}\\ &= \dim[I_{(2b^2-1)(\bar{m}-1)\mathbb{X}}]_{(6b^2-2b-3)(\bar{m}-1)-1},\end{aligned}$$

which is zero by (7.12), a contradiction. □

8. Declarations

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(M.V. Catalisano) Scuola Politecnica, Università di Genova, Genoa, Italy
Email address: mariavirginia.catalisano@unige.it

(G. Favacchio) Dipartimento di Ingegneria, Università degli studi di Palermo, Viale delle Scienze, 90100 Palermo, Italy
Email address: giuseppe.favacchio@unipa.it

(E. Guardo) Dipartimento di Matematica e Informatica, Università di Catania, Catania, Italy
Email address: guardo@dmi.unict.it

(Y.S. Shin) Department of Mathematics, Sungshin Women's University, Seoul, 02844, Republic of Korea
Email address: ysshin@sungshin.ac.kr