On freely generated E-subrings

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Abstract

In this paper we prove, without assuming Schanuel's conjecture, that the E-subring generated by a real number not definable without parameters in the real exponential field is freely generated. We also obtain a similar result for the complex exponential field.

1 Introduction

The second author proved in [10] that the E-subring generated by π, modulo Schanuel's Conjecture, is isomorphic to the free exponential ring on π. Recall that:

Schanuel's Conjecture (SC) Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then $\mathbb{Q}(\lambda_1, \ldots, \lambda_n, e^{\lambda_1}, \ldots, e^{\lambda_n})$ has transcendence degree (t.d.) at least n over $\mathbb{Q}$.

This is the major open problem in transcendence theory and played an important role also in decidability issues. The most important one is the decidability of the real exponential field proved by Macintyre and Wilkie in [8] modulo Schanuel's Conjecture.

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In this paper we generalize the result from [10] to any real number not definable without parameters in the real exponential field, without using Schanuel's Conjecture:

Theorem 1.1. *Let τ be a real number not definable without parameters in the real exponential field, then the E-subring of $\mathbb{R}$ generated by τ is isomorphic to the free E-ring on τ.*

In order to prove this result we prove a version of Schanuel's Conjecture for elements not definable without parameters in o-minimal expansions of the real exponential field following Wilkie's ideas. We wish to thank here Alex Wilkie for his help on this.

For other results on the connection between undefinability in o-minimal expansions of the real exponential field and Schanuel's Conjecture see also [3]. In particular if Schanuel's Conjecture is true, then π is not definable without parameters in the real exponential field, so Theorem 1.1 implies the result for π in [10] (although the technique is the similar).

2 Free E-ring

Here we recall some basic facts about E-rings:

Definition 2.1. *An exponential ring, or E-ring, is a pair (R, E) with R a commutative ring with 1 and $E : R \to \mathcal{U}(R)$ a morphism of the additive group of R into the multiplicative group of units of R satisfying:*

$$E(x + y) = E(x) \cdot E(y) \text{ for all } x, y \in R, \text{ and } E(0) = 1.$$

To set up notation and recall some basic properties needed later, it is useful to review the construction of the free E-ring on a set of generators $X_1, \ldots, X_m$, denoted by $[X_1, \ldots, X_m]^E$. The notion of freeness is a well known, abstract mathematical concept, which applies to a wide variety of situations and in particular to the category of E-rings. See for example [4].

We construct by recursion three sequences:

1. $(R_k, +, \cdot)_{k \geq -1}$ are rings;
2. $(B_k, +)_{k \geq 0}$ are torsion free abelian groups,
3. $(E_k)_{k \geq -1}$ are partial E-morphisms.

Step 0: We define
$R_{-1} = \{0\}$,
$R_0 = \mathbb{Z}[X_1, \ldots, X_m]$ as ring,
$B_0 = \mathbb{Z}[X_1, \ldots, X_m]$ as additive group and
$E_{-1}(0) = 1$.

Inductive step:
Suppose $k \geq 0$ and R_{k-1}, R_k, B_k and E_{k-1} have been defined in such a way that:

$$R_k = R_{k-1} \oplus B_k,\ E_{k-1} : (R_{k-1}, +) \to (\mathcal{U}(R_k), \cdot),$$

where $\mathcal{U}(R_k)$ denotes the set of units in R_k.

Let

$$t : (B_k, +) \to (t^{B_k}, \cdot)$$

be a formal isomorphism. Define

$$R_{k+1} = R_k[t^{B_k}] \text{ (as group ring over } R_k).$$

Therefore

$$R_k \text{ is a subring of } R_{k+1}$$

and as additive group

$$R_{k+1} = R_k \oplus B_{k+1},$$

where B_{k+1} is the R_k-submodule of R_{k+1} freely generated by t^b, with $b \in B_k$ and $b \neq 0$.

We define

$$E_k : (R_k, +) \to (\mathcal{U}(R_{k+1}), \cdot) \text{ as follows}$$

$E_k(x) = E_{k-1}(r) \cdot t^b$, for $x = r + b$, $r \in R_{k-1}$ and $b \in B_k$.

In this way we construct a chain of partial E-ring

$$R_0 \subset R_1 \subset R_2 \cdots \subset R_k \subset \cdots$$

Then the free E- ring is:

$$[X_1, \ldots, X_m]^E = \lim_k R_k = \bigcup_{k=0}^{\infty} R_k$$

and the E-ring morphism defined on $[X_1, \ldots, X_m]^E$ is the following:

$$E(x) = E_k(x) \text{ if } x \in R_k,\ k \in \mathbb{N}.$$

Notice that at each step R_{k+1} as additive group is the direct sum $B_0 \oplus B_1 \oplus \ldots \oplus B_{k+1}$. Moreover, as an additive group $[X_1, \ldots, X_m]^E$ can be considered as $B_0 \oplus B_1 \oplus \ldots \oplus B_{k+1} \oplus \ldots$.

Recall that for all k the group ring R_{k+1} can be viewed in the following different ways

$$R_{k+1} \cong R_0[t^{B_0 \oplus B_1 \oplus \ldots \oplus B_k}];$$

$$R_{k+1} \cong R_1[t^{B_1 \oplus \ldots \oplus B_k}];$$

$$\ldots$$

$$\ldots$$

$$R_{k+1} \cong R_k[t^{B_k}].$$

Moreover, $[X_1, \ldots, X_m]^E = R_0[t^{B_0 \oplus B_1 \oplus \ldots \oplus B_k \ldots}]$.

For other interesting constructions in the category of E-rings we refer the reader to [5].

3 Schanuel's Conjecture and o-minimality

We start by recalling the definition of definability and o-minimality and we review some properties of o-minimal structures which will be useful later.

Definition 3.1. *Let $\mathcal{M} = (M, \cdots)$ be a structure and let $\mathcal{L}_{\mathcal{M}}$ be the first-order language of $\mathcal{M}$. If $C \subseteq M$, then $\mathcal{L}_{\mathcal{M}}(C)$ is $\mathcal{L}_{\mathcal{M}}$ expanded with a constant symbol for each element of C. We say that a subset A of M^n is $\mathcal{L}_{\mathcal{M}}(C)$-definable if there is an $\mathcal{L}_{\mathcal{M}}(C)$-formula $\varphi(\overline{x})$ such that*

$$A = \{\overline{a} \in M^n : \mathcal{M} \models \varphi(\overline{a})\}.$$

In this situation, we also say that A is definable (in $\mathcal{M}$ with parameters in C or in $\mathcal{M}$ over C); and if $C = \emptyset$, we say that A is definable without parameters.

Definition 3.2. *An o-minimal structure $\mathcal{M} = (M, <, \cdots)$ is an expansion of a totally ordered set such that every subset of M which is definable in $\mathcal{M}$ (possibly with parameters) is a finite union of intervals with end points in $M \cup \{-\infty, +\infty\}$.*

This class of ordered structures is very well behaved both from a model theoretic and geometric point of view (it has interesting connections to semi-algebraic and sub-analytic geometry). Two important examples of o-minimal structures over $(\mathbb{R}, <)$ that will appear below are the real exponential field $\mathbb{R}_{\exp}$ and the real exponential field with restricted analytic functions $\mathbb{R}_{\mathrm{an,\,exp}}$. For the o-minimality of these structures see [11] and [6] respectively.

Below we will require a model theoretic (and also geometric) property of o-minimal structures that we now describe.

Definition 3.3. *Let $\mathcal{M} = (M, <, \cdots)$ be an o-minimal structure and let $\mathcal{L}_{\mathcal{M}}$ be the first-order language of $\mathcal{M}$. Consider $a \in M$ and $C \subseteq M$.*

1. *We say that a is model theoretically algebraic over C, denoted $a \in \mathrm{acl}(C)$, if there is an $\mathcal{L}_{\mathcal{M}}$-formula $\phi(x, y)$ (without parameters) such that for some tuple $\overline{c}$ of elements of C we have that $\phi(a, \overline{c})$ is true in $\mathcal{M}$ and the set of solutions of $\phi(x, \overline{c})$ in $\mathcal{M}$ is finite.*

2. *We say that a is in the definable closure of C, denoted $a \in \mathrm{dcl}(C)$, if there is an $\mathcal{L}_{\mathcal{M}}$-formula $\psi(x, y)$ (without parameters) such that for some tuple $\overline{c}$ of elements of C we have that a is the only solution of $\psi(a, \overline{c})$ in $\mathcal{M}$.*

Observe that on ordered structures we clearly have $\mathrm{acl}(-) = \mathrm{dcl}(-)$. In o-minimal structures, by [9] Theorem 4.1 and remarks after the statement of Theorem 4.2, the model theoretic algebraic closure operation satisfies all the usual axioms for closure operations, including Steinitz exchange, and so has an associated well-defined theory of bases, independence and dimension.

Note also, that in $\mathcal{M}$, since acl is the same as dcl, then $a \in \mathrm{acl}(C)$ if and only if there is an $\mathcal{L}_{\mathcal{M}}$-definable function f in $\mathcal{M}$ (without parameters) such that for some tuple $\overline{c}$ of elements of C, we have $f(\overline{c}) = a$.

There are many cases where Schanuel's Conjecture has been proved true: James Ax in [1] proved the power series and the differential fields version of the conjecture, and Ricardo Bianconi in [2] proved a version of the conjecture for the infinitesimal elements in any ultrapower of $\mathbb{C}$. Also Wilkie, in his unpublished notes ([12] and [13]), proved versions of the conjecture for elements not definable without parameters in o-minimal expansions of the real exponential field $\mathbb{R}_{\exp}$. The idea of the proof is to use o-minimality of the exponential field (and o-minimal expansions of it) to reduce the problem to Ax's differential field version of Schanuel's Conjecture proved in [1], which we recall

Theorem 3.4 (SD)**.** *Let K be a field and D a derivation of K with constants $C \supseteq \mathbb{Q}$. Let $y_1, \ldots, y_n, z_1, \ldots, z_n \in K^*$ such that:*

- $Dy_j = \frac{Dz_j}{z_j}$ *for* $j = 1, \ldots, n$;
- $Dy_1, \ldots, Dy_n$ *are linearly independent over* $\mathbb{Q}$.

Then

$$t.d._C C(y_1, \ldots, y_n, z_1, \ldots, z_n) \geq n + 1.$$

By Wilkie's suggestion, we will follow the same ideas and prove yet other versions of the conjecture for elements not definable without parameters in o-minimal expansions of the real exponential field.

In order to prove this we introduce some notations and we recall some properties of o-minimal structure $\mathbb{R}_{\exp} = (\mathbb{R}, 0, 1, +, \cdot, \exp, <)$ and its first-order language $\mathcal{L}_{\exp} = \{0, 1, +, \cdot, \exp, <\}$.

Let τ be a real number not definable in $\mathbb{R}_{\exp}$ without parameters. Consider the operation $\mathrm{cl} : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ of "closure under $\mathcal{L}_{\exp}$-definable (without parameters) functions" i.e., acl or dcl as defined above. As we pointed out, this is an algebraic closure operation satisfying all the usual axioms, including Steinitz exchange, and so has an associated well-defined theory of bases, independence and dimension. In particular, $\dim\tau = 1$ (by hypothesis) and there exists a set $B \subseteq \mathbb{R}$ such that $\{\tau\} \cup B$ is a basis (for this closure operation). Let C be the domain of the elementary submodel of $\mathbb{R}_{\exp}$ generated by B. Then $\tau \notin C$, but for any $a \in \mathbb{R}$ there exists an $\mathcal{L}_{\exp}(C)$-definable function $\theta : \mathbb{R} \to \mathbb{R}$ such that $\theta(\tau) = a$. Now notice that by o-minimality and non definability of τ we have:

- θ is differentiable on an open interval containing τ;
- if $\psi : \mathbb{R} \to \mathbb{R}$ is an $\mathcal{L}_{\exp}(C)$-definable function such that $\psi(\tau) = a$, then θ and ψ agree (and hence so do their derivatives) on an open interval containing τ.

It follows that there is a well defined function $\delta_\tau : \mathbb{R} \to \mathbb{R} : a \mapsto \frac{d\theta}{dx}(\tau)$ where $\theta : \mathbb{R} \to \mathbb{R}$ is an $\mathcal{L}_{\exp}(C)$-definable function such that $\theta(\tau) = a$. It is now routine to check that δ_τ is a derivation on the field $\mathbb{R}$, with field of constants C. Further, we also clearly have that

- $\delta_\tau(\log(a)) = \frac{\delta_\tau(a)}{a}$ for every positive $a \in \mathbb{R}$;

(and also $\delta_\tau(\exp(a)) = \delta_\tau(a)\exp(a)$ for any $a \in \mathbb{R}$ and $\delta_\tau(\tau) = 1$).

Now we have all the ingredients to prove the following result

Theorem 3.5. *Let τ and C be as above. Let $\alpha_1, \ldots, \alpha_n$ be real numbers such that $\delta_\tau(\alpha_1), \ldots, \delta_\tau(\alpha_n)$ are linearly independent over $\mathbb{Q}$. Then*

$$t.d._C C(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}) \geq n + 1.$$

Proof: In order to apply Ax's differential field version of Schanuel's Conjecture (SD), we let $y_i = \alpha_i$ and $z_i = e^{\alpha_i}$ for each $i = 1, \ldots, n$. Then

- $\delta_\tau(y_i) = \frac{\delta_\tau(z_i)}{z_i}$ for all $i = 1, \ldots, n$;
- the $\delta_\tau(y_i)$ are $\mathbb{Q}$-linearly independent.

Thus the conditions of (SD) hold and so the conclusion follows.

□

There is a complex version of Theorem 3.5. For this let τ be a real number not definable in the o-minimal structure $\mathcal{M} = \langle \mathbb{R}, +, \cdot, \exp, \sin_{\restriction[-2,2]}, <, 0, 1 \rangle$ without parameters (this structure is o-minimal since it is a reduct of the o-minimal structure $\mathbb{R}_{\mathrm{an},\exp}$).

Working in this structure, let C and $\delta_\tau : \mathbb{R} \to \mathbb{R}$ be constructed as above. Now define

$$\partial_\tau : \mathbb{C} \to \mathbb{C}$$

$$z \mapsto \delta_\tau(\mathrm{Re}z) + i\delta_\tau(\mathrm{Im}z).$$

It is now routine to check that ∂_τ is a derivation on the field $\mathbb{C}$, with field of constants $K := C(i)$, the algebraic closure of the real closed field C. Further, we also clearly have that

- $\partial_\tau(\log_{|U}(z)) = \frac{\partial_\tau(z)}{z}$ for every $z \in \mathbb{C}$ where U is a bounded open ball in $\mathbb{C}$

(and also $\partial_\tau(\exp_{|U}(z)) = \partial_\tau(z)\exp_{|U}(z)$ for any $z \in \mathbb{C}$ where U is a bounded open ball in $\mathbb{C}$ and $\partial_\tau(\tau) = 1$).

Theorem 3.6. *Let τ and K be as above. Let $\alpha_1, \ldots, \alpha_n$ be complex numbers such that $\partial_\tau(\alpha_1), \ldots, \partial_\tau(\alpha_n)$ are linearly independent over $\mathbb{Q}$. Then*

$$t.d._K K(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}) \geq n + 1.$$

Proof: In order to apply Ax's differential field version of Schanuel's Conjecture (SD), we let $y_i = \alpha_i$ and $z_i = e^{\alpha_i}$ for each $i = 1, \ldots, n$. Then

- $\partial_\tau(y_i) = \frac{\partial_\tau(z_i)}{z_i}$ for all $i = 1, \ldots, n$;
- the $\partial_\tau(y_i)$ are $\mathbb{Q}$-linearly independent.

Thus the conditions of (SD) hold and so the conclusion follows.

□

4 The main result

Here we prove our main result using the ideas and the techniques from [10] together with the versions of Schanuel's Conjecture for real and complex numbers not definable without parameters in o-minimal expansions of the real exponential field proved in the last section.

4.1 E-subring of $\mathbb{R}$ generated by τ

Below we use the notation introduced in the previous sections.

Theorem 4.1. *Let $[x]^E$ be the free E-ring generated by $\{x\}$ and let R be the E-subring of $\mathbb{R}$ generated by τ, where τ is a real number not definable in $\mathbb{R}_{\exp}$ without parameters. Then the E-morphism φ,*

$$\varphi : [x]^E \to R$$

$$x \mapsto \tau.$$

is an E-isomorphism.

Proof: We will prove that at each step the kernel of the restriction of the E-morphism φ is trivial. We use the construction of the E-free ring $[x]^E$ as $\bigcup R_k$ where R_k are the partial E-rings.
k = 0: Recall that $R_0 = \mathbb{Z}[x]$. We have $\varphi(\mathbb{Z}[x]) = \mathbb{Z}[\tau]$. We are considering τ not definable in $\mathbb{R}_{\exp}$ without parameters, so it is transcendental over $\mathbb{Q}$, hence it follows immediately that $\ker\varphi$ is trivial.
k = 1: Recall that $R_1 = \mathbb{Z}[x][t^{(x)}]$ where (x) is the ideal generated by x. We want to define the kernel at step one. From the construction of free E-ring, we have to identify the polynomials

$$P(x) \in \mathbb{Z}[x][t^{(x)}],$$

such that $P(\tau) = 0$, where τ is the real not definable in $\mathbb{R}_{\exp}$ without parameters. Let N be the highest power of τ which appears in P. We consider all powers of τ, so we have

$$\tau, \tau^2, \ldots, \tau^N.$$

The elements $\delta_\tau(\tau) = 1, \delta_\tau(\tau^2) = 2\tau, \ldots, \delta_\tau(\tau^N) = N\tau^{N-1}$ are linearly independent over $\mathbb{Q}$. Otherwise τ would be a zero of a polynomial over $\mathbb{Q}$ which would be a contradiction. So, from Theorem 3.5 we have that

$$t.d._C C(\tau, \ldots, \tau^N, e^\tau, \ldots, e^{\tau^N}) \geq N + 1.$$

From undefinability of τ, we have that it is transcendental over C, and on the other hand the elements $\tau, \tau^2, \ldots, \tau^N$, are algebraic over $C(\tau)$. So,

$$t.d._C C(\tau, \ldots, \tau^N, e^{\tau}, \ldots, e^{\tau^N}) = N + 1.$$

This implies (since $\mathbb{Q} \subseteq C$) that

$$t.d.\mathbb{Q}(\tau, \ldots, \tau^N, e^{\tau}, \ldots, e^{\tau^N}) = N + 1.$$

Hence we have $P(\tau) = 0$ if and only if the polynomial $P(x)$ is identically zero.
Inductive step. We suppose that the statement is true for $k-1$ and we prove the result for k, that is we suppose that for any polynomial

$$P(x) \in R_{k-1} = R_{k-2}[t^{B_{k-2}}],$$

$P(\tau) = 0$ if and only if P is the polynomial identically zero. Now we have to characterize the polynomials

$$P(x) \in R_k = R_{k-1}[t^{B_{k-1}}],$$

such that

$$P(\tau) = 0.$$

We define

$$\Delta_0 = \{\tau, \ldots, \tau^N\},$$
$$\Delta_1 = \{e^{\tau}, \ldots, e^{\tau^N}\}, \text{ i.e. } \Delta_1 = e^{\Delta_0}.$$

We have to distinguish two cases:

Case k even: In this case we define more generally

$$\Delta_{2j} = \Delta_0 \Delta_{2j-1} = \{\mu\delta : \mu \in \Delta_0, \delta \in \Delta_{2j-1}\}, \text{ with } j = 1, \ldots, \frac{k}{2},$$

and

$$\Delta_{2j+1} = e^{\Delta_{2j}}, \text{ with } j = 1, \ldots, \frac{k}{2} - 1.$$

Also, for a given Δ_i, we define

$$\delta_\tau \Delta_i = \{\delta_\tau(\alpha) : \alpha \in \Delta_i\}.$$

By inductive hypothesis, the collection of elements $\delta_\tau\Delta_0, \delta_\tau\Delta_1, \ldots, \delta_\tau\Delta_k$ is linearly independent over $\mathbb{Q}$. Otherwise τ would be a zero of a polynomial in R_{k-1}. So applying Theorem 3.5 we have that:

$$t.d._C C(\Delta_0, \Delta_1, \ldots, \Delta_k, e^{\Delta_0}, e^{\Delta_1}, \ldots, e^{\Delta_k}) \geq \sum_{i=0}^{k} |\Delta_i| + 1.$$

Now observe the following:

- Δ_0 is algebraically dependent over $C(\tau)$;
- $\Delta_{2j+1} = e^{\Delta_{2j}}$, for $j = 0, \dots, \frac{k}{2} - 1$, so we have some repetitions among the elements added to C;
- Δ_{2j}'s, for $j = 1, \dots, \frac{k}{2}$, are algebraically dependent over $C(\Delta_0, \Delta_1, \Delta_3, \dots, \Delta_{2j-1})$.

So we have

$$t.d._C C(\Delta_0, \Delta_1, \dots, \Delta_k, e^{\Delta_0}, e^{\Delta_1}, \dots, e^{\Delta_k}) = 1 + (\sum_{j=0} |\Delta_{2j+1}|) + (\sum_{j=0} |e^{\Delta_{2j+1}}|),$$

and this implies that

$$t.d._C C(\Delta_0, \Delta_1, \dots, \Delta_k, e^{\Delta_0}, e^{\Delta_1}, \dots, e^{\Delta_k}) = \sum_{i=0}^{k} |\Delta_i| + 1,$$

since $|\Delta_{2j}| = |e^{\Delta_{2j+1}}| = |\Delta_{2j+1}|$, and we have

$$t.d.\mathbb{Q}(\Delta_0, \Delta_1, \dots, \Delta_k, e^{\Delta_0}, e^{\Delta_1}, \dots, e^{\Delta_k}) = \sum_{i=0}^{k} |\Delta_i| + 1.$$

Thus, the identity $P(\tau) = 0$ is true if and only if the polynomial P is identically zero.

Case k odd: The proof for k odd follows the lines of the previous case for k even, but we have to pay attention to the indices.

In this case we define

$$\Delta_{2j} = \Delta_0 \Delta_{2j-1} \text{ and } \Delta_{2j+1} = e^{\Delta_{2j}}, \text{ with } j = 1, \dots, \frac{k-1}{2}.$$

Also, for a given Δ_i, we define

$$\delta_\tau \Delta_i = \{\delta_\tau(\alpha) : \alpha \in \Delta_i\}.$$

By the inductive hypothesis, the collection of elements $\delta_\tau \Delta_0, \delta_\tau \Delta_1, \dots, \delta_\tau \Delta_k$ is linearly independent over $\mathbb{Q}$. Otherwise τ would be a zero of a polynomial in R_{k-1}. So applying Theorem 3.5 we have that:

$$t.d._C C(\Delta_0, \Delta_1, \dots, \Delta_k, e^{\Delta_0}, e^{\Delta_1}, \dots, e^{\Delta_k}) \geq \sum_{i=0}^{k} |\Delta_i| + 1.$$

Now observe the following:

- Δ_0 is algebraically dependent over $C(\tau)$;
- $\Delta_{2j+1} = e^{\Delta_{2j}}$, for $1, \ldots, \frac{k-1}{2}$, so we have some repetitions among the elements added to C;
- Δ_{2j}'s, for $1, \ldots, \frac{k-1}{2}$, are algebraically dependent over $C(\Delta_0, \Delta_1, \Delta_3, \ldots, \Delta_{2j-1})$.

So we have

$$t.d._C C(\Delta_0, \Delta_1, \ldots, \Delta_k, e^{\Delta_0}, e^{\Delta_1}, \ldots, e^{\Delta_k}) = 1 + (\sum_{j=0} |\Delta_{2j+1}|) + (\sum_{j=0} |e^{\Delta_{2j+1}}|),$$

and this implies that

$$t.d._C C(\Delta_0, \Delta_1, \ldots, \Delta_k, e^{\Delta_0}, e^{\Delta_1}, \ldots, e^{\Delta_k}) = \sum_{i=0}^{k} |\Delta_i| + 1$$

since $|\Delta_{2j}| = |e^{\Delta_{2j+1}}| = |\Delta_{2j+1}|$. In particular, since $\mathbb{Q} \subseteq C$ we have

$$t.d.\mathbb{Q}(\Delta_0, \Delta_1, \ldots, \Delta_k, e^{\Delta_0}, e^{\Delta_1}, \ldots, e^{\Delta_k}) = \sum_{i=0}^{k} |\Delta_i| + 1.$$

So we have that the identity $P(\tau) = 0$ is true if and only if the polynomial P is identically zero. Now the proof is completed.

$\square$

4.2 E-subring of $\mathbb{C}$ generated by $\tau, i\tau$

Consider the o-minimal structure $\mathcal{M} = \langle \mathbb{R}, +, \cdot, \exp, \sin_{\restriction [-2,2]}, <, 0, 1 \rangle$. Then we have the following result, in whose proof we use the notation introduced in the proof of Theorem 3.6.

Theorem 4.2. *Let $[x, y]^E$ be the free E-ring generated by $\{x, y\}$. Let τ be a real number not definable in $\mathcal{M}$ without parameters. Then the E-morphism:*

$$\psi : [x, y]^E \to (\mathbb{C}, +, \cdot, \exp)$$

$$x \mapsto \tau$$

$$y \mapsto i\tau,$$

is a monomorphism.

Proof: The proof is very similar to the proof of Theorem 4.1, so we will give only the initial step.
k = 0: Recall that $R_0 = \mathbb{Z}[x, y]$. We have $\varphi(\mathbb{Z}[x, y]) = \mathbb{Z}[\tau, i\tau]$. We are considering τ a real number not definable in $\mathcal{M}$ without parameters so it is transcendental over $\mathbb{Q}$ and hence $\ker\varphi$ is trivial.
k = 1: We want to define the kernel at step one. So we recall the construction of free E-ring, we have to identify the polynomials

$$P(x, y) \in \mathbb{Z}[x, y][t^{(x,y)}],$$

such that $P(\tau, i\tau) = 0$, where (x, y) is the ideal generated by x, and y, and τ is the real not definable in $\mathcal{M}$ without parameters. Let N be the highest power of τ which appears in P, and we consider all possible monomials, both real and complex, which can be constructed from i, τ^n, with $n \leq N$. So we have

$$\tau, \dots, \tau^N, i\tau, \dots, i\tau^N.$$

The elements $\partial_\tau(\tau) = 1, \dots, \partial_\tau(\tau^N) = N\tau^{N-1}, \partial_\tau(i\tau) = i, \dots, \partial_\tau(i\tau^N) = Ni\tau^{N-1}$ are linearly independent over $\mathbb{Q}$. Otherwise τ would be a zero of a polynomial over $\mathbb{Q}$ which would be a contradiction. So from Theorem 3.6 we have that

$$t.d._K K(\tau, \dots, \tau^N, i\tau, \dots, i\tau^N, e^\tau, \dots, e^{\tau^N}, e^{i\tau}, \dots, e^{i\tau^N}) \geq 2N + 1.$$

From undefinability of τ we have that it is transcendental over K, and on the other hand the elements $\tau, \dots, \tau^N, i\tau, \dots, i\tau^N$ are algebraic over $K(\tau)$. So we have

$$t.d._K K(\tau, \dots, \tau^N, i\tau, \dots, i\tau^N, e^\tau, \dots, e^{\tau^N}, e^{i\tau}, \dots, e^{i\tau^N}) = 2N + 1,$$

and this implies (since $\mathbb{Q} \subseteq K$) that

$$t.d.\mathbb{Q}(\tau, \dots, \tau^N, i\tau, \dots, i\tau^N, e^\tau, \dots, e^{\tau^N}, e^{i\tau}, \dots, e^{i\tau^N}) = 2N + 1.$$

So we have $P(\tau, i\tau) = 0$ if and only if the polynomial $P(x, y)$ is identically zero.

For the inductive step we continue as in Theorem 4.1 using Theorem 3.6 instead of Theorem 3.5.

$\square$

Remark 4.3. More generally if we consider

$$\psi : [x, y]^E \to (\mathbb{C}, +, \cdot, \exp)$$

$$x \mapsto \alpha\tau$$

$$y \mapsto i\beta\tau,$$

where α, β are real algebraic numbers which are linearly independent over $\mathbb{Q}$ and τ, as before, then $\ker\psi$ is trivial. To see this it is enough to observe at step one that the elements $\alpha\tau, \ldots, \alpha^N\tau^N$, $\beta\tau, \ldots, \beta^N\tau^N$, $i\beta\tau, \ldots, i\beta^N\tau^N$ are linearly independent over $\mathbb{Q}$ and also their derivations are linearly independent over $\mathbb{Q}$. Thus we can apply the Theorem 3.6 and we obtain at step one that the kernel is trivial. The inductive step is the same of the previous proof.

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