Locally definable groups in o-minimal structures

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Abstract

In this paper we develop the theory of locally definable groups in o-minimal structures generalizing in this way the theory of definable groups.

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1 Introduction

Throughout this paper, $\mathcal{N}$ will be an o-minimal structure and definable means $\mathcal{N}$-definable (possibly with parameters). By definition, an o-minimal structure is a structure $\mathcal{N} = (N, <, (c)_{c\in\mathcal{C}}, (f)_{f\in\mathcal{F}}, (R)_{R\in\mathcal{R}})$ over a dense totally ordered set $(N, <)$ with no end points, where $\mathcal{C}$ is a collection of constants, $\mathcal{F}$ is a collection of functions from the cartesian products of N into N and $\mathcal{R}$ is a collection relations in the cartesian products of N, such that every definable subset of N is a finite union of points and intervals with end points in $N \cup \{-\infty, +\infty\}$. The definable sets of $\mathcal{N}$ are the subsets of the cartesian products of N whose elements satisfy a first-order logic formula in the language $\{=, <, (c)_{c\in\mathcal{C}}, (f)_{f\in\mathcal{F}}, (R)_{R\in\mathcal{R}}\}$. The first-order formulas in this language are, roughly, the formulas that one can write down using these symbols, using symbols for variables, parameters from N, the logic connectives $\wedge$ (and), $\vee$ (or) and $\neg$ (not) and the quantifiers $\forall$ (for all) and $\exists$ (there exists). For example, a real closed field $(N, <, 0, 1, +, \cdot)$ is an o-minimal structure and definable sets in this real closed field are, by the Tarski-Seidenberg theorem, the semi-algebraic sets i.e., sets of the form $\{x \in N^k : f_1(x) = \cdots = f_l(x) = 0,\ g_1(x) > 0, \dots, g_m(x) > 0\}$ where $f_1, \dots, f_l, g_1, \dots, g_m \in N[X_1, \dots, X_k]$. O-minimal structures have turned out to be a wide ranging model theoretic generalization of semi-algebraic and sub-analytic geometry. For the basic theory of o-minimal structures we refer the reader to [vdd], and for basic semi-algebraic geometry we refer to [BCR].

Given a real closed field $(N, <, 0, 1, +, \cdot)$ one often studies real algebraic groups in N and algebraic groups in the algebraic closure $N[\sqrt{-1}]$ of N. After identifying $N[\sqrt{-1}]$ with N^2 one can also study these groups in the category of semi-algebraic groups with semi-algebraic homomorphisms i.e., in the category whose objects are groups with underlying set a semi-algebraic set and group operations and group homomorphisms semi-algebraic maps. A semi-algebraic map is a map between semi-algebraic sets whose graph is a semi-algebraic set. More generally, given an arbitrary o-minimal structure $\mathcal{N}$, one can study the category of definable groups with definable homomorphisms. This is the category whose objects are groups with underlying set a definable set and group operations and group homomorphisms definable maps. A definable map is a map between definable sets whose graph is a definable set. The study of definable groups began with [p1] and has since then grown into a well developed branch of mathematics (see the references).

When studying definable groups one often makes use of certain groups which are not definable and are called in the literature $\bigvee$-definable groups (see [pst2]). Roughly, these are groups whose underlying sets are unions of definable sets and the graphs of the group operations are unions of definable sets. In a real closed field these sets, when equipped with a natural topology, are called in [dk] locally semi-algebraic spaces. For this reason, we prefer to call $\bigvee$-definable groups locally definable groups since the groups that we will study here will be equipped with a topology such that in the semi-algebraic case they are locally semi-algebraic spaces. Furthermore, as we shall see in Section 2 when we introduce the exact definitions, what we call here a locally definable group is a small modification of what is called in [pst2] a $\bigvee$-definable group. In [pst2] $\bigvee$-definable groups are defined with a restriction on the size of the parameter set and with no restriction on the size of the cover by definable subsets. Here we require that locally definable groups have a countable subcover by definable subsets. This is not a big restriction since all the important examples are of this form and this constraint allows us to prove many results which otherwise would be impossible to verify.

Let us mention a few examples where locally definable groups have occurred in connection with the theory of definable groups. In [e], we prove the Lie-Kolchin-Mal'cev theorem for solvable definable groups. This theorem says that given a solvable definable group G, the commutator subgroup $G^{(1)}$ of G and the smallest definable subgroup $d(G^{(1)})$ of G containing $G^{(1)}$ are nilpotent. The commutator subgroup $G^{(1)}$ is a locally definable subgroup of G. In [pst2], Peterzil and Starchenko show that if G is a solvable definable group which is definably compact (the o-minimal analogue of semi-algebraically complete), then G is abelian by finite. The proof of this result given in [pst2] uses the groups of definable homomorphisms between definable abelian groups. The group of definable homomorphisms between two definable abelian groups is a locally definable group. In [ps] (see also [s]), Peterzil and Steinhorn construct certain definably compact, abelian definable groups which are not the direct product of one-dimensional definably compact, abelian definable groups. In a sense, these definable groups are constructed by first giving their o-minimal universal covers and their o-minimal fundamental groups. These o-minimal universal covers and these o-minimal fundamental groups are locally definable groups. In [pps2], Peterzil, Pillay and Starchenko use locally definable groups to show that if a definable group is not nilpotent by finite, then the group structure interprets a field.

Since definable groups (e.g., semi-algebraic, real algebraic and algebraic groups in arbitrary real closed fields) are locally definable groups and as we saw above locally definable groups appear quite often attached to definable groups it is natural to ask for a complete development of the theory of locally definable groups. In this paper we develop such theory in a more systematic way continuing what was started in [pst2]. More precisely, we generalise to locally definable groups all the basic theory of definable groups from [p1], [e] and [pps1]. The proof techniques are the same but locally definable groups are more complicated. Hence, we include proofs as complete as possible in order to clarify some details which are not immediately obvious.

We now describe the structure and the main results of the paper. In Section 2, the definition of locally definable groups is introduced, examples are presented, and we prove a basic result for locally definable groups: the existence of a locally definable topological structure making the group operations and locally definable homomorphisms between such locally definable groups continuous. This result is known as property (TOP) and is proved exactly as in [pst2]. In a real closed field, a locally definable group equipped with this locally definable topology is a locally semi-algebraic space.

In Section 3, we define the notion of connectedness for locally definable groups following [pst2]. Inspired by the theory of locally semi-algebraic spaces from [dk] we introduce the notion of compatible locally definable subgroups and, we show that any locally definable group has a unique connected, compatible, locally definable subgroup of maximal dimension. We end Section 3 with the proof of the descending chain condition (DCC) for compatible locally definable subgroups.

Note that the notion of compatible locally definable subgroups is the main novelty of the paper and the crucial notion of the whole theory: there is no uniqueness of connected locally definable subgroups of a locally definable group; there is no DCC for arbitrary locally definable subgroups of a locally definable group; in general the quotient of a locally definable group by a normal locally definable subgroup is not a locally definable group and so on.

In Section 4, we prove that the quotient of a locally definable group by a compatible locally definable normal subgroup is a locally definable group such that there is a locally definable section to the locally definable quotient. This result is used to develop group extension theory and group cohomology theory in the category of locally definable groups with locally definable homomorphisms. In Section 5, we use the theory of Section 4 to describe solvable locally definable groups. More precisely, we show that any such group has a maximal, normal, definably connected definable subgroup with no definably compact parts (see [e]) whose quotient is a definably compact, locally definable solvable group. Unlike in the definable case, the definably compact, locally definable solvable groups are not necessarily abelian (see Example 5.4).

Section 6 contains some basic results about locally definable transitive actions of locally definable groups on locally definable sets. These facts will be used in Section 7 where centerless locally definable groups with no compatible locally definable normal abelian subgroups of positive dimension (i.e., centerless locally definably semi-simple locally definable groups) are shown to be locally definable open and closed subgroups of definably semi-simple definable groups. We also show that centerless, connected locally definable solvable groups are locally definable open and closed subgroups of definably connected definable solvable groups.

We end the paper with Section 8 where we include some applications of our previous results: the existence of strong definable choice for locally definable groups and the existence of compatible locally definable abelian subgroups of positive dimension of locally definable groups of positive dimension (this is known as property (AB)). Furthermore, we also prove that if there is a connected locally definable group which is not nilpotent, then a real closed field in definable in $\mathcal{N}$.

Some other results on definable groups that could be proved also for locally definable groups, such as the Lie-Kolchin-Mal'cev theorem, were omitted to avoid making the paper too long. Similarly, we do not treat here the theory of locally definable rings.

In this paper we will see two main examples of locally definable groups: (i) the locally definable groups of dimension zero and (ii) the locally definable groups which are the subgroups of (type) definable groups. For details see Example 2.2 below. It is an open question if these are the only building blocks of locally definable groups. Our work here reduces this question to the case of connected, definably compact, locally definable abelian groups.

Finally we point out that there are two properties of definable abelian groups, property (TOR) and (DIV), whose analogue for locally definable abelian groups we were unable to prove or disprove here. Namely, we do not know if for every connected locally definable abelian group over A, the torsion points are defined over A and, if every connected locally definable abelian group is divisible.

2 Locally definable groups

Here the definition of locally definable groups is introduced and examples are presented. The main result is property (TOP) for locally definable groups and the property of large locally definable subsets of locally definable groups.

2.1 Locally definable groups

Recall that, by [pst2], if $\mathcal{N}$ is $\aleph_1$-saturated, then a group $\mathcal{Z} = (\mathcal{Z}, \cdot)$ is a $\bigvee$-*definable group* over $A \subseteq N$ where $|A| < \aleph_1$, if there is a collection $\{Z_i : i \in I\}$ of definable subsets of N^n, all definable over A such that: (i) $\mathcal{Z} = \cup\{Z_i : i \in I\}$; (ii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$ and (iii) the restriction of the group multiplication to $Z_i \times Z_j$ is a definable map into N^n.

We modify this definition slightly in the following way.

Definition 2.1 Assume that $\mathcal{N}$ is $\aleph_1$-saturated. A group $(\mathcal{Z}, \cdot)$ is a *locally definable group over* A with $A \subseteq N$ and $|A| < \aleph_1$ if there is a collection $\{Z_i : i \in I\}$ of definable subsets of N^n, all definable over A such that: (i) $\mathcal{Z} = \cup\{Z_i : i \in I\}$; (ii) there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $\mathcal{Z} = \cup\{Z_i : i \in I_0\}$; (iii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$ and (iv) the restriction of the group multiplication to $Z_i \times Z_j$ is a definable map into N^n.

As in the $\bigvee$-definable case, if $\mathcal{Z}$ is a locally definable group over A as above and $\mathcal{Z} = \cup\{Y_j : j \in J\}$ with each Y_j definable over B, $|B| < \aleph_1$, then by saturation the following hold: (i) every Y_j is contained in some Z_i and (ii) there is $J_0 \subseteq J$ with $|J_0| < \aleph_1$ and $\mathcal{Z} = \cup\{Y_j : j \in J_0\}$. *For this reason we will always assume from now on that* $|I| < \aleph_1$. Note however that this assumption will not be necessary until Section 4.

Given $\mathcal{M}$ an $\aleph_1$-saturated elementary extension of $\mathcal{N}$, then $\mathcal{Z}(M) = \cup\{Z_i(M) : i \in I\}$ is also a locally definable group over A. Moreover, if $\mathcal{Z}$ is a definable set, then $(\mathcal{Z}, \cdot)$ is a definable group.

We will assume from now on that $\mathcal{N}$ *is an* $\aleph_1$*-saturated o-minimal structure.*

If $\mathcal{Z} = \cup\{Z_i : i \in I\}$ is a locally definable group over A, we define the *dimension* of $\mathcal{Z}$ by $\dim \mathcal{Z} = \max\{\dim Z_i : i \in I\}$ and we say that $z \in \mathcal{Z}$ is *generic (over A)* if $\dim(z/A) = \dim \mathcal{Z}$. For the notion of dimension of a definable set see [vdd] or [p1]; for the notion of dimension of an element over a set of parameters see [p1] or [pst2].

Example 2.2 The following are the two main examples of locally definable groups over A, with $A \subseteq N$ and $|A| < \aleph_1$.

(1) The locally definable groups over A of dimension zero: Let $\{Z_i : i \in I\}$ be a collection of finite subsets of N^k all of which are defined over A such that for all $i, j \in I$ there is $k \in I$ with $Z_i \cup Z_j \subseteq Z_k$ and $(\mathcal{Z}, \cdot)$ is an abstract group, where $\mathcal{Z} = \cup\{Z_i : i \in I\}$, and there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $\mathcal{Z} = \cup\{Z_i : i \in I_0\}$. Then $(\mathcal{Z}, \cdot)$ is a locally definable group over A of dimension zero.

(2) The locally definable groups over A which are the subgroups of (type) definable groups: Let $(G, \cdot)$ be a (type) definable group over $B \subseteq A$; let $\{Z_i : i \in I\}$ be a collection of definable subsets of G all of which are defined over A such that for all $i, j \in I$ there is $k \in I$ with $Z_i \cup Z_j \subseteq Z_k$, $(\mathcal{Z}, \cdot)$ is a subgroup of $(G, \cdot)$, where $\mathcal{Z} = \cup\{Z_i : i \in I\}$, and there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $\mathcal{Z} = \cup\{Z_i : i \in I_0\}$. Then $(\mathcal{Z}, \cdot)$ is a locally definable group over A.

The proof of [pst2] Proposition 2.2 also shows the following theorem.

Theorem 2.3 *Let $\mathcal{Z} \subseteq N^k$ be a locally definable group over A. Then there is a uniformly definable family $\{V_s : s \in S\}$ of definable subsets of $\mathcal{Z}$ defined over A and containing the identity element of $\mathcal{Z}$ and there is a unique topology τ on $\mathcal{Z}$ such that: (i) $\{V_s : s \in S\}$ is a basis for the τ-open neighbourhoods of the identity element of $\mathcal{Z}$; (ii) $(\mathcal{Z}, \tau)$ is a topological group and (iii) every generic element of $\mathcal{Z}$ has an open definable neighbourhood $U \subseteq N^k$ such that $U \cap \mathcal{Z}$ is τ-open and the topology which $U \cap \mathcal{Z}$ inherits from τ agrees with the topology it inherits from N^k.*

In Theorem 2.3, by a uniformly definable family $\{V_s : s \in S\}$ of definable subsets of $\mathcal{Z}$ defined over A we mean that S is definable over A and there is a definable subset of $N^k \times S$ over A such that the fiber over s is V_s for each $s \in S$.

As in [pst2] Lemma 2.6 we see that the following result holds.

Theorem 2.4 *Let $\mathcal{Z}$ be a locally definable group over A and $\mathcal{W}$ a locally definable subgroup of $\mathcal{Z}$ over A. Then the following holds: (i) the τ-topology on $\mathcal{W}$ is the subspace topology induced by the τ-topology on $\mathcal{Z}$; (ii) $\mathcal{W}$ is closed in $\mathcal{Z}$ in the τ-topology and (iii) $\mathcal{W}$ is open in $\mathcal{Z}$ in the τ-topology if and only if* $\dim \mathcal{W} = \dim \mathcal{Z}$.

The proof of Theorem 2.3 gives more information which we single out in the following corollary. This will be used in Section 7.

Corollary 2.5 *Let $\mathcal{Z}$ be a locally definable group over A and let $\{V_s : s \in S\}$ be the basis for the τ-open neighbourhoods of the identity element of $\mathcal{Z}$. Then we can choose $\{V_s : s \in S\}$ such that there is a uniformly definable family $\{\phi_s : s \in S\}$ of definable homeomorphisms $\phi_s : V_s \longrightarrow U_s$ where U_s is an open definable subset of N^m and m is the dimension of $\mathcal{Z}$.*

Proof. Let $\{Z_i : i \in I\}$ be as in Definition 2.1. Fix Z_i containing a generic g of $\mathcal{Z}$ over A. By the proof of Theorem 2.3 (see [pst2] Proposition 2.2), each V_s is of the form $g^{-1} \cdot (W_s \cap Z_i)$ where $\{W_s : s \in S\}$ is a uniformly definable basis for the open neighbourhoods of g in the standard topology on N^k with k such that $\mathcal{Z} \subseteq N^k$. Moreover, the τ-topology is independent of the choice of g and Z_i such that $g \in Z_i$. Thus we may replace Z_i by a cell of dimension m of a cell decomposition of Z_i and so, by [vdd] Chapter III (2.7), there is a definable homeomorphism $\phi_i : Z_i \longrightarrow U_i$ over A where U_i is an open definable subset of N^m.

To finish define ϕ_s by $\phi_s(v) = \phi_{i|W_s \cap Z_i}(gv)$ and use Theorem 2.3 to conclude that ϕ_s is a definable homeomorphism. $\square$

A homomorphism $\alpha : \mathcal{Z} \longrightarrow \mathcal{X}$ between locally definable groups over A is called a *locally definable homomorphism over* A if for every definable subset $Z \subseteq \mathcal{Z}$ defined over A, the restriction $\alpha_{|Z}$ is a definable map over A. The following remark is easy to show.

Remark 2.6 Let $\alpha : \mathcal{Z} \longrightarrow \mathcal{X}$ be a locally definable homomorphism over A between locally definable groups over A and let $\mathcal{Y}$ be a locally definable subgroup of $\mathcal{X}$ over A. Then $\alpha(\mathcal{Z})$ is a locally definable group over A and $\alpha^{-1}(\mathcal{Y})$ is a locally definable subgroup of $\mathcal{Z}$ over A.

The proof of [pst2] Lemma 2.8 gives the following theorem.

Theorem 2.7 *Any locally definable homomorphism between locally definable groups is a continuous locally definable homomorphism with respect to the τ-topology.*

Theorems 2.3, 2.4 and 2.7 will be called property (TOP) for locally definable groups since they generalise the corresponding property for definable groups.

From now on, whenever we use topological notions on a locally definable group, we are referring to the τ-topology.

2.2 Large locally definable subsets

Definition 2.8 A set $\mathcal{Z}$ is a *locally definable set over* A where $A \subseteq N$ and $|A| < \aleph_1$ if there is a collection $\{Z_i : i \in I\}$ of definable subsets of N^n, all definable over A such that: (i) $\mathcal{Z} = \cup\{Z_i : i \in I\}$; (ii) there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $\mathcal{Z} = \cup\{Z_i : i \in I_0\}$; (iii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$.

A map $\alpha : \mathcal{Z} \longrightarrow \mathcal{X}$ between locally definable sets over A is called a *locally definable map over* A if for every definable subset $Z \subseteq \mathcal{Z}$ defined over A, the restriction $\alpha_{|Z}$ is a definable map over A.

By saturation, if $\mathcal{Z}$ is a locally definable set over A as above and $\mathcal{Z} = \cup\{Y_j : j \in J\}$ with each Y_j definable over B, $|B| < \aleph_1$, then the following hold: (i) every Y_j is contained in some Z_i and (ii) there is $J_0 \subseteq J$ with $|J_0| < \aleph_1$ and $\mathcal{Z} = \cup\{Y_j : j \in J_0\}$. *For this reason we will always assume from now on that* $|I| < \aleph_1$. Note that as in the case of locally definable groups, this assumption will not be necessary until Section 4.

Also, if $\mathcal{M}$ is an $\aleph_1$-saturated elementary extension of $\mathcal{N}$, then $\mathcal{Z}(M) = \cup\{Z_i(M) : i \in I\}$ is also a locally definable set over A.

Remark 2.9 Let $\alpha : \mathcal{Z} \longrightarrow \mathcal{X}$ be a locally definable map over A between locally definable sets over A and let $\mathcal{Y}$ be a locally definable subset of $\mathcal{X}$ over A. Then $\alpha(\mathcal{Z})$ is a locally definable set over A and $\alpha^{-1}(\mathcal{Y})$ is a locally definable subset of $\mathcal{Z}$ over A.

If $\mathcal{Z} = \cup\{Z_i : i \in I\}$ is a locally definable set over A, we define the *dimension* of $\mathcal{Z}$ by $\dim \mathcal{Z} = \max\{\dim Z_i : i \in I\}$ and we say that $z \in \mathcal{Z}$ is *generic (over A)* if $\dim(z/A) = \dim \mathcal{Z}$.

Definition 2.10 We say that a locally definable set $\mathcal{V}$ over A is a *large* locally definable subset over A of a locally definable set $\mathcal{X}$ over A if every generic point of $\mathcal{X}$ over A belongs to $\mathcal{V}$.

The next result is proved in the same way as its definable analogue in [p1] Lemma 2.4.

Proposition 2.11 *Let $\mathcal{X}$ be a locally definable group over A. If $\mathcal{V}$ is a large locally definable subset of $\mathcal{X}$ over A, then there is a locally definable subset $\{x_s : s \in S\}$ of $\mathcal{X}$ over A such that $\mathcal{X} = \cup\{x_s\mathcal{V} : s \in S\}$.*

Proof. Let $\mathcal{K}$ be the prime model of $\mathrm{Th}_A(\mathcal{N})$ and suppose that $\mathcal{X} = \cup\{X_i : i \in I\}$ and $\mathcal{V} = \cup\{V_j : j \in J\}$. Let $i \in I$, $a \in X_i$ and let $c \in X_i$ be a generic point of $\mathcal{X}$ over K such that $\mathrm{tp}(c/Ka)$ is finitely satisfiable in K. Then c is a generic point of $\mathcal{X}$ over Ka (see the proof of [p1] Lemma 2.4). Note that, since $\{X_i :\in I\}$ is a directed system, for every $a \in \mathcal{X}$, there is $i \in I$ such that $a \in X_i$ and there is $c \in X_i$ generic of $\mathcal{X}$ over K with $\mathrm{tp}(c/Ka)$ is finitely satisfiable in K.

Since $\mathcal{V}$ is a large locally definable subset of $\mathcal{X}$ over A, the set $\mathcal{V}a^{-1}$ is a large locally definable subset of $\mathcal{X}$ over $A \cup \{a\}$. Therefore, by definition, $c \in V_j a^{-1}$ and $a \in c^{-1}V_j$ for some $j \in J$. Since $\mathrm{tp}(c/Ka)$ is finitely satisfiable over K, there is $b \in X_i(K)$ such that $a \in b^{-1}V_j$ for some $j \in J$. Therefore, by the compactness theorem, for each $i \in I$, there are $b_1, \ldots, b_{r_i} \in X_i(K)$ and $j_1, \ldots, j_{r_i} \in J$ such that for every $a \in X_i$, $a \in \cup\{(b_l)^{-1}V_{j_l} : l = 1, \ldots, r_i\}$.

Let $S = \{(i,l) : i \in I,\ l = 1, \ldots, r_i\}$ and for $s = (i,l) \in S$, let x_s be the element $(b_l)^{-1}$ with $b_l \in X_i(K)$ as above. Then $\mathcal{X} = \cup\{x_s\mathcal{V} : s \in S\}$. Also, $\{x_s : s \in S\}$ is a locally definable subset of $\mathcal{X}$ over A since each x_s is defined over A and $\{x_s : s \in S\}$ is the union of the collection of all finite subsets of $\{x_s : s \in S\}$. □

Corollary 2.12 *Let $\mathcal{X}$ be a locally definable group over A. If $\{V_j : j \in J\}$ is the collection of all open definable subsets of $\mathcal{X}$ over A, then $\mathcal{X} = \cup\{V_j : j \in J\}$.*

Proof. Suppose that $\mathcal{X} = \cup\{X_i : i \in I\}$ and, by Theorem 2.3, let $\{V_s : s \in S\}$ be the uniformly definable basis for the τ-open neighbourhoods of the identity element of $\mathcal{X}$. For each $i \in I$, the set Y_i of all $x \in X_i$ such that there is $s \in S$ with $xV_s \subseteq X_i$ is a definable open subset of X_i over A. Furthermore, by Theorem 2.3, every generic point of $\mathcal{X}$ over A belongs to some Y_i. Thus $\mathcal{Y} = \cup\{Y_i : i \in I\}$ is a large locally definable subset of $\mathcal{X}$ over A. By Proposition 2.11, there is a locally definable subset $\{x_s : s \in S\}$ of $\mathcal{X}$ over A such that $\mathcal{X} = \cup\{x_sY_i : s \in S,\ i \in I\} \subseteq \cup\{V_j : j \in J\}$. □

Corollary 2.12 implies that in the semi-algebraic case every locally definable group is a locally semi-algebraic space.

3 The descending chain condition

Here we introduce the notion of compatible locally definable subgroups of locally definable groups. The main results are the existence and uniqueness of the compatible connected component of a locally definable group and the descending chain condition for compatible locally definable subgroups.

3.1 Connectedness

Definition 3.1 Let $\mathcal{X}$ be a locally definable group over A and let $\mathcal{Z}$ be a locally definable subgroup (resp., subset) of $\mathcal{X}$ over A. We say that $\mathcal{Z}$ is a *compatible locally definable subgroup (resp., subset)* if for every open definable subset X of $\mathcal{X}$ over A, the set $\mathcal{Z} \cap X$ is a definable subset of $\mathcal{X}$ over A.

For example, if $\mathcal{Z}$ is a definable subgroup (resp., subset) of $\mathcal{X}$ over A, then $\mathcal{Z}$ is a compatible locally definable subgroup (resp., subset) of $\mathcal{X}$.

Lemma 3.2 *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be locally definable groups over A. The following hold:*

(i) $\mathcal{X}$ is a compatible locally definable subgroup of $\mathcal{X}$ over A;

(ii) if $\mathcal{Z}$ is a compatible locally definable subgroup of $\mathcal{X}$ over A and $\mathcal{Y}$ is a locally definable subgroup of $\mathcal{X}$ over A containing $\mathcal{Z}$, then $\mathcal{Z}$ is a compatible locally definable subgroup of $\mathcal{Y}$ over A;

(iii) if $\mathcal{Z}$ is a compatible locally definable subgroup of $\mathcal{Y}$ over A and $\mathcal{Y}$ is a compatible locally definable subgroup of $\mathcal{X}$ over A, then $\mathcal{Z}$ is a compatible locally definable subgroup of $\mathcal{X}$ over A.

Proof. (i) is obvious. For (ii), let U be an open definable subset of $\mathcal{Y}$ over A. Then, by Corollary 2.12 and first-order logic compactness theorem, there is an open definable subset V of $\mathcal{X}$ over A such that $U \subseteq V$. But then $U \cap \mathcal{Z} = U \cap (V \cap \mathcal{Z})$ is definable over A. For (iii), let U be an open definable subset of $\mathcal{X}$ over A. Then $U \cap \mathcal{Y}$ is an open definable subset of $\mathcal{Y}$ over A. Hence $U \cap \mathcal{Z} = (U \cap \mathcal{Y}) \cap \mathcal{Z}$ is definable over A. $\Box$

Lemma 3.3 *Let $\alpha : \mathcal{Z} \longrightarrow \mathcal{X}$ be a locally definable map over A between locally definable groups over A. If $\mathcal{Y}$ is a compatible locally definable subset of $\mathcal{X}$ over A, then $\alpha^{-1}(\mathcal{Y})$ is a compatible locally definable subset of $\mathcal{Z}$ over A.*

Proof. Let Z be an open definable subset of $\mathcal{Z}$ over A. Then $\alpha(Z)$ is a definable subset of $\mathcal{X}$ over A and, by Corollary 2.12 and first-order logic compactness theorem, there is an open definable subset X of $\mathcal{X}$ over A such that $\alpha(Z) \subseteq X$. But clearly $Z \cap \alpha^{-1}(\mathcal{Y}) = \alpha_{|Z}^{-1}(X \cap \mathcal{Y})$. Thus, since $\mathcal{Y}$ is compatible, $X \cap \mathcal{Y}$ is definable. Hence $\alpha_{|Z}^{-1}(X \cap \mathcal{Y})$ is definable since $\alpha_{|Z}$ is definable. $\Box$

Lemmas 3.2 and 3.3 will be used quite often in the paper without mentioning it. Of course the analogue of this lemma for images under locally definable maps fails: let $\alpha : \mathcal{Z} \longrightarrow \mathcal{X}$ be the inclusion map where $\mathcal{Z} = \{z \in N :$ there is $n \in \mathbb{N}$ such that $-n < z < n\}$, $\mathcal{X} = (N, +)$ and $\mathcal{N}$ is an $\aleph_1$-saturated model of the theory of the ordered additive group of real numbers.

Lemma 3.4 *Let $\mathcal{X}$ be a locally definable group over A. If $\mathcal{Z}$ is a compatible locally definable subgroup of $\mathcal{X}$ over A and X an open definable subset of $\mathcal{X}$ over A, then the equivalence relation on X given by $x \simeq y$ if and only if $x\mathcal{Z} = y\mathcal{Z}$ is definable over A.*

Proof. Let $\theta : X \times X \longrightarrow \theta(X \times X)$ be the map given by $\theta(x, y) = x^{-1}y$. Then, by definition of locally definable groups and saturation, θ is a definable map over A and $\theta(X \times X)$ is an open definable subset over A. Since $\mathcal{Z}$ is a compatible locally definable subgroup of $\mathcal{X}$ over A, the set $Z = \theta(X \times X) \cap \mathcal{Z}$ is a definable subset of $\mathcal{Z}$ over A. But, for all $x \in X$, we have $x\mathcal{Z} \cap X = xZ \cap X$. Thus the equivalence relation on X given by $x \simeq y$ if and only if $x\mathcal{Z} = y\mathcal{Z}$ is definable since $x \simeq y$ if and only if there is $z \in Z$ such that $y = xz$. □

The next result is the generalization of [pst2] Lemma 2.15 (i).

Proposition 3.5 *Let $\mathcal{Z}$ be a locally definable group over A and let $\mathcal{W}$ be a compatible locally definable subgroup of $\mathcal{Z}$ over A. Then the following are equivalent: (i) $\mathcal{W}$ is open in $\mathcal{Z}$; (ii) $\dim \mathcal{W} = \dim \mathcal{Z}$ and (iii) $(\mathcal{Z} : \mathcal{W}) < \aleph_1$.*

Proof. By Theorem 2.4, $\mathcal{W}$ is open in $\mathcal{Z}$ if and only if $\dim \mathcal{W} = \dim \mathcal{Z}$. On the other hand, if $(\mathcal{Z} : \mathcal{W}) < \aleph_1$, then by compactness we clearly have $\dim \mathcal{W} = \dim \mathcal{Z}$.

Suppose that $\dim \mathcal{W} = \dim \mathcal{Z}$. We must show that $(\mathcal{Z} : \mathcal{W}) < \aleph_1$, i.e., we must show that there is a locally definable subset $\{z_s : s \in S\}$ of $\mathcal{Z}$ over A such that $\mathcal{Z} = \cup\{z_s\mathcal{W} : s \in S\}$. To start with we have $\mathcal{Z} = \cup\{z\mathcal{W} : z \in \mathcal{Z}\}$.

Let Z be an open definable subset of $\mathcal{Z}$ over A. We must show that Z is covered by finitely many cosets of $\mathcal{W}$ all defined over A. By Lemma 3.4, the equivalence relation on Z given by $x \simeq y$ if and only if $x\mathcal{W} = y\mathcal{W}$ is definable over A. But since $x\mathcal{W} = y\mathcal{W}$ if and only if $x\mathcal{W} \cap Z = y\mathcal{W} \cap Z$, we see that the equivalence classes of $\simeq$ in Z have dimension $\dim \mathcal{W} = \dim \mathcal{Z}$. Therefore, there are finitely many equivalence classes of $\simeq$ in Z for otherwise, by [vdd] Chapter IV (1.5), the definable set Z would have dimension greater than $\dim \mathcal{Z}$, which is a contradiction. So there are finitely many elements $u_1, \dots, u_{r_Z}$ of Z defined over A such that $Z \subseteq \cup\{u_l\mathcal{W} : l = 1, \dots, r_Z\}$.

Let $\{V_j : j \in J\}$ be the collection of all open definable subsets of $\mathcal{Z}$ over A. Let $S = \{(j, l) : j \in J,\ l = 1, \dots, r_{V_j}\}$ and for $s = (j, l) \in S$, let z_s be the element u_l obtained as above with $Z = V_j$. Then by Corollary 2.12, $\mathcal{Z} = \cup\{z_s\mathcal{W} : s \in S\}$. Also $\{z_s : s \in S\}$ is a locally definable subset of $\mathcal{Z}$ over A since each z_s is defined over A and $\{z_s : s \in S\}$ is the union of the collection of all finite subsets of $\{z_s : s \in S\}$. □

The following corollary of the proof of Proposition 3.5 will be used quite often.

Corollary 3.6 *Let $\mathcal{Z}$ be a locally definable group over A and let $\mathcal{W}$ be a compatible locally definable subgroup of $\mathcal{Z}$ over A. If $(\mathcal{Z} : \mathcal{W}) < \aleph_1$, then there is a locally definable subset $\{z_s : s \in S\}$ of $\mathcal{Z}$ over A such that $\mathcal{Z} = \cup\{z_s\mathcal{W} : s \in S\}$ (disjoint union).*

The following definition is the analogue of [pst2] Definition 2.12.

Definition 3.7 Let $\mathcal{Z}$ be a locally definable group over A. We say that a set $Z \subseteq \mathcal{Z}$ is *connected* if there is no definable subset $U \subseteq \mathcal{Z}$ over A such that $U \cap Z$ is a nonempty proper subset of Z which is closed and open in the topology induced on Z by $\mathcal{Z}$.

The next remark can be proved in exactly the same way as [pst2] Lemmas 2.13 and 2.14.

Remark 3.8 Let $\mathcal{Z}$ be a locally definable group over A. Then the following hold:

(1) Every definable open subset $Z \subseteq \mathcal{Z}$ over A can be partitioned into finitely many connected definable subsets of $\mathcal{Z}$ over A.

(2) There is a locally definable subgroup $\mathcal{Z}'$ of $\mathcal{Z}$ over A which is connected and such that $\dim \mathcal{Z}' = \dim \mathcal{Z}$.

As pointed out in [pst2], the connected locally definable subgroups given by Remark 3.8 (2) are not unique. In fact, let $\mathcal{N}$ be a non standard model of the theory of the ordered additive group of real numbers, $\mathcal{Z} = (N^2, +)$, $\mathcal{Z}' = \{(x, y) \in N^2$: there exists $n \in \mathbb{N}$ such that $-n < x < n\}$ and $\mathcal{Z}'' = \{(x, y) \in N^2$: there exists $n \in \mathbb{N}$ such that $-n < y < n\}$. Then $\mathcal{Z}'$ and $\mathcal{Z}''$ are two distinct connected locally definable subgroups of $\mathcal{Z}$ over $\mathbb{Z}$.

Nevertheless, we have the following generalization of [pst2] Lemma 2.15 (iii).

Proposition 3.9 *Let $\mathcal{Z}$ be a locally definable group over A. Then there is a unique connected compatible locally definable normal subgroup $\mathcal{Z}^0$ of $\mathcal{Z}$ over A with dimension* $\dim \mathcal{Z}$. *Moreover, the following hold: (i) $\mathcal{Z}^0$ contains all connected locally definable subgroups of $\mathcal{Z}$ over A and (ii) $\mathcal{Z}^0$ is the smallest compatible locally definable subgroup of $\mathcal{Z}$ over A such that $(\mathcal{Z} : \mathcal{Z}^0) < \aleph_1$.*

Proof. Let $\{Z_k : k \in K\}$ be the collection of all open definable subsets of $\mathcal{Z}$ over A. By Corollary 2.12 and definition of locally definable groups, we may assume that each Z_k contains the identity 1 of $\mathcal{Z}$ and $\mathcal{Z} = \cup\{Z_k : k \in K\}$. By Remark 3.8 (1), each Z_k can be partitioned into finitely many connected components. For each such Z_k, let Z_k^0 be the connected component of Z_k which contains 1.

We claim that $\mathcal{Z}^0 = \cup\{Z_k^0 : k \in K\}$ is a compatible locally definable subgroup of $\mathcal{Z}$ over A. Indeed, given $i, j \in K$, we have $Z_i \cup Z_j \subseteq Z_k$ for some $k \in K$, hence $Z_i^0 \cup Z_j^0 \subseteq Z_k$. But $Z_i^0 \cup Z_j^0$ is a connected set which contains 1, hence it must be contained in Z_k^0. Similarly, $Z_i^0 \cdot Z_j^0$ and $(Z_i^0)^{-1}$ are contained in some Z_k^0. Thus $\mathcal{Z}^0$ is a locally definable subgroup of $\mathcal{Z}$ over A which, by construction, is obviously compatible, connected and $\dim \mathcal{Z}^0 = \dim \mathcal{Z}$.

By Proposition 3.5, we have $(\mathcal{Z} : \mathcal{Z}^0) < \aleph_1$ and so, by Corollary 3.6, $\mathcal{Z} = \cup\{z_s\mathcal{Z}^0 : s \in S\}$ (disjoint union) for some locally definable subset $\{z_s : s \in S\}$ of $\mathcal{Z}$ over A. Thus to show that $\mathcal{Z}^0$ is normal, it is enough to show that for each z_s with $s \in S$, $z_s\mathcal{Z}^0(z_s)^{-1} = \mathcal{Z}^0$. But this is obvious since, for every Z_i^0, the definably connected definable set $z_sZ_i^0(z_s)^{-1}$ over A is of the form Z_j^0.

As $\mathcal{Z} = \cup\{z_s\mathcal{Z}^0 : s \in S\}$ (disjoint union) for some locally definable subset $\{z_s : s \in S\}$ of $\mathcal{Z}$ over A, we see that $\mathcal{Z}^0$ contains all connected locally definable subgroups of $\mathcal{Z}$ over A.

By Proposition 3.5, if $\mathcal{W}$ is a compatible locally definable subgroup of $\mathcal{Z}$ such that $(\mathcal{Z} : \mathcal{W}) < \aleph_1$, then $\dim \mathcal{W} = \dim \mathcal{Z}$. Let $\mathcal{W}^0$ be the compatible, connected locally definable subgroup of $\mathcal{W}$ over A such that $\dim \mathcal{W}^0 = \dim \mathcal{W}$, obtained from $\mathcal{W}$ in the same way as we obtained $\mathcal{Z}^0$ from $\mathcal{Z}$. Then, by Lemma 3.2 (iii), $\mathcal{W}^0$ is a compatible connected locally definable subgroup of $\mathcal{Z}$ over A such that $\dim \mathcal{W}^0 = \dim \mathcal{Z}$ and so, by (i), $\mathcal{W}^0 \subset \mathcal{Z}^0$. Hence, by Proposition 3.5, $\mathcal{W}^0$ is open in $\mathcal{Z}^0$. Therefore, again by Proposition 3.5, we have $(\mathcal{Z}^0 : \mathcal{W}^0) < \aleph_1$ and so, by Corollary 3.6, $\mathcal{Z}^0 = \cup\{z_s\mathcal{W}^0 : s \in S\}$ (disjoint union) for some locally definable subset $\{z_s : s \in S\}$ of $\mathcal{Z}^0$ over A. But both $\mathcal{W}^0$ and $\mathcal{Z}^0$ are connected, so $|S| = 1$, $\mathcal{W}^0 = \mathcal{Z}^0$ and $\mathcal{Z}^0 \subseteq \mathcal{W}$. □

Corollary 3.10 *Let $\mathcal{Z}$ be a locally definable group over A and suppose that Z is a definably connected definable group over A which is a subgroup of $\mathcal{Z}$ and* $\dim Z = \dim \mathcal{Z}$. *Then $Z = \mathcal{Z}^0$.*

Proof. Note that $Z \subseteq \mathcal{Z}$ is connected in the sense of Definition 3.7. Thus, by Proposition 3.9, $Z \subseteq \mathcal{Z}^0$. Since by Proposition 3.5 $(\mathcal{Z} : Z) < \aleph_1$, by Proposition 3.9 again, $\mathcal{Z}^0 \subseteq Z$. So we must have $Z = \mathcal{Z}^0$. □

Definition 3.11 Let $\mathcal{X}$ be a locally definable group over A. Suppose that Z is a definable subset of $\mathcal{X}$ over A. We say that Z is *indecomposable* if for every locally definable subgroup $\mathcal{Y}$ of $\mathcal{X}$ over A, the condition $Z \subseteq x_1\mathcal{Y} \cup \ldots \cup x_n\mathcal{Y}$ implies that $Z \subseteq x_i\mathcal{Y}$ for some i.

Part of the next result, namely the part about the definable subset Z, is proved in [pps2] Theorem 2.4 for $\mathcal{X}$ a definable group. Clearly, the same proof holds if $\mathcal{X}$ is a locally definable group. This part of the result, is considered in [pps2] as the o-minimal analogue of the Zilber's indecomposability theorem. Our version here is slightly stronger because of the indecomposability assumption and is more similar to the Zilber's indecomposability theorem.

Proposition 3.12 *If $\mathcal{X}$ is a locally definable group over A and $\{Z_s : s \in S\}$ with $|S| < \aleph_1$ is a collection of indecomposable definable subsets of $\mathcal{X}$ over A containing* 1*, then the locally definable subgroup $\mathcal{Z}$ of $\mathcal{X}$ over A generated by $\{Z_s : s \in S\}$ is a connected locally definable subgroup of $\mathcal{X}$ over A and there are $\alpha_1, \ldots, \alpha_m \in S$ and $\epsilon_1, \ldots, \epsilon_m \in \{-1, 1\}$ such that the definable set $Z = Z_{\alpha_1}^{\epsilon_1} \cdots Z_{\alpha_m}^{\epsilon_m}$ contains an open definable neighbourhood of* 1 *in $\mathcal{Z}$ over A.*

Proof. For $s \in S$, let $\mathcal{Z}_s$ be the locally definable subgroup of $\mathcal{X}$ over A generated by Z_s. We will show that $\mathcal{Z}_s$ is connected. In fact, since $\mathcal{Z}_s = \{z_r\mathcal{Z}_s^0 : r \in R_s\}$ (disjoint union) with $|R_s| < \aleph_1$, it follows that there is a finite subset R'_s of R_s such that $Z_s = \cup\{z_r\mathcal{Z}_s^0 : r \in R'_s\}$ (disjoint union). But Z_s is indecomposable and contains 1. Hence $Z_s \subseteq \mathcal{Z}_s^0$ and $\mathcal{Z}_s \subseteq \mathcal{Z}_s^0$. This proves that $\mathcal{Z}_s$ is a connected locally definable subgroup of $\mathcal{X}$ over A. By Proposition 3.9 we have $\mathcal{Z}_s \subseteq \mathcal{Z}^0$ for all $s \in S$. But this implies that $\mathcal{Z}$ is connected. The rest, as we mentioned above, is the same as [pps2] Theorem 2.4. □

3.2 The descending chain condition

We do not have a general descending chain condition (DCC) for locally definable subgroups. However, we have DCC for compatible locally definable subgroups.

Proposition 3.13 *Let $\mathcal{Z}$ be a locally definable group over A. If $\{\mathcal{Z}^s : s \in S\}$ is a decreasing sequence of compatible locally definable subgroups of $\mathcal{Z}$ over B with $A \subseteq B$, then $\cap\{\mathcal{Z}^s : s \in S\} = \cap\{\mathcal{Z}^s : s \in S_0\}$ for some $S_0 \subseteq S$ with $|S_0| < \aleph_1$ and this intersection is a compatible locally definable subgroup of $\mathcal{Z}$ over B.*

Proof. For each $s \in S$, let $k_s = \dim \mathcal{Z}^s$. Since $\{k_s : s \in S\} \subseteq \{0, \ldots, \dim \mathcal{Z}\}$, there are $k_1 < \ldots < k_m$ in $\{0, \ldots, \dim \mathcal{Z}\}$ and there are disjoint subsets $S_1, \ldots, S_m$ of S such that $S = S_1 \cup \cdots \cup S_m$ and for each $l \in \{0, \ldots, m\}$, if $s \in S_l$ then $\dim \mathcal{Z}^s = k_l$. Therefore, since we want to determine $\cap\{\mathcal{Z}^s : s \in S\}$, we may assume without loss of generality that for all $s \in S$, we have $\dim \mathcal{Z}^s = r$. It follows from Proposition 3.9, that for all $s \in S$, the connected component of $\mathcal{Z}^s$ is the same compatible locally definable subgroup $\mathcal{V}$ over B.

Since $\{\mathcal{Z}^s : s \in S\}$ is a decreasing sequence of compatible locally definable subgroups of $\mathcal{Z}$ over B, we can totally order S by $s \leq s'$ if and only if $\mathcal{Z}^{s'} \subseteq \mathcal{Z}^s$. Let s_0 be the first element of S (we can assume, without loss of generality that s_0 exists). Then there is a decreasing sequence $\{U_s : s \in S\}$ of locally definable subsets of $\mathcal{Z}^{s_0}$ over B containing the identity element such that, for each $s \in S$, $|U_s| < \aleph_1$ and $\mathcal{Z}^s = \cup\{u\mathcal{V} : u \in U_s\}$. For each $s \in S$, the set U_s is locally definable subsets of $\mathcal{Z}^{s_0}$ over B by Corollary 3.6.

Since $\cup\{U_s : s \in S\} = U_{s_0}$ and $|U_{s_0}| < \aleph_1$, there is $S_0 \subseteq S$ such that $|S_0| < \aleph_1$ and $\{U_s : s \in S\}= \{U_s : s \in S_0\}$. Let $U = \cap\{U_s : s \in S_0\}$. Then U is a nonempty (contains the identity) locally definable subset of $\mathcal{Z}^{s_0}$ over B with $|U| < \aleph_1$. The set U is locally definable subsets of $\mathcal{Z}^{s_0}$ over B since its elements are defined over B and U is the union of the collection of all finite subsets of U.

Let $\mathcal{W} = \cup\{u\mathcal{V} : u \in U\}$. Then $\mathcal{W}$ is a compatible locally definable subgroup of $\mathcal{Z}$ over B such that $\mathcal{W} = \cap\{\mathcal{Z}^s : s \in S\}$. □

From Proposition 3.13 we get the following very useful results.

Corollary 3.14 *Let $\mathcal{Z}$ be a locally definable group over A. If $\{\mathcal{Z}^s : s \in S\}$ is a collection of compatible locally definable subgroups of $\mathcal{Z}$ over B with $A \subseteq B$, then $\cap\{\mathcal{Z}^s : s \in S\} = \cap\{\mathcal{Z}^s : s \in S_0\}$ for some $S_0 \subseteq S$ with $|S_0| < \aleph_1$ and this intersection is a compatible locally definable subgroup of $\mathcal{Z}$ over B.*

Proof. Let $\alpha : \kappa \longrightarrow S$ be an enumeration of S. We define a decreasing sequence $\{\mathcal{X}^\beta : \beta < \kappa\}$ of compatible locally definable subgroups of $\mathcal{Z}$ over B inductively as follows: $\mathcal{X}^0 = \mathcal{Z}^{\alpha(0)}$; for $\beta = \gamma + 1$ we put $\mathcal{X}^\beta = \mathcal{X}^\gamma \cap \mathcal{Z}^{\alpha(\gamma+1)}$ and for β a limit ordinal we put $\mathcal{X}^\beta = \cap\{\mathcal{X}^\gamma : \gamma < \beta\} \cap \mathcal{Z}^{\alpha(\beta)}$ (by Proposition 3.13, this a compatible locally definable subgroup of $\mathcal{Z}$ over B).

To finish the proof of the corollary, note that $\cap\{\mathcal{X}^\beta : \beta < \kappa\} = \cap\{\mathcal{Z}^s : s \in S\}$ and apply Proposition 3.13. □

Corollary 3.15 *Suppose that $\mathcal{Z}$ is a locally definable group over A and $S \subseteq \mathcal{Z}$ is a locally definable subset over B with $A \subseteq B$. Then $C_{\mathcal{Z}}(S) = \{z \in \mathcal{Z} :$ for all $s \in S,\ zs = sz\}$, the centraliser of S in $\mathcal{Z}$, is a compatible locally definable subgroup over B. In fact there is $S_0 \subseteq S$ such that $|S_0| < \aleph_1$ and $C_{\mathcal{Z}}(S) = C_{\mathcal{Z}}(S_0)$. In particular, the centre $Z(\mathcal{Z}) = C_{\mathcal{Z}}(\mathcal{Z})$ of $\mathcal{Z}$ is a compatible locally definable normal subgroup of $\mathcal{Z}$ over A.*

Proof. Let W be a definable subset of S over B and let V be an open definable subset of $\mathcal{Z}$ over B. Since multiplication restricted to $V \cup W$ is a definable map over B, the set $\{z \in V :$ for all $w \in W,\ zw = wz\}$ is a definable subset of $\mathcal{Z}$ over B. But this set is the same as $C_{\mathcal{Z}}(W) \cap V$. Thus $C_{\mathcal{Z}}(W)$ is a compatible locally definable subgroup of $\mathcal{Z}$ over B. Hence, by Corollary 3.14, so is $C_{\mathcal{Z}}(S)$. □

We end this section with the following result whose proof is very similar to that of its analogue for groups of finite Morley rank. However, we use the topology on locally definable groups to simplify the arguments.

Corollary 3.16 *Let $\mathcal{X}$ be a connected locally definable group over A. Then following hold: (i) every locally definable subgroup of $\mathcal{X}$ over A of dimension zero is contained in $Z(\mathcal{X})$; (ii) if $Z(\mathcal{X})$ has dimension zero, then $\mathcal{X}/Z(\mathcal{X})$ is centerless. In particular, if $\mathcal{X}$ is nilpotent of positive dimension, then $Z(\mathcal{X})$ has positive dimension.*

Proof. Let $\mathcal{Y}$ be a locally definable subgroup of $\mathcal{X}$ over A of dimension zero. By Corollary 3.15, $C_{\mathcal{X}}(\mathcal{Y})$ is a compatible locally definable subgroup of $\mathcal{X}$ over A. Clearly, it is enough to show that $\dim C_{\mathcal{X}}(\mathcal{Y}) = \dim \mathcal{X}$. In fact, since $\mathcal{X}$ is connected, by Proposition 3.9, we would get $C_{\mathcal{X}}(\mathcal{Y}) = \mathcal{X}$ and so $\mathcal{Y} \subseteq Z(\mathcal{X})$.

But for each $y \in \mathcal{Y}$, the map $\sigma_y : \mathcal{X} \longrightarrow \mathcal{Y}$ over A given by $\sigma_y(x) = xyx^{-1}$ is continuous and its restriction to any definable subset is definable. Since $\mathcal{Y}$ has dimension zero, $\{y\}$ is open in $\mathcal{Y}$ and so there is an open definable subset V_y of $\mathcal{X}$ over A containing 1 such that $V_y \subseteq (\sigma_y)^{-1}(y) = C_{\mathcal{X}}(y)$. Since $|\mathcal{Y}| < \aleph_1$, by the compactness theorem, there is $x \in \mathcal{X}$ such that $x \neq 1$, $\dim(x/A) = \dim \mathcal{X}$ and $x \in \cap\{V_y : y \in \mathcal{Y}\} \subseteq C_{\mathcal{X}}(\mathcal{Y})$. So $\dim C_{\mathcal{X}}(\mathcal{Y}) = \dim \mathcal{X}$ as required.

Suppose that $\dim Z(\mathcal{X}) = 0$. Let $z \in Z_2(\mathcal{X})$. Then the map $\mathrm{ad}(z) : \mathcal{X} \longrightarrow Z(\mathcal{X})$ given by $\mathrm{ad}(z)(x) = z^{-1}x^{-1}zx$ is a locally definable homomorphism over A and so its kernel $C_{\mathcal{X}}(z)$ is a compatible locally definable subgroup of $\mathcal{X}$ over A of dimension $\dim \mathcal{X}$. Hence, since $\mathcal{X}$ is connected, by Proposition 3.9, we have $C_{\mathcal{X}}(z) = \mathcal{X}$ and so $z \in Z(\mathcal{X})$. Thus $Z_2(\mathcal{X}) \subseteq Z(\mathcal{X})$ and $\mathcal{X}/Z(\mathcal{X})$ is centerless. □

4 Locally definable group extensions

The main result of this section is the existence of a locally definable quotient of a locally definable group by a compatible locally definable normal subgroup. These quotients come equipped also with locally definable sections.

Definition 4.1 Let $\mathcal{X}$ be a locally definable group over A. We say that $(\mathcal{X}, i, j)$ is a *locally definable extension* of $\mathcal{Y}$ by $\mathcal{Z}$ over A if we have an exact sequence

$$1 \to \mathcal{Z} \xrightarrow{i} \mathcal{X} \xrightarrow{j} \mathcal{Y} \to 1$$

in the category of locally definable groups with locally definable homomorphisms over A. If $(\mathcal{X}, i, j)$ is a locally definable extension of $\mathcal{Y}$ by $\mathcal{Z}$ over A and $\mathcal{X}$ is abelian, we say that $(\mathcal{X}, i, j)$ is a *locally definable abelian extension* of $\mathcal{Y}$ by $\mathcal{Z}$ over A. A *locally definable section* over A is a locally definable map $s : \mathcal{Y} \longrightarrow \mathcal{X}$ over A such that $j(s(y)) = y$ for all $y \in \mathcal{Y}$.

Note: Below we will sometimes assume that $\mathcal{Z} \trianglelefteq \mathcal{X}$ and write $(\mathcal{X}, j)$ for $(\mathcal{X}, i, j)$.

The next result is proved by adapting the corresponding result from [e] for definable groups.

Theorem 4.2 *Let $\mathcal{X}$ be a locally definable group over A and let $\mathcal{Z}$ be a normal compatible locally definable subgroup of $\mathcal{X}$ over A. Then we have a locally definable extension $1 \to \mathcal{Z} \to \mathcal{X} \xrightarrow{j} \mathcal{Y} \to 1$ over A of locally definable groups over A with a locally definable section $s : \mathcal{Y} \longrightarrow \mathcal{X}$ over A.*

Proof. Let $\{X_i : i \in I\}$ be the collection of all open definable subsets of $\mathcal{X}$ over A. By Corollary 2.12 we have $\mathcal{X} = \cup\{X_i : i \in I\}$. For each $i \in I$, by Lemma 3.4, the equivalence relation on X_i given by $x \simeq y$ if and only if $x\mathcal{Z} = y\mathcal{Z}$ is definable over A. Thus, the argument in [e] Theorem 2.5 shows that for each $i \in I$, there is a large definable subset U_i of X_i over A and there is a definable function $l_i = (l_{i,1}, \ldots, l_{i,m}) : U_i \longrightarrow X_i$ over A such that for each $x \in U_i$, there is $z \in x\mathcal{Z} \cap X_i$ with $z = l_i(x)$ and for all $y \in U_i$, we have $x\mathcal{Z} = y\mathcal{Z}$ if and only if $l_i(x) = l_i(y)$. Here m is such that $\mathcal{X} \subseteq N^m$. Clearly, we may replace without loss of generality each U_i by its interior in $\mathcal{X}$.

Let $\mathcal{U} = \cup\{U_i : i \in I\}$. Then $\mathcal{U}$ is a large locally definable subset of $\mathcal{X}$ over A. Thus we can use Proposition 2.11 instead of its definable analogue [p1] Lemma 2.4 and the argument in the proof of [e] Theorem 2.5 to show that for each $i \in I$, there is a definable function $l_i : X_i \longrightarrow X_i$ over A extending $l_i : U_i \longrightarrow X_i$ such that for each $x \in X_i$, there is $z \in x\mathcal{Z} \cap X_i$ with $z = l_i(x)$ and for all $y \in X_i$, we have $x\mathcal{Z} = y\mathcal{Z}$ if and only if $l_i(x) = l_i(y)$.

We now define a locally definable map $j : \mathcal{X} \longrightarrow \mathcal{X}$ over A such that for all $u, v \in \mathcal{X}$, we have $u\mathcal{Z} = v\mathcal{Z}$ if and only if $j(u) = j(v)$. For this, let $\kappa \leq \aleph_0$ be an enumeration of I. Clearly we may assume that, for all $\alpha, \beta \in \kappa$, we have $\alpha \leq \beta$ if and only if $X_\alpha \subseteq X_\beta$. For $x \in X_0$, put $l'_0(x) = l_0(x)$; suppose that l'_γ has been defined on X_γ and $\beta = \gamma + 1$. Then we define l'_β on X_β by $l'_\beta(x) = l_\beta(x)$ if $x\mathcal{Z} \cap X_\gamma = \emptyset$ or $l'_\beta(x) = l'_\gamma(y)$ for some (equivalently, for all) $y \in X_\gamma$ such that $y \in x\mathcal{Z}$. Now take $j = \cup\{l'_\beta : \beta < \kappa\}$.

Clearly, by construction, if $u, v \in \mathcal{X}$, then $u\mathcal{Z} = v\mathcal{Z}$ if and only if $j(u) = j(v)$. We need to show that j is a locally definable map over A. For $\gamma < \kappa$, let $\theta_\gamma : X_{\gamma+1} \times X_\gamma \longrightarrow \theta_\gamma(X_{\gamma+1} \times X_\gamma)$ be the definable map given by $\theta_\gamma(x, y) = x^{-1}y$. Then, since $\mathcal{Z}$ is compatible, $Z_\gamma = \theta_\gamma(X_{\gamma+1} \times X_\gamma) \cap \mathcal{Z}$ is a definable set over A. Furthermore, for all $x \in X_{\gamma+1}$ we have $x\mathcal{Z} \cap X_\gamma = \emptyset$

if and only if $xZ_\gamma \cap X_\gamma = \emptyset$. Thus, by induction, for all $\gamma < \kappa$, $j_{|X_\gamma} = l'_\gamma$ is definable over A. Hence j is a locally definable map over A.

If $\mathcal{Y} = j(\mathcal{X})$ then, by Remark 2.9, $\mathcal{Y}$ is a locally definable group over A with group operation given by $xy = j(j^{-1}(x)j^{-1}(y))$. The locally definable section $s : \mathcal{Y} \longrightarrow \mathcal{X}$ over A is just the inclusion of $j(\mathcal{X})$ in $\mathcal{X}$. □

Corollary 4.3 *Suppose that* $1 \to \mathcal{Z} \to \mathcal{X} \xrightarrow{l} \mathcal{Y} \to 1$ *is a locally definable extension over A. Then there is a locally definable section* $t : \mathcal{Y} \longrightarrow \mathcal{X}$ *over A.*

Proof. Since $\mathcal{Z}$ is a normal locally definable subgroup of $\mathcal{X}$ over A, we have a locally definable extension $1 \to \mathcal{Z} \to \mathcal{X} \xrightarrow{j} \mathcal{V} \to 1$ over A with a locally definable section $s : \mathcal{V} \longrightarrow \mathcal{X}$ over A. Define a map $h : \mathcal{V} \longrightarrow \mathcal{Y}$ by $h(v) = l(s(v))$ for all $v \in \mathcal{V}$. Clearly, h is a locally definable map over A which is a bijection. Hence its inverse is also a locally definable map over A. Now define $t : \mathcal{Y} \longrightarrow \mathcal{X}$ by $t(y) = s(h^{-1}(y))$ for all $y \in \mathcal{Y}$. Then t is a locally definable map over A and $l(t(y)) = l(s(h^{-1}(y))) = h(h^{-1}(y)) = y$ for all $y \in \mathcal{Y}$. □

Observe that in the proof of Theorem 4.2 we never used the fact that $\mathcal{Z}$ is normal in $\mathcal{X}$. Hence the following holds.

Corollary 4.4 *Let* $\mathcal{X}$ *be a locally definable group over A and let* $\mathcal{Z}$ *be a compatible locally definable subgroup of* $\mathcal{X}$ *over A. Then we have locally definable maps* $j : \mathcal{X} \longrightarrow \mathcal{Y}$ *and* $s : \mathcal{Y} \longrightarrow \mathcal{X}$ *over A between locally definable sets over A such that, for all* $u, v \in \mathcal{X}$*, we have* $j(u) = j(v)$ *if and only if* $u\mathcal{Z} = v\mathcal{Z}$*, and* $j(s(y)) = y$ *for all* $y \in \mathcal{Y}$*.*

With the previous results available, the next definition and the remarks that follow, one can develop group cohomology theory and group extension theory in the category of locally definable groups in exactly the same way as we did in the category of definable groups in [e] Sections 3.2, 3.4 and 3.5. Since these results are purely algebraic, to avoid unnecessary repeatition, we will refer to [e] when needed.

Definition 4.5 Two locally definable extensions $1 \to \mathcal{Z} \xrightarrow{i} \mathcal{X} \xrightarrow{j} \mathcal{Y} \to 1$ and $1 \to \mathcal{Z} \xrightarrow{i'} \mathcal{X}' \xrightarrow{j'} \mathcal{Y} \to 1$ over A are *locally definably equivalent* over A if there is a locally definable isomorphism $\varphi : \mathcal{X} \longrightarrow \mathcal{X}'$ over A such that

$$\begin{array}{ccccc} & & \mathcal{X} & & \\ & {\scriptstyle i}\nearrow & & \searrow{\scriptstyle j} & \\ 1 \to \mathcal{Z} & & \downarrow{\scriptstyle \varphi} & & \mathcal{Y} \to 1 \\ & {\scriptstyle i'}\searrow & & \nearrow{\scriptstyle j'} & \\ & & \mathcal{X}' & & \end{array}$$

is a commutative diagram.

Remark 4.6 Suppose that we have a locally definable extension $1 \to \mathcal{Z} \to \mathcal{X} \xrightarrow{l} \mathcal{Y} \to 1$ over A and $\mathcal{V} \trianglelefteq \mathcal{Y}$ is a compatible locally definable subgroup over A. Then $\mathcal{W} = l^{-1}(\mathcal{V}) \trianglelefteq \mathcal{X}$ and $\mathcal{Z} \trianglelefteq \mathcal{W}$. Moreover, if we have a locally definable extension $1 \to \mathcal{V} \to \mathcal{Y} \xrightarrow{j} \mathcal{U} \to 1$ over A, then we have locally definable extensions $1 \to \mathcal{W} \to \mathcal{X} \xrightarrow{j \circ l} \mathcal{U} \to 1$ and $1 \to \mathcal{Z} \to \mathcal{W} \xrightarrow{l_{|\mathcal{W}}} \mathcal{V} \to 1$ over A.

Remark 4.7 Suppose that we have a locally definable extension $1 \to \mathcal{Z} \to \mathcal{X} \xrightarrow{l} \mathcal{Y} \to 1$ and $\mathcal{Z} \trianglelefteq \mathcal{W} \trianglelefteq \mathcal{X}$ is a compatible locally definable normal subgroup over A. If we have a locally definable extension $1 \to \mathcal{W} \to \mathcal{X} \xrightarrow{k} \mathcal{U} \to 1$ over A, then we have locally definable extensions $1 \to \mathcal{Z} \to \mathcal{W} \xrightarrow{l_{|\mathcal{W}}} \mathcal{V} \to 1$, $1 \to \mathcal{V} \to \mathcal{Y} \xrightarrow{j} \mathcal{U} \to 1$ over A such that $j \circ l = k$.

The results we prove below will be very useful later on. They are about the invariance of notions such as definably compact and connected under locally definable extensions.

Corollary 4.8 *Suppose that* $1 \to \mathcal{Z} \to \mathcal{X} \xrightarrow{j} \mathcal{Y} \to 1$ *is a locally definable extension of locally definable groups over A. Then the following holds: (i) if* $\mathcal{X}$ *is connected, then* $\mathcal{Y}$ *is connected ; (ii) if* $\mathcal{Z}$ *is connected, then* $\mathcal{X}$ *is connected if and only if* $\mathcal{Y}$ *is connected.*

Proof. Suppose that $\mathcal{Y} = \cup\{y_s\mathcal{Y}^0 : s \in S\}$ (disjoint union) for some locally definable subset $\{y_s : s \in S\}$ of $\mathcal{Y}$. Let $\mathcal{U} = j^{-1}(\mathcal{Y}^0)$. Then $\mathcal{U}$ is a locally definable subgroup of $\mathcal{X}$ and $\mathcal{X} = \cup\{x_s\mathcal{U} : s \in S\}$ (disjoint union) where for each $s \in S$, we have $j(x_s) = y_s$. So, if $\mathcal{Y}$ is not connected, then $\mathcal{X}$ is not connected.

By the above, it remains to show that, if $\mathcal{Z}$ and $\mathcal{Y}$ are connected, then $\mathcal{X}$ is connected. Suppose that $\mathcal{X} = \cup\{x_s\mathcal{X}^0 : s \in S\}$ for some locally definable subset $\{x_s : s \in S\}$ of $\mathcal{X}$. Since $\mathcal{Z}$ is connected, we have $\mathcal{Z} \subseteq \mathcal{X}^0$. But then, $\mathcal{Y} = \cup\{j(x_s)j(\mathcal{X}^0) : s \in S\}$ and so $\mathcal{Y} = j(\mathcal{X}^0)$. Hence, for each $y_s = j(x_s) \in \mathcal{Y}$ we have $x_s\mathcal{Z} = j^{-1}(y_s) \subseteq \mathcal{X}^0$ i.e., $x_s \in \mathcal{X}^0$. Thus $\mathcal{X} = \mathcal{X}^0$. $\square$

Definition 4.9 We say that a locally definable group $\mathcal{X}$ over A is *definably compact* if for every definable continuous map $\sigma : (a, b) \subseteq [-\infty, +\infty] \longrightarrow \mathcal{X}$ over A the limits $\lim_{t\to a^+} \sigma(t)$ and $\lim_{t\to b^-} \sigma(t)$ exist in $\mathcal{X}$.

This definition is similar to its definable analogue in [ps]. The proof of the next result is exactly the same as that of its definable analogue in [e] Lemma 3.14.

Corollary 4.10 *Suppose that* $1 \to \mathcal{Z} \to \mathcal{X} \xrightarrow{j} \mathcal{Y} \to 1$ *is a locally definable extension over A. Then* $\mathcal{X}$ *is definably compact if and only if* $\mathcal{Y}$ *and* $\mathcal{Z}$ *are definably compact.*

5 The solvable case

The main result of this section is the classification of solvable locally definable groups up to definably compact, solvable locally definable groups.

We start with the analogue of [ps] Theorem 1.2. For this we just make sure here that it also holds for locally definable groups. The argument is the same but we will require the following lemma.

Lemma 5.1 *Let* $\mathcal{X}$ *be a locally definable group over A and let Z be a definable subset of X over A. Then the closure* $\overline{Z}$ *of Z in* $\mathcal{X}$ *is a definable subset of* $\mathcal{X}$ *over A.*

Proof. By Corollary 2.12 there is an open definable subset U of $\mathcal{X}$ over A such that $Z \subseteq U$ and with $1 \in U$. Let $\{U_s : s \in S\}$ be the uniformly definable basis for the τ-open neighbourhoods of 1. We can assume that $U_s \subseteq U$ for all $s \in S$. We have that $z \in \overline{Z}$ if and only if for all $s \in S$, there is $y \in U_s$ such that $zy \in Z$. Thus $z \in \overline{Z}$ if and only if for all $s \in S$, there are $y \in U_s$ and $x \in Z$ such that $z = xy^{-1}$. □

In the proof of the next theorem and later on we will make use of the notion of infinitesimals just like in [pps2] and [pst2]. Consider a fixed elementary extension $\mathcal{N}^*$ of $\mathcal{N}$ which is $|\mathrm{Th}_N(\mathcal{N})|^+$-saturated. For $a \in N^k$ we let $\mathcal{V}_a$ denote the intersection of all open definable subsets of $(N^*)^k$ defined over N and containing a; we call this the *infinitesimal neighbourhood* of a. One can verify that for all our purposes the properties of $\mathcal{V}_a$ are independent of the choice of $\mathcal{N}^*$.

Similarly, given a locally definable group $\mathcal{X}$ over A and $a \in \mathcal{X}$, we let $\mathcal{V}_a(\mathcal{X})$ denote the intersection of all τ-open definable subsets of $\mathcal{X}(N^*)$ defined over N and containing a; we call this the *infinitesimal neighbourhood of a in the τ-topology on $\mathcal{X}$*. Note that, by Theorem 2.3, this definition of $\mathcal{V}_a(\mathcal{X})$ is equivalent to that given in [pst2].

Theorem 5.2 *Let $\mathcal{X}$ be a locally definable group over A which is not definably compact. Then $\mathcal{X}$ has a torsion-free definable subgroup over A of dimension one.*

Proof. By Corollary 2.12 let $\mathcal{X} = \cup\{X_i : i \in I\}$ where $\{X_i : i \in I\}$ is the collection of all open definable subsets of $\mathcal{X}$ over A. Let $\sigma : (a, b) \subseteq (-\infty, +\infty) \longrightarrow \mathcal{X}$ be a definable continuous injective map over A such that $\lim_{t \to b^-} \sigma(t)$ does not exist in $\mathcal{X}$. Let $J = \sigma(a, b)$ with the order induced from (a, b) by σ. For $s \in J$, let $J^{>s} = \{x \in J : x > s\}$.

Define a relation $T' \subseteq \mathcal{X} \times \mathcal{X}$ by $(\alpha, \beta) \in T' \cap X_i \times X_j$ if and only if for every $s \in J$ and open definable neighbourhood V of β over A contained in X_j, there is $t \in J$ and an open definable neighbourhood U of α over A contained in X_i such that $U \cdot J^{>t} \subseteq V \cdot J^{>s}$. This is a compatible locally definable subset of $\mathcal{X} \times \mathcal{X}$ over A. In fact, for each $i, j \in I$, the set $T' \cap X_i \times X_j$ is a definable set over A. Now define the relation $T \subseteq \mathcal{X} \times \mathcal{X}$ by $(\alpha, \beta) \in T$ if and only if $(\alpha, \beta) \in T'$ and $(\beta, \alpha) \in T'$. Clearly T is a compatible locally definable subset of $\mathcal{X} \times \mathcal{X}$ over A.

Let $\mathcal{K}$ be the prime model of $\mathrm{Th}_A(\mathcal{N})$. Let $J^\infty = \{x \in J(N^*) : x > s$ for all $s \in J(K)\}$. Here $\mathcal{N}^*$ is a fixed elementary extension of $\mathcal{N}$ which is $|\mathrm{Th}_N(\mathcal{N})|^+$-saturated. Then, just like in [ps] Lemma 3.4, the compactness theorem implies that $(\alpha, \beta) \in T$ if and only if $\mathcal{V}_\alpha(\mathcal{X}) \cdot J^\infty = \mathcal{V}_\beta(\mathcal{X}) \cdot J^\infty$ where $\mathcal{V}_\gamma(\mathcal{X})$ is the infinitesimal neighborhood of γ in the τ-topology on $\mathcal{X}$. Hence, just like in [ps] Lemma 3.6, it follows that T is an equivalence relation on $\mathcal{X}$ and the T-equivalence class of 1 is a compatible locally definable subgroup $\mathcal{Y}$ of $\mathcal{X}$ over A. By the second paragraph of the proof of [ps] Lemma 3.7, we have $\mathcal{Y} \subseteq \cap\{\overline{P_s} : s \in J\} \subseteq \overline{P_s}$ where for $s \in J$, we set $P_s = \{y \cdot x^{-1} : x, y \in J^{>s}\}$. Since for each $s \in J$, the set P_s is a definable subset of $\mathcal{X}$ over A, by Lemma 5.1, the closure $\overline{P_s}$ of P_s in $\mathcal{X}$ is also definable over A. Thus, by saturation, there is X_i such that $\mathcal{Y} \subseteq \overline{P_s} \subseteq X_i$ and, since $\mathcal{Y}$ is compatible, it follows that $\mathcal{Y}$ is a definable subgroup of $\mathcal{X}$ over A. The rest of the argument in the proof of [ps] Lemma 3.7 shows that $\mathcal{Y}$ has dimension less than or equal to one. Moreover, [ps] Lemma 3.8 shows that $\mathcal{Y}$ has dimension one and [ps] Lemma 3.9 shows that $\mathcal{Y}$ is torsion-free. $\square$

Below we mention definable solvable groups with no definably compact parts. These were introduced and classified in [e].

Theorem 5.3 *Suppose that $\mathcal{U}$ is a connected locally definable solvable group over A. Then $\mathcal{U}$ has compatible locally definable normal subgroups $\mathcal{V}$ and W over A such that $\mathcal{U}/\mathcal{V}$ is a definably compact locally definable solvable group over A, $\mathcal{V} = \mathcal{X} \times W$, W is a definable solvable group with no definably compact parts and $\mathcal{X}$ is a connected, definably compact, normal, compatible locally definable subgroup of $\mathcal{U}$ over A of maximal dimension.*

Proof. This is obtained in exactly the same way as its definable analogue [e] Theorem 5.8. In fact, all the results required in the proof of [e] Theorem 5.8 have an analogue for locally definable groups which are proved in exactly the same way. $\square$

Unlike in the definable case where all definably connected, definably compact, definable solvable groups are abelian (see [e] or [pst2])) we have the following example.

Example 5.4 Let $\mathcal{N}$ be a non-standard model of the theory of the ordered field of real numbers. Let $\mathcal{X} = (\mathcal{X}, *)$ be the locally definable group over

$\mathbb{Z}$ given by $\mathcal{X} = \{(a,x) :$ there exists $n \in \mathbb{N}$ such that $-n \leq a \leq n$ and $\frac{1}{n} \leq x \leq n\}$ and $(a,x) * (b,y) = (a+bx, xy)$.

It is easy to see that $\mathcal{X}$ is in fact a well defined locally definable solvable group over $\mathbb{Z}$ which is not abelian. Moreover, $\mathcal{X}$ is connected and definably compact.

Definition 5.5 Suppose that $\mathcal{U}$ is a connected locally definable solvable group over A and let W be the compatible locally definable normal subgroup of $\mathcal{U}$ over A given by Theorem 5.3. Then $\mathcal{U}/W$ is a definably compact, connected, locally definable group over A which we shall call the *definably compact part of* $\mathcal{U}$ over A.

We end this section with the following observation which will be used later.

Proposition 5.6 *Let $\mathcal{X}$ be a connected, locally definable solvable group over A. Then there is a compatible locally definable nilpotent normal subgroup $U(\mathcal{X})$ of $\mathcal{X}$ over A such that $\mathcal{X}/U(\mathcal{X})$ is a connected, locally definable solvable group over A with centre of dimension zero.*

Proof. If the centre of $\mathcal{X}$ has dimension zero, then take $U(\mathcal{X}) = 1$. Suppose otherwise and consider the central series $1 \trianglelefteq Z_1(\mathcal{X}) \trianglelefteq Z_2(\mathcal{X}) \cdots$. By dimension considerations, there is a minimal $n \in \mathbb{N}$ such that for all $i \leq n$, we have $\dim Z_i(\mathcal{X}) > 0$ and $\dim Z_{n+1}(\mathcal{X}) = \dim Z_n(\mathcal{X})$. In this case take $U(\mathcal{X}) = Z_n(\mathcal{X})$. Note that, in both cases, by Corollary 4.8, $\mathcal{X}/U(\mathcal{X})$ is connected. $\square$

6 Locally definable homogeneous spaces

Here we shall present some basic results about locally definable transitive actions of locally definable groups on locally definable sets. These facts will be used in Section 7.

Definition 6.1 An action $\alpha : \mathcal{X} \times \mathcal{S} \longrightarrow \mathcal{S}$ of a locally definable group $\mathcal{X}$ over A on a locally definable set $\mathcal{S}$ over A is called a *locally definable action over A* if α is a locally definable map over A.

Example 6.2 Let $\mathcal{X}$ be a locally definable group over A and let $\mathcal{Z}$ be a compatible locally definable subgroup of $\mathcal{X}$ over A. Then Corollary 4.4 shows that the canonical action $\mathcal{X} \times \mathcal{X}/\mathcal{Z} \longrightarrow \mathcal{X}/\mathcal{Z}$ given by multiplication on the left is a locally definable action over A.

As in the proof of Corollary 3.15 we get the following remark.

Remark 6.3 Let $\alpha : \mathcal{X} \times \mathcal{S} \longrightarrow \mathcal{S}$ be a locally definable action over A. If $\mathcal{S}_0 \subseteq \mathcal{S}$ is a locally definable subset over A, then $\mathrm{Fix}_{\mathcal{S}_0}(\mathcal{X}) = \{x \in \mathcal{X} : \alpha(x, s) = s \text{ for all } s \in \mathcal{S}_0\}$ is a compatible locally definable subgroup of $\mathcal{X}$ over A.

The proof of the next result is obtained by adapting that of [pst2] Proposition 2.2.

Theorem 6.4 *Let $\alpha : \mathcal{X} \times \mathcal{S} \longrightarrow \mathcal{S}$ be a transitive locally definable action over A where $\mathcal{S} \subseteq N^k$. Fix $s_0 \in \mathcal{S}$. Then there is a uniformly definable family $\{U_c : c \in C\}$ of definable subsets of $\mathcal{S}$ defined over A and containing s_0 and there is a unique topology σ on $\mathcal{S}$ such that: (i) $\{U_c : c \in C\}$ is a basis for the σ-open neighbourhoods of s_0; (ii) α is a topological action and (iii) every generic element of $\mathcal{S}$ has an open definable neighbourhood $U \subseteq N^k$ such that $U \cap \mathcal{S}$ is σ-open and the topology which $U \cap \mathcal{S}$ inherits from σ agrees with the topology it inherits from N^k.*

Proof. The uniqueness of σ is clear since a basis of neighborhoods for generic points is determined in advance.

Write $\mathcal{S} = \cup\{S_j : j \in J\}$ and fix S_j of maximal dimension and a generic $s \in S_j$ over A. Fix also a uniformly definable basis $\{W_c : c \in C\}$ for the open definable neighborhoods of s in the standard topology on N^k. For each $c \in C$ define $U_c = \alpha(x_s^{-1}, W_c \cap S_j)$ where $s_0 = \alpha(x_s^{-1}, s)$. Then $\{U_c : c \in C\}$ is a definable family of subsets of $\mathcal{S}$ containing s_0. We take as a basis of open sets in $\mathcal{S}$ the collection $\{\alpha(x^{-1}, U_c) : c \in C,\ x \in \mathcal{X}\}$ and call this topology σ.

By maximality of $\dim S_j$ and genericity of s, we have $\mathcal{V}_s \cap S_j = \mathcal{V}_s \cap S_i$ for every $i \in J$ such that $s \in S_i$. Here, for $r \in \mathcal{S}$, $\mathcal{V}_r$ is the infinitesimal neighborhood of r in N^k. Thus, $\mathcal{V}_s \cap S_j$ is independent of the choice of S_j and equals $\mathcal{V}_s \cap \mathcal{S}$. As in [pst2] Claim 2.3, we see that for every generic r in $\mathcal{S}$, we have $\alpha(x_r^{-1}, \mathcal{V}_r \cap \mathcal{S}) = \alpha(x_s^{-1}, \mathcal{V}_s \cap \mathcal{S}) = \alpha(\mathcal{V}_{x_s^{-1}}(\mathcal{X}), s)$ where for $x \in \mathcal{X}$, we denote by $\mathcal{V}_x(\mathcal{X})$ the infinitesimal neighborhood of x in the τ-topology on $\mathcal{X}$. Therefore, given a generic r in $\mathcal{S}$, its basis of neighborhoods in the σ-topology is also a basis of neighborhoods in the topology which N^k induces on $\mathcal{S}$.

For $r \in \mathcal{S}$, write $\mathcal{V}_r(\mathcal{S})$ for $\alpha(x_r^{-1}, \alpha(x_s^{-1}, \mathcal{V}_s \cap \mathcal{S}))$. Arguing as in the proof of [pst2] Claim 2.4, we get that for every $r \in \mathcal{S}$, $\alpha(x_r^{-1}, \alpha(x_s^{-1}, \mathcal{V}_s(\mathcal{S}))) = \alpha(x_s^{-1}\mathcal{V}_{x_s^{-1}}(\mathcal{X}), r)$. This can be used to show just like in [pst2] Claim 2.5 the following: (i) $\alpha_| : \mathcal{V}_1(\mathcal{X}) \times \mathcal{V}_{s_0}(\mathcal{S}) \longrightarrow \mathcal{V}_{s_0}(\mathcal{S})$ is a transitive action; (ii) for every $x \in \mathcal{X}$ and $r \in \mathcal{S}$, $\alpha(\mathcal{V}_x(\mathcal{X}), \mathcal{V}_r(\mathcal{S})) \subseteq \alpha(1, \mathcal{V}_{\alpha(x,r)}(\mathcal{S})), \alpha(\mathcal{V}_{xx_r}(\mathcal{X}), s_0)$. These claims imply that α is continuous as required. $\Box$

Corollary 6.5 *Let $\alpha : \mathcal{X} \times \mathcal{S} \longrightarrow \mathcal{S}$ be a transitive locally definable action over A and let $\{U_c : c \in C\}$ be the uniformly definable basis for the σ-open neighbourhoods of s_0. Then we can choose $\{U_c : c \in C\}$ such that there is a uniformly definable family $\{\psi_c : c \in C\}$ of definable homeomorphisms $\psi_c : U_c \longrightarrow V_c$ where V_c is an open definable subset of N^m and m is the dimension of $\mathcal{S}$.*

Proof. This is proved as Corollary 2.5 but using Theorem 6.4 instead of Theorem 2.3. $\Box$

As the reader can easily verify, all the results of Subsection 2.2 have an analogue for locally definable sets over A on which there is a transitive locally definable action over A by a locally definable group over A. We call such locally definable sets *locally definable homogeneous spaces over A*.

Let $\mathcal{S}$ be a locally definable homogeneous space over A. We define the notion of compatible locally definable subsets of $\mathcal{S}$ over A just like in Definition 3.1. Similarly we define the notion of connected subsets of $\mathcal{S}$ just like in Definition 3.7. As the reader can easily verify, we have an analogue of Remark 3.8 and, arguing as in the proof of Proposition 3.9, we see that given $s \in \mathcal{S}$ there is a unique connected compatible locally definable subset of $\mathcal{S}$ over A containing s and of dimension $\dim \mathcal{S}$.

7 The centerless case

The goal here is the classification of connected, locally definable groups with no non trivial locally definable abelian normal subgroups.

7.1 Lie algebras of locally definable groups

In this subsection, we will assume that $\mathcal{N}$ is an o-minimal expansion of a real closed field. The goal here is to develop Lie theory for locally definable groups. After introducing the main notions, the proofs are the same as those for the definable case treated in [pps1] Sections 2.3 and 2.4. Hence, we single out only the main results and refer constantly to [pps1] for details.

Given locally definable homogeneous spaces $\mathcal{S}$ and $\mathcal{E}$ over A and a locally definable map $h : \mathcal{S} \longrightarrow \mathcal{E}$ over A we can use Corollary 6.5 to define notions from differential calculus just like in [pps1] 1.1.2. In particular, for $s \in \mathcal{S}$ we have the *tangent vector space* $T_s(\mathcal{S})$ of $\mathcal{S}$ at s and the *differential map* $\mathrm{d}_s(h) : T_s(\mathcal{S}) \longrightarrow T_{h(s)}(\mathcal{E})$ when h is differentiable at s. Note that $T_s(\mathcal{S})$ and $T_{h(s)}(\mathcal{E})$ are finite dimensional vector spaces over the underlying real closed field of $\mathcal{N}$ and $\mathrm{d}_s(h)$ is a linear map. Below we will use freely these notions and refer the reader to [pps1] for details.

Theorem 7.1 *Let $\alpha : \mathcal{X} \times \mathcal{S} \longrightarrow \mathcal{S}$ be a transitive locally definable action over A. Then α is a locally definable C^p-map.*

Proof. Clearly if (g, s) is a generic point of $\mathcal{X} \times \mathcal{S}$ (over A), then α is a locally definable C^p-map at (g, s). Thus, if we choose g generic in $\mathcal{X}$ over A, x and s, by the analogue of [pps1] Claim 2.9, $\alpha(g, s)$ is generic in $\mathcal{S}$ over A. Hence, $(xg^{-1}, \alpha(g, s))$ is generic in $\mathcal{X} \times \mathcal{S}$ and α is a locally definable C^p-map at (x, s) since $\alpha(x, s) = \alpha(xg^{-1}, \alpha(g, s))$. □

Corollary 7.2 *Let $\mathcal{Z}$ be a locally definable group over A and $p \in \mathbb{N}$. Then the group operations on $\mathcal{Z}$ are locally definable C^p-maps. Furthermore, if $\alpha : \mathcal{Z} \longrightarrow \mathcal{X}$ is a locally definable homomorphism over A between locally definable groups over A, then α is a locally definable C^p-homomorphism.*

Proof. The first part of the corollary follows from Theorem 7.1.

On the other hand, if g is a generic point of $\mathcal{Z}$ (over A), then α is a locally definable C^p-map at g. Thus, if we choose g generic in $\mathcal{Z}$ over A and z, it follows that (zg, g^{-1}) is generic in $\mathcal{Z} \times \mathcal{Z}$ and α is a locally definable C^p-map at z because $\alpha(z) = \alpha(zg)\alpha(g^{-1})$. □

Lemma 7.3 *Let $\mathcal{Z}$ be a locally definable group over A and $p \in \mathbb{N}$. If $\mathcal{W}$ is a compatible locally definable subgroup of $\mathcal{Z}$ over A, then the inclusion $i : \mathcal{W} \longrightarrow \mathcal{Z}$ is locally definable C^p-immersion.*

Proof. Consider the locally definable C^p-action $\alpha : \mathcal{W} \times \mathcal{Z} \longrightarrow \mathcal{Z}$ given by multiplication on the left. Then, taking into account Remark 6.3, we see that the analogue of [pps1] Corollary 2.16 holds. Therefore, as in [pps1] Lemma 2.17 we see that the inclusion homomorphism $i : \mathcal{W} \longrightarrow \mathcal{Z}$ is a C^p-immersion. □

The following observation will be used below.

Remark 7.4 Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a locally definable differentiable map over A between locally definable homogeneous spaces over A. If $\mathcal{X}$ is connected and $\mathrm{d}_x(f) = 0$ for every $x \in \mathcal{X}$, then f is a constant map.

In fact, on each open definable subset U of $\mathcal{X}$ over A, f takes only finitely many values, each on an open and closed definable subset of U over A. But $\mathcal{X}$ is connected. So f is constant on $\mathcal{X}$.

Note that, by Lemma 7.3, if $\mathcal{W}$ is a locally definable subgroup of $\mathcal{Z}$ over A, then the tangent space $T_1(\mathcal{W})$ of $\mathcal{W}$ at 1 can be identified with a subspace of the tangent space $T_1(\mathcal{Z})$ of $\mathcal{Z}$ at 1 and $T_1(\mathcal{W}) = T_1(\mathcal{W}^0)$. Moreover, if $\dim \mathcal{W} = \dim \mathcal{Z}$, then $T_1(\mathcal{W}) = T_1(\mathcal{Z})$. As we mentioned before, these tangent spaces are finite dimensional vector spaces over the underlying real closed field of $\mathcal{N}$.

Proposition 7.5 *Let $\mathcal{Z}$ and $\mathcal{X}$ be locally definable groups over A, $f : \mathcal{Z} \longrightarrow \mathcal{X}$ a locally definable homomorphism over A and $g : \mathcal{X} \longrightarrow \mathcal{X}$ a locally definable automorphism over A. Let $\mathcal{U}$ and $\mathcal{V}$ be compatible locally definable subgroups of $\mathcal{X}$ over A. Then the following holds: (i) $\mathcal{U}^0 = \mathcal{V}^0$ if and only if $T_1(\mathcal{U}) = T_1(\mathcal{V})$; (ii) $T_1(f^{-1}(\mathcal{U})) = (\mathrm{d}_1(f))^{-1}(T_1(\mathcal{U}))$ and (iii) $\mathrm{d}_1(g)$ is an automorphism of the vector space $T_1(\mathcal{X})$.*

Proof. For this, we need the locally definable version of [pps1] Theorem 2.19 (with definable subgroups replaced by compatible locally definable subgroups). To get this, we argue as in [pps1] using the locally definable version of [pps1] Corollary 2.16 and Remark 7.4 instead of [pps1] Claim 1.7.

With the locally definable version of [pps1] Theorem 2.19 and Example 6.2, the proof of the proposition is exactly as in the definable case. For more details see [pps1] 2.20, 2.21 and 2.22. □

Definition 7.6 Let $\mathcal{X}$ be a locally definable group over A. For $x \in \mathcal{X}$, let $a(x) : \mathcal{X} \longrightarrow \mathcal{X}$ be the inner automorphism $a(x)(z) = x^{-1}zx$. Let $\mathrm{Ad}(x) = \mathrm{d}_1(a(x))$. By Proposition 7.5 (iii), $\mathrm{Ad}(x) \in \mathrm{Aut}(T_1(\mathcal{X}))$ and hence $\mathrm{Ad} : \mathcal{X} \longrightarrow \mathrm{Aut}(T_1(\mathcal{X}))$ is a locally definable homomorphism over A.

Let $\mathrm{ad} = \mathrm{d}_1(\mathrm{Ad})$. Then we have $\mathrm{ad} : T_1(\mathcal{X}) \longrightarrow \mathrm{End}(T_1(\mathcal{X}))$ and on the tangent space $T_1(\mathcal{X})$ we can define a binary operation $[\ ,\ ]$ as $[\zeta, \chi] = \mathrm{ad}(\zeta)(\chi)$. As in [pps1] Claim 2.27, $(T_1(\mathcal{X}), [\ ,\ ])$ is a Lie algebra called the *Lie algebra* of $\mathcal{X}$.

With Proposition 7.5 available for locally definable groups, we easily get the locally definable version of [pps1] Theorem 2.24. Hence, with the above definition of the Lie algebra of a locally definable group, we immediately get the locally definable analogues of all the results of [pps1] Section 2.4. To avoid unnecessary repeation, we shall not write them down here and we will refer to [pps1] when needed.

7.2 The centerless case

For the reader's convinience we recall here the notions of open transitive rectangular boxes, orthogonal open transitive rectangular boxes and unidimensional open rectangular boxes. For details see [pps1] Subsection 1.3.

An open interval $I \subseteq N$ is *transitive* if for all $a, b \in I$ there are open definably homeomorphic subintervals O_a, O_b of I, containing a and b respectively, and a definable homeomorphism $f : O_a \longrightarrow O_b$ with $f(a) = b$. An open rectangular box $I_1 \times \cdots \times I_n$ is *transitive* if all intervals I_k are transitive.

Two open transitive intervals $I, J \subseteq N$ are *nonorthogonal* if there is a definable homeomorphism between some open subintervals $I_0 \subseteq I$ and $J_0 \subseteq J$. Two open transitive intervals $I, J \subseteq N$ are *orthogonal* if they are not nonorthogonal. Two open rectangular boxes $I_1 \times \cdots \times I_k$ and $J_1 \times \cdots \times J_s$ are *orthogonal* if each I_i,$i = 1, \ldots, k$, is orthogonal to every J_j, $j = 1, \ldots, s$.

Finally, an open rectangular box $I_1 \times \cdots \times I_n$ is *unidimensional* if all intervals I_i are transitive and pairwise nonorthogonal.

By exactly the same argument as in [pps1] Lemmas 1.27 and 1.28, we see that the following remark holds.

Remark 7.7 For every locally definable group $\mathcal{X}$ over A there is an open definable neighbourhood U_s of 1 in $\mathcal{X}$ over A such that $\phi_s(U_s)$ is an open transitive rectangular box.

Definition 7.8 We say that a locally definable group $\mathcal{X}$ over A is *unidimensional* if there is an open definable neighbourhood U_s of 1 in $\mathcal{X}$ over A such that $\phi_s(U_s)$ is a unidimensional open rectangular box.

Theorem 7.9 *Let $\mathcal{X}$ be a locally definable group over A which is connected and centerless. Then $\mathcal{X}$ is the direct product of compatible unidimensional locally definable subgroups over A.*

Proof. Using Remark 7.7, the proof of this result is exactly the same as that of its definable analogue [pps1] Theorem 3.1. In this proof one only uses basic facts about the notion of orthogonality, DCC and the fact that $\mathcal{X}$ is centerless and connected. □

The proof of the next theorem is a modification of the corresponding result [pps1] Theorem 3.2 for definable groups. We explain how the argument goes.

Theorem 7.10 *Let $\mathcal{X}$ be a unidimensional locally definable group over A which is connected and centerless. Then there is a definable real closed field R over A and a locally definable linear group $\mathcal{H} < GL(n, R)$ over A such that $\mathcal{X}$ is locally definably isomorphic to $\mathcal{H}$ over A.*

Proof. By Remark 7.7, there is an open transitive interval U such that $e = (d, \ldots, d)$ for some $d \in U$, where $e = \phi_s(1)$, 1 is the identity of $\mathcal{X}$ and $1 \in U_s$. Moreover, if $B = U^n$ where $n = \dim \mathcal{X}$ then $\phi_s^{-1}(B)$ is an open definable neighbourhood of 1 over A. Let $\rho : U \longrightarrow B$ be the continuous injection defined as $\rho(x) = (x, d, \ldots, d)$. Let $U^+ = \{b \in U : b > d\}$, for $b \in U^+$ let $U_b = \{c \in U : d < c < b\}$. Let $Y_b = C_{\mathcal{X}}(\overline{U_b})$ where $\overline{U_b} = \phi_s^{-1}(\rho(U_b)))$. Clearly, $\{Y_b : b \in U^+\}$ is a sequence of compatible locally definable subgroups of $\mathcal{X}$ over N such that if $b' < b$ then $Y_b \subseteq Y_{b'}$. Therefore, by DCC, $\{C_{\mathcal{X}}(Y_b) : b \in U^+\}$ is a sequence of compatible locally definable subgroups of $\mathcal{X}$ over N such that if $b' < b$ then $C_{\mathcal{X}}(Y_{b'}) \subseteq C_{\mathcal{X}}(Y_b)$.

Let $Y = \cup\{Y_b : b \in U^+\}$. Then $C_{\mathcal{X}}(Y) = \cap\{C_{\mathcal{X}}(Y_b) : b \in U^+\}$. Hence by DCC, there is subset $\{b_s : s \in S\}$ of U^+ with $|S| < \aleph_1$ and such that $C_{\mathcal{X}}(Y) = \cap\{C_{\mathcal{X}}(Y_{b_s}) : s \in S\}$.

By saturation, there is $b \in U^+$ such that $b < b_s$ for all $s \in S$. Then $\overline{U_b} \subseteq C_{\mathcal{X}}(Y_b) \subseteq C_{\mathcal{X}}(Y_{b_s})$ for all $s \in S$, so $\overline{U_b} \subseteq C_{\mathcal{X}}(Y_b) \subseteq C_{\mathcal{X}}(\cup\{Y_{b_s} : s \in S\})$ and $Y \subseteq C_{\mathcal{X}}(\overline{U_b}) = Y_b$. Thus, Y is a compatible locally definable subgroup of $\mathcal{X}$ over a set of cardinality less than $\aleph_1$. Since $\mathcal{X}$ is centerless and connected, $\dim Y < \dim \mathcal{X}$ (otherwise, by Propositions 3.5 and 3.9, $Y = \mathcal{X}$ and $\overline{U_b} \subseteq Z(\mathcal{X})$). Hence, B cannot be covered by finitely many left cosets of Y, and arguing as in [pps1] 3.2.1, there is a definable real closed field R over N on some open subinterval of U. But then, there is one such definable real closed field on some open subinterval of U defined over A.

Let V be a definable subset of U_s over A such that $\phi_s(V) \subseteq R^n$. As in [pps1] Claim 3.12, there is an open definable subset V_0 of V over A containing 1 such that, group multiplication and inversion are C^1-maps (with respect to R) from $V_0 \times V_0$ and V_0, respectively, into V. Furthermore, just like in [pps1] Claim 3.13, for every locally definable endomorphism h of $\mathcal{X}$ over A, there is a definable open subset D of V over A containing 1 such that $h(D) \subseteq V$ and h is C^1 on D with respect to R. Like in [pps1] Subsection 1.1.2, we can define the tangent space of $\mathcal{X}$ at 1, denoted $T_1(\mathcal{X})$.

For each $x \in \mathcal{X}$, consider the locally definable automorphism $a(x) : \mathcal{X} \longrightarrow \mathcal{X}$ over A given by $a(x)(z) = xzx^{-1}$. Let $\mathrm{Ad} = \mathrm{d}_1(a(x))$. Then, after fixing a basis for $T_1(\mathcal{X})$, the map $\mathrm{Ad} : \mathcal{X} \longrightarrow GL(n, R)$ is a locally definable homomorphism over A. Since $\mathcal{X}$ is connected and centerless, by the analogue of [pps1] Claim 3.14, $\mathrm{Ad}(x)$ is a locally definable injective homomorphism over A. Note that, since $\mathcal{X}$ is connected, two locally definable automorphisms of $\mathcal{X}$ over A are equal if and only if they coincide on a definable open neighbourhood of 1 over A. Hence, [opp] Lemma 3.2 holds in our case and therefore, [pps1] Claim 3.14 also holds. □

Corollary 7.11 *Let $\mathcal{Y}$ be a connected, centerless, locally definable solvable group over A. Then $\mathcal{Y}$ is a direct product of compatible locally definable groups $\mathcal{Y}_1, \ldots, \mathcal{Y}_k$ over A and there are definable real closed fields $R_1, \ldots, R_k$ over A such that, for each $i \in \{1, \ldots, k\}$, there is a solvable definable subgroup G_i of $GL(n_i, R_i)$ over A such that $\mathcal{Y}_i$ is locally definably isomorphic to an open and closed locally definable subgroup of G_i over A.*

Proof. By Theorems 7.9 and 7.10, $\mathcal{Y}$ is a direct product of locally definable subgroups $\mathcal{Y}_1, \ldots, \mathcal{Y}_k$ over A such that for each $i \in \{1, \ldots, k\}$, there is a definable real closed field R_i over A and a locally definable subgroup $\mathcal{Z}_i < GL(n_i, R_i)$ over A locally definably isomorphic to $\mathcal{Y}_i$ over A. By DCC, let $G_i = d(\mathcal{Z}_i)$ be the smallest definable subgroup of $GL(n_i, R_i)$ over A containing $\mathcal{Z}_i$. By [e] Lemma 6.7, G_i is solvable. □

7.3 The locally definably semi-simple case

Definition 7.12 We say that a locally definable group $\mathcal{X}$ over A is *locally definably semi-simple* if $\mathcal{X}$ has no compatible locally definable normal abelian subgroups over A of dimension bigger than zero.

We will say that a locally definable group $\mathcal{X}$ over A is *locally definably simple* if $\mathcal{X}$ has no compatible locally definable normal subgroups over A.

Note that a locally definably semi-simple locally definable group has centre of dimension zero. The following fact is the analogue of [pps1] Theorem 2.34. The proof is the same.

Theorem 7.13 *Suppose that $\mathcal{N}$ is an expansion of a real closed field and let $\mathcal{X}$ be a connected locally definable group over A. Then $\mathcal{X}$ is locally definably semi-simple if and only if its Lie algebra* $\mathbf{x}$ *is semi-simple.*

The next result is the analogue of [pps1] Theorem 2.36. Again, the proof is similar.

Theorem 7.14 *Suppose that $\mathcal{N}$ is an expansion of a real closed field and let $\mathcal{X}$ be a connected, centerless locally definable group over A. Then $\mathcal{X}$ is locally definably simple if and only if its Lie algebra* $\mathbf{x}$ *is simple.*

Using Theorems 7.13, 7.14 and some Lie algebra theory like in [pps1] Theorem 2.38, we get the next result.

Theorem 7.15 *Suppose that $\mathcal{N}$ is an expansion of real closed field. If $\mathcal{X}$ is a connected, centerless, locally definably semi-simple locally definable group over A, then $\mathcal{X}$ is the direct product of locally definably simple, compatible locally definable subgroups over A.*

We are now ready to prove our main result on locally definably semi-simple locally definable groups.

Theorem 7.16 *Suppose that $\mathcal{X}$ is a connected, locally definable group over A with no non trivial locally definable abelian normal subgroups over A. Then $\mathcal{X}$ is a direct product of compatible locally definable groups $\mathcal{X}_1, \ldots, \mathcal{X}_k$ over A and there are definable real closed fields $R_1, \ldots, R_k$ over A such that, for each $i \in \{1, \ldots, k\}$, the following holds: (i) there is an R_i-semialgebraic subgroup G_i of $GL(n_i, R_i)$ which is G_i-definably simple with definably connected component G_i^0 definably simple and such that $\mathcal{X}_i$ is locally definably isomorphic to a locally definable open and closed subgroup of G_i; (ii) $\mathcal{X}_i$ is locally definably simple.*

Proof. By Theorems 7.9 and 7.10, $\mathcal{X}$ is a direct product of locally definable subgroups $\mathcal{X}_1, \ldots, \mathcal{X}_m$ over A such that for each $i \in \{1, \ldots, m\}$ there is a definable real closed field R_i over A and locally definable subgroups $\mathcal{Y}_i < GL(n_i, R_i)$ over A locally definably isomorphic to $\mathcal{X}_i$ over A. Clearly, each $\mathcal{X}_i$ (and hence each $\mathcal{Y}_i$) is connected. Similarly, each $\mathcal{X}_i$ (resp., $\mathcal{Y}_i$) has no non trivial locally definable abelian normal subgroups over A. Clearly, it is now sufficient to prove the theorem for each $\mathcal{Y}_i$. So let $\mathcal{Y} \in \{\mathcal{Y}_1, \ldots, \mathcal{Y}_m\}$ and let R be the corresponding definable real closed field. Clearly, we may assume that $\mathcal{N}$ is an expansion of the real closed field R.

Since $\mathcal{Y}$ is locally definably semi-simple and centerless, by Theorem 7.13, the Lie algebra $\mathbf{y}$ of $\mathcal{Y}$ is semi-simple. By Theorem 7.10, $\mathrm{Ad} : \mathcal{Y} \longrightarrow \mathcal{Z} < GL(n, R)$ where $n = \dim \mathcal{Y}$, is a locally definable isomorphism over A. Let $G = \mathrm{Aut}(\mathbf{y})$. Clearly, G is a definable group. Moreover, by the analogue of [pps1] Claim 2.29, $\mathcal{Z}$ is a locally definable subgroup of G. As $\mathcal{Y}$ is connected, so is $\mathcal{Z}$. Similarly, $\mathcal{Z}$ is locally definably semi-simple and centerless. Since $\mathbf{y}$ is semi-simple, $\dim G = \dim \mathbf{y} = \dim \mathcal{Y} = \dim \mathcal{Z}$. So $\mathcal{Z}$ is an open and closed locally definable subgroup of G over A. Hence, the Lie algebra of G is the same as that of $\mathcal{Z}$ which is the same as that of $\mathcal{Y}$. So by [pps1] Theorem 2.34, G is a definably semi-simple definable group and by [pps1] Claim 1.3, G^0 and G are R-semialgebraic groups.

Suppose that $\mathcal{Z}$ is not locally definably simple. Then by Theorem 7.15, $\mathcal{Z}$ is a direct product of locally definably simple compatible locally definable subgroups over A. Hence, an induction on $\dim \mathcal{Z}$, Theorem 7.14 and the argument in the last paragraph ends the proof of the theorem. □

Corollary 7.17 *Let $\mathcal{X}$ be a locally definably simple locally definable group over A. Then there is a definable real closed field R over A and an R-semialgebraic linear group G which is G-definably simple and such that $\mathcal{X}$ is locally definably isomorphic over A to an open and closed locally definable subgroup of G over A.*

8 Some corollaries

Here we will include several corollaries of our previous results. All of these have an analogue in the definable case.

Corollary 8.1 *Let $\mathcal{U}$ be a locally definable group over A and let $T = \{T(x) : x \in X\}$ be a definable family of non empty definable subsets of $\mathcal{U}$ over A. Then there is a definable function $t : X \longrightarrow \mathcal{U}$ over A such that for all $x, y \in X$ we have $t(x) \in T(x)$ and if $T(x) = T(y)$ then $t(x) = t(y)$ (i.e., t is a strong definable choice for T).*

Proof. Let $D = \cup\{T(x) : x \in X\}$. Then D is a definable subset of $\mathcal{U}$ over A. Suppose that $\mathcal{U}$ is definably compact. By Lemma 5.1, the closure $\overline{D}$ of D in $\mathcal{U}$ is a closed definable subset of $\mathcal{U}$, hence $\overline{D}$ is definably compact. Let $\overline{T} = \{\overline{T(x)} : x \in X\}$, where $\overline{T(x)}$ is the closure of $T(x)$ in $\mathcal{U}$. Then $\overline{T}$ is a definable family of non empty definably compact definable subsets of $\overline{D}$ over A. By [e] Lemma 7.1, there is a strong definable choice $l : X \longrightarrow \overline{D}$ over A for the definable family $\{\overline{T(x)} : x \in X\}$. Let O be the definable neighbourhood of 1 in $\mathcal{U}$ over A which has strong definable choice given by the analogue of [e] Lemma 2.3. And consider the definable family $S = \{S(x) : x \in X\}$ of non empty definable subsets of O over A where $S(x) = \{z \in O : l(x)z \in l(x)O \cap T(x)\}$. Note that, if $T(x) = T(y)$ then $S(x) = S(y)$. Let s be a strong definable choice for S over A. Then clearly, $t : X \longrightarrow \mathcal{U}$ given by $t(x) = l(x) \cdot s(x)$ is a strong definable choice for $\{T(x) : x \in X\}$ over A.

We now verify the result for $\mathcal{U}$ a locally definable solvable group over A. By Theorems 4.2 and 5.3 we have a locally definable extension $1 \rightarrow \mathcal{V} \rightarrow \mathcal{U} \xrightarrow{l} \mathcal{U}/\mathcal{V} \rightarrow 1$ over A with a locally definable section $s : \mathcal{U}/\mathcal{V} \longrightarrow \mathcal{U}$ over A with $\mathcal{V} = \mathcal{X} \times W$, W a definable solvable group, $\mathcal{X}$ and $\mathcal{U}/\mathcal{V}$ definably compact locally definable solvable groups over A. By the analogue of [e] Proposition

3.23, $\mathcal{U}$ is locally definably isomorphic over A with a locally definable group over A with domain $\mathcal{V} \times \mathcal{U}/\mathcal{V}$. Since W is definable, the result holds for W by [e] Theorem 7.2. By the last paragraph, the result holds for $\mathcal{X}$ and $\mathcal{U}/\mathcal{V}$. Therefore, by [e] Fact 2.2 (iii), the result holds for $\mathcal{U}$.

Finally, let $\mathcal{U}$ be an arbitrary locally definable group over A. Let $\mathcal{Z}$ be a connected, compatible, locally definable normal solvable subgroup of $\mathcal{U}$ over A of maximal dimension. Then by Corollary 4.8 and Remark 4.6, $\mathcal{U}/\mathcal{Z}$ has no non trivial connected, compatible locally definable normal solvable subgroup over A. In particular, $\mathcal{U}/\mathcal{Z}$ locally definably semi-simple.

By what we have proved in the last paragraph, $\mathcal{Z}$ has strong definable choice. Also, by Theorem 7.16, [e] Remark 2.4 and [e] Fact 2.2 (iii), the result holds for $\mathcal{U}/\mathcal{Z}$. By Theorem 4.2, we have a locally definable extension $1 \rightarrow \mathcal{Z} \rightarrow \mathcal{U} \xrightarrow{\iota} \mathcal{U}/\mathcal{Z} \rightarrow 1$ over A with a locally definable section $s : \mathcal{U}/\mathcal{Z} \longrightarrow \mathcal{U}$ over A. By analogue of [e] Proposition 3.23, $\mathcal{U}$ is locally definably isomorphic over A with a locally definable group over A with domain $\mathcal{Z} \times \mathcal{U}/\mathcal{Z}$. So by [e] Fact 2.2 (iii) again the result holds for $\mathcal{U}$. □

We are now ready to prove the analogue of property (AB) for definable groups (i.e., [p1] Corollary 2.15 (see also [p2] Proposition 5.6)).

Corollary 8.2 *Let $\mathcal{X}$ be a locally definable group over A of positive dimension. Then $\mathcal{X}$ has a compatible locally definable abelian subgroup over A of positive dimension.*

Proof. Clearly we may assume that $\mathcal{X}$ has no proper compatible locally definable subgroups over A of positive dimension. In fact, if $\mathcal{W}$ is a proper compatible locally definable subgroup of $\mathcal{X}$ over A of positive dimension with a compatible locally definable abelian subgroup $\mathcal{Z}$ over A of positive dimension, then by Lemma 3.2 (iii), $\mathcal{Z}$ is a compatible locally definable abelian subgroup of $\mathcal{X}$ over A of positive dimension.

By assumption on $\mathcal{X}$, the centre $Z(\mathcal{X})$ has dimension zero. Let $\mathcal{Y} = \mathcal{X}/Z(\mathcal{X})$. Then $\mathcal{Y}$ is a locally definable group over A of positive dimension and by Corollary 3.16, $\mathcal{Y}$ is centerless. Furthermore, $\mathcal{Y}$ also has no proper compatible locally definable subgroups over A of positive dimension. In particular, $\mathcal{Y}$ is locally definably simple.

By Corollary 7.17, $\mathcal{Y}$ is locally definably isomorphic over A to an open and closed locally definable subgroup $\mathcal{U}$ over A of a definable group G over

A. By [p1] Corollary 2.15, G has a definable abelian subgroup H over A of positive dimension. Then $\mathcal{U} \cap H$ is a locally definable abelian subgroup of $\mathcal{U}$ over A of positive dimension (since $\mathcal{U}$ is open in G). So $\mathcal{Y}$ has a locally definable abelian subgroup $\mathcal{V}$ over A of positive dimension. But by DCC, $Z(C_{\mathcal{Y}}(\mathcal{V}))$ is a compatible locally definable abelian subgroup of $\mathcal{Y}$ over A containing $\mathcal{V}$. So $\mathcal{Y} = Z(C_{\mathcal{Y}}(\mathcal{V}))$ and $\mathcal{Y}$ is abelian and centerless, which is a contradiction. $\square$

Corollary 8.3 *Suppose that $\mathcal{U}$ is a connected, locally definable group over A of dimension one. Then $\mathcal{U}$ is abelian, and either $\mathcal{U}$ is torsion-free and locally definably totally ordered or $\mathcal{U}$ is a definably compact definable group.*

Proof. By (AB) $\mathcal{U}$ is abelian. Suppose that $\mathcal{U}$ is definable. Then by Theorem 5.3, $\mathcal{U}$ is either torsion-free and definably ordered or $\mathcal{U}$ is definably compact. So we may assume that $\mathcal{U}$ is not definable and is definably compact.

Suppose that $\mathcal{U} = \{U_i : i \in I\}$ and $\{U_i : i \in I\}$ is the collection of all open definable subsets of $\mathcal{U}$ over A. Then, for every finite subset J of I, the set $\cup\{U_i : i \in J\}$ can be definably totally ordered. So $\mathcal{U}$ is locally definably totally ordered as a set and $\mathcal{U} \setminus \{0\}$ has two connected components $\mathcal{U}^-$ and $\mathcal{U}^+$. The argument in the proof of [r] Proposition 3, shows that $\mathcal{U}$ is torsion-free. $\square$

Corollary 8.4 *Let $\mathcal{X}$ be a connected, locally definable group over A. If $\mathcal{X}$ is not nilpotent, then a real closed field is definable in $\mathcal{N}$.*

Proof. If $\mathcal{X}$ is not solvable, let $\mathcal{Z}$ be a connected, compatible, locally definable normal solvable subgroup of $\mathcal{Z}$ over A of maximal dimension. Then by Corollary 4.8 and Remark 4.6, $\mathcal{X}/\mathcal{Z}$ is connected and has no non trivial connected, compatible locally definable normal solvable subgroup over A. In particular, $\mathcal{X}/\mathcal{Z}$ is a connected locally definably semi-simple, locally definable group over A and hence, the result follows from Theorem 7.16. On the other hand, if $\mathcal{X}$ is solvable but not nilpotent, then by Proposition 5.6, $\mathcal{Y} = \mathcal{X}/U(\mathcal{X})$ is a connected locally definable solvable group over A with centre of dimension zero. Therefore, by Corollary 3.16, $\mathcal{Y}/Z(\mathcal{Y})$ is a connected, centerless, locally definable solvable group over A. Thus, in this case the result follows from Corollary 7.11. $\square$

The definable analogue of Corollary 8.4 is much stronger. It says that if $G = (G, \cdot)$ is a definable group which is not abelian by finite, then a real closed field is definable in the structure $(N, <, G, \cdot)$ (see [e] or [pst2]). In fact, by [pst2], a field is interpretable in the structure $(G, \cdot)$.

References

[BCR] J.Bochnak, M. Coste and M-F. Roy *Real Algebraic Geometry* Springer-Verlag 1998.

[dk] H.Delfs and M.Knebusch *Locally semi-algebraic spaces* Springer Verlag 1985.

[vdd] L. van den Dries, *Tame Topology and o-minimal structures* Cambridge University Press 1998.

[e] M.Edmundo *Solvable groups definable in o-minimal structures* J. Pure Appl. Algebra 185 (2003) 103-145.

[opp] M.Otero, Y.Peterzil and A.Pillay *On groups and rings definable in o-minimal expansions of real closed fields* Bull. London Math. Soc. 28 (1996) 7-14.

[pst1] Y.Peterzil and S.Starchenko *A trichotomy theorem for o-minimal structures* Proc. London Math. Soc. 77 (1998) 481-523.

[pst2] Y.Peterzil and S.Starchenko *Definable homomorphisms of abelian groups definable in o-minimal structures* Ann. Pure Appl. Logic 101 (1) (1999) 1-27.

[ps] Y.Peterzil and C.Steinhorn *Definable compacteness and definable subgroups of o-minimal groups* J. London Math. Soc. 59 (2)(1999) 769-786.

[pps1] Y.Peterzil, A.Pillay and S.Starchenko *Definably simple groups in o-minimal structures* Trans. Amer. Math. Soc. 352 (10)(2000) 4397-4419.

[pps2] Y.Peterzil, A.Pillay and S.Starchencko *Simple algebraic groups over real closed fields* Trans. Amer. Math. Soc. 352 (10) (2000) 4421-4450.

[pps3] Y.Peterzil, A.Pillay and S.Starchencko *Linear groups definable in o-minimal structures* J. Algebra 247 (2002) 1-23.

[p1] A.Pillay *On groups and fields definable in o-minimal structures* J. Pure Appl. Algebra 53 (1988) 239-255.

[p2] A.Pillay *First order topological structures and theories* J. Symbolic Logic 52 (1988) (3) 763-778.

[r] V.Razenj *One-dimensional groups over an o-minimal structure* J. Pure Appl. Algebra 53 (1991) 269-277.

[s] A.Strzebonski *Euler characteristic in semialgebraic and other o-minimal groups* J. Pure Appl. Algebra 96 (1994) 173-201.