A remark on divisibility of definable groups

Mário J. Edmundo *
CMAF Universidade de Lisboa
Av. Prof. Gama Pinto 2
1649-003 Lisboa, Portugal
edmundo@cii.fc.ul.pt

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Abstract

We show that if G is a definably compact, definably connected definable group defined in an arbitrary o-minimal structure, then G is divisible. Furthermore, if G is defined in an o-minimal expansion of a field, $k \in \mathbb{N}$ and $p_k : G \longrightarrow G$ is the definable map given by $p_k(x) = x^k$ for all $x \in G$, then for all $x \in G$, we have $|(p_k)^{-1}(x)| \geq k^r$ where $r > 0$ is the maximal dimension of abelian definable subgroups of G.

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1 On divisibility of definable groups

We work over an arbitrary o-minimal structure $\mathcal{N}$ and definable means $\mathcal{N}$-definable (possibly with parameters). We are interested here in understanding the subset $(p_m)^{-1}(x)$ of a definably compact definably connected definable group G where $x \in G$ and, for each $m \in \mathbb{N}$, $p_m : G \longrightarrow G$ is the definable map that sends z to z^m.

A definable group is a group whose underlying set is a definable set and the graphs of the group operations are definable sets. The notion of definably compact is the analogue of the notion of semi-algebraically complete and was introduced Peterzil and Steinhorn in [8]. The theory of definable groups, which includes real algebraic groups and semi-algebraic groups, began with Anand Pillay's paper [12] and has since then grown into a well developed branch of mathematics (see for example [8], [9] and [10]). In the literature there are many interesting results about definable groups which have an analogue in the theory of Lie groups. Among these we have the following properties:

(TOP) every definable group G has a unique definable topological structure such that the group operations are continuous and the definable homomorphisms are also continuous;

(DCC) the descending chain condition for definable subgroups of a definable group G;

(QT) existence in the category of definable groups of the quotient of a definable group by a definable normal subgroup together with the existence of a corresponding definable section;

(AB) every definable group G of positive dimension has a definable abelian subgroup of positive dimension.

Properties (TOP), (DCC) and (AB) were proved in [12]. Property (QT) is from [2]. In the paper [13] by Strebonski the following property is proved using the o-minimal Euler characteristic for definable sets defined by van den Dries [1] using the cell decomposition theorem. This is an Euler-Grothendieck characteristic for the definable category in the sense of [6].

(TOR) If G is a definable group, then for all $m \in \mathbb{N}$, the subgroup $G[m]$ of

m-torsion points of G is a finite definable subgroup.

Property (TOR) has the following consequence:

(DIV) If G is a definably connected definable *abelian* group, then G is divisible, i.e., for all $m \in \mathbb{N}$, the definable homomorphism $p_m : G \longrightarrow G$ is surjective.

In fact, let G be a definably connected definable abelian group and $m \in \mathbb{N}$. Since $G[m]$ is a finite definable subgroup, mG is a definable subgroup of G of dimension $\dim G$. By [12], $(G : mG) < \aleph_0$. But since G is definably connected, again by [12], we must have $mG = G$ as required.

In the paper [7] Margarita Otero observes that with the results from o-minimal algebraic topology available in [3] together with classical arguments one can prove that *if $\mathcal{N}$ is an o-minimal expansion of a field, then a definably connected, definably compact definable group is divisible.* Here we generalise this result in the following way:

Theorem 1.1 *If G is a definably compact, definably connected definable group defined in an arbitrary o-minimal structure, then G is divisible. Furthermore, if G is defined in an o-minimal expansion of a field, $k \in \mathbb{N}$ and $p_k : G \longrightarrow G$ is the definable map given by $p_k(x) = x^k$ for all $x \in G$, then for all $x \in G$, we have $|(p_k)^{-1}(x)| \geq k^r$ where $r > 0$ is the maximal dimension of abelian definable subgroups of G.*

The proof of this theorem does not require o-minimal algebraic topology and follows at once from the next Proposition.

Proposition 1.2 *Let G be a definably compact, definably connected definable group defined in an arbitrary o-minimal structure. Then there exists a definably compact, definably connected definable abelian subgroup B of G which is unique up to conjugation such that:*

(i) $\dim B$ *is the maximal dimension of abelian definable subgroups of G and*

(ii) $G = \cup\{gBg^{-1} : g \in G\}$.

From Proposition 1.2 we get Theorem 1.1 in the following way. By Proposition 1.2, for $x \in G$, there is $b \in B$ and $g \in G$ such that $x = gbg^{-1}$. By

(DIV) let $a \in B$ be such that $a^k = b$. Then $(gag^{-1})^k = ga^kg^{-1} = gbg^{-1} = x$. Thus G is divisible.

Assume now that $\mathcal{N}$ is an o-minimal expansion of a field and $l = \dim B$. Then by the proof of the structure theorem in [4] (or [3]), the degree of p_k in B is k^l, and so $|(p_k)^{-1}(b)| \geq k^l$ for all $b \in B$. On the other hand, for $x \in G$, by Proposition 1.2, there is $b \in B$ and $g \in G$ such that $x = gbg^{-1}$. By (DIV) let $a \in B$ be such that $a^k = b$. Then $(gag^{-1})^k = ga^kg^{-1} = gbg^{-1} = x$. Thus $|(p_k)^{-1}(x)| \geq k^l$ for all $x \in G$.

For the proof of Proposition 1.2 we require the following claims.

Claim 1.3 *Let G be a group, H a subgroup of G and Z a normal subgroup of G contained in H and in the center of G. If $G/Z = \cup\{gZ(H/Z)g^{-1}Z : g \in G\}$, then $G = \cup\{gHg^{-1} : g \in G\}$.*

Proof. Let $a \in G$. Then there is $g \in G$ and $h \in H$ such that $aZ = gZhZg^{-1}Z$. So, there are $z_1, z_2, z_3 \in Z$ such that $a = gz_1hz_2g^{-1}z_3$. Since $Z \subseteq Z(G)$, we get $a = g(z_1hz_2z_3)g^{-1}$. Hence, $G = \cup\{gHg^{-1} : g \in G\}$ as required. □

Claim 1.4 *Let G be a group and let H and K be subgroups of G containing a normal subgroup Z of G which is contained in the center of G. If H/Z and K/Z are conjugate subgroups of G/Z, then H and K are conjugates subgroups of G.*

Proof. Suppose that $H/Z = gZ(K/Z)g^{-1}Z$. If $h \in H$, then there are $z_1, z_2, z_3 \in Z$ and $k \in K$ such that $h = gz_1kz_2g^{-1}z_3$. Since $Z \subseteq Z(G)$ and $Z \subseteq K$, we have $h = g(z_1kz_2z_3)g^{-1} \in gKg^{-1}$ and $H \subseteq gKg^{-1}$. Similarly, if $k \in K$, then there are $z \in Z$ and $h \in H$ such that $gkg^{-1} = hz \in H$ (since $Z \subseteq H$) and so $gKg^{-1} \subseteq H$. In conclusion, $H = gKg^{-1}$ as required. □

Proof of Proposition 1.2. By [2] Corollary 4.8 or [9] Corollary 5.4, G is abelian or $G/Z(G)^0$ is definably semi-simple. In the first case the result follows from (DIV) and the main theorem of [4] (or [3]). So assume that $G/Z(G)^0$ is definably semi-simple. The quotient of $G/Z(G)^0$ by its finite center is by [10] the direct product $H_1 \times \cdots \times H_k$ of subgroups such that each

H_i is definably isomorphic to an R_i-semialgebraic subgroup of $\mathrm{GL}(n_i, R_i)$ where R_i is a real closed field definable in $\mathcal{N}$. Moreover, by the proof of [11] Theorem 5.1, each H_i is a definably simple, definably compact, definably connected R_i-definable group defined over the empty set and so $H_i(\mathbb{R})$ is a compact, connected, simple Lie group.

If T_i is the maximal torus of $H_i(\mathbb{R})$, then T_i is definable in $\mathbb{R}$ as the definably connected component of $Z(C(T_i))$ (using the DCC on definable groups and the fact that any definable group in the field of real numbers is a Lie group). Also, since $\dim T_i$ is the maximal dimension of abelian compact Lie subgroups of $H_i(\mathbb{R})$, by the definability of o-minimal dimension (see [1]), we can assume that T_i is defined over the emptyset. As $H_i(\mathbb{R})$ is covered by the conjugates of T_i, and the real closed field R_i is elementarily equivalent to $\mathbb{R}$, H_i is covered by the conjugates (by elements of H_i) of $T_i(R_i)$. A similar argument and definability of o-minimal dimension shows that $\dim T_i$ is the maximal dimension of abelian definable subgroups of H_i. Furthermore, if C_i is a definably connected definable abelian subgroup of H_i such that $\dim C_i = \dim T_i$, then C_i is a conjugate of $T_i(R_i)$. Indeed, by definability of o-minimal dimension, there is a first-order formula $\phi(u, v)$ over the empty set such that for every $r \in R_i$, the formula $\phi(u, r)$ defines a definably connected definable abelian subgroup of H_i of dimension $\dim T_i$ and C_i is defined by $\phi(u, s)$ for some $s \in R_i$. (For the fact that "definably connected" is first-order see [5] Theorem 0.3). Now since the first-order sentence saying that for all r the subgroup of $H_i(\mathbb{R})$ defined by $\phi(u, r)$ is a conjugate of T_i is true in $\mathbb{R}$, it must also hold in R_i. Hence, C_i is a conjugate of $T_i(R_i)$ as required.

To simplify the notation, we will from now on use T_i to denote $T_i(R_i)$ for each $i = 1, \dots, k$.

Let A be the minimal (by DCC) definable subgroup of $G/Z(G)^0$ such that $A/Z(G/Z(G)^0) = T_1 \times \cdots \times T_k$. Then $\dim \mathrm{A} = \dim(T_1 \times \cdots \times T_k)$ and by Claim 1.3, $G/Z(G)^0 = \cup\{gZ(G)^0 A g^{-1} Z(G)^0 : g \in G\}$. Note also that A is definably connected. In fact, if A^0 is the definably connected component of A, then by [2] Lemma 3.15 its quotient by the finite center of $G/Z(G)^0$ is a definably connected subgroup of $T_1 \times \cdots \times T_k$ of maximal dimension. Thus this quotient is $T_1 \times \cdots \times T_k$ and so $A = A^0$.

If B is the minimal definable subgroup of G such that $B/Z(G)^0 = A$, then $\dim B = \dim Z(G)^0 \cdot \dim A$ and by Claim 1.3, $G = \cup\{gBg^{-1} : g \in G\}$. Also by the argument above B is definably connected. Since B is solvable definably compact, it is abelian (by [2] Corollary 4.8 or [9] Corollary 5.4).

It remains to show that B is unique up to conjugation and $\dim B$ is the maximal dimension of abelian definable subgroups of G. Let B_1 be a definably connected definable abelian subgroup of G. By [2] Lemma 3.15, $A_1 = B_1/Z(G)^0$ and $A_1/Z(G/Z(G)^0)$ are definably connected definable abelian subgroups of $G/Z(G)^0$ and $H_1 \times \cdots \times H_k$ respectively. Since $\dim T_i$ is the maximal dimension of abelian definable subgroups of H_i, it follows that $\dim A_1 \leq \dim A$. So $\dim B_1 = \dim(B_1 \cap Z(G)^0) \cdot \dim A_1 \leq \dim B$.

Assume now that $\dim B_1 = \dim B$. Then necessarily $\dim(B_1 \cap Z(G)^0) = \dim Z(G)^0$ and $\dim A_1 = \dim A$. The first condition implies that $Z(G)^0 \subseteq B_1$ and so, by Claim 1.4, we need to show that A_1 is conjugate to A in order to conclude that B_1 is conjugate to B. For this, for each $i = 1, \dots, k$, consider the projection C_i of $A_1/Z(G/Z(G)^0)$ into H_i. By [2] Lemma 3.15, each C_i is a definably connected definable abelian subgroup of H_i. Since $A_1/Z(G/Z(G)^0)$ is contained in $C_1 \times \cdots \times C_k$ and $\dim A_1 = \dim A$, we must have $\dim C_i = \dim T_i$ for each i. Thus, since $C_1 \times \cdots \times C_k$ is definably connected, $A_1/Z(G/Z(G)^0) = C_1 \times \cdots \times C_k$. Now as each C_i is a conjugate of T_i, it follows that $C_1 \times \cdots \times C_k$ is a conjugate of $T_1 \times \cdots \times T_k$ and so by Claim 1.4 A_1 is a conjugate of A as required.

References

[1] L. van den Dries, *Tame topology and o-minimal structures* Cambridge University Press 1998.

[2] M.Edmundo *Solvable groups definable in o-minimal structures* J. Pure Appl. Algebra 185 (2003) 103–145.

[3] M.Edmundo *O-minimal cohomology and definably compact definable groups* RAAG preprint n. 24 (2004) (http://ihp-raag.org/).

[4] M.Edmundo and M.Otero *Definably compact abelian groups* J. Math. Logic 4 (2) (2004) 163–180.

[5] J.Knight, A.Pillay and C. Steinhorn *Definable sets in ordered structures II* Trans. Amer. Math. Soc. 295 (1986) 593-605.

[6] J. Krajicek and T. Scanlon *Combinatorics with definable sets: Euler characteristics and Grothendieck rings* Bull. Symbolic Logic 6 (2000) 311–330.

[7] M.Otero *On divisibility in definable groups* RAAG preprint n. 147 (2004) (http://ihp-raag.org/).

[8] Y.Peterzil and C.Steinhorn *Definable compacteness and definable subgroups of o-minimal groups* J. London Math. Soc. 59 (2) (1999) 769–786.

[9] Y.Peterzil and S.Starchenko *Definable homomorphisms of abelian groups definable in o-minimal structures* Ann. Pure Appl. Logic 101 (1) (1999) 1–27.

[10] Y.Peterzil, A.Pillay and S.Starchenko *Definably simple groups in o-minimal structures* Trans. Amer. Math. Soc. 352 (10)(2000) 4397–4419.

[11] Y.Peterzil, A.Pillay and S.Starchenko *Linear groups definable in o-minimal structures* J. Algebra 247 (2002) 1-23.

[12] A.Pillay *On groups and fields definable in o-minimal structures* J. Pure Appl. Algebra 53 (1988) 239–255.

[13] A.Strzebonski *Euler characteristic in semialgebraic and other o-minimal groups* J. Pure Appl. Algebra 96 (1994) 173–201.