

# Structure theorems for o-minimal expansions of groups \*

Mario J. Edmundo †  
The Mathematical Institute  
24-29 St Giles  
OX1 3LB Oxford, U.K  
edmundo@maths.ox.ac.uk

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## Abstract

Let  $\mathcal{R}$  be an o-minimal expansion of an ordered group  $(R, 0, 1, +, <)$  with distinguished positive element 1. We first prove that the following are equivalent: (1)  $\mathcal{R}$  is semi-bounded, (2)  $\mathcal{R}$  has no poles, (3)  $\mathcal{R}$  cannot define a real closed field with domain  $R$  and order  $<$ , (4)  $\mathcal{R}$  is eventually linear and (5) every  $\mathcal{R}$ -definable set is a finite union of cones. As a corollary we get that  $Th(\mathcal{R})$  has quantifier elimination and universal axiomatization in the language with symbols for the ordered group operations, bounded  $\mathcal{R}$ -definable sets and a symbol for each definable endomorphism of the group  $(R, 0, +)$ .

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# 1 Introduction

Let  $\mathcal{R} = (R, 0, 1, +, <, \dots)$  be an o-minimal expansion of an ordered group  $(R, 0, +, <)$  (where  $1 > 0$ ). The structure  $\mathcal{R}$  will be fixed throughout, definable will mean definable in  $\mathcal{R}$  with parameters.

For the rest of the paper we assume the readers familiarity with the basic results of o-minimality, namely those that can be found in the first few chapters of [vdd] or, in [kps] and [ps].

The structure of the paper is as follows: below we introduce the main concepts to be used, we state our results and illustrate these by presenting some examples. In section 2 we prove part of the theorem stated in the abstract, namely  $(1) \Rightarrow (2)$  and  $(2) \Leftrightarrow (3)$  ( $(3) \Rightarrow (4)$  and  $(5) \Rightarrow (1)$  are immediate). In section 3 we introduce the important notion of linear dimension, which we use in section 4 where we prove the structure theorem for eventually linear  $\mathcal{R}$  (in particular  $(4) \Rightarrow (5)$ ).

Associated to  $\mathcal{R}$  we have the division ring of all definable endomorphisms  $\Lambda := \Lambda(\mathcal{R})$ , which is a subring of the division ring  $\Lambda_{p.e} := \Lambda_{p.e}(\mathcal{R})$ , of all (germs at 0 of) definable partial endomorphisms; below  $\Lambda_{p.e}^0 := \Lambda_{p.e}^0(\mathcal{R})$ , denotes the subdivision ring of  $\Lambda_{p.e} := \Lambda_{p.e}(\mathcal{R})$ , of all (germs at 0 of) 0-definable partial endomorphisms.

Note that the map  $\Lambda \rightarrow R, \lambda \rightarrow \lambda(1)$  is an embedding of ordered groups. We will often identify  $\lambda \in \Lambda$  with  $\lambda(1) \in R$  and  $v = (v_1, \dots, v_n) \in \Lambda^n$  with  $v(1) := (v_1(1), \dots, v_n(1)) \in R^n$ .

We will use the following notation throughout:

**Notation:** Let  $X \subseteq R^n$  and  $f, g : X \rightarrow R \cup \{-\infty, +\infty\}$  be definable, with  $f \leq g$ . Then  $\Gamma(f)$  denotes the graph of  $f$  and  $(f, g)_X := \{(x, y) \in X \times R : f(x) < y < g(x)\}$ .

Also,  $\pi : R^n \rightarrow R^{n-1}$  always denotes the projection onto the first  $n - 1$  coordinates. And  $|\cdot|$  denotes the ‘‘sup-norm’’ on the ordered group  $(R, 0, +, <)$ .

**Definition 1.1** [Cones] A  $k$ -cone  $C \subseteq R^n$  is a definable set of the form

$$\left\{ b + \sum_{i=1}^k v_i(t_i) : b \in B, t_1, \dots, t_k \in R^{>0} \right\}$$

where  $B \subseteq R^n$  is a bounded definable set and  $v_1, \dots, v_k \in \Lambda^n$  are *linearly independent* i.e.,  $\forall t_1, \dots, t_k \in R, \sum_{i=1}^k v_i(t_i) = 0$  iff  $t_1 = \dots = t_k = 0$ .

We say that  $C$  is *normalised* if for each  $x \in C$  there are unique  $b \in B, t_1, \dots, t_k \in R^{>0}$  such that  $x = b + \sum_{i=1}^k v_i(t_i)$ . We use the notation

$$C := B + \sum_{i=1}^k v_i(t_i)$$

for this.

Poston's result [Po2] (fact 1.2 for expansions of  $(\mathbb{R}, 0, 1, +, <)$ , the additive group of the reals) led to Miller-Starchenko Growth Dichotomy for o-minimal expansions of ordered groups

**Fact 1.2** [ms] [*Growth Dichotomy*] For  $\mathcal{R}$  exactly one of the following holds:

(1) there is a 0-definable binary operation  $\cdot$  such that  $(R, 0, 1, +, \cdot, <)$  is a real closed field or;

(2) for every definable function  $f : R \rightarrow R$  there is  $\lambda \in \Lambda$  such that  $\lim_{x \rightarrow +\infty} [f(x) - \lambda(x)] \in R$ .

Here we study o-minimal structures  $\mathcal{R}$  satisfying condition (2).

**Definition 1.3**  $\mathcal{R}$  is called *linearly bounded* if for every definable function  $f : R \rightarrow R$  there exists  $\lambda \in \Lambda$  such that for all  $x$  large enough we have  $|f(x)| \leq \lambda(x)$ .

According to [ms] we have the following

**Fact 1.4** [ms] For an o-minimal expansion  $\mathcal{R}$  of an ordered abelian group the following are equivalent:

(1) There is no 0-definable binary operation  $\cdot$  such that  $(R, 0, 1, +, \cdot, <)$  is a real closed field;

(2) For every definable function  $f : R \rightarrow R$  there is  $\lambda \in \Lambda$  such that  $\lim_{x \rightarrow +\infty} [f(x) - \lambda(x)] \in R$ .

(3)  $\mathcal{R}$  is linearly bounded.

- (4) For any definable  $f : A \times R \rightarrow R$ , with  $A \subseteq R^m$  there exist  $\lambda_1, \dots, \lambda_l \in \Lambda$  such that for every  $a \in A$ , there is an  $i \in \{1, \dots, l\}$  (depending on  $a$ ) with  $\lim_{x \rightarrow +\infty} [f(a, x) - \lambda_i(x)] \in R$ .
- (5) For any definable  $f : A \times R \rightarrow R$ , with  $A \subseteq R^m$  the set  $\{b \in R : \exists a \in A, \lim_{x \rightarrow +\infty} \Delta f(a, x) = b\}$  is finite, where  $\Delta f(a, x) := f(a, x + 1) - f(a, x)$ .
- (6) Every  $\lambda \in \Lambda$  is 0-definable, and  $\Lambda(\mathcal{R}')$  is canonically isomorphic to  $\Lambda$  (as an ordered division ring) for every  $\mathcal{R}' \equiv \mathcal{R}$ .

**Definition 1.5** We say that  $\mathcal{R}$  is *semi-bounded* if every definable set is already definable in the reduct  $(R, 0, 1, +, <, (B_i)_{i \in I}, (\lambda)_{\lambda \in \Lambda})$ , where  $(B_i)_{i \in I}$  is the collection of all bounded definable sets.

We say that  $\mathcal{R}$  is *eventually linear* if for every definable function  $f : R \rightarrow R$  there is  $\lambda \in \Lambda$  and  $c \in R$  such that ultimately  $f(x) = \lambda(x) + c$ .

Finally we say that  $\mathcal{R}$  has no *poles* if there is no definable bijection between a bounded and an unbounded definable set.

Fact 1.6 below, our main result (see theorem 4.2), is a general version of a theorem proved by Y.Peterzil in [Pe1], for o-minimal expansions of  $(\mathbb{R}, 0, 1, +, <)$ , the additive group of the reals.

**Fact 1.6** For an o-minimal expansion  $\mathcal{R}$  of an ordered abelian group the following are equivalent:

- (1)  $\mathcal{R}$  is semi-bounded.
- (2)  $\mathcal{R}$  has no poles.
- (3) In  $\mathcal{R}$  we cannot define a real closed field whose universe is an unbounded subinterval of  $R$  and whose ordering agrees with  $<$ .
- (4)  $\mathcal{R}$  is eventually linear.
- (5) [*Structure theorem for semi-bounded  $\mathcal{R}$* ]: Any definable set  $X \subseteq R^n$  can be partitioned into finitely many definable normalised cones.

And if  $X \subseteq R^n$  is definable and  $\mathcal{F}$  is a finite collection of definable functions from  $X$  into  $R$ , then there is a partition  $\mathcal{C}$  of  $X$  into finitely

many definable normalised cones such that for each cone  $C \in \mathcal{C}$ , each  $f \in \mathcal{F}$  respects  $C$  i.e., if  $C \in \mathcal{C}$ , is a  $k$ -cone and  $C := B + \sum_{i=1}^k v_i(t_i)$ , there are  $\mu_1, \dots, \mu_k \in \Lambda$  such that for all  $b \in B, t_1, \dots, t_k \in R > 0$

$$f(b + \sum_{i=1}^k v_i(t_i)) = f|_B(b) + \sum_{i=1}^k \mu_i(t_i).$$

Moreover, this result combined with the following fact from [lp] gives us fact 1.8 below.

**Fact 1.7** [lp] Let  $\mathcal{V} := (V, +, <, a, (d)_{d \in D}, (P)_{P \in \mathcal{P}})$  be an expansion of an ordered vector space  $(V, +, <, (d)_{d \in D})$  over an ordered division ring  $D$  by predicates  $P \in \mathcal{P}$  on a bounded subset  $[-a, a]^n$ , such that  $\mathcal{P}$  contains predicates for all subsets of  $[-a, a]^n$  which are  $a$ -definable in the vector space structure. Then  $Th(\mathcal{V})$  has quantifier elimination in its language.

**Fact 1.8** [Relative quantifier elimination for semi-bounded  $\mathcal{R}$ ] Suppose that  $\mathcal{R}$  is semi-bounded. Then  $Th(\mathcal{R})$  has quantifier elimination and a universal axiomatisation in the language  $\mathcal{L}_{sb}(\mathcal{R})$  consisting of  $0, 1, +, -, <$ , a symbol for each element of  $\Lambda$ , a symbol for each bounded definable set of  $\mathcal{R}$ .

**Proof.** To prove quantifier elimination, by the structure theorem (fact 1.6) it's enough to show that  $m$ -cones are quantifier free definable, but this follows from fact 1.7. The universal axiomatisation follows from: (1) the quantifier elimination, which implies model completeness and so a  $\forall\exists$ -axiomatisation; (2) existence of definable Skolem functions in  $\mathcal{R}$ , and (3) the fact that, by the structure theorem every definable function is given piecewise by an  $\mathcal{L}_{sb}(\mathcal{R})$ -term.  $\square$

In fact 1.6, the proof of (5)  $\Rightarrow$  (1) is immediate from the definitions, that of (1)  $\Rightarrow$  (2) works in general and was given in [pss]. Here we present for completeness, an obvious modification of the proof in [Pe1], a short version of that in [pss]. (2)  $\Rightarrow$  (3) is easy; The proof of (3)  $\Rightarrow$  (4) given in [Pe1] for the case  $R = \mathbb{R}$  uses the following two facts:

- [Pe1] An o-minimal expansion of  $(\mathbb{R}, 0, 1, +, <)$  is eventually linear iff it has no poles.

- [mpp] An o-minimal expansion of  $(\mathbb{R}, 0, 1, +, <)$  which is not *linear* (i.e. there is a definable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  which is not piecewise linear) defines a field in some subinterval (it may be bounded), whose ordering agrees with  $<$ .

Here a proof of the general version of the first fact is given using Peterzil's proof for the reals and ideas from [ms]. The general case of the second fact follows from the Peterzil-Starchenko Dichotomy [PeS].

(4)  $\Rightarrow$  (5) is the difficult part. [Pe1] gives a proof of it for the reals using the so called "partition condition:"

- Let  $\mathcal{N}$  be an o-minimal structure over the reals. We say that  $\mathcal{N}$  satisfies the partition condition if, for all  $\mathcal{N}$ -definable open set  $U \subseteq \mathbb{R}^n$  and  $\mathcal{N}$ -definable functions  $f : U \rightarrow \mathbb{R}$ , there are open connected sets  $V_1, \dots, V_l \subseteq U$  (not necessarily definable in  $\mathcal{N}$ ) such that  $\dim_{\mathcal{N}}(U \setminus (V_1 \cup \dots \cup V_l)) < n$  and such that  $f$  is analytic on each of the  $V_i$ 's.

R. Poston (in [Po1]) gives the proof for the reals without this partition condition. But his proof doesn't work in general since he assumes the following:

In any o-minimal structure  $\mathcal{N}$  over the reals:

- any  $\mathcal{N}$ -definable partial endomorphism of  $(\mathbb{R}, 0, +)$  is the left multiplication by some  $r \in \mathbb{R}$ .

An essential tool developed in section 3 for the proof of (4)  $\Rightarrow$  (5) is the following notion (which also appears in [Po1] and [Pe1] for the reals).

**Definition 1.9** Let  $Z \subseteq \mathbb{R}^n$  be a definable set. The *linear dimension* of  $Z$  is defined by

$$ldim Z := \max\{k : Z \text{ contains a } k\text{-cone}\}.$$

**Fact 1.10** [*Linear dimension*] We have the following properties (for all definable  $X, Y, Z \subseteq \mathbb{R}^n$  with  $Y \subseteq X$  and all normalised  $k$ -cones  $C$ ):

- (1)  $ldim C = k$ ;
- (2)  $ldim Y \leq ldim X \leq n$ ;

$$(3) \text{ ldim} X \cup Z = \max\{\text{ldim} X, \text{ldim} Z\};$$

Moreover, if  $\mathcal{R}$  is eventually linear, it follows from the structure theorem that if  $f : X \rightarrow R^n$  is a definable injective function then  $\text{ldim} X = \text{ldim} f(X)$ .

Inside the class of semi-bounded o-minimal expansions of  $(R, 0, 1, +, <)$  lives the class of linear o-minimal expansions of  $(R, 0, 1, +, <)$ .

**Definition 1.11** A definable function  $F : U \subseteq R^n \rightarrow R$  is *piecewise linear*, if we can partition  $U$  into finitely many definable sets  $U_1, \dots, U_k$  such that  $F$  is *linear* on each of them i.e., given  $x, y \in U_i$  and  $t \in R^n$ , if  $x + t, y + t \in U_i$ , then  $F(x + t) - F(x) = F(y + t) - F(y)$ .

$\mathcal{R}$  is *linear* if every definable function  $f : (a, b) \rightarrow R$  is piecewise linear, and for every  $\bar{c}$ -definable partial endomorphism  $f : (-a, a) \rightarrow R$  there is a 0-definable partial endomorphism  $g : (-b, b) \rightarrow R$  with  $(-b, b)$  a  $\bar{c}$ -definable interval contained in  $(-a, a)$  such that  $f|_{(-b, b)} = g|_{(-b, b)}$ .

These structures were extensively studied in [lp], where the following was established:

**Fact 1.12** [lp] For an o-minimal expansion  $\mathcal{R}$  of an ordered abelian group the following are equivalent:

- (1) Every definable function  $F : U \subseteq R^n \rightarrow R$  is piecewise linear.
- (2)  $\mathcal{R}$  is linear.
- (3) There exist no definable binary operations  $\oplus, \otimes : I^2 \rightarrow I$  on an interval  $I = (-a, a)$ , and a positive element  $1 \in I$  such that  $(I, <_I, 0, 1, \oplus, \otimes)$  is an ordered real closed field. (Where  $<_I$  denotes  $<$  restricted to  $I$ ).
- (4) Let  $\Sigma_{p.e.}^0 := \Sigma_{p.e.}^0(\mathcal{R})$  be the set of all 0-definable partial endomorphisms of  $(R, 0, +, <)$ . Then every definable set is already definable in  $(R, 0, 1, +, <, (\sigma)_{\sigma \in \Sigma_{p.e.}^0})$ .
- (5) There is an elementary extension of  $\mathcal{R}$  which is a reduct of an ordered vector space over  $\Lambda_{p.e.}^0$ .

In [lp] the following result is also established

**Fact 1.13** [*Relative quantifier elimination for linear  $\mathcal{R}$* ] Suppose that  $\mathcal{R}$  is linear. Then  $Th(\mathcal{R})$  has quantifier elimination and a universal axiomatisation in the language  $\mathcal{L}_l(\mathcal{R})$  consisting of  $0, 1, +, -, <$ , a symbol for each element of  $\Lambda_{p,e}^0$  and a symbol for each algebraic point of  $\mathcal{R}$ .

We now include some examples: in example 1.14 we have a linear  $\mathcal{R}$ , in example 1.16 a semi-bounded not linear  $\mathcal{R}$  and in example 1.17 a linearly bounded not semi-bounded  $\mathcal{R}$ .

**Example 1.14** Let  $D$  be an ordered division ring, and let  $\mathcal{R} := (R, 0, 1, +, <, (d)_{d \in D})$  be an ordered vector space over  $D$ , where  $d$  represents scalar multiplication by  $d$ . Then  $Th(\mathcal{R})$  has quantifier elimination,  $\mathcal{R}$  is *linear* and  $\Lambda(\mathcal{R}) = D$ .

**Remark 1.15** [PeS] (1) If  $\mathcal{D}$  is a non commutative ordered division ring considered as an ordered vector space over itself, then  $\mathcal{D}$  is maximally o-minimal, i.e., if  $X \subseteq D^n$  is not linear then  $(\mathcal{D}, X)$  is not o-minimal.

(2) If  $\mathcal{R}$  is not linear then  $\Lambda$  is a field (and by the example below it can be any ordered field).

**Question:**(Posed by Steinhorn) Is (the “smallest” non trivial ordered divisible abelian group)  $(\mathbb{Q}, 0, 1, +, <)$  maximally o-minimal?

**Example 1.16** Let  $\overline{\mathcal{R}} := (R, 0, 1, +, \cdot, <, (\lambda_s)_{s \in S})$  be a real closed field, where  $S$  is a subfield of  $R$  and  $\lambda_s$  is left multiplication by  $s$ . Let  $\mathcal{R} := (R, 0, 1, +, <, (\lambda_s)_{s \in S}, (B_i)_{i \in I})$  where  $(B_i)_{i \in I}$  is the collection of all bounded 0-definable sets of  $\overline{\mathcal{R}}$ , and  $\underline{\mathcal{R}} := (R, 0, 1, +, <, (\lambda_s)_{s \in S})$ . Then

(1)  $\mathcal{R}$  is semi-bounded and is interdefinable with  $(R, 0, 1, +, <, (\lambda_s)_{s \in S}, *)$  where  $*$  is the restriction of  $\cdot$  to  $[-1, 1] \times [-1, 1]$ . Also  $\Lambda(\mathcal{R}) = (S, 0, 1, +, \cdot, <)$ .

(2)  $\mathcal{R}$  is the unique strong reduct properly between  $\underline{\mathcal{R}}$  and  $\overline{\mathcal{R}}$ , i.e., every 0-definable set of  $\underline{\mathcal{R}}$  (resp. of  $\mathcal{R}$ ) is 0-definable in  $\mathcal{R}$  (resp. in  $\overline{\mathcal{R}}$ ), and there is a 0-definable set of  $\overline{\mathcal{R}}$  (resp. of  $\mathcal{R}$ ) which is not 0-definable in  $\mathcal{R}$  (resp. in  $\underline{\mathcal{R}}$ ) and moreover any other structure satisfying the above has the same 0-definable sets as  $\mathcal{R}$ . This is proved in [pss] using model theory and also in [Pe2] using the version  $\mathcal{R} = \mathbb{R}$  of the structure theorem and then “eliminating parameters”.

(3) In [pss] it is proved that if  $\mathcal{M}$  is a model of  $Th(\mathcal{R})$  then  $M = G \oplus M'$  as an abelian group, where  $M'$  is the equivalence class of 0 for the equivalence



relation  $x \sim y \iff \exists n \in \mathbb{N}(|x - y| \leq n)$ , and is a real closed ring (i.e., an ordered integral domain which obeys the intermediate value theorem for polynomials) with  $a \cdot b := n^2(\frac{a}{n} * \frac{b}{n})$  if  $a, b < n$ . And  $G$  is a maximal subgroup of  $M$  for which  $\pi|_G : G \rightarrow M/\sim$  is injective. This characterization of the models of  $Th(\mathcal{R})$  is then used to prove quantifier elimination for  $Th(\mathcal{R})$ .

**Example 1.17** [Pe2] Let  $\mathcal{M} := (\mathbb{R}, <, 0, 1, +, (x \mapsto \ln(e^x + 1)))$ .  $\mathcal{M}$  is o-minimal since it is a reduct of  $(\mathbb{R}, <, 0, 1, +, \cdot, e^x)$  which is o-minimal (see [w]).  $\mathcal{M}$  is clearly not eventually linear, in [Pe2] its shown that  $\mathcal{M}$  is linearly bounded.

**Remark 1.18** [Po2] Consider  $g(x) := e^{-x}|_{[0,+\infty)}$  and  $h(x) := e^{-x^2}|_{[0,+\infty)}$ . Therefore, on  $(0, 1]$ ,  $g^{-1}(x) = -\ln(x)$ . Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f|_{[1,+\infty)}(x) := g(x - 1),$$

$$f|_{(0,1)}(x) := h \circ g^{-1}(x),$$

$$f|_{(-\infty,0]}(x) := 0.$$

It is clear that  $f$  is linearly bounded and  $Im(f)$  is bounded, and that  $(\mathbb{R}, <, 0, 1, +, f)$  which is (in terms of definable sets) the same as  $(\mathbb{R}, 0, 1, +, <, g, h)$ , which is o-minimal as  $(\mathbb{R}, <, 0, 1, +, \cdot, e^x)$  is o-minimal (see [w]). But in  $(\mathbb{R}, <, 0, 1, +, f)$  one can define on  $[0, +\infty)$ ,  $g^{-1} \circ h(x) = -\ln(e^{-x^2}) = x^2$ , so the above structure is not linearly bounded.

## 2 No poles

### 2.1 Semi-bounded implies no poles

**Proposition 2.1** *If  $\mathcal{R}$  is semi-bounded then  $\mathcal{R}$  has no poles.*

**Proof.** Suppose that  $\mathcal{R}$  is semi-bounded, and that there is an  $\mathcal{R}$ -definable bijection  $\sigma : (a, b) \rightarrow (c, d)$  where  $a, b \in R$  and either  $c = -\infty$  or  $d = +\infty$ . Then  $\sigma$  is definable in a reduct of  $\mathcal{R}$  of the form

$$\mathcal{R}_\sigma := (R, 0, 1, +, <, (\lambda)_{\lambda \in \Lambda}, (B_i)_{i \in I})$$

for some finite collection  $(B_i)_{i \in I}$  of bounded definable subsets  $B_i \subseteq R^{m_i}$  (for some positive integers  $m_i$ ).

Let  $\mathcal{R}'$  be an  $|R|^+$ -saturated elementary extension of  $\mathcal{R}_\sigma$  (hence,  $\mathcal{R}'$  contains “infinity” elements). Then  $R' = (R', 0, +, (\lambda)_{\lambda \in \Lambda})$  is a vector space over  $\Lambda(\mathcal{R}_\sigma)$ . Define a subspace of  $R'$  by  $S = \{s \in R' : |s| < r \text{ for some } r \in R\}$ . Let  $T$  be the complement of  $S$  in  $R'$  (as a vector space). The order in  $\mathcal{R}'$  is the lexicographical order in the vector-space direct sum of  $S$  and  $T$ , and the interpretation of any  $B_i$  in  $\mathcal{R}'$  is contained in  $S$ . On the other hand, any automorphism  $\tau$  of the ordered vector space  $(T, 0, +, (\lambda)_{\lambda \in \Lambda})$  induces in a natural way an automorphism of the ordered vector space  $(R', 0, +, (\lambda)_{\lambda \in \Lambda})$  which fixes all the elements of  $S$ . But the interpretation of  $\sigma$  in  $\mathcal{R}'$  must be a bijection between  $(a, b) \subseteq S$  and  $(c, d) \not\subseteq S$ . Therefore  $\tau$  cannot respect  $\sigma$ . Hence,  $\sigma$  cannot be definable.  $\square$

We now need some definitions and lemmas from [ms].

## 2.2 The group of definable germs at $+\infty$

Here  $\mathcal{R}$  is an expansion of an o-minimal ordered group (not necessarily eventually linear).

**Definition 2.2** Let  $f, g : R \rightarrow R$  be  $\mathcal{R}$ -definable functions. We say that they have the same *germ* (at  $+\infty$ ) if ultimately  $f = g$ . This is an equivalence relation, and we will identify  $f$  with its germ. The set  $\mathcal{G}$  of all germs is an ordered group with the obvious addition and order defined using the monotonicity theorem.

Consider the equivalence relation on  $\mathcal{G}^* := \mathcal{G} \setminus \{0\}$  given by  $E(f, g)$  iff either: there exist  $r, s \in \Lambda$  with  $r, s \geq 1$  such that  $|f| \leq r |g|$  and  $|g| \leq s |f|$ ; or,  $\lim_{x \rightarrow +\infty} f(x) \in R^*$  and  $\lim_{x \rightarrow +\infty} g(x) \in R^*$  (where  $R^* := R \setminus \{0\}$ ).

Let  $v : \mathcal{G}^* \rightarrow v(\mathcal{G}^*)$  denote the quotient map. For  $f \in \mathcal{G}$  we set  $v(f) := 0$  if  $\lim_{x \rightarrow +\infty} f(x) \in R^*$ .

$v(\mathcal{G}^*)$  can be totally ordered by  $v(f) < v(g)$  if  $v(f) \neq v(g)$  and  $|f| > s |g|$  for all  $s \in \Lambda^{>0}$ . Then  $v(f) < 0$  if  $\lim_{x \rightarrow +\infty} |f(x)| = +\infty$ , and  $v(f) > 0$  if  $\lim_{x \rightarrow +\infty} f(x) = 0$ . We extend  $v$  to  $\mathcal{G}$  by putting  $v(0) = +\infty$ .

**Notation:** Given  $f : R \rightarrow R$  definable and  $x, y \in R$ , we put  $\Delta_y f(x) = f(x + y) - f(x)$ . For  $y = 1$ , the subscript will be suppressed. Given  $f : A \times R \rightarrow R$  definable with  $A \subseteq R^m$ , and  $x, y \in R, a \in A$  we put  $\Delta_y f(a, x) = f(a, x + y) - f(a, x)$ .

The following lemma is easy to prove.

**Lemma 2.3** *Let  $f, g \in \mathcal{G}$ .*

- (1) *If  $|f| \geq |g|$ , then  $v(f) \leq v(g)$ .*
- (2)  *$v(f) = v(-f)$ .*
- (3)  *$v(\Delta f) = v(\Delta |f|) = v(|\Delta f|)$ .*
- (4)  *$v(rf) = v(f)$  for all  $r \in \Lambda^*$  (where  $\Lambda^* = \Lambda \setminus \{0\}$ ).*
- (5)  *$v(f + g) \geq \min\{v(f), v(g)\}$ , with equality if  $v(f) \neq v(g)$ .*

□

**Proposition 2.4** *Let  $f : R \rightarrow R$  be definable.*

- (1) *The set  $\{y \in R : \lim_{x \rightarrow +\infty} \Delta_y f(x) \in R\}$  is a definable subgroup of  $(R, 0, +)$ .*
- (2) *The function  $y \mapsto Lf(y) := \lim_{x \rightarrow +\infty} \Delta_y f(x)$  is a definable homomorphism from the above subgroup into  $(R, 0, +)$ .*
- (3) *If  $\lim_{x \rightarrow +\infty} \Delta f(x) = r \in R$  then  $r \in \Lambda$  and  $Lf(y) = r(y)$  for all  $y \in R$ .*

**Proof.** (1) and (2) follows from the fact that if  $y, z, a, b \in R$  are such that

$$\lim_{x \rightarrow +\infty} [f(x + y) - f(x)] = a$$

and

$$\lim_{x \rightarrow +\infty} [f(x + z) - f(x)] = b,$$

then

$$\lim_{x \rightarrow +\infty} [f(x + y + z) - f(x)] = a + b$$

and

$$\lim_{x \rightarrow +\infty} [f(x - y) - f(x)] = -a.$$

(3) follows from (2).

□

**Remark 2.5** Suppose that  $f : R \rightarrow R$  is definable. It follows from the above proposition that: If  $\lim_{x \rightarrow +\infty} |\Delta f(x)| > r$  for all  $r \in \Lambda(\mathcal{R})$ , then  $\lim_{x \rightarrow +\infty} |\Delta f(x)| = +\infty$ ; moreover,  $\lim_{x \rightarrow +\infty} |\Delta_y f(x)| = +\infty$  for all  $y \in R^*$ . And if  $\lim_{x \rightarrow +\infty} |\Delta f(x)| < r$  for all  $r \in \Lambda^{>0}$ , then  $\lim_{x \rightarrow +\infty} \Delta_y f(x) = 0$  for all  $y \in R$ .

**Lemma 2.6** Let  $f, g \in \mathcal{G}$  with  $g$  (ultimately) increasing and  $f \geq g$ . Then

$$\lim_{x \rightarrow +\infty} \Delta f(x) \geq \lim_{x \rightarrow +\infty} \Delta g(x) \geq 0.$$

**Proof.** The right hand inequality holds because  $g$  is increasing. We are done if  $v(\Delta f) < 0$ , so assume that  $v(\Delta f) \geq 0$ . If  $f - g$  is increasing then the left hand inequality holds. If  $f - g$  is not increasing, then it is strictly decreasing to some  $c \in R$ , and thus  $\lim_{x \rightarrow +\infty} [\Delta(f - g)(x)] = 0$ .  $\square$

**Proposition 2.7** Let  $\mathbf{x}$  denote the germ of  $id_R$ . Let  $f \in \mathcal{G}$ . Then:

- (1)  $v(f) < v(\mathbf{x})$  iff  $v(\Delta f) < 0$ .
- (2)  $v(f) > v(\mathbf{x})$  iff  $v(\Delta f) > 0$ .
- (3)  $v(f) = v(\mathbf{x})$  iff  $v(\Delta f) = 0$  iff  $f = r(\mathbf{x}) + u$  for some  $r \in \Lambda^*$  and  $u \in \mathcal{G}$  with  $v(u) > v(\mathbf{x})$ .
- (4) If  $g \in \mathcal{G}$  and  $v(f) = v(g) < 0$ , then  $f = r(g) + u$ , for some  $r \in \Lambda^*$  and some  $u \in \mathcal{G}$  with  $v(u) > v(g)$ .

**Proof.** (1) and (2) follows immediately from the remark and lemma 2.6. For (3), note that by lemma 2.6  $r := \lim_{x \rightarrow +\infty} \Delta f(x)$  exists; put  $u(x) := f(x) - r(x)$ . For (4) use  $v(f \circ g^{-1}) = v(\mathbf{x})$ .  $\square$

We are now ready to prove:

**Proposition 2.8**  $\mathcal{R}$  has no poles iff  $\mathcal{R}$  is eventually linear.

**Proof.** The direction  $\Leftarrow$  is immediate. For  $\Rightarrow$ , suppose not and let  $f : R \rightarrow R$  be definable eventually nonlinear. We may assume  $f$  is eventually nonlinear and positive as  $x$  approaches  $+\infty$ . If  $v(f) > v(\mathbf{x})$  then  $v(\Delta f) > 0$

and  $\Delta f$  defines a pole; if  $v(f) < v(\mathbf{x})$  then  $v(f^{-1}) > v(\mathbf{x})$  for the definable (for large enough  $x$ ) inverse of  $f$ , and we conclude as above; if  $v(f) = v(\mathbf{x})$  then by the proposition 2.7 we have some definable function  $u$  which is not the zero function (otherwise  $f$  would be eventually linear) with  $v(u) > v(\mathbf{x})$ .  
 $\square$

### 3 Subcones and linear dimension

#### 3.1 Lemma on subcones

**Proposition 3.1** *Suppose that  $\mathcal{R}$  is eventually linear.*

- (1) *Let  $f : (a, +\infty) \rightarrow B$  be a definable function where  $B \subseteq R^n$  is a bounded definable set. Then  $f$  is eventually constant.*
- (2) *Let  $X \subseteq R^n$  be a definable set. Then  $X$  doesn't contain an  $m$ -cone with  $m > 0$  iff  $X$  is bounded.*
- (3) *If  $f : B \rightarrow R^m$  is a definable function, where  $B \subseteq R^n$  is bounded then  $f(B)$  is bounded.*

**Proof.** (1) The coordinate function of  $f$  are eventually linear functions whose image is bounded, therefore they must be eventually constant.

(2) One way of the equivalence is obvious. For the other, by existence of definable Skolem functions, if  $X$  is unbounded there is a definable function from  $R^{>0}$  into  $X$  which picks an element of norm at least  $t$  for each  $t \in R^{>0}$ . By eventual linearity this defines a 1-cone contained in  $X$ .

Its clear that (3) follows from (2).  $\square$

**Notation:** For  $\lambda_1, \dots, \lambda_m \in \Lambda^n \setminus \{0\}$ , we set:

$$\langle \lambda_1, \dots, \lambda_m \rangle := \left\{ \sum_{i=1}^m \lambda_i(t_i) \in R^n \mid t_1, \dots, t_m \in R \right\},$$

$$\langle \lambda_1, \dots, \lambda_m \rangle^{\geq 0} := \left\{ \sum_{i=1}^m \lambda_i(t_i) : t_1, \dots, t_m \in R^{\geq 0} \right\},$$

and

$$\langle \lambda_1, \dots, \lambda_m \rangle^{>0} := \left\{ \sum_{i=1}^m \lambda_i(t_i) : t_1, \dots, t_m \in R^{>0} \right\}.$$

**Lemma 3.2** (1) For all  $\lambda, \lambda_1, \dots, \lambda_m \in \Lambda^n \setminus \{0\}$ , we have that  $\langle \lambda \rangle \cap \langle \lambda_1, \dots, \lambda_m \rangle \neq \{0\}$  if and only if  $\langle \lambda \rangle \subseteq \langle \lambda_1, \dots, \lambda_m \rangle$ .

(2) Suppose that  $\lambda \in \Lambda^n$  and  $\lambda_1, \dots, \lambda_m \in \Lambda^n$  are linearly independent, and there exist  $s > 0$  such that  $\lambda(s) \in \langle \lambda_1, \dots, \lambda_m \rangle^{\geq 0}$ . Let  $I_s^{>0} := \{i \in \{1, \dots, m\} : s_i > 0\}$  where  $\lambda(s) = \sum_{i=1}^m \lambda_i(s_i)$  and let  $k = i_s^{>0} := \text{number of elements in } I_s^{>0}$ . Then  $\langle \lambda \rangle^{>0} \subseteq \langle \{\lambda_i\}_{i \in I_s^{>0}} \rangle^{>0}$

(3) For all  $\lambda, \lambda_1, \dots, \lambda_m \in \Lambda^n$  with  $\lambda_1, \dots, \lambda_m$  linearly independent, we have:

$$\langle \lambda \rangle^{\geq 0} \subseteq \langle \lambda_1, \dots, \lambda_m \rangle^{\geq 0} \iff \langle \lambda \rangle^{\geq 0} \cap \langle \lambda_1, \dots, \lambda_m \rangle^{\geq 0} \neq \{0\}$$

**Proof.** (1) The set  $\{t \in R \mid \exists t_1, \dots, t_m [\lambda(t) = \sum_{i=1}^m \lambda_i(t_i)]\}$  is a definable subgroup of  $(R, 0, +, <)$ .

(2) Is by induction on  $k$ : Suppose, that for some  $s > 0$   $i_s^{>0} = 0$ . Then  $\lambda(s) = 0$  implies that  $\lambda = 0$  and  $\langle \lambda \rangle = \langle \emptyset \rangle = \{0\}$ .

Suppose that the result holds for all  $\lambda, \lambda_1, \dots, \lambda_m \in \Lambda^n$  for which there is  $s > 0$  satisfying  $i_s^{>0} = l$  for all  $l < k$ .

Consider  $\lambda, \lambda_1, \dots, \lambda_m \in \Lambda^n$  for which there is  $s > 0$  with  $i_s^{>0} = k$ . Then  $\langle \lambda \rangle \subseteq \langle \{\lambda_i\}_{i \in I_s^{>0}} \rangle$ , by (1) above, and there is a definable continuous map  $R^{>0} \rightarrow R^k$ ,  $t \mapsto (t_1, \dots, t_k)$  where  $\lambda(t) = \sum_{i \in I_s^{>0}} \lambda_i(t_i)$ . If the result doesn't hold, then there must exist  $t > 0$  such that  $i_t^{>0} < i_s^{>0}$ . But then by the inductive hypothesis  $i_s^{>0} < k$ .

(3) Follows from (2). □

**Notation:** Let  $C =: B + \sum_{i=1}^m \lambda_i(t_i)$  be a  $m$ -cone. We will write

$$\langle C \rangle := \langle \lambda_1, \dots, \lambda_m \rangle,$$

$$\langle C \rangle^{\geq 0} := \langle \lambda_1, \dots, \lambda_m \rangle^{\geq 0},$$

and

$$\langle C \rangle^{>0} := \langle \lambda_1, \dots, \lambda_m \rangle^{>0}.$$

**Remark 3.3** In lemma 3.4 below, we do not assume anything about  $\mathcal{R}$ , we only need to work in the reduct  $(R, 0, 1, <, +, C, C')$  of  $\mathcal{R}$ . This reduct is eventually linear by proposition 2.1 and proposition 2.8.

**Lemma 3.4** [*Lemma on sub cones*] If  $C' = B' + \sum_{i=1}^{m'} w_i(t_i)$  and  $C = B + \sum_{i=1}^m v_i(t_i)$  are definable normalised cones such that  $C' \subseteq C \subseteq R^n$  then

- (1) Each  $w_j \in \langle C \rangle^{\geq 0}$  (as an element of  $R^n$ ),
- (2)  $\langle C' \rangle \subseteq \langle C \rangle$  (and hence  $m' \leq m$ ).

**Proof.** (1) Fix  $c \in C'$  then  $\forall u > 0, c + w_j(u) \in C' \subseteq C$ , so there exist a unique  $b \in B$  and unique  $t_1, \dots, t_m \in R^{>0}$  such that  $c + w_j(u) = b + \sum_{i=1}^m v_i(t_i)$ . This gives the following definable functions  $R^{>0} \rightarrow B, u \mapsto b(u)$  and for each  $i \in \{1, \dots, m\}, R^{>0} \rightarrow R, u \mapsto t_i(u)$ .

We have the following: (1) By proposition 3.1  $b(u)$  is eventually constant, and equal say  $b$ . (2) By eventual linearity and the fact that each for  $i \in \{1, \dots, m\} t_i(u) > 0$  for all  $u > 0$ , we see that for each  $i \in \{1, \dots, m\} t_i(u)$  is ultimately linear and non decreasing (otherwise it would be zero).

Let  $U > 0$  be large enough such that both 1 and 2 holds for all  $u > U$ . Fix  $u_1 > U$  and take  $u_2 > u_1$  large enough so that for each  $i \in \{1, \dots, m\} t_i(u_2) \geq t_i(u_1)$ . Then

$$w_j(u_2 - u_1) = \sum_{i=1}^m v_i(t_i(u_2) - t_i(u_1)) \in \langle v_1, \dots, v_m \rangle^{\geq 0},$$

and by lemma 3.2  $\langle w_j \rangle^{\geq 0} \subseteq \langle v_1, \dots, v_m \rangle^{\geq 0}$  and in particular

$$w_j \in \langle v_1, \dots, v_m \rangle^{\geq 0} = \langle C \rangle^{\geq 0}.$$

- (2) This follows from (1). □

## 3.2 Linear dimension

Note that in proposition 3.5 we do not assume anything about  $\mathcal{R}$  unlike in [Po1] where  $\mathcal{R}$  is a linearly bounded expansion of the additive group of real numbers.

**Proposition 3.5** (1) If  $X \subseteq Y \subseteq R^n$  are definable sets then  $ldim(X) \leq ldim(Y) \leq n$ .

(2) If  $C$  is a definable  $m$ -cone then  $ldim(C) = m$ .

(3) For all definable  $X_1, \dots, X_k \subseteq R^n$ ,

$$ldim(X_1 \cup \dots \cup X_k) = \max\{ldim X_1, \dots, ldim X_k\}.$$

**Proof.** (1) is trivial, (2) follows from the lemma on sub cones 3.4 and (3) follows from the next lemma.  $\square$

**Lemma 3.6** (1) <sub>$n$</sub>  For all  $k$  and all definable  $X_1, \dots, X_k \subseteq (R^{>0})^n$  such that  $ldim(X_1 \cup \dots \cup X_k) = n$  there exists  $i \in \{1, \dots, k\}$  such that  $ldim(X_i) = n$ .

(2) <sub>$n$</sub>  For all definable sets  $X \subseteq (R^{>0})^n$  with  $\dim X \leq n - 1$  we have  $ldim((R^{>0})^n \setminus X) = n$ .

**Proof.** We will use induction on  $n$ . (1)<sub>1</sub>, (2)<sub>1</sub> follow by o-minimality.

(1) <sub>$n-1$</sub>   $\Rightarrow$  (2) <sub>$n$</sub> : Assume (1) <sub>$n-1$</sub> , and (2) <sub>$l$</sub>  for all  $l \leq n$ . We will prove (2) <sub>$n$</sub>  by induction on  $\dim X$ : If  $\dim X = 0$  the result is clear. So suppose the result holds for all definable  $Y \subseteq (R^{>0})^n$  with  $\dim Y = l \leq n - 1$ .

Let  $X \subseteq (R^{>0})^n$  be definable with  $\dim X = l + 1$ . If  $l + 1 < n - 1$ , then  $\dim \pi(X) \leq n - 2$  and by (2) <sub>$n-1$</sub>   $ldim((R^{>0})^{n-1} \setminus \pi(X)) = n - 1$ , which implies that  $ldim((R^{>0})^n \setminus X) = n$ .

So we may assume that  $\dim X = n - 1$ . By cell decomposition, the induction hypothesis and the above argument, we may also assume that  $X$  is a finite union of cells  $X_1, \dots, X_k$  each of which of dimension  $n - 1$ , and for each  $i \in \{1, \dots, k\}$ ,  $X_i = \Gamma(f_i)$  for some definable  $f_i : \pi(X_i) \rightarrow R$ .

The proof now proceeds by sub induction on  $k$ : Suppose  $k = 1$ . Then by (1) <sub>$n-1$</sub>  we have either  $ldim[(R^{>0})^{n-1} \setminus \pi(X_1)] = n - 1$ , which implies  $ldim[(R^{>0})^n \setminus X_1] = n$ , or we have  $ldim[\pi(X_1)] = n - 1$ . Then by considering the restriction of  $f_1$  to the  $n - 1$ -cone contained in  $\pi(X_1)$  and by considering a suitable linear transformation we may assume without loss of generality that  $\pi(X_1) = (R^{>0})^{n-1}$ . Let

$$H := \{(x_1, \dots, x_{n-1}, h + \sum_{i=1}^{n-1} \lambda_i(x_i)) : x_1, \dots, x_{n-1} \in (R^{>0})^{n-1}\},$$



be an hyper plane in  $(R^{>0})^n$  such that  $(R^{>0})^n \setminus H$  has two definable connected components each of which of linear dimension  $n$ , and such that  $\pi(H)$  has linear dimension  $n - 1$ .

Let

$$X_{1,<} := \{(x_1, \dots, x_{n-1}) \in \pi(X_1) : f_1(x_1, \dots, x_{n-1}) < h + \sum_{i=1}^{n-1} \lambda_i(x_i)\},$$

$$X_{1,=} := \{(x_1, \dots, x_{n-1}) \in \pi(X_1) : f_1(x_1, \dots, x_{n-1}) = h + \sum_{i=1}^{n-1} \lambda_i(x_i)\}$$

and

$$X_{1,>} := \{(x_1, \dots, x_{n-1}) \in \pi(X_1) : f_1(x_1, \dots, x_{n-1}) > h + \sum_{i=1}^{n-1} \lambda_i(x_i)\}.$$

Then by  $(1)_{n-1}$  we have three sub cases:

- (1)  $ldim(X_{1,<}) = n - 1$  : If  $C \subseteq X_{1,<}$  is an  $n - 1$ -cone, then

$$\{(x_1, \dots, x_{n-1}, x) : (x_1, \dots, x_{n-1}) \in C, x > h + \sum_{i=1}^{n-1} \lambda_i(x_i)\}$$

is contained in  $(R^{>0})^n \setminus X_1$  and contains an  $n$ -cone.

- (2)  $ldim(X_{1,=}) = n - 1$  : If  $C \subseteq X_{1,=}$  is an  $n - 1$ -cone then

$$\{(x_1, \dots, x_{n-1}, x + 1) : (x_1, \dots, x_{n-1}) \in C, x > h + \sum_{i=1}^{n-1} \lambda_i(x_i)\}$$

is contained in  $(R^{>0})^n \setminus X_1$  and contains an  $n$ -cone.

- (3)  $ldim(X_{1,>}) = n - 1$  : If  $C \subseteq X_{1,>}$  is an  $n - 1$ -cone then

$$\{(x_1, \dots, x_{n-1}, x) : (x_1, \dots, x_{n-1}) \in C, h + \sum_{i=1}^{n-1} \lambda_i(x_i) > x > 0\}$$

is contained in  $(R^{>0})^n \setminus X_1$  and contains an  $n$ -cone.

Assume that the result is proved for all  $X$  which is union of less than  $k$  cells of dimension  $n - 1$ . And let  $X$  be as above a union of  $k$  cells of dimension  $n - 1$ . By the sub induction hypothesis there is an  $n$ -cone  $D$  contained in  $(R^{>0})^n \setminus X_1 \cup \dots \cup X_{k-1}$ . Now the same argument as above works if one substitutes every where  $(R^{>0})^n$  by  $D$ ,  $f_1$  by  $f_k$  and  $X_1$  by  $X_k$ . This completes the sub induction.

$(2)_n \Rightarrow (1)_n$  : Without loss of generality we may assume that  $k = 2$  and  $X_1$  and  $X_2$  are disjoint. Since  $\text{ldim}(X_1 \cup X_2) = n$  we may also assume that  $X_1 \cup X_2 = (R^{>0})^n$ .

Let  $X$  be the boundary of  $X_1$  and  $X_2$ , then  $\dim X \leq n - 1$ ; By  $(2)_n$  we conclude that  $X_1$  or  $X_2$  contains an  $n$ -cone.  $\square$

## 4 The structure theorem

### 4.1 The structure theorem

**Notation:** Let  $D = B + \sum_{i=1}^m v_i(t_i) \subseteq R^n$  be a normalised  $m$ -cone, and let  $f : D \rightarrow R$  be a definable function. By

$$\forall b \in B, \forall t_1, \dots, t_m > 0, f(b + \sum_{i=1}^m v_i(t_i)) = f|_B(b) + \sum_{i=1}^m \lambda_i(t_i)$$

(for some  $\lambda_1, \dots, \lambda_m \in \Lambda$ ) we mean that  $f$  has a (unique since  $D$  is normalised) extension  $\tilde{f}$  to  $\{b + \sum_{i=1}^m v_i(t_i) : b \in B, t_1, \dots, t_m \geq 0\}$  and

$$\forall b \in B, \forall t_1, \dots, t_m > 0, \tilde{f}(b + \sum_{i=1}^m v_i(t_i)) = \tilde{f}|_B(b) + \sum_{i=1}^m \lambda_i(t_i)$$

(for some  $\lambda_1, \dots, \lambda_m \in \Lambda$ ).

The following fact will be readily assumed in the proof of the structure theorem.

**Lemma 4.1** *Let  $D = B + \sum_{i=1}^m v_i(t_i) \subseteq R^n$  be a normalised  $m$ -cone, and let  $f : D \rightarrow R$  be a definable function such that*

$$\forall b \in B, \forall t_1, \dots, t_m > 0, f(b + \sum_{i=1}^m v_i(t_i)) = f|_B(b) + \sum_{i=1}^m \lambda_i(t_i)$$

(for some  $\lambda_1, \dots, \lambda_m \in \Lambda$ ). Then for any cone  $D' = B' + \sum_{i=1}^{m'} v'_i(t_i) \subseteq D$ ,

$$\forall b' \in B', \forall t'_1, \dots, t'_m > 0, f(b' + \sum_{i=1}^{m'} v'_i(t'_i)) = f|_{B'}(b') + \sum_{i=1}^{m'} \mu_i(t'_i)$$

(for some  $\mu_1, \dots, \mu_m \in \Lambda$ ).

Furthermore, if  $\lambda_1 = \dots = \lambda_m = 0$  then  $\mu_1 = \dots = \mu_m = 0$ .

**Proof.** By the lemma on sub cones each  $v'_j \in \langle v_1, \dots, v_m \rangle$  and  $f$  is linear in the direction of each  $v_i$ , so  $f$  is linear in the direction of each  $v'_j$ . Furthermore, each  $\mu_j$  is simply the relevant linear combination of  $\lambda_1, \dots, \lambda_m$ , so if  $\lambda_1 = \dots = \lambda_m = 0$  then each  $\mu_j = 0$ .  $\square$

**Theorem 4.2** ( *The structure theorem* ) Let  $\mathcal{R}$  be an o-minimal eventually linear structure. Then

- (1)<sub>n</sub> Any definable set  $X \subseteq R^n$  can be partitioned into finitely many definable normalised cones.
- (2)<sub>n</sub> If  $X \subseteq R^n$  is definable and  $f_1, \dots, f_k : X \rightarrow R$  is a finite collection of definable functions then  $X$  can be partitioned into finitely many definable normalised cones such that for each cone,  $B + \sum_{i=1}^m v_i(t_i)$ , there exist  $\lambda_{1j}, \dots, \lambda_{mj} \in \Lambda(\mathcal{R})$  such that

$$f_j(b + \sum_{i=1}^m v_i(t_i)) = f_j|_B(b) + \sum_{i=1}^m \lambda_{ij}(t_i),$$

$$j \in \{1, \dots, k\}.$$

Let  $X \subseteq R^n$  be definable. We prove the result by parallel induction on  $n$ . (1)<sub>0</sub>, (2)<sub>0</sub>, (1)<sub>1</sub> and (2)<sub>1</sub> are trivial.

#### 4.2 (2)<sub>n-1</sub>, (1)<sub>n</sub> $\Rightarrow$ (2)<sub>n</sub> ( $n \geq 1$ )

First note that it is enough to prove (2)<sub>n</sub> for just one function,  $f$ , since the statement will then follow by induction on  $k$  using lemma 4.1.

We proceed by sub induction on  $ldim(X)$ . If  $ldim(X) = 0$  then  $X$  is bounded and the result is immediate.

Assume that  $ldim(X) = m + 1$  and that the result holds for all lower values of  $ldim(X)$ .

By  $(1)_n$ , and the sub induction hypothesis we reduce to the case that  $X$  is a normalised  $m + 1$ -cone say  $B' + \sum_{i=1}^{m+1} v_i(t_i)$ . Let  $\{v_1, \dots, v_{m+1}, v_{m+2}, \dots, v_n\}$  be a definable (i.e each  $v_i \in \Lambda^n$ ) basis of  $R^n$  containing  $\{v_1, \dots, v_{m+1}\}$ . Consider the linear isomorphism  $L$  of  $R^n$  defined by  $L(v_i) := e_{n-i}$  for  $i \in \{1, \dots, n-1\}$ , where  $e_i$ 's are the standard basis vectors. Clearly if we prove the result for the  $m + 1$ -cone  $L(X)$  we get the result for  $X$ . So we may assume that  $X$  is of the form  $B + \sum_{i=n-m}^n e_i(t_i)$ .

Take  $(x_1, \dots, x_n)$  to be the coordinates in  $X$ , and let  $\pi : R^n \rightarrow R^{n-1}$  be the projection onto the first  $n - 1$  coordinates.

Let

$$X_0 := \{(\bar{x}, x) \in X : \exists \delta > 0 \forall y \in (-\delta, \delta) \forall z \in (x - \delta, x + \delta)$$

$$\Delta_y f(\bar{x}, z) = \Delta_y f(\bar{x}, x)\}$$

be the definable set on which  $f$  is locally linear in the last variable,  $x_n$ ; let  $K, K' : \pi(X) \rightarrow R$  be the (well defined by eventual linearity ) definable functions given by  $K(\bar{x}) := \inf\{x \geq 0 : \forall z \geq x, (\bar{x}, z) \in X_0\}$  and  $K'(\bar{x}) := 0$ .

### 4.3 Proof on $(K, +\infty)_{\pi(X)}$

For each  $\bar{x} \in \pi(X)$  we have  $f(\bar{x}, x) = c_{\bar{x}} + \lambda_{\bar{x}}(x)$  for some definable function  $c : \pi(X) \rightarrow R$   $\bar{x} \mapsto c_{\bar{x}}$  and some  $\lambda_{\bar{x}} \in \Lambda$ , for all  $x > K(\bar{x})$ . But then for each  $\bar{x} \in \pi(X)$   $\lim_{t \rightarrow +\infty} \Delta f(\bar{x}, t) = \lambda_{\bar{x}}$ , and by fact 1.4 we can partition  $\pi(X)$  into finitely many definable sets on which  $\lambda_{\bar{x}}$  is constant.

Let  $A \subseteq \pi(X)$  be one such definable set on which  $\lambda_{\bar{x}}$  equals, say  $\lambda$ .

Apply  $(2)_{n-1}$  with  $A$  in place of  $X$ ,  $k = 1$ , and  $c$  in place of  $f_1$ . Then we will have finitely many normalised  $l$ -cones  $\tilde{A} = \tilde{B} + \sum_{i=1}^l v_i(t_i)$  (with  $l$  depending on  $\tilde{A}$ ) and corresponding  $\tilde{\lambda}_1, \dots, \tilde{\lambda}_l \in \Lambda$  such that

$$f(b + \sum_{i=1}^l v_i(t_i), x) = c|_{\tilde{B}}(b) + \sum_{i=1}^l \tilde{\lambda}_i(t_i) + \lambda(x)$$

on each  $(K|_{\tilde{A}}, +\infty)_{\tilde{A}}$ . By  $(2)_{n-1}$  one can partition  $\tilde{A}$  into normalised cones such that if  $\tilde{A} := \overline{B} + \sum_{i=1}^k w_i(t_i)$  is one such cone then

$$K(b + \sum_{i=1}^k w_i(t_i)) = K|_{\overline{B}}(b) + \sum_{i=1}^k \mu_i(t_i)$$

for some  $\mu_1, \dots, \mu_k \in \Lambda$ . By lemma 4.1 we have

$$f(b + \sum_{i=1}^k w_i(t_i), x) = c|_{\overline{B}}(b) + \sum_{i=1}^k \zeta_i(t_i) + \lambda(x)$$

on  $(K|_{\overline{A}}, +\infty)_{\overline{A}}$  for some  $\zeta_1, \dots, \zeta_k \in \Lambda$ .

But  $(K|_{\overline{A}}, +\infty)_{\overline{A}}$  is the  $k + 1$ -cone  $\Gamma(K|_{\overline{B}}) + \sum_{i=1}^{k+1} u_i(t_i)$  where for  $i = 1, \dots, k$   $u_i := (w_i, \mu_i)$  and  $u_{k+1} := e_n$ . But then we have

$$\begin{aligned} f((b, K|_{\overline{B}}(b)) + \sum_{i=1}^{k+1} u_i(t_i)) = \\ c|_{\overline{B}}(b) + \lambda(K|_{\overline{B}}(b)) + \sum_{i=1}^k (\zeta_i + \lambda\mu_i)(t_i) + \lambda(t_{k+1}). \end{aligned}$$

So the result holds on  $(K, +\infty)_{\pi(X)}$ .

**Remark:** Note that the above proof also shows the following weak form of  $(2)_n$ :

- $(2)'_n$  : If  $X' \subseteq R^n$  and  $f : X' \rightarrow R$  are definable and  $ldim(X') = k$ , then  $X'$  contains a normalised  $k$ -cone,  $B' + \sum_{i=1}^k v_i(t_i)$ , such that  $f(b + \sum_{i=1}^k v_i(t_i)) = f|_{B'}(b) + \sum_{i=1}^k \mu_i(t_i)$  for some  $\mu_1, \dots, \mu_k \in \Lambda$ .

This weak form of  $(2)_n$  will be the key fact which will allow us to proceed with the proof. In fact R.Poston uses this to avoid the use of the “partition condition” in the next step.

#### 4.4 Proof on $X \setminus X_0$

In fact  $ldim(X \setminus X_0) \leq m$ . Suppose not, apply the above remark to  $X \setminus X_0$  and get an  $m + 1$ -cone  $C \subseteq X \setminus X_0 \subseteq X$ . By the lemma on sub cones we will have  $\langle C \rangle \subseteq \langle X \rangle$ . But these vector spaces have the dimension  $m + 1$ , therefore they are equal. Since  $e_n \in \langle X \rangle$ ,  $f$  is linear in  $x_n$  inside some cone contained in  $X \setminus X_0$ , a contradiction.

## 4.5 Proof on $X_0 \cap (K', K)_{\pi(X)}$

**Lemma 4.3** [On projection of normalised cones] *If  $C$  is a normalised  $m$ -cone in  $R^n$  such that  $e_n \in \langle C \rangle$ . Then  $C = (L', L)_{\pi(C)}$  for some definable functions  $L', L : \pi(C) \rightarrow R \cup \{-\infty, +\infty\}$ , with  $L' < L$ .*

**Proof.** Assume  $C = B + \sum_{i=1}^m \lambda_i(t_i)$ .

*Claim (1):* If  $b, b' \in B$  and  $b - b' \in \langle C \rangle$  then  $b = b'$ .

*Proof of Claim (1):* Let  $s_1, \dots, s_m \in R$  be such that  $b - b' = \sum_{i=1}^m \lambda_i(s_i)$ . Now take  $c = b + \sum_{i=1}^m \lambda_i(t_i) \in C$  with  $t_i > 0$  such that  $t_i + s_i > 0$  for all  $i \in \{1, \dots, m\}$ . Then  $c = b + \sum_{i=1}^m \lambda_i(t_i) = b' + \sum_{i=1}^m \lambda_i(t_i + s_i)$ . Since  $C$  is normalised  $b = b'$ .

*Claim (2):* Let  $x, x' \in C$  with corresponding  $b, b' \in B$ . If  $\pi(x) = \pi(x')$  then  $b = b'$ .

*Proof of Claim (2):* We can write  $x = (\pi(x), y) = (\pi(x), 0) + e_n(y) = b + \sum_{i=1}^m \lambda_i(t_i)$  and  $x' = (\pi(x'), y') = (\pi(x'), 0) + e_n(y') = b' + \sum_{i=1}^m \lambda_i(t'_i)$ .

Then  $b - b' = \sum_{i=1}^m \lambda_i(t'_i - t_i) + e_n(y - y') \in \langle C \rangle$  (since  $e_n \in \langle C \rangle$ ). Therefore by (1)  $b = b'$ .  $\square$

*Claim (3):* For each  $x \in C$  the definable set  $C_{\pi(x)} := \{y \in R : (\pi(x), y) \in C\}$  is an open interval.

*Proof of Claim (3):* First notice the following: If  $v, v_1, \dots, v_m \in \Lambda^n$ , with  $v_1, \dots, v_m$  linearly independent and  $v \in \langle v_1, \dots, v_m \rangle$  then for all  $b, c \in R^n$ ,  $b + \langle v \rangle \cap c + \langle v_1, \dots, v_m \rangle^{>0}$  is open and connected in  $b + \langle v \rangle$ : To see this consider a linear isomorphism  $L$  of  $R^n$  that sends each  $v_i$  to  $e_i$ .

Let  $x \in C$ . Then by (2)  $C_{\pi(x)} = \pi_n((b + \langle C \rangle^{>0}) \cap ((\pi(x), 0) + \langle e_n \rangle))$  for some  $b \in B$  and where  $\pi_n : R^n \rightarrow R$  is the projection onto the last coordinate. Its now clear that  $C_{\pi(x)}$  is an open interval.  $\square$

To finish the proof consider the definable functions  $L(\pi(x)) := \sup\{y \in R : (\pi(x), y) \in C\}$  and  $L'(\pi(x)) := \inf\{y \in R : (\pi(x), y) \in C\}$ .  $\square$

We can now proceed. By  $(1)_n$  and the sub induction hypothesis it is sufficient to prove the result on a normalised  $m + 1$ -cone contained in  $X_0 \cap (K', K)_{\pi(X)}$ . Let  $D = B + \sum_{i=1}^{m+1} v_i(t_i)$  be one such. By the lemma on sub cones 3.4  $\langle X \rangle = \langle D \rangle$  so  $e_n \in \langle D \rangle$ . By the lemma on projection

of normalised cones  $D = (L', L)_{\pi(D)}$  for some definable functions  $L', L : \pi(D) \rightarrow R$ . (Note that  $K' \leq L' < L \leq K$ ).

Since for each  $\bar{x} \in \pi(D)$   $f(\bar{x}, -) : (L'(\bar{x}), L(\bar{x})) \rightarrow R$ , is linear, we can write  $f(\bar{x}, x) = c(\bar{x}) + g(\bar{x}, y)$  for some definable function  $c : \pi(D) \rightarrow R$  and a definable partial endomorphism  $g(\bar{x}, -) : (-M(\bar{x}), M(\bar{x})) \rightarrow R$  with  $g(\bar{x}, 0) = 0$ , where  $M(\bar{x}) := \frac{L(\bar{x}) - L'(\bar{x})}{2}$  and where  $y = x - \frac{L(\bar{x}) - L'(\bar{x})}{2}$ .

**Lemma 4.4** [Main Lemma] *There is a partition of  $D$  into finitely many normalised cones such that for each  $m+1$ -cone  $D''$  in this partition there is a  $\lambda \in \Lambda$  such that for all  $\bar{x} \in \pi(D'')$ ,  $g(\bar{x}, y) = \lambda(y)$  for all  $y \in (-M(\bar{x}), M(\bar{x}))$ .*

Notice that this enough to finish the proof of  $(2)_n$  on  $X_0 \cap (K', K)_{\pi(X)}$ : on each  $D''$  we have

$$\begin{aligned} f(\bar{x}, x) &= c(\bar{x}) + g(\bar{x}, x - (\frac{L(\bar{x}) - L'(\bar{x})}{2})) \\ &= c(\bar{x}) + \lambda(\frac{L'(\bar{x}) - L(\bar{x})}{2}) + \lambda(x). \end{aligned}$$

Now let  $h(\bar{x}) := c(\bar{x}) + \lambda(\frac{L'(\bar{x}) - L(\bar{x})}{2})$ . By  $(2)_{n-1}$  there is a decomposition of  $\pi(D'')$  into normalised cones such that if  $A := B + \sum_{i=1}^k v_i(t_i)$  is one such, then we have

$$h(b + \sum_{i=1}^k v_i(t_i)) = h|_B(b) + \sum_{i=1}^k \lambda_i(t_i)$$

on  $A$ , for some  $\lambda_1, \dots, \lambda_k \in \Lambda$ .

Let  $\tilde{A} := \{(\bar{x}, x) \in D'' : \bar{x} \in A\}$ . By  $(1)_n$  we can decompose  $\tilde{A}$  into normalised cones. Let  $C := E + \sum_{i=1}^l w_i(t_i)$  be one such cone. Then  $\pi(C) = \pi(E) + \sum_{i=1}^l \pi w_i(t_i)$  is a normalised sub cone of  $A$ . Therefore by lemma 4.1 we have

$$h(\pi(e) + \sum_{i=1}^l \pi w_i(t_i)) = h|_{\pi(E)}(\pi(e)) + \sum_{i=1}^l \zeta_i(t_i)$$

on  $\pi(C)$ , for some  $\zeta_1, \dots, \zeta_l \in \Lambda$ . But from this it follows that

$$f(e + \sum_{i=1}^l w_i(t_i)) = h|_{\pi(E)}(\pi(e)) + \lambda(e') + \sum_{i=1}^l (\zeta_i + \lambda w_{i,n})(t_i)$$

where  $e := (\pi(e), e')$  and for each  $i = 1, \dots, l$   $w_i := (\pi w_i, w_{i,n})$ .

The proof of the Main Lemma follows from the following lemmas:

**lemma A:** *Let  $C = B + \sum_{i=1}^k v_i(t_i) \subseteq R^n$  be a normalised  $k$ -cone. If  $v \in \langle C \rangle$  then there is an  $k$ -cone  $C' = B' + \sum_{i=1}^k v_i(t_i) \subseteq C$  such that  $\text{ldim}(C \setminus C') < k$  and for all  $z \in C'$  we have  $z + v \in C$ .*

**Proof.** Since  $v \in \langle C \rangle$  we have  $v = \sum_{j=1}^k v_j(s_j)$  for some  $s_1, \dots, s_k \in R$ . Let  $J^- = \{j : s_j < 0\}$  and  $J^+ = \{j : s_j > 0\}$ . We want a normalised  $k$ -cone  $C' \subseteq C$  of the form  $C' := B' + \sum_{i=1}^k v_i(t_i)$  for some bounded definable set  $B'$ , such that if  $z \in C'$  then  $z + v \in C$ . Take

$$B' := \{b + \sum_{j \in J^-} v_j(-s_j) : b \in B\}.$$

If  $z = b' + \sum_{i=1}^k v_i(t_i) \in C'$  then

$$\begin{aligned} z + v &= (b' + \sum_{j \in J^-} v_j(s_j)) + (\sum_{j \in J^+} v_j(s_j) + \sum_{i=1}^k v_i(t_i)). \\ &= b + (\sum_{j \in J^+} v_j(s_j) + \sum_{i=1}^k v_i(t_i)) \in C. \end{aligned}$$

Now we must show that  $\text{ldim}(C \setminus C') < k$ . Let  $z = b + \sum_{i=1}^k v_i(t_i) \in C$  then

$$z \notin C' \iff \exists j \in J^-, t_j \leq -s_j \iff z \in C_j$$

where  $C_j$  is the normalised  $k-1$ -cone  $C_j := B_j + \sum_{i=1, i \neq j}^k v_i(t_i)$  with  $B_j := \{b + v_j(t) : b \in B, 0 < t \leq -s_j\}$ , for each  $j \in J^-$ .  $\square$

We now proceed with the proof of the Main Lemma: Since  $e_n \in \langle D \rangle$  we can apply the lemma A to get an  $m+1$ -cone  $D' \subseteq D$  such that  $\text{ldim}(D \setminus D') < m+1$ , and for all  $z \in D', z + e_n \in D$ . Therefore in  $D'$  the following function

$$\tilde{\Delta}g(z) := (g(z + v_1) - g(z), \dots, g(z + v_m) - g(z), g(z + e_n) - g(z))$$

is well defined. Let

$$D'' := \{z \in D' : \exists \delta > 0 \forall z' \in B(z, \delta) \cap D', \tilde{\Delta}g(z) = \tilde{\Delta}g(z')\},$$



where  $B(z, \delta) := \{z'' \in R^n : |z'' - z| < \delta\}$ .

**Lemma B:**  $\text{ldim}(D' \setminus D'') < m + 1$ .

**Proof.** Suppose not, then by  $(2)'_n$  there is an  $m + 1$ -cone  $C = E + \sum_{i=1}^{m+1} u_i(t_i) \subseteq D' \setminus D''$ , on which we have  $g(e + \sum_{i=1}^{m+1} u_i(t_i)) = g|_E(e) + \sum_{i=1}^{m+1} \mu_i(t_i)$  for some  $\mu_1, \dots, \mu_{m+1} \in \Lambda$ . By the lemma on sub cones

$$\langle u_1, \dots, u_{m+1} \rangle = \langle v_1, \dots, v_{m+1} \rangle.$$

And so  $e_n = \sum_{i=1}^{m+1} u_i(s_i)$  for some  $s_1, \dots, s_{m+1} \in R$  and for each  $k \in \{1, \dots, m + 1\}$  we have  $v_k = \sum_{i=1}^{m+1} u_i(s_{i,k})$  for some  $s_{1,k}, \dots, s_{m+1,k} \in R$ . Apply the lemma A  $m + 1$  times to get an  $m + 1$ -cone  $C' \subseteq C$  such that for all  $j \in \{1, \dots, m\}$  if  $z \in C'$  then  $z + v_j \in C$  and  $z + e_n \in C$ . But then a simple computation shows that for all  $z \in C'$

$$\tilde{\Delta}g(z) = \left( \sum_{i=1}^{m+1} \mu_i(s_{i,1}), \dots, \sum_{i=1}^{m+1} \mu_i(s_{i,m}), \sum_{i=1}^{m+1} \mu_i(s_i) \right).$$

This contradicts the fact that  $C' \subseteq D' \setminus D''$ .  $\square$

**Proof of the Main Lemma:** By sub induction  $(2)_n$  holds in  $D \setminus D''$ . Now,  $\tilde{\Delta}g|_{D''}$  is continuous and locally constant, so  $D''$  is by o-minimality a finite union of definable cells on each of which  $\tilde{\Delta}g$  is constant. By sub induction we may assume that  $D''$  is one such cell.

Apply  $(2)'_n$  to  $D''$ . And let  $C = E + \sum_{i=1}^{m+1} u_i(t_i) \subseteq D''$  be an  $m + 1$ -cone on which  $g(e + \sum_{i=1}^{m+1} u_i(t_i)) = g|_E(e) + \sum_{i=1}^{m+1} \mu_i(t_i)$  for some  $\mu_1, \dots, \mu_{m+1} \in \Lambda$ . Let  $C' \subseteq C$  be an  $m + 1$ -cone obtained as above by applying the lemma A  $m + 1$  times. Then

$$\tilde{\Delta}g(z) = \left( \sum_{i=1}^{m+1} \mu_i(s_{i,1}), \dots, \sum_{i=1}^{m+1} \mu_i(s_{i,m}), \sum_{i=1}^{m+1} \mu_i(s_i) \right),$$

(where for each  $k \in \{1, \dots, m + 1\}$   $s_{1,k}, \dots, s_{m+1,k} \in R$  are such that  $v_k = \sum_{i=1}^{m+1} u_i(s_{i,k})$ , and  $s_1, \dots, s_{m+1} \in R$  are such that  $e_n = \sum_{i=1}^{m+1} u_i(s_i)$ ), since that equality holds in  $C' \subseteq D''$ .

Note that, if  $v \in \Lambda^n$  and  $v_1, \dots, v_l \in \Lambda^n$  are linearly independent and there are  $t_1, \dots, t_l \in R$  such that  $v = \sum_{i=1}^l v_i(t_i)$ , then there are  $\tau_1, \dots, \tau_l \in \Lambda$

such that for each  $i = 1, \dots, l$  we have  $\tau_i(1) = t_i$  : consider the mappings  $t \mapsto \tau_i(t)$  where we write  $v(t) := \sum_{i=1}^l v_i(\tau_i(t))$  , these are well defined definable endomorphisms.

Since, we have  $e_n = \sum_{i=1}^{m+1} u_i(s_i)$ , let  $\sigma_1, \dots, \sigma_{m+1} \in \Lambda$  be such that for each  $i = 1, \dots, m+1$  we have  $\sigma_i(1) = s_i$  and let  $\lambda = \sum_{i=1}^{m+1} \mu_i(\sigma_i)$ . To finish the proof of the Main Lemma we only need to show that for each  $\bar{x} \in \pi(D'')$  we have  $g(\bar{x}, y) = \lambda(y)$  for all  $y \in (-M(\bar{x}), M(\bar{x}))$ . Now since for each  $\bar{x} \in \pi(D'')$   $g(\bar{x}, 1) = g(\bar{x}, 0+1) - g(\bar{x}, 0) = \lambda$  we will be finished by the next lemma.

**Lemma C:** *Let  $\epsilon > 0$ , and let  $\mu, \lambda : (-\epsilon, \epsilon) \rightarrow R$  be partial definable endomorphisms.*

- (1) *If there are  $a$  and  $b$  such that  $0 \leq a < b < \epsilon$  and  $\mu(a) = \mu(b) = 0$  then for all  $x \in [a, b]$   $\mu(x) = 0$ .*
- (2) *If there is  $0 < a < \epsilon$  such that  $\mu(a) = \lambda(a)$ , then  $\mu = \lambda$ .*

**Proof.** (1) : Let  $Z := \{x \in [a, b] : \mu(x) = 0\}$ . Then  $Z$  is infinite since for each  $q \in \mathbb{Q} \cap [0, 1]$  we have

$$\mu((1-q)(a) + q(b)) = (1-q)\mu(a) + q\mu(b) = 0.$$

Therefore by o-minimality it contains an interval  $(\alpha, \beta)$ . Let

$$L := \{\alpha \in (a, b) : \exists \alpha < y < b \forall \alpha \leq t \leq y, \mu(t) = 0$$

$$\text{and } \forall \delta > 0 \exists \alpha - \delta < s < \alpha \mu(s) \neq 0\}.$$

Then

$$L \text{ is finite} \iff L = \emptyset \iff \alpha = a.$$

But by o-minimality  $L$  must be finite. So  $a = \alpha$ , similarly  $\beta = b$ .

(2) : Let  $\alpha := \sup\{x \in (0, \epsilon) : (\mu - \lambda)(y) = 0 \forall y \in [0, x]\}$  (by (1) the above set is nonempty). If  $\alpha < \epsilon$  then there is  $\beta \in (\alpha, \epsilon)$  such that  $\beta - \alpha < \alpha$ , and

$$(\mu - \lambda)(\beta) = (\mu - \lambda)((\beta - \alpha) + \alpha) = 0,$$

a contradiction. □

## 4.6 $(2)_n \Rightarrow (1)_{n+1}$

By cell decomposition we may assume that  $X$  is a cell, for example of the form  $X = (f, g)_{\pi(X)}$ , where  $f, g : \pi(X) \rightarrow R$  are definable functions; (for the other cases the proof is similar).

Apply  $(2)_n$  to  $f, g : \pi(X) \rightarrow R$ . Let  $D = B + \sum_{i=1}^m v_i(t_i)$  be a normalised cone in the resulting decomposition of  $\pi(X)$ , and let

$$E := \{x \in X : \pi(x) \in D\}.$$

It is sufficient to prove the result for this set.

For some  $\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_m \in \Lambda$  we have that  $\forall b \in B \forall t_1, \dots, t_m > 0$ ,

$$f\left(b + \sum_{i=1}^m v_i(t_i)\right) = f|_B(b) + \sum_{i=1}^m \lambda_i(t_i)$$

and

$$g\left(b + \sum_{i=1}^m v_i(t_i)\right) = g|_B(b) + \sum_{i=1}^m \mu_i(t_i).$$

Since  $f < g$ , it is easy to verify that  $f|_B(b) < g|_B(b)$  and so each  $\lambda_i \leq \mu_i$ .

And

$$E = \left\{ \left( b + \sum_{i=1}^m v_i(t_i), u \right) : b \in B, t_1, \dots, t_m > 0, \right.$$

$$\left. f|_B(b) + \sum_{i=1}^m \lambda_i(t_i) < u < g|_B(b) + \sum_{i=1}^m \mu_i(t_i) \right\}$$

$$= E_0 \cup E_1 \cup E_2$$

where:

$$E_0 := \left\{ \left( b + \sum_{i=1}^m v_i(t_i), u \right) : b \in B, t_1, \dots, t_m > 0, \right.$$

$$\left. f|_B(b) + \sum_{i=1}^m \lambda_i(t_i) < u < g|_B(b) + \sum_{i=1}^m \lambda_i(t_i) \right\}$$

which is the normalised  $m$ -cone

$$(f|_B, g|_B)_B + \sum_{i=1}^m (v_i, \lambda_i)(t_i);$$

and,

$$E_1 := \left\{ \left( b + \sum_{i=1}^m v_i(t_i), u \right) : b \in B, t_1, \dots, t_m > 0, \right. \\ \left. u = g|_B(b) + \sum_{i=1}^m \lambda_i(t_i) \right\},$$

which is the normalised  $m$ -cone  $\Gamma(g|_B) + \sum_{i=1}^m (v_i, \lambda_i)(t_i)$  and

$$E_2 := \left\{ \left( b + \sum_{i=1}^m v_i(t_i), u \right) : b \in B, t_1, \dots, t_m > 0, \right.$$

$$\left. g|_B(b) + \sum_{i=1}^m \lambda_i(t_i) < u < g|_B(b) + \sum_{i=1}^m \mu_i(t_i) \right\},$$

$$= \Gamma(g|_B) + \left\{ \left( \sum_{i=1}^m v_i(t_i), u \right) : t_1, \dots, t_m > 0, \right.$$

$$\left. \sum_{i=1}^m \lambda_i(t_i) < u < \sum_{i=1}^m \mu_i(t_i) \right\},$$

which is the disjoint union of the sets

$$E_2^k := \Gamma(g|_B) + \left\{ \left( \sum_{i=1}^m v_i(t_i), u \right) : b \in B, t_1, \dots, t_m > 0, \right.$$

$$\left. \sum_{i=1}^{k-1} \mu_i(t_i) + \sum_{i=k}^m \lambda_i(t_i) < u < \sum_{i=1}^k \mu_i(t_i) + \sum_{i=k+1}^m \lambda_i(t_i) \right\}$$

with  $k \in \{1, \dots, m\}$ , which (except when  $\lambda_k = \mu_k$ , in which case  $E_2^k$  is empty) is the normalised  $m + 2$ -cone

$$\Gamma(g|_B) + \sum_{i=1}^{k-1} (v_i, \mu_i)(t_i) + (v_k, \lambda_k)(t) + (v_k, \mu_k)(t') + \sum_{i=k+1}^m (v_i, \lambda_i)(t_i)$$

(since  $\lambda_k(t_k) < u < \mu_k(t_k)$  for all  $t_k > 0$ , there are unique  $t, t' > 0$  such that  $t + t' = t_k$  (and so  $v_k(t) + v_k(t') = v_k(t_k)$ ) and  $u = \lambda_k(t) + \mu_k(t')$ ).

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