Moduli of Germs of Legendrian Curves

António Araújo and Orlando Neto

Abstract

We construct the generic component of the moduli space of the germs of Legendrian curves with generic plane projection topologicaly equivalent to a curve $y^n = x^m$.

1 Introduction

Zariski [7] initiated the construction of the moduli of plane curve singularities. Delorme [1] organized in a systematic way the ideas of Zariski, obtaining general results o the case of curves with one characteristic exponent in the generic case (see also [6]). Greuel, Laudal and Pfister (see the bibliography of [2]) stratified the space versal deformations of plane curves, constructing moduli spaces on each stratum.

In this paper we initiate the study of the moduli of Legendrian curve singularities. We construct the moduli space of generic irreducible Legendrian singularities with equisingularity type equal to the topological type of the plane curve $y^n = x^m$, $(n, m) = 1$. Our method is based on the analysis of the action of the group of infinitesimal contact transformations on the set of Puiseux expansions of the germs of plane curves.

In section 2 we associate to each pair of positive integers n, m such that $(n, m) = 1$ a semigroup $\Gamma(n, m)$. We show that the semigroup of a generic element of this equisingularity class equals $\Gamma(n, m)$. In section 3 we classify the infinitesimal contact transformations on a contact threefold and study its action on the Puiseux expansion of a plane curve. In section 4 we discuss some simple examples of moduli of germs of Legendrian curves. In section 5 we show that the generic components of the moduli of germs of Legendrian curves with fixed equisingularity class are the points of a Zariski open subset of a weighted projective space.

2 Plane curves versus Legendrian curves

Let Λ be the germ at o of an irreducible space curve. A local parametrization $\imath : (\mathbb{C}, 0) \to (\Lambda, o)$ defines a morphism $\imath^*$ from the local ring $\mathcal{O}_{\Lambda,o}$ into its normalization $\mathbb{C}\{t\}$. The semigroup of Λ equals the set Γ of the orders of the series that belong to the image of $\imath^*$. There is an integer k such that

$l \in \Gamma$ for all $l \geq k$. The smallest integer k with this property is denoted by c and called the *conductor* of Γ.

Let C be the germ at the origin of a singular irreducible plane curve C parametrized by

$$x = t^n, \qquad y = \sum_{i=m}^{\infty} a_i t^i, \tag{1}$$

with $a_m \neq 0$ and $(n, m) = 1$. The pair (n, m) determines the topological type of C.

Let M be a complex manifold of dimension n. The cotangent bundle $\pi_M : T^*M \to M$ of M is endowed of a canonical 1-form θ. The differential form $(d\theta)^{\wedge n}$ never vanishes on M. Hence $d\theta$ is a symplectic form on T^*M. Given a system of local coordinates $(x_1, \ldots, x_n)$ on an open set U of X, there are holomorphic functions $\xi_1, \ldots, \xi_n$ on $\pi_M^{-1}(U)$ such that $\theta \mid_{\pi_M^{-1}(U)} = \xi_1 dx_1 + \cdots + \xi_n dx_n$.

Let X be a complex threefold. Let Ω_X^k denote the sheaf of differential forms of degree k on X. A local section of Ω_X^1 is called a *contact form* if $\omega \wedge d\omega$ never vanishes. Let $\mathcal{L}$ be a subsheaf of the sheaf Ω_X^1. The sheaf $\mathcal{L}$ is called a *contact structure* on X if $\mathcal{L}$ is locally generated by a contact form. A pair $(X, \mathcal{L})$, where $\mathcal{L}$ is a contact structure on X, is called a *contact threefold.* Let $(X_i, \mathcal{L}_i)$, $i = 1, 2$, be two contact threefolds. A holomorphic map $\varphi : X_1 \to X_2$ is called a *contact transformation* if $\varphi^*\mathcal{L}_2 = \mathcal{L}_1$.

Let $\mathbb{P}^*\mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{P}^1 = \{(x, y, (\xi : \eta)) : x, y, \xi, \eta \in \mathbb{C},\ (\xi, \eta) \neq (0, 0)\}$ be the projective cotangent bundle of $\mathbb{C}^2$. Let $\pi : \mathbb{P}^*\mathbb{C}^2 \to \mathbb{C}^2$ be the canonical projection. Let U and V be the open sets of $\mathbb{P}^*\mathbb{C}^2$ defined respectively by $\eta \neq 0$ and $\xi \neq 0$. Set $p = -\xi/\eta$, $q = -\eta/\xi$. The sheaf $\mathcal{L}$ defined by $\mathcal{L} \mid_U = \mathcal{O}_U(dy - pdx)$ and $\mathcal{L} \mid_V = \mathcal{O}_V(dx - qdy)$ is a contact structure on $\mathbb{P}^*\mathbb{C}^2$. By the Darboux theorem every contact threefold is locally isomorphic to $(U, \mathcal{O}_U(dy - pdx))$. We call *infinitesimal contact transformation* to a germ of a contact transformation $\Phi : (U, 0) \mapsto (U, 0)$.

A curve Λ on a contact manifold $(X, \mathcal{L})$ is called *Legendrian* if the restriction of ω to the regular part of Λ vanishes for each section ω of $\mathcal{L}$. Let $C = \{f = 0\}$ be a plane curve. Let Λ be the closure on $\mathbb{P}^*\mathbb{C}^2$ of the graph of the Gauss map $G : \{a \in C : df(a) \neq 0\} \to \mathbb{P}^1$ defined by $G(a) = \langle df(a) \rangle$. The set Λ is a Legendrian curve. We call Λ the *conormal* of the curve C. If C is irreducible and parametrized by (1) then Λ is parametrized by

$$x = t^n, \qquad y = \sum_{i=m}^{\infty} a_i t^i, \qquad p = \frac{dy}{dx} = \sum_{i=m}^{\infty} \frac{i}{n} a_i t^{i-n}. \tag{2}$$

Given a Legendrian curve Λ of $\mathbb{P}^*\mathbb{C}^2$ such that Λ does not contain any fibre of π, $\pi(\Lambda)$ is a plane curve. Moreover, Λ equals the conormal of $\pi(\Lambda)$.

Let $(X, \mathcal{L})$ be a contact threefold. A holomorphic map $\varphi : (X, o) \to (\mathbb{C}^2, 0)$ is called a *Legendrian map* if $D\varphi(o)$ is surjective and the fibers of φ are smooth Legendrian curves. The map φ is Legendrian if and only if there is a contact transformation $\psi : (X, o) \to (\mathbb{P}^*\mathbb{C}^2, (0, 0, (0 : 1))$ such that $\varphi = \pi\psi$.

Let (Λ, o) be a Legendrian curve of X. Let $C_o(\Lambda)$ be the tangent cone of Λ at o. We say that a Legendrian map $\varphi : (X, o) \to (\mathbb{C}^2, 0)$ is *generic* relatively to (Λ, o) if it verifies the transversality condition $T_o\varphi^{-1}(0) \cap C_o(\Lambda) = \{0\}$. We say that a Legendrian curve (Λ, o) of $\mathbb{P}^*\mathbb{C}^2$ is in strong generic position if $\pi : (\mathbb{P}^*\mathbb{C}^2, o) \to (\mathbb{C}^2, \pi(o))$ is generic relatively to (Λ, o). The Legendrian curve Λ parametrized by (2) is in *strong generic position* if and only if $m \geq 2n + 1$. Given a Legendrian curve (Λ, o) of a contact threefold X there is a contact transformation $\psi : (X, o) \to (\mathbb{P}^*\mathbb{C}^2, (0, 0, (0 : 1))$ such that $(\psi(\Lambda), o)$ is in strong generic position (cf [3], section 1).

We say that two germs of Legendrian curves are equisingular if their images by generic Legendrian maps have the same topological type.

3 Infinitesimal Contact Transformations

Let m be the maximal ideal of the ring $\mathbb{C}\{x, y, p\}$. Let $\mathcal{G}$ denote the group of infinitesimal contact transformations Φ such that the derivative of Φ leaves invariant the tangent space at the origin of the curve $\{y = p = 0\}$. Let $\mathcal{J}$ be the group of infinitesimal contact transformations

$$(x, y, p) \mapsto (x + \alpha, y + \beta, p + \gamma) \tag{3}$$

such that $\alpha, \beta, \gamma, \partial\alpha/\partial x, \partial\beta/\partial y, \partial\gamma/\partial p \in m$. Set $\mathcal{H} = \{\Psi_{\lambda,\mu} : \lambda, \mu \in \mathbb{C}^*\}$, where

$$\Psi_{\lambda,\mu}(x, y, p) = \left(\lambda x, \mu y, \frac{\mu}{\lambda} p\right). \tag{4}$$

Let $\mathcal{P}$ denote the group of *paraboloidal contact transformations* (see [4])

$$(x, y, p) \mapsto (ax + bp, y - \frac{1}{2}acx^2 - \frac{1}{2}bdp^2 - bcxp, cx + dp), \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1. \tag{5}$$

The contact transformation (5) belongs to $\mathcal{G}$ if and only if $c = 0$. The paraboloidal contact transformation

$$(x, y, p) \mapsto (-p, y - xp, x) \tag{6}$$

Is called the *Legendre transformation.*

Theorem 1 *The group $\mathcal{J}$ is an invariant subgroup of $\mathcal{G}$. Moreover, the quotient $\mathcal{G}/\mathcal{J}$ is isomorphic to $\mathcal{H}$.*

Proof. If $H \in \mathcal{H}$ and $\Phi \in \mathcal{J}$, $H\Phi H^{-1} \in \mathcal{J}$. Hence it is enough to show that each element of $\mathcal{G}$ is a composition of elements of $\mathcal{H}$ and $\mathcal{J}$. Let $\Phi \in \mathcal{G}$ be the infinitesimal contact transformation $(x, y, p) \mapsto (x', y', p')$. There is $\varphi \in \mathbb{C}\{x, y, p\}$ such that $\varphi(0) \neq 0$ and

$$dy' - p'dx' = \varphi(dy - pdx). \tag{7}$$

Composing Φ with $H \in \mathcal{H}$ we can assume that $\varphi(0) = 1$. Let $\hat{\Phi}$ be the germ of the symplectic transformation $(x, y, p; \eta) \mapsto (x', y', -\eta p'; \varphi^{-1}\eta)$. Notice that $\hat{\Phi}(0, 0; 0, 1) = (0, 0; 0, 1)$. Since $D\hat{\Phi}(0, 0; 0, 1)$ leaves invariant the linear subspace μ generated by $(0, 0; 0, 1)$, $D\hat{\Phi}(0, 0; 0, 1)$ induces a linear symplectic transformation on the linear symplectic space $\mu^{\perp}/\mu$. There is a paraboloidal contact transformation P such that $D\hat{P}(0, 0; 0, 1)$ equals $D\hat{\Phi}(0, 0; 0, 1)$ on $\mu^{\perp}/\mu$. Since $D(\hat{P}^{-1}\hat{\Phi})(0, 0; 0, 1)$ induces the identity map on $\mu^{\perp}/\mu$, $P^{-1}\Phi$ is an infinitesimal contact transformation of the type $(x, y, p) \mapsto (x+\alpha, y', p+\gamma)$, where

$$\frac{\partial \alpha}{\partial x}, \frac{\partial \alpha}{\partial p}, \frac{\partial \gamma}{\partial x}, \frac{\partial \gamma}{\partial p} \in m. \tag{8}$$

Set $\beta = y' - y$. It follows from (7) and (8) that $(\partial\beta/\partial y)(0) = 0$. Hence $P^{-1}\Phi \in \mathcal{J}$. Since Φ and $P^{-1}\Phi \in \mathcal{G}$, $P \in \mathcal{G}$. Therefore p is the composition of an element of $\mathcal{H}$ and an element of $\mathcal{J}$. q.e.d.

Theorem 2 *Let $\alpha \in \mathbb{C}\{x, y, p\}, \beta_0 \in \mathbb{C}\{x, y\}$ be power series such that*

$$\alpha, \beta_0, \frac{\partial \beta_0}{\partial y} \in m. \tag{9}$$

There are $\beta, \gamma \in \mathbb{C}\{x, y, p\}$ such that $\beta - \beta_0 \in (p)$, $\gamma \in m$ and α, β, γ define an infinitesimal contact transformation Φ_{α,β_0} of type (3). The power series β and γ are uniquely determined by these conditions. Moreover, (3) belongs to $\mathcal{J}$ if and only if

$$\frac{\partial \alpha}{\partial x}, \frac{\partial \beta_0}{\partial x}, \frac{\partial^2 \beta_0}{\partial x \partial p} \in m. \tag{10}$$

The function β is the solution of the Cauchy problem

$$\left(1 + \frac{\partial \alpha}{\partial x} + p\frac{\partial \alpha}{\partial y}\right)\frac{\partial \beta}{\partial p} - p\frac{\partial \alpha}{\partial p}\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial p}\frac{\partial \beta}{\partial x} = p\frac{\partial \alpha}{\partial p}. \tag{11}$$

with initial condition $\beta - \beta_0 \in (p)$.

Proof. The map (3) is a contact transformation if and only if there is $\varphi \in \mathbb{C}\{x, y, p\}$ such that $\varphi(0) \neq 0$ and

$$d(y + \beta) - (p + \gamma)d(x + \alpha) = \varphi(dy - pdx). \qquad (12)$$

The equation (12) is equivalent to the system

$$\frac{\partial \beta}{\partial p} = (p + \gamma)\frac{\partial \alpha}{\partial p} \qquad (13)$$

$$\varphi = 1 + \frac{\partial \beta}{\partial y} - (p + \gamma)\frac{\partial \alpha}{\partial y} \qquad (14)$$

$$-p\varphi = \frac{\partial \beta}{\partial x} - (p + \gamma)(1 + \frac{\partial \alpha}{\partial x}). \qquad (15)$$

By (14) and (15),

$$\frac{\partial \beta}{\partial x} - (p + \gamma)\left(1 + \frac{\partial \alpha}{\partial x} + p\frac{\partial \alpha}{\partial y}\right) + p\left(1 + \frac{\partial \beta}{\partial y}\right) = 0, \qquad (16)$$

By (13) and (16), (11) holds.

By the Cauchy-Kowalevsky theorem there is one and only one solution β of (11) such that $\beta - \beta_0 \in (p)$. It follows from (16) that

$$\gamma = \left(1 + \frac{\partial \alpha}{\partial x} + p\frac{\partial \alpha}{\partial y}\right)^{-1}\left(\frac{\partial \beta}{\partial x} + p\left(\frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial x} - p\frac{\partial \alpha}{\partial y}\right)\right). \qquad (17)$$

Since $\partial\beta_0/\partial y \in m$, $\partial\beta/\partial y \in m$. By (14), $\varphi(0) \neq 0$.
(ii) Since $\partial\beta_0/\partial x \in m$, $\partial\beta/\partial x \in m$. By (17), $\gamma \in m$. By (17),

$$\frac{\partial \gamma}{\partial p} \in \left(\frac{\partial^2 \beta}{\partial x \partial p} + \frac{\partial \beta}{\partial y} - \frac{\partial \alpha}{\partial x}, p\right).$$

By (9) and (10), $\partial\gamma/\partial p \in m$. q.e.d.

Corollary 3 *The elements of $\mathcal{J}$ are the infinitesimal contact transformations Φ_{α,β_0} such that α, β_0 verify (9) and (10).*

Lemma 4 *Given $\lambda \in \mathbb{C}$ and $w \in \Gamma(m, n)$ such that $w \geq m + n$, there are α, β_0 verifying the conditions of theorem 2 such that $\imath^*(\beta - p\alpha) = \lambda t^w + \cdots$.*

Proof. By (19) there is $b \in \mathbb{C}\{x, y, p\}$ such that $\imath^* b = \lambda t^w + \cdots$, $b = \sum_{k\geq 0} b_k p^k$ and $v(b_k) \geq v(b) - v(x) - kv(p) + 1$. Set $\alpha = -\partial b/\partial p$, $\beta_0 = b_0$. Set $\alpha = \sum_{k\geq 0} \alpha_k p^k$, $\beta = \sum_{k\geq 0} \beta_k p^k$, where $\alpha_k, \beta_k \in \mathbb{C}\{x, y\}$. By (11),

$$k\beta_k + \sum_{j=1}^{k-1} j\beta_j \left(\frac{\partial \alpha_{k-j}}{\partial x} + \frac{\partial \alpha_{k-j-1}}{\partial y}\right) =$$

$$= \quad (k-1)\alpha_{k-1} + k\alpha_k \frac{\partial \beta_0}{\partial x} + \sum_{j=1}^{k-1} j\alpha_j \left(\frac{\partial \beta_{k-j}}{\partial x} + \frac{\partial \beta_{k-j-1}}{\partial y} \right),$$

for $k \geq 1$. Since $\alpha_l = -(l+1)b_{l+1}$ for $l \geq 1$, $v(\alpha_j p^k) \geq w+1$, if $j \leq k-2$. Moreover, $v(\alpha_{k-1}p^k) \geq w+1-n$ and $v(\alpha_k p^k) \geq w+1-m$. Therefore

$$k\beta_k p^k + \sum_{j=1}^{k-1} j\beta_j \left(\frac{\partial \alpha_{k-j}}{\partial x} + \frac{\partial \alpha_{k-j-1}}{\partial y} \right) p^k \equiv (k-1)\alpha_{k-1}p^k + (k-1)\alpha_{k-1}\frac{\partial \beta_1}{\partial x},$$

mod $\left(t^{w+1}\right)$ for $k \geq 1$. We show by induction in k that

$$k\beta_k p^k \equiv (k-1)\alpha_{k-1}p^k \quad \bmod (t^{w+1}), \text{for } k \geq 1.$$

Hence $\beta - p\alpha \equiv b$ mod (t^{w+1}). q.e.d. There is an action of $\mathcal{J}$ into the set of germs of plane curves C such that the tangent cone to the conormal of C equals $\{y = p = 0\}$. Given $\Phi \in \mathcal{J}$ we associate to C the image by $\pi\Phi$ of the conormal of C. Given integers n, m such that $(m,n) = 1$ and $m \geq 2n+1$, $\mathcal{J}$ acts on the series of type (1). Given an infinitesimal contact transformation (3) there is $s \in \mathbb{C}\{t\}$ such that $s^n = t^n + \alpha$ and for each $i \geq 1$

$$s^i = t^i \left(1 + \frac{i}{n}\frac{\alpha(t)}{t^n} + \frac{i}{n}\left(\frac{i}{n} - 1\right)\left(\frac{\alpha(t)}{t^n}\right)^2 + \cdots \right).$$

Lemma 5 *If $v(\beta_0) \geq v(\alpha) + v(p)$, the contact transformation* (3) *takes* (1) *into the plane curve parametrized by $x = s^n$, $y = y(s) + \beta(s) - p(s)\alpha(s) + \varepsilon$, where $v(\varepsilon) \geq 2v(\alpha) + m - 2n$.*

Proof. Since $t^i = s^i - (i/n)t^{i-n}\alpha(t) + (i(i-n)/n^2)\alpha(t)^2 t^{i-2n} + \cdots$,

$$y(t) = \sum_{i \geq m} a_i s^i - \alpha(t) \sum_{i \geq m} \frac{i}{n} a_i t^{i-m} + \varepsilon' = y(s) - \alpha(t)p(t) + \varepsilon',$$

$$p(t)\alpha(t) = p(s)\alpha(t) - \alpha(t)^2 \sum_{i \geq m} (\frac{i}{n})^2 a_i t^{i-2m} + \varepsilon'' = p(s)\alpha(s) + \varepsilon''',$$

where $v(\varepsilon'), v(\varepsilon''), v(\varepsilon''') \geq 2v(\alpha) + m - 2n$. q.e.d.

4 Examples

Example 6 *If m odd all plane curves topologicaly equivalent to $y^2 = x^m$ are analyticaly equivalent to $y^2 = x^m$ (cf. [7]). Hence all Legendrian curves with generical plane projection $y^2 = x^m$ are contact equivalent to the conormal of $y^2 = x^m$.*

Example 7 *Let m, s, ϵ be positive integers. Assume that $m = 3s + \epsilon$, $1 \leq \epsilon \leq 2$. Let $C_{3,m,\nu}$ be the plane curve parametrized by*

$$x = t^3, \quad y = t^m + t^{m+3\nu+\epsilon-3}.$$

By [7] a plane curve topologically equivalent to $y^3 = x^m$ is analyticaly equivalent to $y^3 = x^m$ or to one of the curves $C_{3,m,\nu}$, $1 \leq \nu \leq s-1$. The infinitesimal contact transformation

$$(x, y, p) \mapsto (x - 2p, y + p^2, p)$$

takes the plane curve $C_{3,m,s-1}$ into the plane curve C' parametrized by

$$3x = 3t^3 - mt^{m-3} - \cdots, \quad y = t^m.$$

By Lemma 5, the curve C' admits a parametrization of the type $x = s^3$, $y = s^m + \delta$, where $v(\delta) \geq m + 3s + \epsilon - 6$. By [7], the curve C' is analyticaly equivalent to the plane curve $y^3 = x^m$.

The semigroup of the conormal of the plane curve $y^3 = x^m$ equals $\Gamma_{3,m,0} = \langle 3, m-3 \rangle$. The semigroup of the conormal of the curve $C_{3,m,\nu}$ equals $\Gamma_{3,m,\nu} = \langle 3, m-3, m+3\nu+\epsilon \rangle$, $1 \leq \nu \leq s-1$. The map from $\{0, 1, \ldots, s-2\}$ into $\mathcal{P}(\mathbf{N})$ that takes ν into $\Gamma_{3,m,\nu}$ is injective. Hence there are $s-1$ analytic equivalence classes of plane curves topologicaly equivalent to $y^3 = x^m$ and $s-2$ equivalence contact classes of Legendrian curves with generical plane projection $y^3 = x^m$. In this case the semigroup of a curve is an analytic invariant that classifies the contact equivalence classes of Legendrian curves. We will see that in the general case there are no discrete invariants that can classify the contact equivalence classes of Legendrian curves.

Given a plane curve

$$x = t^3, \quad y = t^m + \sum_{i \geq m+\epsilon} a_i t, \tag{18}$$

the semigroup of the conormal of (18) equals $\Gamma_{3,m,1}$ if and only if $a_{m+\epsilon} \neq 0$. It is therefore natural to call $\Gamma(3, m) := \Gamma_{3,m,1}$ the generic semigroup of the family of Legendrian curves with generic plane projection $y^3 = x^m$.

5 The generic semigroup of an equisingularity class of irreducible Legendrian curves

We will associate to a pair (n,m) such that $m \geq 2n+1$ and $(m,n)=1$ a semigroup $\Gamma(n,m)$. Let $\langle k_1,\ldots,k_r\rangle$ be the submonoid of $(N,+)$ generated by $k_1,\ldots,k_r$. Let c be the conductor of the semigroup of the plane curve (1). Set $\Gamma_c = \langle n\rangle \cup \{c,c+1,...\}$. We say that the trajectory of $k \geq c$ equals $\{k,k+1,...\}$. Let us assume that we have defined Γ_j and the trajectory of j for some $j \in \langle n,m-n\rangle \setminus \Gamma_c$, $j \geq m$. Let i be the biggest element of $\langle n,m-n\rangle \setminus \Gamma_j$. Let $\sharp_i$ be the minimum of the cardinality of the set of monomials of $\mathbb{C}[x,y,p]$ of valuation i and the cardinality of $\{i,i+1,\ldots\}\setminus\Gamma_j$. Let ω_i be the $\sharp_i$-th element of $\{i,i+1,\ldots\}\setminus\Gamma_j$. We call *trajectory* of i to the set $\tau_i = \{i,i+1,\ldots,\omega_i\}\setminus\langle n\rangle$. Set $\Gamma_i = \tau_i \bigcup \Gamma_j$. Set $\Gamma(n,m)=\Gamma_{m-n}$. The main purpose of this section is to prove theorem 9. Let us show that

$$\omega_i \leq i+n-2. \tag{19}$$

If $\omega_i \geq i+n-1$, $\Gamma_i \supset \{i,\ldots,i+n-1\}$. Hence $\Gamma_i \supset \{i,i+1,\ldots\}$ and $i \geq c$. Therefore (19) holds.

Let $X=t^n$, $Y=\sum_{i\geq 0} a_{m+i}t^{m+i}$, $P=\sum_{i\geq 0}(\mu+i)a_{m+i}t^{m-n+i}$ be power series with coefficients in the ring $\mathbb{Z}[a_m,\ldots,a_{c-1},\mu]$. Given $J=(i,j,l)\in\mathbb{N}^3$, set $v(J)=v(x^iy^jp^l)$. Let $\mathcal{N}=\{J\in\mathbb{N}^3 : j+l\geq 1 \text{ and } v(J)\leq c-1\}$. Let $\Upsilon=(\Upsilon_{J,k})$, $J\in\mathcal{N}$, $m\leq k\leq c-1$ be the matrix such that

$$X^iY^jP^l \equiv \sum_{k=m}^{c-1} \Upsilon_{J,k}t^k \qquad (mod\ (t^c)). \tag{20}$$

Since $\partial Y/\partial\mu = 0$ and $X\partial P/\partial\mu = Y$,

$$\frac{\partial X^iY^jP^l}{\partial\mu} = lX^{i-1}Y^{j+1}P^{l-1} \qquad \text{and} \qquad \frac{\partial\Upsilon_{J,k}}{\partial\mu} = l\Upsilon_{\partial J,k}, \tag{21}$$

where $\partial(i+1,j,l+1)=(i,j+1,l)$. Moreover,

$$\Upsilon_{J,k} = \sum_{\alpha\in A(k)}\sum_{\gamma\in G(\alpha,l)} \frac{j!\ l!}{(\alpha-\gamma)!\gamma!}\ a^\alpha\mu^\gamma, \tag{22}$$

where $A(k)=\{\alpha=(\alpha_m,...,\alpha_{c-1}) : |\alpha|=j+l \text{ and } \sum_{s=m}^{c-1} s\alpha_s = k-(i-l)n\}$, $G(\alpha,l)=\{\gamma : |\gamma|=l \text{ and } 0\leq\gamma\leq\alpha\}$ and $\mu^\gamma=\prod_{s=m}^{c-1}(\mu-m+s)^{\gamma_s}$. Let us prove (22). We can assume that $i=l$. Since $G(\alpha,N)=\{\alpha\}$ and $X^NP^N=\sum_{k\geq 0}t^k\sum_{\alpha\in A(k)}(N!/\alpha!)\mu^\alpha a^\alpha$, (22) holds for $J=(N,0,N)$. Let us show by induction in j that (22) holds when $j+l=N$. Set $e_s=(\delta_{s,r})$, $0\le s,r\le N$. Given $\gamma\in G(\alpha,l-1)$, set $\gamma_{(s)}=\gamma+e_s$. Set $\Delta_s^\gamma=1$ if $\gamma_{(s)}\le\alpha$. Otherwise, set $\Delta_s^\gamma=0$. Since

$$\begin{aligned}\frac{1}{l}\sum_{\gamma\in G(\alpha,l)}\frac{j!l!}{(\alpha-\gamma)!\gamma!}\frac{\partial\mu^\gamma}{\partial\mu} &= \sum_{\gamma\in G(\alpha,l-1)}\sum_{s=m}^{c-1}\frac{j!(l-1)!}{(\alpha-\gamma_{(s)})!\gamma_{(s)}!}(\gamma_s+1)\Delta_s^\gamma\mu^\gamma\\ &= \sum_{\gamma\in G(\alpha,l-1)}\frac{j!(l-1)!}{(\alpha-\gamma)!\gamma!}\mu^\gamma\sum_{s=m}^{c-1}(\alpha_s-\gamma_s)\\ &= \sum_{\gamma\in G(\alpha,l-1)}\frac{(j+1)!(l-1)!}{(\alpha-\gamma)!\gamma!}\mu^\gamma,\end{aligned}$$

the induction step follows from (21). We will consider in the polynomial ring $\mathbb{C}[a_m,\dots,a_{c-1}]$ the order $a^\alpha<a^\beta$ if there is an integer q such that $\alpha_q<\beta_q$ and $\alpha_i=\beta_i$ for $i\ge q+1$. Set $\omega(P)=\sup\{i: a_i \text{ occurs in } P\}$.

Lemma 8 *Let $M,N,q\in\mathbb{Z}$ such that $0\le M\le N$ and $q+N\ge 0$. If $\lambda=(\lambda_{l,k})$, where $M\le l\le N$, $k\ge 0$, $\lambda_{l,k}=\Upsilon_{J,k}$ and $J=(q+l,N-l,l)$, the minors of λ with $N-M+1$ columns different from zero do not vanish at $\mu=m$.*

Proof. One can assume that $q=0$. When we multiply the left-hand side of (20) by P the coefficients of Υ are shifted and multiplied by an invertible matrix. Hence one can assume that $M=0$. Set $Z=(Z_{j,k})$, where $Z_{j,k}=\binom{j}{k}\mu^{j-k}$, $0\le j,k\le N$. Notice that Z is lower diagonal, $\det(Z)=1$ and

$$\frac{\partial Z_{j,k}}{\partial\mu} = jZ_{j-1,k} = (k+1)Z_{j,k+1}. \tag{23}$$

Let us show that

$$Z^{-1}\lambda=\lambda|_{\mu=0}. \tag{24}$$

Since $\lambda_{N,k}$ is a polynomial of degree N in the variable μ with coefficients in the ring $\mathbb{Z}[a_m,\dots,a_{c-1}]$, there are polynomials $\mathcal{Z}_{i,k}\in\mathbb{Q}[a_m,\dots,a_{c-1}]$ such that $\lambda_{N,k}=\sum_{i=0}^N\binom{N}{i}\mathcal{Z}_{i,k}\mu^{N-i}$. Set $\mathcal{Z}=(\mathcal{Z}_{i,k})$, $0\le i\le N$, $0\le k\le c-1$. Since $Z|_{\mu=0}=Id$, it is enough to show that $Z\mathcal{Z}=\lambda$. By construction,

$$\lambda_{j,k}=\sum_{i=0}^N Z_{j,i}\mathcal{Z}_{i,k} \tag{25}$$

when $j=N$. By (21) and (23) statement (25) holds for all j. Remark that

$$\lambda_{l,v(J)+k}|_{\mu=0}=0 \qquad \text{if and only if} \qquad k<l. \tag{26}$$

Let $\theta_{l,k}$ be the leading monomial of $\lambda_{l,k}$. When $k \geq l$,

$$\theta_{l,v(J)+k}=a_m^{N-1}a_{m+k} \qquad \text{if} \qquad l=0, \tag{27}$$

$$\theta_{l,v(J)+k}=a_m^{N-l}a_{m+1}^{l-1}a_{m+k-l+1} \qquad \text{if} \qquad l\geq 1. \tag{28}$$

Let us prove (28). Set $\alpha_0=j$, $\alpha_1=l-1$, $\alpha_{k-l+1}=1$ and $\alpha_s=0$ otherwise. By (22), $\alpha\in A(k)$ and there is one and only one $\gamma\in G(\alpha,j)$ such that $\gamma_0=0$, the tuple $\overline{\alpha}$ given by $\overline{\alpha}_0=0$ and $\overline{\alpha}_i=\alpha_i$ if $i\neq 0$. Since

$$\sum_{\gamma\in G(\alpha,l)}\frac{j!l!\mu^{\gamma}}{(\alpha-\gamma)!\gamma!}\equiv\frac{j!l!\mu^{\overline{\alpha}}}{(\alpha-\overline{\alpha})!\overline{\alpha}!}\equiv l\prod_{s=0}^{c-m-1}s^{\overline{\alpha}_s}=(k-l+1)l \mod \mu,$$

the coefficient of $a_m^{N-l}a_{m+1}^{l-1}a_{k-l+1}$ does not vanish. By (22), $\alpha_{k-l+r}\neq 0$ for some $r>1$ implies that $\gamma_0>0$ for all $\gamma\in G(\alpha,l)$. Hence (28) holds.

Let λ' be the square submatrix of λ with columns $g(i)+Nm$, $0\leq g(0)<\cdots<g(N)$. By (24), $\det(\lambda'|_{\mu=0})=\det(Z^{-1}\lambda')=\det(Z)^{-1}\det\lambda'=\det\lambda'$. Hence $\det\lambda'$ does not depend on μ and $\det(\lambda'|_{\mu=m})=\det(\lambda'|_{\mu=0})$. Set $\det(\lambda')=\sum_\pi sgn(\pi)\lambda_\pi$, where $\lambda_\pi=\prod_{i=0}^N\lambda'_{i,\pi(i)}$. If $\lambda_\pi\neq 0$, let θ_π be the leading monomial of λ_π.

Let ε be the following permutation of $\{0,\ldots,N\}$. Assume that ε is defined for $0\leq i\leq l-1$. Let p_l and q_l be respectively the maximum and the minimum of $\{0,\ldots,N\}\setminus\varepsilon(\{0,\ldots,l-1\})$. If $\lambda_{l+1,q_l}=0$, set $\varepsilon(l)=q_l$. Otherwise, set $\varepsilon(l)=p_l$. Let us show that (26) implies that $\lambda_\varepsilon\neq 0$. It is enough to show that $\lambda_{i,q_i}\neq 0$ for all i. Since $g(0)\geq 0$, $\lambda_{0,q_0}\neq 0$. Assume that $l\geq 1$ and $\lambda_{i,q_i}\neq 0$ for $0\leq i\leq l-1$. Hence $g(q_{l-1})\geq l-1$. If $\lambda_{l,q_{l-1}}\neq 0$ then $\lambda_{l,q_l}\neq 0$. If $\lambda_{l,q_{l-1}}=0$ then $\varepsilon(l-1)=q_{l-1}$. Therefore $g(q_l)=g(q_{l-1}+1)\geq g(q_{l-1})+1\geq l$ and $\lambda_{l,q_l}\neq 0$.

Let us show that θ_ε is the leading monomial of $\det(\lambda'|_{\mu=0})$. Let π be a permutation of $\{1,\ldots,N\}$. Assume that $\pi(i)=\varepsilon(i)$ if $0\leq i\leq l-1$ and $\pi(l)\neq\varepsilon(l)$. If $\lambda_{l,q_{l-1}}=0$ then $\pi(l)\neq q_l$ and $\lambda_\pi=0$. If $\lambda_{l,q_{l-1}}\neq 0$ then $\pi(l)\neq p_l$ and $\omega(\prod_{i=l}^N\lambda_{i,\pi(i)})<\omega(\prod_{i=l}^N\lambda_{i,\varepsilon(i)})$. Therefore $\lambda_\pi<\lambda_\varepsilon$. q.e.d. The semigroup of the legendrian curve (2) only depends on $(a_m,\ldots,a_{c-1})$. We will denote it by $\Gamma_{(a_m,\ldots,a_{c-1})}$.

Theorem 9 *There is a dense Zariski open subset U of $\mathbb{C}^{c-m}$ such that if $(a_m,\ldots,a_{c-1})\in U$, $\Gamma_{(a_m,\ldots,a_{c-1})}=\Gamma(n,m)$.*

Proof. Since U is defined by the non vanishing of several determinants, it is enough to show that $U \neq \emptyset$. Let $j \in \langle n, m-n \rangle$, $j \geq m$. Set $q = \sharp(\tau_j)$. Assume that we associate to j a family of triples $I_1, \ldots, I_q \in \mathcal{N}$ such that $v(I_s) \geq j$, $1 \leq s \leq q$, and if E is the linear subspace of $\mathbb{C}[a_m, \ldots, a_{c-1}]\{t\}$ spanned by $\Upsilon_{I_s,k}|_{\mu=m}$, $1 \leq s \leq q$, $v(E) = \tau_j \cup \{\infty\}$. Let i be the biggest element of $\langle n, m-n \rangle \backslash \Gamma_j$. Assume that $\tau_i \cap \tau_j \neq \emptyset$. Hence τ_i contains τ_j. Since $v(E) = \tau_j \cup \{\infty\}$ and $\sharp(\tau_j) = q$, the determinant D' of the matrix $(\Upsilon_{I_s,k})$, $1 \leq s \leq q$, $k \in \tau_j$, does not vanish at $\mu = m$. In order to prove the theorem it is enough to show that there are $I_{q+1}, \ldots, I_{q+\sharp_i} \in \mathcal{N}$ such that $v(I_s) = i$, $q+1 \leq s \leq q+\sharp_i$, and the determinant D of the matrix $(\Upsilon_{I_s,k})$, $1 \leq s \leq q+\sharp_i$, $k \in \tau_i$, does not vanish at $\mu = m$. Set $I_{q+s+1} = (M-s, s, N-s)$, $M \leq s \leq N$, where $i = v(x^M p^N)$. By (26), (27) and (28),

$$g(\Upsilon_{I_s,k}) < g(\Upsilon_{I_r,k}) \qquad \text{if} \;\; k \geq i \;\; \text{and} \;\; s \leq q < r. \tag{29}$$

Set $\lambda' = (\Upsilon_{I_s,k})$, $q+1 \leq s \leq q+\sharp_i$, $k \in \tau_i \setminus \tau_j$. By lemma 8, $\det(\lambda'|_{\mu=m}) \neq 0$. Set $\Upsilon_\varepsilon = \prod_{s=1}^{q+\sharp_i} \Upsilon_{I_s,\varepsilon(i)}$ for each bijection $\varepsilon : \{1, \ldots, q+\sharp_i\} \to \tau_i$. By (29), $g(\Upsilon_\varepsilon) < g(D'\lambda'|_{\mu=m})$ if $\varepsilon(\{q+1, \ldots, q+\sharp_i\}) \neq \tau_i \setminus \tau_j$. Since

$$D'\lambda'|_{\mu=m} = \sum_{\varepsilon(\{q+1,\ldots,q+\sharp_i\}) = \tau_i \setminus \tau_j} sign(\varepsilon) \Upsilon_\varepsilon,$$

the product of the leading monomials of $D'|_{\mu=m}$ and $\lambda'|_{\mu=m}$ is the leading monomial of $D|_{\mu=m}$. q.e.d.

6 The moduli

Set $s = s(n,m) = \inf(\Gamma(n,m) \backslash \langle n, m-n \rangle)$. We say that (1) is in *Legendrian short form* if $a_m = 1$ and if $a_i \neq 0$ and $i \in \Gamma(n,m)$, $i \in \{m, s(n,m)\}$.

If $n = 2$ or if $n = 3$ and $m \in \{7, 8\}$, $\Gamma(n,m) = \langle n, m-n \rangle \supset \{m, \ldots\}$ and $x = t^n$, $y = t^m$ is the only curve in Legendrian normal form such that the semigroup of its conormal equals $\Gamma(n,m)$. If $n = 3$ and $m \geq 10$ or if $n \geq 4$, $\langle n, n-m \rangle \not\supset \{m, \ldots, m+n-1\}$ and $s(m,n) \in \{m, \ldots, m+n-1\}$.

Lemma 10 *If* (1) *is in Legendrian normal form,* $\Gamma(n,m) \neq \langle n, m-n \rangle$ *and the semigroup of the conormal of* (1) *equals* $\Gamma(n,m)$*,* $a_{s(n,m)} \neq 0$*.*

Proof. Each $f \in \mathbb{C}\{x, y, p\}$ is congruent to a linear combination of the series

$$y, \; nxp - my, \; x^i, \; p^j, \qquad v(x^i), v(p^j) \leq s \tag{30}$$

modulo (t^s). Since the series (30) have different valuations, one of these series must have valuation s, $s \in \Gamma(n,m) \setminus \langle n, m-n \rangle$ and $nxp - my = sa_s t^s + \cdots$, $a_s \neq 0$. q.e.d. Let $\mathcal{X}_{n,m}$ denote the set of plane curves (1) such that (1) is in Legendrian normal form and the semigroup of the conormal of (1) equals $\Gamma(n,m)$. Let W_n be the group of n-roots of unity. There is an action of W_n on $\mathcal{X}_{n,m}$ that takes (1) into $x = t^n$, $y = \sum_{i \geq m} \theta^{i-m} a_i t^i$, for each $\theta \in W_n$. The quotient $\mathcal{X}_{n,m}/W_n$ is an orbifold of dimension equal to the cardinality of the set $\{m, ...\} \setminus (\Gamma(n,m) \setminus \{s(n,m)\})$.

Theorem 11 *The set of isomorphism classes of generic Legendrian curves with equisingularity type* (n,m) *is isomorphic to* $\mathcal{X}_{n,m}/W_n$.

Proof. Let Λ be a germ of an irreducible Legendrian curve. There is a Legendrian map π such that $\pi(\Lambda)$ has maximal contact with the curve $\{y = 0\}$ and the tangent cone of the conormal of Λ equals $\{y = p = 0\}$. Moreover, we can assume that $\pi(\Lambda)$ has a parametrization of type (1), with $a_m = 1$. Assume that there is $i \in \Gamma(m,n)$ such that $i \neq m, s(m,n)$ and $a_i \neq 0$. Let k be the smallest integer i verifying the previous condition. By lemmata 4 and 5 there are $a \in \mathbb{C}\{x,y,p\}$ and $\Phi \in \mathcal{J}$ such that $i^* a = a_k t^k + \cdots$ and Φ takes (1) into the plane curve $x = s^n$, $y = y(s) - a(s) + \delta$, where $v(\delta) \geq 2v(a) + m - 2n$. Hence we can assume that $a_i = 0$ if $i \in \Gamma(m,n)$, $i \neq m, s(m,n)$, and i is smaller then the conductor σ of the plane curve (1). There is a germ of diffeomorfism ϕ of the plane that takes the curve (1) into the curve $x = t^n$, $y = \sum_{i=m}^{\sigma-1} a_i t^i$ (cf. [7]). This curve is in Legendrian normal form. The diffeomorphism ϕ induces an element of $\mathcal{G}$.

Let Φ be a contact transformation such $\Phi(\mathcal{X}) = \mathcal{X}$. Since the tangent cone of the conormal of an element of $\mathcal{X}$ equals $\{y = p = 0\}$, $\Phi \in \mathcal{G}$. By theorem 1, $\Phi = \Psi\Psi_{\lambda,\mu}$, where $\Psi \in \mathcal{J}$ and $\lambda, \mu \in \mathbb{C}^*$. Moreover, $\lambda \in W_n$ and $\mu = \lambda^m$. By lemmata 4 and 5, $\Psi = Id$. q.e.d.

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A. Araújo and O. Neto
CMAF, Universidade de Lisboa,
Av. Gama Pinto, 2 1649, Lisboa
Portugal
ant.arj@gmail.com orlando60@gmail.com