Self-avoiding fractional Brownian motion - The Edwards model

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Abstract

In this work we extend Varadhan's construction of the Edwards polymer model to the case of fractional Brownian motions in $\mathbb{R}^d$, for any dimension $d \geq 2$, with arbitrary Hurst parameters $H \leq 1/d$.

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1 Introduction

In recent years the fractional Brownian motion has become an object of intense study due to its special properties, such as short/long range dependence and self-similarity, leading to proper and natural applications in different fields. In particular, the specific properties of fractional Brownian motion paths have been used e.g. in the modelling of polymers. For the self-intersection properties of sample paths see e.g. [GRV03], [HN05], [HN07], [HNS08], [Ros87], and for the intersection properties with other independent fractional Brownian motion see e.g. [NOL07], [OSS11] and references therein. Comments on the relevance of fractional Brownian motion for polymer modelling, in particular with $H = 1/3$ for polymers in a compact or collapsed phase, can e.g. be found in [BC95].

The fractional Brownian motion on $\mathbb{R}^d$, $d \geq 1$, with Hurst parameter $H \in (0,1)$ is a d-dimensional centered Gaussian process $B^H = \{B_t^H : t \geq 0\}$ with covariance function

$$\mathbb{E}(B_t^{H,i} B_s^{H,j}) = \frac{\delta_{ij}}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad i,j = 1,\dots,d,\ s,t \geq 0.$$

An informal but suggestive definition of self-intersection local time of a fractional Brownian motion B^H is given in terms of an integral over a Dirac δ-function

$$L(T) = \int_0^T dt \int_0^t ds\, \delta(B^H(t) - B^H(s)),$$

intended to measure the amount of time the process spends intersecting itself in a time interval $[0,T]$. A rigorous definition may be given by approximating the δ-function by the heat kernel

$$p_\varepsilon(x) := \frac{1}{(2\pi\varepsilon)^{d/2}} e^{-\frac{|x|^2}{2\varepsilon}}, \quad x \in \mathbb{R}^d, \varepsilon > 0,$$

which leads to the approximated self-intersection local time

$$L_\varepsilon(T) := \int_0^T dt \int_0^t ds\, p_\varepsilon(B^H(t) - B^H(s)). \tag{1}$$

The main problem is then the removal of the approximation, that is, $\varepsilon \searrow 0$.

In the classic Brownian motion case ($H = 1/2$), $L_\varepsilon(T)$ converges in L^2 only for $d = 1$. To ensure the existence of a limiting process for higher dimensions one must center the approximated self-intersection

$$L_{\varepsilon,c}(T) := L_\varepsilon(T) - \mathbb{E}(L_\varepsilon(T)). \tag{2}$$

For the case of the planar Brownian motion this is sufficient to ensure the L^2-convergence of (2) as ε tends to zero [Var69], but for $d \geq 3$ a further multiplicative renormalization $r(\varepsilon)$ is required to yield a limiting process, now as a limit in law of

$$r(\varepsilon)\left(L_\varepsilon(T) - \mathbb{E}(L_\varepsilon(T))\right). \tag{3}$$

Through a different approximation, this has been shown in [CY87], [Yor85].

Extending Varadhan's results to the planar fractional Brownian motion, Rosen in [Ros87] shows that, for $1/2 < H < 3/4$, the centered approximated self-intersection local time converges in L^2 as ε tends to zero.

This result, as well as all the above quoted ones for the classic Brownian motion, have been extended by Hu and Nualart in [HN05] to any d-dimensional fractional Brownian motion with $H < 3/4$. More precisely, Hu and Nualart have shown that for $H < 1/d$ the approximated self-intersection local time (1) always converges in L^2. For $1/d \leq H < 3/(2d)$, a L^2-convergence result still holds, but now for the centered approximated self-intersection local time (2). In this case,

$$\mathbb{E}(L_\varepsilon(T)) = \begin{cases} TC_{H,d}\varepsilon^{-d/2+1/(2H)} + o(\varepsilon), & \text{if } 1/d < H < 3/(2d) \\ \frac{T}{2H(2\pi)^{d/2}}\ln(1/\varepsilon) + o(\varepsilon), & \text{if } H = 1/d \end{cases}, \tag{4}$$

where $C_{H,d}$ is a positive constant which depends of H and d. In particular, for $1/d \leq H < \min\{3/(2d), 2/(d+1)\}$, an explicit integral representation for the mean square limiting process $L_c(T)$ as an Itô integral is even obtained in [HNS08]. For $3/(2d) \leq H < 3/4$, a multiplicative renormalization factor $r(\varepsilon)$ is required in [HN05] to prove the convergence in distribution of the random variable (3) to a normal law as ε tends to zero.

To model polymers by Brownian paths Edwards [Edw65] proposed to suppress self-intersections by a factor

$$\exp\left(-gL(T)\right),$$

with $g > 0$. For planar Brownian motion Varadhan [Var69] showed that the expectation value $\mathbb{E}(L_\varepsilon(T))$ has a logarithmic divergence but after its subtraction the centered $L_{\varepsilon,c}(T)$ converges in L^2, with a suitable rate of convergence.

From this, Varadhan could conclude the integrability of $\exp(-gL_c(T))$, thus giving a proper meaning to the Edwards model. For more details see also [Sim74]. In the three-dimensional case this is clearly much more difficult [Bol93], [Wes80].

In this work we extend Varadhan's construction to arbitrary spatial dimension $d \geq 2$ and Hurst parameters $H \leq 1/d$. In Section 2 we collect from [HN05] the necessary information on fractional Brownian motion and its self-intersection local time. The core results are obtained in Section 3 and can be summarized as follows:

- The Edwards model allows an extension to fractional Brownian motion for $Hd \leq 1$. For $Hd < 1$, $d \geq 2$, $\exp(-gL(T))$ is shown to exist for all $g \geq 0$, see Theorem 2 (ii).

- In the limiting case $dH = 1$, $d \geq 3$, we show that

$$\exp(-gL_c(T))$$

is an integrable function for sufficiently small $g \geq 0$, see Theorem 2 (i). Central to the proof is the estimate for the rate of convergence provided in Proposition 1, using various helpful estimates from [HN05].

It is well-known that investigations of the end-to-end length of Brownian paths with excluded volume as in the Edwards model play a crucial role in polymer physics, see e.g. the reviews [vdHK01], [PV02] and references therein. A tentative to extend these to fractional Brownian motion has recently been undertaken in [BOS11].

2 Preliminaries

As shown in [HN05], given a d-dimensional fractional Brownian motion B^H with Hurst parameter $H \in (0,1)$, for each $\varepsilon > 0$ the approximated self-intersection local time (1) is a square integrable random variable with

$$\mathbb{E}(L_\varepsilon^2(T)) = \frac{1}{(2\pi)^d} \int_{\mathcal{T}} d\tau \, \frac{1}{((\lambda+\varepsilon)(\rho+\varepsilon)-\mu^2)^{d/2}},$$

where

$$\mathcal{T} := \{(s,t,s',t') : 0 < s < t < T, 0 < s' < t' < T\}$$

and for each $\tau = (s,t,s',t') \in \mathcal{T}$,

$$\lambda(\tau) := (t-s)^{2H}, \quad \rho(\tau) := (t'-s')^{2H}, \tag{5}$$

and

$$\mu(\tau) := \frac{1}{2}\left[|s-t'|^{2H} + |s'^{2H} - t|^{2H} - |t-t'|^{2H} - |s-s'|^{2H}\right]. \tag{6}$$

Furthermore, for each $\varepsilon, \gamma > 0$ is

$$\mathbb{E}(L_\varepsilon(T)L_\gamma(T)) - \mathbb{E}(L_\varepsilon(T))\mathbb{E}(L_\gamma(T)) = \tag{7}$$

$$\frac{1}{(2\pi)^d}\int_{\mathcal{T}} d\tau \left(\frac{1}{((\lambda+\varepsilon)(\rho+\gamma)-\mu^2)^{d/2}} - \frac{1}{((\lambda+\varepsilon)(\rho+\gamma))^{d/2}}\right) =: E_{\varepsilon\gamma}. \tag{8}$$

Note that the integral in (8) is also well-defined for all $\varepsilon, \gamma \geq 0$ (however it might be infinite). Hence, using this integral representation, we can extend $E_{\varepsilon\gamma}$ to general $\varepsilon, \gamma \geq 0$. This is contrast to (7) to which in general we cannot give sense to for $\varepsilon = 0$ and/or $\gamma = 0$.

From (8) one can easily derive that a necessary and sufficient condition for convergence of $L_{\varepsilon,c}(T) = L_\varepsilon(T) - \mathbb{E}(L_\varepsilon(T))$ to a limiting process $L_c(T)$ in L^2 as $\varepsilon \searrow 0$ is that $E_{00} < \infty$. As shown in [HN05, Lemma 11], the integral E_{00} is finite if and only if $dH < 3/2$.

3 Results and Proofs

Proposition 1 *Assume that $(d+1)H < 3/2$, $d \geq 2$. Then there exists a positive constant $K(T)$ such that*

$$\mathbb{E}\left((L_{\varepsilon,c}(T) - L_c(T))^2\right) \leq K(T)\,\varepsilon^{1/2}$$

for all $\varepsilon > 0$.

Proof. Using (8), a simple calculation and taking the limit $\gamma \searrow 0$ yields

$$\mathbb{E}\left((L_{\varepsilon,c}(T) - L_c(T))^2\right) = (E_{\varepsilon\varepsilon} - E_{\varepsilon 0}) + (E_{00} - E_{\varepsilon 0})$$

with

$$E_{\varepsilon\varepsilon} - E_{\varepsilon 0} = \frac{d}{2(2\pi)^d}\int_{\mathcal{T}} d\tau\,(\lambda+\varepsilon)\int_0^\varepsilon dx \left(\frac{1}{((\lambda+\varepsilon)(\rho+x))^{d/2+1}} - \frac{1}{((\lambda+\varepsilon)(\rho+x)-\mu^2)^{d/2+1}}\right) \leq 0.$$

Hence

$$\begin{aligned}\mathbb{E}\left((L_{\varepsilon,c}(T)-L_c(T))^2\right) &\le E_{00}-E_{\varepsilon 0}\\ &= \frac{d}{2(2\pi)^d}\int_{\mathcal{T}} d\tau\, \rho \int_0^{\varepsilon} dx\, \left(\frac{1}{(\delta+x\rho)^{d/2+1}} - \frac{1}{((\lambda+x)\rho)^{d/2+1}}\right), \quad (9)\end{aligned}$$

where $\delta := \lambda\rho - \mu^2$. Thus it is sufficient to establish a suitable upper bound for (9). Technically, this will follow closely the proof of Lemma 11 in [HN05], based on the decomposition of the region $\mathcal{T}$ into three subregions

$$\mathcal{T} \cap \{s < s'\} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3,$$

where

$$\begin{aligned}\mathcal{T}_1 &:= \{(t,s,t',s') : 0 < s < s' < t < t' < T\},\\ \mathcal{T}_2 &:= \{(t,s,t',s') : 0 < s < s' < t' < t < T\},\\ \mathcal{T}_3 &:= \{(t,s,t',s') : 0 < s < t < s' < t' < T\}.\end{aligned}$$

Each substitution of $\mathcal{T}$ in (9) by a subregion $\mathcal{T}_i$, $i = 1, 2, 3$, yields a different case and for each particular case we will then establish a suitable upper bound.

As in [HN05], we will denote by k a generic positive constant which may be different from one expression to another one. We set $D := d + 1$.

Subregion $\mathcal{T}_1$: We do the change of variables $a := s' - s$, $b := t - s'$, and $c = t' - t$ for $(t,s,t',s') \in \mathcal{T}_1$. Thus, on $\mathcal{T}_1$, for the functions λ, ρ, and μ defined in (5) and (6) we have

$$\begin{aligned}&\lambda(t,s,t',s') =: \lambda_1(a,b,c) = (a+b)^{2H}, \quad \rho(t,s,t',s') =: \rho_1(a,b,c) = (b+c)^{2H}\\ &\mu(t,s,t',s') =: \mu_1(a,b,c) = \frac{1}{2}\left[(a+b+c)^{2H} + b^{2H} - c^{2H} - a^{2H}\right].\end{aligned}$$

On the region $\mathcal{T}_1$ one can bound (9) by the first term only, and to estimate the latter we shall use Lemma 5 below, yielding

$$\rho_1 \int_0^{\varepsilon} dx\, \frac{1}{(\delta_1 + x\rho_1)^{(D+1)/2}} \le A\varepsilon^{1/2}\rho_1^{1/2}\delta_1^{-D/2}.$$

From [HN05, eq. (59)],

$$\delta_1 \ge k(a+b)^H(b+c)^H a^H c^H \ge k(abc)^{4H/3},$$

we deduce

$$\int_{[0,T]^3} da\,db\,dc\,\delta_1^{-D/2} \le k \int_{[0,T]^3} da\,db\,dc\,(abc)^{-2DH/3} < \infty,$$

because $DH < 3/2$. In conclusion the part of (9) stemming from integration over $\mathcal{T}_1$ is of order $\varepsilon^{1/2}$.

On the subregions $\mathcal{T}_i$, $i = 2, 3$, we have to consider the difference

$$\Xi_i^\varepsilon := \rho_i \int_0^\varepsilon dx \left(\frac{1}{(\delta_i + x\rho_i)^{(D+1)/2}} - \frac{1}{((\lambda_i + x)\rho_i)^{(D+1)/2}} \right), \quad \varepsilon > 0.$$

Subregion $\mathcal{T}_2$: In this case we do the change of variables $a := s' - s$, $b := t' - s'$, and $c = t - t'$ for $(t, s, t', s') \in \mathcal{T}_2$. That is, on $\mathcal{T}_2$ we will have

$$\lambda(t,s,t',s') =: \lambda_2(a,b,c) = b^{2H}, \quad \rho(t,s,t',s') =: \rho_2(a,b,c) = (a+b+c)^{2H}$$
$$\mu(t,s,t',s') =: \mu_2(a,b,c) = \frac{1}{2}\left[(b+c)^{2H} + (a+b)^{2H} - c^{2H} - a^{2H}\right].$$

In this case we decompose the corresponding integral (9) over the regions $\{b \ge \eta a\}$, $\{b \ge \eta c\}$, and $\{b < \eta a, b < \eta c\}$, for some fixed but arbitrary $\eta > 0$. We have by (16), see Appendix,

$$\begin{aligned} \int_{b \ge \eta a} da\,db\,dc\,\Xi_2^\varepsilon &\le C\varepsilon^{1/2} \int_{b \ge \eta a} da\,db\,dc\,\rho_2^{1/2}\,(\lambda_2\rho_2)^{-D/2} \\ &\le k\varepsilon^{1/2} \int_{b \ge \eta a} \frac{da\,db\,dc}{(a+b+c)^{DH} b^{DH}}. \end{aligned}$$

If $DH < 1$, the integral is finite. If $1 < DH < 3/2$, then by Young inequality

$$\begin{aligned} \int_{b \ge \eta a} da\,db\,dc\,\Xi_2^\varepsilon &\le k\varepsilon^{1/2} \int_0^T \int_0^T \frac{da\,dc}{(a+c)^{DH}} \int_{\eta a}^T db\,b^{-DH} \\ &\le k\varepsilon^{1/2} \int_0^T da\,a^{-4DH/3+1} \int_0^T dc\,c^{-2DH/3} < \infty. \end{aligned}$$

In the case $DH = 1$ we have

$$\int_{b \ge \eta a} da\,db\,dc\,\Xi_2^\varepsilon \le k\varepsilon^{1/2} \int_0^T dc\,c^{-2/3} \int_0^T da\,a^{-1/3} \ln(T/(\eta a)) < \infty.$$

The case $b \geq \eta c$ can be treated analogously.

To handle the case $b < \eta a$ and $b < \eta c$ we first observe that since $2H < 3/D \leq 1$ we have

$$\begin{aligned} \mu_2 &= \frac{1}{2}\left(a^{2H}\left(\left(1+\frac{b}{a}\right)^{2H}-1\right)+c^{2H}\left(\left(1+\frac{b}{c}\right)^{2H}-1\right)\right) \\ &\leq k\left(a^{2H-1}+c^{2H-1}\right)b \end{aligned}$$

for sufficiently small $\eta > 0$. Hence, together with (15), see Appendix, and the fact that $2H < 1$ we obtain

$$\begin{aligned} &\int_{b<\eta a, b<\eta c} da\, db\, dc\, \Xi_2^{\varepsilon} \leq C\varepsilon^{1/2} \int_{b<\eta a, b<\eta c} da\, db\, dc\, \rho_2^{1/2} \mu_2^2 \left(\lambda_2 \rho_2\right)^{-(D+2)/2} \\ &\leq k\varepsilon^{1/2} \int_{b<\eta a, b<\eta c} da\, db\, dc\, \left(a^{4H-2}+c^{4H-2}\right)(a+b+c)^{-2H-DH} b^{2-2H-DH} \\ &\qquad \leq k\varepsilon^{1/2} \int_{b<\eta a, b<\eta c} da\, db\, dc\, b^{-DH}(a+b+c)^{-2H-DH} \\ &\qquad\qquad \times \left(a^{(2-D/3)H} b^{DH/3} + c^{(2-D/3)H} b^{DH/3}\right) \\ &\quad \leq k\varepsilon^{1/2} \int_{[0,T]^3} da\, db\, dc\, b^{-DH}(a+b+c)^{-2H-DH} a^{(2-D/3)H} b^{DH/3} \\ &\qquad\qquad \leq k\varepsilon^{1/2} \int_{[0,T]^3} da\, db\, dc\, b^{-2DH/3} c^{-2DH/3} a^{-2DH/3} < \infty, \end{aligned}$$

because $DH < 3/2$.

Subregion $\mathcal{T}_3$: We do the change of variables $a := t - s$, $b := s' - t$, and $c = t' - s'$ for $(t, s, t', s') \in \mathcal{T}_3$. Thus, on $\mathcal{T}_3$, we have

$$\begin{aligned} &\lambda(t,s,t',s') =: \lambda_3(a,b,c) = a^{2H}, \quad \rho(t,s,t',s') =: \rho_3(a,b,c) = c^{2H} \\ &\mu(t,s,t',s') =: \mu_3(a,b,c) = \frac{1}{2}\left[(a+b+c)^{2H} + b^{2H} - (b+c)^{2H} - (a+b)^{2H}\right]. \end{aligned}$$

In this case we decompose the corresponding integral (9) over the regions $\{a \geq \eta_1 b, c \geq \eta_2 b\}$, $\{a < \eta_1 b, c < \eta_2 b\}$, $\{a \geq \eta_1 b, c < \eta_2 b\}$, and $\{a < \eta_1 b, c \geq \eta_2 b\}$ for some fixed but arbitrary $\eta_1, \eta_2 > 0$. By symmetry it suffices to consider the first three regions. Using (16), see Appendix, we obtain

$$\int_{a\geq\eta_1 b, c\geq\eta_2 b} da\, db\, dc\, \Xi_3^\varepsilon \leq C\varepsilon^{1/2} \int_{a\geq\eta_1 b, c\geq\eta_2 b} da\, db\, dc\, \rho_3^{1/2} \left(\lambda_3\rho_3\right)^{-D/2}$$
$$\leq k\varepsilon^{1/2} \int_0^T db \int_{\eta_1 b}^T \frac{da}{a^{DH}} \int_{\eta_2 b}^T \frac{dc}{c^{DH}} \leq k\varepsilon^{1/2} \int_0^T \frac{db}{b^{2DH-2}} < \infty.$$

For the region $\{a < \eta_1 b, c < \eta_2 b\}$, we observe that since $2H < 3/D \leq 1$, we can conclude from (15), see Appendix, together with [HN05, eq. (55)], i.e., $\mu_3 \leq k b^{2H-2} ac$, that

$$\begin{aligned} \Xi_3^\varepsilon &\leq C\varepsilon^{1/2}\rho_3^{1/2}\mu_3^2\left(\lambda_3\rho_3\right)^{-(D+2)/2} \\ &\leq k\varepsilon^{1/2} b^{4H-4} a^{2-2H-DH} c^{2-2H-DH} \leq k\varepsilon^{1/2} a^{-2DH/3} c^{-2DH/3} b^{-2DH/3}, \end{aligned}$$

which is integrable. Finally, we consider the case $\{a \geq \eta_1 b, c < \eta_2 b\}$. For $\eta_1, \eta_2 > 0$ small enough we have

$$\begin{aligned} \mu_3 &= \frac{1}{2}\left((a+b)^{2H}\left(\left(1+\frac{c}{a+b}\right)^{2H} - 1\right) - b^{2H}\left(\left(1+\frac{c}{b}\right)^{2H} - 1\right)\right) \\ &\leq k\left((a+b)^{2H-1} + b^{2H-1}\right)c = kb^{2H-1}\left(\left(1+\frac{a}{b}\right)^{2H-1} + 1\right)c \leq kb^{2H-1}c, \end{aligned}$$

where again we have used $2H < 1$. Then using (15) we obtain

$$\int_{a\geq\eta_1 b, c<\eta_2 b} da\, db\, dc\, \Xi_3^\varepsilon \leq C\varepsilon^{1/2} \int_{a\geq\eta_1 b, c<\eta_2 b} da\, db\, dc\, \rho_3^{1/2}\mu_3^2 \left(\lambda_3\rho_3\right)^{-(D+2)/2}$$
$$\leq k\varepsilon^{1/2} \int_{a\geq\eta_1 b} da\, db\, b^{4H-2} a^{-2H-DH} \int_0^{\eta_2 b} dc\, c^{2-2H-DH}$$
$$\leq k\varepsilon^{1/2} \int_0^T da\, a^{-2H-DH} \int_0^{a/\eta_2} db\, b^{-DH+2H+1} \leq k\varepsilon^{1/2} \int_0^T da\, a^{-2DH+2},$$

which is finite because $DH < 3/2$. ■

Theorem 2 *Assume that $dH = 1$, $d \geq 3$. Then for all $0 \leq g \leq H(2\pi)^{1/(2H)}/2T$ the function*

$$\exp(-gL_c(T)) \tag{10}$$

is integrable.

Remark 3 *The case $Hd < 1$ is simpler since then*

$$L(T) := \lim_{\varepsilon \searrow 0} L_\varepsilon(T)$$

exists in L^2 and $L(T)$ is nonnegative. Therefore for $g \geq 0$, $\exp(-gL(T))$ is integrable. Actually, in [HNS08] a much stronger result is proved, namely for any $p < \frac{1}{Hd}$, $\mathbb{E}(\exp(L(T)^p)) < \infty$, and there exists $\lambda_0 > 0$ such that $\mathbb{E}(\exp(\lambda L(T)^{1/(Hd)}) < \infty$, for any $0 \leq \lambda < \lambda_0$.

Proof. In all cases we have a logarithmic divergence of $\mathbb{E}(L_\varepsilon(T))$ as $\varepsilon \searrow 0$, see (4). Combining this moderate divergence with the rate of convergence provided in Proposition 1, the proof for integrability of the function in (10) for small enough non-negative g follows very close along the lines of [Var69, proof of Step 3]. More precisely, by (4), for $0 < \varepsilon \leq 1$ there exists a positive constant k such that

$$L_{\varepsilon,c}(T) \geq -\mathbb{E}(L_\varepsilon(T)) \geq -k - \frac{T}{2H(2\pi)^{d/2}} |\ln(\varepsilon)|.$$

For any constant $N \geq k + T|\ln(\varepsilon)|/(2H(2\pi)^{d/2})$ one has

$$\begin{aligned} \mathbb{P}(L_c(T) \leq -N) &= \mathbb{P}(L_c(T) - L_{\varepsilon,c}(T) \leq -N - L_{\varepsilon,c}(T)) \\ &\leq \mathbb{P}\left(|L_{\varepsilon,c}(T) - L_c(T)| \geq N - k - \frac{T}{2H(2\pi)^{d/2}} |\ln(\varepsilon)| \right). \end{aligned}$$

An application of Chebyshev's inequality together with Proposition 1 yields

$$\mathbb{P}(L_c(T) \leq -N) \leq \frac{\mathbb{E}(|L_{\varepsilon,c}(T) - L_c(T)|^2)}{\left(N - k - \frac{T}{2H(2\pi)^{d/2}} |\ln(\varepsilon)|\right)^2} \leq K \frac{\varepsilon^{1/2}}{\left(N - k - \frac{T}{2H(2\pi)^{d/2}} |\ln(\varepsilon)|\right)^2}.$$

In particular, for

$$\varepsilon = \exp\left(-\frac{H(2\pi)^{d/2}}{T}(N-k)\right)$$

one obtains

$$\mathbb{P}(L_c(T) \leq -N) \leq \frac{4K}{(N-k)^2} \exp\left(-\frac{H(2\pi)^{d/2}}{2T}(N-k)\right).$$

Hence, taking $N = (n-1)/g$, $g > 0$, one finds

$$\begin{aligned}\mathbb{E}\left(\exp(-gL_c(T))\right) &\leq 1 + \sum_{n=1}^{\infty} e^n\, \mathbb{P}\left(L_c(T) \leq -\frac{n-1}{g}\right) \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{4K}{\left(\frac{n-1}{g} - k\right)^2} \exp\left(n - \frac{H(2\pi)^{d/2}}{2T}\left(\frac{n-1}{g} - k\right)\right),\end{aligned}$$

which converges whenever $g \leq \frac{H(2\pi)^{d/2}}{2T} = \frac{H(2\pi)^{1/(2H)}}{2T}$. ■

Remark 4 *(i) In the case $d = 2$, $H = 1/2$ a rate of convergence similarly as in Proposition 1 is obtained in [Var69, Step 2]. Since in this case the divergence of $\mathbb{E}(L_\varepsilon(T))$ also is logarithmic, the considerations in [Var69] imply integrability of (10) for $d = 2$ and $H = 1/2$. The range of admissible g, however, is a little smaller, because the speed of convergence obtained in that case is a little slower.*
(ii) One might be tempted to use the scaling property of fractional Brownian motions, which yields

$$L_c(T) = k^{dH-2} L_c(kT) \quad \textit{for all} \quad k > 0,$$

to remove the upper restriction on g. Unfortunately, this does not work, since the upper bound has a reciprocal dependence on T.

Appendix

The following lemma is an immediate consequence of the Cauchy–Schwartz inequality.

Lemma 5 *Let $0 < \alpha, \beta < \infty$ and $1/2 < m < \infty$, then there exists a positive constant A such that*

$$\int_0^\varepsilon dx\, (\alpha + \beta x)^{-m} \leq A\varepsilon^{1/2} \alpha^{-m+1/2} \beta^{-1/2}.$$

For $i = 2, 3$ we set

$$\xi_i(x) := \frac{1}{(\delta_i + x\rho_i)^{(D+1)/2}} - \frac{1}{((\lambda_i + x)\rho_i)^{(D+1)/2}}, \quad x \geq 0.$$

The following lemma is a generalization of estimates (56) and (57) obtained in [HN05, Lemma 10].

Lemma 6 *For $i = 2, 3$ there exists a positive constant B such that*

$$\xi_i(x) \leq B\mu_i^2((\lambda_i + x)\rho_i)^{-(D+1)/2-1}, \tag{11}$$

$$\xi_i(x) \leq B\left((\lambda_i + x)\,\rho_i\right)^{-(D+1)/2}, \tag{12}$$

for all $x \geq 0$.

Proof. Estimate (11) implies estimate (12). Indeed, according to [Hu01, Lemma 3 (2)], for some suitable constant $0 < k < 1$,

$$\lambda_i\rho_i - \mu_i^2 = \delta_i \geq k\lambda_i\rho_i.$$

Since λ_i, ρ_i are positive, this implies that

$$\mu_i^2 \leq (1-k)\lambda_i\rho_i \leq (1-k)(\lambda_i + x)\rho_i, \tag{13}$$

for all $x \geq 0$. Thus, assuming (11), (12) follows from (13).

Therefore, the proof amounts to prove (11). Given

$$\xi_i(x) = \left(\left(1 - \frac{\mu_i^2}{(\lambda_i + x)\rho_i}\right)^{-(D+1)/2} - 1\right)((\lambda_i + x)\rho_i)^{-(D+1)/2} \tag{14}$$

observe that due to (13)

$$0 \leq \frac{\mu_i^2}{(\lambda_i + x)\rho_i} \leq 1 - k < 1.$$

Hence let us consider the function

$$[0, 1-k] \ni y \mapsto f(y) := (1-y)^{-(D+1)/2} - 1 \in [0, \infty).$$

Since $f(0) = 0$ and f' is continuous on $(0, 1-k)$ with a continuous continuation to $[0, 1-k]$, there exists a positive constant B such that

$$f(y) \leq \max_{z\in[0,1-k]} |f'(z)|y \leq By \quad \text{for all} \quad y \in [0, 1-k].$$

Applying this inequality to (14) yields the required estimate (11). ∎

Lemma 7 *For $i = 2, 3$ there exists a positive constant C such that*

$$\Xi_i^\varepsilon \leq C\varepsilon^{1/2}\rho_i^{1/2}\mu_i^2\left(\lambda_i\rho_i\right)^{-(D+2)/2}, \tag{15}$$

$$\Xi_i^\varepsilon \leq C\varepsilon^{1/2}\rho_i^{1/2}\left(\lambda_i\rho_i\right)^{-D/2}, \tag{16}$$

for all $\varepsilon > 0$.

Proof. Recall that

$$\Xi_i^\varepsilon = \rho_i \int_0^\varepsilon dx\, \xi_i(x), \quad i = 2, 3.$$

Hence (15) and (16) follow from (11) and (12), respectively, together with Lemma 5. ■

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