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Quantitative aspects on the ill-posedness of the Prandtl and hyperbolic Prandtl equations

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Abstract. We address the Prandtl equations and a physically meaningful extension known as hyperbolic Prandtl equations. For the extension, we show that the linearised model around a non-monotonic shear flow is ill-posed in any Sobolev spaces. Indeed, shortly in time, we generate solutions that experience a dispersion relation of order $\sqrt[3]{k}$ in the frequencies of the tangential direction, akin the pioneering result of Gérard-Varet, D., Dormy, E.: On the ill-posedness of the Prandtl equation. J. Am. Math. Soc., 23(2), 591–609 (2010) for Prandtl (where the dispersion was of order $\sqrt{k}$). We emphasise, however, that this growth rate does not imply (a-priori) ill-posedness in Gevrey-class m, with $m > 3$. We relate these aspects to the original Prandtl equations in Gevrey-class m, with $m > 2$: We show that the result in Gérard-Varet, D., Dormy, E.: On the ill-posedness of the Prandtl equation. J. Am. Math. Soc., 23(2), 591–609 (2010) determines a dispersion relation of order $\sqrt{k}$ for a short time proportional to $\ln(\sqrt{k})/\sqrt{k}$. Therefore, the ill-posedness in Gérard-Varet, D., Dormy, E.: On the ill-posedness of the Prandtl equation. J. Am. Math. Soc., 23(2), 591–609 (2010) in its generality is momentarily constrained to Sobolev spaces rather than extending to the Gevrey classes.

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1. Introduction

1.1. Presentation of the problem

The aim of this paper is twofold. First, we address the following hyperbolic extension of the linearised Prandtl equations (cf. [1,8,21,28,29])

$$\begin{cases} (\partial_t+1)\big(\partial_t u + U_{\rm s}\partial_x u + v\ U_{\rm s}'\big) - \partial_y^2 u = 0, & (t,x,y)\in \quad (0,T)\times\mathbb{T}\times\mathbb{R}_+,\\ \partial_x u + \partial_y v = 0 & (0,T)\times\mathbb{T}\times\mathbb{R}_+,\\ (u,u_t)|_{t=0} = (u_{\rm in}, u_{t,{\rm in}}) & \mathbb{T}\times\mathbb{R}_+,\\ u|_{y=0}=0,\quad v_{|y=0}=0 & (0,T)\times\mathbb{T}. \end{cases} \tag{1.1}$$

The main unknown is the scalar component $u = u(t, x, y)$ of the velocity field $(u, v)^T$, while $v = v(t, x, y)$ is formally determined by the divergence-free condition

$$v(t,x,y) = -\int_0^y \partial_x u(t,x,z)\mathrm{d}z.$$

The function $U_s = U_s(y)$ is prescribed and defines a suitable shear flow $(U_s(y), 0)$ around which the model have been linearised. For our main result, we will assume that U_s is a general function in $W^{4,\infty}_\alpha(\mathbb{R}_+)$, characterised by a loss of monotonicity at a specific point $y = a > 0$. Further details about this aspect are elaborated in Sect. 1.4 and Theorem 1.1. We aim to develop an ill-posedness theory of (1.1) in Sobolev spaces akin to the one in the seminal paper of Gérard-Varet and Dormy [12], for the linearisation of the Prandtl equations around a shear flow. Our main result concerns a *norm inflation* phenomenon in

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Sobolev regularities: some initial data that are of unitary size in H^m along the variable $x \in \mathbb{T}$ generate solutions of inflated size δ^{-1} after a time $t = \delta > 0$ arbitrarily small (cf. Theorem 1.1).
In [8], we proved that System (1.1) is locally well-posed when the initial data have regularity Gevrey-class 3 along $x \in \mathbb{T}$. With this work, we aim to provide further insights on the well- or ill-posedness issue when the initial data are Gevrey-class m, $m > 3$.
Secondly, our objective is to present some remarks on the ill-posed nature of the original Prandtl equations when linearised around a non-monotonic shear flow (cf. System (1.5)). The pioneering work conducted by Gérard-Varet and Dormy [12] demonstrated that the linearised Prandtl equations are ill-posed within Sobolev spaces. They constructed solutions that exhibit a dispersion relation of order $\sqrt{k}$ in the frequencies $k \in \mathbb{Z}$. If such dispersion persists over time, it indicates that the problem is ill-posed in a more regular function setting, specifically in any Gevrey class of order m, where $m > 2$. We show, however, that the result presented in [12] only holds for a brief duration, contingent on the frequencies (a time proportional to $\ln(\sqrt{k})/\sqrt{k}$). Consequently, the ill-posedness of the linearised Prandtl equations in any Gevrey class m, with $m > 2$, remains an unresolved matter at the moment (cf. Sect. 1.5 and our second main result, Theorem 1.6).

1.2. A brief overview on the analysis of the Prandtl equations

System (1.1) shares several similarities with the classical Prandtl system, whose leading equation is always on the state variable $u = u(t, x, y)$

$$\partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u = 0, \qquad (t, x, y) \in (0, T) \times \mathbb{T} \times \mathbb{R}_+$$

and replaces the first equation in (1.1) (with initial data only on u). Nowadays, the Prandtl model belongs to the mathematical and physical folklore, predicting the dynamics of boundary layers in fluid mechanics. Numerous mathematical investigations have been performed during the past decades, in order to reveal the underlying instabilities of the solutions. In this paragraph, we shall briefly recall some of the results about the related well- and ill-posedness issues. For further analytical problems like asymptotic behaviour, boundary-layer separations, boundary-layer expansions, homogenisation and more refined models, we refer, for instance, to [7,9,11,13,18,20,27,30] and references therein.
Regarding the well-posedness of the classical Prandtl equations, there are two branches of assumptions that one might impose on the initial data: they are monotonic in the vertical variable (hence they do not allow for non-degenerate critical points), or they belong to suitable function spaces that control all derivatives along the tangential variable (i.e. analytic or Gevrey initial data).
In case the initial data are monotonic, the system is known to be well-posed in standard functions spaces, such as Sobolev and Hölder. Roughly speaking, this monotonicity precludes certain instabilities of the solutions (at least locally in time), such as the so-called boundary-layer separations. An analytic result in this direction was shown by Oleinik (see e.g. [25,26]) using the Crocco transform. More recently, the result was renovated by Masmoudi and Wong [24], and Alexandre, Wang, Xu and Yang [3] without employing the latter.
On the other hand, requiring the initial data to be highly regular suffices as well for the well-posedness of the Prandtl equations. Caflish and Sammartino [31] proved local-in-time well-posedness for initial data that are analytic in all directions. In reality, a refinement elaborated in [23] showed that the analyticity condition is only required in the tangential variable.
Motivated by findings on ill-posedness, the set of functions spaces allowing for local existence theory was enlarged from analytic to the Gevrey scale. We mention here the works of Gérard-Varet and Masmoudi [14] in the class 7/4, Li and Yang [22] for small initial data in the class 2, and Dietert and Gérard-Varet [10] achieving the well-posedness also for large data in the Gevrey class 2.
The latter regularity property is strongly suggested to be the borderline case. Indeed, smooth solutions might blow-up without analyticity or monotonicity assumptions as shown in [32]. But more drastically,

Gérard-Varet and Dormy proved in [12] the ill-posedness of the linearised Prandtl system in any Sobolev space by constructing solutions which experience a norm inflation as $e^{\sigma\sqrt{k}t}$ (for short times, c.f. the discussion in Sect. 1.5), with $\sigma > 0$. These results are extended in [15,17] by Gérard-Varet and Nguyen, and Guo and Nguyen, to non-existence results of weak solutions for the linear Prandtl system in the energy space. The latter works additionally shed light on the fully nonlinear system culminating in proving Lipschitz ill-posedness.
In sum, the well-posedness theory obtained in the literature reflects in general the physically observed instabilities related to the Prandtl equations. One of the questions which arise is if extended models as (1.1) inhere better properties as a result of the finite speed propagation. In [8] we provided a positive answer to this dilemma, enlarging the Gevrey scale from the class 2 to the class 3. With the current work, we show, however, that System (1.1) is still not desirable for the well-posedness in Sobolev spaces.

1.3. Some physical aspects of the model

System (1.1) formally arises as a (linearised) model for the boundary layers of the incompressible Navier–Stokes equations, whose Cauchy stress tensor is "delayed" through a first-order Taylor expansion:

$$\partial_t U + U\cdot\nabla U + \nabla P = \operatorname{div}\mathbb{S}, \qquad \mathbb{S}(t+\tau_r,\cdot) \approx \mathbb{S}(t,\cdot) + \tau_r\partial_t\mathbb{S}(t,\cdot) = \mu\frac{\nabla U(t,\cdot)+\nabla U(t,\cdot)^T}{2}. \quad (1.2)$$

Specifically, by differentiating the equation for U with respect to time, multiplying by τ_r, and adding this to the original equation, we obtain

$$(\tau_r\partial_t + 1)\big(\partial_t U + U\cdot\nabla U\big) + \nabla\tilde{P} = \operatorname{div}\big(\tau_r\partial_t\mathbb{S} + \mathbb{S}\big) = \mu\Delta U,$$

where $\tilde{P}$ represents a suitably defined new pressure term. This formulation is known as the hyperbolic extension of the Navier–Stokes equations. We refer to [2,8] and references therein for a deeper discussion on the physical motivations and derivation of the model. We point out that this delayed relation on $\mathbb{S}$ was introduced in fluid dynamics by Carrassi and Morro [4], inspired by the celebrated work of Cattaneo [5,6] on heat diffusion. Moreover, System (1.2) represents somehow a simplified version of the celebrated Oldroyd-B model (or Maxwell equations, cf. for instance [19]):

$$\begin{aligned}\partial_t U + U\cdot\nabla U - \nu\Delta u + \nabla P &= \mu_1\operatorname{div}\mathcal{T},\\ \partial_t\mathcal{T} + u\cdot\nabla\mathcal{T} + \gamma\mathcal{T} - \nabla U\mathcal{T} - (\nabla U)^T\mathcal{T} &= \mu_2\frac{\nabla U(t,\cdot)+\nabla U(t,\cdot)^T}{2},\end{aligned}$$

where the constant γ represents the time scale for the elastic relaxation. Equation (1.2) neglects nevertheless several terms of Oldroyd-B, in particular the convective and upper convective derivatives. We might therefore interpret our results as precursors for the analytical understanding of boundary layers of viscoelastic fluids.

1.4. Main result

For the sake of comparison, we employ a similar Ansatz as the one of Gérard-Varet and Dormy [12]. All along the present manuscript, we denote by $W^{s,\infty}_\alpha$, with $s \geq 0$ and $\alpha > 0$, the following weighted Sobolev spaces in $\mathbb{R}_+ = [0,\infty)$

$$W^{s,\infty}_\alpha = W^{s,\infty}_\alpha(\mathbb{R}_+) := \ \big\{ f \in W^{s,\infty}(\mathbb{R}_+) \text{ such that } y\in\mathbb{R}_+ \mapsto e^{\alpha y}f(y) \text{ is also in } \ W^{s,\infty}(\mathbb{R}_+)\big\},$$

endowed with the norm $\|f\|_{W^{s,\infty}_\alpha} := \|e^{\alpha\cdot}f\|_{W^{s,\infty}}$. For the entirety of this manuscript $\alpha > 0$ is fixed and might be imposed equal to 1 (we keep the notation $W^{s,\infty}_\alpha$, mainly because of [12]). Since our solutions are periodic in the tangential direction $x \in \mathbb{T}$, we also set $H^m W^{s,\infty}_\alpha = H^m(\mathbb{T}, W^{s,\infty}_\alpha(\mathbb{R}_+))$, $m \geq 0$ and also $\mathcal{C}^\omega W^{0,\infty}_\alpha = \mathcal{C}^\omega(\mathbb{T}, W^{0,\infty}_\alpha(\mathbb{R}_+))$, making use of the Fourier transform in $x \in \mathbb{T}$:

$$H^m W^{0,\infty}_\alpha = \left\{ f = \sum_{k\in\mathbb{Z}} f_k(y) \mathrm{e}^{ikx} \quad (x,y) \in \mathbb{T}\times\mathbb{R}_+, \quad \|f\|_{H^m W^{s,\infty}_\alpha} := \left(\sum_{k\in\mathbb{Z}} (1+k^2)^m \|f_k\|^2_{W^{s,\infty}_\alpha} \right)^{\frac{1}{2}} < \infty \right\},$$

$$\mathcal{C}^\omega W^{0,\infty}_\alpha = \left\{ f = \sum_{k\in\mathbb{Z}} f_k(y) \mathrm{e}^{ikx} \quad (x,y) \in \mathbb{T}\times\mathbb{R}_+, \quad \sup_{k\in\mathbb{Z}} \mathrm{e}^{\sigma|k|} \|f_k\|_{W^{s,\infty}_\alpha} < \infty, \quad \text{for a given} \quad \sigma > 0 \right\}.$$

Our main result expresses the ill-posedness of System (1.1) in $H^m W^{0,\infty}_\alpha$ when the shear flow $U_s(y)$ is non-monotonic. It develops around a family of solutions that are of unitary size in Sobolev spaces at $t = 0$ and experience an inflation of the norm after any short time $t = \delta > 0$.

Theorem 1.1. *Assume that the shear flow $U_s = U_s(y)$ is in $W^{4,\infty}_\alpha(0,\infty)$ and satisfies the structural relations $U_s(a) = U_s'(a) = 0$ together with $U_s''(a) \neq 0$, for a given $a \in (0,\infty)$. Then for all $m \geq 0$, $\mu \in [0, 1/3)$ and small time $\delta > 0$, there exists a pair of initial data*

$$(u_{\mathrm{in}}, u_{t,\mathrm{in}}) \in \mathcal{C}^\omega(\mathbb{T}, W^{1,\infty}_\alpha(\mathbb{R}_+)) \times \mathcal{C}^\omega(\mathbb{T}, W^{0,\infty}_\alpha(\mathbb{R}_+)) \hookrightarrow H^m(\mathbb{T}, W^{1,\infty}_\alpha(\mathbb{R}_+)) \times H^m(\mathbb{T}, W^{0,\infty}_\alpha(\mathbb{R}_+))$$
$$\text{with} \quad \|u_{\mathrm{in}}\|_{H^m W^{1,\infty}_\alpha} + \|u_{t,\mathrm{in}}\|_{H^m W^{0,\infty}_\alpha} \leq 1,$$

that generates a global-in-time smooth solution u of (1.1), which satisfies

$$\sup_{0\leq t\leq\delta} \|u(t)\|_{H^{m-\mu} W^{0,\infty}_\alpha} \geq \frac{1}{\delta}. \tag{1.3}$$

Remark 1.2. We allow the solutions u in Theorem 1.1 and the suitable family of initial data to take values in $\mathbb{C}$. Nevertheless, since the shear flow as well as all other coefficients (1.1) are real-valued, taking the real (or imaginary part, respectively) provide (non-trivial) real-valued solutions which undergo the same instability mechanism.
The parameter $\mu \in [0, 1/3)$ allows to enlarge the Sobolev space for the values of the solutions. As long as the regularity index of $H^{m-\mu}$ differs less than 1/3 from the original space H^m (a reminiscent of the Gevrey-class 3 regularity), the problem is still ill-posed. An analogous threshold was shown in [12] with $\mu \in [0, 1/2)$ (reminiscent of Gevrey-class 2).

As for its homologous problem in [12], the instability process described by Theorem 1.1 is mainly enabled by a meaningful family of initial data, which highly oscillate in the tangential variable $x \in \mathbb{T}$. More precisely, at high frequencies $k \gg 1$, we select initial data $(u_{\mathrm{in}}, u_{t,\mathrm{in}})$ on the eigenspace of e^{ikx} and determine a suitable non-trivial asymptotic of the (t,y)-dependent Fourier coefficients as frequencies $k \to \infty$ (more details are provided starting from Sect. 2). We show that the corresponding solutions grow as $\mathrm{e}^{\sigma \sqrt[3]{k} t}$ at least for a very short time depending on the frequencies (cf. Sect. 1.5, Proposition 2.1 and Lemma 2.3). Moreover, as $k \to \infty$, the profiles in $y \in \mathbb{R}_+$ of the initial data relate to a suitable "spectral condition" for the following ordinary differential equation:

Lemma 1.3. *There exists a complex number $\gamma \in \mathbb{C}$ with $\mathrm{Im}(\gamma) < 0$ and a complex solution $W : \mathbb{R} \to \mathbb{C}$ of the ordinary differential equation*

$$\gamma\big(\gamma - z^2\big)^2 \frac{d}{dz} W(z) + \frac{d^3}{dz^3}\Big[\big(\gamma - z^2\big) W(z)\Big] = 0, \qquad z \in \mathbb{R}, \tag{1.4}$$

such that $\lim_{z\to-\infty} W(z) = 0$ and $\lim_{z\to+\infty} W(z) = 1$.

The ODE (1.4) is similar to its homologous (1.7) in [12] (cf. also here (3.18)). They do present, however, some technical differences that are mainly due to the fact that the leading operator of Prandtl is a heat equation, whereas the leading term in (1.1) is a wave equation (at high frequencies $k \gg 1$).

Remark 1.4. Theorem 1.1 remains valid even when the equation for u in (1.1) is replaced by

$$(\tau_\mathrm{r}\partial_t + 1)\big(\partial_t u + U_s(y)\partial_x u + vU_s'(y)\big) - \partial_y^2 u = 0,$$

where $\tau_\mathrm{r} > 0$ represents a positive retardation time. Furthermore, Theorem 1.1 also applies for $\tau_\mathrm{r} = 0$, a case which directly follows from the renowned result of Dormy and Gérard-Varet [12]. It is important to note, however, that the nature of the solutions causing the inflation differs significantly between the cases $\tau_\mathrm{r} > 0$ and $\tau_\mathrm{r} = 0$. For a general $\tau_\mathrm{r} > 0$ and a given frequency $k \in \mathbb{N}$, we infer that the norm of these solutions grows exponentially as $\exp\Big\{Ct\frac{\sqrt[3]{k}}{\sqrt{\tau_\mathrm{r}}}\Big\}$, as we will show in our forthcoming analysis for the case of $\tau_r = 1$. This indicates that these solutions cease to exist as $\tau_\mathrm{r} \to 0$. In contrast, for $\tau_\mathrm{r} = 0$, Dormy and Gérard-Varet constructed solutions that exhibit a rapid growth over time as $\exp\{Ct\sqrt{k}\}$ for fixed frequencies $k \in \mathbb{N}$. These particular solutions of Dormy and Gérard-Varet demonstrate therefore the ill-posedness of the Prandtl equations ($\tau_r = 0$) in Sobolev spaces.

1.5. A quantitative result about the ill-posedness of the Prandtl equation

The pioneering work of Dietert and Gérard-Varet [10] established the local-in-time well-posedness of the Prandtl equations, when the initial data have Gevrey class 2 regularity along the tangential direction $x \in \mathbb{T}$. Before their result, the exponent of Gevrey-class 2 was attained only in the special setting prescribed by [22], in light also of the previous investigation of Gérard-Varet and Dormy [12] on the ill-posedness of Prandtl in Sobolev spaces. Indeed, for any sufficiently large frequency $k \gg 1$, Gérard-Varet and Dormy succeeded in establishing solutions of the linearised Prandtl equations around a non-monotonic shear flow $(U_s(y), 0)$

$$\partial_t u + U_s(y)\partial_x u + vU_s'(y) - \partial_y^2 u = 0, \qquad \partial_x u + \partial_y v = 0, \qquad (t, x, y) \in (0, T) \times \mathbb{T} \times \mathbb{R}_+, \tag{1.5}$$

experiencing a dispersion relation proportional to $\sqrt{k}$. Their result was later on strengthened by further investigations on related problems, we report for instance Gérard-Varet and Nguyen [15] and Ghoul et al. [16], emphasising that there exist solutions experiencing growth of order $\mathrm{e}^{\sqrt{k}t}$ in the frequencies:

- *The authors ... construct $\mathcal{O}(k^{-\infty})$ approximate solutions that grow like $\mathrm{e}^{\sqrt{k}t}$ for high frequencies k in x,*
- its linearisation around a special background flow has unstable solutions of similar form, but with $\sigma_k \sim \lambda\sqrt{k}$, for $k \gg 1$ arbitrarily large and some positive $\lambda \in \mathbb{R}_+$.

Because of this growth rate, one may wonder why Gérard-Varet and Dormy addressed the ill-posedness in Sobolev spaces rather than in any Gevrey-class m, with $m > 2$. With this section, we want to highlight some quantitative aspects related to the proof in [12] that play a fundamental role in the present manuscript as well. Furthermore, in our case we build solutions of the hyperbolic extension (1.1) with a dispersive relation of order $\sqrt[3]{k}$; thus, these remarks reveal also our choice of Sobolev rather than Gevrey-class m, with $m > 3$.
Roughly speaking, the most compelling reason resides in the maximal lifespan for which the mentioned growths occur: it is proportional to $t \sim \ln(k)/\sqrt{k}$ for the Prandtl equation and to $t \sim \ln(k)/\sqrt[3]{k}$ for the hyperbolic extension. To be more specific, we state the following corollary of Theorem 1 in [12], when applied to the autonomous system (1.5).

Corollary 1.5. (due to Theorem 1 in [12]) *Let $U_s \in W^{4,\infty}_\alpha(\mathbb{R}_+)$ and assume that $U_s'(a) = 0$ and $U_s''(a) \neq 0$ for some $a > 0$. Denote by T the semigroup of (1.5) for analytic solutions* (cf. Proposition 1 in [12]). *There exists $\sigma > 0$, such that*

$$\sup_{0\le t\le\delta} \big\| e^{-\sigma t\sqrt{|\partial_x|}} T(t)\big\|_{\mathcal{L}(H^m W^{0,\infty}_\alpha, H^{m-\mu} W^{0,\infty}_\alpha)} = +\infty,$$

for all $\delta > 0$, $m \ge 0$ and $\mu \in [0, 1/2)$.

The result of Gérard-Varet and Dormy implies in particular that T does not extend to an operator in the Sobolev space $H^m W^{0,\infty}_\alpha$.

Our aim is to present a refined version of Corollary 1.5, which provides indeed an explicit lower bound of the semigroup T_k (the projection of T on the eigenspace generated by e^{ikx}), associated to

$$\partial_t u_k + ikU_s(y)u_k + v_k U_s'(y) - \partial_y^2 u_k = 0, \qquad iku_k + \partial_y v = 0, \qquad (t,y) \in (0,T)\times\mathbb{R}_+, \tag{1.6}$$

with $v_k = u_k = 0$ in $y = 0$, and an explicit upper bound for the time in which the instability occurs. Our main result reads as follows:

Theorem 1.6. *Let $U_s \in W^{4,\infty}_\alpha(\mathbb{R}_+)$ and assume that $U_s'(a) = 0$ and $U_s''(a) \neq 0$ for some $a > 0$. Then there exists $\sigma_0 > 0$ such that for all $\sigma \in [0, \sigma_0)$ and $k \in \mathbb{N}$, the following inequality*

$$\sup_{0\le t\le \frac{1}{2(\sigma_0-\sigma)}\frac{\ln(k)}{\sqrt{k}}} e^{-\sigma t k^{\frac12}} \left\| T_k(t)\right\|_{\mathcal{L}(W^{0,\infty}_\alpha)} > \mathcal{C}_\sigma k^{\frac12}, \tag{1.7}$$

holds true for a constant $\mathcal{C}_\sigma$, which depends uniquely upon σ and σ_0.

Theorem 1.6 implies in particular Corollary 1.5. Indeed, at any small time $t = \delta > 0$, we may consider a sufficiently large frequency $\mathbf{k} \gg 1$ with $\ln(\mathbf{k})/\sqrt{\mathbf{k}} \le 2(\sigma_0 - \sigma)\delta$, so that

$$\begin{aligned}\sup_{0\le t\le\delta}\left\|\mathrm{e}^{-\sigma t\sqrt{|\partial_x|}}T(t)\right\|_{\mathcal{L}(H^m W^{0,\infty}_\alpha, H^{m-\mu}W^{0,\infty}_\alpha)} &= \sup_{0\le t\le\delta}\sup_{k\in\mathbb{Z}} k^{-\mu}\mathrm{e}^{-\sigma t k^{\frac12}}\left\|T_k(t)\right\|_{\mathcal{L}\left(W^{0,\infty}_\alpha, W^{0,\infty}_\alpha\right)}\\ &\ge \sup_{k\ge\mathbf{k}} k^{\frac12-\mu} = +\infty.\end{aligned}$$

Remarkably, Theorem 1.6 provides an explicit inflation of the norms at any frequency $k \in \mathbb{N}$, as well as a related maximum lifespan. This time is proportional to $t \sim \ln(k)/\sqrt{k}$ and was somehow already observed in the proof of [12] (cf. pag. 602 in [12], where the authors obtained a contradiction considering a time $t \gg \frac{\mu}{\sigma-\sigma_0}|\ln(\varepsilon)|\sqrt{\varepsilon}$, with $\varepsilon = 1/k$).

On a more technical level, the proof in [12] develops around the following ansatz for a velocity field $u(t,x,y) = \mathrm{e}^{ik(-U_s(a)+\tau k^{-1/2})t+ikx}\hat{u}_k(y)$, where $\tau \in \mathbb{C}$ and $\sigma_0 := \mathcal{I}m(\tau) < 0$. This function u experiences therefore a-priori a growth as $\mathrm{e}^{\sigma_0\sqrt{k}t}$, at any time $t > 0$. However, we shall emphasise that this flow is a solution $u^{\mathrm{fr}} = u^{\mathrm{fr}}_k(t,y)\mathrm{e}^{ikx}$ of a "forced" version of the Prandtl equations, which depends on a non-trivial remainder $r_k = \mathcal{R}_k(t,y)\mathrm{e}^{ikx}$ (cf. (4.2) in [12], with $\varepsilon = 1/k$ and $u_\varepsilon = u^{\mathrm{fr}}$). Hence, in order to transfer the instability of u^{fr} to a homogeneous solution u of Prandtl, the authors invoked the Duhamel's identity

$$u^{\mathrm{fr}}_k(t,y) - u_k(t,y) = \int_0^t T_k(t-s)(\mathcal{R}_k(s))(y)\mathrm{d}s,$$

(cf. the identity below (4.2) in [12], where $\tilde{U}_\varepsilon = u^{\mathrm{fr}}_k$ and $U_\varepsilon = u_k$). Roughly speaking, u_k behaves similarly as u^{fr}_k, as long as the integral at the r.h.s of Duhamel is sufficiently small. Theorem 1.6 translates this smallness relation in terms of the semigroup T_k and the maximal lifespan $t = \frac{1}{2(\sigma_0-\sigma)}\frac{\ln(k)}{\sqrt{k}}$. We obtain a similar result also to the hyperbolic extension (1.1) (cf. Sect. 2 and Lemma 2.3).

Remark 1.7. Finally, we shall remark that the norm inflation of Theorem 1.6 is inefficient in providing ill-posedness in $\mathcal{G}^m_\sigma W^{0,\infty}_\alpha$ (Gevrey-class m in $x \in \mathbb{T}$), with $m > 2$. The analytic semigroup $T(t)$ never extends to an operator from $\mathcal{G}^m_\sigma W^{0,\infty}_\alpha$ into $\mathcal{G}^m_\eta W^{0,\infty}_\alpha$ (with a smaller radius of regularity $\eta \in [0,\sigma)$) if and only if

$$\sup_{k\in\mathbb{Z}}\sup_{0\le t\le\delta} \mathrm{e}^{(\eta-\sigma)k^{\frac1m}}\|T_k(t)\|_{\mathcal{L}(W^{0,\infty}_\alpha)} = +\infty, \tag{1.8}$$

for any $\delta > 0$. Inequality (1.7) does not automatically imply (1.8), since it would require that (1.7) holds true at a time $t \sim \frac{\sigma-\eta}{\sigma}k^{\frac1m-\frac12}$, which satisfies $\mathrm{e}^{(\eta-\sigma)k^{\frac1m}+\sigma t k^{\frac12}} \sim 1$. The upper bound $t \le \frac{1}{2(\sigma_0-\sigma)}\frac{\ln(k)}{\sqrt{k}}$ is hence too restrictive for the ill-posedness of the linearised Prandtl equations in any Gevrey-class m, with $m > 2$.

The main goal of this paper is to address the ill-posedness of the hyperbolic extension (1.1) in Sobolev spaces. We, however, provide a short proof of Theorem 1.6, as further support of the pioneeric work of Gérard-Varet and Dormy [12] (cf. Sect. 4).

2. Outline of the proof

In this section, we illustrate the general principles that we set as basis of our proof, and we postpone the technical details to the remaining paragraphs. Our proof develops along three major axes:

(i) We project the main equation (1.1) into frequency eigenspaces in $x \in \mathbb{T}$ (cf. (2.1)) and we perturb the resulting equations with some meaningful forcing terms (cf. (2.2) and Proposition 2.1). This ensures a specific exponential growth (typical of Gevrey-class 3 regularities) on certain inhomogeneous solutions. In particular, this exponential growth holds true, globally in time.

(ii) Locally in time, we transpose the growth described in (i) to the original homogeneous equation. Contrarily to (i), this holds uniquely for a very short time, which depends in particular upon the frequency of each eigenspace (cf. Lemma 2.3).

(iii) Finally, making use of the homogeneous solutions built in (ii), we derive an inflation of the Sobolev norms of the solutions of (1.1) as described by Theorem 1.1.

We postpone the major aspects of part (i) to Sect. 3 and we devote this section to some underlying remarks and the details of parts (ii) and (iii).

(i) The inhomogeneous equation and the global-in-time growth of Gevrey-3 type

We first formally project the main equation (1.1) to the eigenspace of each positive frequency $k \in \mathbb{N}$ in the x-variable, by means of the Ansatz

$$u(t,x,y) = \mathrm{e}^{ikx} u_k(t,y), \quad v(t,x,y) = k\mathrm{e}^{ikx} v_k(t,y), \quad (t,x,y) \in (0,T) \times \mathbb{T} \times \mathbb{R}_+.$$

We hence look for a suitable family of (t,y)-dependent profiles u_k, which solve the homogeneous system

$$\begin{cases} (\partial_t + 1)\big(\partial_t u_k + ikU_{\mathrm{s}} u_k + k v_k U_{\mathrm{s}}'\big) - \partial_y^2 u_k = 0 & (t,y) \in \quad (0,T) \times \mathbb{R}_+, \\ i u_k + \partial_y v_k = 0 & \qquad\quad (0,T) \times \mathbb{R}_+, \\ (u_k, \partial_t u_k)|_{t=s} = (u_{k,\mathrm{in}}, u_{k,t,\mathrm{in}}) & \qquad\qquad \mathbb{R}_+, \\ u_k|_{y=0} = 0, \quad v_k|_{y=0} = 0 & \qquad\quad (0,T). \end{cases} \tag{2.1}$$

Since this equation is (roughly) a linear damped wave equation at each fixed frequency $k \in \mathbb{N}$, we infer that any initial data $(u_{k,\mathrm{in}}, u_{k,t,\mathrm{in}}) \in W^{1,\infty}_\alpha \times W^{0,\infty}_\alpha$ generates a unique global-in-time weak solution $u_k \in L^\infty(0,T; W^{0,\infty}_\alpha)$, for any lifespan $T > 0$. Furthermore, we also infer that u_k can be written in terms of a semigroup T_k

$$u_k(t) = T_k(t)\,(u_{k,\mathrm{in}}, u_{k,t,\mathrm{in}})\,, \quad \text{with} \quad T_k(t) : W^{1,\infty}_\alpha \times W^{0,\infty}_\alpha \to W^{0,\infty}_\alpha,$$

for any $T > 0$ and any $t \in (0,T)$. Certainly, u_k satisfies additional regularities; however, our central goal is to estimate u_k in $L^\infty(0,T; W^{0,\infty}_\alpha)$ and to determine a suitable growth of $\|u_k(t)\|_{W^{0,\infty}_\alpha}$ as time increases. As depicted in part (i), we momentarily allow for perturbation of equation (2.1) by means of a general forcing term $f_k \in L^\infty(0,T; W^{0,\infty}_\alpha)$. To avoid confusion in the notation, we set this inhomogeneous version in the state variable $u_k^{\mathrm{fr}} = u_k^{\mathrm{fr}}(t,y)$, which reflects a "forced" version of the hyperbolic Prandtl equation:

$$\begin{cases} (\partial_t + 1)\big(\partial_t u_k^{\mathrm{fr}} + ikU_{\mathrm{s}} u_k^{\mathrm{fr}} + kv_k^{\mathrm{fr}} U_{\mathrm{s}}'\big) - \partial_y^2 u_k^{\mathrm{fr}} = f_k, & (t,y) \in \quad (0,T)\times\mathbb{R}_+, \\ iu_k^{\mathrm{fr}} + \partial_y v_k^{\mathrm{fr}} = 0 & (0,T)\times\mathbb{R}_+, \\ (u_k^{\mathrm{fr}}, \partial_t u_k^{\mathrm{fr}})|_{t=0} = (u_{k,\mathrm{in}}, u_{k,t,\mathrm{in}}) & \mathbb{R}_+, \\ u_k|_{y=0} = 0, \quad v_k|_{y=0} = 0 & (0,T). \end{cases} \tag{2.2}$$

We hence aim to determine a meaningful non-trivial forcing term f_k, such that (2.2) admits a solution u_k^{fr} of (2.2), whose norm $\|u_k^{\mathrm{fr}}(t)\|_{W_\alpha^{0,\infty}}$ experiences an exponential growth in time as $\mathrm{e}^{\sigma_0 t k^{1/3}}$, for a suitable constant $\sigma_0 > 0$. This growth holds true at any time $t \in (0,T)$, as described by the following proposition.

Proposition 2.1. *Assume that the shear flow $U_s = U_s(y)$ is in $W_\alpha^{4,\infty}(0,\infty)$ and satisfies the relations $U_s(a) = U_s'(a) = 0$ together with $U_s''(a) \neq 0$, for a given $a \in (0,\infty)$. For any positive frequency $k \in \mathbb{N}$, there exist two non-trivial profiles $\mathbb{U}_k$, $\mathcal{R}_k : \mathbb{R}_+ \to \mathbb{C}$ in $W_\alpha^{2,\infty}(\mathbb{R}_+)$ and $W_\alpha^{0,\infty}(\mathbb{R}_+)$, respectively, and there exists a complex number $\tau \in \mathbb{C}$ only depending on U_s with negative imaginary part $\mathcal{I}m(\tau) < 0$, such that the initial data and the forcing term*

$$u_{k,\mathrm{in}}(y) = \mathbb{U}_k(y), \quad u_{k,t,\mathrm{in}}(y) = i\tau k^{\frac{1}{3}} \mathbb{U}_k(y), \quad f_k(t,y) = -k^{\frac{4}{3}} e^{i\tau k^{\frac{1}{3}} t} \mathcal{R}_k(y), \quad (t,y) \in (0,T)\times\mathbb{R}_+, \tag{2.3}$$

generate a global-in-time solution $u_k^{\mathrm{fr}} \in L^\infty(0,T; W_\alpha^{0,\infty})$ of (2.2), which can be written explicitly in the following form

$$u_k^{fr}(t,y) = e^{i\tau k^{\frac{1}{3}} t} \mathbb{U}_k(y), \quad v_k^{fr}(t,y) = k e^{i\tau k^{\frac{1}{3}} t} \mathbb{V}_k(y), \quad \textit{with} \quad \mathbb{V}_k(y) := -i \int_0^y \mathbb{U}_k(z)\mathrm{d}z. \tag{2.4}$$

The sequences $(\mathbb{U}_k)_{k\in\mathbb{N}}$ is in $W_\alpha^{2,\infty}(\mathbb{R}_+)$ and it is uniformly bounded from above and below in $W_\alpha^{0,\infty}(\mathbb{R}_+)$, namely there exist two constants $c, \mathcal{C} > 0$ such that

$$0 < c \leq \|\mathbb{U}_k\|_{W_\alpha^{0,\infty}(\mathbb{R}_+)} \leq \mathcal{C} < \infty. \tag{2.5}$$

Furthermore, at high frequencies, the sequence $(k^{\frac{4}{3}} \mathcal{R}_k)_{k\in\mathbb{N}}$ is uniformly bounded in $W_\alpha^{0,\infty}(\mathbb{R}_+)$: there exists a constant $\mathcal{C}_{\mathcal{R}} > 0$, which depends uniquely on the shear flow U_s, such that

$$k^{\frac{4}{3}} \|\mathcal{R}_k\|_{W_\alpha^{0,\infty}} \leq \mathcal{C}_{\mathcal{R}} < \infty, \qquad \textit{for any } k \in \mathbb{N}. \tag{2.6}$$

Remark 2.2. Denoting by $\sigma_0 := -\mathcal{I}m(\tau) > 0$, Proposition 2.1 implies that $\|u_k^{\mathrm{fr}}(t)\|_{W_\alpha^{0,\infty}} \sim \mathrm{e}^{\sigma_0 k^{1/3} t}$, at any $t \in (0,T)$, thanks to the explicit form (2.4) of the forced solution u_k^{fr}. Indeed, invoking also the uniform estimates from below in (2.5), we gather

$$\|u_k^{\mathrm{fr}}(t)\|_{W_\alpha^{0,\infty}(\mathbb{R}_+)} = \mathrm{e}^{-\mathcal{I}m(\tau) k^{\frac{1}{3}} t} \|\mathbb{U}_k\|_{W_\alpha^{0,\infty}(\mathbb{R}_+)} \geq c\, \mathrm{e}^{\sigma_0 k^{\frac{1}{3}} t}, \qquad \text{for any } t \in (0,T). \tag{2.7}$$

Unfortunately, this method of determining an explicit solution as in (2.4) seems to work uniquely for the inhomogeneous system (2.2) and it is somehow inefficient with its homogeneous counterpart (1.1) (thus with the original system). This is mainly due to the sequence of remainders $(\mathcal{R}_k)_{k\in\mathbb{N}}$, which define non-trivial forces $(f_k)_{k\in\mathbb{N}}$ in (2.4). From (2.4) and (2.5), $\|f_k(t)\|_{W_\alpha^{0,\infty}}$ does not vanish (a-priori) as $k \to \infty$. We will determine indeed an explicit form of $\mathcal{R}_k$ (cf. (3.30)), which implies in particular

$$\lim_{k\to\infty} k^{\frac{4}{3}} \mathcal{R}_k(y) = iH(y-a) U_s'''(y) \neq 0, \quad \text{for all} \quad y > 0 \quad \text{with} \quad y \neq a,$$

where H stands for the Heaviside step function. The contribution of f_k to the global-in-time instability seems to be not negligible. However, we show in (ii) that we can still obtain a similar result with the homogeneous equation, at least for a very short time that depends on the frequencies.

Since the proof of Proposition 2.1 is rather technical, we postpone it to Sect. 3. We anticipate that it follows a similar approach as the one used in [12]: we plug the ansatz (2.3) to the main equations (2.2), we analyse the asymptotic limit as frequencies $k \to \infty$ and we reduce the problem to a "spectral condition" on a related ODE. We shall hence devote the remaining parts of this section to address the details of (ii) and (iii) and thus to the proof of Theorem 1.1.

(ii) An instability of the homogeneous equation at a short time $t \sim \ln(k)/\sqrt[3]{k}$

Based on the result given by Proposition 2.1, we aim to obtain a similar instability for the homogeneous system (2.1), which will lead in (iii) to the ill-posedness of (1.1) in Sobolev spaces. We first invoke the following Duhamel's formula, which relates a general forced solution u_k^{fr} of (2.2) with the semigroup T_k of the homogeneous system (2.1):

$$u_k^{\mathrm{fr}}(t,y) = \underbrace{T_k(t)(u_{k,\mathrm{in}}, u_{k,t,\mathrm{in}})(y)}_{=: u_k(t,y)} + \int_0^t T_k(t-s)(0, f_k(s))(y)\mathrm{d}s. \tag{2.8}$$

Here we shall interpret u_k as the unique solution of the homogeneous problem (2.1), with same initial data of u_k^{fr}. Roughly speaking, in order to transfer the instability of u_k^{fr} to u_k, we need to ensure that the integral on the r.h.s. of (2.8) remains sufficiently small. With the following lemma, we translate this condition directly as a property of the semigroup $T_k(t)$ at a time t, which is (at maximum) proportional to $\ln(k)/\sqrt[3]{k}$ (thus a time that vanishes as the frequency $k \to \infty$).

Lemma 2.3. *Assume that the shear flow $U_s = U_s(y)$ is in $W^{3,\infty}_\alpha(0,\infty)$ and satisfies the relations $U_s(a) = U_s'(a) = 0$ together with $U_s''(a) \neq 0$, for a given $a \in (0,\infty)$. Let $\tau \in \mathbb{C}$ be as in Proposition 2.1 and denote by $\sigma_0 := -\mathcal{I}m(\tau) > 0$. For any $k \in \mathbb{N}$ and any $\sigma \in (0,\sigma_0)$ the following inequality holds true*

$$\sup_{0 \le t \le \frac{1}{3(\sigma_0-\sigma)}\frac{\ln(k)}{\sqrt[3]{k}}} e^{-\sigma t k^{\frac{1}{3}}} \left\|T_k(t)\right\|_{\mathcal{L}\left(W^{1,\infty}_\alpha \times W^{0,\infty}_\alpha, W^{0,\infty}_\alpha\right)} > \mathcal{C}_\sigma k^{\frac{1}{3}} \quad \textit{with} \quad \mathcal{C}_\sigma := \frac{c}{2}\frac{(\sigma_0-\sigma)}{(\sigma_0-\sigma)+\mathcal{C}_R}, \tag{2.9}$$

where the constants c and $\mathcal{C}_R$ are as in Proposition 2.1.

Remark 2.4. Before establishing the proof of Lemma 2.3, some remarks are here in order. Inequality (2.9) is written in terms of the semigroup T_k. It implies, however, that there exist two profiles $u_{\mathrm{in,k}} \in W^{1,\infty}_\alpha$ and $u_{t,\mathrm{in,k}} \in W^{0,\infty}_\alpha$ with $\|u_{\mathrm{in,k}}\|_{W^{1,\infty}_\alpha} + \|u_{t,\mathrm{in,k}}\|_{W^{0,\infty}_\alpha} \le 1$ (which may differ with respect to the ones of Proposition 2.1), such that the generated homogeneous solution u_k of (2.1) satisfies

$$\sup_{0 \le t \le \frac{1}{3(\sigma_0-\sigma)}\frac{\ln(k)}{\sqrt[3]{k}}} \mathrm{e}^{-\sigma t k^{\frac{1}{3}}} \left\|u_k(t)\right\|_{W^{0,\infty}_\alpha} > \mathcal{C}_\sigma k^{\frac{1}{3}}. \tag{2.10}$$

Unfortunately this inequality presents a major disadvantage: it is unclear at what time the inflation $\left\|u_k(t)\right\|_{W^{0,\infty}_\alpha} \ge \mathcal{C}_\sigma k^{\frac{1}{3}} \mathrm{e}^{\sigma t k^{1/3}}$ holds true. This is deeply in contrast with the instability (2.7) of u_k^{fr}, which is indeed satisfied globally in time. Certainly, (2.10) is not achieved at $t = 0$ because of the initial data; however, the inflation may occur at a time t very close to the origin (for which $\mathrm{e}^{-\sigma t k^{1/3}} \sim 1$). Furthermore, also in case that the inflation occurs at the largest time $t = \frac{1}{3(\sigma_0-\sigma)}\frac{\ln(k)}{\sqrt[3]{k}}$, we may obtain at best that $\|u_k(t)\|_{W^{0,\infty}_\alpha} \ge \mathcal{C}_\sigma k^{\frac{\sigma_0}{3(\sigma_0-\sigma)}}$, which somehow implies only an inflation of Sobolev type. In other words, because of the time limitation of estimate (2.9), we deal in this work only with ill-posedness in Sobolev spaces and our approach seems to be inconclusive in Gevrey-class m, with $m > 3$.

Proof of Lemma 2.3. Assume by contradiction that there exists a frequency $k \in \mathbb{N}$, so that

$$\sup_{0 \le t \le \frac{1}{3(\sigma_0-\sigma)} \frac{\ln(k)}{\sqrt[3]{k}}} \mathrm{e}^{-\sigma t k^{\frac{1}{3}}} \big\|T_k(t)\big\|_{\mathcal{L}} \le \frac{c}{2} \frac{(\sigma_0-\sigma)}{(\sigma_0-\sigma)+\mathcal{C}_R} k^{\frac{1}{3}}, \tag{2.11}$$

where we have used the abbreviation $\mathcal{L} := \mathcal{L}\big(W^{1,\infty}_\alpha \times W^{0,\infty}_\alpha, W^{0,\infty}_\alpha\big)$. We consider the global-in-time solution $u^{\mathrm{fr}}_k(t,y) = \mathrm{e}^{i\tau k^{1/3} t}\mathbb{U}_k(y)$ provided by Proposition 2.1, which by uniqueness (at a fixed frequency) also satisfies the Duhamel's relation

$$u^{\mathrm{fr}}_k(t,y) = T_k(t)\big(\mathbb{U}_k, i\tau k^{\frac{1}{3}}\mathbb{U}_k\big)(y) + \int_0^t T_k(t-s)(0, -k^{\frac{4}{3}}\mathrm{e}^{i\tau k^{\frac{1}{3}} s}\mathcal{R}_k(s))(y)\mathrm{d}s,$$

thanks to (2.8). Hence, applying the $W^{0,\infty}_\alpha$-norm to this identity and making use of the triangular inequality, we gather that

$$\begin{aligned}
\|T_k(t)\big(\mathbb{U}_k, i\tau k^{\frac{1}{3}}\mathbb{U}_k\big)\|_{W^{0,\infty}_\alpha} &\ge \|u^{\mathrm{fr}}_k(t)\|_{W^{0,\infty}_\alpha} - \int_0^t \|T_k(t-s)(0, -k^{\frac{4}{3}}\mathrm{e}^{i\tau k^{\frac{1}{3}} s}\mathcal{R}_k(s))\|_{W^{0,\infty}_\alpha}\mathrm{d}s \\
&\ge c\,\mathrm{e}^{\sigma_0 k^{\frac{1}{3}} t} - \int_0^t \|T_k(t-s)\|_{\mathcal{L}} k^{\frac{4}{3}} \|\mathcal{R}_k(s)\|_{W^{0,\infty}_\alpha} \mathrm{e}^{\sigma_0 s k^{\frac{1}{3}}}\mathrm{d}s \\
&\ge c\,\mathrm{e}^{\sigma_0 k^{\frac{1}{3}} t} - \int_0^t \|T_k(t-s)\|_{\mathcal{L}}\, \mathrm{e}^{-\sigma k^{\frac{1}{3}}(t-s)} k^{\frac{4}{3}} \|\mathcal{R}_k(s)\|_{W^{0,\infty}_\alpha} \mathrm{e}^{-(\sigma_0-\sigma)(t-s)k^{\frac{1}{3}}}\mathrm{d}s \mathrm{e}^{\sigma_0 t k^{\frac{1}{3}}},
\end{aligned}$$

where we have estimated $\|u^{\mathrm{fr}}_k(t)\|_{W^{0,\infty}_\alpha}$ with the inequality in (2.7). Hence, applying (2.11) and invoking the uniform bound (2.6) on the forcing term $\mathcal{R}_k$, we obtain

$$\|T_k(t)\big(\mathbb{U}_k, i\tau k^{\frac{1}{3}}\mathbb{U}_k\big)\|_{W^{0,\infty}_\alpha} \ge c\,\mathrm{e}^{\sigma_0 k^{\frac{1}{3}} t} - \frac{c}{2}\frac{(\sigma_0-\sigma)}{(\sigma_0-\sigma)+\mathcal{C}_R} k^{\frac{1}{3}} \mathcal{C}_R \int_0^t \mathrm{e}^{-(\sigma_0-\sigma)(t-s)k^{\frac{1}{3}}}\mathrm{d}s \mathrm{e}^{\sigma_0 k^{\frac{1}{3}} t}.$$

Multiplying both l. and r.h.s. by $\mathrm{e}^{-\sigma_0 k^{1/3} t}/c$ and calculating explicitly the integral on the r.h.s, we get

$$\frac{\mathrm{e}^{-\sigma_0 k^{\frac{1}{3}} t}}{c} \|T_k(t)\big(\mathbb{U}_k, i\tau k^{\frac{1}{3}}\mathbb{U}_k\big)\|_{W^{0,\infty}_\alpha} \ge 1 - \frac{1}{2}\frac{\mathcal{C}_R}{(\sigma_0-\sigma)+\mathcal{C}_R}\left(1 - \mathrm{e}^{-(\sigma_0-\sigma)k^{\frac{1}{3}} t}\right).$$

Recasting $\mathrm{e}^{-\sigma_0 k^{\frac{1}{3}} t} = \mathrm{e}^{-(\sigma_0-\sigma)k^{\frac{1}{3}} t}\mathrm{e}^{-\sigma k^{\frac{1}{3}} t}$ on the l.h.s., we apply once more (2.11), to deduce that

$$\frac{1}{2}\frac{(\sigma_0-\sigma)}{(\sigma_0-\sigma)+\mathcal{C}_R}\mathrm{e}^{-(\sigma_0-\sigma)k^{\frac{1}{3}} t} k^{\frac{1}{3}} \ge 1 - \frac{1}{2}\frac{\mathcal{C}_R}{(\sigma_0-\sigma)+\mathcal{C}_R}\left(1 - \mathrm{e}^{-(\sigma_0-\sigma)k^{\frac{1}{3}} t}\right),$$

for any time $t \in \big[0, \frac{1}{3(\sigma_0-\sigma)}\frac{\ln(k)}{\sqrt[3]{k}}\big]$. We hence set $t = \frac{1}{3(\sigma_0-\sigma)}\frac{\ln(k)}{\sqrt[3]{k}}$, which in particular implies that $\mathrm{e}^{-(\sigma_0-\sigma)k^{1/3} t} = k^{-1/3}$, so that

$$\frac{1}{2}\frac{(\sigma_0-\sigma)}{(\sigma_0-\sigma)+\mathcal{C}_R} \ge 1 - \frac{1}{2}\frac{\mathcal{C}_R(1-k^{-\frac{1}{3}})}{(\sigma_0-\sigma)+\mathcal{C}_R}.$$

By bringing the last term on the r.h.s to the l.h.s., we finally obtain that

$$\frac{1}{2} = \frac{1}{2}\frac{(\sigma_0-\sigma)+\mathcal{C}_R}{(\sigma_0-\sigma)+\mathcal{C}_R} \ge \frac{1}{2}\frac{(\sigma_0-\sigma)+\mathcal{C}_R(1-k^{-\frac{1}{3}})}{(\sigma_0-\sigma)+\mathcal{C}_R} \ge 1,$$

which is indeed a contradiction. This concludes the proof of Lemma 2.3. □

(ii) Proof of Theorem 1.1 and the inflation of the Sobolev norms

Thanks to Lemma 2.3, we are now in the condition to conclude the proof of Theorem 1.1. Let σ be a fixed value in $(0,\sigma_0) = (0, -\mathcal{I}m(\tau))$ and let $\delta > 0$ be an arbitrary short time. We first recall (2.9)

$$\sup_{0\leq t\leq \frac{1}{3(\sigma_0-\sigma)}\frac{\ln(k)}{\sqrt[3]{k}}} \mathrm{e}^{-\sigma t k^{\frac{1}{3}}} \big\|T_k(t)\big\|_{\mathcal{L}} > \mathcal{C}_\sigma k^{\frac{1}{3}}, \tag{2.12}$$

for any $k\in\mathbb{N}$, where $\mathcal{L}$ abbreviates $\mathcal{L}\big(W^{1,\infty}_\alpha \times W^{0,\infty}_\alpha, W^{0,\infty}_\alpha\big)$. Recalling that $\mu\in[0,1/3)$, we consider a general frequency $k\in\mathbb{N}$ that satisfies

$$\frac{1}{3(\sigma_0-\sigma)}\frac{\ln(k)}{\sqrt[3]{k}} \leq \delta, \quad \text{and} \quad \mathcal{C}_\sigma k^{\frac{1}{3}-\mu} \geq \frac{1}{\delta}.$$

By multiplying (2.12) with $k^{-\mu}$, we remark that there exists a time $t_\delta\in(0,\delta)$, such that

$$k^{-\mu}\big\|T_k(t_\delta)\big\|_{\mathcal{L}} \geq k^{-\mu}\mathrm{e}^{-\sigma t_\delta k^{\frac{1}{3}}}\big\|T_k(t_\delta)\big\|_{\mathcal{L}} > \mathcal{C}_\sigma k^{\frac{1}{3}-\mu} \geq \frac{1}{\delta}.$$

Next we write the above estimate in terms of a specific solution $u_k = u_k(t,y)$ of the homogeneous system (2.1): we consider two profiles $u_{\rm in,k}$ and $u_{t,\rm in,k}$ in $W^{1,\infty}_\alpha$ and $W^{0,\infty}_\alpha$, which satisfy

$$\|u_{k,\mathrm{in},k}\|_{W^{1,\infty}_\alpha} + \|u_{t,\mathrm{in},k}\|_{W^{0,\infty}_\alpha} \leq 1 \quad \text{and} \quad k^{-\mu}\big\|u_k(t_\delta)\big\|_{W^{0,\infty}_\alpha} > \frac{1}{\delta}, \quad \text{where} \quad u_k(t) := T_k(t)(u_{\mathrm{in},k}, u_{t,\mathrm{in},k}).$$

We hence set a solution $u = u(t,x,y)$ with of the original system (1.1) by means of

$$u(t,x,y) := \frac{1}{(1+k^2)^{m/2}}\mathrm{e}^{ikx}u_k(t,y), \qquad (t,x,y)\in(0,T)\times\mathbb{T}\times\mathbb{R}_+.$$

In particular, the initial data of u satisfy

$$\|u(0,\cdot)\|_{H^mW^{1,\infty}_\alpha} + \|\partial_t u(0,\cdot)\|_{H^mW^{0,\infty}_\alpha} = \|u_{k,\mathrm{in},k}\|_{W^{1,\infty}_\alpha} + \|u_{t,\mathrm{in},k}\|_{W^{0,\infty}_\alpha} \leq 1.$$

On the other hand, the following inflation of the Sobolev norm holds true at $t = t_\delta$:

$$\big\|u(t_\delta)\big\|_{H^{m-\mu}W^{0,\infty}_\alpha} = k^{-\mu}\big\|u_k(t_\delta)\big\|_{W^{0,\infty}_\alpha} > \frac{1}{\delta}.$$

This concludes the proof of Theorem 1.1.

3. Proof of Proposition 2.1

Without loss of generality, we may assume that $U_s''(a) < 0$. Contrarily, we might set $\tilde{U}_s(y) := -U_s(y)$, $\tilde{u}(t,x,y) := -u(t,-x,y)$, $\tilde{v}(t,x,y) := v(t,-x,y)$ and $\tilde{f}(t,x,y) := -f(t,-x,y)$. Thus, $(\tilde{u},\tilde{v})$ is solution of

$$(\partial_t+1)\big(\partial_t\tilde{u} + \tilde{U}_{\rm s}\partial_x\tilde{u} + \tilde{v}\tilde{U}_{\rm s}'\big) - \partial_y^2\tilde{u} = \tilde{f}, \qquad \text{with} \quad \tilde{U}_s''(a) < 0.$$

For simplicity, we denote by $\varepsilon = \varepsilon(k) := 1/k > 0$ the inverse of a positive frequency $k\in\mathbb{N}$, so that $\varepsilon\to 0$ when $k\to\infty$. Furthermore, throughout our proof, we will repeatedly use the following abuse of notation: we interchange any index k of a general function with its corresponding index ε, for instance

$$u_\varepsilon = u_k, \quad v_\varepsilon = v_k, \quad \mathbb{V}_\varepsilon = \mathbb{V}_k, \quad \mathcal{R}_\varepsilon = \mathcal{R}_k,$$

and so on.

We aim to construct a solution $(u_\varepsilon(t,x,y), v_\varepsilon(t,x,y))$ of (2.2) with a forcing term $f_\varepsilon(t,x,y)$ of the form

$$u_\varepsilon(t,x,y) = \mathrm{e}^{i\frac{\omega(\varepsilon)t+x}{\varepsilon}}\mathbb{U}_\varepsilon(y), \quad v_\varepsilon(t,x,y) = \frac{1}{\varepsilon}\mathrm{e}^{i\frac{\omega(\varepsilon)t+x}{\varepsilon}}\mathbb{V}_\varepsilon(y), \quad f_\varepsilon(t,x,y) = -\frac{1}{\varepsilon^{\frac{4}{3}}}\mathrm{e}^{i\frac{\omega(\varepsilon)t+x}{\varepsilon}}\mathcal{R}_\varepsilon(y), \tag{3.1}$$

for a suitable $\omega(\varepsilon) \in \mathbb{C}$ that we will soon determine and suitable profiles $\mathbb{U}_\varepsilon, \mathbb{V}_\varepsilon, \mathcal{R}_\varepsilon : [0, \infty) \to \mathbb{C}$. We momentarily take for given that ω is $\mathcal{O}(\varepsilon^{\frac{2}{3}})$ and $\mathcal{R}_\varepsilon \in W^{0,\infty}_\alpha(\mathbb{R}_+)$, for any $\varepsilon > 0$, with $\|\mathcal{R}_\varepsilon\|_{W^{0,\infty}_\alpha}$ in $\mathcal{O}(\varepsilon^{\frac{4}{3}})$ (for the impatient reader, check (3.4) and (3.30)).
In reality, the profile $\mathbb{U}_\varepsilon$ is redundant, since the divergence-free condition implies that

$$\partial_x u_\varepsilon + \partial_y v_\varepsilon = 0 \quad \Rightarrow \quad \frac{i}{\varepsilon} e^{i\frac{\omega(\varepsilon)t+x}{\varepsilon}} \mathbb{U}_\varepsilon(y) + \frac{1}{\varepsilon} e^{i\frac{\omega(\varepsilon)t+x}{\varepsilon}} \mathbb{V}'_\varepsilon(y) = 0 \quad \Rightarrow \quad \boxed{\mathbb{U}_\varepsilon(y) = i\mathbb{V}'_\varepsilon(y)}.$$

We hence plug the expressions (3.1) into the main equation (1.1). We obtain that $\mathbb{U}_\varepsilon$ and $\mathbb{V}_\varepsilon$ satisfies

$$e^{i\frac{\omega(\varepsilon)t+x}{\varepsilon}} \left\{ \left(\frac{i\omega(\varepsilon)}{\varepsilon} + 1 \right) \left(\frac{i\omega(\varepsilon)}{\varepsilon} \mathbb{U}_\varepsilon(y) + \frac{i}{\varepsilon} U_s(y) \mathbb{U}_\varepsilon(y) + \frac{1}{\varepsilon} U'_s(y) \mathbb{V}_\varepsilon(y) \right) - \mathbb{U}''_\varepsilon(y) \right\} = -\frac{1}{\varepsilon^{\frac{4}{3}}} e^{i\frac{\omega(\varepsilon)t+x}{\varepsilon}} \mathcal{R}_\varepsilon(y).$$

Dividing by $e^{i\frac{\omega(\varepsilon)t+x}{\varepsilon}}$ and multiplying by $-\varepsilon^{4/3}$ we get

$$\left(\frac{i\omega(\varepsilon)}{\varepsilon^{\frac{2}{3}}} + \varepsilon^{\frac{1}{3}} \right) \left(\omega(\varepsilon)(-i\mathbb{U}_\varepsilon(y)) + U_s(y)(-i\mathbb{U}_\varepsilon(y)) - U'_s(y)\mathbb{V}_\varepsilon(y) \right) + \varepsilon^{\frac{4}{3}} \mathbb{U}''_\varepsilon(y) = \mathcal{R}_\varepsilon(y).$$

Thus recasting the above relation uniquely in terms of $\mathbb{V}_\varepsilon$ (with $\mathbb{V}'_\varepsilon = -i\mathbb{U}_\varepsilon$), we finally derive the following ordinary differential equation:

$$\left(\frac{i\omega(\varepsilon)}{\varepsilon^{\frac{2}{3}}} + \varepsilon^{\frac{1}{3}} \right) \Big((\omega(\varepsilon) + U_s(y)) \mathbb{V}'_\varepsilon(y) - U'_s(y)\mathbb{V}_\varepsilon(y) \Big) + i\varepsilon^{\frac{4}{3}} \mathbb{V}'''_\varepsilon(y) = \mathcal{R}_\varepsilon(y), \qquad y > 0. \tag{3.2}$$

We shall remark that for any $\mathbb{V}_\varepsilon$ satisfying (3.2) in distributional sense, in reality $\mathbb{V}'_\varepsilon$ belongs to $W^{2,\infty}_\alpha(\mathbb{R}_+)$ and $\mathbb{V}_\varepsilon \in W^{3,\infty}(\mathbb{R}_+)$ ($\mathbb{V}_\varepsilon$ does not need to decay as $y \to \infty$), for any $\varepsilon > 0$, since $U_s \in W^{4,\infty}_\alpha$ and $\mathcal{R}_\varepsilon \in W^{0,\infty}_\alpha$. Formally, by sending $\varepsilon \to 0$, denoting the asymptotic $\mathbb{V}_\varepsilon \to v_a$, and recalling that the limit $\lim_{\varepsilon\to 0} \varepsilon^{-2/3}\omega(\varepsilon) \neq 0$, we obtain the equation

$$U_s(y) v'_a(y) - U'_s(y) v_a(y) = 0, \qquad y > 0. \tag{3.3}$$

Of course this limit is only formal, we shall soon analyse the difference between a solution $\mathbb{V}_\varepsilon$ of (3.2) and a solution v_a of (3.3). Since $U_s(a) = U'_s(a) = 0$, for a given $a \in (0, \infty)$, (3.3) admits a non-trivial weak solution given by

$$\boxed{v_a(y) = H(y-a)U_s(y)}.$$

We remark that v_a belongs to $W^{2,\infty}_\alpha(\mathbb{R}_+) \cap W^{3,\infty}_\alpha(\mathbb{R}_+ \backslash \{a\})$, since $U'_s(a) = U_s(a) = 0$. However, v'''_a behaves as a Dirac delta distribution in $y = a$, since $U''_s(a) \neq 0$.
We next set a complex number $\tau \in \mathbb{C}\backslash\{0\}$ and we introduce the following Ansatz:

$$\boxed{\omega(\varepsilon) = \varepsilon^{\frac{2}{3}} \tau}. \tag{3.4}$$

We hence aim to determine $\mathbb{V}_\varepsilon$ of (3.2), making use of the following perturbation of v_a, depending on τ:

$$\boxed{\begin{aligned} \mathbb{V}_\varepsilon(y) &:= v_a(y) + \varepsilon^{\frac{2}{3}} \tau H(y-a) + \varepsilon^{\frac{2}{3}} V\left(\frac{y-a}{\sqrt[3]{\varepsilon}} \right) \Phi(y) \\ &= \Big(U_s(y) + \varepsilon^{\frac{2}{3}} \tau \Big) H(y-a) + \varepsilon^{\frac{2}{3}} V\left(\frac{y-a}{\sqrt[3]{\varepsilon}} \right) \Phi(y), \end{aligned}} \tag{3.5}$$

where Φ is a fixed smooth function with compact support in $(0, \infty)$, which is identically equal to 1 on a neighbourhood of $y = a$. Hence, the unknowns are momentarily the profile $V : \mathbb{R} \to \mathbb{C}$ and the complex number $\tau \in \mathbb{C}$. We remark that the expansion (3.5) is similar to its homologous (2.5) in [12] for the Prandtl equations. There are, however, major differences in the exponents of the terms in ε. These eventually lead to different dispersion rates: of order $\sqrt[3]{k} = \varepsilon^{-1/3}$ for System (1.1) and of order $\sqrt{k} = \varepsilon^{-1/2}$ for the Prandtl equation.

Remark 3.1. Since $\mathbb{V}_\varepsilon$ belongs to $W^{3,\infty}$, the profile V must cancel the singularities in $y = a$ of the functions $v_a(y)$ and $\varepsilon^{\frac{2}{3}}\tau H(y-a)$. We will show that V is indeed in $W^{3,\infty}(\mathbb{R}_+\backslash\{0\}) \cap L^\infty(\mathbb{R}_+)$ and V' behave as a Dirac delta distribution $-\tau\delta_a$ in $y = a$.

Replacing (3.4) and (3.5) into (3.2), we gather the following equation for the profile V:

$$\begin{aligned}\Big(i\tau+\varepsilon^{\frac13}\Big)\Big\{\Big(\varepsilon^{\frac23}\tau+U_s(y)\Big)\Big(v_a'(y)+\varepsilon^{\frac23}\tau\delta_a(y)+\varepsilon^{\frac23}\frac{d}{dy}\Big[\Phi(y)V\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\Big]\Big)+\\ -U_s'(y)\Big(v_a(y)+\varepsilon^{\frac23}\tau H(y-a)+\varepsilon^{\frac23}\Phi(y)V\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\Big)\Big\}+i\varepsilon^{\frac43}\mathbb{V}_\varepsilon'''(y)=\mathcal{R}_\varepsilon(y).\end{aligned}\tag{3.6}$$

We first remark that several terms of (3.6) cancel out thanks to the definition of v_a in (3.3) and the conditions on the shear flow $U_s(a) = U_s'(a) = 0$. Indeed, (3.6) can be recasted as

$$\begin{aligned}&\Big(i\tau+\varepsilon^{\frac13}\Big)\Big\{\varepsilon^{\frac23}\tau\Big(\underbrace{v_a'(y)-H(y-a)U_s'(y)}_{=\delta_a(y)U_s(y)=0}\Big)+\varepsilon^{\frac43}\tau^2\delta_a(y)+\tau\varepsilon^{\frac43}\frac{d}{dy}\Big[\Phi(y)V\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\Big]+\\ &+\Big(\underbrace{U_s(y)v_a'(y)-U_s'(y)v_a(y)}_{=0}\Big)+\varepsilon^{\frac23}\tau\underbrace{U_s(y)\delta_a(y)}_{y=0}+\varepsilon^{\frac23}U_s(y)\frac{d}{dy}\Big[\Phi(y)V\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\Big]-\varepsilon^{\frac23}U_s'(y)\Phi(y)V\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\Big\}+\\ &+i\varepsilon^{\frac43}v_a'''(y)+i\tau\varepsilon^2\delta_a''(y)+i\varepsilon^2\frac{d^3}{dy^3}\Big[\Phi(y)V\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\Big]=\mathcal{R}_\varepsilon(y).\end{aligned}$$

Next, we divide the above relation with $\varepsilon^{4/3} > 0$ and we remark that $v_a'''(y) = U_s''(a)\delta_a(y)+H(y-a)U_s'''(y)$. Hence, we are left with the following identity, which shall be intended in terms of distributions $\mathcal{D}'(0,\infty)$:

$$\begin{aligned}&\Big(i\tau+\varepsilon^{\frac13}\Big)\Big\{\tau^2\delta_a(y)+\tau\frac{d}{dy}\Big[\Phi(y)V\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\Big]+\varepsilon^{-\frac23}U_s(y)\frac{d}{dy}\Big[\Phi(y)V\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\Big]-\varepsilon^{-\frac23}U_s'(y)\Phi(y)V\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\Big\}+\\ &+i\Big(U_s''(a)\delta_a(y)+H(y-a)U_s'''(y)\Big)+i\tau\varepsilon^{\frac23}\delta_a''(y)+i\varepsilon^{\frac23}\frac{d^3}{dy^3}\Big[\Phi(y)V\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\Big]=\varepsilon^{-\frac43}\mathcal{R}_\varepsilon(y).\end{aligned}\tag{3.7}$$

We next apply (3.7) to an appropriate test function $\varphi \in \mathcal{D}(0,\infty)$. We first consider a general test function $\psi \in \mathcal{D}(\mathbb{R})$ having $\operatorname{supp}\psi \subseteq (-a/\sqrt[3]{\varepsilon},\infty)$ is valid. Hence we set $\varphi(y) = \psi((y-a)/\sqrt[3]{\varepsilon})$, for any $y > 0$. By applying (3.7) to this specific test function φ, we get the identity

$$\begin{aligned}&\Big(i\tau+\varepsilon^{\frac13}\Big)\Big\{\tau^2\psi(0)-\tau\int_0^\infty\Phi(y)V\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\psi'\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\frac{\mathrm{d}y}{\sqrt[3]{\varepsilon}}-\varepsilon^{-\frac23}\int_0^\infty\Phi(y)V\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\frac{\mathrm{d}}{\mathrm{d}y}\Big(U_s(y)\psi\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\Big)\mathrm{d}y+\\ &-\varepsilon^{-\frac13}\int_0^\infty U_s'(y)\Phi(y)V\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\psi\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\frac{\mathrm{d}y}{\sqrt[3]{\varepsilon}}\Big\}+iU_s''(a)\psi(0)+i\int_a^\infty U_s'''(y)\psi\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\mathrm{d}y+i\tau\psi''(0)+\\ &-i\int_0^\infty\Phi(y)V\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\psi'''\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\frac{\mathrm{d}y}{\sqrt[3]{\varepsilon}}=\varepsilon^{-\frac43}\int_0^\infty\mathcal{R}_\varepsilon(y)\psi\Big(\frac{y-a}{\sqrt[3]{\varepsilon}}\Big)\mathrm{d}y.\end{aligned}$$

We perform a change of variables with $\tilde z := (y-a)/\sqrt[3]{\varepsilon} \in (-a/\sqrt[3]{\varepsilon}, \infty)$. Hence $y = a+\sqrt[3]{\varepsilon}\tilde z$ and $\mathrm{d}y = \sqrt[3]{\varepsilon}\mathrm{d}\tilde z$, so that

$$\begin{aligned}&\Big(i\tau+\varepsilon^{\frac13}\Big)\Big\{\tau^2\psi(0)-\tau\int_{-\frac{a}{\sqrt[3]{\varepsilon}}}^{\infty}\Phi(a+\sqrt[3]{\varepsilon}\tilde z)V(\tilde z)\psi'(\tilde z)\mathrm{d}\tilde z-\varepsilon^{-\frac23}\int_{-\frac{a}{\sqrt[3]{\varepsilon}}}^{\infty}\Phi(a+\sqrt[3]{\varepsilon}\tilde z)V(\tilde z)\frac{d}{\mathrm{d}\tilde z}\Big(U_s(a+\sqrt[3]{\varepsilon}\tilde z)\psi(\tilde z)\Big)\mathrm{d}\tilde z+\\&-\varepsilon^{-\frac13}\int_{-\frac{a}{\sqrt[3]{\varepsilon}}}^{\infty}U_s'(a+\sqrt[3]{\varepsilon}\tilde z)\Phi(a+\sqrt[3]{\varepsilon}\tilde z)V(\tilde z)\psi(\tilde z)\mathrm{d}\tilde z\Big\}+iU_s''(a)\psi(0)+i\sqrt[3]{\varepsilon}\int_0^{\infty}U_s'''(a+\sqrt[3]{\varepsilon}\tilde z)\psi(\tilde z)\mathrm{d}\tilde z+\\&+i\tau\psi''(0)-i\int_{-\frac{a}{\sqrt[3]{\varepsilon}}}^{\infty}\Phi(a+\sqrt[3]{\varepsilon}\tilde z)V(\tilde z)\psi'''(\tilde z)\mathrm{d}\tilde z=\frac{1}{\varepsilon}\int_{-\frac{a}{\sqrt[3]{\varepsilon}}}^{\infty}\mathcal{R}_\varepsilon(a+\sqrt[3]{\varepsilon}\tilde z)\psi(\tilde z)\mathrm{d}\tilde z.\end{aligned}$$

In particular, we deduce that V is distributional solution in $\mathcal{D}'(-a/\sqrt[3]{\varepsilon}, \infty)$ of the following equation:

$$\begin{aligned}&\Big(i\tau+\varepsilon^{\frac13}\Big)\Big\{\tau^2\delta_0(\tilde z)+\tau\frac{\mathrm{d}}{\mathrm{d}z}\Big(\Phi(a+\sqrt[3]{\varepsilon}\tilde z)V(\tilde z)\Big)+\varepsilon^{-\frac23}U_s(a+\sqrt[3]{\varepsilon}\tilde z)\frac{\mathrm{d}}{\mathrm{d}\tilde z}\Big[\Phi(a+\sqrt[3]{\varepsilon}\tilde z)V(\tilde z)\Big]+\\&-\varepsilon^{-\frac13}U_s'(a+\sqrt[3]{\varepsilon}\tilde z)\Phi(a+\sqrt[3]{\varepsilon}\tilde z)V(\tilde z)\Big\}+iU_s''(a)\delta_0(\tilde z)+i\varepsilon^{\frac13}H(\tilde z)U_s'''(a+\sqrt[3]{\varepsilon}\tilde z)+i\tau\delta_0''(\tilde z)+\\&+i\frac{\mathrm{d}^3}{\mathrm{d}z^3}\Big[\Phi(a+\sqrt[3]{\varepsilon}\tilde z)V(\tilde z)\Big]=\varepsilon^{-1}\mathcal{R}_\varepsilon(a+\sqrt[3]{\varepsilon}\tilde z).\end{aligned}\tag{3.8}$$

Remark that although V does not depend on $\varepsilon > 0$, the remainder $\mathcal{R}_\varepsilon$ is still unknown and will be chosen to absorb all the ε-dependences. We now send ε towards 0 in order to determine an equation for the profile V in $\mathcal{D}'(\mathbb{R})$. We recall that we assume $\mathcal{R}_\varepsilon \in \mathcal{O}(\varepsilon^{\frac43})$ in $W^{0,\infty}_\alpha$, that both $U_s(a) = U_s'(a) = 0$ and that $U_s \in W^{4,\infty}(0,\infty)$, hence

$$\begin{aligned}\lim_{\varepsilon\to0}\varepsilon^{-\frac13}U_s'(a+\sqrt[3]{\varepsilon}\tilde z)&=\lim_{\varepsilon\to0}\frac{U_s'(a+\sqrt[3]{\varepsilon}\tilde z)-U_s'(a)}{\sqrt[3]{\varepsilon}}=\tilde zU_s''(a),\\\lim_{\varepsilon\to0}\varepsilon^{-\frac23}U_s(a+\sqrt[3]{\varepsilon}\tilde z)&=\lim_{\varepsilon\to0}\frac{U_s(a+\sqrt[3]{\varepsilon}\tilde z)-U_s(a)-U_s'(a)\sqrt[3]{\varepsilon}\tilde z}{\varepsilon^{\frac23}}=\frac{\tilde z^2}{2}U_s''(a),\end{aligned}$$

for any $z > -a/\sqrt[3]{\varepsilon}$. Since the shear flow U_s is in $W^{4,\infty}_\alpha(0,\infty)$, the convergence is uniform in any compact set of $(-a/\sqrt[3]{\varepsilon}, \infty)$. Hence, as $\varepsilon > 0$ vanishes, recalling that $\Phi(y) \equiv 1$ on a neighbourhood of $y = a$, we are left with the following identity for the profile V:

$$\tau\Big\{\tau^2\delta_0(\tilde z)+\Big(\tau+\frac{\tilde z^2}{2}U_s''(a)\Big)V'(\tilde z)-\tilde zU_s''(a)V(\tilde z)\Big\}+U_s''(a)\delta_0(\tilde z)+\tau\delta_0''(\tilde z)+V'''(\tilde z)=0 \text{ in } \mathcal{D}'(\mathbb{R}).\tag{3.9}$$

In particular, we aim to determine a profile $V \in W^{3,\infty}(\mathbb{R}_+\backslash\{0\})$ (decaying to 0 as $\tilde z \to \pm\infty$), which is solution of the following linear ordinary differential equation away from the origin

$$\boxed{\tau\Big\{\Big(\tau+\frac{\tilde z^2}{2}U_s''(a)\Big)V'(\tilde z)-\tilde zU_s''(a)V(\tilde z)\Big\}+V'''(\tilde z)=0,\qquad \tilde z\in\mathbb{R}\setminus\{0\},}\tag{3.10}$$

fulfilling also the following jump relations at the origin:

$$\boxed{\begin{aligned}[V]_{|\tilde z=0}&=\lim_{h\to0+}\Big(V(\tilde z+h)-V(\tilde z-h)\Big)=-\tau,\\ [V']_{|\tilde z=0}&=\lim_{h\to0+}\Big(V'(\tilde z+h)-V'(\tilde z-h)\Big)=0,\\ [V'']_{|\tilde z=0}&=\lim_{h\to0+}\Big(V''(\tilde z+h)-V''(\tilde z-h)\Big)=-U_s''(a).\end{aligned}}\tag{3.11}$$

We next remark that the function $\tilde{z} \in \mathbb{R}\backslash\{0\} \to \tau + U_s''(a)\tilde{z}^2/2$ is non-decaying solution of (3.10). Furthermore, $\tilde{z} \in \mathbb{R} \to H(\tilde{z})(\tau + U_s''(a)\tilde{z}^2/2)$ is a distributional solution of (3.9) and it satisfies the jump conditions of (3.11). We can thus get rid of the singularity at the origin by superposition introducing the function

$$\tilde{V}(\tilde{z}) = V(\tilde{z}) + H(\tilde{z})\Big(\tau + \frac{U_s''(a)}{2}\tilde{z}^2\Big), \qquad \tilde{z} \in \mathbb{R}. \tag{3.12}$$

The new profile $\tilde{V}$ shall be determined in $W^{3,\infty}_{\text{loc}}(\mathbb{R})$, satisfying the ODE

$$\begin{cases} \tau\Big\{\Big(\tau + \frac{\tilde{z}^2}{2}U_s''(a)\Big)\tilde{V}'(\tilde{z}) - \tilde{z}U_s''(a)\tilde{V}(\tilde{z})\Big\} + \tilde{V}'''(\tilde{z}) = 0, \qquad \tilde{z} \in \mathbb{R}, \\ \lim\limits_{\tilde{z}\to-\infty} \tilde{V}(\tilde{z}) = 0, \qquad \lim\limits_{\tilde{z}\to+\infty} \Big(\tilde{V}(\tilde{z}) - \Big(\tau + \frac{U_s''(a)}{2}\tilde{z}^2\Big)\Big) = 0. \end{cases}$$

Next, recalling that we seek for a $\tau \in \mathbb{C}$ with $\mathcal{I}m(\tau) < 0$, we introduce the function

$$\tilde{W}(\tilde{z}) := \frac{\tilde{V}(\tilde{z})}{\tau + U_s''(a)\frac{\tilde{z}^2}{2}}, \qquad \tilde{z} \in \mathbb{R}. \tag{3.13}$$

We thus get that $W \in W^{3,\infty}(\mathbb{R})$ shall satisfy

$$\boxed{\tau\Big(\tau + U_s''(a)\frac{\tilde{z}^2}{2}\Big)^2\tilde{W}'(\tilde{z}) + \frac{\mathrm{d}^3}{\mathrm{d}\tilde{z}^3}\Big[\Big(\tau + U_s''(a)\frac{\tilde{z}^2}{2}\Big)\tilde{W}(\tilde{z})\Big] = 0, \qquad \tilde{z} \in \mathbb{R},} \tag{3.14}$$

with boundary conditions

$$\lim_{\tilde{z}\to-\infty} \tilde{W}(\tilde{z}) = 0, \qquad \lim_{\tilde{z}\to+\infty} \tilde{W}(\tilde{z}) = 1.$$

Finally, we perform the following change of variables

$$\tau = \Big(\frac{|U_s''(a)|}{2}\Big)^{\frac{1}{3}}\gamma, \qquad \tilde{z} = \Big(\frac{2}{|U_s''(a)|}\Big)^{\frac{1}{3}}z, \qquad W(z) = \tilde{W}\Big(\sqrt[3]{\frac{2}{|U_s''(a)|}}z\Big), \tag{3.15}$$

which leads to

$$\gamma\Big(\gamma + \operatorname{sgn} U_s''(a)z^2\Big)^2 W'(z) + \frac{d^3}{dz^3}\Big[\Big(\gamma + \operatorname{sgn} U_s''(a)z^2\Big)W(z)\Big] = 0, \qquad z \in \mathbb{R}.$$

Recalling that we assume $U_s''(a) < 0$, we finally obtain

$$\boxed{\gamma\big(\gamma - z^2\big)^2\frac{d}{dz}W(z) + \frac{d^3}{dz^3}\Big[\big(\gamma - z^2\big)W(z)\Big] = 0, \qquad z \in \mathbb{R}} \tag{3.16}$$

with boundary conditions

$$\lim_{z\to-\infty} W(z) = 0, \qquad \lim_{z\to+\infty} W(z) = 1. \tag{3.17}$$

3.1. The spectral condition

The goal of this section is to show that the ordinary differential equation (3.16), together with its boundary conditions (3.17), admits a smooth solution $W = W(z)$ for a suitable $\gamma \in \mathbb{C}$, whose imaginary part is negative. We proceed with a similar procedure as the one used by Gérard-Varet and Dormy [12]. We shall, however, remark that (3.16) inherently differs from its counterpart of [12] (cf. (1.7) in [12]), since the authors derived an ODE of the form

$$\big(\gamma - z^2\big)^2\frac{d}{dz}W(z) + i\frac{d^3}{dz^3}\Big[\big(\gamma - z^2\big)W(z)\Big] = 0, \qquad z \in \mathbb{R}. \tag{3.18}$$

Indeed, note that by dividing (3.16) by γ, the leading third derivative in (3.16) is multiplied by $1/\gamma$, while in (3.18) the third derivative relates uniquely to i. Nevertheless, our aim is to find a $\gamma \in \mathbb{C}$ with $\mathcal{I}m(\gamma) < 0$, thus (3.16) and (3.18) share eventually similarities in terms of the behaviour of their solutions. This is exploited in details in what follows.

To begin with, we consider the auxiliary eigenvalue problem (cf. (3.2) in [12]):

$$Af(\tilde{x}) := \frac{1}{\tilde{x}^2+1} f''(\tilde{x}) + \frac{6\tilde{x}}{(\tilde{x}^2+1)^2} f'(\tilde{x}) + \frac{6}{(\tilde{x}^2+1)^2} f(\tilde{x}) = \alpha\, f(\tilde{x}), \qquad \tilde{x} \in \mathbb{R}. \tag{3.19}$$

We aim to build the solution W in terms of the eigenfunction f. In order to do so, let us first recall certain of its underlying properties. The domain $D(A)$ of the operator A is defined by making use of the weighted spaces

$$\begin{aligned}
\mathcal{L}^2 &:= \Big\{ f \in L^2_{\mathrm{loc}}(\mathbb{R}),\ \int_{\mathbb{R}} (\tilde{x}^2+1)^4 |f(\tilde{x})|^2 d(\tilde{x}) < +\infty \Big\},\\
\mathcal{H}^1 &:= \Big\{ f \in H^1_{\mathrm{loc}}(\mathbb{R}),\ \int_{\mathbb{R}} (\tilde{x}^2+1)^4 |f(\tilde{x})|^2 d(\tilde{x}) + \int_{\mathbb{R}} (\tilde{x}^2+1)^3 |f'(\tilde{x})|^2 d(\tilde{x}) < +\infty \Big\},\\
D(A) &:= \Big\{ f \in \mathcal{H}^1,\ Af \in \mathcal{L}^2 \Big\}.
\end{aligned}$$

Furthermore, the function f admits an extension on a simply connected domain of $\mathbb{C}$ and decays exponentially along suitable sectors.

Lemma 3.2. *The problem* (3.19) *admits a positive eigenvalue $\alpha > 0$ and an eigenfunction $f \in D(A)$. Secondly, the function f admits a complex extension, which still satisfies* (3.19) *and is holomorphic in the simply connected domain*

$$\mathbb{C} \setminus \big\{ i\beta \in \mathbb{C} \big|\ \beta > \sqrt{\alpha} \text{ or } \beta < -\sqrt{\alpha} \big\}.$$

Furthermore, in the sectors $\arg(\tilde{x}) \in (-\pi/4+\delta, \pi/4-\delta)$ *and* $\arg(\tilde{x}) \in (3\pi/4+\delta, 5\pi/4-\delta)$ *for $\delta > 0$, it satisfies the inequality*

$$|f(\tilde{x})| \le C_\delta \exp\left(-\frac{\sqrt{\alpha}\, |\tilde{x}|^2}{4} \right). \tag{3.20}$$

Proof. This lemma follows from Proposition 2 in [12]. Indeed defining $\theta := -\alpha^{1/2}$ and $\tilde{z} = \alpha^{1/4}\tilde{x}$, $Y(\tilde{z}) = f(\tilde{x})$, the function Y is solution of

$$(\theta - \tilde{z}^2) Y''(\tilde{z}) - 6\tilde{z} Y'(\tilde{z}) + ((\theta - \tilde{z}^2)^2 - 6) Y(\tilde{z}) = 0,$$

which corresponds to equation (3.5) in [12]. Thanks to Proposition 2 in [12], Y is holomorphic in the domain $U_\theta := \mathbb{C} \setminus \{ i\beta \in \mathbb{C} \mid \beta > \sqrt{|\theta|} \text{ or } \beta < -\sqrt{|\theta|} \}$ and satisfies

$$|Y(\tilde{z})| \le C_\delta \exp\Big(-\frac{|\tilde{z}|^2}{4} \Big)$$

in the sectors $\arg(\tilde{z}) \in (-\pi/4+\delta, \pi/4-\delta)$ and $\arg(\tilde{z}) \in (3\pi/4+\delta, 5\pi/4-\delta)$. The result follows by a direct substitution. □

We are now in the condition of defining the complex number $\gamma \in \mathbb{C}$ of (3.16) in terms of the positive eigenvalue α of Lemma 3.2. Recalling that the eigenvalue α is positive, we set

$$\boxed{\gamma := \alpha^{\frac{1}{3}} \mathrm{e}^{-i\frac{2\pi}{3}}} \quad \Rightarrow \quad \mathcal{I}m(\gamma) = -\frac{\sqrt{3}}{2} \alpha^{\frac{1}{3}} < 0. \tag{3.21}$$

This also implies that the complex number $\tau = (|U_s''(a)|/2)^{1/3}\gamma$ in (3.15) has negative imaginary part, i.e.

$$\boxed{\tau := \left(\frac{|U_s''(a)|}{2}\alpha\right)^{\frac{1}{3}} \mathrm{e}^{-i\frac{2\pi}{3}}} \quad \Rightarrow \quad \mathcal{I}m(\tau) = -\frac{\sqrt{3}}{2}\left(\frac{|U_s''(a)|}{2}\alpha\right)^{\frac{1}{3}} < 0.$$

We next introduce the change of variable w.r.t. (3.19),

$$z = \alpha^{\frac{1}{6}}\mathrm{e}^{i\frac{\pi}{6}}\tilde{x} \in \Omega := \mathbb{C} \setminus \left\{\mathrm{e}^{i2\pi/3}\beta \in \mathbb{C},\ \text{with } \beta > \alpha^{2/3} \text{ or } \beta < -\alpha^{2/3}\right\},$$

and the function $X : \Omega \to \mathbb{C}$ by means of

$$X(z) = f(\tilde{x}) = f(\alpha^{-\frac{1}{6}}\mathrm{e}^{-i\frac{\pi}{6}}z). \tag{3.22}$$

Remark 3.3. We shall remark that X decays exponentially as $z \in \mathbb{R}$ converges towards $\pm\infty$. Indeed, in virtue of (3.20), we gather that

$$|X(z)| \le C\exp\left(-\sqrt{\alpha}\,\frac{|\alpha^{-\frac{1}{6}}\mathrm{e}^{-i\frac{\pi}{6}}z|^2}{4}\right) = C\exp\left(-\frac{\alpha^{\frac{1}{6}}|z|^2}{4}\right) = C\exp\left(-\frac{\sqrt{|\gamma|}|z|^2}{4}\right). \tag{3.23}$$

The corresponding derivatives of X are trivially computed:

$$X'(z) = \alpha^{-\frac{1}{6}}\mathrm{e}^{-i\frac{\pi}{6}}f'(\alpha^{-\frac{1}{6}}\mathrm{e}^{-i\frac{\pi}{6}}z),$$
$$X''(z) = \alpha^{-\frac{1}{3}}\mathrm{e}^{-i\frac{\pi}{3}}f''(\alpha^{-\frac{1}{6}}\mathrm{e}^{-i\frac{\pi}{6}}z).$$

We can now recast the relation (3.19) of the eigenfunction f in terms of X and its derivatives. First

$$\begin{aligned}
&\alpha X(z) = \alpha f(\alpha^{-\frac{1}{6}}\mathrm{e}^{-i\frac{\pi}{6}}z) = Af(\alpha^{-\frac{1}{6}}\mathrm{e}^{-i\frac{\pi}{6}}z)\\
&= \frac{1}{\alpha^{-\frac{1}{3}}\mathrm{e}^{-i\frac{\pi}{3}}z^2+1}f''(\alpha^{-\frac{1}{6}}\mathrm{e}^{-i\frac{\pi}{6}}z) + \frac{6\alpha^{-\frac{1}{6}}\mathrm{e}^{-i\frac{\pi}{6}}z}{(\alpha^{-\frac{1}{3}}\mathrm{e}^{-i\frac{\pi}{3}}z^2+1)^2}f'(\alpha^{-\frac{1}{6}}\mathrm{e}^{-i\frac{\pi}{6}}z) + \frac{6}{(\alpha^{-\frac{1}{3}}\mathrm{e}^{-i\frac{\pi}{3}}z^2+1)^2}f(\alpha^{-\frac{1}{6}}\mathrm{e}^{-i\frac{\pi}{6}}z)\\
&= \frac{\alpha^{\frac{1}{3}}\mathrm{e}^{i\frac{\pi}{3}}}{\alpha^{-\frac{1}{3}}\mathrm{e}^{-i\frac{\pi}{3}}z^2+1}X''(z) + \frac{6z}{(\alpha^{-\frac{1}{3}}\mathrm{e}^{-i\frac{\pi}{3}}z^2+1)^2}X'(z) + \frac{6}{(\alpha^{-\frac{1}{3}}\mathrm{e}^{-i\frac{\pi}{3}}z^2+1)^2}X(z)\\
&= \frac{\alpha^{\frac{2}{3}}\mathrm{e}^{i\frac{2\pi}{3}}}{z^2-\alpha^{\frac{1}{3}}\mathrm{e}^{-i\frac{2\pi}{3}}}X''(z) + \frac{6\alpha^{\frac{2}{3}}\mathrm{e}^{i\frac{2\pi}{3}}z}{(z^2-\alpha^{\frac{1}{3}}\mathrm{e}^{-i\frac{2\pi}{3}})^2}X'(z) + \frac{6\alpha^{\frac{2}{3}}\mathrm{e}^{i\frac{2\pi}{3}}}{(z^2-\alpha^{\frac{1}{3}}\mathrm{e}^{-i\frac{2\pi}{3}})^2}X(z).
\end{aligned}$$

We next multiply both left- and right-hand sides with $\alpha^{-\frac{2}{3}}\mathrm{e}^{-i\frac{2\pi}{3}}(z^2-\alpha^{\frac{1}{3}}\mathrm{e}^{-i\frac{2\pi}{3}})^2$, so that

$$\alpha^{\frac{1}{3}}\mathrm{e}^{-i\frac{2\pi}{3}}(z^2-\alpha^{\frac{1}{3}}\mathrm{e}^{-i\frac{2\pi}{3}})^2X(z) = (z^2-\alpha^{\frac{1}{3}}\mathrm{e}^{-i\frac{2\pi}{3}})X''(z) + 6zX'(z) + 6X(z).$$

Recalling that $\gamma = \alpha^{\frac{1}{3}}\mathrm{e}^{-i\frac{2\pi}{3}}$ from (3.21), we finally obtain the following relation on X,

$$(z^2-\gamma)^2X(z) = (z^2-\gamma)X''(z) + 6zX'(z) + 6X(z), \qquad z \in \Omega,$$

which can also be written as

$$\gamma(\gamma-z^2)^2X(z) = -\frac{\mathrm{d}^3}{\mathrm{d}z^3}\Big[(\gamma-z^2)\int_{-\infty}^{z} X(s)\mathrm{d}s\Big].$$

In this last identity, we have used that X is in $\mathbb{R}$ integrable, thanks to (3.22) and the exponential decay (3.20). We hence aim to define the function $W : \mathbb{R} \to \mathbb{C}$ as

$$\boxed{W(z) := \frac{\int_{-\infty}^{z} X(s)ds}{\int_{\mathbb{R}} X(s)ds}}. \tag{3.24}$$

For this definition, we need therefore that X has average in $\mathbb{R}$ different from 0. Assume by contradiction that this were false. Then the function $F : \mathbb{R} \to \mathbb{C}$ defined by

$$F(z) := (\gamma - z^2) \int_{-\infty}^{z} X(s)ds$$

decays exponentially as $z \to \pm\infty$. Furthermore, F satisfies

$$F^{(4)} + \gamma(\gamma - z^2)F''(z) + 2\gamma F(z) = 0.$$

Multiplying the equation by $\overline{F''(z)}$ and integrating over $\mathbb{R}$, we gather

$$\gamma \int_{\mathbb{R}} (\gamma - z^2)|F''(z)|^2 \mathrm{d}z - \int_{\mathbb{R}} |F'''(z)|^2 \mathrm{d}z - 2\gamma \int_{\mathbb{R}} |F'(z)|^2 \mathrm{d}z = 0.$$

Hence extrapolating the imaginary part

$$-\frac{\mathcal{I}m\,(\gamma^2)}{\mathcal{I}m\,(\gamma)} \int_{\mathbb{R}} |F''(z)|^2 \mathrm{d}z + \int_{\mathbb{R}} \left(z^2|F''(z)|^2 + 2|F'(z)|^2\right) dz = 0.$$

Since $\mathcal{I}m(\gamma) < 0$, cf. eq. (3.21), one has that $F' \equiv 0$ which implies that also $F \equiv 0$, from the exponential decay. We conclude that X is identically null and so also the holomorphic function f in (3.19). Since f is an eigenfunction, this brings a contradiction.

3.2. The velocity profile and the remainder

In virtue of the previous sections, we are now in the condition to determine both $\mathbb{U}_k$ and the remainder $\mathcal{R}_k$, introduced in Proposition 2.1. We first recast the profile V in (3.5) in terms of the complex number $\tau = \gamma(|U_s''(a)|/2)^{1/3}$ and the meaningful solution W of the ordinary differential equation (3.16)–(3.17). Thanks to (3.12), (3.13) and (3.15), we have in particular that

$$\begin{aligned} V(\tilde{z}) &= -\Big(\tau + \frac{U_s''(a)}{2}\tilde{z}^2\Big)H(\tilde{z}) + \Big(\tau + \frac{U_s''(a)}{2}\tilde{z}^2\Big)\tilde{W}(\tilde{z}) \\ &= \left(\tau + \frac{U_s''(a)}{2}\tilde{z}^2\right)\left(W\left(\sqrt[3]{\frac{|U_s''(a)|}{2}}\tilde{z}\right) - H(\tilde{z})\right), \qquad \text{for all} \quad \tilde{z} \in \mathbb{R}. \end{aligned} \tag{3.25}$$

As predicted in Remark 3.1, we observe that V has a jump in $\tilde{z} = 0$ (W is smooth in $\mathbb{R}$) and its derivative behaves as the Dirac delta $-\tau\delta(\tilde{z})$ near the origin. Furthermore, V and its derivatives decay exponentially as $z \to \pm\infty$, since from (3.17), (3.23) and (3.24) we have that

$$\begin{aligned} \big|W(z) - 1\big| &\le \frac{\int_z^\infty |X(s)|ds}{\int_{\mathbb{R}} |X(s)|ds} &&\le C_W \exp\left\{-\frac{\sqrt{|\gamma|}|z|^2}{4}\right\} && z \ge 0, \\ \big|W(z)\big| &\le \frac{\int_{-\infty}^z |X(s)|ds}{\int_{\mathbb{R}} |X(s)|ds} &&\le C_W \exp\left\{-\frac{\sqrt{|\gamma|}|z|^2}{4}\right\} && z < 0, \\ \big|W'(z)\big| &\le \frac{|X(z)|}{\int_{\mathbb{R}} |X(s)|ds} &&\le C_W \exp\left\{-\frac{\sqrt{|\gamma|}|z|^2}{4}\right\} && z \in \mathbb{R}, \end{aligned} \tag{3.26}$$

for an harmless constant $C_W > 0$, that depends only upon the function W. In particular, the profile $\mathbb{V}_\varepsilon$ in (3.5) can be written in terms of W and τ, by means of

$$\boxed{\begin{aligned}\mathbb{V}_\varepsilon(y) &= \Big(U_s(y) + \varepsilon^{\frac{2}{3}}\tau\Big)H(y-a) + \varepsilon^{\frac{2}{3}} V\left(\frac{y-a}{\varepsilon^{1/3}}\right)\Phi(y)\\ &= U_s(y)H(y-a) + \varepsilon^{\frac{2}{3}}\tau H(y-a) +\\ &\quad + \Phi(y)\left(\varepsilon^{\frac{2}{3}}\tau + \frac{U_s''(a)(y-a)^2}{2}\right)\left\{W\left(\sqrt[3]{\frac{|U_s''(a)|}{2\varepsilon}}(y-a)\right) - H(y-a)\right\}.\end{aligned}} \tag{3.27}$$

Next, we determine the profile $\mathbb{U}_k$ of Proposition 2.1, recalling that $\varepsilon = \varepsilon(k) = 1/k$ and $\mathbb{U}_k = \mathbb{U}_\varepsilon = i\mathbb{V}_\varepsilon'(y)$:

$$\boxed{\mathbb{U}_k(y) = iU_s'(y)H(y-a) + \frac{i}{\sqrt[3]{k}}\mathcal{U}_{1,k}(y) + \frac{i}{\sqrt[3]{k}}\mathcal{U}_{2,k}(y),} \tag{3.28}$$

with profiles $\mathcal{U}_{1,k}$ and $\mathcal{U}_{2,k}$ given by

$$\begin{aligned}\mathcal{U}_{1,k}(y) &= \Phi(y)\left(\tau + \frac{U_s''(a)}{2}\big(k^{\frac{1}{3}}(y-a)\big)^2\right)\sqrt[3]{\frac{|U_s''(a)|}{2}}W'\left(\sqrt[3]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{3}}(y-a)\right)\\ \mathcal{U}_{2,k}(y) &= \Bigg(U_s''(a)k^{\frac{1}{3}}(y-a)\Phi(y) +\\ &\quad + \frac{\Phi'(y)}{\sqrt[3]{k}}\left(\tau + \frac{U_s''(a)}{2}(k^{\frac{1}{3}}(y-a))^2\right)\Bigg)\left\{W\left(\sqrt[3]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{3}}(y-a)\right) - H(y-a)\right\}.\end{aligned}$$

The profile $\mathbb{U}_k$ is in $W^{2,\infty}_\alpha(\mathbb{R}_+)$ for all $k \in \mathbb{N}$. Furthermore it is uniformly bounded from below and above in $W^{0,\infty}_\alpha(\mathbb{R}_+)$, since it converges towards $iU_s'(y)H(y-a)$ in $W^{0,\infty}_\alpha(\mathbb{R}_+)$, as $k \to \infty$. Indeed, introducing the change of variable $z = \frac{U_s''(a)}{2}k^{\frac{1}{3}}(y-a)$, recalling that $\tau = (U_s''(a)/2)^{1/3}\gamma$ and invoking (3.26)

$$\begin{aligned}\left\|\frac{i}{\sqrt[3]{k}}\mathcal{U}_{1,k}(y)\right\|_{W^{0,\infty}_\alpha} &\le \frac{1}{\sqrt[3]{k}}\|\Phi\|_\infty C_W e^{\alpha a}\left(\frac{|U_s''(a)|}{2}\right)^{\frac{2}{3}}\sup_{z\in\mathbb{R}}\big(|\gamma| + |z|^2\big)\exp\left\{\frac{\alpha}{\sqrt[3]{k}}\left(\frac{2}{|U_s''(a)|}\right)^{\frac{1}{3}}z - \frac{\sqrt{|\gamma|}|z|^2}{4}\right\}\\ &\lesssim \frac{1}{\sqrt[3]{k}},\end{aligned}$$

as well as

$$\begin{aligned}\left\|\frac{i}{\sqrt[3]{k}}\mathcal{U}_{2,k}(y)\right\|_{W^{0,\infty}_\alpha} &\le \frac{1}{\sqrt[3]{k}}C_W e^{\alpha a}\sup_{z\in\mathbb{R}}\left(2\|\Phi\|_\infty|z| + \frac{\|\Phi'\|_\infty}{\sqrt[3]{k}}\big(|\gamma|+|z|^2\big)\right)\exp\left\{\frac{\alpha}{\sqrt[3]{k}}\left(\frac{2}{|U_s''(a)|}\right)^{\frac{1}{3}}z - \frac{\sqrt{|\gamma|}|z|^2}{4}\right\}\\ &\lesssim \frac{1}{\sqrt[3]{k}}.\end{aligned}$$

These values converge towards 0 as $k \to \infty$.
To determine the remainder $\mathcal{R}_\varepsilon$, we now take the difference between Eq. (3.8) for a positive $\varepsilon > 0$ and (3.9) multiplied by i. We hence decompose $\varepsilon^{-1}\mathcal{R}_\varepsilon$ into

$$\begin{aligned}\varepsilon^{-1}\mathcal{R}_\varepsilon(a + \sqrt[3]{\varepsilon}\tilde z) &= i\varepsilon^{\frac{1}{3}}U_s'''(a + \sqrt[3]{\varepsilon}\tilde z)H(\tilde z) + \mathfrak{R}_{1,\varepsilon}(a + \sqrt[3]{\varepsilon}\tilde z) +\\ &\quad + \mathfrak{R}_{2,\varepsilon}(a + \sqrt[3]{\varepsilon}\tilde z) + \mathfrak{R}_{3,\varepsilon}(a + \sqrt[3]{\varepsilon}\tilde z) + i\frac{d^3}{d\tilde z^3}\left[\big(\Phi(a + \sqrt[3]{\varepsilon}\tilde z) - 1\big)V(\tilde z)\right],\end{aligned}$$

where the functions $\mathfrak{R}_{1,\varepsilon}$ and $\mathfrak{R}_{2,\varepsilon}$ stand for

$$\begin{aligned}
\mathfrak{R}_{1,\varepsilon}(a+\sqrt[3]{\varepsilon}\tilde{z}) &:= \varepsilon^{\frac{1}{3}}\Big\{\tau^2\delta(\tilde{z}) + \Big(\tau+\varepsilon^{-\frac{2}{3}}U_s(a+\sqrt[3]{\varepsilon}\tilde{z})\Big)\frac{d}{dt}\Big[\Phi(a+\sqrt[3]{\varepsilon}\tilde{z})V(\tilde{z})\Big]+\\
&\qquad - \varepsilon^{-\frac{1}{3}}U_s'(a+\sqrt[3]{\varepsilon}\tilde{z})\Phi(a+\sqrt[3]{\varepsilon}\tilde{z})V(\tilde{z})\Big\},\\
\mathfrak{R}_{2,\varepsilon}(a+\sqrt[3]{\varepsilon}\tilde{z}) &:= i\tau\Big\{\varepsilon^{-\frac{2}{3}}U_s(a+\sqrt[3]{\varepsilon}\tilde{z})\frac{d}{d\tilde{z}}\Big[\Phi(a+\sqrt[3]{\varepsilon}\tilde{z})V(\tilde{z})\Big] - \frac{U_s''(a)}{2}\tilde{z}^2\Phi(a+\sqrt[3]{\varepsilon}\tilde{z})V'(\tilde{z})+\\
&\qquad - \varepsilon^{-\frac{1}{3}}U_s'(a+\sqrt[3]{\varepsilon}\tilde{z})\Phi(a+\sqrt[3]{\varepsilon}\tilde{z})V(\tilde{z}) + \tilde{z}U_s''(a)\Phi(a+\sqrt[3]{\varepsilon}\tilde{z})V(\tilde{z})\Big\},\\
\mathfrak{R}_{3,\varepsilon}(a+\sqrt[3]{\varepsilon}\tilde{z}) &:= i\tau\Big(\Phi(a+\sqrt[3]{\varepsilon}\tilde{z})-1\Big)\Big\{\Big(\tau+\frac{U_s''(a)}{2}\tilde{z}^2\Big)V'(\tilde{z}) - \tilde{z}U_s''(a)V(\tilde{z})\Big\}.
\end{aligned} \tag{3.29}$$

By plugging the formula (3.25) in the definition of $\mathfrak{R}_{1,\varepsilon}$, we can recast this function in terms of W (for the sake of a compact presentation, we omit the variables in each function, resulting in $W = W(\sqrt[3]{\frac{|U_s''(a)|}{2}}\tilde{z})$, $H = H(\tilde{z})$, $\Phi = \Phi(a+\sqrt[3]{\varepsilon}\tilde{z})$, $\Phi' = \Phi'(a+\sqrt[3]{\varepsilon}\tilde{z})$ and so on):

$$\begin{aligned}
\mathfrak{R}_{1,\varepsilon}(a+\sqrt[3]{\varepsilon}\tilde{z}) &= \varepsilon^{\frac{1}{3}}\big(W-H\big)\Phi\Big\{\Big(\tau+\varepsilon^{-\frac{2}{3}}U_s\Big)U_s''(a)\tilde{z} - \varepsilon^{-\frac{1}{3}}U_s'\Big(\tau+\frac{U_s''(a)}{2}\tilde{z}^2\Big)\Big\}+\\
&\quad + \varepsilon^{\frac{2}{3}}\big(W-H\big)\Phi'\Big(\tau+\varepsilon^{-\frac{2}{3}}U_s\Big)\Big(\tau+\frac{U_s''(a)}{2}\tilde{z}^2\Big) + W'\sqrt[3]{\frac{|U_s''(a)|}{2}}\Phi\Big(\tau+\varepsilon^{-\frac{2}{3}}U_s\Big)\Big(\tau+\frac{U_s''(a)}{2}\tilde{z}^2\Big).
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathfrak{R}_{2,\varepsilon}(a+\sqrt[3]{\varepsilon}\tilde{z}) &= i\tau\big(W-H\big)\Phi\Big\{\Big(\varepsilon^{-\frac{2}{3}}U_s - \frac{U_s''(a)}{2}\tilde{z}^2\Big)U_s''(a)\tilde{z} - \Big(\varepsilon^{-\frac{2}{3}}U_s' - U_s''(a)\tilde{z}\Big)\Big(\tau+\frac{U_s''(a)}{2}\tilde{z}^2\Big)\Big\}+\\
&\quad + i\tau\big(W-H\big)\Phi'\varepsilon^{-\frac{1}{3}}U_s\Big(\tau+\frac{U_s''(a)}{2}\tilde{z}^2\Big) + i\tau W'\Phi\sqrt[3]{\frac{|U_s''(a)|}{2}}\Big(\varepsilon^{-\frac{2}{3}}U_s - \frac{U_s''(a)}{2}\tilde{z}^2\Big)\Big(\tau+\frac{U_s''(a)}{2}\tilde{z}^2\Big),
\end{aligned}$$

as well as

$$\mathfrak{R}_{3,\varepsilon}(a+\sqrt[3]{\varepsilon}\tilde{z}) = i\tau\big(\Phi(y)-1\big)\Big(\tau+\frac{U_s''(a)}{2}\tilde{z}^2\Big)^2\sqrt[3]{\frac{|U_s''(a)|}{2}}W'.$$

Hence, we recall that $\varepsilon = \varepsilon(k) = 1/k$, for any positive frequency $k\in\mathbb{N}$, and we relabel $\mathfrak{R}_{1,k}(y) := \mathfrak{R}_{1,\varepsilon}(y)$, $\mathfrak{R}_{2,k}(y) := \mathfrak{R}_{2,\varepsilon}(y)$ and $\mathfrak{R}_{3,k}(y) := \mathfrak{R}_{3,\varepsilon}(y)$, so that

$$\begin{aligned}
\mathfrak{R}_{1,k}(y) &= k^{-\frac{1}{3}}\Big(W\Big(\sqrt[3]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{3}}(y-a)\Big) - H(y-a)\Big)\Phi(y)\Big\{\Big(\tau+k^{\frac{2}{3}}U_s(y)\Big)U_s''(a)k^{\frac{1}{3}}(y-a)+\\
&\qquad - k^{\frac{1}{3}}U_s'(y)\Big(\tau+\frac{U_s''(a)}{2}\big(k^{\frac{1}{3}}(y-a)\big)^2\Big)\Big\}+\\
&\qquad + k^{-\frac{2}{3}}\Big(W\Big(\sqrt[3]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{3}}(y-a)\Big) - H(y-a)\Big)\Phi'(y)\Big(\tau+k^{\frac{2}{3}}U_s(y)\Big)\Big(\tau+\frac{U_s''(a)}{2}\big(k^{\frac{1}{3}}(y-a)\big)^2\Big)+\\
&\qquad + W'\Big(\sqrt[3]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{3}}(y-a)\Big)\sqrt[3]{\frac{|U_s''(a)|}{2}}\Phi(y)\Big(\tau+k^{\frac{2}{3}}U_s(y)\Big)\Big(\tau+\frac{U_s''(a)}{2}\big(k^{\frac{1}{3}}(y-a)\big)^2\Big),
\end{aligned}$$

as well as

$$\begin{aligned}\mathfrak{R}_{2,k}(y)=&i\tau\left(W\left(\sqrt[3]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{3}}(y-a)\right)-H(y-a)\right)\Phi(y)\Bigg\{\left(k^{\frac{2}{3}}U_s(y)-\frac{U_s''(a)}{2}\left(k^{\frac{1}{3}}(y-a)\right)^2\right)U_s''(a)k^{\frac{1}{3}}(y-a)+\\&-\left(k^{\frac{1}{3}}U_s'(y)-U_s''(a)k^{\frac{1}{3}}(y-a)\right)\left(\tau+\frac{U_s''(a)}{2}\left(k^{\frac{1}{3}}(y-a)\right)^2\right)\Bigg\}+\\&+i\tau\left(W\left(\sqrt[3]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{3}}(y-a)\right)-H(y-a)\right)\Phi'(y)k^{-\frac{1}{3}}U_s(y)\left(\tau+\frac{U_s''(a)}{2}\left(k^{\frac{1}{3}}(y-a)\right)^2\right)+\\&+i\tau W'\left(\sqrt[3]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{3}}(y-a)\right)\Phi(y)\sqrt[3]{\frac{|U_s''(a)|}{2}}\left(k^{\frac{2}{3}}U_s(y)-\frac{U_s''(a)}{2}\left(k^{\frac{1}{3}}(y-a)\right)^2\right)\left(\tau+\frac{U_s''(a)}{2}\left(k^{\frac{1}{3}}(y-a)\right)^2\right),\end{aligned}$$

and finally

$$\mathfrak{R}_{3,\varepsilon}(y)=i\tau\big(\Phi(y)-1\big)\left(\tau+\frac{U_s''(a)}{2}(k^{\frac{1}{3}}(y-a))^2\right)^2\sqrt[3]{\frac{|U_s''(a)|}{2}}W'\left(\sqrt[3]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{3}}(y-a)\right).$$

We obtain therefore the following expression for the remainder $\mathcal{R}_k$:

$$\boxed{\begin{aligned}\mathcal{R}_k(y)=&\,ik^{-\frac{4}{3}}U_s'''(y)H(y-a)+k^{-1}\mathfrak{R}_{1,k}(y)+k^{-1}\mathfrak{R}_{2,k}(y)+k^{-1}\mathfrak{R}_{3,k}(y)+\\&+\frac{i}{k^2}\frac{d^3}{dy^3}\left[\big(\Phi(y)-1\big)\left(\tau+\frac{U_s''(a)}{2}\left(k^{\frac{1}{3}}(y-a)\right)^2\right)W\left(\sqrt[3]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{3}}(y-a)\right)\right].\end{aligned}} \quad (3.30)$$

We shall now remark that the remainders $\mathfrak{R}_{1,k}$ and $\mathfrak{R}_{2,k}$ vanishes in $W^{0,\infty}_\alpha$ as $\mathcal{O}(k^{-\frac{1}{3}})$, as $k\to\infty$, thanks to the exponential decay of $W-H$ and W'. Furthermore $\mathfrak{R}_{3,k}$ and the last term in (3.30) are identically null on a neighbourhood of $y=a$ (independent on the frequencies), therefore they exponentially decay as $k\to\infty$ because of the corresponding exponential decay of $W-H$ and all derivatives W', W'' and W'''. The exponential decay of W'' and W''' arises coupling the one of W' and Eq. (3.18) of W. This conclude the proof of Proposition 2.1.

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4. Appendix: Proof of Theorem 1.6

In this section we outline the proof of Theorem 1.6. Similarly to the procedure introduced in Sect. 2, we address at any frequency $k \in \mathbb{N}$ the following "forced" Prandtl equation

$$\begin{cases} \partial_t u_k^{\mathrm{fr}} + ikU_{\mathrm{s}} u_k^{\mathrm{fr}} + kv_k^{\mathrm{fr}} U_{\mathrm{s}}' - \partial_y^2 u_k^{\mathrm{fr}} = f_k, & (t,y) \in \quad (0,T) \times \mathbb{R}_+, \\ iu_k^{\mathrm{fr}} + \partial_y v_k^{\mathrm{fr}} = 0 & (0,T) \times \mathbb{R}_+, \\ u_k^{\mathrm{fr}}|_{t=0} = u_{k,\mathrm{in}} & \mathbb{R}_+, \\ u_k|_{y=0} = 0, \quad v_k|_{y=0} = 0 & (0,T), \end{cases} \tag{A.1}$$

and we denote by $T_k(t) : W_\alpha^{0,\infty} \to W_\alpha^{0,\infty}$, for any $t \in (0,T)$, the semigroup generated by the homogeneous system ($f_k \equiv 0$). Our starting point is the following proposition, that somehow recasts the main result of Gérard-Varet [12] in terms of solutions of (A.1).

Proposition A.1. *Let $U_s \in W_\alpha^{4,\infty}(\mathbb{R}_+)$, $U_s'(a) = 0$ and $U_s''(a) \neq 0$, for a given $a \in (0,\infty)$. There exists a complex number $\tau \in \mathbb{C}$ with $\mathcal{I}m(\tau) < 0$, such that, for any $k \in \mathbb{N}$, there exist $\mathbb{U}_k \in W_\alpha^{1,\infty}(\mathbb{R}_+)$ and $\mathcal{R}_k \in W_\alpha^{0,\infty}(\mathbb{R}_+)$ so that*

$$u_{k,\mathrm{in}}(y) = \mathbb{U}_k(y), \quad f_k(t,y) = -k\, e^{i\tau k^{\frac{1}{2}} t} \mathcal{R}_k(y), \quad (t,y) \in (0,T) \times \mathbb{R}_+, \tag{A.2}$$

generate a global-in-time solution $u_k^{\mathrm{fr}} \in L^\infty(0,T; W_\alpha^{0,\infty})$ of (A.1), which can be written explicitly as

$$u_k^{\mathrm{fr}}(t,y) = e^{i\tau k^{\frac{1}{2}} t} \mathbb{U}_k(y), \quad v_k^{\mathrm{fr}}(t,y) = k\, e^{i\tau k^{\frac{1}{2}} t} \mathbb{V}_k(y), \quad \text{with} \quad \mathbb{V}_k(y) := -i \int_0^y \mathbb{U}_k(z) dz. \tag{A.3}$$

Moreover there exist three constants $c, \mathcal{C}, \mathcal{C}_{\mathcal{R}} > 0$ such that for any $k \in \mathbb{N}$

$$0 < c \leq \| \mathbb{U}_k \|_{W_\alpha^{0,\infty}(\mathbb{R}_+)} \leq \mathcal{C} < \infty, \qquad k \| \mathcal{R}_k \|_{W_\alpha^{0,\infty}} \leq \mathcal{C}_{\mathcal{R}}. \tag{A.4}$$

The proof of Proposition A.1 is similar to the one of Proposition 2.1. In essence, it was performed in [12], from Sect. 2 to Section 4.1 (although formulated differently). We shall here report uniquely the exact form of $\mathbb{U}_k$ and $\mathcal{R}_k$ in terms of the solution W of the ODE (3.18):

$$\boxed{\mathbb{U}_k(y) = iU_s'(y)H(y-a) + \frac{i}{\sqrt[4]{k}}\mathcal{U}_{1,k}(y) + \frac{i}{\sqrt[4]{k}}\mathcal{U}_{2,k}(y)} \tag{A.5}$$

with

$$\mathcal{U}_{1,k}(y) := \left(\tau + \frac{U_s''(a)(k^{\frac{1}{4}}(y-a))^2}{2}\right)\sqrt[4]{\frac{|U_s''(a)|}{2}}W'\left(\sqrt[4]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{4}}(y-a)\right)\Phi(y),$$

$$\mathcal{U}_{2,k}(y) := \Bigg\{U_s''(a)k^{\frac{1}{4}}(y-a)\Phi(y) + \\ + \Phi'(y)\left(\tau + \frac{U_s''(a)}{2}\left(k^{\frac{1}{4}}(y-a)\right)^2\right)\Bigg\}\left(W\left(\sqrt[4]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{4}}(y-a)\right) - H(y-a)\right),$$

while

$$\boxed{\begin{aligned}\mathcal{R}_k(y) &= ik^{-1}H(y-a)U_s'''(y) + k^{-\frac{3}{4}}\mathfrak{R}_{1,k}(y) + k^{-\frac{3}{4}}\mathfrak{R}_{2,k}(y)\\ &+ ik^{-\frac{3}{2}}\frac{d^3}{dy^3}\left[(\Phi(y)-1)\left(\tau + \frac{U_s''(a)}{2}\left(k^{\frac{1}{4}}(y-a)\right)^2\right)\left(W\left(\sqrt[4]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{4}}(y-a)\right) - H(y-a)\right)\right].\end{aligned}} \tag{A.6}$$

The function $\mathfrak{R}_{1,k}$ stands for

$$\mathfrak{R}_{1,k}(y) = \Phi(y)\mathcal{F}_{1,k}(y)W'\left(\sqrt[4]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{4}}(y-a)\right) + \\ + \Big\{\Phi(y)\mathcal{F}_{2,k}(y) + \Phi'(y)\mathcal{F}_{3,k}(y)\Big\}\left(W\left(\sqrt[4]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{4}}(y-a)\right) - H(y-a)\right),$$

with

$$\mathcal{F}_{1,k}(y) = \sqrt[4]{\frac{|U_s''(a)|}{2}}\left(k^{\frac{1}{2}}(U_s(y)-U_s(a)) - \frac{U_s''(a)}{2}\left(k^{\frac{1}{4}}(y-a)\right)^2\right)\left(\tau + \frac{U_s''(a)}{2}\left(k^{\frac{1}{4}}(y-a)\right)^2\right),$$

$$\mathcal{F}_{2,k}(y) = \left(k^{\frac{1}{2}}(U_s(y)-U_s(a)) - \frac{U_s''(a)}{2}\left(k^{\frac{1}{4}}(y-a)\right)^2\right)U_s''(a)k^{\frac{1}{4}}(y-a) \\ - \left(k^{\frac{1}{4}}U_s'(y) - U_s''(a)k^{\frac{1}{4}}(y-a)\right)\left(\tau + \frac{U_s''(a)}{2}\left(k^{\frac{1}{4}}(y-a)\right)^2\right),$$

$$\mathcal{F}_{3,k}(y) = \left(\tau + k^{\frac{1}{2}}(U_s(y)-U_s(a))\right)\left(\tau + \frac{U_s''(a)}{2}\left(k^{\frac{1}{4}}(y-a)\right)^2\right).$$

The function $\mathfrak{R}_{2,k}$ corresponds to

$$\mathfrak{R}_{2,k}(y) = (\Phi(y)-1)\left(\tau + \frac{U_s''(a)}{2}\left(k^{\frac{1}{4}}(y-a)\right)^2\right)^2\sqrt[4]{\frac{|U_s''(a)|}{2}}W'\left(\sqrt[4]{\frac{|U_s''(a)|}{2}}k^{\frac{1}{4}}(y-a)\right).$$

We next set $\mathcal{C}_\sigma$ in Theorem 1.6 as

$$\mathcal{C}_\sigma := \frac{\mathcal{C}}{2}\frac{(\sigma_0-\sigma)}{(\sigma_0-\sigma)+\mathcal{C}_R}.$$

Assume by contradiction that there exists a frequency $k \in \mathbb{N}$, so that

$$\sup_{0 \le t \le \frac{1}{2(\sigma_0-\sigma)} \frac{\ln(k)}{\sqrt{k}}} \mathrm{e}^{-\sigma t k^{\frac{1}{2}}} \big\| T_k(t) \big\|_{\mathcal{L}(W^{0,\infty}_\alpha)} \le \frac{c}{2} \frac{(\sigma_0-\sigma)}{(\sigma_0-\sigma)+\mathcal{C}_R} k^{\frac{1}{2}}, \tag{A.7}$$

We consider the global-in-time solution $u_k^{\mathrm{fr}}(t,y) = \mathrm{e}^{i\tau k^{1/2} t} \mathbb{U}_k(y)$ provided by Proposition A.1, which by uniqueness also satisfies the Duhamel's relation

$$u_k^{\mathrm{fr}}(t,y) = T_k(t)\big(\mathbb{U}_k\big)(y) + \int_0^t T_k(t-s)(-k \mathrm{e}^{i\tau k^{\frac{1}{2}} s} \mathcal{R}_k(s))(y) \mathrm{d}s,$$

Hence, applying the $W^{0,\infty}_\alpha$-norm to this identity and applying the triangular inequality, we gather that

$$\begin{aligned} \|T_k(t)\big(\mathbb{U}_k\big)\|_{W^{0,\infty}_\alpha} &\ge \|u_k^{\mathrm{fr}}(t)\|_{W^{0,\infty}_\alpha} - \int_0^t \|T_k(t-s)(-k\mathrm{e}^{i\tau k^{\frac{1}{2}} s}\mathcal{R}_k(s))\|_{W^{0,\infty}_\alpha} \mathrm{d}s \\ &\ge c\mathrm{e}^{\sigma_0 k^{\frac{1}{2}} t} - \int_0^t \|T_k(t-s)\|_{\mathcal{L}} k \|\mathcal{R}_k(s)\|_{W^{0,\infty}_\alpha} \mathrm{e}^{\sigma_0 s k^{\frac{1}{2}}} \mathrm{d}s \\ &\ge c\mathrm{e}^{\sigma_0 k^{\frac{1}{2}} t} - \int_0^t \|T_k(t-s)\|_{\mathcal{L}} \mathrm{e}^{-\sigma k^{\frac{1}{2}}(t-s)} k \|\mathcal{R}_k(s)\|_{W^{0,\infty}_\alpha} \mathrm{e}^{-(\sigma_0-\sigma)(t-s)k^{\frac{1}{2}}} \mathrm{d}s \mathrm{e}^{\sigma_0 t k^{\frac{1}{2}}}, \end{aligned}$$

where we have estimated $\|u_k^{\mathrm{fr}}(t)\|_{W^{0,\infty}_\alpha}$ with (A.4). Hence applying (A.7) and invoking the uniform bound on $\mathcal{R}_k$, we obtain

$$\|T_k(t)\big(\mathbb{U}_k\big)\|_{W^{0,\infty}_\alpha} \ge c\mathrm{e}^{\sigma_0 k^{\frac{1}{2}} t} - \frac{c}{2} \frac{(\sigma_0-\sigma)}{(\sigma_0-\sigma)+\mathcal{C}_R} k^{\frac{1}{2}} \mathcal{C}_R \int_0^t \mathrm{e}^{-(\sigma_0-\sigma)(t-s)k^{\frac{1}{2}}} \mathrm{d}s \mathrm{e}^{\sigma_0 k^{\frac{1}{2}} t}.$$

Multiplying both l. and r.h.s. by $\mathrm{e}^{-\sigma_0 k^{1/2} t}/c$ and calculating explicitly the integral on the r.h.s., we obtain

$$\frac{\mathrm{e}^{-\sigma_0 k^{\frac{1}{2}} t}}{c} \|T_k(t)\big(\mathbb{U}_k\big)\|_{W^{0,\infty}_\alpha} \ge 1 - \frac{1}{2} \frac{\mathcal{C}_R}{(\sigma_0-\sigma)+\mathcal{C}_R} \left(1 - \mathrm{e}^{-(\sigma_0-\sigma)k^{\frac{1}{2}} t}\right).$$

Recasting $\mathrm{e}^{-\sigma_0 k^{\frac{1}{2}} t} = \mathrm{e}^{-(\sigma_0-\sigma)k^{\frac{1}{2}} t} \mathrm{e}^{-\sigma k^{\frac{1}{2}} t}$ on the l.h.s., we apply once more (A.7), to deduce that

$$\frac{1}{2} \frac{(\sigma_0-\sigma)}{(\sigma_0-\sigma)+\mathcal{C}_R} \mathrm{e}^{-(\sigma_0-\sigma)k^{\frac{1}{2}} t} k^{\frac{1}{2}} \ge 1 - \frac{1}{2} \frac{\mathcal{C}_R}{(\sigma_0-\sigma)+\mathcal{C}_R} \left(1 - \mathrm{e}^{-(\sigma_0-\sigma)k^{\frac{1}{2}} t}\right),$$

for any time $t \in \big[0, \frac{1}{2(\sigma_0-\sigma)} \frac{\ln(k)}{\sqrt{k}}\big]$. We hence set $t = \frac{1}{2(\sigma_0-\sigma)} \frac{\ln(k)}{\sqrt{k}}$, which in particular implies that $\mathrm{e}^{-(\sigma_0-\sigma)k^{1/2} t} = k^{-1/2}$, so that

$$\frac{1}{2} \frac{(\sigma_0-\sigma)}{(\sigma_0-\sigma)+\mathcal{C}_R} \ge 1 - \frac{1}{2} \frac{\mathcal{C}_R(1-k^{-\frac{1}{2}})}{(\sigma_0-\sigma)+\mathcal{C}_R}.$$

By bringing the last term on the r.h.s. to the l.h.s., we finally obtain that

$$\frac{1}{2} = \frac{1}{2} \frac{(\sigma_0-\sigma)+\mathcal{C}_R}{(\sigma_0-\sigma)+\mathcal{C}_R} \ge \frac{1}{2} \frac{(\sigma_0-\sigma)+\mathcal{C}_R(1-k^{-\frac{1}{2}})}{(\sigma_0-\sigma)+\mathcal{C}_R} \ge 1,$$

which is a contradiction. This concludes the proof of Theorem 1.6. ☐

Remark 4.1. Let us notice that if we relax the boundary conditions $\mathbb{U}_{\varepsilon|y=0} = \mathbb{V}_{\varepsilon|y=0} = 0$ and allow both the shear flow U_s and the solutions of the Prandtl equations to belong to a wider space, namely $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+)$, we can obtain an explicit solution of Prandtl (with small but not null value in $y = 0$), which behaves as $\mathrm{e}^{\sigma_0\sqrt{k}t}$, at any time $t > 0$, but does not decay as $y \to \infty$. By setting indeed

$$\Phi(y) \equiv 1 \quad \text{and} \quad U_s(y) := \frac{(y-a)^2}{2},$$

we remark that the forcing term $\mathcal{R}_k$ in (A.6) vanishes, thus $u(t,x,y) := \mathrm{e}^{i\tau\sqrt{k}t+ikx}\mathbb{U}_k(y)$ (with $\mathbb{U}_k$ as in (A.5)) is an exact solution of the Prandtl equations. However, $\mathbb{U}_k(y) \sim iU_s'(y) = (y-a)$ as $y \gg 1$, namely the profile $\mathbb{U}_k$ behaves similarly as a couette flow, which does not decay as $y \to \infty$.

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