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Perazzo 3-folds and the weak Lefschetz property

Luca Fiorindo [a,1], Emilia Mezzetti [a,2], Rosa M. Miró-Roig [b,*,3]

[a] *Dipartimento di Matematica e Geoscienze, Università di Trieste, Via Valerio 12/1, 34127 Trieste, Italy*
[b] *Facultat de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via des les Corts Catalanes 585, 08007 Barcelona, Spain*

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We deal with Perazzo 3-folds in $\mathbb{P}^4$, i.e. hypersurfaces $X = V(f) \subset \mathbb{P}^4$ of degree d defined by a homogeneous polynomial $f(x_0, x_1, x_2, u, v) = p_0(u, v)x_0 + p_1(u, v)x_1 + p_2(u, v)x_2 + g(u, v)$, where p_0, p_1, p_2 are algebraically dependent but linearly independent forms of degree $d-1$ in u, v, and g is a form in u, v of degree d. Perazzo 3-folds have vanishing hessian and, hence, the associated graded Artinian Gorenstein algebra A_f fails the strong Lefschetz Property. In this paper, we determine the maximum and minimum Hilbert function of A_f and we prove that if A_f has maximal Hilbert function it fails the weak Lefschetz Property while it satisfies the weak Lefschetz Property when it has minimum Hilbert function. In addition, we classify all Perazzo 3-folds in $\mathbb{P}^4$ such that A_f has minimum Hilbert function.

* Corresponding author.
E-mail addresses: luca.fiorindo@dima.unige.it (L. Fiorindo), mezzette@units.it (E. Mezzetti), miro@ub.edu (R.M. Miró-Roig).
[1] Current address: Dipartimento di Matematica (DIMA), Università degli Studi di Genova-ITALIA, Via Dodecaneso 35, 16146, Genova.
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1. Introduction

Having vanishing hessian is an elementary property of hypersurfaces that are cones. Conversely in every projective space $\mathbb{P}^N$ with $N \geq 4$ there exist classes of examples of hypersurfaces with vanishing hessian that are not cones. This was first proved by P. Gordan and M. Noether ([12]), who disproved a claim by O. Hesse: any hypersurface $X \subset \mathbb{P}^N$ with vanishing hessian is a cone ([14], [15]). Let X be the hypersurface in $\mathbb{P}^N$ defined by a polynomial $f(x_0, \dots, x_N)$. Gordan and Noether realized that X being a cone is equivalent to the condition that the partial derivatives of f are K-linearly dependent, while X has vanishing hessian if and only if they are K-algebraically dependent. They also gave a complete description of the hypersurfaces in $\mathbb{P}^4$, not cones, with vanishing hessian. Subsequent contributions were given by several authors. We refer to [26] for an exhaustive bibliography.

J. Watanabe in [29], and in [21] in collaboration with T. Maeno, established the following strict connection between the Lefschetz properties of Artinian Gorenstein algebras and hypersurfaces with vanishing hessian. We recall that an Artinian K-algebra A has the strong Lefschetz Property (respectively, the weak Lefschetz Property) if for a general linear form $L \in [A]_1$, the morphism $\times L^k : [A]_t \longrightarrow [A]_{t+k}$ has maximal rank for all integers $t \geq 0$ and $k \geq 1$ (respectively, the morphism $\times L : [A]_t \longrightarrow [A]_{t+1}$ has maximal rank for all integers $t \geq 0$).

Given a homogeneous polynomial f in $N + 1$ variables, we denote by A_f the quotient of the differential operators' ring by the annihilator of f; it is a standard Artinian Gorenstein algebra whose socle degree d coincides with the degree of f. In addition to the classical hessian, one defines the hessians of f of order t, for $0 \leq t \leq d$. Then, A_f fails the strong Lefschetz Property if and only if the hessian of f of order t vanishes for some t with $1 \leq t \leq \lfloor \frac{d}{2} \rfloor$. In particular, the hypersurfaces with vanishing hessian all fail the strong Lefschetz Property. A natural question is then if they have or fail the weak Lefschetz Property. This question was considered by R. Gondim in [11], and he found examples of both types.

In this article, we consider the case of $\mathbb{P}^4$, where the classification of hypersurfaces with vanishing hessian not cones is complete. Following the terminology introduced by Gondim in [11], a hypersurface in $\mathbb{P}^4$ of degree $d \geq 3$ is a Perazzo hypersurface if, using homogeneous coordinates x_0, x_1, x_2, u, v, it has equation of the form $f = p_0x_0 + p_1x_1 + p_2x_2 + g$, where p_0, p_1, p_2 are algebraically dependent but linearly independent forms of degree $d - 1$ in u, v, and g is a form in u, v of degree d. Perazzo hypersurfaces have vanishing hessian. On the other hand, according to [30], [31] and [32], any hypersurface of degree d, with $3 \leq d \leq 5$ of $\mathbb{P}^4$ not cone with vanishing hessian is a Perazzo hypersurface. In general, as proved by Gordan and Noether in [12], all forms with vanishing hessian, not cones, are elements of $K[u, v][\Delta]$ where Δ is a Perazzo polynomial of the form $p_0x_0 + p_1x_1 + p_2x_2$ (see [32, Theorem 7.3]). If $d = 3$ for such an f clearly A_f fails the weak Lefschetz Property. For $d = 4$ Gondim proved that the Artinian Gorenstein algebra of every Perazzo 3-fold has the weak Lefschetz Property.

Here we study the weak Lefschetz Property for Artinian Gorenstein algebras associated to Perazzo 3-folds of any degree $d \geq 3$. First we consider the possible Hilbert functions HF_{A_f} of A_f. In Propositions 3.7 and 3.8 we prove that they have a maximum and a minimum, coinciding only if $d = 3, 4$. We then study the weak Lefschetz Property for the algebras A_f whose Hilbert function attains the upper or the lower bound. Our main results, contained in Theorems 4.1 and 4.3, say that A_f has the weak Lefschetz Property if HF_{A_f} is minimum but it fails the weak Lefschetz Property if HF_f is maximum. For $d = 5$ this exhausts all possibilities; for $d \geq 6$ we give examples proving that for intermediate values of the Hilbert function both possibilities occur (see Example 4.5). For further results on this topic see [1].

We then focus our attention on Artinian Gorenstein algebras A_f having minimum Hilbert function; using the theory of Waring rank for forms in 2 variables, we are able to obtain in Theorem 5.4 a complete list of these Perazzo 3-folds. The classification is in terms of the position of the linear space π generated by p_0, p_1, p_2 in $\mathbb{P}(K[u, v]_{d-1})$, with respect to the secant varieties of the rational normal curve C_{d-1}. It results that, to ensure that A_f has minimal Hilbert function, π has to meet C_{d-1}, and there are three possibilities: either π is an osculating plane to C_{d-1}, or it is tangent to π and meets the curve again in a second point, or the intersection $\pi \cap C_{d-1}$ consists of three distinct points. We conclude with a geometrical study of the polar and Gauss maps associated to these 3-folds.

Next we outline the structure of this article. In Section 2, we recall the notions of strong and weak Lefschetz Property of an Artinian Gorenstein algebra, and of higher order hessians of a form. Then we state the theorem of J. Watanabe establishing a link between the failure of the strong Lefschetz Property for Artinian Gorenstein algebras and the vanishing of some hessian (Theorem 2.5). We give some examples illustrating these notions. In Section 3, we define Perazzo hypersurfaces and we study the h-vectors $h = (h_0,\ h_1,\ \ldots, h_{d-1},\ h_d)$ of the associated Artinian Gorenstein algebras in the case of $\mathbb{P}^4$. We relate them to the ranks of some block matrices composed of catalecticant matrices. In Propositions 3.7 and 3.8 we find the minimum and the maximum h-vector of these algebras for any degree $d \geq 4$. In Section 4, we study if the weak Lefschetz Property holds for Artinian Gorenstein algebras associated to Perazzo hypersurfaces. We prove that, for $d \geq 5$, the algebras A_f whose h-vector is maximum always fail the weak Lefschetz Property (Theorem 4.1), while the algebras whose h-vector is minimum always have it (Theorem 4.3). In Section 5, we give a full classification of the Perazzo 3-folds of degree $d \geq 5$ whose associated Artinian Gorenstein algebra has minimum h-vector. This is done using the stratification of $\mathbb{P}^{d-1}$ via the Waring rank of forms of degree $d - 1$ in two variables. Finally we study the geometry of these 3-folds, in terms of the image and fibres of their polar and Gauss map.

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2. Lefschetz properties and Artinian Gorenstein algebras

In this section we fix notation, we recall the definition of weak/strong Lefschetz Property and we briefly discuss general facts on Artinian Gorenstein algebras needed in next sections.

Throughout this work K will be an algebraically closed field of characteristic zero. Given a standard graded Artinian K-algebra $A = R/I$ where $R = K[x_0, x_1, \dots, x_n]$ and I is a homogeneous ideal of R, we denote by $HF_A : \mathbb{Z} \longrightarrow \mathbb{Z}$ with $HF_A(j) = \dim_K [A]_j$ its Hilbert function. Since A is Artinian, its Hilbert function is captured in its *h-vector* $h = (h_0, h_1, \dots, h_e)$ where $h_i = HF_A(i) > 0$ and e is the last index with this property. The integer e is called the *socle degree of A*.

2.1. Lefschetz properties

Definition 2.1. Let $A = R/I$ be a graded Artinian K-algebra. We say that A has the *weak Lefschetz Property* (WLP, for short) if there is a linear form $L \in [A]_1$ such that, for all integers $i \geq 0$, the multiplication map

$$\times L : [A]_i \longrightarrow [A]_{i+1}$$

has maximal rank, i.e. it is injective or surjective. (We will often abuse notation and say that the ideal I has the WLP.) In this case, the linear form L is called a Lefschetz element of A. If for the general form $L \in [A]_1$ and for an integer j the map $\times L : [A]_{j-1} \longrightarrow [A]_j$ does not have maximal rank, we will say that the ideal I fails the WLP in degree j.

A has the *strong Lefschetz Property* (SLP, for short) if there is a linear form $L \in [A]_1$ such that, for all integers $i \geq 0$ and $k \geq 1$, the multiplication map

$$\times L^k : [A]_i \longrightarrow [A]_{i+k}$$

has maximal rank. Such an element L is called a strong Lefschetz element for A.

A has the *strong Lefschetz Property in the narrow sense* if there exists an element $L \in [A]_1$ such that the multiplication map

$$\times L^{e-2i} : [A]_i \longrightarrow [A]_{c-i}$$

is bijective for $i = 0, \cdots, [e/2]$ being e the socle degree of A.

At first glance the problem of determining whether an Artinian standard graded K-algebra A has the WLP seems a simple problem of linear algebra, but instead it has proven to be extremely elusive. Part of the great interest in the WLP stems from the ubiquity of its presence and there are a long series of papers determining classes of Artinian algebras holding/failing the WLP but much more work remains to be done (see, for instance, [7] and [22]). The first result in this direction is due to Stanley [27] and Watanabe [28] and it asserts that the WLP holds for an Artinian complete intersection ideal generated by powers of linear forms.

Example 2.2. (1) The ideal $I = (x_1^3, x_1^3, x_2^3, x_1x_2x_3) \subset K[x_1, x_2, x_3]$ fails to have the WLP, because for any linear form $L = ax_1 + bx_2 + cx_3$ the multiplication map

$$\times L : [k[x_1, x_2, x_3]/I]_2 \cong K^6 \longrightarrow [k[x_1, x_2, x_3]/I]_3 \cong K^6$$

is neither injective nor surjective. More details on this example can be found in [4, Example 3.1].

(2) The ideal $I = (x_1^3, x_2^3, x_3^3, x_1^2x_2) \subset K[x_1, x_2, x_3]$ has the WLP. Since the h-vector of R/I is (1,3,6,6,4,1), we only need to check that the map $\times L : [R/I]_i \longrightarrow [R/I]_{i+1}$ induced by $L = x_1 + x_2 + x_3$ is surjective for $i = 2, 3, 4$. This is equivalent to check that $[R/(I, L)]_i = 0$ for $i = 3, 4, 5$. Obviously, it is enough to check the case $i = 3$. We have

$$\begin{aligned}
[R/(I, L)]_3 &\cong [K[x_1, x_2, x_3]/(x_1^3, x_2^3, x_3^3, x_1^2x_2, x_1 + x_2 + x_3)]_3 \\
&\cong [K[x_1, x_2]/(x_1^3, x_2^3, x_1^3 + 3x_1^2x_2 + 3x_1x_2^2 + x_2^3, x_1^2x_2)]_3 \\
&\cong [k[x_1, x_2]/(x_1^3, x_2^3, x_1^2x_2, x_1x_2^2)]_3 = 0
\end{aligned}$$

which proves what we want.

It is worthwhile to point out that the weak Lefschetz Property implies the unimodality of the Hilbert function. If a graded Artinian K-algebra A has the SLP in the narrow sense, then the Hilbert function of A is unimodal and symmetric. Finally, if a graded Artinian K-algebra A has a symmetric Hilbert function, the notion of the SLP on A coincides with the one in the narrow sense. In this work, we will deal with Artinian Gorenstein algebras A. It is well known that A has symmetric Hilbert function. So, in the subsequent sections, the strong Lefschetz Property will be used in the narrow sense.

2.2. Artinian Gorenstein ideals

In this subsection, we will characterize the Lefschetz elements for graded Artinian Gorenstein algebras A. Given $R = K[x_0, \cdots, x_n]$, we denote by $S = K[y_0, \cdots, y_n]$ the ring of differential operators on R, i.e., $y_i = \frac{\partial}{\partial x_i}$. For any homogeneous polynomial $f \in R_d$, we define

$$\operatorname{Ann}_S(f) := \{p \in S \mid p(f) = 0\} \subset S.$$

It is well known that $A = S/\operatorname{Ann}_S(f)$ is a standard graded Artinian Gorenstein K-algebra. Conversely, the theory of inverse systems developed by Macaulay gives the following characterization of standard graded Artinian Gorenstein K-algebras.

Proposition 2.3. *Set $R = K[x_0, \cdots, x_n]$ and let $S = K[y_0, \cdots, y_n]$ be the ring of differential operators on R. Let $A = S/I$ be a standard Artinian graded K-algebra. Then, A is Gorenstein if and only if there is $f \in R$ such that $A \cong S/Ann_S(f)$. Moreover, isomorphic Gorenstein algebras are defined by forms equal up to a linear change of variables in R.*

Under the hypothesis of the above proposition we have that the degree of f coincides with the socle degree of A.

Definition 2.4. Let $f \in K[x_0, \cdots, x_n]$ be a homogeneous polynomial and let $A = S/\operatorname{Ann}_S(f)$ be the associated Artinian Gorenstein algebra. Fix $\mathcal{B} = \{w_j \mid 1 \le j \le h_t := \dim A_t\} \subset A_t$ be an ordered K-basis. The t-th (relative) *Hessian matrix* of f with respect to $\mathcal{B}$ is defined as the $h_t \times h_t$ matrix:

$$\mathrm{Hess}_f^t = (w_i w_j(f))_{i,j}.$$

The t-th *Hessian of f with respect to $\mathcal{B}$* is

$$\mathrm{hess}_f^t = \det(\mathrm{Hess}_f^t).$$

The 0-th Hessian is just the polynomial f and, in the case $\dim A_1 = n + 1$, the 1st Hessian, with respect to the standard basis, is the classical Hessian. It is worthwhile to point out that the definition of Hessians and Hessian matrices of order t depends on the choice of a basis of A_t but the vanishing of the t-th Hessian is independent of this choice.

We end this preliminary section with a result due to Watanabe which establishes a useful link between the failure of Lefschetz properties and the vanishing of higher order Hessians.

Theorem 2.5. *Let $f \in K[x_0, \cdots, x_n]$ be a homogeneous polynomial of degree d and let $A = S/\operatorname{Ann}_S(f)$ be the associated Artinian Gorenstein algebra. $L = a_0y_0 + \cdots + a_ny_n \in A_1$ is a strong Lefschetz element of A if and only if $\mathrm{hess}_f^t(a_0, \cdots, a_n) \neq 0$ for $t = 1, \cdots, [d/2]$. More precisely, up to a multiplicative constant, $\mathrm{hess}_f^t(a_0, \cdots, a_n)$ is the determinant of the dual of the multiplication map $\times L^{d-2t} : [A]_t \longrightarrow [A]_{d-t}$.*

Proof. See [29, Theorem 4] and [21, Theorem 3.1]. □

Example 2.6. To illustrate Watanabe's theorem, we consider Ikeda's example of an Artinian Gorenstein algebra of codimension 4 failing WLP (see [18, Example 4.4]). We take

$$f = x_0x_2^3x_3 + x_1x_2x_3^3 + x_0^3x_1^2 \in K[x_0, x_1, x_2, x_3].$$

Let $S = K[y_0, y_1, y_2, y_3]$ be the ring of differential operators on R. We compute $\operatorname{Ann}_S(f)$ and we get:

$$\mathrm{Ann}_S(f) = \langle y_0y_3^2, y_1^2y_3, y_0y_1y_3, y_0^2y_3, y_1y_2^2, y_0y_2^2 - y_1y_3^2, y_1^2y_2, y_0y_1y_2, y_0^2y_2, y_1^3, y_3^4, y_2^2y_3^2,$$
$$y_2^4, y_0^2y_1^2 - 2y_2^3y_3, y_0^3y_1 - 2y_2y_3^3, y_0^4\rangle.$$

The h-vector of $A = S/\mathrm{Ann}_S(f)$ is: (1, 4, 10, 10, 4, 1). We will apply the above criterion to check that A fails the WLP in degree 3. To this end, we consider a K-basis of $[A]_2$:

$$\{y_0^2, y_1^2, y_2^2, y_3^2, y_0y_1, y_0y_2, y_0y_3, y_1y_2, y_1y_3, y_2y_3\}.$$

We get

$$\mathrm{Hess}_f^2 = \begin{pmatrix} 0 & 12x_0 & 0 & 0 & 6x_1 & 0 & 0 & 0 & 0 & 0 \\ 12x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6x_3 & 6x_2 & 0 & 0 & 6x_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6x_3 & 6x_2 & 6x_1 \\ 6x_1 & 0 & 0 & 0 & 6x_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 6x_2 \\ 0 & 0 & 6x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6x_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6x_2 & 0 & 0 & 0 & 0 & 0 & 6x_3 \\ 0 & 0 & 6x_0 & 6x_1 & 0 & 6x_2 & 0 & 0 & 6x_3 & 0 \end{pmatrix}.$$

For any $(a_0, a_1, a_2, a_3) \in K^4$, we have $\mathrm{hess}_f^2(a_0, a_1, a_2, a_3) = 0$. So, for any $L \in [A]_1$, the multiplication map $\times L : [A]_2 \longrightarrow [A]_3$ has zero determinant. Therefore, it is not bijective and we conclude that A fails the WLP.

3. Perazzo 3-folds and the h-vector of the associated Gorenstein algebra

The goal of this section is to get upper and lower bounds for the h-vector of a standard graded Artinian Gorenstein algebra associated to a Perazzo 3-fold X in $\mathbb{P}^4$. So, let us start recalling its definition.

Definition 3.1. Fix $N \geq 4$. A *Perazzo* hypersurface $X \subset \mathbb{P}^N$ of degree d is a hypersurface defined by a form $f \in K[x_0, \cdots, x_n, u_1 \cdots, u_m]$ of the following type:

$$f = x_0p_0 + x_1p_1 + \cdots + x_np_n + g$$

where $n + m = N$, $n, m \geq 2$, $p_i \in K[u_1, \cdots, u_m]_{d-1}$ are algebraically dependent but linearly independent and $g \in K[u_1, \cdots, u_m]_d$.

It is worthwhile to point out that usually Perazzo hypersurfaces are assumed to be reduced and irreducible (see, for instance, [11, Definition 3.12]). We will insert these hypotheses if it is required.

Example 3.2. As a first example of Perazzo hypersurface we have the cubic 3-fold in $\mathbb{P}^4$ of equation:

$$f(x_0, x_1, x_2, u, v) = x_0u^2 + x_1uv + x_2v^2.$$

It is a cubic hypersurface with vanishing hessian but not a cone. So, it provides the first counterexample to Hesse's claim: any hypersuface $X \subset \mathbb{P}^N$ with vanishing hessian is a cone ([14] and [15]).

Hesse's claim, which is true for quadrics, was studied by Gordan and Noether in [12] for hypersurfaces of degree $d \geq 3$. They proved it is true for $N \leq 3$ but it is false for any $N \geq 4$. More precisely they gave a complete classification of the hypersurfaces with vanishing hessian for $N = 4$ and a series of examples of hypersurfaces with vanishing hessian not cones for any $N \geq 5$. Subsequently Perazzo in [25] described all cubic hypersurfaces with vanishing hessian for $N = 4, 5, 6$. The results of Gordan-Noether and of Perazzo have been recently considered and rewritten in modern language by many authors [3], [6], [9], [19], [10], [30] and [32].

Remark 3.3. We recall that the hypersurface defined by a polynomial f has vanishing hessian if and only if the partial derivatives of f are algebraically dependent, and it is a cone if and only if they are linearly dependent. It follows that the Perazzo hypersurfaces, introduced in Definition 3.1, have all vanishing first hessian and in general are not cones.

In $\mathbb{P}^4$ the Gordan-Noether classification states that, for degree $d \leq 5$, the hypersurfaces not cones with vanishing hessian are all Perazzo hypersurfaces, while for degree $d > 5$, a form of degree d with vanishing hessian, not cone, is an element of $K[u, v][\Delta]$ where Δ is a Perazzo polynomial of the form $p_0x_0 + p_1x_1 + p_2x_2$ (see [12] and [32, Theorem 7.3]).

In [21] Maeno and Watanabe found a connection between the vanishing of higher order hessians and Lefschetz properties, in particular with the SLP; then Gondim in [11] studied the WLP for some hypersurfaces with vanishing hessian.

In this note, we will concentrate our attention on Perazzo 3-folds X in $\mathbb{P}^4$ and our first goal will be to determine the maximum and minimum h-vector for the Gorenstein Artinian algebras associated to them. We will use the following notations: $R = K[x_0, x_1, x_2, u, v]$ is the polynomial ring in 5 variables, $S = K[y_0, y_1, y_2, U, V]$ is the ring of differential operators on R, and a Perazzo 3-fold $X \subset \mathbb{P}^4$ of degree d is defined by a form

$$f = x_0p_0(u, v) + x_1p_1(u, v) + x_2p_2(u, v) + g(u, v) \in R_d. \tag{3.1}$$

If $d = 3$, the corresponding algebras have all the same h-vector, and precisely $(1, 5, 5, 1)$. In fact, by Remark 3.3, X not being a cone implies $h_1 = h_2 = 5$. So, from now on, we will assume that $d \geq 4$ and we write

$$\begin{aligned} p_0(u,v) &= \textstyle\sum_{i=0}^{d-1} \binom{d-1}{i} a_i u^{d-1-i} v^i, \\ p_1(u,v) &= \textstyle\sum_{i=0}^{d-1} \binom{d-1}{i} b_i u^{d-1-i} v^i, \\ p_2(u,v) &= \textstyle\sum_{i=0}^{d-1} \binom{d-1}{i} c_i u^{d-1-i} v^i, \text{ and} \\ g(u,v) &= \textstyle\sum_{i=0}^{d} \binom{d}{i} g_i u^{d-i} v^i. \end{aligned} \tag{3.2}$$

For any $2 \le k \le \lfloor \frac{d+1}{2} \rfloor$, we define the matrices:

$$\mathcal{A}_k := \begin{pmatrix} a_0 & a_1 & \cdots & a_{k-1} \\ a_1 & a_2 & \cdots & a_k \\ \vdots & \vdots & & \vdots \\ a_{d-k} & a_{d-k+1} & \cdots & a_{d-1} \end{pmatrix}, \quad \mathcal{B}_k := \begin{pmatrix} b_0 & b_1 & \cdots & b_{k-1} \\ b_1 & b_2 & \cdots & b_k \\ \vdots & \vdots & & \vdots \\ b_{d-k} & b_{d-k+1} & \cdots & b_{d-1} \end{pmatrix},$$

$$\mathcal{C}_k := \begin{pmatrix} c_0 & c_1 & \cdots & c_{k-1} \\ c_1 & c_2 & \cdots & c_k \\ \vdots & \vdots & & \vdots \\ c_{d-k} & c_{d-k+1} & \cdots & c_{d-1} \end{pmatrix}, \text{ and} \quad \mathcal{G}_k := \begin{pmatrix} g_0 & g_1 & \cdots & g_k \\ g_1 & g_2 & \cdots & g_{k+1} \\ \vdots & \vdots & & \vdots \\ g_{d-k} & g_{d-k+1} & \cdots & g_d \end{pmatrix}.$$

The matrices $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$ and $\mathcal{G}_k$ are the building blocks of the matrices M_k, N_k and N'_k that will play an important role in the proof of our main results. They are defined as follows:

$$M_k := (\mathcal{A}_k|\mathcal{B}_k|\mathcal{C}_k), \qquad N_k := \begin{pmatrix} \mathcal{A}_{k+1} \\ \hline \mathcal{B}_{k+1} \\ \hline \mathcal{C}_{k+1} \end{pmatrix} \text{ and } \quad N'_k := \begin{pmatrix} \mathcal{A}_{k+1} \\ \hline \mathcal{B}_{k+1} \\ \hline \mathcal{C}_{k+1} \\ \hline \mathcal{G}_k \end{pmatrix}.$$

Remark 3.4. (1) The matrices N_k and M_{k+1} contain the same 3 blocks of size $(d-k) \times (k+1)$.

(2) Since $M_k = N^t_{d-k}$, we have $\operatorname{rank} M_k = \operatorname{rank} N_{d-k}$.

(3) As we will see in the proof of Propositions 3.7 and 3.8, the h-vector of $S/\operatorname{Ann}_S(f)$ is minimal if and only if for all k, $2 \le k \le \lfloor \frac{d}{2} \rfloor$, $\operatorname{rank} M_k = \operatorname{rank} N'_k = 3$.

Proposition 3.5. *Let $f = x_0p_0(u,v)+x_1p_1(u,v)+x_2p_2(u,v)$ be a form of degree d defining a Perazzo 3-fold in $\mathbb{P}^4$. Let $h = (h_0, h_1, \dots, h_d)$ be its h-vector. Then $h_0 = h_d = 1, h_1 = h_{d-1} = 5$ and, for $2 \le i \le d-2$, $h_i = 4i + 1 - m_i - n_i$, where $m_i = 3i - \operatorname{rank} M_i$ and $n_i = i + 1 - \operatorname{rank} N_i$.*

Proof. Recall that the h-vector of an Artinian Gorenstein algebra is symmetric and, hence, we only have to compute h_i for $0 \le i \le \lfloor \frac{d}{2} \rfloor$. We have

$$h_i = \dim A_i = \dim S_i - \dim \operatorname{Ann}_S(f)_i = \binom{4+i}{i} - \dim \operatorname{Ann}_S(f)_i.$$

So, we have to compute $\dim \operatorname{Ann}_S(f)_i$ for any i, $0 \le i \le \lfloor \frac{d}{2} \rfloor$. Since $p_0(u,v)$, $p_1(u,v)$ and $p_2(u,v)$ are K-linearly independent, we have $\dim \operatorname{Ann}_S(f)_1 = 0$ and, hence, $h_1 = 5$.

We observe that, for any $i \geq 2$, $\mathrm{Ann}_S(f)_i$ contains $(y_0, y_1, y_2)^{i-k}(U, V)^k$, for $0 \leq k \leq i-2$. Therefore

$$\dim A_i \leq \binom{4+i}{i} - \sum_{k=0}^{i-2}(k+1)\binom{i-k+2}{2} = 4i+1.$$

We have to compute the numbers

$$m_i = \dim(\mathrm{Ann}_S(f)_i \cap (y_0, y_1, y_2)(U, V)^{i-1}),$$
$$n_i = \dim(\mathrm{Ann}_S(f)_i \cap (U, V)^i),$$

and we will get

$$\dim A_i = 4i + 1 - m_i - n_i. \tag{3.3}$$

This can be done because there are no linear dependence relations between the two parts, given the bi-homogeneous nature of f with respect to the two groups of variables x_0, x_1, x_2 and u, v.

To compute m_i we consider a general polynomial of degree i in $(y_0, y_1, y_2)(U, V)^{i-1}$:

$$(\alpha_0 U^{i-1} + \alpha_1 U^{i-2}V + \cdots + \alpha_{i-1}V^{i-1})y_0 + (\beta_0 U^{i-1} + \cdots + \beta_{i-1}V^{i-1})y_1 + (\gamma_0 U^{i-1} + \cdots)y_2.$$

It belongs to $\mathrm{Ann}_S(f)_i$ if and only if

$$\alpha_0 p_{0,u^{i-1}} + \alpha_1 p_{0,u^{i-2}v} + \cdots + \alpha_{i-1}p_{0,v^{i-1}} + \beta_0 p_{1,u^{i-1}} + \cdots + \gamma_0 p_{2,u^{i-1}} + \cdots + \gamma_{i-1}p_{2,v^i} = 0.$$

The partial derivatives of p_0, p_1, p_2 appearing in the above expression have degree $d-i$; setting equal to zero the coefficients of the $d-i+1$ monomials in u, v, we get a homogeneous linear system of $d-i+1$ equations in the $3i$ unknowns $\alpha_0, \ldots, \alpha_{i-1}, \beta_0, \ldots, \beta_{i-1}, \gamma_0, \ldots, \gamma_{i-1}$. The matrix of the coefficients is M_i, therefore $m_i = 3i - \mathrm{rank}\, M_i$, and we are done.

To compute n_i we consider a general polynomial of degree i in U, V:

$$\delta_0 U^i + \delta_1 U^{i-1}V + \cdots + \delta_i V^i$$

and we impose that it belongs to $\mathrm{Ann}_S(f)_i$. We get

$$(\delta_0 p_{0,u^i} + \delta_1 p_{0,u^{i-1}v} + \cdots + \delta_i p_{0,v^i})x_0 + (\delta_0 p_{1,u^i} + \cdots)x_1 + (\delta_0 p_{2,u^i} + \cdots)x_2 = 0.$$

Looking at the coefficients of x_0, x_1, x_2 and then the coefficients of the monomials in u, v of degree $d-i-1$ and $d-i$, we get a homogeneous linear system of $3(d-i)+(d-i+1)$ equations in $i+1$ unknowns, whose matrix of the coefficients is N_i. We conclude that $n_i = i + 1 - \mathrm{rank}\, N_i$. The proof is complete. □

Remark 3.6. We observe that the expression for h_i can also be written in the form $h_i = \operatorname{rank} M_i + \operatorname{rank} N_i$. In fact, if we write a unique linear system to compute the dimension of the space $\operatorname{Ann}_S(f)_i \cap [(y_0, y_1, y_2)(U,V)^{i-1} + (U,V)^i]$, the matrix of this linear system results to be $\left(\begin{array}{c|c} 0 & N_i \\ \hline M_i & 0 \end{array}\right)$.

In the general case, when f is as in (3.1) with $g \neq 0$, equality (3.3) is not necessarily true, but only the inequality $\dim A_i \geq 4i + 1 - m_i - n_i$ holds true. An explicit example is provided by the form $f = x_0 u^9 + x_1 u^8 v + x_2 v^9 + u^5 v^5$.

On the other hand, in this more general situation the matrix associated to the linear system to be considered to compute h_i is $\left(\begin{array}{c|c} 0 & N_i \\ \hline M_i & \mathcal{G}_i \end{array}\right)$. This implies, for every index i, the series of inequalities

$$\operatorname{rank} M_i + \operatorname{rank} N_i \leq h_i \leq \operatorname{rank} M_i + \operatorname{rank} N_i'.$$

Clearly, every time $\operatorname{rank} N_i = \operatorname{rank} N_i'$, we obtain a relation as in Proposition 3.5. This is obviously the case when $g = 0$. It is also the case if one of the polynomials p_0, p_1, p_2 is general enough. Indeed, we observe that N_i has maximal rank if and only if its columns are linearly independent; so if the rank of one of the matrices $\mathcal{A}_{i+1}$, $\mathcal{B}_{i+1}$, $\mathcal{C}_{i+1}$ is computed by the number of columns then N_i has maximal rank. This happens if one of the polynomials p_0, p_1, p_2 is general enough in view of [16, Proposition 3.4].

Proposition 3.7. *Let $d \geq 4$. The maximum h-vector of the Artinian Gorenstein algebras $S/Ann_S(f)$ associated to the Perazzo 3-folds of degree d in $\mathbb{P}^4$ is:*

(1) *If $d = 4t - 1$ then*

$$h_i = \begin{cases} 4i + 1 \text{ for } 0 \leq i \leq t \\ 4t + 1 \text{ for } t + 1 \leq i \leq 2t - 1 \\ \text{symmetry;} \end{cases}$$

(2) *If $d = 4t$ then*

$$h_i = \begin{cases} 4i + 1 \text{ for } 0 \leq i \leq t \\ 4t + 2 \text{ for } t + 1 \leq i \leq 2t \\ \text{symmetry;} \end{cases}$$

(3) *If $d = 4t + 1$ then*

$$h_i = \begin{cases} 4i + 1 \text{ for } 0 \leq i \leq t \\ 4t + 3 \text{ for } t + 1 \leq i \leq 2t \\ \text{symmetry;} \end{cases}$$

(4) *If* $d = 4t + 2$ *then*

$$h_i = \begin{cases} 4i + 1 \text{ for } 0 \le i \le t \\ 4t + 4 \text{ for } t + 1 \le i \le 2t + 1 \\ \text{symmetry.} \end{cases}$$

Proof. Let f be a form of degree d as in (3.1) with p_0, p_1, p_2, g as in (3.2). Being the h-vector symmetric, we only have to compute h_i for $0 \le i \le \frac{d}{2}$.

In view of Proposition 3.5 and Remark 3.6, the maximal Hilbert function is obtained when m_i, n_i are minimal for any i, i.e. when the ranks of the matrices M_i, N_i' are as large as possible.

Clearly $\operatorname{rank} M_i \le \min\{3i, d - i + 1\}$. Therefore

$$\operatorname{rank} M_i \le \begin{cases} 3i \text{ for } i \le \frac{d+1}{4}; \\ d - i + 1 \text{ for } i \ge \frac{d+1}{4}. \end{cases}$$

Regarding N_i', we observe that, in our situation, $i + 1 \le 3(d - i) + (d - i + 1)$, so always $\operatorname{rank} N_i' \le i + 1$.

This gives upper bounds on h_i depending on the class of congruence of the degree d modulo 4, that are precisely those in the statement of this Proposition.

To conclude the proof, we claim that these bounds are achieved. To this end, we observe that, in view of the expressions (3.2), the columns of the matrices $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{G}_i$ contain up to a constant the coefficients of the partial derivatives of order $i - 1$ of p_0, p_1, p_2, g respectively. But, if p_0, p_1, p_2, g are general enough, then, by [16, Proposition 3.4], for any i their partial derivatives of order $i - 1$ are as linearly independent as possible in $K[u, v]_{d-1-i}$. This means that the ranks of the matrices $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{G}_i$ are as large as possible. This proves our claim. □

A class of explicit examples of polynomials such that the bound in Proposition 3.7 is attained can be found in [8, Example 3.20].

To determine the minimum h-vector for the Gorenstein Artinian algebra associated to a Perazzo 3-fold X in $\mathbb{P}^4$ we need first to recall some results about the growth of the Hilbert function of standard graded K-algebras and to fix some additional notation.

Given integers $n, d \ge 1$, we define the *d-th binomial expansion of* n as

$$n = \binom{\eta_d}{d} + \binom{\eta_{d-1}}{d-1} + \cdots + \binom{\eta_e}{e}$$

where $\eta_d > \eta_{d-1} > \cdots > \eta_e \ge e \ge 1$ are uniquely determined integers (see [5, Lemma 4.2.6]). We write

$$n^{<d>} = \binom{\eta_d + 1}{d + 1} + \binom{\eta_{d-1} + 1}{d} + \cdots + \binom{\eta_e + 1}{e + 1}, \text{ and}$$

$$n_{<d>} = \binom{\eta_d - 1}{d} + \binom{\eta_{d-1} - 1}{d-1} + \cdots + \binom{\eta_e - 1}{e}.$$

The numerical functions $H : \mathbb{N} \longrightarrow \mathbb{N}$ that are Hilbert functions of graded standard K-algebras were characterized by Macaulay, [5]. Indeed, given a numerical function $H : \mathbb{N} \longrightarrow \mathbb{N}$ the following conditions are equivalent:

(i) There exists a standard graded K-algebra A with Hilbert function H,
(ii) $H(0) = 1$ and $H(t+1) \leq H(t)^{<t>}$ for all $t \geq 1$.

Notice that condition (ii) imposes strong restrictions on the Hilbert function of a standard graded K-algebra and, in particular, it bounds its growth. As an application of Macaulay's theorem, we have:

Proposition 3.8. *Let $d \geq 4$. Let $R = K[x_0, x_1, x_2, u, v]$ and $S = K[y_0, y_1, y_2, U, V]$ be the ring of differential operators on R. The minimum h-vector of the Artinian Gorenstein algebras $A = S/Ann_S(f)$ associated to the Perazzo 3-folds of degree d in $\mathbb{P}^4$ is:*

$$(1,\ 5,\ 6,\ 6,\ \cdots\ 6,\ 6,\ 5,\ 1).$$

Proof. The proof proceeds as follows: we first prove that the cited t-uple is less than any possible h-vector associated to a Perazzo 3-fold, with respect to the termwise order; then, we give examples of Perazzo forms that have this h-vector. Let

$$h_A = (h_0,\ h_1,\ h_2,\ h_3,\ \cdots\ h_{d-2},\ h_{d-1},\ h_d)$$

be the h-vector of A. First of all we observe that, arguing as in the proof of Proposition 3.7, we get that $6 \leq h_2 \leq 9$ which, together with the fact that the h-vector of any standard graded Artinian Gorenstein algebra is symmetric, gives us that a lower bound for h_A looks like

$$(1,\ 5,\ 6,\ h_3,\ \cdots\ h_{d-2},\ 6,\ 5,\ 1).$$

This concludes the first step for $d \leq 5$. We will now prove that if $d \geq 6$, for any i, $3 \leq i \leq d-3$, $h_i \geq 6$.

First we assume $d \geq 8$. If $h_j \leq 5$ for some $5 \leq j \leq d-3$, using Macaulay's inequality $h_{t+1} \leq h_t^{<t>}$ for all $t \geq 1$, we get that $h_i \leq 5$ for all $i \geq j$ contradicting the fact that $h_{d-2} = 6$. Therefore, $h_j \geq 6$ for all $5 \leq j \leq d-2$ and, by symmetry, we also have $h_3, h_4 \geq 6$.

For $d = 6, 7$, we must show that $h_3 \geq 6$. This last equality follows after a straightforward computation which shows that $(y_0, y_1, y_2)^3 \oplus (y_0, y_1, y_2)^2(U, V) \subset \operatorname{Ann}_S(f)_3$, $\dim\{(\alpha_0 U^2 + \alpha_1 UV + \alpha_2 V^2)y_0 + (\beta_0 U^2 + \beta_1 UV + \beta_2 V^2)y_1 + (\gamma_0 U^2 + \gamma_1 UV + \gamma_2 V^2)y_2 \in \operatorname{Ann}_S(f)_3\} \leq 6$ and $\dim\{\delta_0 U^3 + \delta_1 U^2 V + \delta_2 UV^2 + \delta_3 V^3 \in \operatorname{Ann}_S(f)_3\} \leq 1$. Therefore, $h_3 = \dim S_3/\operatorname{Ann}_S(f)_3 \geq \binom{7}{4} - 29 = 6$.

Summarizing, we have got that for any $d \geq 4$ and for $2 \leq i \leq d-2$, it holds $h_i \geq 6$. To finish the proof it suffices to give an example with $h_i = 6$, for any i, $2 \leq i \leq d-2$. We take the homogeneous polynomial of degree d:

$$f(x_0, x_1, x_2, u, v) = u^d x_0 + u^{d-1} v x_1 + v^d x_2.$$

It is easy to check that it has the desired h-vector. □

Remark 3.9. From Proposition 3.5 it follows that $A = S/\operatorname{Ann}_S(f)$ has minimum h-vector $(1, 5, 6, 6, \ldots, 6, 6, 5, 1)$ if and only if $\operatorname{rank} M_i = \operatorname{rank} N_i' = 3$ for any i with $2 \leq i \leq \lfloor \frac{d}{2} \rfloor$. We note that none of these ranks can be strictly less than 3 due to the assumption that p_0, p_1, p_2 are linearly independent.

Remark 3.10. From Propositions 3.7 and 3.8, it follows that for $d = 4$ the unique possible h-vector is $(1, 5, 6, 5, 1)$. Instead, for $d = 5$, we can obtain only the maximal h-vector $(1, 5, 7, 7, 5, 1)$, and the minimal h-vector $(1, 5, 6, 6, 5, 1)$. For bigger values of d, also some intermediate cases are a priori possible.

4. Perazzo 3-folds and the WLP

From Theorem 2.5 and Remark 3.3, since the Perazzo 3-folds have vanishing first hessian, it follows that the associated algebras A fail the strong Lefschetz Property. In particular the map

$$\times L^{d-2} : [A]_1 \longrightarrow [A]_{d-1}$$

is not an isomorphism for every $L \in [A]_1$. The goal of this section is to analyze whether the Artinian Gorenstein algebra A associated to a Perazzo 3-fold $X \subset \mathbb{P}^4$ has the WLP. If $d = 3$, clearly A fails also WLP. But Gondim has proved that, for any Perazzo 3-fold of degree 4, A has the WLP ([11], Theorem 3.5). More precisely, we will see that, in any degree $d \geq 5$, WLP holds when A has minimum h-vector and fails when it has maximum h-vector.

Theorem 4.1. *Let $X \subset \mathbb{P}^4$ be a Perazzo 3-fold of degree $d \geq 5$ and equation*

$$f = x_0 p_0(u, v) + x_1 p_1(u, v) + x_2 p_2(u, v) + g(u, v) \in R_d = K[x_0, x_1, x_2, u, v]_d.$$

Let $S = K[y_0, y_1, y_2, U, V]$ be the ring of differential operators on R. If $A = S/\operatorname{Ann}_S(f)$ has maximum h-vector, then A fails WLP.

Proof. According to the parity of the socle degree of A, we distinguish two cases.
Case 1: d is odd. Write $d = 2r + 1$. To show that A fails WLP, we will prove that for any $L \in [A]_1$, the multiplication map

$$\times L : [A]_r \longrightarrow [A]_{r+1}$$

is not bijective. By Theorem 2.5, it is enough to see the vanishing of the r-th Hessian hess_f^r of $f = x_0p_0(u,v) + x_1p_1(u,v) + x_2p_2(u,v) + g(u,v)$ with respect to a suitable K-basis $\mathcal{B}$ of $[A]_r$. First we can notice that a basis $\mathcal{B}$ made of classes with a monomial representative always exists. So, Hess_f^r is just a submatrix of dimension $h_r \times h_r$ of the following matrix:

$$\Big(\frac{\partial^{2r} f}{\partial u^\alpha \partial v^\beta \partial x_0^\gamma \partial x_1^\delta \partial x_2^\eta}\Big)_{\alpha+\beta+\gamma+\delta+\eta=2r}$$

where monomials are lexicographic ordered (for simplicity). Knowing that f is linear in the variables x_0, x_1, x_2, the above matrix can be partially computed as:

$$\left(\begin{array}{cccccccccc}
\frac{\partial^{2r} f}{\partial u^{2r}} & \frac{\partial^{2r} f}{\partial u^{2r-1}\partial v} & \cdots & \frac{\partial^{2r} f}{\partial u^{r}\partial v^{r}} & \frac{\partial^{2r-1} p_0}{\partial u^{2r-1}} & \frac{\partial^{2r-1} p_0}{\partial u^{2r-2}\partial v} & \cdots & \frac{\partial^{2r-1} p_2}{\partial u^{r}\partial v^{r-1}} & 0 & \cdots & 0 \\
\frac{\partial^{2r} f}{\partial u^{2r-1}\partial v} & \frac{\partial^{2r} f}{\partial u^{2r-2}\partial v^2} & \cdots & \frac{\partial^{2r} f}{\partial u^{r-1}\partial v^{r+1}} & \frac{\partial^{2r-1} p_0}{\partial u^{2r-2}\partial v} & \frac{\partial^{2r-1} p_0}{\partial u^{2r-3}\partial v^2} & \cdots & \frac{\partial^{2r-1} p_3}{\partial u^{r-1}\partial v^{r}} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^{2r} f}{\partial u^{r}\partial v^{r}} & \frac{\partial^{2r} f}{\partial u^{r-1}\partial v^{r+1}} & \cdots & \frac{\partial^{2r} f}{\partial u^{r-1}\partial v^{2r}} & \frac{\partial^{2r-1} p_0}{\partial u^{r-1}\partial v^{r}} & \frac{\partial^{2r-1} p_0}{\partial u^{r-2}\partial v^{r+1}} & \cdots & \frac{\partial^{2r-1} p_2}{\partial u^{2r-1}} & 0 & \cdots & 0 \\
\hline
\frac{\partial^{2r-1} p_0}{\partial u^{r-1}\partial v^{r}} & \cdots & \cdots & \frac{\partial^{2r-1} p_0}{\partial u^{r-1}\partial v^{r}} & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^{2r-1} p_2}{\partial u^{r}\partial v^{r-1}} & \cdots & \cdots & \frac{\partial^{2r-1} p_2}{\partial u^{2r-1}} & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\
\hline
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0
\end{array}\right)$$

The three vertical (respectively, horizontal) blocks are composed respectively by $r+1$, $3r$, $\binom{r+4}{4} - (4r+1)$ columns (respectively, rows). Thus every possible choice of a $h_r \times h_r$ submatrix turns out to have at least one all zero sub-submatrix of size $(h_r - (r+1)) \times (h_r - (r+1))$. We now use the hypothesis of A to have maximum h-vector and Proposition 3.7 to obtain that $h_r = 2r+3$. We have just proved that Hess_f^r, matrix of dimension $(2r+3) \times (2r+3)$, has one all zero submatrix of dimension $(r+2) \times (r+2)$: this implies that hess_f^r identically vanishes.

Case 2: d is even. Write $d = 2r+2$. Note that, since the h-vector is maximum, then $h_r = h_{r+1} = h_{r+2}$. Using again the hessian criterion of Watanabe's (Theorem 2.5), we will check that for any $L \in [A]_1$, the multiplication map

$$\times L^2 : [A]_r \longrightarrow [A]_{r+2}$$

is not bijective. This implies that for any $L \in [A]_1$, the multiplication map

$$\times L : [A]_r \longrightarrow [A]_{r+1}$$

is not bijective and, hence, A fails the WLP.

Same adapted argument of the previous case can be used also here. In fact, the matrix to be considered is Hess_f^r which is now of size $(2r+4) \times (2r+4)$ which is even bigger than the previous case. Thus, as discussed above, its determinant is always zero. □

In contrast with the last result we have that if an Artinian Gorenstein algebra A associated to a Perazzo 3-fold has minimum h-vector, then A has the WLP. Our proof uses Green's theorem that we recall for sake of completeness.

Theorem 4.2. *Let $A = R/I$ be an Artinian graded algebra and let $L \in A_1$ be a general linear form. Let h_t be the entry of degree t of the h-vector of A. Then the degree t entry h'_t of the h-vector of $R/(I, L)$ satisfies the inequality:*

$$h'_t \leq (h_t)_{<t>} \text{ for all } t \geq 1.$$

Proof. See [13, Theorem 1]. □

Theorem 4.3. *Let $X \subset \mathbb{P}^4$ be a Perazzo 3-fold of degree $d \geq 5$ and equation*

$$f = x_0 p_0(u, v) + x_1 p_1(u, v) + x_2 p_2(u, v) + g(u, v) \in R = K[x_0, x_1, x_2, u, v]_d.$$

Let $S = K[y_0, y_1, y_2, U, V]$ be the ring of differential operators on R. If $A = S/\operatorname{Ann}_S(f)$ has minimum h-vector, then A has WLP.

Proof. For $5 \leq d \leq 7$ see next section where a full classification of Perazzo 3-folds with minimal h-vector is given. Assume $d \geq 8$. By the minimality assumption, $h_2 = h_3 = \cdots = h_{d-2} = 6$. By [23, Proposition 2.1], if for a general linear form $L \in [A]_1$, the multiplication map

$$\times L : [A]_2 \longrightarrow [A]_3$$

is bijective, then

$$\times L : [A]_1 \longrightarrow [A]_2$$

is injective, and for all $j \geq 2$,

$$\times L : [A]_j \longrightarrow [A]_{j+1}$$

is surjective, therefore A has the WLP. By the symmetry property of Artinian Gorenstein algebras,

$$\times L : [A]_2 \longrightarrow [A]_3$$

is bijective if and only if

$$\times L : [A]_{d-3} \longrightarrow [A]_{d-2}$$

is bijective. So, let us prove the bijection of this last map. To this end, for a general linear form $L \in [A]_1$, we consider the exact sequence:

$$[A]_{d-3} \longrightarrow [A]_{d-2} \longrightarrow [S/(\mathrm{Ann}_S(f), L)]_{d-2} \longrightarrow 0.$$

It follows that $\times L : [A]_{d-3} \longrightarrow [A]_{d-2}$ is bijective if and only if $[S/(\mathrm{Ann}_S(f), L)]_{d-2} = 0$. Using the hypothesis $d-2 \geq 6$ (and, hence, $h_{d-2} \leq d-2$) and Theorem 4.2 we get

$$\dim[S/(\mathrm{Ann}_S(f), L)]_{d-2} \leq (h_{d-2})_{<d-2>} = 0$$

which proves what we want. □

Remark 4.4. As a consequence of Theorem 4.3, all forms of degree d which define a Perazzo 3-fold with minimum h-vector are examples of forms with zero first order hessian, and all hessians of order t different from zero, for $2 \leq t \leq \lfloor \frac{d}{2} \rfloor$.

For Gorenstein Artinian algebras associated to Perazzo 3-folds X in $\mathbb{P}^4$ and with intermediate h-vector both possibilities occur: there are examples failing WLP and examples satisfying WLP as next example shows.

Example 4.5. 1.- Let $X \subset \mathbb{P}^4$ be the Perazzo 3-fold of equation

$$f(x_0, x_1, x_2, u, v) = u^6 x_0 + (u^2 v^4 + u^4 v^2) x_1 + v^6 x_2 \in K[x_0, x_1, x_2, u, v]_7.$$

Let $S = K[y_0, y_1, y_2, U, V]$ be the ring of differential operators on R. We have

$$\mathrm{Ann}_S(f) = \langle y_0^2, y_1^2, y_2^2, y_0y_1, y_0y_2, y_1y_2, y_0V, y_2U, y_0U^2 + 15y_1U^2 - 15y_1V^2 - y_2V^2, \\ U^3V - UV^3, 15y_1U^4 - y_2V^4, UV^5, V^7, U^7\rangle.$$

Therefore, the Artinian Gorenstein algebra $A = S/Ann_S(f)$ has h-vector: (1, 5, 7, 8, 8, 7, 5, 1). Using Macaulay2 [20] we check that for a general linear form $L \in [A]_1$, the multiplication map

$$\times L : [A]_3 \longrightarrow [A]_4$$

is bijective and, hence, A satisfies the WLP. It does not have the SLP because for any linear form $L \in [A]_1$

$$\times L^3 : [A]_2 \longrightarrow [A]_5$$

is not surjective.

2.- Let $X \subset \mathbb{P}^4$ be the Perazzo 3-fold of equation

$$f(x_0, x_1, x_2, u, v) = u^6x_0 + u^3v^3x_1 + v^6x_2 \in K[x_0, x_1, x_2, u, v]_7.$$

Let $S = K[y_0, y_1, y_2, U, V]$ be the ring of differential operators on R. We have

$$\operatorname{Ann}_S(f) = \langle y_0^2, y_1^2, y_2^2, y_0y_1, y_0y_2, y_1y_2, y_0v, y_2u, 20y_1U^3 - y_2V^3, \\ y_0U^3 - 20y_1V^3, UV^4, U^4V, V^7, U^7\rangle.$$

Therefore, the Artinian Gorenstein algebra $A = S/Ann_S(f)$ has h-vector: (1, 5, 7, 9, 9, 7, 5, 1). Computing the third hessian, since it results to be zero, we get that for any linear form $L \in [A]_1$, the multiplication map

$$\times L : [A]_3 \longrightarrow [A]_4$$

is not bijective and, hence, A fails the WLP.

5. On the classification of certain Perazzo 3-folds of degree at least 5

The goal of this section is to classify all Perazzo 3-folds X in $\mathbb{P}^4$ of degree $d \geq 5$ whose associated Artinian Gorenstein algebra $S/\operatorname{Ann}_S(f)$ has h-vector: $(1, 5, 6, 6, \cdots, 6, 6, 5, 1)$. As a corollary we will also classify all Perazzo 3-folds X in $\mathbb{P}^4$ of degree 5 whose associated Artinian Gorenstein algebra $S/\operatorname{Ann}_S(f)$ has the WLP.

We start the section with some technical lemmas and remarks.

Lemma 5.1. *Let $f_1 = p_0(u,v)x_0+p_1(u,v)x_1+p_2(u,v)x_2$ and $f_2 = q_0(u,v)x_0+q_1(u,v)x_1+q_2(u,v)x_2$ be two Perazzo 3-folds of degree d in $\mathbb{P}^4$ such that $\langle p_0, p_1, p_2\rangle = \langle q_0, q_1, q_2\rangle \subset K[u,v]_{d-1}$. Then, the h-vectors of $S/\operatorname{Ann}_S(f_1)$ and $S/\operatorname{Ann}_S(f_2)$ coincide.*

Proof. By [17, Proposition A7] it is enough to prove that f_1 and f_2 define projectively equivalent 3-folds in $\mathbb{P}^4$. Write $q_0(u,v) = \lambda_0p_0(u,v) + \lambda_1p_1(u,v) + \lambda_2p_2(u,v)$, $q_1 = \mu_0p_0(u,v) + \mu_1p_1(u,v) + \mu_2p_2(u,v)$, $q_3 = \rho_0p_0(u,v) + \rho_1p_1(u,v) + \rho_2p_2(u,v)$. We have

$$\begin{aligned} f_2 &= q_0(u,v)x_0 + q_1(u,v)x_1 + q_2(u,v)x_2 \\ &= (\lambda_0p_0(u,v) + \lambda_1p_1(u,v) + \lambda_2p_2(u,v))x_0 + (\mu_0p_0(u,v) + \mu_1p_1(u,v)+ \\ &\quad + \mu_2p_2(u,v))x_1 + (\rho_0p_0(u,v) + \rho_1p_1(u,v) + \rho_2p_2(u,v))x_2 \\ &= (\lambda_0x_0 + \mu_0x_1 + \rho_0x_2)p_0(u,v) + (\lambda_1x_0 + \mu_1x_1 + \rho_1x_2)p_1(u,v)+ \\ &\quad + (\lambda_2x_0 + \mu_2x_1 + \rho_2x_2)p_2(u,v). \end{aligned}$$

Therefore, f_1 and f_2 define projectively equivalent hypersurfaces in $\mathbb{P}^4$. $\square$

Remark 5.2. We fix integers $d \geq 5$ and $2 \leq k \leq \lfloor \frac{d}{2} \rfloor$. We keep the notations introduced in Section 3. If $\operatorname{rank} M_k = 3$, then $\operatorname{rank} \mathcal{A}_k \leq 3$, $\operatorname{rank} \mathcal{B}_k \leq 3$ and $\operatorname{rank} \mathcal{C}_k \leq 3$.

We will now explain the geometrical meaning of the rank of the matrices $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k$ introduced in Section 3. To this end, we recall some basic facts about symmetric tensors in two variables. For more details see [17], [24] and [2].

Let us fix an integer $t \geq 3$ and consider the vector space $K[u, v]_t$ of forms of degree t. Its elements can also be interpreted as symmetric tensors in two variables; by definition the Waring rank, or symmetric rank, of $p \in K[u, v]_t$ is the minimum integer r such that there exist linear forms $l_1, \dots, l_r \in K[u, v]_1$ such that $p = l_1^t + \cdots + l_r^t$. In particular, a symmetric tensor p has Waring rank 1 if $p = l^t$ for a suitable linear form l, i.e. p is a pure power of degree t.

In the projective space $\mathbb{P}^t$, naturally identified with $\mathbb{P}(K[u, v]_t)$, the set of (equivalence classes of) forms of Waring rank 1 is the image of the t-tuple Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^t$, that is the rational normal curve C_t of degree t. We recall that, for any $r \geq 1$, the r-secant variety of C_t is

$$\sigma_r(C_t) = \overline{\cup_{p_1, \dots, p_r \in C_t} \langle p_1, \dots, p_r \rangle}.$$

Clearly $C_t = \sigma_1(C_t) \subset \sigma_2(C_t) \subset \cdots$, and a general element of $\sigma_r(C_t) \setminus \sigma_{r-1}(C_t)$ is a symmetric tensor of Waring rank r, but if $r > 1$ $\sigma_r(C_t)$ contains also tensors of Waring rank $> r$. The dimension of $\sigma_r(C_t)$ is $\min\{2r - 1, t\}$. Moreover, for any $r < \frac{t+1}{2}$, $\sigma_{r-1}(C_t)$ is the singular locus of $\sigma_r(C_t)$ (see [26, Proposition 1.2.2 and Corollary 1.2.3]).

We recall also that the tangential surface of C_t, TC_t, is the closure of the union of the embedded tangent lines to C_t. The tangent line at the point $l_1^t \in C_t$ is the set of tensors that can be written in the form $l_1^{t-1} l_2$, with l_2 a linear form. Similarly the osculating 3-fold of C_t, $T^2 C_t$, is the closure of the union of the embedded osculating planes to C_t, and the osculating plane at l_1^t is the set of tensors that can be written in the form $l_1^{t-2} m$, with m a form of degree 2. We are now ready to give the desired interpretation of the rank of the matrices introduced in Section 3. We state and prove Proposition 5.3 for the form p_0 and the matrices $\mathcal{A}_k$; the analogous results hold true also for p_1, p_2, and their respectively catalecticant matrices $\mathcal{B}_k$, $\mathcal{C}_k$.

Proposition 5.3. *We fix an integer $d \geq 5$ and we keep the notations introduced in Section 3. It holds:*

(1) *If $\operatorname{rank} \mathcal{A}_k = 1$ for some $2 \leq k \leq \lfloor \frac{d+1}{2} \rfloor$ (and, hence, for all k), then $p_0 = \ell^{d-1}$ for some $\ell \in K[u, v]_1$.*
(2) *If $\operatorname{rank} \mathcal{A}_k = 2$ for some $3 \leq k \leq \lfloor \frac{d+1}{2} \rfloor$ (and, hence, for all k), then either $p_0 = \ell_1^{d-1} + \ell_2^{d-1}$ or $p_0 = \ell_1^{d-2} \ell_2$ for some $\ell_1, \ell_2 \in K[u, v]_1$.*

(3) *If* $\operatorname{rank}\mathcal{A}_k = 3$ *for some* $4 \le k \le \lfloor \frac{d+1}{2} \rfloor$ *(and, hence, for all* k*), then either* $p_0 = \ell_1^{d-1} + \ell_2^{d-1} + (\lambda\ell_1 + \mu\ell_2)^{d-1}$ *or* $p_0 = \ell_1^{d-1} + \ell_2^{d-2}(\lambda\ell_1 + \mu\ell_2)$ *for some* $\ell_1, \ell_2 \in K[u, v]_1$ *and* $\lambda, \mu \in K^*$*.*

Proof. Let r be any integer such that $r + 1 \le k$. From [24, Theorem 1.3], it follows that all the minors of order $r + 1$ of $\mathcal{A}_k$ vanish if and only if $[p_0] \in \sigma_r(C_{d-1})$. For $r = 1$, this gives (1). For $r = 2$, we get that if $\mathcal{A}_k$ has rank 2, then $p_0 \in \sigma_2(C_{d-1})$. From [2, Corollary 26], it follows that either p_0 has Waring rank 2 or $p_0 \in TC_{d-1}$; this proves (2). Similarly, for $r = 3$, $\operatorname{rank}\mathcal{A}_k = 3$ implies that $p_0 \in \sigma_3(C_{d-1})$. So, either the Waring rank of p_0 is 3, or p_0 belongs to the join of C_{d-1} and its tangential surface ([2, Corollary 26]). This proves (3). □

Theorem 5.4. *The Artinian Gorenstein algebra* $S/\operatorname{Ann}_S(f)$ *associated to a Perazzo 3-fold of degree* $d \ge 5$ *has h-vector:* $(1,\ 5,\ 6,\ 6,\ \cdots\ 6,\ 6,\ 5,\ 1)$ *if and only if, after a possible change of coordinates, one of the following cases holds:*

(i) $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + u^{d-2}vx_1 + u^{d-3}v^2x_2 + au^d + bu^{d-1}v + cu^{d-2}v^2$ *with* $a, b, c \in K$*, or*

(ii) $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + u^{d-2}vx_1 + v^{d-1}x_2 + au^d + bu^{d-1}v + cv^d$ *with* $a, b, c \in K$*, or*

(iii) $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + (\lambda u + \mu v)^{d-1}x_1 + v^{d-1}x_2 + au^d + b(\lambda u + \mu v)^d + cv^d$ *with* $\lambda, \mu \in K^*$ *and* $a, b, c \in K$*.*

Proof. As observed in Remark 3.9, the h-vector is minimal if and only if $\operatorname{rank} M_k = \operatorname{rank} N'_k = 3$ for any k. A straightforward computation shows that for any f as in (i), (ii) or (iii) one has $\operatorname{rank} M_k = \operatorname{rank} N'_k = 3$ for any k and, therefore, $S/\operatorname{Ann}_S(f)$ has h-vector $(1,\ 5,\ 6,\ \cdots\ 6,\ 5,\ 1)$. To prove the converse, we first observe that if $\operatorname{rank} M_k = \operatorname{rank} N'_k = 3$ for any k, then the ranks of $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{G}_k$ are all bounded above by 3. We analyze first the various possibilities for p_0, p_1, p_2.

(I) $d \ge 7$ and $\operatorname{rank}\mathcal{A}_k = \operatorname{rank}\mathcal{B}_k = \operatorname{rank}\mathcal{C}_k = 3$ for $4 \le k \le \lfloor \frac{d+1}{2} \rfloor$ and p_0, p_1, p_2 all have Waring rank 3. We use [24, Corollary 1.2]: the spaces of the columns of $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k$ coincide, so there exist linear forms l_1, l_2, l_3 and suitable constants such that

$$p_0 = \lambda_0 l_1^{d-1} + \mu_0 l_2^{d-1} + \nu_0 l_3^{d-1}$$
$$p_1 = \lambda_1 l_1^{d-1} + \mu_1 l_2^{d-1} + \nu_1 l_3^{d-1}$$
$$p_2 = \lambda_2 l_1^{d-1} + \mu_2 l_2^{d-1} + \nu_2 l_3^{d-1}.$$

Since p_0, p_1, p_2 are linearly independent, the matrix $\begin{pmatrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu 2 \end{pmatrix}$ is invertible, so $\langle p_0, p_1, p_2 \rangle = \langle l_0^{d-1}, l_1^{d-1}, l_2^{d-1} \rangle$. In view of Lemma 5.1 in f we can replace p_0, p_1, p_2 with $l_0^{d-1}, l_1^{d-1}, l_2^{d-1}$.

(II) $d \geq 7$ and $\operatorname{rank} \mathcal{A}_k = 3$ for $4 \leq k \leq \lfloor \frac{d+1}{2} \rfloor$, but p_0 has Waring rank strictly > 3. So from Proposition 5.3 (3), p_0 is of the form $\ell_1^{d-1} + \ell_2^{d-2}(\alpha\ell_1 + \beta\ell_2)$ for some $\ell_1, \ell_2 \in K[u,v]_1$ and $\alpha, \beta \in K^*$. So up to the change of variables that sends l_1 into u, and l_2 into v, $p_0 = u^{d-1} + \alpha uv^{d-2} + \beta v^{d-1}$. Then

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & b_0 & b_1 & b_2 & c_0 & c_1 & c_2 \\ 0 & 0 & 0 & & & & & & \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & \alpha & b_{d-3} & b_{d-2} & b_{d-1} & c_{d-3} & c_{d-2} & c_{d-1} \\ 0 & \alpha & \beta & b_{d-2} & b_{d-1} & b_d & c_{d-2} & c_{d-1} & c_d \end{pmatrix}.$$

From $\operatorname{rank} M_3 < 4$ it follows $b_1 = \cdots = b_{d-3} = c_1 = \cdots = c_{d-3} = 0$. Therefore

$$p_1 = b_0 u^{d-1} + b_{d-2} uv^{d-2} + b_{d-1} v^{d-1}, \; p_2 = c_0 u^{d-1} + c_{d-2} uv^{d-2} + c_{d-1} v^{d-1},$$

and we can replace p_0, p_1, p_2 with $u^{d-1}, uv^{d-2}, v^{d-1}$.

(III) $\operatorname{rank} \mathcal{A}_k = 2$ for $3 \leq k \leq \lfloor \frac{d+1}{2} \rfloor$ and p_0 has Waring rank 2, so it can be written $p_0 = u^{d-1} + v^{d-1}$. Then M_3 is as in case (II) with $\alpha = 0$, $\beta = 1$ and

$$M_2 = \begin{pmatrix} 1 & 0 & b_0 & b_1 & c_0 & c_1 \\ 0 & 0 & b_1 & b_2 & c_1 & c_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & b_{d-2} & b_{d-1} & c_{d-2} & c_{d-1} \end{pmatrix}.$$

From $\operatorname{rank} M_2 < 4$ we deduce that

$$\operatorname{rank} \begin{pmatrix} b_1 & b_2 \\ \vdots & \vdots \\ b_{d-3} & b_{d-2} \end{pmatrix} < 2, \; \operatorname{rank} \begin{pmatrix} c_1 & c_2 \\ \vdots & \vdots \\ c_{d-3} & c_{d-2} \end{pmatrix} < 2, \; \operatorname{rank} \begin{pmatrix} b_1 & b_2 & \dots & b_{d-2} \\ c_1 & c_2 & \dots & c_{d-2} \end{pmatrix} < 2.$$

Therefore

$$(b_1, \dots, b_{d-2}) = (\lambda^{d-3}, \lambda^{d-4}\mu, \dots, \mu^{d-3}), \; (c_1, \dots, c_{d-2}) = (\sigma^{d-3}, \sigma^{d-4}\rho, \dots, \rho^{d-3}),$$

for suitable $\lambda, \mu, \sigma, \rho \in K$. We get:

$$p_1 = b_0 u^{d-1} + uv((d-1)\lambda^{d-3}u^{d-3} + \binom{d-1}{2}\lambda^{d-4}\mu u^{d-4}v + \cdots) + b_{d-1}v^{d-1},$$

$$p_2 = c_0 u^{d-1} + uv((d-1)\sigma^{d-3}u^{d-3} + \binom{d-1}{2}\sigma^{d-4}\rho u^{d-4}v + \cdots) + c_{d-1}v^{d-1}.$$

We can also write

$$p_1 = b_0 u^{d-1} + uv\phi_{d-3} + b_{d-1}v^{d-1}, \; p_2 = c_0 u^{d-1} + kuv\phi_{d-3} + c_{d-1}v^{d-1}$$

where ϕ_{d-3} is a form of degree $d-3$ and $k \in K$, because $(b_1, \ldots, b_{d-2})$ and $(c_1, \ldots, c_{d-2})$ are proportional. We can assume $b_0 = c_0 = 0$ and we get $v^{d-1} \in \langle p_0, p_1, p_2 \rangle$, hence $u^{d-1}, uv\phi_{d-3} \in \langle p_0, p_1, p_2 \rangle$. Finally, adding to $uv\phi_{d-3}$ suitable multiples of u^{d-1}, v^{d-1}, we get $(\lambda u + \mu v)^{d-1} \in \langle p_0, p_1, p_2 \rangle$.

(IV) $\operatorname{rank} \mathcal{A}_k = 2$ for $3 \le k \le \lfloor \frac{d+1}{2} \rfloor$ but p_0 has Waring rank > 2, so up to a change of variables $p_0 = u^{d-2}v$.

$$M_2 = \begin{pmatrix} 0 & 1 & b_0 & b_1 & c_0 & c_1 \\ 1 & 0 & b_1 & b_2 & c_1 & c_2 \\ \vdots & \vdots & & & & \\ 0 & 0 & b_{d-2} & b_{d-1} & c_{d-2} & c_{d-1} \end{pmatrix}$$

has rank 3, therefore

$$\operatorname{rank} \begin{pmatrix} b_2 & b_3 & c_2 & c_3 \\ \vdots & \vdots & \vdots & \vdots \\ b_{d-2} & b_{d-1} & c_{d-2} & c_{d-1} \end{pmatrix} < 2,$$

and arguing in a similar way to (III), we conclude that $\langle p_0, p_1, p_2 \rangle$ is either of the form $\langle u^{d-1}, u^{d-2}v, (\lambda u + \mu v)^{d-1} \rangle$, or $\langle u^{d-1}, u^{d-2}v, u^{d-3}v^2 \rangle$.

(V) $\operatorname{rank} \mathcal{A}_k = \operatorname{rank} \mathcal{B}_k = \operatorname{rank} \mathcal{C}_k = 1$, then p_0, p_1, p_2 are all pure powers of degree $d-1$.

(VI) Let π be the 2-plane generated by the polynomials p_0, p_1, p_2. If $d = 5$, $\pi \subset \mathbb{P}^4 = \mathbb{P}(K[u, v]_4)$. The tangential variety TC_4 has codimension 2, so the intersection $\pi \cap TC_4 \neq \emptyset$. If π intersects TC_4 outside its singular locus C_4, up to a change of variables $u^3v \in \pi$ and we conclude as in (IV); otherwise, we are in the situation of (V). If $d = 6$, $\pi \subset \mathbb{P}^5 = \mathbb{P}(K[u, v]_5)$. Now $\sigma_2(C_5)$ has codimension 2 and therefore $\pi \cap \sigma_2(C_5) \neq \emptyset$. Therefore we are either in the situation of (III) or of (IV).

We have proved that for any $d \ge 5$, if f defines a Perazzo 3-fold and $S/\operatorname{Ann}_S(f)$ has minimal h-vector, then the polynomials p_0, p_1, p_2 are as in (i), or (ii), or (iii).

It remains to find out how we can choose the polynomial g in each of the cases. From Proposition 3.5 we deduce that the only condition that g has to satisfy is $\operatorname{Ann}_S(p_0x_0 + p_1x_1 + p_2x_2)_3 = \operatorname{Ann}_S(f - g)_3 = \operatorname{Ann}_S(f)_3$. In other words, we impose that g is annihilated by a system of generators of $\operatorname{Ann}_S(f)_3$.

(i) If $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + u^{d-2}vx_1 + u^{d-3}v^2x_2 + g$, we have that $\operatorname{Ann}_S(f)_3 = \langle V^3 \rangle$ and so $g = g_0u^d + g_1u^{d-1}v + g_2u^{d-2}v^2$.

(ii) If $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + u^{d-2}vx_1 + v^{d-1}x_2 + g$, we have that $\operatorname{Ann}_S(f)_3 = \langle UV^2 \rangle$. This gives that $\sum_{i=2}^{d-1} g_i \binom{d}{i}(k-i)i(i-1)u^{k-i-1}v^{i-2} = 0$, so $g_2 = \cdots = g_{d-1} = 0$. Thus we get $g = g_0u^d + g_1u^{d-1}v + g_dv^d$.

(iii) If $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + (\lambda u + \mu v)^{d-1}x_1 + v^{d-1}x_2 + g$, we have that $\operatorname{Ann}_S(f)_3 = \langle \mu U^2V - \lambda UV^2 \rangle$. Then we have the condition

$$\sum_{i=2}^{d-1}\binom{k-3}{i-1}(\mu g_i-\lambda g_{i+1})u^{d-k-2}v^{i-1}=0 \iff \mu g_i-\lambda g_{i+1}=0,\quad i=1,\dots,k-2.$$

So we can collect g_1 and complete the d-th power to obtain $g=au^d+b(\lambda u+\mu v)^d+cv^d$. □

Remark 5.5. As we noticed in Lemma 5.1, the Hilbert function of the algebra $S/\operatorname{Ann}_S(f)$ depends only on the plane $\pi=\langle p_0,p_1,p_2\rangle\subset\mathbb{P}^{d-1}$ and not on the choice of the three generators. In Theorem 5.4 we have proved that the h-vector is minimal if and only if the plane π is in one of the following positions: it is an osculating plane to the rational normal curve C_{d-1} (case (i)), or it contains the tangent line to C_{d-1} at a point and meets C_{d-1} also at a second point (case (ii)), or it intersects C_{d-1} at three distinct points (case (iii)).

Remark 5.6. In Theorem 5.4, we have obtained a complete characterization of the polynomials f such that $A=S/\operatorname{Ann}_S(f)$ has minimum h-vector for any $d\geq 5$. This allows us to conclude with a direct verification the proof of Theorem 4.3, proving the WLP of these algebras in the cases $5\leq d\leq 7$.

Corollary 5.7. *The Artinian Gorenstein algebra $S/\operatorname{Ann}_S(f)$ associated to a Perazzo 3-fold of degree 5 has the WLP if and only if, after a possible change of coordinates, one of the following cases holds:*

(i) $f(x_0,x_1,x_2,u,v)=u^4x_0+u^3vx_1+u^2v^2x_2+au^5+bu^4v+cu^3v^2\in R_5$ *with* $a,b,c\in K$, *or*
(ii) $f(x_0,x_1,x_2,u,v)=u^4x_0+u^3vx_1+v^4x_2+au^5+bu^4v+cv^5\in R_5$ *with* $a,b,c\in K$, *or*
(iii) $f(x_0,x_1,x_2,u,v)=u^4x_0+(\lambda u+\mu v)^4x_1+v^4x_2+au^5+b(\lambda u+\mu v)^5+cv^5\in R_5$ *with* $\lambda,\mu\in K^*$ *and* $a,b,c\in K$.

Proof. It follows from Theorems 5.4, 4.1, and 4.3. □

Note that, as consequence of the results of Gordan-Noether, Corollary 5.7 gives also a complete classification of threefolds of degree 5 in $\mathbb{P}^4$ with vanishing hessian.

6. Final comments

In this last section, we give a short geometrical description of the hypersurfaces of Theorem 5.4 when $a=b=c=0$.

Case (i) corresponds to the union of the classic cubic Perazzo 3-fold in $\mathbb{P}^4$ of equation: $u^2x_0+uvx_1+v^2x_2=0$ with the non-reduced hyperplane of equation: $u^{d-3}=0$. To describe the other two hypersurfaces, we first recall some known geometric properties of hypersurfaces with vanishing hessian. Let $X=V(f)\subset\mathbb{P}^N$ be such a hypersurface with $\operatorname{hess}_f=0$.

We denote by

$$\nabla_f : \mathbb{P}^N \dashrightarrow (\mathbb{P}^N)^*$$

its *polar map* defined by

$$\nabla_f(p) = \left(\frac{\partial f}{\partial x_0}(p), \frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_N}(p)\right),$$

and by

$$\gamma : X \dashrightarrow (\mathbb{P}^N)^*$$

the restriction of ∇_f to X, i.e. the *Gauss map* of X, associating to each smooth point of X its embedded tangent space. The image of γ is the dual variety X^* of X. Let $Z = \overline{\nabla_f(\mathbb{P}^N)}$ be the closure of the image of the polar map. Then $X^* \subsetneq Z \subsetneq (\mathbb{P}^N)^*$ ([26, Corollary 7.2.8]). Moreover, if $N = 4$, Z is a cone with vertex a line over an irreducible plane curve, and its dual Z^* is a rational plane curve in $\mathbb{P}^4$, naturally identified with the bidual space $(\mathbb{P}^4)^{**}$ ([26, Lemma 7.4.13]).

Let X be a Perazzo hypersurface of degree d in $\mathbb{P}^4$ of equation (3.1). X contains the line $L : x_0 = x_1 = x_2 = 0$ and the plane $\Pi : u = v = 0$. From [26, Sections 7.3 and 7.4], it follows that Π is the singular locus of X with multiplicity $d-1$; moreover, X^* is a scroll surface of degree d, having the line Π^* as directrix. In particular, Π^* is also the vertex of Z, and the general plane ruling of the cone Z meets X^* along a line of the scroll. The curve Z^* is contained in Π and the hyperplanes containing Π cut on X, outside Π, a 1-dimensional family Σ of planes: they are all tangent to Z^* and meet L. If p is general in X, then the fibre of the Gauss map $\gamma^{-1}(\gamma(p))$ is the line $\langle p, p'\rangle$ where p' is the tangency point to Z^* of the plane of the family Σ passing through p.

We now see how this picture specializes if we consider the reduced, irreducible Perazzo 3-fold $X_1 \subset \mathbb{P}^4$ of equation

$$f_1(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + u^{d-2}vx_1 + v^{d-1}x_2,$$

case (ii) in Theorem 5.4. We use coordinates $z_0, \dots, z_4$ in $(\mathbb{P}^4)^*$. The equation of Z, which expresses the algebraic dependence of p_0, p_1, p_2, is $z_1^{d-1} - z_0^{d-2}z_2 = 0$; the one of Z^* is $(d-1)^{d-1}x_0^{d-2}x_2 + (d-2)^{d-2}x_1^{d-1} = 0$. They both represent rational curves of degree $d-1$ with a singular point of multiplicity $d-2$ with only one tangent line.

In case (iii) we have

$$f_2(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + (\lambda u + \mu v)^{d-1}x_1 + v^{d-1}x_2 \text{ with } \lambda, \mu \in K^*.$$

For low values of d we have checked with the help of Macaulay2 ([20]) that Z is a cone over a rational curve of degree $d-1$ with $\frac{(d-2)(d-3)}{2}$ distinct nodes. Its dual Z^* results to be a rational curve of degree $2d-4$. If $d=5$, then Z^* has degree 6 and it has 3 cuspidal points of multiplicity 3 at the fundamental points $[1,0,0],[0,1,0],[0,0,1]$ and one node; if $d=6$, then Z^* has cuspidal points of multiplicity 4 at the fundamental points and 3 nodes; if $d=7$, then Z^* has cuspidal points of multiplicity 5 at the fundamental points and 6 nodes.

Data availability

No data was used for the research described in the article.

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