Scaling behaviour in a coagulation-annihilation model and Lotka-Volterra competition systems ‡

FP da Costa[1,2], JT Pinto[2,3], H van Roessel[4] and R Sasportes[1,2]

[1] Departmento de Ciências e Tecnologia, Universidade Aberta, Lisboa, Portugal
[2] Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, LARSyS, Instituto Superior Técnico, Universidade Técnica de Lisboa, Lisboa, Portugal
[3] Departmento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, Lisboa, Portugal
[4] Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Canada

E-mail: fcosta@uab.pt, jpinto@math.ist.utl.pt, henry.vanroessel@ualberta.ca, rafael@uab.pt

Abstract. In a recent paper Laurençot and van Roessel (2010) studied the scaling behaviour of solutions to a two-species coagulation-annihilation system with total annihilation and equal strength coagulation, and identified cases where self-similar behaviour occur, and others where it does not. In this paper we proceed with the study of this kind of systems by assuming that the coagulation rates of the two different species need not be equal. By applying Laplace transform techniques the problem is transformed into a two dimensional ordinary differential system that can be transformed into a Lotka-Volterra competition model. The long-time behaviour of solutions to this Lotka-Volterra system helps explain the different cases of existence and nonexistence of similarity behaviour, as well as why, in some cases, the behaviour is nonuniversal, in the sense of being dependent of initial conditions.

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1. The coagulation-annihilation system

The two-species coagulation-annihilation system describes the time evolution of the concentration of clusters of two different particle species (A and B, say) in which the

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A-particle clusters [resp., B-particle clusters] undergo coagulation between themselves, symbolically

$$A_\mu + A_\lambda \longrightarrow A_{\mu+\lambda} \quad [\text{resp., } B_\mu + B_\lambda \longrightarrow B_{\mu+\lambda}], \tag{1}$$

but when an A-particle cluster and a B-particle cluster come together, they annihilate each other, and in the simplest such model the annihilation is complete, i.e., for all μ and λ,

$$A_\mu + B_\lambda \longrightarrow \varnothing. \tag{2}$$

These processes, as well as similar ones, have been approached in the literature through a variety of techniques, including Monte Carlo simulations in lattice systems (see, e.g., Hoyuelos and Mártin (1996), Privman et al. (1995), and Sokolov and Blumen (1994)), renormalization group theory (see, e.g., Lee and Cardy (1995)), or mean field rate equations (see references below).

In a mean field description of this process, denoting by $a(t, \mu)$ the concentration of A-clusters of size μ at time t, the differential equation for the rate of change of this concentration due to coagulation equation (1) is

$$\frac{\partial a}{\partial t}(t, \mu) = -K_a(\mu, \lambda)a(t, \mu)a(t, \lambda),$$

and analogously for the rate of change of the B-clusters concentration, with rate coefficientes $K_b(\mu, \lambda)$. The annihilation process (2) contributes with a rate of change, for either $a(t, \mu)$, or $b(t, \lambda)$, given by

$$\frac{\partial a}{\partial t}(t, \mu) = \frac{\partial b}{\partial t}(t, \lambda) = -L(\mu, \lambda)a(t, \mu)b(t, \lambda).$$

When considering the different ways a given cluster of a certain type and size can be created and destroyed these contributions must be taken together and the resulting mathematical model is either a system for a countable number of ordinary differential equations, if the cluster sizes are discrete, $\mu, \lambda \in \mathbb{N}^+$, or a system of integro-differential equations, for continuous cluster sizes, $\mu, \lambda \in \mathbb{R}^+$ (see below).

These models were considered by Ben-Naim and Krapivsky (1995) for discrete cluster sizes and for reaction rates independent of the cluster sizes and all with the same value, $K_a(\mu, \lambda) = K_b(\mu, \lambda) = L(\mu, \lambda) = 2$. In that work, the authors investigated the time evolution of the system and the existence, or nonexistence, of a universal similarity behaviour of the solutions. More recently, Laurençot and van Roessel (2010) considered these same issues in the case of continuous cluster sizes with reaction rates also independent of the cluster sizes but with the coagulation rates $K_a(\mu, \lambda) = K_b(\mu, \lambda) = k$ possibly different from the annihilation kernel $L(\mu, \lambda) = L$, a case that had already been considered by Krapivsky (1993) for the discrete case with $k = 2$.

In this paper we extend the analysis of Laurençot and van Roessel (2010) by considering the possibility of the coagulation rates of A-clusters and of B-clusters to be different from each other, i.e., we consider $K_a(\mu, \lambda) = K_a$, $K_b(\mu, \lambda) = K_b$, and

$L(\mu,\lambda) = L$, where K_a, K_b, L are positive constants, otherwise unrestricted. Using the Laplace transform approach of Laurençot and van Roessel (2010) we obtain a two dimensional system of ordinary differential equations that governs the scaling dynamics of the coagulation-annihilation model. In contradistinction to what was done for the corresponding ordinary differential system in Ben-Naim and Krapivsky (1995) and Laurençot and van Roessel (2010), we study the behaviour of solutions to the ordinary differential system by transforming it into a Lotka-Volterra competition system. The analysis of the equilibria of this competition system and their stability allow us to draw conclusions about the existence, or nonexitence, of scaling behaviour in the coagulation-annihilation model, as well as providing a dynamical systems explanation for the nonuniversality of the scaling behaviour already identified in the above mentioned papers for some parameter regions. The final result of this analysis in presented in Table 4.

2. Scaling behaviour via Laplace transforms

The time evolution of a population of A-clusters and B-clusters, with continuous cluster sizes, undergoing coagulation and annihilation according to the reactions (1) and (2), and with reaction rates independent of the cluster sizes, can be described by the following system of integro-differential equations (see Laurençot and van Roessel (2010))

$$\frac{\partial a}{\partial t}(t,\lambda) = \frac{1}{2}\int_0^\lambda K_a a(t,\lambda-\mu)a(t,\mu)d\mu - a(t,\lambda)\left(\int_0^\infty K_a a(t,\mu)d\mu + \int_0^\infty L b(t,\mu)d\mu\right),$$
$$\frac{\partial b}{\partial t}(t,\lambda) = \frac{1}{2}\int_0^\lambda K_b b(t,\lambda-\mu)b(t,\mu)d\mu - b(t,\lambda)\left(\int_0^\infty K_b b(t,\mu)d\mu + \int_0^\infty L a(t,\mu)d\mu\right).$$

As in Laurençot and van Roessel (2010), we can rescale this system so that a smaller number of free parameters remain (two, in this case): let $\kappa := K_b/K_a$, $J := 2L/K_a$ and rescale the time $t \mapsto 2t/K_a$. Let

$$A(t) := \int_0^\infty a(t,\lambda)d\lambda, \qquad B(t) := \int_0^\infty b(t,\lambda)d\lambda. \tag{3}$$

Then, the coagulation-annihilation system becomes

$$\frac{\partial a}{\partial t}(t,\lambda) = \int_0^\lambda a(t,\lambda-\mu)a(t,\mu)d\mu - a(t,\lambda)\left(2A(t) + JB(t)\right), \tag{4}$$
$$\frac{\partial b}{\partial t}(t,\lambda) = \kappa\int_0^\lambda b(t,\lambda-\mu)b(t,\mu)d\mu - b(t,\lambda)\left(2\kappa B(t) + JA(t)\right), \tag{5}$$

with initial conditions $a(0,\lambda) = a_0(\lambda) \geqslant 0$, $\quad b(0,\lambda) = b_0(\lambda) \geqslant 0$, $\quad \lambda \in \mathbb{R}^+$. The study of the case $J > 0$ and $\kappa = 1$ was done in Laurençot and van Roessel (2010). Here we will consider any positive value of J and κ.

To study the scaling dynamics of (4)-(5) we will follow previous studies of this system and other coagulation type equations and resort to Laplace transforms, a tool

that is well adapted to the convolution structure of (4)-(5). Following Laurençot and van Roessel (2010), define the Laplace transforms by

$$F(t,x) := \mathcal{L}(a(t,\cdot))(x) := \int_0^\infty e^{-\lambda x} a(t,\lambda) d\lambda,$$

$$G(t,x) := \mathcal{L}(b(t,\cdot))(x) := \int_0^\infty e^{-\lambda x} b(t,\lambda) d\lambda,$$

and use (4)-(5) to get the following system for the time evolution of F and G

$$\frac{\partial F}{\partial t}(t,x) = F^2(t,x) - (2A(t) + JB(t))F(t,x),$$
$$\frac{\partial G}{\partial t}(t,x) = \kappa G^2(t,x) - (2\kappa B(t) + JA(t))G(t,x),$$

with initial conditions $F(0,x) = F_0(x) := \mathcal{L}(a_0(\cdot))(x)$, and $G(0,x) = G_0(x) := \mathcal{L}(b_0(\cdot))(x)$. Observe that the system of differential equations for F and G can be written only in terms of F and G themselves, since, from (3), we have $A(t) = F(t,0)$ and $B(t) = G(t,0)$. Assuming we know the behaviour of $A(t)$ and $B(t)$, each of the equations for F and G can be easily solved independently of the other to give

$$F(t,x) = \frac{F_0(x)e^{-P(t)}}{1 - F_0(x)\int_0^t e^{-P(s)}ds}, \quad G(t,x) = \frac{G_0(x)e^{-Q(t)}}{1 - \kappa G_0(x)\int_0^t e^{-Q(s)}ds}, \tag{6}$$

where

$$P(t) := \int_0^t (2A(s) + JB(s))ds \quad \text{and} \quad Q(t) := \int_0^t (2\kappa B(s) + JA(s))ds. \tag{7}$$

Observe that, apart from the factor κ, these are exactly equal to the functions occurring in Laurençot and van Roessel (2010), so, once the long-time dynamic behaviour of the functions $A(t)$ and $B(t)$ is known, exactly the same computations can be reproduced to get the behaviour of the Laplace transforms F and G and, *a posteriori,* the dynamic scaling behaviour of the solutions of (4)-(5).

To get the needed information on the dynamics of the functions $A(t)$ and $B(t)$ first observe that, by integrating equations (4)-(5) over $(0,\infty)$, we easily conclude that A and B must be the solutions to the system

$$\begin{cases} \dfrac{dA}{dt} &= -A^2 - JAB \\ \dfrac{dB}{dt} &= -\kappa B^2 - JAB \end{cases} \tag{8}$$

with initial conditions $A(0) = A_0 := F_0(0) > 0$ and $B(0) = B_0 := G_0(0) > 0$.

What we need to know about the long-time behaviour of solutions to (8) in order to pursue the above program are the rates of convergence to zero of the positive solutions to this system (that positive solutions of (8) necessarily converge to zero is obvious by inspection).

In Laurençot and van Roessel (2010) the particular (symmetric) version of (8) with $\kappa = 1$ was studied by applying a change of variables that allows one to solve for one of

the variables in terms of the other, effectively decoupling the system. That change of variables is also not dissimilar from the one used by Ben-Naim and Krapivsky (1995).

When $\kappa \neq 1$ we could not find a change of variables akin to those, so we need to consider a different approach. The one we use here is of a more geometrical nature: it is based on a change of variables in (8) that transforms it into a Lotka-Volterra competition system, the equilibrium points of which are related to the long-time asymptotic profiles of the solutions to (8). This approach will not provide all the details concerning the constants involved, but will give the correct rates of convergence relevant for the coagulation studies downstream. The final result is presented in Table 1 at the end of next section.

3. A Lotka-Volterra competition system

Let $\widetilde{A}(t) := (t+1)A(t)$ and $\widetilde{B}(t) := (t+1)B(t)$. If (A, B) solves (8) with initial condition $A(0) = A_0 \geqslant 0$, $B(0) = B_0 \geqslant 0$, then $(\widetilde{A}, \widetilde{B})$ is a solution of the system

$$\begin{cases} (t+1)\dfrac{d\widetilde{A}}{dt} &= \widetilde{A} - \widetilde{A}^2 - J\widetilde{A}\widetilde{B} \\ (t+1)\dfrac{d\widetilde{B}}{dt} &= \widetilde{B} - \kappa\widetilde{B}^2 - J\widetilde{A}\widetilde{B}, \end{cases} \tag{9}$$

with the same initial conditions. Let us change the time scale $t \mapsto \tau := \log(t+1)$, and define $x(\tau) := \widetilde{A}(t(\tau))$, $y(\tau) := \kappa\widetilde{B}(t(\tau))$. Then, (x, y) solves the following Lotka-Volterra competition system

$$\begin{cases} \dot{x} &= x(1 - x - \frac{J}{\kappa}y) \\ \dot{y} &= y(1 - y - Jx), \end{cases} \tag{10}$$

where the dot represents the derivative with respect to τ, and the solution satisfies the initial condition $x(0) = A_0, y(0) = \kappa B_0$. The analysis of the convergence to equilibria of solutions to (10) provides information about the rates of convergence to zero of the solutions to the original system (8).

It is clear that (10) has always three nonnegative equilibria, the points $(0, 0)$ which is always unstable, and the points $(1, 0)$ and $(0, 1)$ whose stability depends on the values of the parameters J and κ. Additionally to these equilibria, in some parameter regions there is a single strictly positive equilibrium point, whose stability also depends on where the parameters lie. Finally, in the single parameter point $\kappa = J = 1$ system (10) has a degenerate continuum of equilibria formed by all the points in the straight line segment between $(1, 0)$ and $(0, 1)$.

The study of what happens in all relevant parameter regions and of the global and linearized stability properties (eigenvalues, invariant regions, etc.) of the equilibria in each case is a standard exercise that can be checked in the literature (see, e.g., Murray (2002, Section 3.4)) and so, we will start by stating those properties without proof, and shall deal only with those geometric aspects needed for the application to (8) and not

available in Murray (2002), such as the eigenvectors of (some) of the linearizations, or the dynamics on the center manifold (in the final case).

There are a number of relevant parameter sets that need to be considered, namely

$$\begin{aligned}
\Omega_1 &= \{(J,\kappa) \in \mathbb{R}^{2+} : \kappa > J \text{ and } J < 1\},\\
\Omega_2 &= \{(J,\kappa) \in \mathbb{R}^{2+} : \kappa > J \text{ and } J > 1\},\\
\Omega_3 &= \{(J,\kappa) \in \mathbb{R}^{2+} : \kappa < J \text{ and } J > 1\},\\
\Omega_4 &= \{(J,\kappa) \in \mathbb{R}^{2+} : \kappa < J \text{ and } J < 1\},
\end{aligned}$$

as well as the boundaries $\Omega_{ij} := \partial\Omega_i \cap \partial\Omega_j$, and the point $\Omega_* = \bigcap \Omega_{ij} = \{(1,1)\}$. All these sets are ploted in Figure 1.

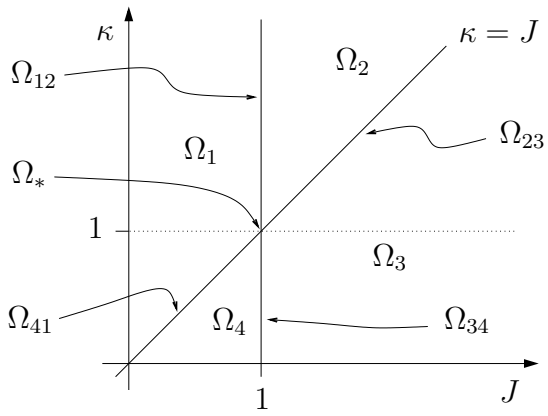

Figure 1. Parameter sets relevant to the study of (10). The dotted line ($\kappa = 1$) corresponds to the parameter case studied in Laurençot and van Roessel (2010).

3.1. Parameters in Ω_1 ($\kappa > J$ and $J < 1$)

In this case, (10) has a unique positive equilibrium $(x_\infty, y_\infty) = (\frac{\kappa - J}{\kappa - J^2}, \frac{\kappa - J\kappa}{\kappa - J^2})$ that is globally exponentially asymptotically stable in the interior of $\mathbb{R}^+ \times \mathbb{R}^+$. Hence, the equilibria lying on the coordinate axis, $(1,0)$ and $(0,1)$, are linearly unstable. So, for all initial conditions, we know that

$$(x(\tau), y(\tau)) = \left(\frac{\kappa - J}{\kappa - J^2}, \frac{\kappa - J\kappa}{\kappa - J^2}\right)(1 + o(1)), \quad \text{as } \tau \to \infty,$$

which, reverting to the original variables, gives

$$\begin{cases} A(t) = \frac{1}{\left(1 + J\frac{\alpha}{\beta}\right)t}(1 + o(1)) \\ B(t) = \frac{\alpha/\beta}{\left(1 + J\frac{\alpha}{\beta}\right)t}(1 + o(1)) \end{cases} \quad \text{as } t \to \infty, \tag{11}$$

where $\alpha = 1 - J$ and $\beta = \kappa - J$. Note that, when $\kappa = 1$ we have $\alpha/\beta = 1$ and (11) reduces to the equation (30) of Laurençot and van Roessel (2010).

3.2. Parameters in Ω_2 ($\kappa > J > 1$)

In this case the point $(1,0)$ is globally asymptotically stable in $\mathbb{R}^+ \times \mathbb{R}^+$. So, as in the previous case, the behaviour of the function $A(t)$ can be immediately read off from that of $x(\tau)$:

$$A(t) = \frac{1}{t}(1 + o(1)) \quad \text{as } t \to \infty.$$

However, in order to have the rate of convergence of $B(t)$, the information provided by the convergence of $y(\tau)$ is not yet sufficient: in fact, what $y(\tau) \to 0$ implies is that $B(t)$ converges to zero faster than $1/t$. Thus, when applied to the study of self-similar behaviour of the coagulation-annihilation equations, this case is certain to lead us into the kind of trouble already faced in Laurençot and van Roessel (2010) when one of the functions was converging to zero at a faster rate than $1/t$.

Even if the result proves to be irrelevant for self-similar studies we will complete the analysis by identifying the rate at which $B(t)$ converge to zero. This requires information about the rate of convergence of $y(\tau)$ to zero. This is the kind of information provided by the linearization, when applicable. Linearizing (10) about $(1,0)$ we obtain the jacobian matrix

$$J(1,0) = \begin{bmatrix} -1 & -\frac{J}{\kappa} \\ 0 & 1-J \end{bmatrix},$$

which, when $J \neq 2$, has the eigenpairs

$$\lambda_1 = -1,\ v_1 = (1,0)^\mathsf{T}, \qquad \lambda_2 = 1 - J,\ v_2 = (\frac{J}{\kappa}, J-2)^\mathsf{T}.$$

Hence, we can write

$$\begin{bmatrix} x \\ y \end{bmatrix}(\tau) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \Phi\left(\alpha_1 e^{-\tau}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 e^{(1-J)\tau}\begin{bmatrix} \frac{J}{\kappa} \\ J-2 \end{bmatrix}\right), \tag{12}$$

where Φ is the topological conjugacy provided by the Hartman-Grobman theorem. Remember that $\Phi(0) = 0$ and $D\Phi(0) = \mathrm{id}$. So, when $\tau \to \infty$ we can write

$$\begin{bmatrix} x \\ y \end{bmatrix}(\tau) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_1 e^{-\tau}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 e^{(1-J)\tau}\begin{bmatrix} \frac{J}{\kappa} \\ J-2 \end{bmatrix} + \text{h.o.t.}, \tag{13}$$

and, in particular, we obtain

$$y(\tau) = \alpha_2(J-2)e^{(1-J)\tau} + \text{h.o.t.}, \quad \text{as } \tau \to \infty. \tag{14}$$

Note that α_2 depends on the initial data (A_0, B_0) in an unknown way, and also that $\alpha_2(J-2) > 0$. Now, remembering that $\tau = \log(t+1)$, and observing that

$$\frac{e^{(1-J)\log(t+1)}}{t+1} = \frac{(t+1)^{1-J}}{t+1} = \frac{1}{(t+1)^J} = \frac{1}{t^J}(1 + o(1)),\ \text{as } t \to \infty, \tag{15}$$

we conclude that

$$B(t) = \frac{\text{const.}}{t^J}(1 + o(1)) \qquad \text{as } t \to \infty, \tag{16}$$

where "const." is a positive constant depending on the initial data (A_0, B_0) in an unspecified way.

When $J = 2$ we need to compute the generalized eigenpair of the jacobian matrix $J(1, 0)$, which is

$$\lambda_2 = -1,\ v_2 = (1, 0)^{\mathsf{T}}.$$

Now, instead of (12) we have

$$\begin{bmatrix} x \\ y \end{bmatrix}(\tau) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \Phi\left(\alpha_1 e^{-\tau}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 e^{-\tau}\begin{bmatrix} \tau \\ -\frac{\kappa}{2} \end{bmatrix}\right), \tag{17}$$

and thus

$$y(\tau) = (\text{const.})e^{-\tau} + \text{h.o.t.}, \quad \text{as } \tau \to \infty,$$

which implies, via (15), that (16) also holds with $J = 2$.

Summarizing: in the case $\kappa > J > 1$ the asymptotic behaviour of positive solutions to (8) is

$$\begin{cases} A(t) = \frac{1}{t}(1 + o(1)) \\ B(t) = \frac{\text{const.}}{t^J}(1 + o(1)) \end{cases} \quad \text{as } t \to \infty. \tag{18}$$

3.3. Parameters in Ω_4 ($\kappa < J < 1$)

In this case the point $(0, 1)$ is globally asymptotically stable in $\mathbb{R}^+ \times \mathbb{R}^+$ and we could proceed with the analysis as in the previous section. However, we can avoid it by observing that this case is easily reduced to the previous one by changing $x \leftrightarrow y$ and $\frac{J}{\kappa} \leftrightarrow J$. Hence, we can read out the asymptotic behaviour directly from (18):

$$\begin{cases} A(t) = \frac{\text{const.}}{t^{J/\kappa}}(1 + o(1)) \\ B(t) = \frac{1}{\kappa t}(1 + o(1)) \end{cases} \quad \text{as } t \to \infty. \tag{19}$$

Note that, due to the factor κ in the definition of y, the expression for $B(t)$ in (19) is not exactly that of $A(t)$ in (18).

3.4. Parameters in Ω_3 ($\kappa < J$ and $J > 1$)

This case corresponds to the situation when there is a unique saddle point in the interior of $\mathbb{R}^+ \times \mathbb{R}^+$, with the same coordinates as (x_∞, y_∞) in Section 3.1, and the two equilibria $(1, 0)$ and $(0, 1)$ are asymptotically stable points whose basins of attraction in $\mathbb{R}^+ \times \mathbb{R}^+$ are open sets $\Omega_{(1,0)}$ and $\Omega_{(0,1)}$, respectively, and whose separatrix $\omega := \overline{\Omega_{(1,0)}} \cap \overline{\Omega_{(0,1)}}$ is the stable manifold of (x_∞, y_∞). It is easy to conclude that ω is the straight line $y = \frac{y_\infty}{x_\infty}x$, since in points of this line the slope of the vetor field of (10) is equal to the slope of the line, $\frac{dy}{dx} = \dot{y}/\dot{x} = y_\infty/x_\infty$ (see Figure 2).
So, taking into consideration the results in Sections 3.1, 3.2 and 3.3, we have

- If $B_0 = \frac{\alpha}{\beta}A_0$, then the solution behaves as in (11),

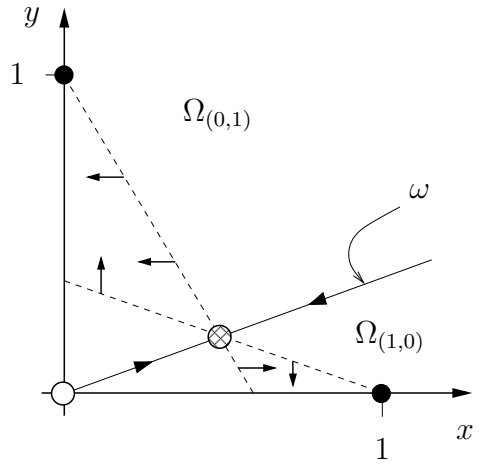

Figure 2. Equilibria of (10) for parameters in Ω_3. Circles represent equilibria: full circles are exponentially asymptotically stable equilibria, the shaded circle is a saddle point, and the white circle is an unstable equilibrium. The dashed lines are the nullclines of (10) and the arrows represent the flow on the corresponding line. The straight line ω is the separatrix of the basins of attraction of the stable equilibria.

- If $B_0 < \frac{\alpha}{\beta} A_0$, then the solution behaves as in (18),
- If $B_0 > \frac{\alpha}{\beta} A_0$, then the solution behaves as in (19).

This dependence on the initial conditions was already observed in Laurençot and van Roessel (2010) for the particular case when $\kappa = 1$. The present geometric picture helps us to understand its origin a little better.

3.5. Parameters in the boundary lines $\kappa = J$ or $J = 1$

We start by observing that, here again, the case $\kappa = J$ can be reduced to the case $J = 1$ by changing the variables $x \leftrightarrow y$ and $J/\kappa \leftrightarrow J$. Thus, we shall only study this last case.

3.5.1. Parameters in Ω_* ($\kappa = J = 1$). This degenerate case has the whole line segment between $(1, 0)$ and $(0, 1)$ as equilibria. By a quick inspection of the vector field of (10) we conclude that every positive solution to (10) will converge to a point (x_∞, y_∞) of this line segment. Furthermore, one easily concludes that the function $m(t) := y(t)/x(t)$ satisfies $\dot{m} = \frac{\dot{y}x - y\dot{x}}{x^2} \equiv 0$, and this means that the orbits of (10) are straight lines with slope $m = y_0/x_0$. Thus, the limit point (x_∞, y_∞) is obtained as the intersection of the straight lines $y = 1 - x$ and $y = \frac{y_0}{x_0} x$, which implies that $x_\infty = \frac{x_0}{x_0 + y_0}$ and $y_\infty = \frac{y_0}{x_0 + y_0}$.

Hence, reverting to the original variables, we conclude that

$$\begin{cases} A(t) = \frac{A_0}{(A_0 + B_0)t}(1 + o(1)) \\ B(t) = \frac{B_0}{(A_0 + B_0)t}(1 + o(1)) \end{cases} \qquad \text{as } t \to \infty. \tag{20}$$

3.5.2. Parameters in Ω_{34} ($\kappa < J = 1$). In this case the equilibrium $(0,1)$ is globally asymptotically stable (in $\mathbb{R}^+ \times \mathbb{R}^+$). Using the remark in Section 3.3 and observing that all computations in Section 3.2 still hold with $J = 1$, we conclude that the result in Section 3.3 is valid also for this case:

$$\begin{cases} A(t) = \frac{\text{const.}}{t^{1/\kappa}}(1 + o(1)) \\ B(t) = \frac{1}{\kappa t}(1 + o(1)) \end{cases} \quad \text{as } t \to \infty. \tag{21}$$

3.5.3. Parameters in Ω_{12} ($\kappa > J = 1$). In this case the equilibrium $(1,0)$ is globally asymptotically stable (in $\mathbb{R}^+ \times \mathbb{R}^+$). Again, as in Section 3.2, the behaviour of the component $A(t)$ of the solution to (8) can be immediately read off from the fact that $x(\tau) \to 1$ as $\tau \to \infty$, but the precise rate of decay of $B(t)$ requires a little more effort. As in Section 3.2, we start by linearizing (10) about the equilibrium solution $(1,0)$ to obtain the jacobian matrix

$$J(1,0) = \begin{bmatrix} -1 & -\frac{1}{\kappa} \\ 0 & 0 \end{bmatrix},$$

which has eigenpairs

$$\lambda_1 = -1,\ v_1 = (1,0)^{\mathsf{T}}, \qquad \lambda_2 = 0,\ v_2 = (1,-\kappa)^{\mathsf{T}}.$$

The existence of a zero eigenvalue implies that the nonlinear dynamics cannot be analyzed via the linearized system using the Hartman-Grobman theorem, but we need to investigate the dynamics on the center manifold of $(1,0)$. This manifold is tangent to the eigenspace correspondent to $\lambda_2 = 0$. To simplify the computations let us change variables from (x,y) to $(\widetilde{x},\widetilde{y})$ in such a way that $(1,0)$ is mapped to $(0,0)$, the eigenspace correspondent to $\lambda_1 = -1$ becomes the $\widetilde{x}$-axis and the eigenspace correspondent to $\lambda_2 = 0$ becomes the $\widetilde{y}$-axis. It is easy to see that $\widetilde{x} := x - 1 + y/\kappa, \widetilde{y} := y$ fulfills these requirements. System (10) becomes

$$\begin{cases} \dot{\widetilde{x}} &= -\widetilde{x} - \widetilde{x}^{\,2} - (\kappa-1)\widetilde{y}^{\,2}/\kappa^2 \\ \dot{\widetilde{y}} &= -\widetilde{x}\widetilde{y} - (\kappa-1)\widetilde{y}^{\,2}/\kappa. \end{cases} \tag{22}$$

By standard methods (see, e.g., Carr (1981)) we conclude that, up to second order, the center manifold is the graph of a function $\phi(\widetilde{y}) = -\frac{\kappa-1}{\kappa^2}\widetilde{y}^{\,2} + O(\widetilde{y}^{\,3})$. Thus, substituting $\widetilde{x} = \phi(\widetilde{y})$ in the $\widetilde{y}$-equation in (22) we conclude that $\widetilde{y}$ satisfies $\dot{\widetilde{y}} = -\frac{\kappa-1}{\kappa}\widetilde{y}^{\,2} + \frac{\kappa-1}{\kappa^2}\widetilde{y}^{\,3} + O(\widetilde{y}^{\,4})$. Thus, knowing that $\widetilde{y} \to 0$ as $\tau \to \infty$, we can write this equation as

$$\frac{d}{d\tau}\frac{\kappa}{(\kappa-1)\widetilde{y}} = \frac{\dot{\widetilde{y}}}{-\frac{\kappa-1}{\kappa}\widetilde{y}^{\,2}} = 1 + o(1), \qquad \text{as } \tau \to \infty.$$

From this we can deduce that

$$\frac{\kappa}{(\kappa-1)\tau\widetilde{y}} = 1 + o(1), \qquad \text{as } \tau \to \infty,$$

and, reverting to the original variables, we conclude that $B(t) \sim \frac{1}{(\kappa-1)t\log(t+1)}$ as $t \to \infty$.

Thus, in this case we have

$$\begin{cases} A(t) = \frac{1}{t}(1+o(1)) \\ B(t) = \frac{1}{(\kappa-1)t\log t}(1+o(1)) \end{cases} \qquad \text{as } t \to \infty, \tag{23}$$

and this concludes the analysis.

The results in this section are summarized in Table 1. Observe that, in all cases, when $t \to \infty$ solutions of (8) converge to zero at least as fast as t^{-1}.

Table 1. Asymptotic behaviour of the solutions of (8) as $t \to \infty$. The following notation is used: $\alpha = 1 - J$, $\beta = \kappa - J$, and "const." denote positive constants dependent on the initial data (A_0, B_0).

Parameter set	Initial data	Asymptotic behaviour, as $t \to \infty$, of	
		$A(t)$	$B(t)$
$\kappa > J, J < 1$	$A_0, B_0 > 0$	$\big(1 + J\frac{\alpha}{\beta}\big)^{-1} t^{-1}$	$\frac{\alpha}{\beta}\big(1 + J\frac{\alpha}{\beta}\big)^{-1} t^{-1}$
$\kappa > J = 1$	$A_0, B_0 > 0$	t^{-1}	$(\kappa-1)^{-1}(t\log t)^{-1}$
$\kappa \geqslant J > 1$	$A_0, B_0 > 0$	t^{-1}	$(\text{const.})t^{-J}$
$\kappa < J, J > 1$	$B_0 < \alpha A_0/\beta$	t^{-1}	$(\text{const.})t^{-J}$
	$B_0 = \alpha A_0/\beta$	$\big(1 + J\frac{\alpha}{\beta}\big)^{-1} t^{-1}$	$\frac{\alpha}{\beta}\big(1 + J\frac{\alpha}{\beta}\big)^{-1} t^{-1}$
	$B_0 > \alpha A_0/\beta$	$(\text{const.})t^{-J/\kappa}$	$\kappa^{-1} t^{-1}$
$\kappa < J \leqslant 1$	$A_0, B_0 > 0$	$(\text{const.})t^{-J/\kappa}$	$\kappa^{-1} t^{-1}$
$\kappa = J < 1$	$A_0, B_0 > 0$	$(1-\kappa)^{-1}(t\log t)^{-1}$	$\kappa^{-1} t^{-1}$
$\kappa = J = 1$	$A_0, B_0 > 0$	$\frac{A_0}{A_0+B_0} t^{-1}$	$\frac{B_0}{A_0+B_0} t^{-1}$

4. On the scaling behaviour

As pointed out in section 2 just after equation (7), once we have the results in Table 1 we can apply exactly the same approach as in Laurençot and van Roessel (2010) to obtain the values of the limits of $\xi(t)F(t, \frac{x}{\zeta(t)})$ and of $\xi(t)G(t, \frac{x}{\zeta(t)})$ as $t \to \infty$, for some convenient, strictly increasing, functions $\xi(t), \zeta(t)$. From these limits, the self-similar behaviour of the solutions is obtained by the properties of the Laplace transform.

Following Laurençot and van Roessel (2010), we consider the functions

$$\mathscr{A}(t) := \exp\Big(\int_0^t A(s)ds\Big) \quad \text{and} \quad \mathscr{B}(t) := \exp\Big(\int_0^t B(s)ds\Big). \tag{24}$$

From the definitions (7) of $P(t)$ and $Q(t)$ we easily conclude that

$$e^{P(t)} = \mathscr{A}(t)^2 \mathscr{B}(t)^J \quad \text{and} \quad e^{Q(t)} = \mathscr{B}(t)^{2\kappa} \mathscr{A}(t)^J. \tag{25}$$

To get the long-time behaviour of $\mathscr{A}$ and $\mathscr{B}$ we can, in (8), divide the A-equation by A and the B-equation by B and integrate between $t = 0$ and t to get

$$\frac{A_0}{A(t)} = \mathscr{A}(t)\mathscr{B}(t)^J \quad \text{and} \quad \frac{B_0}{B(t)} = \mathscr{B}(t)^\kappa \mathscr{A}(t)^J. \tag{26}$$

Inverting this relation we obtain

$$\mathscr{A}(t)^{J^2-\kappa} = \Big(\frac{A(t)}{A_0}\Big)^{\kappa}\Big(\frac{B_0}{B(t)}\Big)^{J} \quad \text{and} \quad \mathscr{B}(t)^{J^2-\kappa} = \Big(\frac{B(t)}{B_0}\Big)\Big(\frac{A_0}{A(t)}\Big)^{J}. \tag{27}$$

If we now plug the long-time behaviour of A and B into the expressions (27) we obtain all the information we need. In fact, denoting the asymptotic behaviour as $t \to \infty$ of these several functions by

$$\begin{cases} A(t) \sim A_\infty t^p \\ B(t) \sim B_\infty t^q \end{cases}, \qquad \begin{cases} \mathscr{A}(t) \sim \mathscr{A}_\infty t^{\widetilde{a}} \\ \mathscr{B}(t) \sim \mathscr{B}_\infty t^{\widetilde{b}} \end{cases} \tag{28}$$

we can use (27) and the behaviour collected in Table 1 to conclude the exponents are those presented in Table 2. Also in Table 2 we present the result for the case $\kappa = J = 1$ which have to be analyzed using (20) and (24) (with an arbitrary $t_0 > 0$ instead of 0 in the lower limit of integration), since (27) gives no information in this case.

Table 2. Exponents of the long-time behaviour (28) of the functions (A, B) and $(\mathscr{A}, \mathscr{B})$ obtained from the results in Table 1.

Case	Parameter sets	p	q	$\widetilde{a}$	$\widetilde{b}$
(i)	$\kappa > J,\ J < 1$ $\kappa < J,\ J > 1$ (with initial conditions $B_0 = \alpha A_0/\beta$)	-1	-1	$\dfrac{J-\kappa}{J^2-\kappa}$	$\dfrac{J-1}{J^2-\kappa}$
(ii)	$\kappa \geqslant J > 1$ $\kappa < J,\ J > 1$ (with initial conditions $B_0 < \alpha A_0/\beta$)	-1	$-J$	1	0
(iii)	$\kappa < J \leqslant 1$ $\kappa < J,\ J > 1$ (with initial conditions $B_0 > \alpha A_0/\beta$)	$-\dfrac{J}{\kappa}$	-1	0	$1/\kappa$
(iv)	$\kappa = J = 1$	-1	-1	$\frac{A_0}{A_0+B_0}$	$\frac{B_0}{A_0+B_0}$

For the only two situations for which the asymptotic behaviour of either A or B is not asymptotically a power law (when $\kappa > J = 1$, and when $\kappa = J < 1$) we can still use (27) and the results in Table 1 to conclude the results presented in Table 3.

Table 3. Long-time behaviour of the functions (A, B) and $(\mathscr{A}, \mathscr{B})$ when $\kappa > J = 1$, and $\kappa = J < 1$.

Parameter	Asymptotic behaviour, as $t \to \infty$, of			
set	$A(t)$	$B(t)$	$\mathscr{A}(t)^{\lvert\kappa-1\rvert}$	$\mathscr{B}(t)^{\lvert\kappa-1\rvert}$
$\kappa > J = 1$	$\dfrac{1}{t}$	$\dfrac{(\kappa-1)^{-1}}{t\log t}$	$\dfrac{A_0^\kappa/B_0}{\kappa-1}\dfrac{t^{\kappa-1}}{\log t}$	$\dfrac{\kappa-1}{A_0/B_0}\log t$
$\kappa = J < 1$	$\dfrac{(1-\kappa)^{-1}}{t\log t}$	$\dfrac{1}{\kappa t}$	$\dfrac{1-\kappa}{\kappa B_0/A_0}\log t$	$\dfrac{(\kappa B_0)^{1/\kappa}}{(1-\kappa)A_0}\dfrac{t^{(1-\kappa)/\kappa}}{\log t}$

Now, let us point out how to use this information. We will consider only the behaviour of F, the study of G being analogous.

Let $\zeta(t)$ be any positive function such that $\zeta(t) \to \infty$ as $t \to \infty$. For all fixed x, we can use (25) to write the numerator of the expression for F in (6) as

$$F_0\Big(\frac{x}{\zeta(t)}\Big)e^{-P(t)} \sim F_0(0)e^{-P(t)} = A_0\mathscr{A}(t)^{-2}\mathscr{B}(t)^{-J}, \ \text{ as } t \to \infty. \tag{29}$$

Now, making $x = 0$ in the expression for F in (6) and using the first equalities in (25) and (26) we conclude that

$$\int_0^t e^{-P(s)}ds = \frac{1}{A_0}\Big(1 - \frac{1}{\mathscr{A}(t)}\Big),$$

from which we can write,

$$\begin{aligned}
1 - F_0\Big(\frac{x}{\zeta(t)}\Big)\int_0^t e^{-P(s)}ds &= \frac{1}{A_0}\Big[F_0(0) - F_0\Big(\frac{x}{\zeta(t)}\Big) + F_0\Big(\frac{x}{\zeta(t)}\Big)\frac{1}{\mathscr{A}(t)}\Big] \\
&= \frac{1}{A_0}\left[\frac{x}{\zeta(t)}\frac{F_0(0) - F_0\Big(\frac{x}{\zeta(t)}\Big)}{\frac{x}{\zeta(t)}} + F_0\Big(\frac{x}{\zeta(t)}\Big)\frac{1}{\mathscr{A}(t)}\right] \\
&\sim \frac{1}{A_0}\left[-\frac{x}{\zeta(t)}F_0'(0) + A_0\frac{1}{\mathscr{A}(t)}\right], \ \text{ as } t \to \infty \\
&= \frac{1}{A_0\zeta(t)}\Big[\|a_0\|_1 x + \frac{A_0\zeta(t)}{\mathscr{A}(t)}\Big], \ \text{ as } t \to \infty,
\end{aligned} \tag{30}$$

where the asymptotic behaviour was obtained by applying Lagrange's mean value theorem when $\zeta(t) \to \infty$, and in the last term $\|a_0\|_1 := \int_0^\infty \lambda a_0(\lambda)d\lambda = -F_0'(0)$.

Gathering (29) and (30), we can write the equality for F in (6) as follows

$$\frac{1}{A_0\zeta(t)}F\Big(t, \frac{x}{\zeta(t)}\Big) \sim \frac{A_0\mathscr{A}(t)^{-2}\mathscr{B}(t)^{-J}}{\|a_0\|_1 x + \frac{A_0\zeta(t)}{\mathscr{A}(t)}}, \ \text{ as } t \to \infty,$$

which, can be rearranged to

$$\frac{\mathscr{A}(t)}{A_0\zeta(t)}\,\frac{\mathscr{A}(t)\mathscr{B}(t)^J}{A_0}\,F\Big(t, \frac{x}{\zeta(t)}\Big) \sim \frac{1}{\|a_0\|_1 x + \frac{A_0\zeta(t)}{\mathscr{A}(t)}}, \ \text{ as } t \to \infty. \tag{31}$$

In order to be able to invert the Laplace transform and the resulting object to be a positive function, we need to have $\frac{A_0\zeta(t)}{\mathscr{A}(t)} \to 1$ as $t \to \infty$. Furthermore, remember that (31) was deduced using the assumption $\zeta(t) \to \infty$ as $t \to \infty$. Thus, the component $a = \mathcal{L}^{-1}(F)$ of the solution to the coagulation-annihilation system (4)-(5) exhibits a scaling behaviour if $\mathscr{A}(t) \to \infty$ as $t \to \infty$, in which case, and without loss of generality, we can choose the similarity scaling as $\zeta(t) = \mathscr{A}(t)/A_0$ in (31). This choice leads to

$$\frac{\mathscr{A}(t)\mathscr{B}(t)^J}{A_0}\,F\Big(t, \frac{x}{\mathscr{A}(t)/A_0}\Big) \sim \frac{1}{\|a_0\|_1 x + 1}, \ \text{ as } t \to \infty. \tag{32}$$

Recalling that F is the Laplace transform of a, which means that

$$F(t, z) = \int_0^\infty e^{-\varsigma z}a(t, \varsigma)d\varsigma,$$

we can write

$$F\Big(t, \frac{x}{\mathscr{A}(t)/A_0}\Big) = \int_0^\infty e^{-\mu x} a(t, \tfrac{\mathscr{A}(t)}{A_0}\mu)\tfrac{\mathscr{A}(t)}{A_0} d\mu,$$

and thus

$$\frac{\mathscr{A}(t)\mathscr{B}(t)^J}{A_0} F\Big(t, \frac{x}{\mathscr{A}(t)/A_0}\Big) = \int_0^\infty e^{-\mu x}\Big(\tfrac{\mathscr{A}(t)^2\mathscr{B}(t)^J}{A_0^2} a(t, \tfrac{\mathscr{A}(t)}{A_0}\mu)\Big) d\mu. \tag{33}$$

Since the Laplace transform of $\frac{e^{-\varsigma/\alpha}}{\alpha}$ is $\frac{1}{\alpha z+1}$, equations (32) and (33) imply that, in a weak sense,

$$\lim_{t\to\infty} \frac{\mathscr{A}(t)^2\mathscr{B}(t)^J}{A_0^2} a(t, \tfrac{\mathscr{A}(t)}{A_0}\mu) = \frac{1}{\|a_0\|_1} e^{-\mu/\|a_0\|_1}. \tag{34}$$

The same conclusion holds true if, instead of taking the exact expressions for $\mathscr{A}(t)$ and $\mathscr{B}(t)$ in the left-hand side of (34) we use their asymptotic form as $t \to \infty$.

Inspecting Tables 2 and 3 we conclude that (34) holds, and a exhibits self-similar behaviour, when the parameters (J, κ) are in Cases (i), (ii), and (iv), and in the boundary sets $\Omega_{1,2} \cup \Omega_{4,1}$.

An entirely analogous argument can be applied to the equation for G in (6). The result for the Laplace transform limit is

$$\frac{\mathscr{B}(t)^\kappa \mathscr{A}(t)^J}{B_0} G\Big(t, \frac{x}{\mathscr{B}(t)^\kappa/B_0}\Big) \sim \frac{1}{\|b_0\|_1 x + 1}, \quad \text{as } t \to \infty, \tag{35}$$

valid in the cases when $\mathscr{B}(t) \to \infty$ as $t \to \infty$, and from this we similarly conclude that, in a weak sense,

$$\lim_{t\to\infty} \frac{\mathscr{B}(t)^{2\kappa}\mathscr{A}(t)^J}{B_0^2} b(t, \tfrac{\mathscr{B}(t)^\kappa}{B_0}\mu) = \frac{1}{\|b_0\|_1} e^{-\mu/\|b_0\|_1}. \tag{36}$$

As in (34), the same result will be obtained here by using, instead of the exact expressions for $\mathscr{A}(t)$ and $\mathscr{B}(t)$, their asymptotic form as $t \to \infty$.

Again by inspection of the Tables 2 and 3, (36) is valid, and b exhibits self-similar behaviour, when the parameters (J, κ) satisfy Cases (i), (iii), and (iv), and in $\Omega_{1,2} \cup \Omega_{4,1}$.

Both in (34) and in (36) the knowledge of the precise asymptotic behaviour of $\mathscr{A}(t)$ and $\mathscr{B}(t)$, (28), i.e., knowing both the scaling exponents and the multiplicative factors, is necessary for getting all the details of the scaling behaviour. However, without further effort, an inspection of the results in Tables 2 and 3 allow us to compute the dynamical exponents involved in the scaling behaviour (34) and (36), which are physically more interesting than the precise multiplicative constants.

Clearly, by not using the precise multiplicative factors $\mathscr{A}_\infty$ and $\mathscr{B}_\infty$, the exact constants in the similarity limit will be lost and the kind of information obtained will not be as precise as in (34) and (36), but something like

$$\lim_{t\to\infty} \psi_a(t) a(t, \varphi_a(t)\mu) = C_1^a e^{-\mu C_2^a}, \qquad \lim_{t\to\infty} \psi_b(t) b(t, \varphi_b(t)\mu) = C_1^b e^{-\mu C_2^b} \tag{37}$$

with appropriate scaling functions $\psi_{a,b}, \varphi_{a,b}$ and positive constants $C_1^{a,b}, C_2^{a,b}$.

Table 4 collects the results about the scaling functions $\psi_{a,b}$ and $\varphi_{a,b}$ in (37) that can be gathered from the data in Tables 2 and 3.

Table 4. Scaling functions $\psi_{a,b}(t)$ and $\varphi_{a,b}(t)$ for the similarity behaviour (37) of the a and b cluster distributions.

Parameter sets	$\psi_a(t)$	$\varphi_a(t)$	$\psi_b(t)$	$\varphi_b(t)$
$\kappa > J,\ J < 1$ $\kappa < J,\ J > 1$ (with initial conditions $B_0 = \alpha A_0/\beta$)	$t\varphi_a(t)$	$t^{\frac{J-\kappa}{J^2-\kappa}}$	$t\varphi_b(t)$	$t^{\kappa\frac{J-1}{J^2-\kappa}}$
$\kappa \geqslant J > 1$ $\kappa < J,\ J > 1$ (with initial conditions $B_0 < \alpha A_0/\beta$)	$t\varphi_a(t)$	t	—	—
$\kappa < J \leqslant 1$ $\kappa < J,\ J > 1$ (with initial conditions $B_0 > \alpha A_0/\beta$)	—	—	$t\varphi_b(t)$	t
$\kappa > J = 1$	$t\varphi_a(t)$	$\dfrac{t}{(\log t)^{\frac{1}{\kappa-1}}}$	$(t\log t)\varphi_b(t)$	$(\log t)^{\frac{\kappa}{\kappa-1}}$
$\kappa = J < 1$	$(t\log t)\varphi_a(t)$	$(\log t)^{\frac{1}{1-\kappa}}$	$t\varphi_b(t)$	$\dfrac{t}{(\log t)^{\frac{\kappa}{1-\kappa}}}$
$\kappa = J = 1$	$t\varphi_a(t)$	$t^{\frac{A_0}{A_0+B_0}}$	$t\varphi_b(t)$	$t^{\frac{B_0}{A_0+B_0}}$

5. Conclusion

In this paper we studied the self-similar behaviour of solutions to a two-species coagulation-annihilation cluster system with constant, but possibly distinct, reaction coefficients, (4)-(5). It consists of a generalization of the model studied in Laurençot and van Roessel (2010), for which the coagulation rates were constant and *equal*.

The approach starts with the application of Laplace transforms in the original system (4)-(5). This leads to the expressions (6) for the Laplace transforms of the cluster size distributions of each of the species, as functions of time and of the Laplace variable. These expressions involve the zero moments (3) of the cluster distributions of each of the species (i.e., their L^1-norm, or, in physical terms, the total amount of clusters of each species). In order to use (6) to get information on the similarity behaviour of the original cluster system (4)-(5), one needs to have information about the rates of convergence to zero of the moments (3). This is done by deriving an ordinary differential system for the time evolution of these quantities, and by a change of variables that transforms that system into a Lotka-Volterra competition system (10) an analysis of which provides the sought after information.

Considering the different regions in the parameter space for which the moments (3) have distinct rates of convergence to zero, we can use these rates and the expressions (6) to obtain the formulas (34) and (36) for the similarity limits of the two different cluster species, which are valid when the integrals (24) of the appropriate moments diverge to infinity as $\tau \to \infty$. Using the asymptotic rates of divergence of those functions (24) in

(34) and (36) we obtain the exponents governing the similarity behaviour of solutions for all interesting parameter cases, including those obtained in Laurençot and van Roessel (2010) as a very particular case.

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References

Ben-Naim E, Krapivsky P 1995 Kinetics of aggregation-annihilation processes *Phys. Rev. E* **52** 6066–6070.

Carr J 1981 *Applications of Center Manifold Theory* (*Applied Mathematical Sciences* vol 35) (New York: Springer-Verlag).

Hoyuelos M, Mártin, H O 1996 Annihilation and coagulation reactions in low-dimensional substrata: effects of probability of reaction and short range interactions *Langmuir* **12** 61–69.

Krapivsky P 1993 Nonuniversality and breakdown of scaling in two-species aggregation with annihilation *Physica A* **198** 135–149.

Laurençot P, van Roessel H 2010 Nonuniversal self-similarity in a coagulation-annihilation model with constant kernels *J. Phys. A: Math. Theor.* **43** 455210.

Lee B P, Cardy J 1995 Renormalization group study of the $A + B \to \varnothing$ diffusion-limited reaction *J. Stat. Phys.* **80** 5/6 971–1007.

Murray J 2002 *Mathematical Biology, I: An Introduction* 3rd Edition (*Interdisciplinary Applied Mathematics* vol. 17) (New York: Springer).

Privman V, Cadilhe A M R, Lawrence Glasser M 1995 Exact solutions of anisotropic diffusion-limited reactions with coagulation and annihilation *J. Stat. Phys.* **81** 5/6, 881–899.

Sokolov I M, Blumen A 1994 Kinetics in coagulation-annihilation processes *Phys. Rev. E* **50** 3, 2335–2338.