Proceedings of the Tenth Islamic Countries Conference on Statistical Sciences (ICCS-X), Volume I,
The Islamic Countries Society of Statistical Sciences, Lahore: Pakistan, (2010): 82–96.

BIB DESIGNS WITH REPEATED BLOCKS: REVIEW AND PERSPECTIVES

Teresa Azinheira Oliveira
CEAUL and DCeT, Universidade Aberta, Rua Fernão Lopes nº9, 2ºdto, 1000-132 Lisboa, Portugal
E-mail: toliveir@univ-ab.pt

ABSTRACT

Experimental Design plays an important role on establishing an interface between Applied Mathematics and statistical applications in several fields, like Agriculture, Industry, Genetics, Biology and Education Sciences. The goal of any Experimental Design is to obtain the maximum amount of information for a given experimental effort, to allow comparisons between varieties and to control for sources of random variability. Randomized block designs are used to control for these sources. A Balanced Incomplete Block Design (BIB Design) is a randomized block design with number of varieties greater than the block size and with all pairs of varieties occurring equally often along the blocks. The Fisher related information of a balanced block design will remain invariant whether or not the design has repeated blocks. This fact can be used theoretically to build a large number of non-isomorphic designs for the same set of design parameters, which could be used for many different purposes both in experimentations and surveys from finite populations. The original and most important method on the construction of BIB Designs with repeated blocks (BIBDR) is due to Hedayat and Li (1979): the trade-off method. Since then, many authors and researchers have been paying particular attention to the construction of BIBDR, but still some unsolved problems remain. This issue will be briefly reviewed and new results on the existence and construction of BIBDR, as well as several unsolved problems for further research will be presented.

Keywords: Experimental Design, BIBD, BIBD with Repeated Blocks.

1. INTRODUCTION AND REMARKS CONNECTED WITH LITERATURE

An experimental design is a set of rules by which the varieties (treatments) to be used in the experiment are assigned to experimental units, so to produce valid results as efficiently as possible. The importance of incomplete block designs is very well known in experiments involving a big number of varieties and a large class of incomplete block designs consists of the so-called balanced incomplete block designs (BIB Designs).

Yates (1936a) formally introduced BIB Designs in agricultural experiments and since that time several challenging problems concerning the construction, non-existence and combinatorial properties of BIB Designs have been posed. A BIB Design is a binary incomplete block design for v varieties in b blocks of size k, so that each variety occur exactly r times along the blocks and every pair of varieties concur in exactly λ blocks. The five integers v,b,r,k,λ are the parameters of the BIBD, and they are not independent.

Applications of BIB Designs are known for long, not only in Agriculture but also for example in Genetics, Industry and Education Sciences. See Raghavarao et al. (1986), Oliveira and Sousa (2002), Gosh and Shrivastava (2001) and Yang (1985).

More recently, applications of BIB Designs are present in fields such as: biotechnology, for example on microarray experiments, Groβmann and Schwabe (2007); Feeding Consume Sciences, Wakeling and Buck (2001); Electronic Engineering, Wireless and Communications problems, Camarda, and Fiume, (2007), Boggia et al. (2009); Computer Science and Code Theory, along with connections with Cryptography, Chakrabarti (2006).

Raghavarao and Padgett (2005) presented an entire chapter devoted to BIB Designs applications. The authors explain the importance and connections of BIB Designs to the following statistical areas: Finite Sample Support and Controlled Sampling, Randomized Response Procedure, Balanced Incomplete Cross Validation, Group Testing, Fractional Plans, Box-Behnken Designs, Intercropping Experiments, Validation Studies, Tournament and Lotto Designs, and Balanced Half-Samples.

BIB Designs are optimal for a number of criteria under the usual homocedastic linear additive model and this optimality is not affected if the design admits block repetition. In fact, several reasons may lead to the consideration of block repetition in the design: the costs of the design implementation may differ using or not block repetition, the experimenter may consider that some treatment combinations are preferable to others or that some treatment combinations must be avoid, see Foody and Hedayat (1977) and Hedayat and Hwang (1984). Also many experiments are affected by the loss of observations, and even in extreme cases all the information contained in one or more blocks is lost. If there are no repeated blocks in the design some of the elementary treatment contrasts are no longer estimable. To avoid these situations, repetitions of blocks in the design are recommended, since it allow rely on the information contained in another block, with the same composition of the lost block and overcoming the missing data problem.

Another reason for using BIBDR in a big number of applications is that, besides the variance expression for comparisons for each design being the same, the number of comparisons of block effects with the same variance is different for BIB Designs considering or not block repetition. BIB designs with repeated blocks have some block contrasts with minimum variance, Ghosh and Shrivastava (2001), Oliveira et al. (2006) and Mandal et al. (2008).

The important role of BIB Designs with repeated blocks in Experimental Design and in Controlled Sampling contexts is explained in Foody and Hedayat (1977) and Wynn (1977).

Important details, namely from the point of view of existence conditions, construction, particular characteristics and applications of BIBDR, as well as some topics for further research can be found for example in Van Lint (1973), Foody and Hedayat (1977), Hedayat and Li (1979), Hedayat and Hwang (1984), Raghavarao et al. (1986), Hedayat et al. (1989), Gosh and Shrivastava (2001). Raghavarao et al. (1986) considered a linear model that will distinguish designs with different support sizes, and they observed the relevance of this model in intercropping experiments and market research.

Besides many authors pay special attention to BIB Designs with repeated blocks, this is a very challenging area since there are yet many research questions with no answer. A review on important known results, some examples and unsolved problems will be presented.

2. IMPORTANT CONCEPTS ON BIB DESIGNS WITH REPEATED BLOCKS

2.1 BIB Designs and the Existence Problem

A BIB Design with parameters v,b,r,k,λ is denoted by BIBD(v,b,r,k,λ), and the following are the necessary but not sufficient conditions for the existence of a BIBD:

$$vr = bk\,;\; r(k-1) = \lambda(v-1)\,;\; b \geq v\,.$$

The condition $b \geq v$ is known as Fisher inequality.

For cases k=3 and k=4 to all λ and for k=5 and λ=1,4,20 Hanani (1961) proved that conditions $\lambda(v-1) \equiv 0(\mathrm{mod}(k-1))$ and $\lambda v(v-1) \equiv 0(\mathrm{mod}\, k(k-1))$ are sufficient for the existence of a BIB Design.

In some particular cases, even when a set (v, k, λ) satisfies the necessary conditions for the existence of a BIB Design, the actual design does not exist or its existence may be unknown. For these situations Hedayat et al. (1995) introduced two classes of designs: Contingently Balanced Incomplete Block (C-BIB) Designs and Virtually Balanced Incomplete Block (V-BIB) Designs. The authors show that both of these classes can be constructed by a sequential search algorithm.

2.2 The Existence of BIB Designs with Repeated Blocks

Consider a particular BIB design, and let B be a specific block randomly selected in this design. Let x_i, i = 0, 1, ..., k be the number of blocks apart from B itself, which have exactly i varieties in common with B.

It is known that for BIB designs with repeated blocks the following conditions hold:

$$\sum_{i=0}^{k}\binom{i}{0}x_i = b-1; \quad \sum_{i=1}^{k}\binom{i}{1}x_i = k(r-1); \quad \sum_{i=2}^{k}\binom{i}{2}x_i = \binom{k}{2}(\lambda-1)$$

In Sousa and Oliveira (2004) these conditions are developed to obtain a bound to the number of blocks, so that the design admits block repetition. Subtracting the third expression to the second and after some algebraic operations we have:

$$x_0 + \sum_{i=2}^{k}\frac{ix_i(i-3)}{2} + k(r-1) - \frac{k(k-1)(\lambda-1)}{2} + \sum_{i=2}^{k}x_i = b-1$$

Solving in order to b it becomes:

$$\text{(i)} \quad b = x_0 + \sum_{i=3}^{k}x_i\left[\frac{(i-1)(i-2)}{2}\right] + 1 + k(r-1) - \frac{k(k-1)(\lambda-1)}{2}$$

and since $x_0+\sum_{i=3}^{k} x_i[\frac{(i-1)(i-2)}{2}]\geq 0$, then: $b\geq 1+k(r-1)-k(k-1)(\lambda-1)/2$

Developing (i):

$$x_0+\sum_{i=3}^{k-1} x_i[\frac{(i-1)(i-2)}{2}]=b-1-k(r-1)+\frac{k(k-1)(\lambda-1)}{2}-\frac{(k-1)(k-2)}{2}x_k$$

then: $x_k \leq \dfrac{2b-2-2k(r-1)+k(k-1)(\lambda-1)}{(k-1)(k-2)}$

Since x_k denotes the number of blocks identical to block B in the design, then the design just admits repeated blocks if there is at least one block in the design identical to block B, which means, if $x_k\geq 1$. So, to $k\geq 3$:

$$1\leq x_k \leq \frac{2b-2-2k(r-1)+k(k-1)(\lambda-1)}{(k-1)(k-2)}$$

And, solving last inequality in order to b:

$$\text{(ii)} \qquad b\geq \frac{(k-1)(k-2)}{2}+1+k(r-1)-\frac{k(k-1)(\lambda-1)}{2}$$

If there exists a BIBD obeying the parameters (v,b,r,k,λ) then the number of blocks, b, should obey inequality (ii) so that the design admits block repetition.

2.3 Classification into Families

Hedayat and Hwang (1984) divided the collection of all BIB designs with $\lambda\geq 2$ into three mutually exclusive and exhaustive families:

Family 1 consists of all BIBD(v,b,r,k,λ) whose parameters (b,r,λ) have a common integer divisor, t>1, and there exists one or more BIB($v,b/t,r/t,k,\lambda/t$) designs.

Family 2 consists of all BIBD(v,b,r,k,λ) whose parameters (b,r,λ) have one or more common integer divisors greater than one but there is no BIB($v,b/t,r/t,k,\lambda/t$) design if t>1 is one of the divisors of b, r, and λ. No member of this family can be obtained by collecting or taking copies of smaller size BIB designs of type BIBD($v,b/t,r/t,k,\lambda/t$).

Family 3 consists of all BIBD(v,b,r,k,λ) whose parameters (b,r,λ) are relatively prime, thus the parameters v,b,r,k,λ for the members of this family are such that the great integer divisor of (b,r,λ) is one. As in family two, in family three no member can be obtained by collecting or taking copies of smaller size BIB designs of type BIBD($v,b/t,r/t,k,\lambda/t$).

2.4 Results on Possible Parameters for BIBDR Existence (K=3,4,5,6)

Considering the particular cases to k=3,4,5,6 and inequality (ii), we have, respectively the inequalities:

$$\lambda\left[(v-5)^2+2\right] \geq 2$$

$$\lambda\left[(v-8)^2-v+24\right] \geq 72$$

$$\lambda\left[(v-13)^2+56\right] \geq 240$$

$$\lambda\left[(v-18)^2-v+162\right] \geq 600$$

Parameters to possible BIBDR to particular cases k=3, 4, 5, 6, considering $5 \leq v \leq 30$, $\lambda \leq 10$ and $b \leq 200$, were obtained, by implementing an algorithm to obey known BIBDR existence conditions.

Considering the integer t, $t \geq 2$, if the BIBD with parameters presented in Tables 1,2,3,4 exits, then it allows block repetition. These Tables also present the design classification into families.

2.5 Blocks Multiplicity and Bounds

If in the BIB Design there are less than b distinct blocks then the design has repeated blocks. The set of all distinct blocks in a BIB Design is called the support of the design, and the design cardinality is represented by b^*. The notation BIBD($v,b,r,k,\lambda|b^*$) is used to denote a BIBD(v,b,r,k,λ) with precisely b^* distinct blocks.

Considering BIBDR it seems important to find the answer to some fundamental questions, like for example which blocks and how many times should it be repeated in the design, and what is the minimum admissible value to b* so that the design admits block repetition. Some more concepts are then crucial.

Table 1 Parameters to possible BIBDR, block size 3, and classification into families

v	b	r	k	λ	Family		v	b	r	k	λ	Family
5	10t	6t	3	3t	1		15	35t	7t	3	t	1
6	10t	5t	3	2t	1		16	80	15	3	2	3
7	7t	3t	3	t	1		16	80t	15t	3	2t	1
8	56	21	3	6	3		17	136	24	3	3	3
8	56t	21t	3	6t	1		17	136t	24t	3	3t	1
9	12t	4t	3	t	1*		18	102	17	3	2	3
10	30	9	3	2	3		18	102t	17t	3	2t	1
10	30t	9t	3	2t	1		19	57t	9t	3	t	1
11	55	15	3	3	3		21	70t	10t	3	t	1
11	55t	15t	3	3t	1		22	154	21	3	2	3
12	44	11	3	2	3		22	154t	21t	3	2t	1
12	44t	11t	3	2t	1		24	184	23	3	2	3
13	26t	6t	3	t	1		24	184t	23t	3	2t	1
14	182	39	3	6	3		25	100t	12t	3	t	1
14	182t	39t	3	6t	1							

Table 2 Parameters to possible BIBDR, block size 4, and classification into families

v	b	r	k	λ	Family		v	b	r	k	λ	Family
5	5t	4t	4	3t	1		16	20t	5t	4	t	1
6	15	10	4	6	3		17	68	16	4	3	3
6	15t	10t	4	6t	1		17	68t	16t	4	3t	1
7	7t	4t	4	2t	1		18	153	34	4	6	3
8	14t	7t	4	3t	1		18	153t	34t	4	6t	1
9	18t	8t	4	3t	1		19	57	12	4	2	3
10	15t	6t	4	2t	1		19	57t	12t	4	2t	1
11	55	20	4	6	3		20	95	19	4	3	3
11	55t	20t	4	6t	1		20	95t	19t	4	3t	1
12	33	11	4	3	3		21	105	20	4	3	3
12	33t	11t	4	3t	1		21	105t	20t	4	3t	1
13	13t	4t	4	t	1*		22	77	14	4	2	3
14	91	26	4	6	3		22	77t	14t	4	2t	1
14	91t	26t	4	6t	1		24	138	23	4	3	3
15	105	28	4	6	3		25	50t	8t	4	t	1
15	105t	28t	4	6t	1		28	63t	9t	4	t	1

Table 3 Parameters to possible BIBDR, block size 5, and classification into families

v	b	r	k	λ	Family	v	b	r	k	λ	Family
6	6t	5t	5	4t	1	17	68	20	5	5	3
7	21	15	5	10	3	17	68t	20t	5	5t	1
7	21t	15t	5	10t	1	19	171	45	5	10	3
9	18	10	5	5	3	19	171t	45t	5	10t	1
9	18t	10t	5	5t	1	20	76	19	5	4	3
10	18	9	5	4	3	20	76t	19t	5	4t	1
10	18t	9t	5	4t	1	21	21t	5t	5	t	1
11	11t	5t	5	2t	1	25	30t	6t	5	t	1
13	39	15	5	5	3	26	130	25	5	4	3
13	39t	15t	5	5t	1	26	130t	25t	5	4t	1
15	21t	7t	5	2t	2*	30	174	29	5	4	3
16	48	15	5	4	3	30	174t	29t	5	4t	1
16	48t	15t	5	4t	1						

* There is no BIBD(15,21,7,5,2), so BIBDR(15,42,14,5,4) belongs to family 2, according to Hedayat and Hwang (1984).

Table 4 Parameters to possible BIBDR, block size 6, and classification into families

v	b	r	k	λ	Family	v	b	r	k	λ	Family
7	7t	6t	6	5t	1	21	14p	4p	6	p	1
9	12t	8t	6	5t	1*	22	77	21	6	5	3
10	15t	9t	6	5t	1	22	77t	21t	6	5t	1
11	11t	6t	6	3t	1	24	92	23	6	5	3
12	22	11	6	5	3	24	92t	23t	6	5t	1
12	22t	11t	6	5t	1	25	100	24	6	5	3
13	26	12	6	5	3	25	100t	24t	6	5t	1
13	26t	12t	6	5t	1	26	65	15	6	3	3
15	35	14	6	5	3	26	65t	15t	6	3t	1
15	35t	14t	6	5t	1	27	117	26	6	5	3
16	16t	6t	6	2t	1	27	117t	26t	6	5t	1
18	51	17	6	5	3	28	126	27	6	5	3
18	51t	17t	6	5t	1	28	126t	27t	6	5t	1
19	57	18	6	5	3	30	145	29	6	5	3
19	57t	18t	6	5t	1	30	145t	29t	6	5t	1

The multiplicity of a block is the number of times the block occurs in the design. If in a BIB Design there are exactly α blocks with multiplicity i, exactly β blocks with multiplicity j,…, and all other blocks with multiplicity 1, then design has multiplicity pattern $\alpha^i \beta^j$.

If in a BIB Design with parameters (v,b,r,k,λ) each block occurs exactly m times ($m>1$), then the design is called a multiple block design of multiplicity m and is denoted by M-BIBD (v,b,r,k,λ).

Let s represents the number of identical blocks in a design. The conditions $\lambda \geq 2$ and $s \leq \lambda$ are necessary for the existence of a design with $b^*<b$. For a BIBD (v,b,r,k,λ) with s identical blocks and $r>\lambda$, Mann (1969) proved that $\frac{r}{k}=\frac{b}{v}\geq s$. This inequality covers the Fisher's inequality (case s=1). Van Lint and Ryser (1972) proved that the support size b* of a BIB Design satisfies $b^*=v$ or b*>v+1. Foody and Hedayat (1977) established new limits to b^*, namely they reach the expression $b^*_{min} \geq \frac{b}{\lambda}$ where b^*_{min} represents the minimal support size to a BIBD based on v and k. Constantine and Hedayat (1983) present the construction of complete designs with blocks of maximal multiplicity and set its relevance in simple random sampling context.

For particular cases with $b<\lambda v$ and using the inequality of Mann (1969), Hedayat et al. (1989) obtained a more restrictive bound to b^*_{min} by considering the information about those blocks in the support that are repeated λ times in the design:

$$b^*_{min} \geq 2\frac{v(v-1)}{k(k-1)}$$

2.6 The Variance for Block Effect Contrasts

Consider a BIBDR $(v, b, r, k, \lambda | b^*)$ with the incidence matrix $N=(n_{ij})$, where n_{ij} is the number of times that the ith variety occur in the jth block, $i=1,2,...,v$, $j=1,2,...,b$. For this design the coefficient matrix D for estimating a vector of block effects $\beta=(\beta_1,\beta_2,...,\beta_b)'$ has the form $D=kI_b-\frac{1}{r}N'N$, where I_b denote $b\times b$ identity matrix.

Consider any two blocks of the design, B_j and $B_{j'}$, which have h varieties in common. Then the (j,j')th element of the matrix $N'N$ is equal to h and the variance of the difference of block effects is given by:

$$Var(\hat{\beta}_j-\hat{\beta}_{j'})=2\sigma^2(v\lambda+k-h)/vk\lambda .$$

Since h can take at most $k+1$ values, namely 0,1,…, k, so there will be at most $k+1$ possible variances for the estimated elementary block effect contrasts.

Note that the variance of estimated contrasts of block effects tends to the minimum value as the number of common treatments between the two blocks increases. For BIBDR we can have some block contrasts with minimum variance and this is one of the reasons for using it in a big number of applications, Oliveira et al. (2006).

3. BRIEF REVISION ON THE CONSTRUCTION METHODS FOR BIB DESIGNS AND BIBDR

Bose (1939) introduced the method of cyclical development of initial blocks, which allows the construction of most of the existing BIB Designs. Since then, several methods have been developed for the construction of BIB Designs. The most important ones are presented in Hinkelmann and Kempthorne (2005). These authors explain some Difference Methods, such as Cyclic Development of Difference Sets, the Method of Symmetrically Repeated Differences and the Formulation in Terms of Galois Field Theory and present some other methods using particular cases like irreducible BIB Designs, complement of BIB Designs, residual BIB Designs and Orthogonal Series. Also in Raghavarao and Padgett (2005) the construction of BIB Designs from Finite Geometries and by the Method of Differences is presented.

Raghavarao (1971) presents the complete list of existing BIB Designs obeying the conditions $v, b \leq 100$ and $r, k \leq 15$. Hinkelmann and Kempthorne (2005) present a list of BIB Designs with $v \leq 25$, $k \leq 11$. Julian et al. (2006), Chapter 3, present the existence results for BIB Designs with small block sizes, k≤9, and point some values of v for which the existence of a BIB Design remains undecided.

Some construction methods have been also developed for the particular BIB Designs with repeated blocks. Van Lint and Ryser (1972) studied this problem and their basic interest was in constructing BIBD (v, b, r, k, λ) with repeated blocks such that parameters b, r, λ were relatively prime. Foody and Hedayat (1977) showed that the combinatorial problem of searching BIBDR was equivalent to the algebraic problem of finding solutions to a set of homogeneous linear equations and presented the construction of BIB designs with $v=8$ and $k=3$ with support sizes 22 to 55.

Hedayat and Li (1979) presented one of the still most important methods for the construction of BIB Designs with variable support sizes: the trade-off method. Using this method Hedayat and Hwang (1984) presented the construction of BIBDR(8,56,21,3,6) and BIBDR(10,30,9,3,2). The basic idea is to trade some blocks with some other blocks, without losing the BIB general characteristics of the design. The resulting design may differ from the original in what concerns the support size.

Hedayat and Hwang (1984) call the following set of blocks a $(v,3)$ trade of volume 4 for any $v \geq 6$, where 6 varieties (x, y, z, u, v, w) appear in each set of four blocks:

I	II
xyz	uvw
xuv	yzw
yuw	xzv
zvw	xyu

If in a design we replace blocks in set I by blocks in set II we do not loose BIB property. We will use this method in our examples.

Trades properties are very important. In Handbook of Combinatorial Designs (2006), Chapter 60, by Hedayat and Khosrovshah, is entirely devoted to Trades and related references.

Several authors have been paying particular attention to the construction and properties of *M*-BIB(7,7,3,3,1) Designs. Raghavarao et al. (1986) presented characteristics for distinguishing among balanced incomplete block designs with repeated blocks and obtained a class of BIB Designs β(7,21,9,3,3) where nine BIB Designs out of ten have repeated blocks. In Gosh and Shrivastava (2001) a method for the BIBDR construction, based on a composition of two semi-regular group divisible designs and respective parameters was presented. The authors present a class of BIB designs with repeated blocks and the example of BIBDR (7,28,12,3,4) construction, as well as a table with the respective multiplicity of occurrence of variance for the estimated elementary contrast of block effects. These authors include the comparison of fifteen BIBD with the same parameters, according to the number of distinct blocks and to the multiplicities of variance of elementary contrasts of the block effect. Fourteen of these designs have repeated blocks. Mandal et al. (2008) present a complete class of BIB Designs β(7,35,15,3,5), with thirty one BIB Designs, where one of them has no block repetition and the remaining thirty are BIB Designs with repeated blocks.

Oliveira and Sousa (2004) presented examples of BIBDR(12,44,11,3,2) with different structures for the same cardinality and Sousa and Oliveira (2004) presented the cardinality analysis for BIBDR (9,24,8,3,2). In the Handbook of Combinatorial Designs (2006), Manthon, R. and Rosa, A. chapter 1, results on BIBD (v,b,r,k,λ) of small order are presented, as well as a list of nonisomorphic such designs for particular values of v. Also a table with admissible parameter sets of nontrivial BIB Designs with $r \leq 41$ and $k \leq v/2$ is presented, as well as references for earlier listings of BIB Designs.

4. BIBDR STRUCTURES: SOME EXAMPLES

We will present some examples of BIBDR with different parameters and also some BIBDR with the same parameters and different support sizes. Finally for BIBDR with the same support size we will present BIB Designs with different structures for repeated blocks.

In example 1 we will consider BIBDR(13,26,8,4,2) from Table II and we present one possible structure, the most trivial one, with all blocks repeated once, b*=13. In example 2 we will consider BIBDR (9,24,8,3,2) from Table I, and we will use the trade-off method to obtain designs with two different cardinalities, b*=18 and b*=20. Finally we will present two possible structures for repeated blocks in BIBDR with the same cardinality .

Example 1

The most trivial way of getting a design from an existing one is simply by repeating the blocks in the original design. According to Calinski and Kageyama (2003) this is so-called juxtaposition method. Consider the varieties 1, …, 9, x, y, z, w. The representation for BIBD (13,13,4,4,1) structure can be:

BIBD(13,13,4,4,1)

1234	1567	189*x*	1*yzw*
258*y*	269*w*	27*xw*	
359*w*	36*xy*	378*z*	
45*xz*	468*w*	479*y*	

We can easily obtain the structure for BIBDR(13,26,8,4,2|b*=13), considering block multiplicity equal 2 and duplicating BIB(13,13,4,4,1). When we wish to compare any two blocks of the design we have $\binom{26}{2} = 325$ possible comparisons, and to any two blocks which have four common varieties we have contrasts with minimum variance.

Example 2

In this example we present BIBDR (9,24,8,3,2) with different cardinalities, b*=18, b*=20 and b*=21. We illustrate the trade-off method application to obtain a possible structure to BIBDR (9,24,8,3,2/*b*=20*) by using BIBDR (*9,24,8,3,2/b*=18*). We present two different structures for repeated blocks considering the BIBDR (9,24,8,3,2/*b*=21*).

Design with 6 repeated blocks — Design with 4 repeated blocks

BIBDR(9,24,8,3,2/*b*=18*) TRADE BIBDR(9,24,8,3,2/*b*=20*)

123	247	359		123	247	358
123	249	359		123	248	359
147	257	367	348→259	147	257	367
149	258	367	359→248	149	259	367
157	268	456	249→358	157	268	456
158	269	456	258→349	158	269	456
168	348	789		168	348	789
169	348	789		169	349	789

Example 3

In this example we will consider the definition of complementary design to illustrate the construction of BIBDR (*9,24,16,6,10/b*=21*), and two different structures for repeated blocks with the same cardinality. We will use results of example 2).

Design with 3 repeated blocks BIBDR (*9,24,8,3,2|b*=21*)

Structure 1				Structure 2		
123	248	357		123	246	357
123	249	358		123	246	358
147	257	368		145	257	367
148	259	369		145	259	369
158	267	456		167	278	478
159	268	456		168	289	479
167	347	789		179	348	568
169	349	789		189	349	569

Definition of complementary design is very important, particularly for cases with large k size. The complement of a BIBD(v, b, r, k, λ) is also a BIB design with b blocks each of size v-k, so the existence of a BIBD with block size k implies the existence of another BIBD with block size v-k.

A simple technique for constructing one BIB Design from another is to replace the varieties in each block by the set of varieties that are not in the block, known as the complementary set of varieties.

To k=6 and using complementary designs of BIBDR (*9,24,8,3,2*) presented in example 2, examples follows to BIBDR (*9,24,16,6,10|b*=21*) structure.
Consider the varieties 1,…,9, and block size equal to six.

BIBDR (9,24,16,6,10/*b*=21*)- Structure 1

456789	135679	124689
456789	135678	124679
235689	134689	124579
235679	134678	124578
234679	134589	123789
234678	134579	123789
234589	125689	123456
234578	125678	123456

Structure 1: We have three blocks repeated twice and18 non-repeated blocks. Any of the nine varieties appear twice in repeated blocks;

BIBDR (*9,24,16,6,10|b*=21*)- Structure 2:

456789	135789	124689
456789	135789	124679
236789	134689	124589
236789	134678	124578
234589	134569	123569
234579	134567	123468
234568	125679	123479
234567	125678	123478

Structure 2: We have three blocks repeated twice and18 non-repeated blocks. Varieties 1,2,4 appear once in repeated blocks, varieties 5,3,6 appear twice in repeated blocks and varieties 7,8,9 appear three times in repeated blocks.

5. CONSIDERATIONS AND PERSPECTIVE RESEARCH

The construction of BIB Designs for some particular parameters combinations still remain unsolved, and so are the questions connected with the respective BIBDR. This is an important issue to develop, as well as the necessary or sufficient conditions for the existence of a BIBDR and given parameters. Bounds for the multiplicity of a block in such a BIBD need also more investigation, since for many BIBDR there are still open questions.

In further research, besides continuing the cardinality analysis for particular BIBDR, it seems also important to investigate the relations between the number of possible structures and the design parameters, considering BIBDR with the same cardinality.

Applications of BIB Designs and BIBDR are known in many diverse fields, but there are yet many others to explore and improve, using this area of knowledge as a new challenge.

REFERENCES

Boggia G., Camarda, P., Cormio, C., and Grieco, L.A. (2009): *Iranian Journal of Electrical & Electronic Engineering, Vol. 5, No. 2*

Bose, R.C. (1939): On the construction of balanced incomplete block designs, Annals of Eugenics. 9 , pp. 358–399.

Calinski, T. and Kageyama, S. (2003): *Block Design: A Randomization Approach,* Volume II: Design, Lecture Notes in Statistics, Springer.

Camarda, P. and Fiume, O. (2007): Collision Free MAC Protocols for Wireless Ad Hoc Networks based on BIBD Architecture. Journal of Communications, Vol. 2, number 7, pp 1-8.

Chakrabarti (2006): A new visual cryptography scheme for STS based Access structures. Journal of Discrete Mathematical Sciences & Cryptography, Vol.9, n 1, pp9-23.

Colbourn and Dinitz (2006): *The CRC Handbook of Combinatorial Designs, CRC Press. Chapman & Hall.*

Constantine, C. M. and Hedayat, A. (1983): *Complete designs with blocks of maximal multiplicity.* Journal of Statistical Planning and Inference, 7, 289-294.

Foody, W, and Hedayat, A. (1977): *On theory and applications of BIB Designs with repeated blocks.* The Annals of Statistics, Vol 5, nº5, 932 - 945.

Großmann, H. and Schwabe, R. (2007): The relationship between optimal designs for microarray and paired comparison experiments. Otto–von–Guericke–Universität Magdeburg, Fakultät für Mathematik, Preprint Nr. 23 / 2007.

Gosh and Shrivastava (2001): *A class of BIB designs with repeated blocks.* Journal of Applied Statistics, Vol.28, nº7, 821-833.

Hanani, H. (1961): *The existence and construction of Balanced Incomplete Block Designs.* The Ann. Math. Statist., Vol 32, 361 - 386.

Hedayat, A. S., and Hwang, H. L. (1984): *BIB(8,56,21,3,6) and BIB(10,30,9,3,2) Designs with Repeated Blocks.* Journal of Combinatorial Theory, Series A 36 , 73 - 91.

Hedayat, A.S., Landgev, I.N. and Tonchev, V.D. (1989): *Results on the support of BIB designs.* Journal of Statistical Planning and Inference, 22, 295-306.

Hedayat, A, and Shuo-Yen R. Li (1979): *The trade off method in the construction of BIB Designs with repeated blocks.* The Annals of Statistics , Vol.7, nº6, 1277- 1287.

Hedayat, A, Stufken, J. and Zhang, W.G. (1995): *Contingently and Virtually Balanced Incomplete Block Designs and their efficiencies under various optimality criteria.* Statistica Sinica 5, 575-591.

Hinkelmann, K. and Kempthorne, O. (2005): *Design and Analysis of Experiments, Advanced Experimental Design*, Vol.2. Wiley Series in Probability and Statistics, Wiley-Interscience.

Mandal, S., Ghosh, D.K., Sharma, R.K. and Bagui, S.C. (2008): *A complete class of balanced incomplete block designs $\beta(7,35,15,3,5)$ with repeated blocks.* Statistics and Probability Letters , Vol 78, Issue 18, 3338-3343.

Mann, H.B. (1969): *A note on balanced incomplete block designs.* The Annals of Math. Statist., Vol.40, nº2, 679-680.

Oliveira, T.A., Sousa, M.F. (2002): *Some correspondence between Diallel Crosses and BIB Designs.* Proceedings of the 17th IWSM, 515-520.

Oliveira, T.A., Sousa, M.F. (2004): *BIBD with repeated block: some analyses on BIBD(12,44,11,3,2) with different structures.* Colloquium Biometryczne, 34a,131-140.

Oliveira, T.A., Ceranka, B., Graczyk, M. (2006): *The variance of the difference of block effects in the balanced incomplete block designs with repeated blocks*, Colloquium Biometryczne, 36, 2006, 115-124.

Raghavarao, D. (1971): *Constructions and Combinatorial Problems in Design of Experiments*, John Wiley and Sons, New York.

Raghavarao, D., Federer, W.T. and Schwager, S.J. (1986): *Characteristics for distinguish among balanced incomplete block designs with repeated blocks*. Journal of Statistical Planning and Inference 13, 151-163.

Raghavarao, D. and Padgett, L.V. (2005): *Block Designs: Analysis, Combinatorics and Applications*. Series on Applied Mathematics, World Scientific Publishing Co.

Sousa, M.F., Oliveira, T.A. (2004): *BIBDR: Some analysis on BIBD(*9,24,8,3,2*) cardinalities*. Colloquium Biometryczne, 34a,161-170.

Van Lint, J.H. (1973): *Block Designs with repeated blocks and* $(b,r,\lambda)=1$. J. Combinatorial Theory Ser A 15, 288 - 309.

Van Lint, J.H. and Ryser , H.J. (1972): *Block Designs with repeated blocks*. Discrete Math., Vol 3, 381 - 396.

Wynn, H.P. (1977): *Convex sets of finite plans*. Ann. Statist.,Vol5,nº2, 414-418.

Wakeling, I.N. and Buck, D. (2001): *Balanced incomplete block designs useful for consumer experimentation*. Food Quality and Preference, 12, 265-268.

Yang, J. (1985): *An Example of the application of a Balanced Incomplete Block Design.* Assessment & Evaluation in Higher Education, 1469-297X, Volume 10, Issue 3, 250 – 253

Yates, F. (1936a). Incomplete randomized blocks, Ann. Eugen. 7, 121–140.