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Holomorphic Bogoliubov functionals for interacting particle systems in continuum

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Abstract

Combinatorial harmonic analysis techniques are used to develop new analytical methods for the study of interacting particle systems in continuum based on a Bogoliubov functional approach. Concrete applications of the methods are presented, namely, conditions for the existence of Bogoliubov functionals, a uniqueness result for Gibbs measures in the high temperature regime. We also propose a new approach to the study of non-equilibrium stochastic dynamics in terms of evolution equations for Bogoliubov functionals.

Keywords: Configuration spaces; Generating functional; Continuous system; Gibbs measure; Stochastic dynamics

1. Introduction

The combinatorial harmonic analysis on configuration spaces introduced and developed in [15,17,19,22] is a natural tool for the study of equilibrium states of continuous systems in terms of the corresponding Bogoliubov or generating functionals. Originally, this class of functionals

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was introduced by N.N. Bogoliubov in [4] to define correlation functions for statistical mechanics systems. In the context of classical statistical mechanics, this class of functionals, as a basic concept, was analyzed by G.I. Nazin. We refer to [27] for historical remarks and references therein. Apart from this specific application, and many others, the Bogoliubov functionals are, by themselves, a subject of interest in infinite-dimensional analysis. This is partially due to the fact that to any probability measure μ defined on the space Γ of locally finite configurations one may associate a Bogoliubov functional

$$L_\mu(\theta) := \int_\Gamma \prod_{x\in\gamma}\big(1+\theta(x)\big)\,d\mu(\gamma),$$

allowing the study of μ through the functional L_μ. Technically, this means that through the Bogoliubov functionals one may reduce measure theory problems to functional analysis ones, yielding a new method in measure theory as well as new applications in functional analysis.

From this standpoint, new perspectives were announced in [19] in the setting of combinatorial harmonic analysis on configuration spaces. The purpose of this work is to carry out these technical improvements.

Of course, the domain of a Bogoliubov functional L_μ depends on the underlying probability measure μ. Conversely, the domain of a Bogoliubov functional L_μ carries special properties over to the probability measure μ. In this work we mainly analyze the class of entire Bogoliubov functionals on a L^1-space (Section 3), which is a natural environment to widen the scope of this work towards Gibbs measures (or equilibrium states). This restriction allows, in particular, to recover the notion of correlation function.

As a side remark, let us mention that in the same setting further progresses, under holomorphy assumptions, are achieved in [23] on a space of continuous functions.

The close relation between probability measures and Bogoliubov functionals is best illustrated by a "dictionary" (cf. G.I. Nazin), relating measure concepts and problems to functional analysis ones. In this "dictionary," the translation of the Dobrushin–Lanford–Ruelle equation, defining Gibbs measures, leads to a functional equation, called the Bogoliubov (equilibrium) equation (Section 4). As a result, through analytical techniques one may derive a uniqueness result for Gibbs measures corresponding to positive potentials in the high temperature-low activity regime (Theorem 26). Although this result does not improve the known uniqueness results for Gibbs measures (see, e.g., [9,31,33]), the proof inspired by the classical work of D. Ruelle presents an alternative and natural treatment of the uniqueness problem.

This work concludes with the presentation of a new method for the study of the non-equilibrium stochastic dynamics of continuous systems based on Bogoliubov functionals (Section 5). This method reformulates the problem in terms of evolution equations for holomorphic functions over an infinite-dimensional space. The scheme proposed is described, for concreteness, in the case of the gradient diffusion dynamics. The existence problem of these dynamics has been well analyzed for the equilibrium case using Dirichlet forms techniques (see, e.g., [1,29,39]). For the non-equilibrium case, the existence problem is essentially open and at the moment all we have is the technically very involved construction of non-equilibrium processes done by J. Fritz in [12], in the case of smooth potentials with finite range and $d \leqslant 4$, or the existence of the short time evolution for correlation functions described by a correlation diffusion hierarchy, see [21] and the references therein.

The dynamical description by Bogoliubov functionals may also be applied to other types of stochastic dynamics of infinite particle systems as well. For a particular type of dynamics, namely, Glauber type dynamics in the continuum, the corresponding evolution equations for Bogoliubov functionals may be solved using the results of [20]. This and other cases are now being studied and will be reported in forthcoming publications.

2. Harmonic analysis on configuration spaces

Let X be a geodesically complete connected oriented Riemannian C^∞-manifold[1] and $\Gamma := \Gamma_X$ the configuration space over X:

$$\Gamma := \big\{\gamma \subset X\colon\ |\gamma \cap K| < \infty \text{ for every compact } K \subset X\big\}.$$

Here $|\cdot|$ denotes the cardinality of a set. As usual we identify each $\gamma \in \Gamma$ with the non-negative Radon measure $\sum_{x\in\gamma}\varepsilon_x \in \mathcal{M}(X)$, where ε_x is the Dirac measure with mass at x, $\sum_{x\in\emptyset}\varepsilon_x$ is, by definition, the zero measure, and $\mathcal{M}(X)$ denotes the space of all non-negative Radon measures on the Borel σ-algebra $\mathcal{B}(X)$. This identification allows to endow Γ with the topology induced by the vague topology on $\mathcal{M}(X)$. We denote the Borel σ-algebra on Γ by $\mathcal{B}(\Gamma)$.

Another description of the measurable space $(\Gamma, \mathcal{B}(\Gamma))$ is also possible. For each $Y \in \mathcal{B}(X)$, let Γ_Y be the space of all configurations contained in Y, $\Gamma_Y := \{\gamma \in \Gamma\colon\ |\gamma \cap (X\setminus Y)| = 0\}$, and let $\Gamma_Y^{(n)}$ be the subset of all n-point configurations, $\Gamma_Y^{(n)} := \{\gamma \in \Gamma_Y\colon\ |\gamma| = n\}$, $n \in \mathbb{N}$, $\Gamma_Y^{(0)} := \{\emptyset\}$. For $n \in \mathbb{N}$, there is a natural surjective mapping of

$$\widetilde{Y^n} := \big\{(x_1, \ldots, x_n)\colon\ x_i \in Y,\ x_i \neq x_j \text{ if } i \neq j\big\}$$

onto $\Gamma_Y^{(n)}$ defined by

$$\begin{aligned} \mathrm{sym}_Y^n : \widetilde{Y^n} &\to \Gamma_Y^{(n)} \\ (x_1, \ldots, x_n) &\mapsto \{x_1, \ldots, x_n\}. \end{aligned} \tag{1}$$

This leads to a bijection between the space $\Gamma_Y^{(n)}$ and the symmetrization $\widetilde{Y^n}/S_n$ of $\widetilde{Y^n}$ under the permutation group S_n over $\{1, \ldots, n\}$, and then to a metrizable topology on $\Gamma_Y^{(n)}$. We denote the corresponding Borel σ-algebra on $\Gamma_Y^{(n)}$ by $\mathcal{B}(\Gamma_Y^{(n)})$. For $\Lambda \in \mathcal{B}(X)$ with compact closure (shortly $\Lambda \in \mathcal{B}_c(X)$), one clearly has $\Gamma_\Lambda = \bigsqcup_{n=0}^\infty \Gamma_\Lambda^{(n)}$. In this case we endow Γ_Λ with the topology of the disjoint union of topological spaces and with the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_\Lambda)$. It is easy to check that $\mathcal{B}(\Gamma_\Lambda)$ is equal to

$$\mathcal{B}(\Gamma_\Lambda) = \sigma\big(\big\{\gamma \in \Gamma_\Lambda\colon\ |\gamma \cap \Lambda'| = n\big\}\big), \quad \Lambda' \in \mathcal{B}_c(X),\ n \in \mathbb{N}_0.$$

The measurable space $(\Gamma, \mathcal{B}(\Gamma))$ is the projective limit of the measurable spaces $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$, $\Lambda \in \mathcal{B}_c(X)$, with respect to the projections

[1] The results stated in Sections 2–4 can be generalized to X being a Polish space. However, in this case, one has to work with bounded sets instead of compact sets as local sets, and all definitions have to be properly adjusted. For a generalization of the notion of local sets see [32].

$$\begin{aligned} p_\Lambda : \Gamma &\to \Gamma_\Lambda \\ \gamma &\mapsto \gamma_\Lambda := \gamma \cap \Lambda. \end{aligned} \tag{2}$$

A measurable function F is called a cylinder function with domain of cylindricity a $\Lambda \in \mathcal{B}_c(X)$ if there is a measurable function $F_\Lambda : \Gamma_\Lambda \to \mathbb{R}$ such that $F = F_\Lambda \circ p_\Lambda$.

Apart from the spaces described above we also consider the space of finite configurations

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \Gamma_X^{(n)}$$

endowed with the topology of disjoint union of topological spaces and with the corresponding Borel σ-algebra denoted by $\mathcal{B}(\Gamma_0)$.

To define the K-transform (cf. [24]), among the functions defined on Γ_0 we distinguish the space $B_{\exp,\mathrm{ls}}(\Gamma_0)$ of all complex-valued exponentially bounded $\mathcal{B}(\Gamma_0)$-measurable functions G with local support, i.e., $G\restriction_{\Gamma_0 \setminus \Gamma_\Lambda} \equiv 0$ for some $\Lambda \in \mathcal{B}_c(X)$ and there are $C_1, C_2 > 0$ such that $|G(\eta)| \leqslant C_1 e^{C_2|\eta|}$ for all $\eta \in \Gamma_0$. The K-transform of any $G \in B_{\exp,\mathrm{ls}}(\Gamma_0)$ is the mapping $KG : \Gamma \to \mathbb{C}$ defined at each $\gamma \in \Gamma$ by

$$(KG)(\gamma) := \sum_{\substack{\eta \subset \gamma \\ |\eta| < \infty}} G(\eta). \tag{3}$$

Note that for every $G \in B_{\exp,\mathrm{ls}}(\Gamma_0)$ the sum in (3) has only a finite number of summands different from zero and thus KG is a well-defined cylinder function. Moreover, given a G described as before, Λ is a domain of cylindricity of KG and for all $\gamma \in \Gamma$ one has $|(KG)(\gamma)| \leqslant C_1 e^{(C_2+1)|\gamma_\Lambda|}$.

Throughout this work the so-called (Lebesgue–Poisson) coherent states $e_\lambda(f)$ of $\mathcal{B}(X)$-measurable functions f, defined by

$$e_\lambda(f, \eta) := \prod_{x \in \eta} f(x), \quad \eta \in \Gamma_0 \setminus \{\emptyset\}, \qquad e_\lambda(f, \emptyset) := 1,$$

will play an essential role (see also Remark 3). This is partially due to the fact that the K-transform of this class of functions coincides with the integrand functions of the Bogoliubov functionals (Section 3). More precisely, for every bounded $\mathcal{B}(X)$-measurable function f with bounded support (shortly $f \in B_{\mathrm{bs}}(X)$) one has $e_\lambda(f) \in B_{\exp,\mathrm{ls}}(\Gamma_0)$, and

$$\bigl(K e_\lambda(f)\bigr)(\gamma) = \prod_{x \in \gamma} \bigl(1 + f(x)\bigr), \quad \gamma \in \Gamma.$$

Besides the K-transform, we also consider the dual operator K^*. Let $\mathcal{M}^1_{\mathrm{fexp}}(\Gamma)$ denote the set of all probability measures μ on $(\Gamma, \mathcal{B}(\Gamma))$ with finite local exponential moments, i.e.,

$$\int_\Gamma e^{\alpha|\gamma_\Lambda|}\, d\mu(\gamma) < \infty \quad \text{for all } \Lambda \in \mathcal{B}_c(X) \text{ and all } \alpha > 0.$$

By the definition of a dual operator, given a $\mu \in \mathcal{M}^1_{\mathrm{fexp}}(\Gamma)$, $K^*\mu =: \rho_\mu$ is a measure defined on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ by

$$\int_{\Gamma_0} G(\eta)\, d\rho_\mu(\eta) = \int_{\Gamma} (KG)(\gamma)\, d\mu(\gamma), \tag{4}$$

for all $G \in B_{\mathrm{exp,ls}}(\Gamma_0)$. The measure ρ_μ is called the correlation measure corresponding to μ. For more details see, e.g., [15,24]. This definition shows, in particular, that[2]

$$B_{\mathrm{exp,ls}}(\Gamma_0) \subset L^1(\Gamma_0, \rho_\mu).$$

Moreover, on the dense set $B_{\mathrm{exp,ls}}(\Gamma_0)$ in $L^1(\Gamma_0, \rho_\mu)$ the inequality $\|KG\|_{L^1(\mu)} \leqslant \|G\|_{L^1(\rho_\mu)}$ holds, allowing an extension of the K-transform to a bounded operator $K : L^1(\Gamma_0, \rho_\mu) \to L^1(\Gamma, \mu)$ in such a way that equality (4) still holds for any $G \in L^1(\Gamma_0, \rho_\mu)$. For the extended operator the explicit form (3) still holds, now μ-a.e. This means, in particular,

$$\big(Ke_\lambda(f)\big)(\gamma) = \prod_{x \in \gamma} \big(1 + f(x)\big), \qquad \mu\text{-a.a. } \gamma \in \Gamma, \tag{5}$$

for all $\mathcal{B}(X)$-measurable functions f such that $e_\lambda(f) \in L^1(\Gamma_0, \rho_\mu)$, cf., e.g., [15].

Remark 1. All the notions described above as well as their relations are graphically summarized in Fig. 1. Having in mind the concrete application in Section 5, let us mention the natural meaning of this figure in the context of an infinite particle system. The state of such a system is described by a probability measure μ on Γ and the functions F on Γ are considered as observables of the system and they represent physical quantities which can be measured. The expected values of the measured observables correspond to the expectation values $\int_\Gamma F(\gamma)\, d\mu(\gamma)$. In this interpretation we call the functions G on Γ_0 quasi-observables, because they are not observables themselves, but can be used to construct observables via the K-transform. In this way one obtains all observables which are additive in the particles, e.g., number of particles, energy.

In the sequel, we fix on $(X, \mathcal{B}(X))$ a non-atomic Radon measure σ, i.e., $\sigma(\{x\}) = 0$ for all $x \in X$, which we assume to be non-degenerate, i.e., $\sigma(O) > 0$ for all non-empty open sets $O \subset X$. Technically, the more challenging case is $\sigma(X) = \infty$.

$$\begin{tikzcd}
F \arrow[rr, leftrightarrow, "{\langle F,\mu\rangle = \int_\Gamma F(\gamma)\,d\mu(\gamma)}"] & & \mu \arrow[d, "K^*"] \\
G \arrow[u, "K"] \arrow[rr, leftrightarrow, "{\langle G,\rho_\mu\rangle = \int_{\Gamma_0} G(\eta)\,d\rho_\mu(\eta)}"'] & & \rho_\mu
\end{tikzcd}$$

Fig. 1.

[2] Throughout this work all L^p-spaces, $p \geqslant 1$, consist of complex-valued functions.

Example 2. The Poisson measure π_σ with intensity σ is the probability measure defined on $(\Gamma, \mathcal{B}(\Gamma))$ by

$$\int_\Gamma \exp\biggl(\sum_{x\in\gamma}\varphi(x)\biggr)\,d\pi_\sigma(\gamma) = \exp\biggl(\int_X \bigl(e^{\varphi(x)}-1\bigr)\,d\sigma(x)\biggr), \quad \varphi \in \mathcal{D},$$

where $\mathcal{D} := C_0^\infty(X)$ denotes the Schwartz space of all infinitely differentiable real-valued functions on X with compact support. The correlation measure corresponding to the Poisson measure π_σ is the so-called Lebesgue–Poisson measure λ_σ (with intensity σ), cf. [8],

$$\lambda_\sigma := \sum_{n=0}^\infty \frac{1}{n!}\sigma^{(n)},$$

where each $\sigma^{(n)}$, $n \in \mathbb{N}$, is the symmetrization of the product measure $\sigma^{\otimes n}$, i.e., the image measure on $\Gamma_X^{(n)}$ of the measure $\sigma^{\otimes n}$ under the mapping sym_X^n defined in (1). For $n = 0$ we set $\sigma^{(0)}(\{\emptyset\}) := 1$.

Remark 3. The following Lebesgue–Poisson measure properties emphasize the role of coherent states. First, $e_\lambda(f) \in L^p(\Gamma_0, \lambda_\sigma)$ whenever $f \in L^p(X, \sigma)$ for some $p \geqslant 1$, and, moreover,

$$\int_{\Gamma_0} \bigl|e_\lambda(f,\eta)\bigr|^p\,d\lambda_\sigma(\eta) = \exp\biggl(\int_X \bigl|f(x)\bigr|^p\,d\sigma(x)\biggr).$$

Second, given a dense subspace $\mathcal{L} \subset L^2(X, \sigma)$, the set $\{e_\lambda(f)\colon f \in \mathcal{L}\}$ is total in $L^2(\Gamma_0, \lambda_\sigma)$. Concerning the second property, let us observe that there is a natural isomorphism between the space $L^2(\Gamma_0, \lambda_\sigma)$ and the symmetric Fock space over $L^2(X, \sigma)$. The image of the classical coherent states in this Fock space (usually denoted by $e(f)$) under this isomorphism gives the elements $e_\lambda(f) \in L^2(\Gamma_0, \lambda_\sigma)$. This justifies the terminology and notation used. For more details see [18].

3. Bogoliubov functionals

For the case $X = \mathbb{R}^d$, $d \in \mathbb{N}$, we refer to [27] and his own references therein.

Definition 4. Let μ be a probability measure on $(\Gamma, \mathcal{B}(\Gamma))$. The Bogoliubov functional L_μ corresponding to μ is a functional defined at each $\mathcal{B}(X)$-measurable function θ by

$$L_\mu(\theta) := \int_\Gamma \prod_{x\in\gamma}\bigl(1+\theta(x)\bigr)\,d\mu(\gamma),$$

provided the right-hand side exists for $|\theta|$.

We note that if $L_\mu(|\theta|) < \infty$, then the product $\prod_{x\in\gamma}(1+\theta(x))$ is μ-a.e. absolutely convergent. For the definition and properties of infinite products see [14].

It is clear that the set of θ for which $L_\mu(|\theta|)$ is finite depends on the measure μ fixed on $(\Gamma, \mathcal{B}(\Gamma))$. Conversely, the same set reflects special properties over the underlying measure μ.

For instance, probability measures μ for which the Bogoliubov functional is well defined on multiples of indicator functions $\mathbb{1}_\Lambda$, $\Lambda \in \mathcal{B}_c(X)$, necessarily have finite local exponential moments, i.e., $\mu \in \mathcal{M}^1_{\mathrm{fexp}}(\Gamma)$. The converse is also true. In fact, for all $\alpha > 0$ and all $\Lambda \in \mathcal{B}_c(X)$ we have

$$\int_\Gamma e^{\alpha|\gamma_\Lambda|}\, d\mu(\gamma) = \int_\Gamma \prod_{x\in\gamma} e^{\alpha \mathbb{1}_\Lambda(x)}\, d\mu(\gamma) = L_\mu\big(\big(e^\alpha - 1\big)\mathbb{1}_\Lambda\big) < \infty.$$

In the sequel, for each probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ and each $\Lambda \in \mathcal{B}_c(X)$, we denote by $\mu^\Lambda := \mu \circ (p_\Lambda)^{-1}$ the image measure on Γ_Λ of the measure μ under the projection p_Λ defined in (2), i.e., μ^Λ is the projection of μ onto Γ_Λ. Given a $\Lambda \in \mathcal{B}_c(X)$, the definition of a Bogoliubov functional L_μ on the space of all functions θ with support contained in Λ reduces to the Bogoliubov functional L_{μ^Λ}:

$$L_\mu(\theta) = \int_\Gamma \prod_{x\in\gamma}\big(1+\theta(x)\big)\, d\mu(\gamma) = \int_\Gamma \prod_{x\in\gamma_\Lambda}\big(1+\theta(x)\big)\, d\mu(\gamma) = L_{\mu^\Lambda}(\theta).$$

Furthermore, one may straightforwardly express the μ-measure of a large class of sets by the Bogoliubov functional L_μ. In fact, given $z_1, \ldots, z_n \in \mathbb{C}$ and a collection of mutually disjoint sets $\Lambda_1, \ldots, \Lambda_n \in \mathcal{B}_c(X)$, $\Delta := \bigsqcup_{i=1}^n \Lambda_i$, $n \in \mathbb{N}$, the above calculation has shown that

$$L_\mu\left(\sum_{i=1}^n z_i \mathbb{1}_{\Lambda_i} - \mathbb{1}_\Delta\right) = \int_\Gamma \prod_{x\in\gamma_\Delta}\left(\sum_{i=1}^n z_i \mathbb{1}_{\Lambda_i}(x)\right) d\mu(\gamma).$$

Since Γ_Δ may be written as the disjoint union

$$\Gamma_\Delta = \bigsqcup_{k_1,\ldots,k_n=0}^\infty \big\{\gamma \in \Gamma_\Delta\colon |\gamma_{\Lambda_i}| = k_i,\ i = 1, \ldots, n\big\},$$

the latter integral is then equal to

$$\sum_{k_1,\ldots,k_n=0}^\infty z_1^{k_1} \ldots z_n^{k_n} \mu\big(\big\{\gamma \in \Gamma\colon |\gamma_{\Lambda_i}| = k_i,\ i = 1, \ldots, n\big\}\big).$$

Heuristically, this means that

$$\begin{aligned} &\mu\big(\big\{\gamma \in \Gamma\colon |\gamma_{\Lambda_i}| = k_i,\ i = 1, \ldots, n\big\}\big) \\ &\quad = \frac{1}{k_1! \ldots k_n!} \frac{\partial^{k_1+\cdots+k_n}}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}} L_\mu\left(\sum_{i=1}^n z_i \mathbb{1}_{\Lambda_i} - \mathbb{1}_{\bigcup_{i=1}^n \Lambda_i}\right)\Bigg|_{z_1=\cdots=z_n=0}. \end{aligned} \tag{6}$$

According to the definition of the σ-algebra $\mathcal{B}(\Gamma)$, the collection of sets appearing in the left-hand side of the informal equality (6) already characterizes the measure μ.

Of course, in order to apply the above procedure we must assume that the Bogoliubov functional L_μ is well defined and differentiable on the class of linear combinations of indicator functions which appears in (6). As the linear space spanned by indicator functions or the spaces of measurable functions are both difficult to handle, throughout this work we will consider Bogoliubov functionals on a $L^1(X,\sigma) =: L^1(\sigma)$ space, for a measure σ defined as before. Furthermore, we will assume for simplicity that the Bogoliubov functionals are entire. We observe that from the viewpoint of particle systems these restrictions are natural. Actually, even stronger properties should be expected.

For a comprehensive presentation of the general theory of holomorphic functionals on Banach spaces see, e.g., [3,7]. We recall the following characterization of entire functionals. A functional $A : L^1(\sigma) \to \mathbb{C}$ is entire on $L^1(\sigma)$ whenever A is locally bounded, and for all $\theta_0, \theta \in L^1(\sigma)$ the mapping $\mathbb{C} \ni z \mapsto A(\theta_0 + z\theta) \in \mathbb{C}$ is entire. Thus, at each $\theta_0 \in L^1(\sigma)$, every entire functional A on $L^1(\sigma)$ has a representation in terms of its Taylor expansion,

$$A(\theta_0 + z\theta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} d^n A(\theta_0; \theta, \ldots, \theta), \quad z \in \mathbb{C},\ \theta \in L^1(\sigma).$$

The next theorem states properties specific for holomorphic functionals A on L^1-spaces and their higher order derivatives $d^n A(\theta_0; \cdot)$.

Theorem 5. *Let A be an entire functional on $L^1(\sigma)$. Then each differential $d^n A(\theta_0; \cdot)$, $n \in \mathbb{N}$, $\theta_0 \in L^1(\sigma)$ is defined by a (symmetric) kernel in $L^\infty(X^n, \sigma^{\otimes n})$ denoted by*

$$\frac{\delta^n A(\theta_0)}{\delta\theta_0(x_1) \ldots \delta\theta_0(x_n)}$$

and called the variational derivative of nth order of A at the point θ_0. In other words,

$$\begin{aligned} d^n A(\theta_0; \theta_1, \ldots, \theta_n) &:= \frac{\partial^n}{\partial z_1 \ldots \partial z_n} A\left(\theta_0 + \sum_{i=1}^n z_i \theta_i\right)\Bigg|_{z_1=\cdots=z_n=0} \\ &=: \int_{X^n} \frac{\delta^n A(\theta_0)}{\delta\theta_0(x_1) \ldots \delta\theta_0(x_n)} \prod_{i=1}^n \theta_i(x_i)\, d\sigma^{\otimes n}(x_1, \ldots, x_n) \end{aligned}$$

for all $\theta_1, \ldots, \theta_n \in L^1(\sigma)$. Moreover, for all $r > 0$

$$\left\| \frac{\delta^n A(\theta_0)}{\delta\theta_0(x_1) \ldots \delta\theta_0(x_n)} \right\|_{L^\infty(X^n, \sigma^n)} \leqslant n! \left(\frac{e}{r}\right)^n \sup_{\|\theta'\|_{L^1(\sigma)} \leqslant r} \left|A(\theta_0 + \theta')\right|. \tag{7}$$

Remark 6. According to Theorem 5, the Taylor expansion of an entire functional A at a point $\theta_0 \in L^1(\sigma)$ may be written in the form

$$A(\theta_0 + \theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} \frac{\delta^n A(\theta_0)}{\delta\theta_0(x_1) \ldots \delta\theta_0(x_n)} \prod_{i=1}^n \theta(x_i)\, d\sigma^{\otimes n}(x_1, \ldots, x_n),$$

for all $\theta \in L^1(\sigma)$. Using the notation

$$\big(D^{|\eta|}A\big)(\theta_0;\eta) := \frac{\delta^n A(\theta_0)}{\delta\theta_0(x_1)\ldots\delta\theta_0(x_n)} \quad \text{for } \eta = \{x_1,\ldots,x_n\} \in \Gamma_X^{(n)},\ n \in \mathbb{N},$$

this means

$$A(\theta_0+\theta) = \int_{\Gamma_0} e_\lambda(\theta,\eta)\big(D^{|\eta|}A\big)(\theta_0;\eta)\,d\lambda_\sigma(\eta),$$

where λ_σ is the Lebesgue–Poisson measure introduced in Example 2. Concerning the estimate (7), we note that A being entire does not insure that for every $r>0$ the supremum on the right-hand side is always finite. This will hold if, in addition, the entire functional A is of bounded type, that is,

$$\forall r>0, \qquad \sup_{\|\theta\|_{L^1(\sigma)}\leqslant r} \big|A(\theta_0+\theta)\big| < \infty, \quad \forall \theta_0 \in L^1(\sigma).$$

For simplicity, throughout this work we will assume this assumption.

Some parts of the proof of Theorem 5 are of a technical nature outside of the context of the paper, but they are standard in the theory of holomorphic functionals on Banach spaces. Because of this, we just present a sketch of these parts conveniently adapted to our aims. Consequences specific for L^1-spaces are presented in more detail.

Proof. According to the Cauchy formula for holomorphic functionals on Banach spaces, each differential $d^nA(\theta_0;\cdot)$ of an entire functional A on $L^1(\sigma)$ is a bounded symmetric n-linear functional on $L^1(\sigma)$.

In particular, for $n=1$, the first order differential $dA(\theta_0;\cdot)$ is a bounded linear functional on $L^1(\sigma)$, insuring that it can be represented by a kernel in $L^\infty(\sigma)$, the so-called first variational derivative $\delta A(\theta_0)/\delta\theta_0(x)$. Furthermore, the (usual) operator norm of the bounded linear functional $dA(\theta_0;\cdot)$ is equal to $\|\delta A(\theta_0)/\delta\theta_0(\cdot)\|_{L^\infty(X,\sigma)}$.

For higher orders, the proof of existence of the corresponding variational derivatives is a straightforward consequence of the following (non-trivial) isometries between the Banach spaces

$$B_n\big(L^1(X,\sigma)\big) \simeq \big(L^1\big(X^n,\sigma^{\otimes n}\big)\big)' \simeq L^\infty\big(X^n,\sigma^{\otimes n}\big), \tag{8}$$

$B_n(L^1(X,\sigma))$ being the space of all bounded n-linear functionals on $L^1(X,\sigma)$. For the proof see, e.g., [6,36,38]. These isometries prove, on the one hand, the existence of the variational derivatives

$$\frac{\delta^n A(\theta_0)}{\delta\theta_0(x_1)\ldots\delta\theta_0(x_n)} \in L^\infty\big(X^n,\sigma^{\otimes n}\big)$$

as kernels for $d^nA(\theta_0;\cdot)$, and, on the other hand, that the operator norm of $d^nA(\theta_0;\cdot) \in B_n(L^1(X,\sigma))$ is given by

$$\left\|\frac{\delta^n A(\theta_0)}{\delta\theta_0(\cdot)\ldots\delta\theta_0(\cdot)}\right\|_{L^\infty(X^n,\sigma^{\otimes n})}.$$

This shows the first part of the theorem. To prove the second one, we observe that by the Cauchy formula, for any $\theta \in L^1(\sigma)$ one has

$$\frac{1}{n!} d^n A(\theta_0; \theta, \dots, \theta) = \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{A(\theta_0 + z\theta)}{z^{n+1}} \, dz$$

for any $r > 0$ and any $n \in \mathbb{N}$. Therefore

$$\left| d^n A(\theta_0; \theta, \dots, \theta) \right| \leqslant n! \sup_{\|\theta'\|_{L^1(\sigma)} \leqslant r} \left| A(\theta_0 + \theta') \right| \left(\frac{\|\theta\|_{L^1(\sigma)}}{r} \right)^n,$$

and an application of the polarization identity extends this inequality to $\theta_1, \dots, \theta_n \in L^1(\sigma)$:

$$\left| d^n A(\theta_0; \theta_1, \dots, \theta_n) \right| \leqslant n! \left(\frac{e}{r} \right)^n \sup_{\|\theta'\|_{L^1(\sigma)} \leqslant r} \left| A(\theta_0 + \theta') \right| \prod_{i=1}^n \|\theta_i\|_{L^1(\sigma)},$$

see, e.g., [7, Theorem 1.7]. □

Remark 7. Observe that the first isometry in (8) is specific of L^1-spaces. The analogous result does not hold neither for other L^p-spaces, nor Banach spaces of continuous functions, or Sobolev spaces.

Theorem 5 stated for Bogoliubov functionals yields the next result. In particular, it gives a rigorous sense to the discussion at the beginning of this section. For the definition of the Poisson measure see Example 2.

Corollary 8. *Let L_μ be a Bogoliubov functional corresponding to some probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$. If L_μ is entire of bounded type on $L^1(\sigma)$, then the measure μ is locally absolutely continuous with respect to the Poisson measure π_σ, i.e., for all $\Lambda \in \mathcal{B}_c(X)$ the measure $\mu^\Lambda = \mu \circ (p_\Lambda)^{-1}$ is absolutely continuous with respect to $\pi_\sigma^\Lambda = \pi_\sigma \circ (p_\Lambda)^{-1}$. Moreover, for all $\Lambda \in \mathcal{B}_c(X)$ one has*

$$\frac{d\mu^\Lambda}{d\pi_\sigma^\Lambda}(\gamma) = e^{\sigma(\Lambda)} \left(D^{|\gamma|} L_\mu \right)(-\mathbb{1}_\Lambda; \gamma) \quad \text{for } \pi_\sigma^\Lambda\text{-a.a. } \gamma \in \Gamma_\Lambda,$$

and for each $r > 0$ there is a constant $C \geqslant 0$ such that

$$\left| \frac{d\mu^\Lambda}{d\pi_\sigma^\Lambda}(\gamma) \right| \leqslant e^{\sigma(\Lambda)} C |\gamma|! \left(\frac{e}{r} \right)^{|\gamma|} \quad \text{for } \pi_\sigma^\Lambda\text{-a.a. } \gamma \in \Gamma_\Lambda^{(n)}.$$

Proof. In Theorem 5 replace A by the functional L_μ and θ_0 by an indicator function $-\mathbb{1}_\Lambda$ for some $\Lambda \in \mathcal{B}_c(X)$. Thus, for all functions $\theta \in L^1(\sigma)$ with support contained in Λ we find

$$L_\mu(\theta) = L_\mu\left(-\mathbb{1}_\Lambda + (\theta + \mathbb{1}_\Lambda)\right) = \int\limits_{\Gamma_\Lambda} \prod_{x \in \eta} \left(1 + \theta(x)\right) \left(D^{|\eta|} L_\mu\right)(-\mathbb{1}_\Lambda; \eta) \, d\lambda_\sigma(\eta).$$

On the other hand, according to the considerations done at the beginning of this section, we also have

$$L_\mu(\theta) = \int_{\Gamma_\Lambda} \prod_{x\in\gamma} \bigl(1+\theta(x)\bigr)\, d\mu^\Lambda(\gamma).$$

Therefore

$$\int_{\Gamma_\Lambda} \prod_{x\in\gamma} \bigl(1+\theta(x)\bigr)\, d\mu^\Lambda(\gamma) = \int_{\Gamma_\Lambda} \prod_{x\in\eta} \bigl(1+\theta(x)\bigr)\bigl(D^{|\eta|}L_\mu\bigr)(-\mathbb{1}_\Lambda;\eta)\, d\lambda_\sigma(\eta)$$

for all functions $\theta \in L^1(\sigma)$ with support contained in Λ. The proof follows by a monotone class argument. □

Since $\mu \in \mathcal{M}^1_{\mathrm{fexp}}(\Gamma)$ whenever the corresponding Bogoliubov functional is well defined on the whole space $L^1(\sigma)$, one can associate the correlation measure $\rho_\mu = K^*\mu$ to a such measure. Equalities (5) and (4) then yield a description of the functional L_μ in terms of the measure ρ_μ:

$$L_\mu(\theta) = \int_\Gamma \bigl(Ke_\lambda(\theta)\bigr)(\gamma)\, d\mu(\gamma) = \int_{\Gamma_0} e_\lambda(\theta,\eta)\, d\rho_\mu(\eta). \tag{9}$$

Within this formalism Theorem 5 states as follows.

Proposition 9. *Let L_μ be an entire Bogoliubov functional of bounded type on $L^1(\sigma)$. Then the measure ρ_μ is absolutely continuous with respect to the Lebesgue–Poisson measure λ_σ and the Radon–Nikodym derivative $k_\mu := d\rho_\mu/d\lambda_\sigma$ is given by*

$$k_\mu(\eta) = \bigl(D^{|\eta|}L_\mu\bigr)(0;\eta) \quad \textit{for } \lambda_\sigma\textit{-a.a. } \eta \in \Gamma_0.$$

Furthermore, for each $r > 0$ there is a constant $C \geqslant 0$ such that

$$\bigl|\bigl(D^{|\eta|}L_\mu\bigr)(0;\eta)\bigr| \leqslant C|\eta|!\left(\frac{e}{r}\right)^{|\eta|} \quad \textit{for } \lambda_\sigma\textit{-a.a. } \eta \in \Gamma_0.$$

In the sequel we call k_μ the correlation function corresponding to μ.

Proof. A straightforward application of Theorem 5 yields

$$L_\mu(\theta) = \int_{\Gamma_0} e_\lambda(\theta,\eta)\bigl(D^{|\eta|}L_\mu\bigr)(0;\eta)\, d\lambda_\sigma(\eta), \quad \theta \in L^1(\sigma),$$

and

$$\bigl|\bigl(D^{|\eta|}L_\mu\bigr)(0;\eta)\bigr| \leqslant C|\eta|!\left(\frac{e}{r}\right)^{|\eta|}, \quad \lambda_\sigma\text{-a.a. } \eta \in \Gamma_0,$$

for some $C \geqslant 0$ depending on r. Expression (9) then allows to identify $k_\mu(\eta)$ with $(D^{|\eta|}L_\mu)(0;\eta)$. □

Remark 10. Proposition 9 shows that the correlation functions $k_\mu^{(n)} := k_\mu \upharpoonright_{\Gamma_X^{(n)}}$ are the Taylor coefficients of the Bogoliubov functional L_μ. In other words, L_μ is the generating functional for the correlation functions $k_\mu^{(n)}$. This was also the reason why N.N. Bogoliubov introduced these functionals. Furthermore, Bogoliubov functionals are also related to the general infinite-dimensional analysis on configuration spaces, cf., e.g., [18]. Namely, through the unitary isomorphism S_λ defined in [18] between the space $L^2(\Gamma_0, \lambda_\sigma)$ and the Bargmann–Segal space one has $L_\mu = S_\lambda(k_\mu)$.

Proposition 11. *For any Bogoliubov functional L_μ entire of bounded type on $L^1(\sigma)$ the following relations between variational derivatives hold*:

$$\left(D^{|\eta|}L_\mu\right)(\theta;\eta) = \int_{\Gamma_0} k_\mu(\eta \cup \xi) e_\lambda(\theta, \xi)\, d\lambda_\sigma(\xi) \quad \text{for } \lambda_\sigma\text{-a.a. } \eta \in \Gamma_0 \tag{10}$$

and, more generally,

$$\left(D^{|\eta|}L_\mu\right)(\theta_1+\theta_2;\eta) = \int_{\Gamma_0} \left(D^{|\eta\cup\xi|}L_\mu\right)(\theta_1;\eta\cup\xi) e_\lambda(\theta_2, \xi)\, d\lambda_\sigma(\xi) \quad \text{for } \lambda_\sigma\text{-a.a. } \eta \in \Gamma_0,$$

for $\theta, \theta_1, \theta_2 \in L^1(\sigma)$.

To prove this result as well as other forthcoming ones the next lemma shows to be useful.

Lemma 12. [11,18,34] *The following equality holds*:

$$\int_{\Gamma_0}\int_{\Gamma_0} G(\eta\cup\xi)H(\xi,\eta)\, d\lambda_\sigma(\eta)\, d\lambda_\sigma(\xi) = \int_{\Gamma_0} G(\eta)\sum_{\xi\subset\eta} H(\xi, \eta\setminus\xi)\, d\lambda_\sigma(\eta)$$

for all positive measurable functions $G : \Gamma_0 \to \mathbb{R}$ and $H : \Gamma_0 \times \Gamma_0 \to \mathbb{R}$.

Proof of Proposition 11. According to Theorem 5, for all $\theta_1, \theta_2, \theta \in L^1(\sigma)$ one has

$$L_\mu(\theta_1+\theta_2+\theta) = \int_{\Gamma_0} \left(D^{|\eta|}L_\mu\right)(\theta_1+\theta_2;\eta)\; e_\lambda(\theta,\eta)\, d\lambda_\sigma(\eta),$$

as well as

$$L_\mu(\theta_1+\theta_2+\theta) = \int_{\Gamma_0} \left(D^{|\eta|}L_\mu\right)(\theta_1;\eta)\; e_\lambda(\theta_2+\theta,\eta)\, d\lambda_\sigma(\eta).$$

The bounds obtained in Theorem 5 allows to apply Lemma 12 to the latter equality yielding

$$\int_{\Gamma_0}\int_{\Gamma_0} \left(D^{|\eta\cup\xi|}L_\mu\right)(\theta_1;\eta\cup\xi)e_\lambda(\theta_2,\xi)\,d\lambda_\sigma(\xi)e_\lambda(\theta,\eta)\,d\lambda_\sigma(\eta).$$

The second stated equality follows by a monotone class argument. By Proposition 9 one sees that (10) is a special case of the derived result for $\theta_1 = 0$ and $\theta_2 = \theta$. □

A particular application of Proposition 11 yields the next two formulas well known in statistical mechanics, see, e.g., [35], and in the theory of point processes, see, e.g., [5].

Corollary 13. *Under the conditions of Proposition* 11*, for all* $\Lambda \in \mathcal{B}_c(X)$ *we have*

$$k_\mu(\eta) = \int_{\Gamma_\Lambda} \frac{d\mu^\Lambda}{d\pi_\sigma^\Lambda}(\eta\cup\gamma)\,d\pi_\sigma^\Lambda(\gamma) \quad \textit{for } \lambda_\sigma\textit{-a.a. } \eta\in\Gamma_\Lambda, \tag{11}$$

and

$$\frac{d\mu^\Lambda}{d\pi_\sigma^\Lambda}(\gamma) = e^{\sigma(\Lambda)}\int_{\Gamma_\Lambda}(-1)^{|\eta|}k_\mu(\gamma\cup\eta)\,d\lambda_\sigma(\eta) \quad \textit{for } \pi_\sigma^\Lambda\textit{-a.a. } \gamma\in\Gamma_\Lambda. \tag{12}$$

Proof. Fixing a $\Lambda \in \mathcal{B}_c(X)$, in Proposition 11 replace both functions θ and θ_1 by the function $-\mathbb{1}_\Lambda$ and θ_2 by $\mathbb{1}_\Lambda$. The expressions for the densities given in Corollary 8 and Proposition 9 complete the proof. □

Remark 14. Corollary 13 may be stated under more general conditions. Given a probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ such that

$$\int_\Gamma |\gamma_\Lambda|^n\,d\mu(\gamma), \qquad \int_{\Gamma_\Lambda} 2^{|\eta|}\,d\rho_\mu(\eta) < \infty$$

for all $\Lambda \in \mathcal{B}_c(X)$ and all $n \in \mathbb{N}$, one can show that μ is locally absolutely continuous with respect to the Poisson measure π_σ if and only if the correlation measure ρ_μ is absolutely continuous with respect to the Lebesgue–Poisson measure λ_σ. Under these conditions, equalities (11) and (12) hold (see, e.g., [15]).

Corollary 15. *Let* L_μ *be an entire Bogoliubov functional of bounded type on* $L^1(\sigma)$*. For any* $\mathcal{B}(\Gamma_0)$*-measurable function* $G:\Gamma_0 \to \mathbb{R}$ *such that there is a* $f \in L^1(\sigma)$ *so that* $|G| \leqslant e_\lambda(f)$ *one has*

$$\int_{\Gamma_0} G(\eta)\left(D^{|\eta|}L_\mu\right)(\theta;\eta)\,d\lambda_\sigma(\eta) = \int_{\Gamma_0}\sum_{\xi\subset\eta} G(\xi)e_\lambda(\theta,\eta\setminus\xi)\,d\rho_\mu(\eta),$$

for all $\theta \in L^1(\sigma)$*.*

According to Proposition 9, the correlation function k_μ of an entire Bogoliubov functional on $L^1(\sigma)$ fulfills the so-called generalized Ruelle bound, that is, for any $0 \leqslant \varepsilon \leqslant 1$ and any $r > 0$ there is some constant $C \geqslant 0$ depending on r such that

$$k_\mu(\eta) \leqslant C\big(|\eta|!\big)^{1-\varepsilon}\left(\frac{e}{r}\right)^{|\eta|}, \quad \lambda_\sigma\text{-a.a. } \eta \in \Gamma_0. \tag{13}$$

In our case, ε is zero. We note that if (13) holds for $\varepsilon = 1$ and for at least one $r > 0$, then condition (13) is the classical Ruelle bound. For a general $0 < \varepsilon \leqslant 1$ one may state the following result.

Proposition 16. [17] *If there are a function $0 \leqslant C \in L^1_{\mathrm{loc}}(X, \sigma)$ and a $0 < \varepsilon \leqslant 1$ such that*

$$k_\mu(\eta) \leqslant \big(|\eta|!\big)^{1-\varepsilon} e_\lambda(C, \eta), \quad \lambda_\sigma\text{-}a.a.\ \eta \in \Gamma_0,$$

then there are constants $c_1 = c_1(\varepsilon)$, $c_2 = c_2(\varepsilon) > 0$ such that

$$\big|L_\mu(\varphi)\big| \leqslant c_1 \exp\big(\|\varphi\|^{1/\varepsilon}_{L^1(c_2 C\sigma)}\big), \quad \varphi \in \mathcal{D}.$$

Furthermore, L_μ is an entire functional of bounded type on $L^1(C\sigma)$.

The definition of a Bogoliubov functional clearly shows that for any probability measure $\mu \in \mathcal{M}^1_{\mathrm{fexp}}(\Gamma)$ L_μ is a normalized functional, that is, $L_\mu(0) = 1$. If, in addition, L_μ is an entire functional on $L^1(\sigma)$, then, according to Corollary 8, for all $\Lambda \in \mathcal{B}_c(X)$ we have

$$\big(D^{|\gamma|}L_\mu\big)(-\mathbb{1}_\Lambda; \gamma) = e^{-\sigma(\Lambda)} \frac{d\mu^\Lambda}{d\pi^\Lambda_\sigma}(\gamma) \geqslant 0, \quad \lambda_\sigma\text{-a.a } \gamma \in \Gamma_\Lambda.$$

These conditions are also sufficient to insure that a generic entire functional on $L^1(\sigma)$ is a Bogoliubov functional corresponding to some measure in $\mathcal{M}^1_{\mathrm{fexp}}(\Gamma)$.

Proposition 17. *Let L be a normalized entire functional of bounded type on $L^1(\sigma)$ such that for all $\Lambda \in \mathcal{B}_c(X)$*

$$\big(D^{|\eta|}L\big)(-\mathbb{1}_\Lambda; \eta) \geqslant 0, \quad \lambda_\sigma\text{-}a.a.\ \eta \in \Gamma_\Lambda. \tag{14}$$

Then there is a unique probability measure $\mu \in \mathcal{M}^1_{\mathrm{fexp}}(\Gamma)$ such that for all $\theta \in L^1(\sigma)$

$$L(\theta) = \int_\Gamma \prod_{x\in\gamma}\big(1 + \theta(x)\big)\, d\mu(\gamma). \tag{15}$$

Remark 18. This result is reminiscent of the classical characterization of completely monotonic functions as Laplace transforms of non-negative measures, see, e.g., [10]. Proposition 17 is substantially different, because one additionally obtains special support properties of the measure, i.e., a measure on Γ. The connection with classical results can be understood in the zero-dimensional analogue. This means that we neglect the spatial structure, i.e., we only look at the distribution of the random variable $\gamma \mapsto |\gamma_\Lambda|$ for a fixed Λ. Moreover, given a measure μ on $\mathbb{N}_0$, the analogue of the Bogoliubov functional is the following function defined on $\mathbb{R}$ by

$$L(k) := \sum_{n=0}^{\infty} (1+k)^n \mu\bigl(\{n\}\bigr). \tag{16}$$

We note that, on the one hand, on the interval $(-1, \infty)$ the function L is a modified Laplace transform, namely, $L(k) = \int_{\mathbb{N}_0} e^{x \ln(1+k)}\, d\mu(x)$. On the other hand, the function L is actually the classical probability generating functional considered, e.g., in [10], shifted by 1. The analogue of Proposition 17 is then a classical characterization result: L is completely monotonic at -1 if and only if (16) holds for a non-negative measure μ on $\mathbb{N}_0$. Another classical result states that continuous completely monotonic functions are automatically holomorphic due to an approximation by Bernstein polynomials. This idea was generalized to point processes by J. Mecke in [26] for functionals that are continuous with respect to monotonic bounded convergence. Within the analytic functions framework (14) is a natural assumption.

Proof of Proposition 17. For any $\Lambda \in \mathcal{B}_c(X)$ let us define the function

$$G_\Lambda(\eta) := \bigl(D^{|\eta|}L\bigr)(-\mathbb{1}_\Lambda; \eta) \geqslant 0, \quad \eta \in \Gamma_\Lambda.$$

For all $\Lambda \in \mathcal{B}_c(X)$ we have

$$\begin{aligned} \int_{\Gamma_\Lambda} G_\Lambda(\eta)\, d\lambda_\sigma(\eta) &= \int_{\Gamma_0} e_\lambda(\mathbb{1}_\Lambda, \eta)\bigl(D^{|\eta|}L\bigr)(-\mathbb{1}_\Lambda; \eta)\, d\lambda_\sigma(\eta) \\ &= L(\mathbb{1}_\Lambda - \mathbb{1}_\Lambda) = L(0) = 1, \end{aligned}$$

allowing to define a family of probability measures μ^Λ on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ by

$$\mu^\Lambda(A) := \int_{\Gamma_\Lambda} \mathbb{1}_A(\eta) G_\Lambda(\eta)\, d\lambda_\sigma(\eta), \quad A \in \mathcal{B}(\Gamma_\Lambda).$$

Similarly, one verifies that the family $(\mu^\Lambda)_{\Lambda \in \mathcal{B}_c(X)}$ is consistent. Therefore, by the version of the Kolmogorov theorem for the projective limit space $(\Gamma, \mathcal{B}(\Gamma))$ [30, Chapter V, Theorem 5.1], there is a unique probability measure μ on Γ such that the measures μ^Λ are the projections of μ. From the definition of G_Λ follows the relation (15) between L and μ for every θ supported in Λ. The L^1-continuity of L and monotone convergence arguments extend this relation to all non-negative functions $\theta \in L^1(\sigma)$. The general relation follows from dominated convergence results. $\square$

4. Bogoliubov equations

Particularly interesting is the characterization of Gibbs measures through the Bogoliubov functionals.

Given a pair potential $\phi: X \times X \to \mathbb{R} \cup \{+\infty\}$, that is, a symmetric measurable function, let $E: \Gamma_0 \to \mathbb{R} \cup \{+\infty\}$ be the energy functional and $W: \Gamma_0 \times \Gamma \to \mathbb{R} \cup \{+\infty\}$ be the interaction energy defined for all $\eta \in \Gamma_0$ and all $\gamma \in \Gamma$ by

$$E(\eta) := \sum_{\{x,y\} \subset \eta} \phi(x, y), \qquad E(\emptyset) := E(\{x\}) := 0$$

and

$$W(\eta, \gamma) := \begin{cases} \sum_{x \in \eta,\ y \in \gamma} \phi(x, y), & \text{if } \sum_{x \in \eta,\ y \in \gamma} |\phi(x, y)| < \infty, \\ +\infty, & \text{otherwise,} \end{cases}$$

respectively. We set $W(\emptyset, \gamma) := W(\eta, \emptyset) := 0$. A grand canonical Gibbs measure (Gibbs measure, for short) corresponding to a pair potential ϕ, the intensity measure σ, and an inverse temperature $\beta > 0$, is usually defined through the Dobrushin–Lanford–Ruelle equation. For convenience, we present here an equivalent definition through the Georgii–Nguyen–Zessin equation ((GNZ)-equation) ([28, Theorem 2], see also [16, Theorem 3.12], [22, Appendix A.1]). More precisely, a probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a Gibbs measure if it fulfills the integral equation

$$\int_{\Gamma} \sum_{x \in \gamma} H(x, \gamma \setminus \{x\})\, d\mu(\gamma) = \int_{\Gamma} \int_{X} H(x, \gamma) e^{-\beta W(\{x\}, \gamma)}\, d\sigma(x)\, d\mu(\gamma) \tag{17}$$

for all positive measurable functions $H: X \times \Gamma \to \mathbb{R}$. In particular, for $\phi \equiv 0$, (17) reduces to the Mecke identity, which yields an equivalent definition of the Poisson measure π_σ [25, Theorem 3.1].

Correlation measures corresponding to Gibbs measures are always absolutely continuous with respect to the Lebesgue–Poisson measure λ_σ. In view of this fact and Remark 14, the framework used throughout this section is restricted to measures $\mu \in \mathcal{M}^1_{\mathrm{fexp}}(\Gamma)$ that are locally absolutely continuous with respect to the Poisson measure π_σ. Furthermore, we shall assume that the corresponding correlation functions k_μ fulfill the so-called Ruelle type bound inequality, that is, there are $a > 0$ and $0 < \varepsilon \leqslant 1$ such that

$$k_\mu(\eta) \leqslant (|\eta|!)^{1-\varepsilon} e_\lambda(a, \eta) = (|\eta|!)^{1-\varepsilon} a^{|\eta|}, \quad \lambda_\sigma\text{-a.a. } \eta \in \Gamma_0.$$

According to Proposition 16, this assumption implies

Assumption 1. There are $c_1, c_2 > 0$ such

$$|L_\mu(\theta)| \leqslant c_1 \exp\left(c_2 \|\theta\|^{1/\varepsilon}_{L^1(\sigma)}\right) \quad \text{for all } \theta \in L^1(\sigma).$$

As a consequence of Proposition 16, the Bogoliubov functional L_μ is entire of bounded type on $L^1(\sigma)$.

To proceed towards the equivalent description of Gibbs measures through Bogoliubov functionals, we consider potentials ϕ fulfilling the following semi-boundedness and integrability conditions:

Assumption 2. $\exists B \geqslant 0$: $\phi(x,y) \geqslant -2B$ for all $x, y \in X$.

Assumption 3. $C(\beta) := \operatorname{ess\,sup}_{x\in X} \int_X |e^{-\beta\phi(x,y)} - 1| \, d\sigma(y) < \infty$.

Let us note that Assumptions 1–3 are essentially weaker than the usual ones for existence results for Gibbs measures, cf., e.g., [35].

Proposition 19. *Given a $\mu \in \mathcal{M}^1_{\mathrm{fexp}}(\Gamma)$ and a pair potential ϕ, assume that Assumptions* 1–3 *are fulfilled. Then μ is a Gibbs measure corresponding to the potential ϕ, the intensity measure σ, and the inverse temperature β if and only if the Bogoliubov functional L_μ corresponding to μ solves the so-called Bogoliubov* (*equilibrium*) *equation,*

$$\frac{\delta L(\theta)}{\delta\theta(x)} = L\big((1+\theta)\big(e^{-\beta\phi(x,\cdot)} - 1\big) + \theta\big), \quad \sigma\text{-a.e.},$$

for all $\theta \in L^1(\sigma)$.

Proof. The holomorphicity of L_μ on $L^1(\sigma)$ implies

$$\begin{aligned} dL_\mu(\theta; f) &= \int_\Gamma \frac{d}{dz} \prod_{x\in\gamma} \big(1 + \theta(x) + zf(x)\big)\bigg|_{z=0} d\mu(\gamma) \\ &= \int_\Gamma \sum_{x\in\gamma} f(x) \prod_{y\in\gamma\setminus\{x\}} \big(1+\theta(y)\big)\, d\mu(\gamma), \quad \theta, f \in L^1(\sigma). \end{aligned} \tag{18}$$

Thus, for a Gibbs measure μ, the (GNZ)-equation yields for the right-hand side of (18)

$$\int_X f(x) \int_\Gamma \prod_{y\in\gamma} \big(1+\theta(y)\big) e^{-\beta W(\{x\},\gamma)}\, d\mu(\gamma)\, d\sigma(x). \tag{19}$$

We claim that

$$e^{-\beta W(\{x\},\gamma)} = \prod_{y\in\gamma} \big(1 + \big(e^{-\beta\phi(x,y)} - 1\big)\big), \tag{20}$$

which proof we postpone to the end. Hence (19) is given by

$$\begin{aligned} &\int_X f(x) \int_\Gamma \prod_{y\in\gamma} \big(1+\theta(y)\big) \prod_{y\in\gamma} \big(1 + \big(e^{-\beta\phi(x,y)} - 1\big)\big)\, d\mu(\gamma)\, d\sigma(x) \\ &\quad = \int_X f(x) \int_\Gamma \prod_{y\in\gamma} \big(\big(1+\theta(y)\big)\big(e^{-\beta\phi(x,y)} - 1\big) + 1 + \theta(y)\big)\, d\mu(\gamma)\, d\sigma(x). \end{aligned}$$

In this way we show that for all $f \in L^1(\sigma)$

$$dL_\mu(\theta; f) = \int_X f(x) L_\mu\big((1+\theta)\big(e^{-\beta\phi(x,\cdot)} - 1\big) + \theta\big)\, d\sigma(x),$$

provided $(1+\theta)(e^{-\beta\phi(x,\cdot)}-1)+\theta \in L^1(X,\sigma)$. Assumption 1 then implies that $L_\mu((1+\theta)\times(e^{-\beta\phi(x,\cdot)}-1)+\theta) \in L^\infty(X,\sigma)$ which completes the first part of the proof. Conversely, the same arguments as before yield,

$$\begin{aligned}
&\int_\Gamma \sum_{x\in\gamma} f(x) \prod_{y\in\gamma\setminus\{x\}} \bigl(1+\theta(y)\bigr)\, d\mu(\gamma) \\
&\quad = dL_\mu(\theta; f) = \int_X f(x) L_\mu\bigl((1+\theta)\bigl(e^{-\beta\phi(x,\cdot)}-1\bigr)+\theta\bigr)\, d\sigma(x) \\
&\quad = \int_X f(x) \int_\Gamma \prod_{y\in\gamma}\bigl(1+\theta(y)\bigr) e^{-\beta W(\{x\},\gamma)}\, d\mu(\gamma)\, d\sigma(x),
\end{aligned}$$

showing that the measure μ fulfills the (GNZ)-equation for the class of functions H of the form

$$H(x,\gamma) = f(x) \prod_{y\in\gamma\setminus\{x\}} \bigl(1+\theta(y)\bigr), \quad \theta, f \in L^1(\sigma).$$

The result follows by a monotone class argument.

To conclude this proof amounts to check the technical problems left open. Due to Assumptions 2 and 3 one has

$$\bigl\|\theta e^{-\beta\phi(x,\cdot)} + e^{-\beta\phi(x,\cdot)} - 1\bigr\|_{L^1(\sigma)} \leqslant e^{2\beta B}\|\theta\|_{L^1(\sigma)} + C(\beta),$$

showing that $(1+\theta)(e^{-\beta\phi(x,\cdot)}-1)+\theta \in L^1(X,\sigma)$.

The infinite product $\prod_{y\in\gamma}(1+|e^{-\beta\phi(x,y)}-1|)$ converges for $\sigma\otimes\mu$-a.a. (x,γ), because Assumption 3 implies that σ-a.e. $\|e^{-\beta\phi(x,\cdot)}-1\|_{L^1(\sigma)} < \infty$ and

$$\int_\Gamma \prod_{y\in\gamma}\bigl(1+\bigl|e^{-\beta\phi(x,y)}-1\bigr|\bigr)\, d\mu(\gamma) < \infty.$$

The absolute convergence of the infinite product in (20) implies the convergence of $\sum_{y\in\gamma}|e^{-\beta\phi(x,y)}-1|$. Hence, either the series $\sum_{y\in\gamma}|\phi(x,y)|$ converges or there is a $y\in\gamma$ such that $\phi(x,y)=+\infty$. In the latter case the infinite product in (20) as well as $e^{-\beta W(\{x\},\gamma)}$ are both zero. For the first case we obtain

$$\prod_{y\in\gamma}\bigl(1+\bigl(e^{-\beta\phi(x,y)}-1\bigr)\bigr) = \exp\Bigl(-\beta\sum_{y\in\gamma}\phi(x,y)\Bigr) = e^{-\beta W(\{x\},\gamma)}. \qquad \square$$

For higher order derivatives the corresponding Bogoliubov equations are defined as follows.

Corollary 20. *Given a $\mu \in \mathcal{M}^1_{\mathrm{fexp}}(\Gamma)$ and a pair potential ϕ, assume that Assumptions* 1–3 *are fulfilled. If μ is a Gibbs measure corresponding to the potential ϕ, the intensity measure σ, and the inverse temperature β, then for all $\theta\in L^1(\sigma)$ the following relation holds*:

$$\bigl(D^n L_\mu\bigr)(\theta;\eta) = e^{-\beta E(\eta)} L_\mu\bigl((1+\theta)\bigl(e^{-\beta W(\eta,\{\cdot\})}-1\bigr)+\theta\bigr), \quad \sigma^{(n)}\text{-a.a. } \eta\in\Gamma_X^{(n)}.$$

Proof. It follows from successive applications of Proposition 19 and the chain rule to the function $L^1(\sigma) \ni \theta \mapsto (1+\theta)(e^{-\beta\phi(x,\cdot)}-1)+\theta \in L^1(\sigma)$. □

Proposition 21. *For any pair potential ϕ and any measure $\mu \in \mathcal{M}^1_{\mathrm{fexp}}(\Gamma)$ under Assumptions* 1–3*, the following equations are equivalent*:

(i) *For all $\theta \in L^1(\sigma)$,*

$$\frac{\delta L_\mu(\theta)}{\delta\theta(x)} = L_\mu\big((1+\theta)\big(e^{-\beta\phi(x,\cdot)}-1\big)+\theta\big) \quad \textit{for } \sigma\textit{-a.a. } x \in X.$$

(ii) *For every $\theta, f \in L^1(\sigma)$,*

$$L_\mu(\theta+f) - L_\mu(\theta) = \int_X f(x) \int_0^1 L_\mu\big((1+\theta+tf)\big(e^{-\beta\phi(x,\cdot)}-1\big)+\theta+tf\big)\,dt\,d\sigma(x).$$

Furthermore, the previous equations imply that

(iii) *For all $\theta, f \in L^1(\sigma)$,*

$$L_\mu(\theta+f) = \int_{\Gamma_0} e_\lambda(f,\eta)e^{-\beta E(\eta)} L_\mu\big((1+\theta)\big(e^{-\beta W(\eta,\{\cdot\})}-1\big)+\theta\big)\,d\lambda_\sigma(\eta).$$

Remark 22. Assumptions 1–3 are not sufficient to insure the existence of the integral on the right-hand side of the equation stated in (iii).

Proof. (i) ⇒ (ii). Since L_μ is entire on $L^1(\sigma)$, one has

$$L_\mu(\theta+f) - L_\mu(\theta) = \int_0^1 \frac{d}{dt} L_\mu(\theta+tf)\,dt$$

and, according to (i),

$$\begin{aligned}\frac{d}{dt} L_\mu(\theta+tf) &= dL_\mu(\theta+tf, f)\\ &= \int_X f(x) L_\mu\big((1+\theta+tf)\big(e^{-\beta\phi(x,\cdot)}-1\big)+\theta+tf\big)\,d\sigma(x).\end{aligned}$$

(ii) ⇒ (i). Assuming (ii), for any $\theta, f \in L^1(\sigma)$ one finds

$$\begin{aligned}\frac{d}{dz} L_\mu(\theta+zf)\bigg|_{z=0} &= \lim_{z\to 0} \frac{L_\mu(\theta+zf) - L_\mu(\theta)}{z}\\ &= \lim_{z\to 0} \int_X f(x) \int_0^1 L_\mu\big((1+\theta+tzf)\big(e^{-\beta\phi(x,\cdot)}-1\big)+\theta+tzf\big)\,dt\,d\sigma(x).\end{aligned}$$

Assumptions 1–3 allow to apply the Lebesgue dominated convergence theorem and thus, interchanging the limit with the integrals and using the continuity of L_μ on $L^1(\sigma)$, to obtain

$$\int_X f(x) L_\mu\big((1+\theta)\big(e^{-\beta\phi(x,\cdot)}-1\big)+\theta\big)\,d\sigma(x).$$

(i) $\Rightarrow$ (iii). The holomorphicity of L_μ straightforwardly leads (Remark 6) to

$$\begin{aligned}
L_\mu(\theta+f) &= \int_{\Gamma_0} e_\lambda(f,\eta)\big(D^{|\eta|}L_\mu\big)(\theta;\eta)\,d\lambda_\sigma(\eta)\\
&= \int_{\Gamma_0} e_\lambda(f,\eta)e^{-\beta E(\eta)}L_\mu\big((1+\theta)\big(e^{-\beta W(\eta,\{\cdot\})}-1\big)+\theta\big)\,d\lambda_\sigma(\eta),
\end{aligned}$$

where the second equality is a consequence of Corollary 20. □

Propositions 19 and 21 lead to a uniqueness result for Gibbs measures corresponding to positive potentials. As a first step towards this purpose, we must introduce additional spaces of functionals. More precisely, for each $\alpha > 0$, let $\mathrm{Ent}_\alpha(L^1(\sigma))$ be the space of all entire functionals L on $L^1(\sigma)$ such that

$$\|L\|_\alpha := \sup_{\theta\in L^1(\sigma)}\big(\big|L(\theta)\big|e^{-\alpha\|\theta\|_{L^1(\sigma)}}\big) < \infty.$$

It is clear that $\|\cdot\|_\alpha$ defines a norm on $\mathrm{Ent}_\alpha(L^1(\sigma))$.

Proposition 23. *With respect to the norm* $\|\cdot\|_\alpha$, $\mathrm{Ent}_\alpha(L^1(\sigma))$ *has the structure of a Banach space.*

Proof. Fixing an $\alpha > 0$, let $(L_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in the space $\mathrm{Ent}_\alpha(L^1(\sigma))$, i.e., $(L_n e^{-\alpha\|\cdot\|_{L^1(\sigma)}})_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space consisting of all complex-valued bounded functions defined on $L^1(\sigma)$ endowed with the supremum norm. By completeness, there is a complex-valued bounded function $\bar{L}$ such that

$$\lim_{n\to\infty}\sup_{\theta\in L^1(\sigma)}\big(\big|L_n(\theta)e^{-\alpha\|\theta\|_{L^1(\sigma)}}-\bar{L}(\theta)\big|\big) = 0. \tag{21}$$

It remains to show that the functional $L(\theta) := \bar{L}(\theta)e^{\alpha\|\theta\|_{L^1(\sigma)}}$, $\theta\in L^1(\sigma)$, is entire on $L^1(\sigma)$. This follows from the Vitali theorem (see, e.g., [13]), since by (21) the sequence $(L_n)_{n\in\mathbb{N}}$ converges pointwisely to L and, by the inequality

$$\big|L_n(\theta)\big| \leqslant \sup_{\theta\in L^1(\sigma)}\big(\big|L_n(\theta)\big|e^{-\alpha\|\theta\|_{L^1(\sigma)}}\big)e^{\alpha\|\theta\|_{L^1(\sigma)}} = \|L_n\|_\alpha e^{\alpha\|\theta\|_{L^1(\sigma)}},\quad n\in\mathbb{N},$$

the sequence $(L_n)_{n\in\mathbb{N}}$ is locally uniformly bounded in $L^1(\sigma)$. □

Note that $L \in \mathrm{Ent}_\alpha(L^1(\sigma))$ implies that L fulfills Assumption 1 for $\varepsilon = 1$. In this way, for pair potentials ϕ fulfilling Assumptions 2 and 3, Proposition 21 has shown that any functional L in $\mathrm{Ent}_\alpha(L^1(\sigma))$ solving the initial value problem

$$\begin{cases} \frac{\delta L(\theta)}{\delta\theta(x)} = L((1+\theta)(e^{-\beta\phi(x,\cdot)}-1)+\theta), & \theta \in L^1(\sigma), \\ L(0) = 1 \end{cases}$$

is a solution of the equation

$$L(\theta) - 1 = \int_X \theta(x) \int_0^1 L\big((1+t\theta)\big(e^{-\beta\phi(x,\cdot)}-1\big)+t\theta\big)\,dt\,d\sigma(x), \quad \theta \in L^1(\sigma).$$

In the sequel we denote by J the linear mapping defined on each space $\mathrm{Ent}_\alpha(L^1(\sigma))$, $\alpha > 0$, by

$$(JL)(\theta) := \int_X \theta(x) \int_0^1 L\big((1+t\theta)\big(e^{-\beta\phi(x,\cdot)}-1\big)+t\theta\big)\,dt\,d\sigma(x),$$

for $L \in \mathrm{Ent}_\alpha(L^1(\sigma)), \theta \in L^1(\sigma)$.

Proposition 24. *Let ϕ be a positive pair potential fulfilling Assumption* 3. *Then, for any $\alpha > 0$, the mapping J defines a bounded linear operator on* $\mathrm{Ent}_\alpha(L^1(\sigma))$. *Moreover, for all $L \in \mathrm{Ent}_\alpha(L^1(\sigma))$,*

$$\|JL\|_\alpha \leqslant \frac{e^{\alpha C(\beta)}}{\alpha}\|L\|_\alpha.$$

Proof. Let $\alpha > 0$ be given. For all $\theta \in L^1(\sigma)$ one has

$$\big|(JL)(\theta)\big| \leqslant \|L\|_\alpha \int_X \big|\theta(x)\big| \int_0^1 e^{\alpha\|(1+t\theta)(e^{-\beta\phi(x,\cdot)}-1)+t\theta\|_{L^1(\sigma)}}\,dt\,d\sigma(x)$$

and, according to the stated assumptions on ϕ,

$$\begin{aligned} &\big\|(1+t\theta)\big(e^{-\beta\phi(x,\cdot)}-1\big)+t\theta\big\|_{L^1(\sigma)} \\ &\quad\leqslant t\int_X \big|\theta(y)\big|e^{-\beta\phi(x,y)}\,d\sigma(y) + \int_X \big|e^{-\beta\phi(x,y)}-1\big|\,d\sigma(y) \\ &\quad\leqslant t\|\theta\|_{L^1(\sigma)} + C(\beta). \end{aligned}$$

Therefore

$$\begin{aligned}\big|(JL)(\theta)\big| &\leqslant \|L\|_\alpha \|\theta\|_{L^1(\sigma)} e^{\alpha C(\beta)} \int_0^1 e^{\alpha t \|\theta\|_{L^1(\sigma)}}\,dt \\ &= \|L\|_\alpha \frac{e^{\alpha C(\beta)}}{\alpha}\big(e^{\alpha\|\theta\|_{L^1(\sigma)}} - 1\big) < \|L\|_\alpha \frac{e^{\alpha C(\beta)}}{\alpha} e^{\alpha\|\theta\|_{L^1(\sigma)}},\end{aligned}$$

showing the required estimate of the norms. □

Corollary 25. *Let $\beta > 0$ be given. Then, under the assumptions of Proposition* 24*, on each space* $\mathrm{Ent}_\alpha(L^1(\sigma))$ *with*

$$\frac{e^{\alpha C(\beta)}}{\alpha} < 1$$

exists a unique solution of the equation

$$L - JL = 1. \tag{22}$$

In particular, for all $\beta > 0$ such that $C(\beta) < e^{-1}$, there is a unique solution of Eq. (22) *on each space* $\mathrm{Ent}_\alpha(L^1(\sigma))$ *for each $\alpha \leqslant e$.*

Proof. According to Proposition 24, one has

$$\|JL\|_\alpha \leqslant \frac{e^{\alpha C(\beta)}}{\alpha}\|L\|_\alpha < \|L\|_\alpha, \quad L \in \mathrm{Ent}_\alpha\big(L^1(\sigma)\big).$$

That is, the operator J is a contraction on $\mathrm{Ent}_\alpha(L^1(\sigma))$. Thus, by the contraction mapping principle, there is a unique solution of Eq. (22), namely, $(1-J)^{-1}1$, with $(1-J)^{-1}$ defined by the von Neumann series $\sum_{n=0}^\infty J^n$. The last assertion follows by minimizing the expression $\alpha^{-1}e^{\alpha C(\beta)}$ in the parameter α. □

As a consequence of Propositions 19 and 21, we have proved the following uniqueness result.

Theorem 26. *Let ϕ be a positive pair potential fulfilling the integrability condition*

$$C(\beta) = \operatorname*{ess\,sup}_{x\in X} \int_X \big|e^{-\beta\phi(x,y)} - 1\big|\,d\sigma(y) < \infty.$$

For each $\beta > 0$ such that $C(\beta) < e^{-1}$ there is at most one Gibbs measure with correlation function fulfilling the Ruelle bound with constant e and corresponding to the potential ϕ, the intensity measure σ, and the inverse temperature β.

5. Stochastic dynamic equations

To deal with the differential structures used below to study a gradient diffusion dynamics of a continuous system, this section begins by recalling a few concepts of the intrinsic geometry on configuration spaces [2,15,22].

5.1. Differential geometry on configuration spaces

Apart from the topological structure, the bijection defined in Section 2 between the spaces $\Gamma_X^{(n)}$ and $\widetilde{X^n}/S_n$ also induces a differentiable structure on $\Gamma_X^{(n)}$ (see (1)). More precisely, given n charts $(h_1, U_1), \ldots, (h_n, U_n)$ of X, where $U_1, \ldots, U_n$ are mutually disjoint open sets in X, one constructs a chart $h_1 \hat{\times} \cdots \hat{\times} h_n$ of $\Gamma_X^{(n)}$ defined on the open set $U_1 \hat{\times} \cdots \hat{\times} U_n$ in $\Gamma_X^{(n)}$,

$$U_1 \hat{\times} \cdots \hat{\times} U_n := \bigl\{\eta = \{x_1, \ldots, x_n\} \in \Gamma_X^{(n)}\colon \exists \iota \in S_n \text{ s.t. } x_{\iota(k)} \in U_k,\ k = 1, \ldots, n\bigr\},$$

by

$$(h_1 \hat{\times} \cdots \hat{\times} h_n)\bigl(\{x_1, \ldots, x_n\}\bigr) := \bigl(h_1(x_{\iota(1)}), \ldots, h_n(x_{\iota(n)})\bigr) \in h_1(U_1) \times \cdots \times h_n(U_n).$$

Each set $\Gamma_X^{(n)}$ endowed with this geometry has the structure of a $n \cdot \dim(X)$-dimensional C^∞-manifold. In this way we have also defined a differentiable structure on Γ_0. For any vector field v on X we have

$$\bigl(\nabla_v^{\Gamma_0} G\bigr)(\eta) = \sum_{x \in \eta} \bigl\langle \bigl(\nabla^{\Gamma_0} G\bigr)(\eta, x), v(x)\bigr\rangle_{T_x X},$$

yielding, in particular,

$$\bigl(\nabla^{\Gamma_0} e_\lambda(\theta)\bigr)(\eta, x) = \nabla^X \theta(x) e_\lambda\bigl(\theta, \eta \setminus \{x\}\bigr), \quad \eta \in \Gamma_0,\ x \in \eta, \tag{23}$$

$\nabla := \nabla^X$ being the gradient on X. For the Laplace–Beltrami operator Δ^{Γ_0} on Γ_0, which is defined by the direct sum of the Laplace–Beltrami operators $\Delta^{\Gamma_X^{(n)}}$ on $\Gamma_X^{(n)}$, we find

$$\bigl(\Delta^{\Gamma_0} e_\lambda(\theta)\bigr)(\eta) = \sum_{x \in \eta} \Delta^X \theta(x) e_\lambda\bigl(\theta, \eta \setminus \{x\}\bigr), \tag{24}$$

where $\Delta := \Delta^X$ denotes the Laplace–Beltrami operator on X.

In the sequel we use the classical notation $C^k(\Gamma_0)$, $k \in \mathbb{N} \cup \{\infty\}$, for the space of all real-valued C^k-functions on Γ_0, and $C_0^k(\Gamma_0)$ for the space of all functions G in $C^k(\Gamma_0)$ with bounded support such that for some $\varepsilon > 0$ one has $G(\eta) = 0$ for all η containing a pair x, y, $x \neq y$, such that $|x - y| \leqslant \varepsilon$.

Through the K-transform one may introduce a differential structure on Γ [15], which coincides with the one introduced in [2] by "lifting" the geometrical structure on the underlying manifold X. For each $G \in C_0^1(\Gamma_0)$,

$$\bigl(\nabla^\Gamma (KG)\bigr)(\gamma, x) := \sum_{\eta \subset \gamma\colon |\eta| < \infty, x \in \eta} \bigl(\nabla^{\Gamma_0} G\bigr)(\eta, x), \quad \gamma \in \Gamma,\ x \in \gamma,$$

and $\Delta^\Gamma := K \Delta^{\Gamma_0} K^{-1}$ on $\mathcal{FP}(C_0^2, \Gamma)$, the set of all twice differentiable cylinder polynomials F with the property that there is a $\varepsilon > 0$ such that $F(\gamma) = 0$ on all γ which contains a pair of points in the domain of cylindricity with distance smaller than ε. Equivalently, all such functions F are of the form $F = KG$, $G \in C_0^2(\Gamma_0)$.

5.2. Non-equilibrium stochastic dynamics equations

For particles in suspension in a liquid, each particle interacts with the molecules of the fluid and the remaining particles in the suspension. At the microscopic level, the time evolution of the whole system is described by Hamiltonian dynamics. In the mesoscopic approximation, the system is described as the result of random perturbations of the particles with dynamics heuristically given by a system of stochastic differential equations

$$\begin{cases} dx_k(t) = -\frac{\beta}{2}\sum_{1\leqslant i\neq k}\nabla V(x_k(t)-x_i(t))\,dt + dW_k(t), & t\geqslant 0,\\ x_k(0)=x_k, \quad k\in\mathbb{N}, \end{cases} \tag{25}$$

for a given starting configuration $\gamma = \{x_k\colon k\in\mathbb{N}\}$. Here W_k, $k\in\mathbb{N}$, is a family of independent standard Brownian motions describing the random perturbations and $V:\mathbb{R}^d\setminus\{0\}\to\mathbb{R}$ is the interaction potential between the particles.

The purpose of this subsection is to investigate the problem heuristically formulated in (25). Let us first fix the framework. On the space $X=\mathbb{R}^d$, $d\in\mathbb{N}$, let us consider the intensity measure $d\sigma(x) = z\,dm(x)$, m being the Lebesgue measure on $\mathbb{R}^d$ and $z>0$ (activity), and an even C^2-function $V:\mathbb{R}^d\setminus\{0\}\to\mathbb{R}$ (potential) such that all the first and second order derivatives of e^{-V} are in $L^1(X,m)$. Accordingly, we may define a translation-invariant pair potential ϕ on $\mathbb{R}^d$ by $\phi(x,y):=V(y-x)$. Concerning V, we may additionally assume the standard Ruelle conditions of superstability, integrability (i.e., Assumption 3), and lower regularity [35], which are sufficient to insure the existence of the corresponding Gibbs measures, cf., e.g., [35, Section 5]. In particular, this includes the class of potentials V which are bounded from below and integrable at infinity, and having a small enough negative part.

Let us note that due to the symmetry in the labels, any solution $(x_k)_k$ in $(\mathbb{R}^d)^{\mathbb{N}}$ of (25) can be interpreted (modulo collapse) as a stochastic process with paths in the configuration space Γ, that is, $\gamma(t):=\{x_k(t)\colon k\in\mathbb{N}\}$. Informally, the generator of this dynamics is given by

$$(HF)(\gamma) := -\frac{1}{2}\big(\Delta^\Gamma F\big)(\gamma) + \frac{\beta}{2}\sum_{x\in\gamma}\sum_{y\in\gamma\setminus\{x\}}\big\langle\nabla_x V(x-y), \nabla^\Gamma F(\gamma,x)\big\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathbb{R}^d$ and β the inverse temperature. Note that in contrast to (25), the generator H is well defined, for example, on $\mathcal{FP}(C_0^2,\Gamma)$.

In the equilibrium dynamics case, the authors in [1] have constructed a solution for a wide class of potentials V. More precisely, for a Gibbs measure μ_{inv} corresponding to V, the same as used in definition (25), it has been shown that H is a positive symmetric operator on the space $L^2(\Gamma,\mu_{\mathrm{inv}})$ associated to the Dirichlet form

$$(HF,F)_{L^2(\mu_{\mathrm{inv}})} = \frac{1}{2}\int_\Gamma \sum_{x\in\gamma}\big|\nabla^\Gamma F(\gamma,x)\big|^2\,d\mu_{\mathrm{inv}}(\gamma).$$

This allows the use of standard Dirichlet forms techniques to construct a diffusion process corresponding to H having μ_{inv} as an invariant (and, moreover, reversible) measure and starting on μ_{inv}-a.a. initial points. In particular, the corresponding semigroup yields a solution $F_t := e^{-Ht}F_0$, $t \geqslant 0$, in $L^2(\Gamma, \mu_{\mathrm{inv}})$ of the Cauchy problem

$$\begin{cases} \frac{d}{dt}F_t = -HF_t, & t \geqslant 0, \\ F_0. \end{cases} \tag{26}$$

For further references see also [1].

An essentially more difficult and interesting question is the non-equilibrium dynamics case. This means, the construction of the dynamics without reference to any invariant measure. In this case, the above scheme does not apply, and the only general result was obtained in [12] for a restrictive class of potentials and $d \leqslant 4$.

In the sequel we describe a new scheme for the construction of the dynamics based on the diagram in Remark 1 (Section 2). For this purpose we shall fix a probability measure μ on Γ as initial distribution. Now, in contrast to the previous situation, we neither assume that the measure μ is an invariant measure nor that it has a density with respect to an invariant one. The idea is then to transform the Cauchy problem (26) using the mappings presented in the aforementioned diagram. More precisely, we rigorously transform the corresponding operator expression by the linear mappings. This can be done under very mild assumptions. At the end, this leads to the Cauchy problem (30) and (32) for Bogoliubov functionals corresponding to the states developed in time.

The starting point for the approach is the description of the operator H in terms of quasi-observables. In fact, as H is well defined, for instance, on $\mathcal{FP}(C_0^2, \Gamma)$, its image under the K-transform yields on the space of quasi-observables the operator $\hat{H} := K^{-1}HK$ acting on functions $G \in C_0^2(\Gamma_0)$ by

$$\begin{aligned}(\hat{H}G)(\eta) = -\frac{1}{2}\big(\Delta^{\Gamma_0}G\big)(\eta) + \frac{\beta}{2}\sum_{x\in\eta}\sum_{y\in\eta\setminus\{x\}} \big\{\big\langle\nabla_x V(x-y), \big(\nabla^{\Gamma_0}G\big)(\eta, x)\big\rangle \\ + \big\langle\nabla_x V(x-y), \big(\nabla^{\Gamma_0}G\big)\big(\eta\setminus\{y\}, x\big)\big\rangle\big\}. \end{aligned} \tag{27}$$

One may thus consider the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}G_t(\eta) = -(\hat{H}G_t)(\eta), & t \geqslant 0,\ \eta \in \Gamma_0, \\ G_0 \in C_0^2(\Gamma_0). \end{cases} \tag{28}$$

Concerning the underlying Kolmogorov equation, note that due to (27) the time derivative of each $G_t\restriction_{\Gamma^{(n)}_{\mathbb{R}^d}}$ depends only on $G_t\restriction_{\Gamma^{(n)}_{\mathbb{R}^d}}$ itself and $G_t\restriction_{\Gamma^{(n-1)}_{\mathbb{R}^d}}$. This means that, in contrast to (26), one may recursively solve (28). However, the difficulty is to show that the solution of (28) is sufficiently regular to allow for a reconstruction of the dynamics on the level of functions on Γ through the K-transform.

According to the aforementioned diagram (Remark 1), we may also describe the dynamics in terms of correlation functions through the dual operator $\hat{H}^*$ of $\hat{H}$ in the sense

$$\int_{\Gamma_0} (\hat{H}G)(\eta)k(\eta)\,d\lambda_m(\eta) = \int_{\Gamma_0} G(\eta)(\hat{H}^*k)(\eta)\,d\lambda_m(\eta).$$

As an aside, let us mention that in the Hamiltonian dynamics case this approach corresponds to the well-known BBGKY-hierarchy, see, e.g., [4]. In our case, this leads informally to

$$\begin{cases} \frac{\partial}{\partial t} k_t^{(n)} = -(\hat{H}^* k_t)^{(n)}, \\ k_0^{(n)}, \quad n \in \mathbb{N}_0, \end{cases}$$

where $k_0^{(n)}$, $n \in \mathbb{N}_0$, are the correlation functions corresponding to the initial distribution μ. The operator $\hat{H}^*$ can be rigorously determined on all correlation functions k fulfilling

$$\left|\Delta k(\eta)\right| + \left|\nabla k(\eta)\right| + k(\eta) \leqslant C_1^{|\eta|} e^{-C_2 E(\eta)}, \quad C_1, C_2 \geqslant 0,\ \lambda_m\text{-a.a. } \eta \in \Gamma_0. \tag{29}$$

This condition was exploited in [20] for an existence result. Hence, writing $\hat{H}^*$ explicitly, one finds

$$\begin{aligned} &-(\hat{H}^* k)^{(n)}(x_1, \ldots, x_n) \\ &\quad = \frac{1}{2} \sum_{l=1}^{n} \Delta_{x_l} k^{(n)}(x_1, \ldots, x_n) + \frac{\beta}{2} \sum_{\substack{l,j=1 \\ k \neq j}}^{n} \Delta V(x_l - x_j) k^{(n)}(x_1, \ldots, x_n) \\ &\qquad + \frac{\beta}{2} \sum_{\substack{l,j=1 \\ l \neq j}}^{n} \left\langle \nabla_{x_l} V(x_l - x_j), \nabla_{x_l} k^{(n)}(x_1, \ldots, x_n) \right\rangle \\ &\qquad + \frac{\beta}{2} \sum_{l=1}^{n} \int_{\mathbb{R}^d} \left\langle \nabla_{x_l} V(x_l - y), \nabla_{x_l} k^{(n+1)}(x_1, \ldots, x_n, y) \right\rangle dy \\ &\qquad + \frac{\beta}{2} \sum_{l=1}^{n} \int_{\mathbb{R}^d} \Delta V(x_l - y) k^{(n+1)}(x_1, \ldots, x_n, y)\, dy. \end{aligned}$$

In theoretical physics the latter is related to the well-known Bogoliubov–Streltsova diffusion hierarchy (see [37]).

The previous construction gives us a way to express the dynamics in terms of Bogoliubov functionals

$$L_t(\theta) := \int_{\Gamma_0} e_\lambda(\theta, \eta) k_t(\eta)\, d\lambda_m(\eta), \quad t \geqslant 0.$$

The idea is summarized in the following informal calculations:

$$\begin{aligned} \frac{\partial}{\partial t} L_t(\theta) &= \int_{\Gamma_0} e_\lambda(\theta, \eta) \left(\frac{\partial}{\partial t} k_t(\eta) \right) d\lambda_m(\eta) = -\int_{\Gamma_0} e_\lambda(\theta, \eta) (\hat{H}^* k_t)(\eta)\, d\lambda_m(\eta) \\ &= -\int_{\Gamma_0} \left(\hat{H} e_\lambda(\theta) \right)(\eta) k_t(\eta)\, d\lambda_m(\eta). \end{aligned}$$

Heuristically, this means that the Bogoliubov functionals L_t, $t \geqslant 0$, are a solution of the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} L_t = -\tilde{H} L_t, \\ L_0, \end{cases} \tag{30}$$

for

$$(\tilde{H}L)(\theta) := \int_{\Gamma_0} \big(\hat{H} e_\lambda(\theta)\big)(\eta) k(\eta)\, d\lambda_m(\eta), \quad t \geqslant 0. \tag{31}$$

The operator $\tilde{H}$ can be rigorously defined. In fact, by (27), for all C^2-functions θ on $\mathbb{R}^d$ with compact support (shortly $\theta \in C_0^2(\mathbb{R}^d)$) we have

$$\begin{aligned} \big(\hat{H} e_\lambda(\theta)\big)(\eta) = -\frac{1}{2}\big(\Delta^{\Gamma_0} e_\lambda(\theta)\big)(\eta) + \frac{\beta}{2} \sum_{x\in\eta} \sum_{y\in\eta\setminus\{x\}} \big\{\big\langle \nabla_x V(x-y), \big(\nabla^{\Gamma_0} e_\lambda(\theta)\big)(\eta, x)\big\rangle \\ + \big\langle \nabla_x V(x-y), \big(\nabla^{\Gamma_0} e_\lambda(\theta)\big)\big(\eta \setminus \{y\}, x\big)\big\rangle\big\}. \end{aligned}$$

Therefore, if the correlation function k in (31) fulfills (29), we find from equalities (23) and (24),

$$\begin{aligned} -(\tilde{H}L)(\theta) &= \frac{1}{2} \int_{\Gamma_0} \sum_{x\in\eta} \Delta\theta(x) e_\lambda\big(\theta, \eta\setminus\{x\}\big) k(\eta)\, d\lambda_m(\eta) \\ &\quad - \frac{\beta}{2} \int_{\Gamma_0} \sum_{x\in\eta} \sum_{y\in\eta\setminus\{x\}} \big\langle \nabla_x V(x-y), \nabla\theta(x)\big\rangle e_\lambda\big(\theta, \eta\setminus\{x\}\big) k(\eta)\, d\lambda_m(\eta) \\ &\quad - \frac{\beta}{2} \int_{\Gamma_0} \sum_{x\in\eta} \sum_{y\in\eta\setminus\{x\}} \big\langle \nabla_x V(x-y), \nabla\theta(x)\big\rangle e_\lambda\big(\theta, \eta\setminus\{x,y\}\big) k(\eta)\, d\lambda_m(\eta) \\ &= \frac{1}{2} \int_{\Gamma_0} \sum_{x\in\eta} \Delta\theta(x) e_\lambda\big(\theta, \eta\setminus\{x\}\big) k(\eta)\, d\lambda_m(\eta) \\ &\quad - \frac{\beta}{2} \int_{\Gamma_0} \sum_{\{x,y\}\subset\eta} \big\langle \nabla_x V(x-y), \nabla\theta(x)\theta(y) - \nabla\theta(y)\theta(x)\big\rangle \\ &\quad \times e_\lambda\big(\theta, \eta\setminus\{x,y\}\big) k(\eta)\, d\lambda_m(\eta) \\ &\quad - \frac{\beta}{2} \int_{\Gamma_0} \sum_{\{x,y\}\subset\eta} \big\langle \nabla_x V(x-y), \nabla\theta(x) - \nabla\theta(y)\big\rangle e_\lambda\big(\theta, \eta\setminus\{x,y\}\big) k(\eta)\, d\lambda_m(\eta). \end{aligned}$$

Given a correlation function k fulfilling (29), we note that by Proposition 16 the functional

$$L(\theta) := \int_{\Gamma_0} e_\lambda(\theta, \eta) k(\eta)\, d\lambda_m(\eta)$$

is in $\mathrm{Ent}_\alpha(L^1(\sigma))$ for some α. An application of Corollary 15 leads then to

$$-(\tilde{H}L)(\theta) = \frac{1}{2}\int_{\mathbb{R}^d} \Delta\theta(x)\frac{\delta L(\theta)}{\delta\theta(x)}\,dx - \frac{\beta}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big\langle \nabla_x V(x-y), \nabla\theta(x)\big(\theta(y)+1\big) - \nabla\theta(y)\big(\theta(x)+1\big)\big\rangle \frac{\delta^2 L(\theta)}{\delta\theta(x)\delta\theta(y)}\,dx\,dy \tag{32}$$

for all $\theta \in C_0^2(\mathbb{R}^d)$.

Up to this point we have worked under assumption (29). As we have already mentioned, it implies that the functional L in (32) is in $\mathrm{Ent}_\alpha(L^1(\sigma))$ for some $\alpha > 0$. In addition, by Proposition 11, the following bound follows from (29):

$$\left|\frac{\delta^2 L(\theta)}{\delta\theta(x)\delta\theta(y)}\right| \leqslant e^{\alpha'\|\theta\|_{L^1(m)}} e^{-cV(x-y)}, \quad \alpha', c > 0,\ m\text{-a.a. } x, y. \tag{33}$$

One might thus consider the operator $\tilde{H}$ acting in a $\mathrm{Ent}_\alpha(L^1(\sigma))$ space for some $\alpha > 0$. Hence it is convenient to replace (29) by a condition just involving functionals from $\mathrm{Ent}_\alpha(L^1(\sigma))$, namely, (33).

To conclude let us consider the functional $\mathcal{L}(\varphi) := L(e^\varphi - 1)$ with $\varphi \in C_0^2(\mathbb{R}^d)$. We observe that if L is a Bogoliubov functional corresponding to a measure μ, then $\mathcal{L}$ is the Laplace transform corresponding to μ, i.e.,

$$\mathcal{L}(\varphi) := \int_\Gamma \exp\big(\langle\gamma, \varphi\rangle\big)\,d\mu(\gamma) = L_\mu\big(e^\varphi - 1\big).$$

In (32) consider $\theta = e^\varphi - 1$ with $\varphi \in C_0^2(\mathbb{R}^d)$, i.e.,

$$\begin{aligned} -(\tilde{H}L)\big(e^\varphi - 1\big) &= \frac{1}{2}\int_{\mathbb{R}^d} \big(\Delta\varphi(x) + \big|\nabla\varphi(x)\big|^2\big) e^{\varphi(x)} \frac{\delta L(\theta)}{\delta\theta(x)}\,dx \\ &\quad - \frac{\beta}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big\langle \nabla_x V(x-y), \nabla\varphi(x) - \nabla\varphi(y)\big\rangle \\ &\quad \times e^{\varphi(x)+\varphi(y)} \frac{\delta^2 L(\theta)}{\delta\theta(x)\delta\theta(y)}\,dx\,dy. \end{aligned}$$

The chain rule then yields

$$\frac{\delta\mathcal{L}(\varphi)}{\delta\varphi(x)} = \frac{\delta L(\theta)}{\delta\theta(x)} \frac{\delta(e^\varphi - 1)(\varphi)}{\delta\varphi(x)} = \frac{\delta L(\theta)}{\delta\theta(x)} e^{\varphi(x)}, \quad m\text{-a.a. } x$$

and

$$\frac{\delta^2\mathcal{L}(\varphi)}{\delta\varphi(x)\delta\varphi(y)} = \frac{\delta}{\delta\varphi(x)}\left(\frac{\delta\mathcal{L}(\varphi)}{\delta\varphi(y)}\right) = e^{\varphi(y)}\frac{\delta}{\delta\varphi(x)}\left(\frac{\delta L(\theta)}{\delta\theta(y)}\right)$$
$$= e^{\varphi(y)+\varphi(x)}\frac{\delta^2 L(\theta)}{\delta\theta(x)\delta\theta(y)}, \quad m\text{-a.a. } x, y.$$

As a consequence, for L_t given by (30) and (32), we have obtained the time evolution equation for the Laplace transform $\mathcal{L}_t(\varphi)$

$$\frac{\partial}{\partial t}\mathcal{L}_t(\varphi) = \frac{\partial}{\partial t}L_t\big(e^{\varphi}-1\big) = -(\tilde{H}L_t)\big(e^{\varphi}-1\big)$$
$$= \frac{1}{2}\int_{\mathbb{R}^d}\big(\Delta\varphi(x) + \big|\nabla\varphi(x)\big|^2\big)\frac{\delta\mathcal{L}_t(\varphi)}{\delta\varphi(x)}\,dx$$
$$- \frac{\beta}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big\langle\nabla_x V(x-y), \nabla\varphi(x) - \nabla\varphi(y)\big\rangle\frac{\delta^2\mathcal{L}_t(\varphi)}{\delta\varphi(x)\delta\varphi(y)}\,dx\,dy,$$

for all $\varphi \in C_0^2(\mathbb{R}^d)$. This equation is related to the Hopf equation from hydrodynamics.

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