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A generalized Clark-Ocone formula

Margarida de FARIA[1], Maria João OLIVEIRA[2] and Ludwig STREIT[1,3]
[1] *CCM, Universidade da Madeira, P 9000 Funchal*
[2] *Univ. Aberta/Grupo de Física Matemática, Av. Prof. Gama Pinto, 2, P 1649-003 Lisboa*
[3] *BiBoS, Universität Bielefeld, D 33501 Bielefeld*

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Abstract— We extend the Clark-Ocone formula to a suitable class of generalized Brownian functionals. As an example we derive a representation of Donsker's delta function as (limit of) a stochastic integral.

1. INTRODUCTION

For suitable functionals φ of Brownian motion, expressed in terms of Itô integrals

$$\varphi = \mathbf{E}\,(\varphi) + \int m(\tau)\,\mathrm{d}B(\tau) \tag{1}$$

the Clark-Ocone formula [5], [14] provides us with an explicit formula for the integrand $m(\cdot)$, given φ. It has become clear that such an expression should be useful in the determination of hedging portfolios, see *e.g.*, [1], [3], [15]. Another application is in the context of determining the quadratic variation process of Brownian martingales, see *e.g.*, [7] for a recent example.

On the other hand it was pointed out in [2] that the conditions on φ are restrictive. It seems desirable to extend the validity of (1) and of the Clark-Ocone formula. A possible setting is that of generalized functionals of white noise as described, *e.g.*, in [8]–[13], [16]. In particular the generalized function space elaborated in [16], or the larger one of [8], retain the probabilistic properties that are required for such a generalization. In [3] one finds an announcement of results in terms of the Potthoff-Timpel [4], [16] space (in the meantime elaborated in [1]), for related results in the space D' of Malliavin calculus see [17]. Here we use the space of [8], which we present in the following section together with extensions of the Skorohod and Itô integrals, the gradient and some further auxiliary notions. Section 3 has our generalization of the Clark-Ocone formula, in Section 4 we translate the result into the language, often useful in practical calculations, of the $S-$transform. A case in point is Donsker's $\delta-$function for which we elaborate the generalized Clark-Ocone formula in Section 5.

2. REGULAR GENERALIZED FUNCTIONS OF WHITE NOISE

2.1. Regular generalized functions

We will recall the definition and some properties of the space $\mathcal{G}^{-1}$ of regular generalized functions of White Noise [8], [9].

Within the Hilbert space $L^2_d(\mathbb{R}) \equiv L^2(\mathbb{R}, \mathbb{R}^d)$, $d \in \mathrm{N}$, of vector valued square integrable functions we consider the space $S_d(\mathbb{R})$ of vector valued Schwartz test functions. The topology on $S_d(\mathbb{R})$ may be given in terms of a system of increasing Hilbertian norms

$$\left|\vec{\xi}\right|_p^2 = \sum_{i=1}^{d} |\xi_i|_p^2, \vec{\xi} = (\xi_1, ..., \xi_d) \in S_d(\mathbb{R}), \xi_i \in S(\mathbb{R}), i = 1, ..., d, p \in \mathrm{N}_0.$$

The basic nuclear triple is thus

$$S_d(\mathbb{R}) \subset L^2_d(\mathbb{R}) \subset S'_d(\mathbb{R}).$$

On $S'_d(\mathbb{R})$ we fix the canonical Gaussian measure μ_d which is determined by the characteristic function

$$C\left(\vec{\xi}\right) \equiv \exp\left(-\frac{1}{2}\sum_{i=1}^{d}\int_{\mathbb{R}} \xi_i^2(t)\,dt\right), \quad \vec{\xi} \in S_d(\mathbb{R}).$$

The space $L^2(S'_d(\mathbb{R}), \mu_d)$ will be briefly denoted by (L^2).

We will denote by $\vec{n}$ the d-tuple $(n_1, ..., n_d)$, $n_i \in \mathrm{N}_0$, and write

$$n = \sum_{i=1}^{d} n_i,$$
$$\vec{n}! = \prod_{i=1}^{d} n_i!.$$

The norm and the inner product in $L^2(\mathbb{R}^n)$ will be denote by $|\cdot|_n$ and $(\cdot,\cdot)_n$, respectively.

Considering square integrable white noise functionals φ for which the chaos expansion

$$\begin{aligned} \varphi(\vec{\omega}) &= \sum_{\vec{n}} <: \vec{\omega}^{\otimes \vec{n}} :, \varphi_{\vec{n}} > \\ &\equiv \sum_{\vec{n}} \int_{\mathbb{R}^n} d^n t \varphi_{\vec{n}}(t_1^1, ..., t_n^d) \prod_{i=1}^{d} : \omega_i^{\otimes n_i} : (t_1^i, ..., t_{n_i}^i) \end{aligned}$$

converges rapidly, *i.e.*,

$$\|\varphi\|_q^2 \equiv \sum_{\vec{n}} (\vec{n}!)^2 2^{qn} |\varphi_{\vec{n}}|_n^2 < \infty,$$

we define the Hilbert space G^1_q as

$$G^1_q = \left\{\varphi \in (L^2) : \|\varphi\|_q^2 < \infty\right\}.$$

The space of test functions $\mathcal{G}^1$ is defined as the projective limit of the spaces G^1_q, $q \in \mathrm{N}_0$,

$$\mathcal{G}^1 = pr - \lim_q G^1_q.$$

Let G^{-1}_{-q} be the dual with respect to (L^2) of G^1_q and $\mathcal{G}^{-1}$ the dual space of $\mathcal{G}^1$ with respect to (L^2). The corresponding bilinear dual pairing $\ll \cdot, \cdot \gg$ is connected to the sesquilinear inner product on (L^2) by

$$\ll F, \varphi \gg = (\bar{F}, \varphi)_{(L^2)}, \text{ if } F \in (L^2).$$

(We shall use the same notation $\ll \cdot,\cdot \gg$ for dual pairings in more general settings such as, *e.g.*, for $L^2(\mathbb{R}) \otimes \mathcal{G}^{\pm 1}$). Since the constant function 1 is in $\mathcal{G}^1$ we may extend the definition of the expectation $\mathbf{E}(\cdot)$ from integrable functions to distributions $\Phi \in \mathcal{G}^{-1}$:

$$\mathbf{E}(\Phi) = \ll \Phi, 1 \gg .$$

From general duality theory it follows that

$$\mathcal{G}^{-1} = \bigcup_{q \geq 0} G^{-1}_{-q};$$

therefore, every distribution is of finite order, *i.e.*, for every $\Phi \in \mathcal{G}^{-1}$ there exists $q \in \mathrm{N}_0$ such that $\Phi \in G^{-1}_{-q}$. It turns out from the definition that the Hilbert space G^{-1}_{-q} can be described as follows:

$$G^{-1}_{-q} = \left\{ \Phi(\vec{\omega}) = \sum_{\vec{n}} <: \vec{\omega}^{\otimes \vec{n}} :, \Phi_{\vec{n}} >, \|\Phi\|^2_{-q} \equiv \sum_{\vec{n}} 2^{-qn} \left|\Phi_{\vec{n}}\right|^2_n < \infty \right\}.$$

Given $\vec{\xi} \in S_d(\mathbb{R})$, let us consider the Wick exponential

$$\begin{aligned} :\!\exp< \vec{\omega}, \vec{\xi} >\!: \; &\equiv \exp\left(< \vec{\omega}, \vec{\xi} > - \tfrac{1}{2} \sum_{i=1}^{d} \int \xi_i^2(t) dt \right) \\ &= \sum_{\vec{n}} \tfrac{1}{\vec{n}!} <: \vec{\omega}^{\otimes \vec{n}} :, \vec{\xi}^{\otimes \vec{n}} >, \; \vec{\omega} \in S'_d(\mathbb{R}). \end{aligned}$$

Since the sum

$$\sum_{\vec{n}} (\vec{n}!)^2 2^{qn} \left| \frac{1}{\vec{n}!} \vec{\xi}^{\otimes \vec{n}} \right|^2_{L^2_d(\mathbb{R}^n)} = \sum_{\vec{n}} 2^{qn} \left| \vec{\xi} \right|^{2n}_{L^2_d(\mathbb{R})}$$

converges if and only if $2^q \left|\vec{\xi}\right|^2_{L^2_d(\mathbb{R})} < 1$, the Wick exponentials are not test functions in $\mathcal{G}^1$, but they are in those G^1_q for which $2^q \left|\vec{\xi}\right|^2_{L^2_d(\mathbb{R})} < 1$. Thus it is still possible to define an S-transform in the space $\mathcal{G}^{-1}$ because every distribution is of finite order. Given $\Phi \in \mathcal{G}^{-1}$, there exists $q \in \mathrm{N}_0$ such that $\Phi \in G^{-1}_{-q}$. For all $\vec{\xi} \in S_d(\mathbb{R})$ with $2^q \left|\vec{\xi}\right|^2_{L^2_d(\mathbb{R})} < 1$ we define the S-transform of Φ as

$$S\Phi(\vec{\xi}) \equiv \ll \Phi, : \exp < \cdot, \vec{\xi} >: \gg = \sum_{\vec{n}} \left(\Phi_{\vec{n}}, \vec{\xi}^{\otimes \vec{n}} \right)_n .$$

This definition extends to complex vectors $\vec{\eta} \in S_{d,c}(\mathbb{R})$ such that $2^q \left|\vec{\eta}\right|^2_{L^2_d(\mathbb{R})} < 1$,

$$S\Phi(\vec{\eta}) \equiv \ll \Phi, : \exp < \cdot, \vec{\eta} >: \gg = \sum_{\vec{n}} \left(\Phi_{\vec{n}}, \vec{\eta}^{\otimes \vec{n}} \right)_n . \qquad (2)$$

Therefore, for $\Phi \in G^{-1}_{-q}$, (2) defines the S-transform for every $\vec{\eta}$ from the open neighborhood of zero, $U_q \equiv \{\vec{\eta} \in S_{d,c}(\mathbb{R}) : 2^q \left|\vec{\eta}\right|^2_{L^2_d(\mathbb{R})} < 1\}$, $q \in \mathrm{N}_0$.

2.2. Generalization of the gradient operator

We begin with the observation that the Hida derivative ∂_t fails to be pointwise defined on the spaces $G^{\pm 1}_{\pm q}$. However we still may consider the gradient $(\partial^i_{\cdot}\cdot)_{1\le i\le d}$ as an operator

$$(\partial^i_{\cdot}\cdot)_{1\le i\le d}: G^{\pm 1}_{\pm q} \to \mathbb{R}^d \otimes L^2(\mathbb{R}) \otimes G^{\pm 1}_{\pm p}.$$

Given ψ a test function from $\mathcal{G}^1$ with kernel functions $\psi_{\vec{n}}$, $n \in \mathrm{N}_0$, we define the operator *gradient of* ψ, $\nabla\psi \equiv (\partial^i_{\cdot}\psi)_{1\le i\le d}$, where, for each $1 \le i \le d$, $\partial^i_{\cdot}\psi$ is the functional from $L^2(\mathbb{R}) \otimes G^1_q$, $q \in \mathrm{N}_0$, characterized by the sequence

$$\psi^i_{\vec{n}}(t,s) \equiv (n_i+1)\psi_{\overrightarrow{n+\delta_i}}(s^1_1,\dots,s^1_{n_1};\dots;s^i_1,\dots,s^i_{n_i},t;\dots;s^d_1,\dots,s^d_{n_d}) \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^n), \quad \vec{n} = (n_1,\dots,n_d).$$

In fact,

$$\begin{aligned}\int_{\mathbb{R}} dt \sum_{\vec{n}} (\vec{n}!)^2 2^{qn} \left|\psi^i_{\vec{n}}(t,\cdot)\right|^2_n &= \sum_{\vec{n}} ((\overrightarrow{n+\delta_i})!)^2 2^{qn} \left|\psi_{\overrightarrow{n+\delta_i}}\right|^2_{n+1} \\ &= 2^{-q} \sum_{\vec{n},n\ge 1} (\vec{n}!)^2 2^{qn} \left|\psi_{\vec{n}}\right|^2_n,\end{aligned}$$

which proves that $\partial^i_{\cdot}\psi \in L^2(\mathbb{R}) \otimes G^1_q$ for every non negative integer number q and, moreover, the continuity of the linear operators $\partial^i_{\cdot}: G^1_q \to L^2(\mathbb{R}) \otimes G^1_q$:

$$\left\|\partial^i_{\cdot}\psi\right\|^2_{L^2(\mathbb{R})\otimes G^1_q} \le 2^{-q}\left\|\psi\right\|^2_{G^1_q}, \psi \in \mathcal{G}^1,$$

for every $q \in \mathrm{N}_0$. Hence,

$$\sum_{i=1}^d \left\|\partial^i_{\cdot}\psi\right\|^2_{L^2(\mathbb{R})\otimes G^1_q} \le d\cdot 2^{-q}\left\|\psi\right\|^2_{G^1_q}, \psi \in \mathcal{G}^1,$$

which proves that the linear operator $\nabla: G^1_q \to \mathbb{R}^d \otimes L^2(\mathbb{R}) \otimes G^1_q$ is continuous.

We extend the operator gradient from test functions on $\mathcal{G}^1$ (introduced above) to distributions on $\mathcal{G}^{-1}$.

Given a regular generalized function Φ from $\mathcal{G}^{-1}$, $\Phi \in G^{-1}_{-q}$ for some $q \in \mathrm{N}_0$, characterized by the sequence $(\Phi_{\vec{n}})$, $n \in \mathrm{N}_0$, $\Phi_{\vec{n}} \in L^2(\mathbb{R}^n)$, consider the functional characterized by the sequence

$$\Phi^i_{\vec{n}}(t,s) \equiv (n_i+1)\Phi_{\overrightarrow{n+\delta_i}}(s^1_1,\dots,s^1_{n_1};\dots;s^i_1,\dots,s^i_{n_i},t;\dots;s^d_1,\dots,s^d_{n_d}) \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^n),$$

$n \in \mathrm{N}_0$.

Using the inequality $2^{-k}k^2 < 2$ for $k = (p-q)n \ge 0$, we have

$$\begin{aligned}\int_{\mathbb{R}} dt \sum_{\vec{n}} 2^{-pn}(n_i+1)^2 \left|\Phi_{\overrightarrow{n+\delta_i}}(\cdot,t,\cdot)\right|^2_n &= \sum_{\vec{n}} 2^{-pn}(n_i+1)^2 \left|\Phi_{\overrightarrow{n+\delta_i}}\right|^2_{n+1} \\ &= 2^p \sum_{\vec{n}} 2^{-q(n+1)} 2^{-(p-q)(n+1)}(n_i+1)^2 \left|\Phi_{\overrightarrow{n+\delta_i}}\right|^2_{n+1} \\ &\le \frac{2^{p+1}}{(p-q)^2} \sum_{\vec{n}} 2^{-q(n+1)} \left|\Phi_{\overrightarrow{n+\delta_i}}\right|^2_{n+1} \\ &= \frac{2^{p+1}}{(p-q)^2} \sum_{\vec{n},n\ge 1} 2^{-qn} \left|\Phi_{\vec{n}}\right|^2_n,\end{aligned} \tag{3}$$

where the above sum is convergent because $\Phi \in G^{-1}_{-q}$. Hence, the sequence $(\Phi^i_{\vec{n}})$, $n \in \mathbb{N}_0$, defines a functional from $L^2(\mathbb{R}) \otimes G^{-1}_{-p}$. Keeping the terminology and the notation introduced above, we will denote this functional by $\partial^i_\cdot \Phi$ and the operator $(\partial^i_\cdot \Phi)_{1\le i\le d}$ will be called the *gradient of* Φ, denoted by $\nabla\Phi$. Using this notation, it follows from (3) that

$$\|\nabla\Phi\|^2_{\mathbb{R}^d\otimes L^2(\mathbb{R})\otimes G^{-1}_{-p}} \le d\frac{2^{p+1}}{(p-q)^2}\|\Phi\|^2_{G^{-1}_{-q}},$$

for every pair $p > q \ge 0$, which proves that the gradient is a bounded linear operator from G^{-1}_{-q} into $\mathbb{R}^d \otimes L^2(\mathbb{R}) \otimes G^{-1}_{-p}$ if $p > q$.

2.3. An extension of the Skorohod and Itô integrals

In [10] the Skorohod integral was discussed in a white noise setting. An extension to certain generalized white noise integrands can be found in [6].

Considering an element Φ from $L^2(\mathbb{R}) \otimes G^{-1}_{-q}$, for some $q \in \mathbb{N}_0$, characterized by the sequence $\Phi_{\vec{n}}(\cdot;\cdot) \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^n)$, $n \in \mathbb{N}_0$, let us consider the functional characterized by the sequence

$$\begin{aligned}\Psi^i_0 &\equiv 0\\ \Psi^i_{\vec{n}} &\equiv \tilde{\Phi}_{\overrightarrow{n-\delta_i}} \in L^2(\mathbb{R}^n), \vec{n} = (n_1,\dots,n_d), n \in N,\end{aligned}$$

where $\tilde{\Phi}_{\overrightarrow{n-\delta_i}}$ denotes the symmetrization of $\Phi_{\overrightarrow{n-\delta_i}}$ in the variables $t, s^i_1, \dots, s^i_{n_i-1}$. Since, for each $1 \le i \le d$,

$$\begin{aligned}\sum_{\vec{n},n\ge1} 2^{-qn}\left|\tilde{\Phi}_{\overrightarrow{n-\delta_i}}\right|^2_n &\le \sum_{\vec{n},n\ge1} 2^{-qn}\left|\Phi_{\overrightarrow{n-\delta_i}}\right|^2_{L^2(\mathbb{R})\otimes L^2(\mathbb{R}^{n-1})} \\ &= 2^{-q}\int_{\mathbb{R}} dt \sum_{\vec{n}} 2^{-qn}\left|\Phi_{\vec{n}}(t;\cdot)\right|^2_n \\ &= 2^{-q}\|\Phi\|^2_{L^2(R)\otimes G^{-1}_{-q}},\end{aligned} \tag{4}$$

the sequence $(\Psi^i_{\vec{n}})$, $n \in \mathbb{N}_0$, defines a distribution from G^{-1}_{-q}. We denote it by $I_i(\Phi)$. For every test function ψ from $\mathcal{G}^1$ with kernel functions $(\psi_{\vec{n}})$, $n \in \mathbb{N}_0$, we have for each $1 \le i \le d$,

$$\begin{aligned}\ll I_i(\Phi), \psi \gg &= \sum_{\vec{n},n\ge1} \vec{n}!\left(\tilde{\Phi}_{\overrightarrow{n-\delta_i}}, \psi_{\vec{n}}\right)_n \\ &= \sum_{\vec{n},n\ge1} \vec{n}!\left(\Phi_{\overrightarrow{n-\delta_i}}, \psi_{\vec{n}}\right)_{L^2(\mathbb{R})\otimes L^2(\mathbb{R}^{n-1})} \\ &= \int_{\mathbb{R}} dt \sum_{\vec{n},n\ge1} (\overrightarrow{n-\delta_i})!\left(\Phi_{\overrightarrow{n-\delta_i}}(t;\cdot), n_i\psi_{\vec{n}}(\cdot,t,\cdot)\right)_{n-1} \\ &= \int_{\mathbb{R}} dt \ll \Phi(t;\cdot), \partial^i_t\psi \gg = \ll \Phi, \partial^i_\cdot \psi \gg,\end{aligned}$$

and $I_i(\Phi)$ is the unique functional from $\mathcal{G}^{-1}$ for which the above equality holds for every test function $\psi \in \mathcal{G}^1$; if $I'(\Phi)$ is another functional in such conditions, it turns out that $\ll I_i(\Phi) - I'(\Phi), \cdot \gg$ is identically equal to zero on $\mathcal{G}^1$.

We may now formulate the

Definition. Given Φ an element from $\mathbb{R}^d \otimes L^2(\mathbb{R}) \otimes G_{-q}^{-1}$, for some $q \in \mathrm{N}_0$, we call generalized Skorohod integral of Φ the distribution on $\mathcal{G}^{-1}$, $I(\Phi)$, defined by the sum

$$I(\Phi) \equiv \sum_{i=1}^{d} I_i(\Phi_i),$$

where, for each $i = 1, ..., d$, $I_i(\Phi_i)$ is the unique regular generalized function from $\mathcal{G}^{-1}$ for which the following equality

$$\langle\langle I_i(\Phi_i), \psi\rangle\rangle = \langle\langle \Phi_i, \partial_\cdot^i \psi\rangle\rangle$$

holds for every test function ψ from $\mathcal{G}^1$.

This definition generalizes the notion of Skorohod integral. In fact, in the particular situation $\Phi \in \mathbb{R}^d \otimes L^2(\mathbb{R}) \otimes D$,

$$D \equiv \left\{ F \in (L^2) : F(\omega) = \sum_{\vec{n}} <: \omega^{\otimes \vec{n}} :, F_{\vec{n}} >, \sum_{\vec{n}} \vec{n}! n \left| F_{\vec{n}} \right|_n^2 < \infty \right\},$$

the generalized Skorohod integral $I(\Phi)$ coincides with the Skorohod integral. In view of the relation between the Skorohod and Itô integral we may add the following remark.

Remark 1. For $t \in \mathbb{R}$, let $\mathcal{F}_t$ denote the σ-algebra generated by the random variables $\{B(s), s \leq t\}$, where $(B_t)_{t\in\mathbb{R}}$ is a d-dimensional Brownian motion. If $F \in \mathbb{R}^d \otimes L^2(\mathbb{R}) \otimes (L^2)$ and it is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$, then the generalized Skorohod integral $I(F)$ is equal to the Itô integral of F. Without the first condition we speak of a *generalized Itô integral.*

Before ending this subsection, we return to the sequence of inequalities (4) which imply

$$\|I_i(\Phi_i)\|_{G_{-q}^{-1}}^2 \leq 2^{-q} \|\Phi_i\|_{L^2(\mathbb{R})\otimes G_{-q}^{-1}}^2,$$

i.e., I is a bounded linear operator from $\mathbb{R}^d \otimes L^2(\mathbb{R}) \otimes \mathrm{G}_{-q}^{-1}$ into G_{-q}^{-1}, $q \in \mathrm{N}_0$.

2.4. Some notations and definitions

For the next sections it is useful to introduce some notations and recall some definitions.

We shall denote by Θ_t the Heaviside function

$$\Theta_t(s) \equiv \begin{cases} 1 \text{ if } s \leq t \\ 0 \text{ if } s > t \end{cases}$$

and also the linear operator given by

$$\Theta_t : f(\cdot) \to \Theta_t(\cdot) f(\cdot).$$

The functional derivatives $\frac{\delta}{\delta f(t)}$, for suitable G, are defined as

$$\lim_{\varepsilon \to 0} \frac{G(f + \varepsilon f_0) - G(f)}{\varepsilon} = \int f_0(t) \frac{\delta G(f)}{\delta f(t)} \, dt.$$

In particular for cylinder functions $G(f) = g\left(\int h(t)f(t)\,\mathrm{d}t\right)$ where g is a differentiable function, then

$$\frac{\delta G(f)}{\delta f(\tau)} = g'\left(\int h(t)f(t)\mathrm{d}t\right) h(\tau). \tag{5}$$

For what follows it is also helpful to define the notion of second quantization of Θ_t, $t \in \mathbb{R}$, defined on the space $\mathcal{G}^{-\infty}$. Recall:

For bounded linear operators A on $L^2(\mathbb{R})$ the linear map which transforms each sequence $(\varphi_{\vec{n}})$, $n \in \mathbb{N}_0$, $\varphi_{\vec{n}} \in L^2(\mathbb{R}^n)$, to the sequence $(A^{\otimes n}\varphi_{\vec{n}})$, $n \in \mathbb{N}_0$, is called the *second quantization of* A. It is denoted by $\Gamma(A)$.

LEMMA 2.1 *For bounded linear operators A on $L^2(\mathbb{R})$, $\Gamma(A)$ is a continuous operator on $\mathcal{G}^{-1}$.*

Proof. Given a bounded linear operator A on $L^2(\mathbb{R})$, for every element $\Phi \in \mathcal{G}^{-1}$ (belonging to G^{-1}_{-q}, for some q), we have,

$$\sum_{\vec{n}} 2^{-pn} \left|A^{\otimes n}\Phi_{\vec{n}}\right|_n^2 \leq \sum_{\vec{n}} 2^{-pn} \|A\|^{2n} \left|\Phi_{\vec{n}}\right|_n^2,$$

for every non negative integer number p. If $\|A\|^2 \leq 2^{p-q}$ for some $p \in \mathbb{N}_0$ the above sum is majorized by $\|\Phi\|^2_{G^{-1}_{-q}}$ and the second quantization of A is a bounded linear operator from G^{-1}_{-q} into G^{-1}_{-p}. In particular, if $\|A\| \leq 1$, $\Gamma(A)$ is a bounded linear operator from $\mathcal{G}^{-1}$ into itself. The lemma is proved.

In particular, for $A = \Theta_t$, for some $t \in \mathbb{R}$, it follows that

$$\|\Gamma(\Theta_t)\Phi\|^2_{G^{-1}_{-q}} \leq \sum_{\vec{n}} 2^{-qn} \left|\Theta_t^{\otimes n}\Phi_{\vec{n}}\right|_n^2 \leq \|\Phi\|^2_{G^{-1}_{-q}},$$

for every $q \in \mathbb{N}_0$. Hence $\Gamma(\Theta_t)$ is a bounded linear operator from the space of regular generalized functions $\mathcal{G}^{-1}$ into itself.

Remark 2. Consider the σ-algebra $\mathcal{F}_t$ generated by the random variables $\{B(s), s \leq t\}$. $\Gamma(\Theta_t)\Phi$ coincides with the conditional expectation for elements Φ from $\mathcal{G}^{-1}$ with respect to $\mathcal{F}_t$, as introduced in [8].

3. THE GENERALIZED CLARK-OCONE FORMULA

Now we are prepared to present the main result of this note. It generalizes the well known Clark-Ocone formula to regular generalized functions of white noise, *i.e.*, to the space $\mathcal{G}^{-1}$.

THEOREM 3.1 ***(Generalized Clark-Ocone Formula)*** *Let Φ be a regular generalized function, $\Phi \in \mathcal{G}^{-1}$. Then it can be written as a generalized Itô integral*

$$\Phi = \mathbf{E}(\Phi) + I(m)$$

with

$$m_i(t) = \Gamma(\Theta_t)\partial_t^i \Phi$$

Proof. We begin by noting that the integrand m is non-anticipating. Let Φ be an arbitrary element from $\mathcal{G}^{-1}$, *i.e.*, $\Phi \in G_{-q}^{-1}$ for some q, characterized by the sequence $(\Phi_{\vec{n}})$, $n \in \mathbb{N}_0$. Hence, for every test function $\psi \in \mathcal{G}^1$ with kernel functions given by $(\psi_{\vec{n}})$, $n \in \mathbb{N}_0$, we have

$$\ll \Phi, \psi \gg = \mathbf{E}(\Phi)\mathbf{E}(\psi) + \sum_{\vec{n}, n \geq 1} \vec{n}!\, (\Phi_{\vec{n}}, \psi_{\vec{n}})_n ,$$

where, for each n-tuple $\vec{n} = (n_1, \ldots, n_d)$ such that $n \geq 1$,

$$(\Phi_{\vec{n}}, \psi_{\vec{n}})_n = \int_{\mathbb{R}^n} \mathrm{d}^n s\, \Phi_{\vec{n}}(\ldots; s_1^i, \ldots, s_{n_i}^i; \ldots)\psi_{\vec{n}}(\ldots; s_1^i, \ldots, s_{n_i}^i; \ldots).$$

Taking each variable s_j^i, $j = 1, \ldots, n_i$, $i = 1, \ldots, d$, in term, in the range $s_j^i \geq \sup_{(l,k)\neq(i,j)} s_k^l$, the above integral can be written as

$$\sum_{i=1}^{d} \sum_{j=1}^{n_i} \int_{\mathbb{R}} \mathrm{d}s_j^i \int_{-\infty}^{s_j^i} \mathrm{d}^{n-1} s\, \Phi_{\vec{n}}(\ldots; s_1^i, \ldots, s_j^i, \ldots, s_{n_i}^i; \ldots)\psi_{\vec{n}}(\ldots; s_1^i, \ldots, s_j^i, \ldots, s_{n_i}^i; \ldots),$$

which is equal to

$$\sum_{i=1}^{d} n_i \int_{\mathbb{R}} \mathrm{d}\tau \int_{-\infty}^{\tau} d^{n-1} s\, \Phi_{\vec{n}}(\ldots; \tau, s_1^i, \ldots, s_{n_i-1}^i; \ldots)\psi_{\vec{n}}(\ldots; \tau, s_1^i, \ldots, s_{n_i-1}^i; \ldots),$$

by the symmetry of $\Phi_{\vec{n}}$ and the kernel function $\psi_{\vec{n}}$ with respect to each n_i-tuple of variables $(s_1^i, \ldots, s_{n_i}^i)$, $i = 1, \ldots, d$. This means,

$$\begin{aligned}
\ll \Phi, \psi \gg \quad -\mathbf{E}(\Phi)\mathbf{E}(\psi) &= \sum_{\vec{n}, n\geq 1} \vec{n}! \sum_{i=1}^{d} n_i \int_{\mathbb{R}} \mathrm{d}\tau \left(\Theta_\tau^{\otimes(n-1)} \Phi_{\vec{n}}(\cdot, \tau, \cdot), \psi_{\vec{n}}(\cdot, \tau, \cdot)\right)_{n-1} \\
&= \sum_{i=1}^{d} \int_{\mathbb{R}} \mathrm{d}\tau \sum_{\vec{n}, n_i\geq 1} \vec{n}! n_i \left(\Theta_\tau^{\otimes(n-1)} \Phi_{\vec{n}}(\cdot, \tau, \cdot), \psi_{\vec{n}}(\cdot, \tau, \cdot)\right)_{n-1} \\
&= \sum_{i=1}^{d} \int_{\mathbb{R}} \mathrm{d}\tau \sum_{\vec{n}, n_i\geq 1} (\overrightarrow{n - \delta_i})! \left(\Theta_\tau^{\otimes(n-1)} n_i \Phi_{\vec{n}}(\cdot, \tau, \cdot), n_i \psi_{\vec{n}}(\cdot, \tau, \cdot)\right)_{n-1} \\
&= \sum_{i=1}^{d} \int_{\mathbb{R}} \mathrm{d}\tau \sum_{\vec{n}} \vec{n}! \left(\Theta_\tau^{\otimes n}(n_i + 1)\Phi_{\overrightarrow{n + \delta_i}}(\cdot, \tau, \cdot), (n_i + 1)\psi_{\overrightarrow{n + \delta_i}}(\cdot, \tau, \cdot)\right)_n \\
&= \sum_{i=1}^{d} \ll \Gamma(\Theta_.)\partial_.^i \Phi, \partial_.^i \psi \gg .
\end{aligned}$$

Therefore,

$$\ll \Phi, \psi \gg = \mathbf{E}(\Phi)\mathbf{E}(\psi) + \sum_{i=1}^{d} \ll I_i(\Gamma(\Theta_.)\partial_.^i \Phi), \psi \gg,$$

for every ψ belongs to $\mathcal{G}^1$, which implies the result. Theorem 3.1 is proved.

4. S-TRANSFORM

In this section, we find an expression to the S-transform of a regular generalized function $\Phi \in \mathcal{G}^{-1}$ which corresponds to the Clark-Ocone formula established above.

THEOREM 4.1 *Given a regular generalized function Φ from $\mathcal{G}^{-1}$ and $q \in \mathbb{N}_0$ such that $\Phi \in G^{-1}_{-q}$, its S-transform is equal to*

$$S\Phi(\vec{\eta}) = \mathbf{E}(\Phi) + \sum_{i=1}^{d} \int_{\mathbb{R}} d\tau\, \eta_i(\tau) \frac{\delta}{\delta \eta_i(\tau)} S(\Phi)(\Theta_\tau \vec{\eta}),$$

for every $\vec{\eta} = (\eta_1, ..., \eta_d) \in U_q$.

Proof. Taking $\Phi \in \mathcal{G}^{-1}$ characterized by the sequence $(\Phi_{\vec{n}})$, $n \in \mathbb{N}_0$, such that $\Phi \in G^{-1}_{-q}$ for some $q \in \mathbb{N}_0$, for every test function $\vec{\eta} = (\eta_1, ..., \eta_d) \in S_{d,c}(\mathbb{R})$ with $2^q |\vec{\eta}|^2_{L^2_d(\mathbb{R})} < 1$ we have

$$S\Phi(\vec{\eta}) = \mathbf{E}(\Phi) + \sum_{i=1}^{d} \ll I_i(\Gamma(\Theta_.)\partial^i_. \Phi), : \exp < \cdot, \vec{\eta} >: \gg . \qquad (6)$$

Here,

$$\sum_{i=1}^{d} \ll I_i(\Gamma(\Theta_.)\partial^i_. \Phi), : \exp < \cdot, \vec{\eta} >: \gg = \sum_{\vec{n}, n \geq 1} \left(\Phi_{\vec{n}}, \vec{\eta}^{\otimes \vec{n}}\right)_n .$$

Using the symmetry of $\Phi_{\vec{n}}$ in each n_i-tuple of variables $(s^i_1, \ldots, s^i_{n_i})$, $i = 1, \ldots, d$, it follows that

$$\begin{aligned}
\left(\Phi_{\vec{n}}, \vec{\eta}^{\otimes \vec{n}}\right)_n &= \int_{\mathbb{R}^n} d^n s \Phi_{\vec{n}}(\ldots; s^i_1, \ldots, s^i_{n_i}; \ldots) \prod_{k=1}^{d} \eta_k(s^k_1) \cdots \eta_k(s^k_{n_k}) \\
&= \sum_{i=1}^{d} \sum_{j=1}^{n_i} \int_{\mathbb{R}} ds^i_j \int_{-\infty}^{s^i_j} d^{n-1} s \Phi_{\vec{n}}(\ldots; s^i_1, \ldots, s^i_{n_i}; \ldots) \prod_{k=1}^{d} \eta_k(s^k_1) \cdots \eta_k(s^k_{n_k}) \\
&= \sum_{i=1}^{d} n_i \int_{\mathbb{R}} d\tau \int_{-\infty}^{\tau} d^{n-1} s \Phi_{\vec{n}}(\ldots; \tau, s^i_1 \ldots, s^i_{n_i - 1}; \ldots) \cdot \\
&\quad \times \eta_i(\tau)\eta_i(s^i_1) \cdots \eta_i(s^i_{n_i-1}) \prod_{k=1, k \neq i}^{d} \eta_k(s^k_1) \cdots \eta_k(s^k_{n_k}) \\
&= \sum_{i=1}^{d} \int_{\mathbb{R}} d\tau\, \eta_i(\tau) \left(n_i \int_{-\infty}^{\tau} d^{n-1} s \Phi_{\vec{n}}(\cdot, \tau, \cdot) \vec{\eta}^{\otimes \overrightarrow{n - \delta_i}} \right).
\end{aligned}$$

Thus

$$\sum_{\vec{n}, n \geq 1} \left(\Phi_{\vec{n}}, \vec{\eta}^{\otimes \vec{n}}\right)_n = \sum_{i=1}^{d} \int_{\mathbb{R}} d\tau\, \eta_i(\tau) \mu_i(\tau),$$

where

$$\mu_i(\tau) \equiv \sum_{\vec{n}, n_i \geq 1} n_i \int_{-\infty}^{\tau} d^{n-1} s \Phi_{\vec{n}}(\cdot, \tau, \cdot) \vec{\eta}^{\otimes \overrightarrow{n - \delta_i}}, \quad \tau \in \mathbb{R}.$$

But, for each τ,

$$\begin{aligned}
\mu_i(\tau) &= \sum_{\vec{n}, n_i \geq 1} \left(n_i \Theta_\tau^{\otimes n-1} \Phi_{\vec{n}}(\cdot, \tau, \cdot), \vec{\eta}^{\otimes \overrightarrow{n - \delta_i}} \right)_{n-1} \\
&= S\left(\Gamma(\Theta_\tau)\partial^i_\tau \Phi\right)(\vec{\eta}).
\end{aligned}$$

Hence, (6) can be written as

$$S\Phi(\vec{\eta}) = \mathbf{E}(\Phi) + \sum_{i=1}^{d} \int_{\mathbb{R}} \mathrm{d}\tau\, \eta_i(\tau) S\Big(\Gamma(\Theta_\tau)\partial_\tau^i \Phi\Big)(\vec{\eta}).$$

Observing that, for each τ,

$$\begin{aligned} S\Big(\Gamma(\Theta_\tau)\partial_\tau^i \Phi\Big)(\vec{\eta}) &= \sum_{\vec{n}} (n_i+1) \int_{\mathbb{R}^n} \mathrm{d}^n s\, \Phi_{\overrightarrow{n+\delta_i}}(\cdot,\tau,\cdot)(\Theta_\tau \vec{\eta})^{\otimes \vec{n}} \\ &= \frac{\delta}{\delta \eta_i(\tau)} S(\Phi)(\Theta_\tau \vec{\eta}), \end{aligned}$$

there follows the required equality

$$S\Phi(\vec{\eta}) = \mathbf{E}(\Phi) + \sum_{i=1}^{d} \int_{\mathbb{R}} \mathrm{d}\tau\, \eta_i(\tau) \frac{\delta}{\delta \eta_i(\tau)} S(\Phi)(\Theta_\tau \vec{\eta}).$$

Theorem 4.1 is proved.

5. AN EXAMPLE

As an application of the above let us consider Φ equal to the Donsker delta function which we may consider defined as a Fourier (Bochner) integral [10]

$$\delta(B(t)-a) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}\lambda\, \mathrm{e}^{\mathrm{i}\lambda(B(t)-a)},$$

with S-transform

$$(S\delta(B(t)-a))(f) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\left(\int_0^t f(s)\,\mathrm{d}s - a\right)^2}{2t} \right), \quad f \in S_c(\mathbb{R}). \tag{7}$$

It is well known that $\delta(B(t)-a)$ is in $\mathcal{G}^{-1}$. From Theorems 3.1 and 4.1 it follows that

$$\delta(B(t)-a) = \mathbf{E}\,(\delta(B(t)-a)) + \int \mathrm{d}B(\tau) m(\tau)$$

with

$$Sm(\tau)(f) = \frac{\delta}{\delta f(\tau)} S(\Phi)(\Theta_\tau f).$$

The functional derivative of (7) is calculated straightforwardly using (5)

$$\left(\frac{\delta}{\delta f(\tau)} S(\Phi)\right)(f) = -\frac{1_{[0,t]}(\tau)}{\sqrt{2\pi t^3}} \left(\int_0^t f(s)\,\mathrm{d}s - a \right) \exp\left(-\frac{\left(\int_0^t f(s)\,\mathrm{d}s - a\right)^2}{2t} \right)$$

(here, $1_{[0,t]}$ denotes the indicator function of the interval $[0,t]$), so that, projecting the f with Θ_τ, we obtain

$$(Sm(\tau))(f) = -\frac{1_{[0,t]}(\tau)}{\sqrt{2\pi t^3}} \left(\int_0^\tau f(s)\,\mathrm{d}s - a \right) \exp\left(-\frac{\left(\int_0^\tau f(s)\,\mathrm{d}s - a\right)^2}{2t} \right).$$

Note that the rhs depends only on

$$\lambda \equiv \int f(s)e(s)\, ds$$

where $e = \frac{1}{\sqrt{\tau}} 1_{[0,\tau]}$ is a unit vector in $L^2(\mathbb{R})$. Consequently, m depends only on the normal random variable

$$x =< \omega, e >= \frac{1}{\sqrt{\tau}} B(\tau)$$

and

$$m(\tau) = h\left(\frac{1}{\sqrt{\tau}} B(\tau)\right)$$

with S-transform

$$\int \frac{dx}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} h(x) e^{\lambda x} e^{-\frac{1}{2}\lambda^2} = -\frac{1_{[0,t]}(\tau)}{\sqrt{2\pi t^3}} \left(\sqrt{\tau}\lambda - a\right) \exp\left(-\frac{\left(\sqrt{\tau}\lambda - a\right)^2}{2t}\right).$$

To obtain m itself we must thus calculate the inverse Laplace transform of

$$q(\lambda) = -\frac{1_{[0,t]}(\tau)}{\sqrt{t^3}} \left(\sqrt{\tau}\lambda - a\right) e^{-\frac{(\sqrt{\tau}\lambda - a)^2}{2t}} e^{\frac{1}{2}\lambda^2}$$

which gives

$$h(x) = -\frac{1_{[0,t]}(\tau)}{\sqrt{t^3}} \left(\frac{t}{t-\tau}\right)^{3/2} (\sqrt{\tau}x - a) \exp\left(-\frac{(\sqrt{\tau}x - a)^2}{2(t-\tau)}\right).$$

Substituting

$$x = \frac{1}{\sqrt{\tau}} B(\tau)$$

we finally obtain

$$m(\tau) = -\frac{1_{[0,t]}(\tau)}{\sqrt{(t-\tau)^3}} (B(\tau) - a) \exp\left(-\frac{(B(\tau) - a)^2}{2(t-\tau)}\right).$$

One notes that $m(\tau)$ is an adapted random variable in (L^2) as long as $\tau < t$, and it permits conventional Itô integration. It is thus not hard to show that, as a limit in $\mathcal{G}^{-1}$,

$$\delta(B(t) - a) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} + \lim_{\varepsilon \to +0} \int_0^{t-\varepsilon} dB(\tau) m(\tau).$$

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