NMS flows on S^3 with no heteroclinic trajectories connecting saddle orbits

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Abstract.   In this paper we find topological conditions for the non existence of heteroclinic trajectories connecting saddle orbits in non singular Morse-Smale flows on S^3.

We obtain the non singular Morse-Smale flows that can be decomposed as connected sum of flows and we show that these flows are those who have no heteroclinic trajectories connecting saddle orbits. Moreover, we characterize these flows in terms of links of periodic orbits.

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1.   Introduction

Morse-Smale flows are structurally stable flows in the set of C^1-vector fields on compact connected manifolds. In dimension three, only the set of periodic orbits of non singular Morse-Smale systems (NMS) on S^3 (M. Wada [8], K. Yano [9]) and $S^2 \times S^1$ (A. Cordero, J. Martínez Alfaro and P. Vindel [5]) have been characterized in terms of links. These characterizations are based on the round handle decomposition (RHD) introduced by Asimov [1] and Morgan [6].

M. Wada [8, Theorem 1] characterizes the links of periodic orbits of NMS flows on S^3 in terms of six operations and a generator, the hopf link. He states that every link obtained by applying these operations corresponds to the set of periodic orbits of a NMS flow on S^3.

The link of periodic orbits of a NMS flow on S^3 is defined by the cores of the round-handles in the round handle decomposition of the manifold. Although there is a 1-1 correspondence between the flow and the round handle decomposition, this is not the case for the link of periodic orbits. Different round handle decompositions can yield the same link (B. Campos, J. Martínez Alfaro and P. Vindel [2]), but the corresponding flows are not topologically equivalent (B. Campos and P. Vindel [3]). So, the link of periodic orbits does not describe completely the flow. Nevertheless,

we can obtain some relevant information from the link such as the presence or not of heteroclinic trajectories connecting saddle orbits.

Next, we see how this paper is organized and the most significant results obtained in the different sections.

In section 3 we carry out iterated sum of tori and we characterize the 3-manifold obtained.

In section 4 we define and characterize the set of the *links coming from iterated sum of tori.* This set is formed by the NMS links of periodic orbits on S^3 associated to round 1-handles that are iterated sum of tori (lemma 6).

In section 5 we obtain a correspondence between flows characterized by links coming from iterated sum of tori and NMS flows on S^3 that are connected sum of flows (theorem 13).

Finally, in section 5.2, we show that such flows have no heteroclinic trajectories connecting saddle orbits (proposition 14). Moreover, we characterize a kind of flows that necessarily have these type of trajectories (proposition 14).

In fact, we divide the set of the flows with unknotted and unlinked saddle orbits in two complementary sets: the set of flows $\mathcal{F}_I(S^3)$, having no heteroclinic orbits and its complement $\bar{\mathcal{F}}_I(S^3)$, the flows containing these type of trajectories.

2. PREVIOUS RESULTS

In this section we collect some previous results that are necessary for the development of the subsequent sections.

2.1. Non Singular Morse Smale flows.

Definition 1. *A non singular Morse-Smale flow (or NMS for short) on $(W, \partial_- W)$ is a flow without fixed points in W which is transverse to ∂W, pointing inward on $\partial_- W$ and outward on $\partial_+ W = \partial W - \partial_- W$, satisfying the following properties:*

(1) The non-wandering set consists entirely of closed orbits.

(2)The Poincaré map of each closed orbit is hyperbolic.

(3) The stable and unstable manifolds of the closed orbits have transversal intersection with each other.

2.2. Round handle decompositions.
The notion of the round handle decomposition was introduced by Asimov [1] and modified by Morgan [6]; it establish a correspondence between NMS flows and round handle decompositions.

Definition 2. *Let X and Y be two n-dimensional manifolds. The manifold X is obtained from Y by attaching a round k-handle if there are disk bundles over S^1, D_s^k and D_u^{n-k-1}, and an embedding $\varphi : (\partial D_s^k \underset{S^1}{\times} D_u^{n-k-1}) \to \partial Y$ such that*

$$X \cong Y \bigcup_{\varphi} \left(D_s^k \oplus D_u^{n-k-1} \right)$$

Definition 3. *A round handle decomposition for $(X, \partial_- X)$ is a filtration*

$$\partial_- X \times I = X_0 \subset X_1 \subset \cdots \subset X_i \subset X_{i+1} \subset \cdots \subset X_N = X \qquad (1)$$

where each X_i is obtained from X_{i-1} by attaching a round handle.

Proposition 1. *[1], [6] If a manifold X admits a round handle decomposition, then there is a NMS flow on X whose closed orbits coincide with the cores of the round handles and the flow is pointing outward on ∂X_i. Conversely, if X has a NMS flow, X admits a round handle decomposition whose core circles are the closed orbits of the flow.*

Theorem 2. *[6] Let X be a compact, orientable and irreducible manifold with ∂X either empty or disjoint union of tori. If X admits a round handle decomposition then ∂X_i is formed by tori.*

For the case of dimension 3, the round handles are diffeomorphic to tori and correspond to 0-handles when there is a repulsive periodic orbit in the core, to 2-handles if there is an attractive periodic orbit in the core and to 1-handles if the orbit in the core is a saddle.

For the study of the round handle decomposition of a compact, orientable 3-manifold M is more convenient to use the fat round handle decomposition introduced by Morgan [6]:

$$\emptyset = M_0 \subset M_1 \subset \cdots \subset M_i \subset M_{i+1} \subset \cdots \subset M_N = M$$

where

$$M_i = \bigcup_{j=1}^{i} C_j \qquad i = 1, 2, ..., N$$

and C_j is, either a 0 or 2-handle, or a fat round handle of the form:

$$C_j = A \times [0,1] \bigcup_{\varphi} D_s \oplus D_u$$

where A is a union of components of ∂M_{i-1}, $D_s \oplus D_u$ is the Whitney sum of disk bundles D_s and D_u over S^1 and the image of $\varphi : (\partial D_s) \oplus D_u \to A \times \{1\}$ meets every component of $A \times \{1\}$.

The fat round handle is obtained from the manifold A by attaching a round 1-handle by means of one or two attaching circles depending on whether the 1-handle is twisted or untwisted, let $c_1 = \varphi(\{-1\} \times \{0\} \times S^1)$ and $c_2 = \varphi(\{+1\} \times \{0\} \times S^1)$ denote these circles. So, while round 1-handles are always disk bundles over S^1, the fat round 1-handles can be different types of manifolds depending on the way the 1-handles are attached.

The attachment of M_{N-1} with the last 2-handle gives the 3-manifold M.

2.3. Links of periodic orbits in S^3. In order to classify the set of closed orbits of a NMS flow on S^3, M. Wada [8] describes the different types of fat round 1-handles for the manifold S^3 . This result, was independently obtained in a different way by K. Yano [9].

Consider C together with $\partial_- C = A \times \{0\}$ and the core γ which is the 0-section of $D_s \oplus D_u$, i.e., the core of $D_s \oplus D_u$. The component C associated to a round 0 or 2-handle is just a solid torus. For 1-handles we have the following result:

Lemma 3. *[8, Lemma 1] The triple $(C, \partial_- C, \gamma)$ associated to a round 1-handle is one of the following types:*

(a) $C \cong T_1 \times [0,1] \# T_2 \times [0,1]$, where T_1 and T_2 are tori, $\partial_- C = T_1 \times \{0\} \cup T_2 \times \{0\}$ and γ is an unknot in C.

(b) $C \cong T^2 \times [0,1] \# \tau_1$, $\partial_- C = T^2 \times \{0\}$ or $\partial_- C = T^2 \times \{0\} \cup \partial \tau_1$ and γ is an unknot in C.

(c) $C \cong \tau_1 \# \tau_2$, where τ_1 and τ_2 are solid tori, $\partial_- C = \partial \tau_1$ and γ is an unknot in C.

(d) $C \cong F \times S^1$ where F is a disk with two holes, $\partial_- C$ is a component or a union of two components of ∂C and $\gamma = * \times S^1$ for some point $*$ in $Int F$.

(e) $C \cong D^2 \times S^1 \setminus Int W$ where W is a tubular neighborhood of the $(2,1)$-cable of $\{0\} \times S^1$ in $D^2 \times S^1$, $\partial_- C = \partial W$ and $\gamma = \{0\} \times S^1$.

The set of periodic orbits of a NMS flow in S^3 is characterized by Wada in terms of indexed links and six operations:

Theorem 4. *[8] Every indexed link which consists of all the closed orbits of a non-singular Morse-Smale flow on S^3 is obtained from $(0,2)$-hopf links by applying the following six operations. Conversely, every indexed link obtained from $(0,2) - links$ by applying the operations is the set of all the closed orbits of some non-singular Morse-Smale flow on S^3.*

OPERATIONS: For given indexed links l_1 and l_2, the six operations are defined as follows. Let $l_1 \cdot l_2$ denote the split sum of the links l_1 and l_2 and $N(k, M)$ a regular neighbourhood of k in M.

1) $I(l_1, l_2) = l_1 \cdot l_2 \cdot u$, where u is an unknot with index 1.

2) $II(l_1, l_2) = l_1 \cdot (l_2 - k_2) \cdot u$, where k_2 is a component of l_2 of index 0 or 2.

3) $III(l_1, l_2) = (l_1 - k_1) \cdot (l_2 - k_2) \cdot u$, where k_1 is a component of l_1 of index 0 and k_2 is a component of l_2 of index 2.

4) $IV(l_1, l_2) = (l_1 \# l_2) \cup m$. The connected sum $(l_1 \# l_2)$ is obtained by composing a component k_1 of l_1 and a component k_2 of l_2, each of which has index 0 or 2. The index of the composed component $k_1 \# k_2$ is equal to either $i(k_1)$ or $i(k_2)$. Finally, m is a meridian of $k_1 \# k_2$ with $i = 1$.

5) $V(l_1)$: *Choose a component k_1 of l_1 of index 0 or 2, and replace $N(k_1,S^3)$ by $D^2 \times S^1$ with three indexed circles in it; $\{0\} \times S^1$, k_2 and k_3. Here, k_2 and k_3 are parallel (p,q)-cables on $\partial N(\{0\} \times S^1, D^2 \times S^1)$, where p is the number of longitudinal turns and q the number of the transverse ones. The indices of $\{0\} \times S^1$ and k_2 are either 0 or 2, and one of them is equal to $i(k_1)$. The index of k_3 is 1.*

6) $VI(l_1)$: *Choose a component k_1 of l_1 of index 0 or 2. Replace $N(k_1, S^3)$ by $D^2 \times S^1$ with two indexed circles in it; $\{0\} \times S^1$ and the $(2,q)$-cable k_2 of $\{0\} \times S^1$. The index of $\{0\} \times S^1$ is 1, and $i(k_2) = i(k_1)$.*

The link of periodic orbits of a NMS flow on S^3 is defined by the cores of the round-handle decomposition of the manifold. We call NMS links on S^3 those links that correspond to the set of periodic orbits of a NMS flow on S^3 and denote this set of links by $\mathcal{L}(S^3)$.

The cases (a), (b) and (c) in lemma (3) define operations I, II and III, respectively. We refer to them as type A operations and we denote by $\mathcal{L}_A(S^3)$ the set of the links obtained by applying only type A operations. For these links, all the components with index 1 are unknotted and unlinked. In these three cases, the round 1-handle C is a connected sum of thick and solid tori and their attachments imply at least one inessential circle.

On the other hand, operations IV, V and VI imply only essential attachments. We refer to them as type B operations.

We want to know which links are associated to round 1-handles that are iterated connected sum of tori. So, we begin by characterizing the 3-manifolds obtained from successive connected sum of tori.

3. Connected sum of tori

Definition 4. *Let M be a 3-dimensional manifold and S a 2-sphere embedded in M which separates the manifold M. Let M_1 and M_2 denote the 3-dimensional manifolds obtained by cutting M along S. Then, the manifold M is the connected sum of M_1 and M_2 and it is denoted by $M_1 \# M_2$.*

Let us apply this definition in order to make connected sums of thick and solid tori.

Let τ_1 and τ_2 denote two solid tori $D^2 \times S^1$. Consider S_1 and S_2 2-spheres embedded in τ_1 and τ_2, respectively. In order to make the connected sum, we have to reverse one of the 2-spheres. Let us assume that S_2 is reversed; then, a 3-ball with a toroidal hole is obtained. The boundary of this 3-ball is identified with S_1 and the torus τ_1 with a toroidal hole is obtained.

Then, $\tau_1 \# \tau_2$ is a solid torus with a toroidal hole (figure 1).

Let $(T \times I)$ denote the product of a 2-torus T by an interval I, called a thick torus.

If we consider a 2-sphere embedded in a thick torus and we reverse it, we obtain a 3-ball with two linked toroidal holes inside; let us refer to them as a *hopf hole* (see
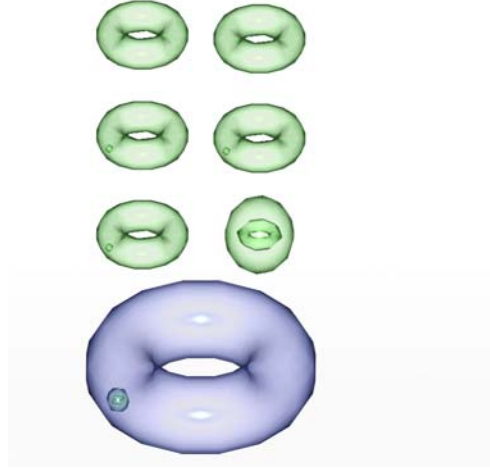
Figure 1: $\tau_1 \# \tau_2$

figure 2). So, the connected sum of two thick tori, $(T_1 \times I)\#(T_2 \times I)$, is a thick torus with a hopf hole.

Finally, the connected sum of a thick torus $(T \times I)$ and a solid torus τ is a thick torus $(T \times I)$ with a toroidal hole (see figure 3).

The realization of $\tau \#(T \times I)$ yields to a solid torus with a hopf hole, because we reverse the 3-ball in the thick torus. When the tori are embedded in S^3 both results are equivalent; so, there is commutativity for the connected sum of tori. The identity element of the connected sum is S^3. These operations verify commutativity and associativity on S^3; so, by iteration, it is easy to prove the following proposition:

Proposition 5. *Let* $\mathbb{T}$ *denote a torus (thick or solid) with toroidal or hopf holes. Then*

$$\mathbb{T} \underset{i=1}{\overset{n}{\#}} \tau_i \underset{j=1}{\overset{m}{\#}} (T_j \times I) = \mathbb{T} \text{ with } n \text{ toroidal holes and } m \text{ hopf holes} \tag{2}$$

Proof. Suppose that $\mathbb{T}$ is a solid torus. We prove by induction that

1. $\tau \underset{i=1}{\overset{n}{\#}} \tau_i = \tau$ with n toroidal holes

   By definition, when we make the connected sum of two solid tori τ and τ_1, we reverse one of them into a 3-ball and identify its boundary with a 2-sphere embedded in the other torus.

   Let us see that this operation does not depend on the torus chosen. If we reverse τ_1 we obtain the solid torus τ with one toroidal hole.

   $$\tau \# \tau_1 = \tau \text{ with 1 toroidal hole}$$
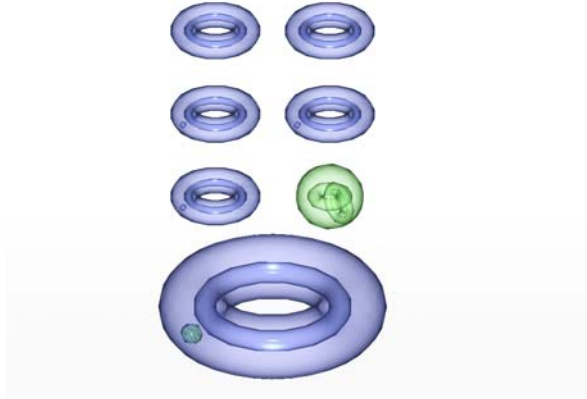
   The same occurs if we reverse τ.
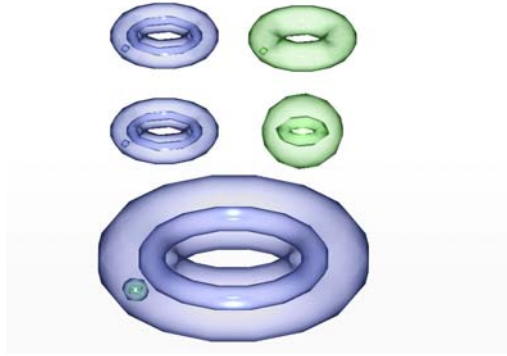
Figure 2: $(T \times I) \# (T \times I)$

Figure 3: $(T \times I) \# \tau$

Now, if the connected sum with another solid torus τ_2 is made, reversing τ_2, a new toroidal hole appears; then, we have a solid torus τ with 2 toroidal holes.

But if we make the connected sum reversing the torus with the toroidal hole, we have a 3-ball with two toroidal holes; then, when identifying its boundary with a 2-sphere embedded in τ_2, we also obtain a solid torus with 2 toroidal holes.

Now, suppose that $n - 1$ connected sum of solid tori have been made and we have that

$$\tau \# \tau_1 \#...\# \tau_{n-1} = \tau \text{ with } n - 1 \text{ toroidal holes}$$

If the connected sum with another solid torus τ_n is made, a new toroidal hole in τ appears; then:

$$\tau \# \tau_1 \#...\# \tau_n = \tau \text{ with } n \text{ toroidal holes}$$

Similarly, it is easy to prove by induction that:

2. $\tau \overset{n}{\underset{i=1}{\#}} (T_i \times I) = \tau$ with n hopf holes

3. $(T \times I) \overset{n}{\underset{i=1}{\#}} \tau_i = (T \times I)$ with n toroidal holes

4. $(T \times I) \overset{n}{\underset{i=1}{\#}} (T_i \times I) = (T \times I)$ with n hopf holes

So, these four equalities can be expressed as 2. ■

Let us remark that, as a consequence of this proposition, each time that the connected sum is made with a solid torus a new toroidal hole appears. If the connected sum is made with a thick torus a hopf hole appears.

In the next section we relate iterated connected sum of tori to the first three operations defined in [8, Theorem 1] .

4. LINKS COMING FROM ITERATED SUM OF TORI

As commented before, Wada [8] describes the different types for the triple $(C, \partial_- C, \gamma)$ associated to a fat round 1-handle on S^3. The results obtained in the previous section allows us to distinguish a special type of links coming from type A operations: the links on S^3 associated to fat round handles that are iterated connected sum of tori. We remark that in order to obtain a link on S^3 the components of the complement of C must be filled with 2-handles in the last step of the filtration in the round handle decomposition. This is equivalent to fill the components of the complement of T with attractive orbits. Moreover, we prove in section 5 that the flow associated to these links is also a connected sum of flows.

Let us refer to this type of links on S^3 as links coming from iterated connected sum of tori. We include the hopf link in this definition, corresponding to 0 iterations. Let us denote by $\mathcal{L}_I(S^3)$ the set of these links and let $\bar{\mathcal{L}}_I(S^3)$ denote the complement of the set $\mathcal{L}_I(S^3)$ into $\mathcal{L}_A(S^3)$.

In the following, h denotes the $(0,2)$-hopf link, d denotes an unknotted and un-linked periodic orbit with index 0 or 2 and u denotes and unknotted periodic orbit with index 1. The power c^p means the split sum of p components of type c.

Lemma 6. *Let l be a NMS link on S^3, then $l \in \mathcal{L}_I(S^3)$ if and only if l can be written as:*

$$l = c^{p+1} \cdot u^p \tag{3}$$

where c is either an unknot d or a hopf link h and $p \geq 0$ is the number of iterations.

Proof. As we have proved in the proposition 5, the fat round 1-handle associated to the iterated connected sum of tori is

$$\mathbb{T} \overset{n}{\underset{i=1}{\#}} \tau_i \overset{m}{\underset{j=1}{\#}} (T_j \times I) = \mathbb{T} \text{ with } n \text{ toroidal holes and } m \text{ hopf holes}$$

We obtain a NMS link on S^3 by attaching 0 or 2 handles. So there are attractive or repulsive orbits just in the core of the toroidal holes and saddle periodic orbits on the boundary of the 3-balls implied in each connected sum. In the case of hopf holes, one orbit must be attractive and the other must be repulsive.

Then, after $p = n + m$ connected sums of tori, the corresponding link l on S^3 can be written as:

$$c \overset{n}{\underset{i=1}{\cdot}} d_i \overset{m}{\underset{j=1}{\cdot}} h_j \overset{m+n}{\underset{k=1}{\cdot}} u_k \tag{4}$$

where c is either a hopf link h or an unknot d. So, the link l is formed by the split sum of hopfs and components of type d and u with the following ratio:

$$l = c^{n+m+1} \cdot u^{n+m}$$

∎

In the following result we characterize the links of the set $\mathcal{L}_I(S^3)$ in terms of Wada operations.

Lemma 7. *Let l be a NMS link on S^3, then $l \in \mathcal{L}_I(S^3)$ if and only if l is obtained from $(0,2)$-hopf links by applying the following operations:*

1. $l = I(l_1, l_2) = l_1 \cdot l_2 \cdot u$

2. $l = II(l_1, h) = l \cdot d \cdot u$

3. $l = III(h, h) = d \cdot d \cdot u$

Proof. As type A operations commute (see [2]), in this proof we refer to combinations of operations and no matter the order of application of them.

Firstly, let us see that if $l \in \mathcal{L}_I(S^3)$, it can be written in terms of the operations given above.

If $l \in \mathcal{L}_I(S^3)$, then l is a $(0,2)$-hopf link or l it is of the form (4):

$$l = c \cdot d^n \cdot h^m \cdot u^{n+m}, \qquad n + m > 0$$

- If c is a hopf link, l has the form:

$$l = h \cdot d^n \cdot h^m \cdot u^{n+m}$$

This link can be written in terms of n operations II and m operations I applied on hopf links:
$$l = I(I(...(II(II(..., h), h), ...h), h)$$

Then, we have:
$$l = I(l_1, l_2)$$

with l_1 and l_2 links obtained from operations I and operations II with the hopf link in the second argument.

- If c is a component d and $m > 1$, we can change the order of the components and then l is expressed as:

$$l = h \cdot d^{n+1} \cdot h^{m-1} \cdot u^{n+m}$$

This link can be written in terms of $n + 1$ operations II and $m - 1$ operations I on hopf links. As above, we have:

$$l = I(l_1, l_2)$$

with l_1 and l_2 links obtained from operations I and operations II with the hopf link in the second argument.

- If c is a component d and $m = 1$, then l is of the form:

$$l = h \cdot d^{n+1} \cdot u^{n+m}$$

In this case l can be written in terms of $n + 1$ operations II on hopf links, so l can be expressed as

$$l = II(l_1, h)$$

where l_1 is a link obtained from operations II on hopfs links in the second argument.

- If c is a component d and $m = 0$, then l is of the form:

$$l = d^{n+1} \cdot u^n$$

In this case, n is different from 0, because for $m + n = 0$, there has been 0 iterations and the link l corresponds to a hopf link.

  - if $n = 1$, the link l is.
    $$l = III(h, h)$$

  - if $n > 1$, the link l can be written in terms of one operation III and $n - 1$ operations II on hopf links. By the commutativity of type A operations, we have:
    $$l = II(l_1, h)$$

    where l_1 is a link obtained from operations II on hopfs links in the second argument and one operation III on two hopfs links.

Conversely, let us see that every link obtained from $(0, 2)$-hopf links by applying the operations given above verifies condition (3) and then, the link comes from iterated sum of tori.

- If $l = III(h, h)$, then:

$$l = d \cdot d \cdot u = c^2 \cdot u \tag{5}$$

  and verifies condition (3). So, $l \in \mathcal{L}_I(S^3)$.

- If $l = II(h, h)$, then:

$$l = h \cdot d \cdot u = c^2 \cdot u^1$$

  and verifies condition (3). So, $l \in \mathcal{L}_I(S^3)$.

  If we iterate n times operation II with h in the second argument, each operation produces one d and one u more, then condition (3) is verified:

$$l = II(II(..., h), h) = h \cdot d^n \cdot u^n = c^{n+1} \cdot u^n \tag{6}$$

  and $l \in \mathcal{L}_I(S^3)$.

  If we combine n operations II with h in the second argument and one operation III on two hopf links, we have

$$l = II(...II(III(h, h), h), h) = d \cdot d \cdot u \cdot d^n \cdot u^n = d^{n+2} \cdot u^{n+1} \tag{7}$$

  As condition (3) is verified, $l \in \mathcal{L}_I(S^3)$.

  In both cases, l can be written as

$$l = II(l_1, h)$$

  where l_1 has been obtained with operations II with h in the second argument or a combination operations II with h in the second argument and one operation III on two hopf links.

- Suppose that $l = I(l_1, l_2)$, where l_1 and l_2 are of the form given in (5), (6), or (7). As the links l_1 and l_2 verify condition (3):

$$\begin{aligned} l_1 &= c^{p+1} \cdot u^p \\ l_2 &= c^{q+1} \cdot u^q \end{aligned}$$

  then $l = I(l_1, l_2)$ is of the form

$$l = l_1 \cdot l_2 \cdot u = c^{p+1} \cdot u^p \cdot c^{q+1} \cdot u^q \cdot u = c^{p+q+2} \cdot u^{p+q+1} \tag{8}$$

  and verifies condition (3). So, $l \in \mathcal{L}_I(S^3)$.

  Similarly, if $l = I(l_1, l_2)$, where l_1 and l_2 are of the form given in (5), (6), (7) or (8), then $l \in \mathcal{L}_I(S^3)$. ∎

Let us remark that operation III can not be iterated to obtain a link in $\mathcal{L}_I(S^3)$; that is, we can combine operations III with operations I and II, but operation III can not appear in the argument of another operation III. In fact, as we see in the following section, these type of links characterize flows with heteroclinic trajectories connecting saddle orbits.

In the following lemma, we characterize the links in the complement of $\mathcal{L}_I(S^3)$:

Lemma 8. *Let l be a NMS link on S^3, then $l \in \bar{\mathcal{L}}_I(S^3)$ if and only if l can be written as:*

$$l = l' \cdot u$$

where $l' \in \mathcal{L}_A(S^3)$.

Proof. ($\Rightarrow$)If $l \in \bar{\mathcal{L}}_I(S^3)$, l is a link in $\mathcal{L}_A(S^3)$ and does not verify condition (3); then, the number of $c's$ is equal or less than the number of $u's$. Therefore

$$l = c^p \cdot u^{p+q}$$

where $q \geq 0$ and $p \geq 2$. Let us apply induction for showing that l can be written as $l' \cdot u$ with $l' \in \mathcal{L}_A(S^3)$.

For $q = 0$:

$$l_0 = c^p \cdot u^p = c^p \cdot u^{p-1} \cdot u = l_{-1} \cdot u$$

and $l_{-1} = c^p \cdot u^{p-1} \in \mathcal{L}_I(S^3) \subset \mathcal{L}_A(S^3)$.

Suppose that it is true for $q = n$:

$$l_n = c^p \cdot u^{p+n} = l_{n-1} \cdot u$$

with $l_{n-1} \in \mathcal{L}_A(S^3)$.

If $l_{n-1} \in \mathcal{L}_A(S^3)$ then $l_{-1} \cdot u \in \mathcal{L}_A(S^3)$ (see [2, Lemma 1]). Therefore, $l_n \in \mathcal{L}_A(S^3)$ and for $q = n + 1$:

$$l_{n+1} = c^p \cdot u^{p+n+1} = c^p \cdot u^{p+n} \cdot u = l_n \cdot u$$

with $l_n \in \mathcal{L}_A(S^3)$.

So, the links that do not come from iterated connected sum of tori can be written as $l' \cdot u$, with $l' \in \mathcal{L}_A(S^3)$.

($\Leftarrow$) From [2, Lemma 1], we know that, if $l' \in \mathcal{L}_A(S^3)$ then the link $l = l'.u \in \mathcal{L}_A(S^3)$. Then, if l contains n orbits of type u, it means that n type A operations have been applied. In each operation, the number of u components increases in 1 and the number of c components can increase in 1 or does not change. So after n operations, it contains a number n of u components and a number of c components equal or less than $n + 1$.

Then, $l' \cdot u$ contains a number of c components equal or less than $n + 1$ and $n + 1$ components of type u. So, $l' \cdot u$ does not verify the condition (3). ■

$>$From [2, Lemma 1], we have the expressions of the links of the set $\bar{\mathcal{L}}_I(S^3)$ in terms of Wada operations:

Lemma 9. *Let l be a NMS link on S^3, then $l \in \bar{\mathcal{L}}_I(S^3)$ if and only if l can be written as:*

1. $l = II(l_1, II(l_2, h)) = l_1 \cdot l_2 \cdot u \cdot u$

2. $l = III(II(l_1, h), l_2) = l_1 \cdot (l_2 - k) \cdot u \cdot u$

3. $l = III(l_1, III(l_2, h)) = (l_1 - k_1) \cdot (l_2 - k_2) \cdot u \cdot u$
   where $l_1, l_2 \in \mathcal{L}_A(S^3)$.

5. Connected sum of NMS flows on S³ and heteroclinic trajectories

In this section we study the NMS flows on S^3 associated with the links coming from type A operations and we obtain the following important results:

1. These flows are connected sum of NMS flows. Moreover, we see that the flows associated to links coming from iterated sum of tori can be successively decomposed as connected sums of basic flows, that is, flows corresponding to $h's$ and $d's$.

2. The flows associated to links in $\mathcal{L}_I(S^3)$ have no heteroclinic trajectories connecting saddle orbits and the flows associated to links in $\bar{\mathcal{L}}_I(S^3)$ have heteroclinic trajectories connecting saddle orbits.

5.1. Connected sum of flows. The definition of the connected sum of flows in 3-manifolds given in [4] is a generalization of the connected sum of flows defined for 2-dimensional manifolds (see [7]).

Given two flows φ_1 and φ_2 on the 3-dimensional manifolds M_1 and M_2, respectively, we can form a new flow $\varphi_1 \# \varphi_2$ on the connected sum $M_1 \# M_2$ in the following way:

Definition 5. *Let φ_1 and φ_2 be NMS flows on the 3-dimensional manifolds M_1 and M_2, respectively. Let $D_i \subset M_i$, $i = 1, 2$ be 3-balls with the boundaries transversal to φ_i everywhere except at the points of the circles σ_i on their equator. Let B_i^+ (B_i^-) be the part of the boundary through which the trajectories of φ_i enter (leave) D_i. By putting the circles σ_i aside we denote the modified flows by $\hat{\varphi}_i$.*

$M_1 \# M_2$ is obtained by identifying the boundaries of the manifolds $M_1 \backslash D_1$ and $M_2 \backslash D_2$ with an orientation-reversing homeomorphism $h : \partial D_1 \to \partial D_2$ such that

$$h\left(B_1^+\right) = B_2^-, \quad h\left(B_1^-\right) = B_2^+, \quad h\left(\sigma_1\right) = \sigma_2$$

Herewith the flows $\hat{\varphi}_1$ and $\hat{\varphi}_2$ give a flow φ on $M_1 \# M_2$ with a saddle orbit at σ_i. The flow φ is called a connected sum of the flows φ_1 and φ_2.

The boundaries of the 3-balls that appears when connected sum of tori are made, are transversal to the respective flows except in the points of the circles σ_i on the equator (see [4]). We use this result to show that the flow is the connected sum of flows.

The attaching of a round 1-handle by means of two inessential circles is equivalent to taking the connected sum of two thick tori. Then, for the case of flows characterized by operation I we have the following result:

Theorem 10. *Let φ_1 and φ_2 be NMS flows on S^3 characterized by the links l_1 and l_2 respectively; let φ be a NMS flow on S^3 characterized by the link l. Then:*

$$\varphi = \varphi_1 \# \varphi_2 \Leftrightarrow l = I(l_1, l_2)$$

Proof. Suppose that $l = I(l_1, l_2)$. The fat round 1-handle associated to operation I is $C \cong T_1 \times [0,1] \# T_2 \times [0,1]$. For $i = 1, 2$, let N_{i-} and N_{i+} be the components of the complement of C in S^3 which bound $T_i \times \{0\}$ and $T_i \times \{1\}$, repectively. If we consider a 2-sphere splitting C as a connected sum of $T_1 \times [0,1]$ and $T_2 \times [0,1]$, it bounds 3-balls on both sides. As $N_{i-} \cup N_{i+} \cong S^3$, we have a round handle decomposition of S^3 with l_i consisting of the cores of the round handles. The flow φ associated to $l = I(l_1, l_2) = l_1 \cdot l_2 \cdot u$ is the connected sum of the flow φ_i associated to l_i, then $\varphi = \varphi_1 \# \varphi_2$.

Now, suppose that $\varphi = \varphi_1 \# \varphi_2$. Then, we have a 2-sphere splitting S^3 as connected sum of two 3-spheres with the respective flow transversal to the boundary except in the points of the circle σ_i on the equator. The 2-sphere bounds 3-balls on both sides. The boundaries of the 3-balls are identified by means of an orientation reversing homeomorphism and a saddle orbit appears in the equator. The link of periodic orbits of φ consists of the set of periodic orbits of φ_1, the set of the periodic orbits of φ_2 and a trivial orbit with index 1 on the circles σ_i on the equator, then:

$$l = l_1 \cdot l_2 \cdot u$$

∎

Thus, a flow characterized by a link $l = I(l_1, l_2)$ is always the connected sum of two flows on S^3.

If a flow is characterized by means of operations II or III, it can also be connected sum of flows.

A link on a solid torus can be completed to a link on S^3 by adding a component k corresponding to an attractive or repulsive orbit in the core of the complement of this torus in S^3, which is another solid torus. Conversely, if l is a link of periodic orbits of a NMS flow on S^3 and k corresponds to an attractive (repulsive) orbit, the link $l - k$ characterizes the remaining flow on a solid torus.

Therefore, the following results are obtained immediately.

Corollary 11. *Let φ and φ_1 be NMS flows on S^3 characterized by the links l and l_1 respectively; let φ_2 be a NMS flow on the solid torus corresponding to a component d. Then:*

$$\varphi = \varphi_1 \# \varphi_2 \Leftrightarrow l = II(l_1, h) = l_1 \cdot d \cdot u$$

Corollary 12. *Let φ be a NMS flow on S^3 characterized by the link l. Let φ_1 and φ_2 be NMS flows on solid tori corresponding to components d_1 and d_2, respectively. Then:*

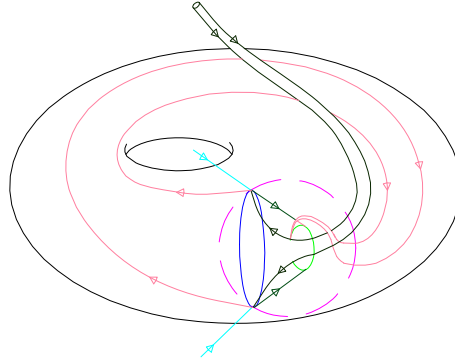$$\varphi = \varphi_1 \# \varphi_2 \Leftrightarrow l = III(h, h) = d_1 \cdot d_2 \cdot u$$

Figure 4: Connected sum of flows for the link $III\,(h, h)$

As commented before, another significant result is that we characterize those flows that can be successively decomposed as connected sums of *basic flows*; for basic flows we mean flows corresponding to $h's$ and $d's$. In this case, we say that the flow is a *decay* of connected sums of flows. These flows are just the flows associated to links coming from iterated sum of tori.

Theorem 13. *Let φ be a NMS flow on S^3 characterized by the link l. Then:*

$$\varphi \text{ is a decay of connected sum of flows} \Leftrightarrow l \in \mathcal{L}_I(S^3)$$

Proof. ($\Leftarrow$) There exists an equivalence between connected sum of tori and connected sum of flows; the 3-balls implied in the connected sum of tori correspond to the 3-balls implied in the connected sum of flows because the boundaries of the 3-balls are transversal to the respective flows except in the points of the circles σ_i on the equator and these circles correspond to the saddle orbits.

As a consequence of the proposition 5, each time that a connected sum is made with a thick torus a hopf hole appears; if the connected sum is made with a solid

torus a new toroidal hole appears. This is equivalent to the connected sum with a basic flow (see figure 5).

Let l be a link in $\mathcal{L}_I(S^3)$ different from the polar flow. Then, from lemma 7, l is one of the following links:

$$l = I(l_1, l_2), \qquad l = II(l_1, h) = l_1 \cdot d \cdot u, \qquad l = III(h, h) = d_1 \cdot d_2 \cdot u$$

where $l_1, l_2 \in \mathcal{L}_I(S^3)$ and, from theorem (10) and its corollaries, the flows associated are the connected sum of the flows associated to l_1 and l_2, l_1 and d, d_1 and d_2, respectively.

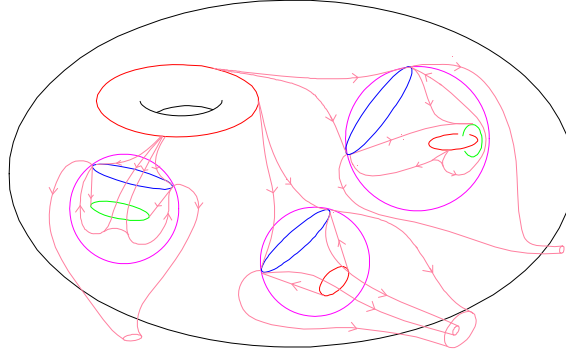As $l_1, l_2 \in \mathcal{L}_I(S^3)$, they can be further decomposed as above until arrive to basic flows.

Figure 5: Connected sum of flows in a thick torus with one repulsive orbit in its core

($\Rightarrow$) If a flow φ is a connected sum of two flows φ_1 and φ_2, the link of periodic orbits of φ is the split sum:

$$l = l_1 \cdot l_2 \cdot u$$

where l_1 and l_2 are the links of periodic orbits of the flows φ_1 and φ_2, respectively, and u represents the saddle orbit on the equator of the 2-sphere implied in the connected sum of flows.

If the flow φ is a decay of connected sum of flows, l_1 and l_2 can also be written as split sums of links and one orbit u:

$$l = l_1 \cdot l_2 \cdot u = (l_{11} \cdot l_{12} \cdot u) \cdot (l_{21} \cdot l_{22} \cdot u) \cdot u$$

and so on until arrive to basic components $h's$ and $d's$. Each connected sum implies three components, one of them a u orbit. So, the number of $u's$ is one less than the number of $h's$ and $d's$. Then, l verifies the condition (3) and $l \in \mathcal{L}_I(S^3)$. ∎

5.2. NMS flows with heteroclinic trajectories connecting saddle orbits .
Let $\mathcal{F}_A(S^3)$ denote the set of NMS flows associated to the links $\mathcal{L}_A(S^3)$ and let $\mathcal{F}_I(S^3)$ denote the set of NMS flows associated to the links $\mathcal{L}_I(S^3)$. In the following result we relate the existence of heteroclinic trajectories connecting saddle orbits to the type of link.

Theorem 14. *Let $\varphi \in \mathcal{F}_A(S^3)$. Then, φ has no heteroclinic trajectories connecting saddle orbits if and only if $\varphi \in \mathcal{F}_I(S^3)$.*

Proof. ($\Rightarrow$) Let us show that if $\varphi \in \bar{\mathcal{F}}_I(S^3)$ then φ has heteroclinic trajectories connecting saddles.

The link associated to a flow $\varphi \in \bar{\mathcal{F}}_I(S^3)$ does not come from iterated connected sum of tori. So, the link can not be written as in lemma 7. Therefore, it implies at least two type A operations and it is obtained by attaching the round 1-handle to a solid torus by means of one essential circle. This solid torus has toroidal or hopf holes inside generated by previous attachments.
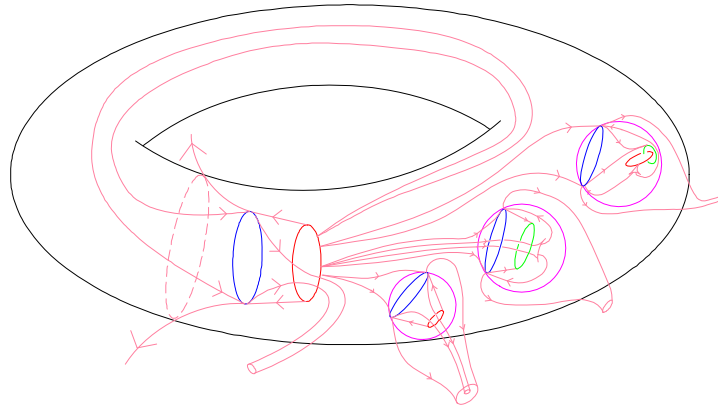

Figure 6: Iterated connected sum of flows in a torus without any atractive or repulsive orbit in his core

There are two possibilities:

1. The round 1-handle is attached to the solid torus (with toroidal or hopf holes inside) by means one essential and one inessential circles (operation III is made).

2. The round 1-handle is attached to the solid torus (with hopf or toroidal holes inside) by means one essential circle and to another thick torus (with hopf or toroidal holes inside) by means of one inessential circle (operation II is made).

In both cases there exists an invariant manifold of one saddle orbit that cuts the boundary of the torus by means of one transversal circle (see figure 6). The

essential circle of the attachment of the new round 1-handle is a longitudinal circle and therefore one invariant manifold of the new saddle cuts the boundary of the solid torus by means of one longitudinal circle; consequently, both circles intersect and heteroclinic trajectories between the two saddles appear.

($\Leftarrow$) As we have proved in theorem 13, the flows characterized by links coming from iterated sum of tori can be decomposed as a decay of connected sum of basic flows. Then, each of them can be isolated by a 3-ball and there are no heteroclinic trajectories connecting saddle orbits (see figures 4, 5 and 6). ■

The connection between links and flows studied in this paper can be summarized in the following table:

$$
\begin{array}{ccccc}
\text{Links:} \ \mathcal{L}_A(S^3) & = & \mathcal{L}_I(S^3) & \cup & \bar{\mathcal{L}}_I(S^3) \\
& & \Updownarrow & & \Updownarrow \\
& & l = c^{p+1} \cdot u^p & & l = l' \cdot u \\
& & \Updownarrow & & \Updownarrow \\
\text{Flows:} \ \mathcal{F}_A(S^3) & = & \mathcal{F}_I(S^3) & \cup & \bar{\mathcal{F}}_I(S^3) \\
& & \Updownarrow & & \Updownarrow
\end{array}
$$

No heteroclinic trajectories connecting saddle orbits	Heteroclinic trajectories connecting saddle orbits
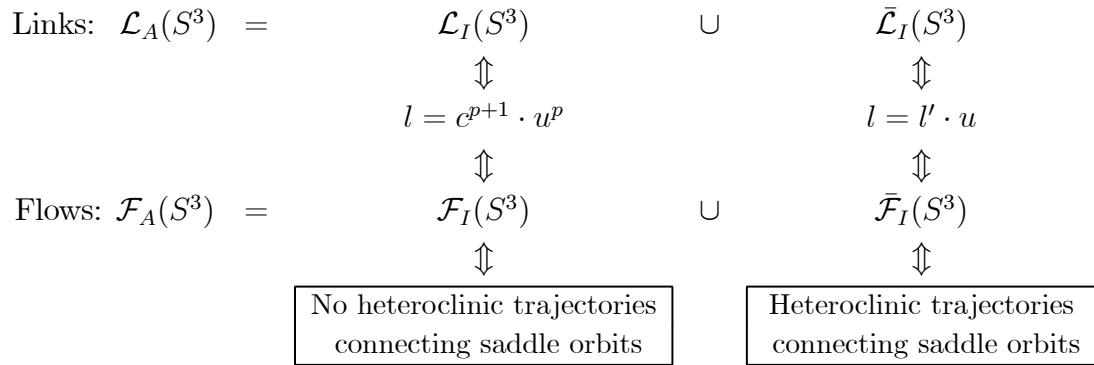
In [4] we showed that NMS flows with one saddle orbit also "live" in general lens spaces. Now, from the results of this paper, this statement can be generalized to the case of flows with more than one saddle orbit.

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