BILINEAR ISOMETRIES ON SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS

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ABSTRACT. Let X, Y, Z be compact Hausdorff spaces and let E_1, E_2, E_3 be Banach spaces. If $T : C(X, E_1) \times C(Y, E_2) \longrightarrow C(Z, E_3)$ is a bilinear isometry which is stable on constants and E_3 is strictly convex, then there exists a nonempty subset Z_0 of Z, a surjective continuous mapping $h : Z_0 \longrightarrow X \times Y$ and a continuous function $\omega : Z_0 \longrightarrow Bil(E_1 \times E_2, E_3)$ such that

$$T(f, g)(z) = \omega(z)(f(\pi_X(h(z)), g(\pi_Y(h(z)))$$

for all $z \in Z_0$ and every pair $(f, g) \in C(X, E_1) \times C(Y, E_2)$. This result generalizes the main theorems in [2] and [6].

1. INTRODUCTION.

Let X be a compact Hausdorff space and E a Banach space. Let $C(X)$ (resp. $C(X, E)$) denote the Banach spaces of all continuous scalar-valued (resp. vector-valued) functions on X endowed with the supremum norm, $\|\cdot\|_\infty$. A bilinear mapping $T : C(X)\times C(Y) \longrightarrow C(Z)$ which satisfies

$$\|T(f, g)\|_\infty = \|f\|_\infty \|g\|_\infty$$

for every $(f, g) \in C(X) \times C(Y)$ is called a *bilinear isometry*.

In [6], Moreno and Rodriguez proved the following bilinear version of the well-known Holsztyński's Theorem on non-surjective linear isometries of $C(X)$-spaces ([5] and, also, [1]):

Let $T : C(X) \times C(Y) \longrightarrow C(Z)$ be a bilinear isometry. Then there exist a closed subset Z_0 of Z, a surjective continuous mapping $h : Z_0 \longrightarrow X \times Y$ and a norm-one continuous function $a \in C(Z)$ such that $T(f, g)(z) = a(z)f(\pi_X(h(z))g(\pi_Y(h(z))$ for all $z \in Z_0$ and every pair $(f, g) \in C(X) \times C(Y)$. The proof of this result rests heavily on the powerful Stone-Weierstrass Theorem. In [3], the authors extend

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these results to certain subspaces of continuous scalar-valued functions, where Stone-Weierstrass Theorem is not applicable.

The concept of bilinear isometry can be naturally extended to the context of spaces of vector-valued continuous functions. Examples of bilinear isometries defined on these spaces can be found, for instance, in [7, Proposition 5.2], where the author provide certain compact spaces X and Banach spaces E for which there exists a bilinear isometry $T : C(X,E) \times C(X,E) \longrightarrow C(Y,E)$.

In this paper we study the conditions under which we can obtain a representation of such bilinear isometries on this vector-valued setting. Thus, given three Banach spaces E_1, E_2 and E_3, we prove that if $T : C(X,E_1) \times C(Y,E_2) \longrightarrow C(Z,E_3)$ is a bilinear isometry which is stable on constants (see Definition 3) and E_3 is strictly convex, then there exists a nonempty subset Z_0 of Z, a surjective continuous mapping $h : Z_0 \longrightarrow X \times Y$ and a continuous function $\omega : Z_0 \longrightarrow Bil(E_1 \times E_2, E_3)$ such that

$$T(f,g)(z) = \omega(z)(f(\pi_X(h(z)), g(\pi_Y(h(z)))$$

for all $z \in Z_0$ and every pair $(f,g) \in C(X,E_1) \times C(Y,E_2)$.

It can be easily checked that this result contains the main theorems in [6] and in [2] (see the concluding remarks at the end of the paper).

2. Notation and previous lemmas.

Let E be a Banach space and let S_E denote the unit sphere of E.

For any $e \in E$, we denote by $\widetilde{e}$ the element of $C(X,E)$ which is constantly equal to e. For any $x \in X$ and $e \in S_E$, let

$$C_{x,e} := \{f \in C(X,E) : 1 = \|f\|_\infty \text{ and } f(x) = e\}.$$

We shall write $Bil(E_1 \times E_2, E_3)$ to denote the space of jointly continuous bilinear mappings between $E_1 \times E_2$ and E_3 endowed with the strong operator topology.

In the sequel we shall assume that $T : C(X,E_1) \times C(Y,E_2) \longrightarrow C(Z,E_3)$ is a bilinear mapping which satisfies

$$\|T(f,g)\|_\infty = \|f\|_\infty \|g\|_\infty$$

for every $(f,g) \in C(X,E_1) \times C(Y,E_2)$, which is to say that T is *bilinear isometry.*

Lemma 1. *Assume* $(x,y) \in X \times Y$ *and* $(e,e') \in S_{E_1} \times S_{E_2}$. *The set*

$$I_{x,y,e,e'} := \{z \in Z : 1 = \|T(f,g)\|_\infty = \|(T(f,g)(z)\|, (f,g) \in C_{x,e} \times C_{y,e'}\}$$

is nonempty.

Proof. For any $f \in C(X, E_1)$ and $g \in C(Y, E_2)$, let us define the following compact subset of Z: $M_{f,g} := \{z \in Z : \|T(f,g)(z)\| \geq \frac{1}{2}\}$. It is apparent that $I_{x,y,e,e'}$ is a closed subset of $M_{f,g}$. Hence, in order to prove that $I_{x,y,e,e'}$ is nonempty, it suffices to check that if $f_1, ..., f_n$ belong to $C_{x,e}$ and $g_1, ..., g_n$ belong to $C_{y,e'}$, then

$$\bigcap_{i,j}\{z \in Z : 1 = \|T(f_i, g_j)\|_\infty = \|(T(f_i, g_j)(z)\|\} \neq \emptyset.$$

Let $f_0 \in C(X, E_1)$ and $g_0 \in C(Y, E_2)$ defined as follows:

$$f_0 := \sum_{i=1}^{n} f_i \quad \text{and} \quad g_0 := \sum_{j=1}^{n} g_i.$$

It is clear that $\|f_0(x)\| = n = \|f_0\|_\infty$ and $\|g_0(y)\| = n = \|g_0\|_\infty$.

Hence, $\|T(f_0, g_0)\|_\infty = \|f_0\|_\infty \cdot \|g_0\|_\infty = n^2$ since T is a bilinear isometry and, consequently, there exists $z_0 \in Z$ such that

$$n^2 = \|T(f_0, g_0)(z_0)\| = \left\|\sum_{i,j} T(f_i, g_j)(z_0)\right\| \leq \sum_{i,j} \|T(f_i, g_j)(z_0)\| \leq n^2.$$

This fact yields $\|T(f_i, g_j)(z_0)\| = 1$ for all i, j, which is to say that

$$z_0 \in \bigcap_{i,j}\{z \in Z : 1 = \|T(f_i, g_j)\|_\infty = \|(T(f_i, g_j)(z)\|\}.$$

□

Lemma 2. *Assume E_3 is strictly convex and fix $(x_0, y_0) \in X \times Y$ and $(e, e') \in S_{E_1} \times S_{E_2}$.*

(1) *If $f(x_0) = 0$ for some $f \in C(X, E_1)$ and $g' \in C_{y_0,e'}$, then $T(f, g')(z) = 0$ for all $z \in I_{x_0,y_0,e,e'}$.*
(2) *If $g(y_0) = 0$ for some $g \in C(Y, E_2)$ and $f' \in C_{x_0,e}$, then $T(f', g)(z) = 0$ for all $z \in I_{x_0,y_0,e,e'}$.*

Proof. (1) Let us choose $z_0 \in I_{x_0,y_0,e,e'}$. Define a linear isometry $T' : C(X, E_1) \longrightarrow C(Z, E_3)$ as $T'(f) := T(f, g')$.

We shall first check that if $f \in C(X, E_1)$ vanishes on an open neighborhood, U, of x_0, then $(T'f)(z_0) = 0$. With no loss of generality, we shall assume that $\|f\|_\infty = 1$.

Let us take $\xi \in C(X)$ such that $1 = |\xi(x_0)| = \|\xi\|_\infty$ and such that its support is included in U. We can now define two functions in $C(X, E_1)$ as follows:

$$g := f + \xi e$$

and

$$h := \frac{1}{2}(g + \xi e).$$

It is clear that $g(x_0) = h(x_0) = \xi(x_0)e$ and that $\|\xi e\|_\infty = \|g\|_\infty = \|h\|_\infty = 1$. Therefore, since $z_0 \in I_{x_0,y_0,e,e'}$, then

$$\|T'(\xi e)(z_0)\| = \|T'(g)(z_0)\| = \|T'(h)(z_0)\| = 1.$$

Now, as $T'(h)(z_0)$ is on the segment which joins $T'(\xi e)(z_0)$ and $T'(g)(z_0)$, the strict convexity of E yields $T'(\xi e)(z_0) = T'(g)(z_0)$, which is to say that $T'(f)(z_0) = 0$.

Let us now define two linear functionals on $C(X, E_1)$ as follows: $\hat{T'}\hat{z_0}(f) := T'(f)(z_0)$ and $\hat{x_0}(f) := f(x_0)$. It is not hard to check that the functions in $C(X, E_1)$ which vanish on a neighborhood of x_0 are dense in the kernel of $\hat{x_0}$, $ker(\hat{x_0})$, which is closed due to the continuity of this functional. Consequently, the above paragraph yields the inclusion $ker(\hat{x_0}) \subseteq ker(\hat{T'}\hat{z_0})$; that is, if $f(x_0) = 0$, then $T'(f)(z_0) = 0$, as was to be proved.

(2) The proof of (2) is similar to (1). □

Definition 2. *For any pair $(x, y) \in X \times Y$, we define the set*

$$I_{x,y} := \bigcup_{(e,e') \in S_{E_1} \times S_{E_2}} I_{x,y,e,e'}.$$

Lemma 3. *Assume E_3 is strictly convex. Let $(x_0, y_0) \in X \times Y$ and suppose that there exist $(\tilde{f}, \tilde{g}) \in C(X, E_1) \times C(Y, E_2)$ which vanish on x_0 and y_0 respectively. Then $T(\tilde{f}, \tilde{g})(z) = 0$ for all $z \in I_{x_0,y_0}$.*

Proof. Assume first that there exist $(f, g) \in C(X, E_1) \times C(Y, E_2)$ which vanish on certain neighborhoods, U and V, of x_0 and y_0 respectively. Then we claim that $T(f, g)(z) = 0$ for all $z \in I_{x_0,y_0}$.

To this end, fix $z_0 \in I_{x_0,y_0}$. Then $z_0 \in I_{x_0,y_0,e,e'}$ for some $(e, e') \in S_{E_1} \times S_{E_2}$. Assume, with no loss of generality, $\|f\|_\infty \leq 1$ and $\|g\|_\infty \leq 1$.

Let us consider $(f_1, g_1) \in C(X) \times C(Y)$ such that $supp(f_1) \subset U$ and $supp(g_1) \subset V$, and $1 = \|f_1\|_\infty = f_1(x_0)$ and $1 = \|g_1\|_\infty = g_1(y_0)$.

It is then clear that $\|f + f_1 e\|_\infty = \|f(x_0) + f_1(x_0)e\| = \|e\| = 1$ and $\|g + g_1 e'\|_\infty = \|g(y_0) + g_1(y_0)e'\| = \|e'\| = 1$. Consequently, since $z_0 \in I_{x_0,y_0,e,e'}$,

$$\|T(f + f_1 e, g + g_1 e')(z_0)\| = 1,$$
$$\|T(f_1 e, g_1 e')(z_0)\| = 1$$

and

$$\left\| T\left(\frac{f}{2} + f_1 e, g + g_1 e'\right)(z_0) \right\| = 1.$$

On the other hand, by Lemma 2, we know that $T(f, g_1 e')(z_0) = T(f_1 e, g)(z_0) = 0$. Therefore

$$\frac{T(f + f_1 e, g + g_1 e')(z_0) + T(f_1 e, g_1 e')(z_0)}{2} =$$

$$= \frac{T(f,g)(z_0)}{2} + T(f_1 e, g_1 e')(z_0) = T\left(\frac{f}{2} + f_1 e, g + g_1 e'\right)(z_0).$$

This means that $T\left(\frac{f}{2} + f_1 e, g + g_1 e'\right)(z_0)$ is on the segment which joins $T(f + f_1 e, g + g_1 e')(z_0)$ and $T(f_1 e, g_1 e')(z_0)$. Hence, since E_3 is strictly convex, $T(f + f_1 e, g + g_1 e')(z_0)$ and $T(f_1 e, g_1 e')(z_0)$ coincide, which is to say, again by Lemma 2, that $T(f,g)(z_0) = 0$.

Let us now take a sequence $(f_n) \in C(X, E_1)$ convergent to $\tilde{f}$ and such that $f_n \equiv 0$ on a certain neighborhood U_n of x_0. Similarly, take a sequence $(g_n) \in C(Y, E_2)$ convergent to $\tilde{g}$ and such that $g_n \equiv 0$ on a certain neighborhood V_n of y_0. Fix $z_0 \in I_{x_0,y_0}$. Then we can define a linear functional on $C(X, E_1) \times C(Y, E_2)$ as follows: $T_{z_0}(f,g) := T(f,g)(z_0)$. It is apparent, from the above paragraph, that $T_{z_0}(f_n, g_n) = 0$ for all $n \in N$. On the other hand, by the Uniform Boundedness Theorem (see, e.g., [4, 11.15 Theorem]), we deduce that $(T_{z_0}(f_n, g_n))$ converges to $T_{z_0}(\tilde{f}, \tilde{g}) = T(\tilde{f}, \tilde{g})(z_0)$. This fact yields $T(\tilde{f}, \tilde{g})(z_0) = 0$.

□

Definition 4. *We say that T is* stable on constants *if, given $(f,g) \in C(X, E_1) \times C(Y, E_2)$ and $z \in Z$, then*

$$\|T(f, \widetilde{e_2})(z)\| = \|T(f, \widetilde{e_2'})(z)\|$$

for every pair $e_2, e_2' \in S_{E_2}$ and

$$\|T(\widetilde{e_1}, g)(z)\| = \|T(\widetilde{e_1'}, g)(z)\|$$

for every pair $e_1, e_1' \in S_{E_1}$.

Lemma 4. *Assume E_3 is strictly convex. Fix $(x_0, y_0) \in X \times Y$ and assume that T is stable on constants.*

(1) *If $f(x_0) = 0$ for some $f \in C(X, E_1)$ (resp. $g(y_0) = 0$ for some $g \in C(Y, E_2)$), then $T(f,g)(z) = 0$ for all $z \in I_{x_0,y_0}$ and all $g \in C(Y, E_2)$ (resp. all $f \in C(X, E_1)$).*

(2) *Furthermore, $T(f,g)(z) = T(\widetilde{f(x_0)}, \widetilde{g(y_0)})(z)$ for all $z \in I_{x_0,y_0}$ and all $(f,g) \in C(X, E_1) \times C(Y, E_2)$.*

Proof. (1) Let us take $(f,g) \in C(X, E_1) \times C(Y, E_2)$ such that $f(x_0) = 0$ and assume, with no loss of generality, that $\|g(y_0)\| = 1$.

Fix $z_0 \in I_{x_0,y_0}$. Then $z_0 \in I_{x_0,y_0,e,e'}$ for some $(e,e') \in S_{E_1} \times S_{E_2}$. By Lemma 2, we know that $T(f, \widetilde{e'})(z_0) = 0$

By Lemma 3, $T(f, g - \widetilde{g(y_0)})(z_0) = 0$, which yields $T(f,g)(z_0) = T(f, \widetilde{g(y_0)})(z_0)$.

Therefore, since T is stable on constants, we have

$$0 = T(f, \widetilde{e'})(z_0) = T(f, \widetilde{g(y_0)})(z_0) = T(f,g)(z_0).$$

(2) Take now a pair $(f,g) \in C(X,E_1) \times C(Y,E_2)$ and define the function $f' := f - \widetilde{f(x_0)}$. Since $f'(x_0) = 0$, then, by (a), $T(f - \widetilde{f(x_0)}, g)(z) = 0$ for all $z \in I_{x_0,y_0}$, which is to say, by the bilinearity of T, that $T(f,g)(z) = T(\widetilde{f(x_0)}, g)(z)$ for all $z \in I_{x_0,y_0}$.

Next, define the function $g' := g - \widetilde{g(y_0)}$. Since $g'(y_0) = 0$, then, again by (a), $T(\widetilde{f(x_0)}, g - \widetilde{g(y_0)})(z) = 0$ for all $z \in I_{x_0,y_0}$, which yields $T(f,g)(z) = T(\widetilde{f(x_0)}, g)(z) = T(\widetilde{f(x_0)}, \widetilde{g(y_0)})(z)$.

3. The main result.

Theorem 1. *Let $T : C(X,E_1) \times C(Y,E_2) \longrightarrow C(Z,E_3)$ be a bilinear isometry which is stable on constants and assume that E_3 is strictly convex. Then there exists a nonempty subset Z_0 of Z, a surjective continuous mapping $h : Z_0 \longrightarrow X \times Y$ and a continuous function $\omega : Z_0 \longrightarrow Bil(E_1 \times E_2, E_3)$ such that $T(f,g)(z) = \omega(z)(f(\pi_X(h(z)), g(\pi_Y(h(z)))$ for all $z \in Z_0$ and every pair $(f,g) \in C(X,E_1) \times C(Y,E_2)$.*

Proof. Let us suppose that (x,y) and (x',y') belong to $X \times Y$ and are distinct. Then we claim that $I_{x,y} \cap I_{x',y'} = \emptyset$. Assume, contrary to what we claim, that there exists $z \in I_{x,y} \cap I_{x',y'}$. Let us suppose, with no loss of generality, that $x \neq x'$.

- If $y \neq y'$, then we can choose $f \in C_{x,e}$ and $g \in C_{y,e'}$ for some $e, e' \in S_E$ with $f(x') = g(y') = 0$. Consequently, $\|T(f,g)(z)\| = 1$, but, by Lemma 3, $T(f,g)(z) = 0$, which is a contradiction.
- If $y = y'$, then we can choose $f \in C_{x,e}$ and $g \in C_{y,e'}$ for some $e, e' \in S_E$ with $f(x') = 0$. Consequently, $\|T(f,g)(z)\| = 1$, but, by Lemma 4, $T(f,g)(z) = 0$, which is a contradiction.

Let us next define a subset Z_0 of Z as follows:

$$Z_0 := \bigcup_{(x,y) \in X \times Y} I_{x,y}$$

Now we can define a linear map ω from Z_0 to $Bil(E_1 \times E_2, E_3)$ as $\omega(z)(e, e') := T(\widetilde{e}, \widetilde{e'})(z)$ where $(e, e') \in E_1 \times E_2$. Hence, by Lemma 4,

$$T(f, g)(z) = T(\widetilde{f(x_0)}, \widetilde{g(y_0)})(z) = \omega(z)(f(x_0), g(y_0))$$

for all $z \in Z_0$ and every pair $(f, g) \in C(X, E_1) \times C(Y, E_2)$.

To prove the continuity of ω, let (z_α) be a net convergent to $z_0 \in Z_0$. Fix $(e, e') \in E_1 \times E_2$. Then $\|\omega(z_\alpha)(e, e') - \omega(z_0)(e, e')\| = \|T(\widetilde{e}, \widetilde{e'})(z_\alpha) - T(\widetilde{e}, \widetilde{e'})(z_0)\|$. Since $(T(\widetilde{e}, \widetilde{e'})(z_\alpha))$ converges to $T(\widetilde{e}, \widetilde{e'})(z_0)$, the continuity of ω is then verified.

Let us next define a mapping $h : Z_0 \longrightarrow X \times Y$ as $h(z) := (x, y)$ where $z \in I_{x,y}$. We claim that h is continuous. To this end, fix $z_0 \in Z_0$ and let $h(z_0) = (x_0, y_0)$. Let U be a neighborhood of x_0 and choose $f \in C(X, E_1)$ such that $1 = \|f\|_\infty = \|f(x_0)\|$ and $\|f\|_\infty < 1$ off U. Let $s(x_0) = sup_{x \in X \setminus U} \|f(x)\|$. It is apparent that $s(x_0) < 1$. In like manner, let V be a neighborhood of y_0 and choose $g \in C(Y, E_2)$ such that $1 = \|g\|_\infty = \|g(y_0)\|$ and $\|g\|_\infty < 1$ off V. Let $s(y_0) = sup_{y \in Y \setminus U} \|g(y)\|$. As above, $s(y_0) < 1$.

Since $h(z_0) = (x_0, y_0)$, then $\|T(f, g)(z_0)\| = \|T(f, g)\|_\infty = 1$. Let $s := max\{s(x_0), s(y_0)\}$ and define the following open neighborhood of z_0:

$$W := \{z \in Z_0 : \|T(f, g)(z)\| > s\}.$$

Fix $z_1 \in W$ and suppose that $h(z_1) := (x_1, y_1)$. Then, by the above representation of T,

$$\begin{aligned}
s < \|T(f, g)(z_1)\| &= \|\omega(z_1)(f(x_1), g(y_1))\| \\
&= \|T(\widetilde{f(x_1)}, \widetilde{g(y_1)})(z_1)\| \\
&\leq \|T(\widetilde{f(x_1)}, \widetilde{g(y_1)})\|_\infty \\
&= \|\widetilde{f(x_1)}\|_\infty \cdot \|\widetilde{g(y_1)}\|_\infty \\
&= \|f(x_1)\| \|g(y_1)\|
\end{aligned}$$

and, consequently, $\|f(x_1)\| > s \geq s(x_0)$ and $\|g(y_1)\| > s \geq s(y_0)$. This yields $x_1 \in U$ and $y_1 \in V$, which is to say that $h(W) \subseteq U \times V$ and the proof is done.

Finally, it is clear that $T(f, g)(z) = \omega(z)(f(\pi_X(h(z)), g(\pi_Y(h(z)))$ □

Concluding remarks.

(1) To be stable on constants can be regarded as a necessary condition in the following sense: Let $T : C(X, E_1) \times C(Y, E_2) \longrightarrow$

$C(Z, E_3)$ be a bilinear isometry which can be written as

$$T(f,g)(z) = \omega(z)(f(\pi_X(h(z)), g(\pi_Y(h(z))$$

for all $z \in Z$ and every pair $(f,g) \in C(X,E_1) \times C(Y,E_2)$, where h is a surjective continuous mapping from Z onto $X \times Y$ and $\omega(z) \in Bil(E_1 \times E_2, E_3)$. Then

$$\|T(f,\widetilde{e})(z)\| = \|\omega(z)(f(\pi_X(h(z)), e)\| = \|f(\pi_X(h(z))\|$$

for all $e \in E_2$ and all $z \in Z$; that is, T is stable on constants.

(2) It is clear that if we assume E_1, E_2 and E_3 to be the field of real or complex numbers, then T is stable on constants. Hence, Theorem 1 is an extension, indeed a vector-valued version, of the main result in [6].

(3) In like manner, Theorem 1 contains the main theorem in [2], by assuming Y to be a singleton and E_2 to be the field of real or complex numbers. Indeed, it is a routine matter to verify that, in this context, Lemma 4 and Theorem 1 remain true even if we do not assume T to be stable on constants.

(4) Typical examples of bilinear isometries can be defined as follows: assume that there exists a continuous surjection $h : X \longrightarrow X \times X$ and let E be a Banach algebra. Then we can define a mapping $T(f,g)(z) := f(\pi_1(h(z))g(\pi_2(h(z))$ for all $z \in X$ and every pair $(f,g) \in C(X,E) \times C(X,E)$. It is apparent that T is a bilinear isometry which is stable on constants.

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