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BIFURCATIONS IN THE TWO IMAGINARY CENTERS PROBLEM

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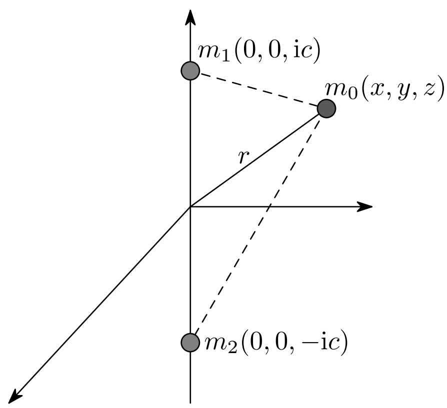
Abstract. In this paper we show that, for a given value of the energy, there is a bifurcation for the two imaginary centers problem. For this value not only the configuration of the orbits changes but also a change in the topology of the phase space occurs.

Keywords: bifurcation, topological configuration, orbital structure

MSC 2010: 37G99

1. Introduction

The two fixed centers problem with imaginary distance serves as a model for the study of the problem of the motion of a satellite of mass m_0 in the gravitational field of the Earth spheroid [2]. We consider two particles of masses m_1 and m_2, respectively, located on two fixed centers on the Z axes, and we denote the distance from the origin to the particle m_i by c_i, $i = 1, 2$.



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The study of the motion of a particle of mass m_0 in the resultant of the force fields of Newtonian attraction of these two particles can be compared with the study of the motion of a satellite in the gravitational field of the Earth. We can identify the first terms of the force function of both problems requiring that

$$m_1 = m_2 = M, \quad c_1 = \mathrm{i}R\sqrt{|I_2|},\; c_2 = -\mathrm{i}R\sqrt{|I_2|}$$

where M is the mass of the Earth, R is its radius and I_2 is the coefficient of the oblateness of the Earth. So, the motion of a satellite in the gravitational field of the oblate Earth that is symmetric with respect to the equatorial plane can be interpreted as the integrable problem of a particle in the field of two fixed centers, with equal masses and located at a purely imaginary distance one from the other [2].

Consider two fixed centers situated at the points $(0, 0, \pm\mathrm{i}c)$. Hamilton's equations of motion of the particle of mass m_0 can be separated by using elliptic coordinates

$$x = \cosh\xi\cos\eta\cos\nu y = \cosh\xi\cos\eta\sin\nu z = \sinh\xi\sin\eta.$$

By introducing the time scale $\mathrm{d}t/\,\mathrm{d}\tau = \cosh^2\xi - \cos^2\eta$ the hamiltonian is

$$H = \frac{1}{\cosh^2\xi - \cos^2\eta}\Big(\frac{p_\xi^2}{2} + \frac{p_\eta^2}{2} + \Big(\frac{1}{\cos^2\eta} - \frac{1}{\cosh^2\xi}\Big)\frac{p_\nu^2}{2} - k\sinh\xi\Big) \tag{1.1}$$

with $k = \gamma m_0 M$ and γ the gravitational constant.

If we fix the value of the energy $H = h$ on the corresponding level of energy, we can write the hamiltonian as a sum of two functions in such a way that the variables ξ and η are separated,

$$H = \frac{H_\xi + H_\eta}{\cosh^2\xi - \cos^2\eta} + h, \tag{1.2}$$

where

$$\begin{aligned} H_\xi &= \frac{p_\xi^2}{2} - \frac{1}{\cosh^2\xi}\frac{p_\nu^2}{2} - k\sinh\xi - h\cosh^2\xi,\\ H_\eta &= \frac{p_\eta^2}{2} - \frac{1}{\cos^2\eta}\frac{p_\nu^2}{2} + h\cos^2\eta. \end{aligned} \tag{1.3}$$

These expressions allow us to study two separated problems and to obtain the phase spaces of these problems for different values of the energy. With the purpose to construct the complete phase space we consider a particular value of the energy $H = h$ which leads to $H_\xi + H_\eta = 0$, that is $H_\eta = -H_\xi$. Then we obtain a compact complete phase space.

2. Phase spaces (ξ, p_ξ), (η, p_η) for $\omega = 0$.

In this section we consider the two separated problems analyzing the hamiltonians H_ξ and H_η for different values of $h < 0$. In particular, we study the case $\omega = 0$, where $\omega = p_\nu^2/2$.

The expressions of the hamiltonians are

$$H_\xi = \frac{p_\xi^2}{2} - k\sinh\xi - h\cosh^2\xi, \tag{2.1}$$

$$H_\eta = \frac{p_\eta^2}{2} + h\cos^2\eta. \tag{2.2}$$

It follows from the analysis of the phase spaces that $\frac{1}{4}k^2/h - h \leqslant H_\xi$ and $h \leqslant H_\eta$. Since $H_\eta = -H_\xi$ we obtain

$$\frac{k^2}{4h} - h \leqslant H_\xi \leqslant -h, \quad h \leqslant H_\eta \leqslant -\frac{k^2}{4h} + h. \tag{2.3}$$

We characterize different cases according to the values of h with respect to the values of k.

(1) For $H_\xi = \frac{1}{4}k^2/h - h$ we obtain only one equilibrium point $(\arg\sinh(-k/2h), 0)$.

(2) For $\frac{1}{4}k^2/h - h < H_\xi < -h$ the domain of p_ξ is $[\mu_1, \mu_2]$ where

$$\mu_{1,2} = \arg\sinh\frac{-k \mp \sqrt{k^2 - 4h(H_\xi + h)}}{2h},$$

and periodic orbits appear around the equilibrium point (see Figure 1).

(3) If $H_\xi = -h$ the domain is $[0, \arg\sinh(-k/h)]$, and there is a unique periodic orbit as the limit of the orbits of the case above.

We make a similar study for the (η, p_η)-phase space:

(1) If $H_\eta = h$ we obtain the equilibrium points $\alpha\pi$, $\alpha \in \mathbb{Z}$.

(2) If $h < H_\eta < -\frac{1}{4}k^2/h + h$ the domain of p_η in the interval $[-\pi, \pi]$ is

$$[-\pi, -\gamma_1] \cup [-\gamma_2, \gamma_2] \cup [\gamma_1, \pi],$$

where $\gamma_1 = \arccos\left(-\sqrt{H_\eta/h}\right)$ and $\gamma_2 = \arccos\left(\sqrt{H_\eta/h}\right)$.

(3) If $H_\eta = -\frac{1}{4}k^2/h + h$ a periodic orbit appears as the limit of the periodic orbits of the previous case.

The relation $H_\xi = -H_\eta$ implies that the limit orbit in the (ξ, p_ξ)-phase space corresponds to the equilibrium points of the (η, p_η)-phase space and vice versa.

Let us remark that a change of sign of the term $\frac{1}{4}k^2/h - h$ corresponds to the bifurcation value $h = -\frac{1}{2}k$ and is related to the appearance or disappearance of two equilibria in the (η, p_η)-phase space.

In the following we describe the phase spaces (ξ, p_ξ) and (η, p_η) for all values of the energy.

For $h < -\frac{1}{2}k$ the phase spaces (ξ, p_ξ) and (η, p_η) can be seen in Figures 1 and 2 respectively.

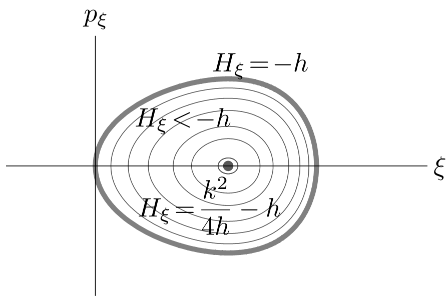

Figure 1. (ξ, p_ξ)-space for $h < -\frac{1}{2}k$.

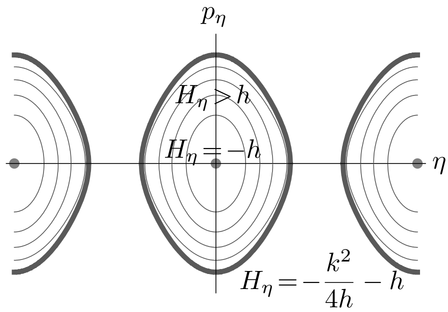

Figure 2. (η, p_η)-space for $h < -\frac{1}{2}k$.

One can observe that in the (η, p_η)-phase space the equilibrium points correspond to $H_\eta = h$; in the (ξ, p_ξ)-phase space this value implies $H_\xi = -h$ and corresponds to the limit orbit. For $H_\eta > h$ in (η, p_η), the orbits are circles with decreasing radii. The corresponding orbits for $H_\xi > -h$ in the (ξ, p_ξ)-phase space have increasing radii. The limit orbit in the (η, p_η)-phase space appears for $H_\eta = -\frac{1}{4}k^2/h + h$ and corresponds to the equilibrium point in the (ξ, p_ξ)-phase space for $H_\xi = \frac{1}{4}k^2/h - h$.

In the case $h = -\frac{1}{2}k$ there exists one equilibrium point in the (ξ, p_ξ)-phase space on the horizontal axis for $H_\xi = 0$, which corresponds to the limit orbit in the (η, p_η)-phase space for $H_\eta = 0$.

For $H_\xi < -h$ the periodic orbits in the (ξ, p_ξ)-phase space have increasing radii until they reach the limit circle for $H_\xi = -h$. Their corresponding orbits in the (η, p_η)-phase space have decreasing radii and their limits are the equilibrium points $(0, 0)$, $(\pm\pi, 0)$. The (ξ, p_ξ) and (η, p_η) phase spaces can be seen in Figures 3 and 4, respectively.

For $h > -\frac{1}{2}k$, the equilibrium point in the (ξ, p_ξ)-phase space corresponds to the limit orbit in the (η, p_η)- phase space.

The (ξ, p_ξ) and (η, p_η) phase spaces can be seen in Figures 5 and 6, respectively.

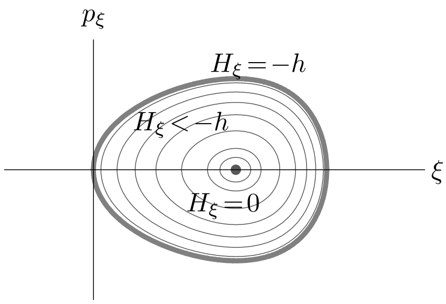

Figure 3. (ξ, p_ξ)-space for $h = -\frac{1}{2}k$.

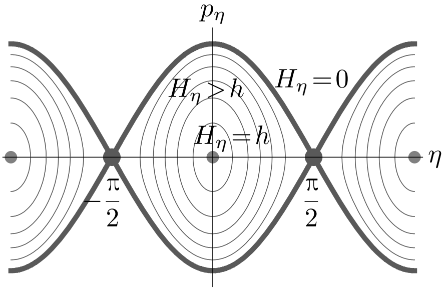

Figure 4. (η, p_η)-space for $h = -\frac{1}{2}k$.

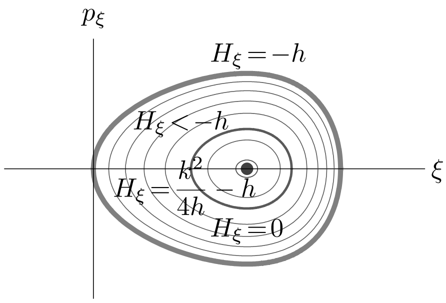

Figure 5. (ξ, p_ξ)-space for $h > -\frac{1}{2}k$.

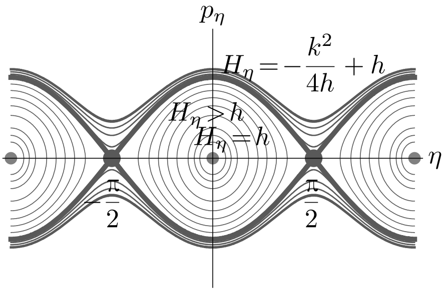

Figure 6. (η, p_η)-space for $h > -\frac{1}{2}k$.

3. Bifurcations for $\omega = 0$.

In this section we build the complete phase space from the separated spaces (ξ, p_ξ) and (η, p_η) studied in the previous section, taking into account the relationship $H_\eta = -H_\xi$. Because of the change of sign of the term $\frac{1}{4}k^2/h - h$ we obtain a bifurcation for $h = -\frac{1}{2}k$. For this value of bifurcation we obtain a change in the topology of the complete phase space and also in the structure of the set of periodic orbits.

Proposition 3.1. *The topology of the phase space of the two imaginary centers problem changes at the energy value $h = -\frac{1}{2}k$.*

P r o o f. For each value of the energy, the phase space of (ξ, p_ξ) is a disk, as we can see in Figures 1, 3, 5. On the other hand, the phase space of (η, p_η) changes its topology for different values of h:

- If $h < -\frac{1}{2}k$ the (η, p_η)-phase space is formed by two isolated disks.
- If $h = -\frac{1}{2}k$ the (η, p_η)-phase space is formed by two disks joined by two points.
- If $h > -\frac{1}{2}k$ the (η, p_η)-phase space is formed by a corona.

We study the complete phase space according to the different values of the energy.

The complete phase space is built taking into account that both two dimensional phase spaces are related by the energy condition $H_\xi = -H_\eta$. Then the orbit obtained for a given value of H_ξ, in the (ξ, p_ξ)-phase space, has to be multiplied by the orbit associated with the value $-H_\eta$ in the (η, p_η)-phase space. So, the different circles obtained when H_η decreases have to be multiplied by the corresponding circles in the (ξ, p_ξ)-phase space when H_ξ increases.

For $h < -\frac{1}{2}k$ the complete phase space is *two copies of* S^3. In this case, the two disks in the (η, p_η)-phase space are isolated and the complete phase space is formed by two copies of the same 3-manifold.

Let us consider one of these disks. Its boundary, corresponding to the greatest value of H_η, is multiplied by the fixed point of the (ξ, p_ξ)-phase space, and one circle is obtained.

As H_η decreases, the circles are multiplied by the corresponding circles in the (ξ, p_ξ)-phase space when H_ξ increases; so, a sequence of fitted 2-tori is obtained. Finally, the fixed point is multiplied by the limit orbit of the (ξ, p_ξ)-phase space and one limit orbit is obtained. When two 2-disks are identified in this way, the result is the three sphere S^3.

So, the complete phase space is formed by two copies of S^3 (see Figure 7). In the figures, the shaded area represents the (η, p_η)-phase space.

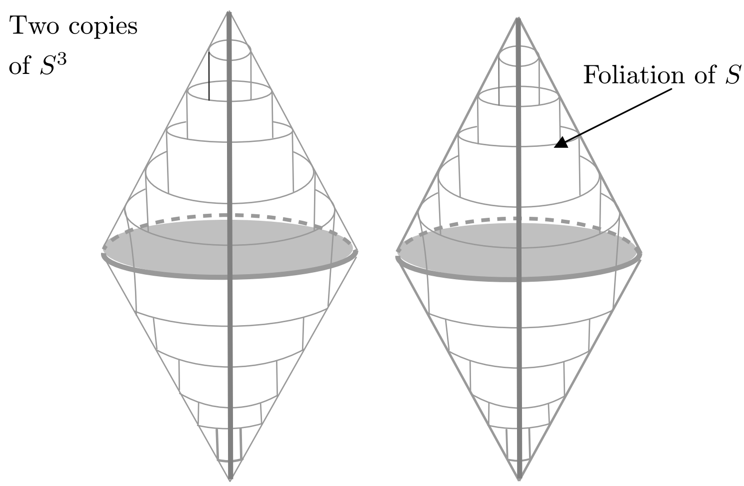

Figure 7. Complete phase space for $h < -\frac{1}{2}k$.

For $h = -\frac{1}{2}k$ the complete phase space is *two copies of* S^3 *joined by two points.*

Indeed, the phase space (η, p_η) in this case is formed by two disks joined by two points (in this phase space $\eta = -\pi$ and $\eta = \pi$ are identified). The complete phase space is built by multiplying the boundary of these two disks by the fixed point of the (ξ, p_ξ)-phase space. The result is two circles joined by two points. Since $H_\eta = -H_\xi$, the different circles obtained when H_η decreases have to be multiplied by the corresponding circles in the (ξ, p_ξ)-phase space when H_ξ increases, obtaining a sequence of 2-tori. Finally, the two fixed points are multiplied by the limit orbit corresponding to the value $H_\xi = -h$ in the (ξ, p_ξ)-phase space, and two limit orbits in the complete phase space are obtained. The conclusion is that the complete phase space in this case is formed by two copies of S^3 joined by two points (see Figure 8).

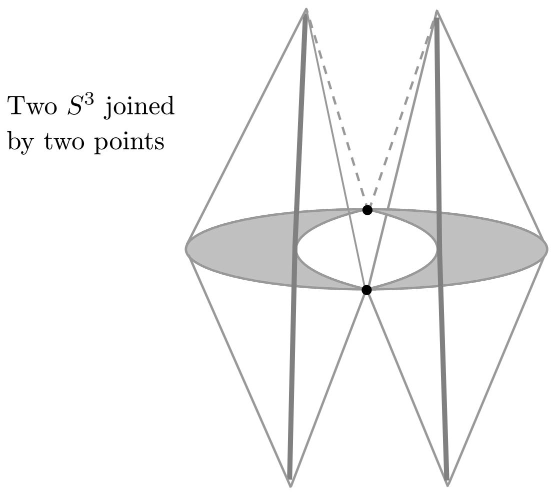

Figure 8. Complete phase space for $h = -\frac{1}{2}k$.

For $h > -\frac{1}{2}k$ the complete phase space is $S^2 \times S^1$. As in the previous cases, the (ξ, p_ξ)-phase space is a disk D^2 and the (η, p_η)-phase space is formed by an annulus.

We proceed as above, the complete phase space is built by multiplying the boundary of this annulus by the fixed point of the (ξ, p_ξ)-phase space obtaining two circles. When $H_\eta = -H_\xi$, the different circles obtained as H_η decreases have to be multiplied by the corresponding circles in the (ξ, p_ξ)-phase space as H_ξ increases. The result is a sequence of 2-tori, where each torus is inside the next one.

The product of the separatrix orbit in the (η, p_η)-phase space by the corresponding S^1 in the (ξ, p_ξ)-phase space yields two 2-tori joined along two different circles, these circles are the result of multiplying the points $(\pm\pi, 0)$ by the corresponding S^1 in the (ξ, p_ξ)-phase space. These circles are saddle orbits.

Finally, the two fixed points $(\pm\pi, 0)$ of the (η, p_η)-phase space are multiplied by the limit orbit in the (ξ, p_ξ)-phase space for the value $H_\xi = -h$ and we get two limit orbits. We conclude that the complete phase space for $h > -\frac{1}{2}k$ is $S^2 \times S^1$ as we see in Figure 9.

Then, the complete phase space changes for $h = -\frac{1}{2}k$. □

We refer to closed curves which are either in the common axis of the invariant 2-dimensional tori or the two 2-tori intersection as NMS periodic orbits (see [3]).

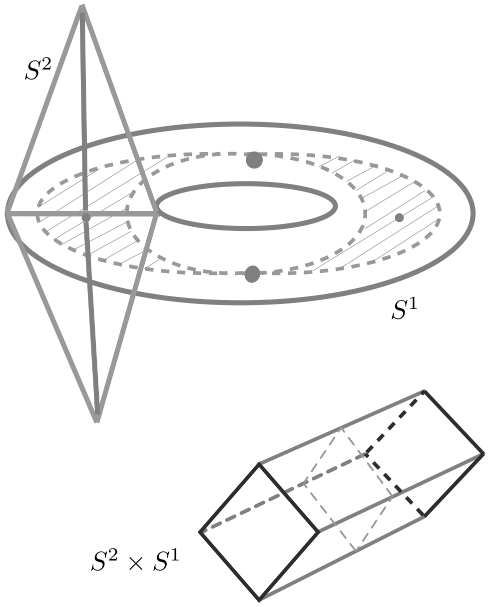

Figure 9. Complete phase space for $h > -\frac{1}{2}k$.

Corollary 3.1. *The NMS periodic orbits of the two imaginary centers problem bifurcate at the energy value* $h = -\frac{1}{2}k$.

Proof. A NMS periodic orbit in the complete phase space is obtained when one point in the (ξ, p_ξ)-phase space is multiplied by an S^1 in the (η, p_η)-phase space or vice versa.

If $h < -\frac{1}{2}k$, the complete phase space is two copies of S^3 and two periodic orbits are linked in each of them, i.e., two hopf links are obtained (see Figure 10).

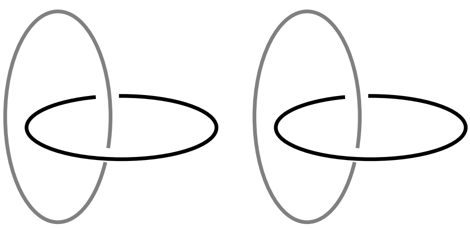

Figure 10. Periodic orbits for $h < -\frac{1}{2}k$.

For $h = -\frac{1}{2}k$, the phase space is two copies of S^3 joined by two points. It implies that two hopf links are joined by two points (see Figure 11).

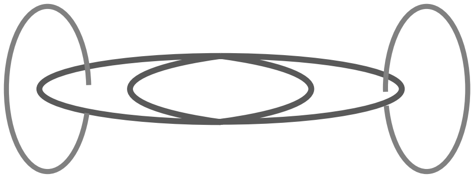

Figure 11. Periodic orbits for $h = -\frac{1}{2}k$.

For the last case $h > -\frac{1}{2}k$, the phase space is $S^2 \times S^1$. The periodic orbits obtained by multiplying the fixed points $(\pm\pi, 0)$ by S^1 are saddle orbits because they are in the intersection of two tori. The other periodic orbits are attractive and repulsive orbits because they are in the core of a solid tori.

The manifold $S^2 \times S^1$ admits local and global orbits depending on whether they are isolated by a 3-cell or not. By construction, we can obtain periodic orbits by multiplying the fixed point in the (ξ, p_ξ)-phase space by the boundary circles of the corona, which are global orbits. The other four orbits, obtained by multiplying the fixed points in the (ξ, p_ξ)-phase space by S^1 are local periodic orbits. Let us observe that the four local orbits are linked to one of the global orbits (see Figure 12).

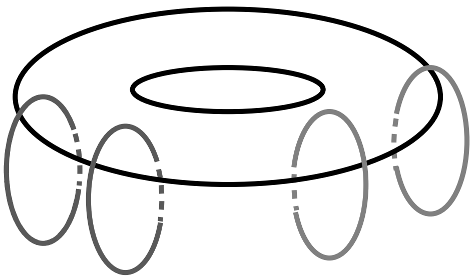

Figure 12. Periodic orbits for $h > -\frac{1}{2}k$.

So, the link of periodic orbits bifurcates for $h = -\frac{1}{2}k$. □

Therefore, $h = -k/2$ is a bifurcation point for both the phase space of the problem and, of course, the link of periodic orbits.

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