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A Statistical Equilibrium Model of Competitive Firms*

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Abstract

We find that the empirical distribution of firm profit rates, measured as returns on assets, is markedly non-Gaussian and reasonably well described by an exponential power (or Subbotin) distribution. Thus we propose a statistical equilibrium model that leads to a stationary Subbotin density in the presence of complex interactions among competitive heterogeneous firms. To investigate the dynamics of firm profitability, we also construct a diffusion process that has the Subbotin distribution as its stationary probability density, leading to a phenomenologically inspired interpretation of variations in the shape parameter of the Subbotin distribution. Our findings have profound implications both for the previous literature on the 'persistence of profits' as well as for understanding competition as a dynamic process. Our main finding is that firms' idiosyncratic efforts and the tendency for competition to equalize profit rates are two sides of the same coin.

Keywords: Maximum entropy principle, diffusion process, stochastic differential equation, competition, return on assets, Subbotin distribution.

JEL codes: C16, L10, D21, E10.

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> Profit is so very fluctuating, that the person who carries on a particular trade, cannot always tell you himself what is the average of his annual profit. It is affected, not only by every variation of price in the commodities which he deals in, but by the good or bad fortune both of his rivals and of his customers, and by a thousand other accidents [...].
>
> Adam Smith (1776, p. 58)

1 Competition and Profitability

We propose a statistical equilibrium model that accounts for the empirical distribution of firm profit rates, which turns out to be well described by a Laplace distribution. Our findings have profound implications both for the specific time evolution of individual firm profitability and the previous 'persistence of profits' literature, as well as for the general understanding of competition as a dynamic process.

The notion of economic competition comes in many forms and varieties, and it is certainly one of the most pervasive concepts in the history of economic thought (see, e.g., Stigler, 1957; Vickers, 1995). The dominant strand of thought, following Cournot, associates (perfect) competition with a particular market form, and emphasizes the efficient allocation of resources at points where prices equal marginal costs (see, e.g., McNulty, 1968). Another important strand of thought originates with Adam Smith's notion of competition as a dynamic process that leads to a tendency for profit rate equalization, which we henceforth label as classical competition.[1] Classical competition essentially describes a negative feedback mechanism. Capital will seek out sectors or industries where the profit rate is higher than the economy-wide average, typically attracting labor, raising output, and reducing prices and profit rates, which in turn provides an incentive for capital to leave the sector, thereby leading to higher prices and profit rates for firms that remain in the sector (see, e.g., Foley, 2006). As a result, classical competition tends to equalize profit rates, yet it simultaneously leads to perpetual changes in technologies and competitive practices. Coupled with continually changing tastes of consumers, and the entry and exit dynamics of rival firms, the very nature of (classical) competition renders a complete elimination of differences in and across sectoral profit rates improbable.

Modeling the process of competition is made all the more difficult by the interactions among firms, which in themselves create a complex environment that feeds back into the destinies of individual companies. One company's

[1] Schumpeter's (1950) theory of innovation and creative destruction, or evolutionary theories of industrial dynamics (see, e.g., the edited volume by Dosi et al., 2000), also highlight the intrinsically dynamic character of economic competition and would be consistent with the notion of 'classical' competition from this viewpoint.

gain is often the loss of others, particularly in situations where resources are limited, for instance when it comes to the hiring of exceptional talent, the retainment and acquisition of clients, or the patenting of new technologies. Positive feedbacks, typically arising from symbiotic relationships and synergetic interactions, further increase the complexity of the competitive environment.

The interactions of competitive firms and their idiosyncratic efforts to stay ahead of the game give rise to an enormous amount of information and complexity that is hard to approach from a deterministic viewpoint. In light of the intricate connections and interactions among business firms, our focus shifts accordingly from a fixed-point equilibrium to the notion of a *statistical equilibrium* in the spirit of Foley (1994). Formally, Foley's statistical equilibrium theory of markets revolves around the *maximum entropy principle* (MEP) of Jaynes (1978). After all, MEP derives the combinatorially most likely (or informationally least biased) distribution of a random variate subject to moment constraints. Thus, instead of considering competitive equilibrium as a situation in which all economic agents face an identical profit rate, our statistical equilibrium model stresses the *stationary distribution of profit rates.*

Approaching the profitability of business firms from a probabilistic perspective is of course not unique to statistical equilibrium modeling, but rather follows a long-standing tradition that revolves around distributional regularities in a wide range of socio-economic variables (see, e.g., Champernowne, 1953; Gibrat, 1931; Kalecki, 1945; Pareto, 1897; Simon, 1955; Steindl, 1965). In order to apply the maximum entropy formalism to any kind of economic phenomenon, one essentially needs to encode the economic content in terms of moment constraints (see, e.g., Castaldi and Milaković, 2007; Foley, 1994; Stutzer, 1996). Hence, modeling classical competition by way of MEP boils down to expressing competition in the form of moment constraints. We take the position that the average profit rate corresponds to a measure of central tendency, while the complex movements of capital in search of profit rate equalization and the resulting feedback mechanisms translate into a generic measure of dispersion around the average. When the number of competitive firms in a decentralized type of market organization is large, probabilistic factors can give rise to statistical regularities in the distribution of profit rates. The distribution of profit rates that can be achieved in the most evenly distributed number of ways under the dispersion constraint is then the statistical equilibrium or maximum entropy distribution of profit rates, and turns out to be an *exponential power* or *Subbotin* distribution.

The Subbotin (1923) distribution has three parameters: a location, a scale, and a shape parameter. Structural differences in the statistical equilibrium model stem from differences in the shape parameter, because operating on the location or scale parameter does not change the qualitative features of the Subbotin distribution. If the shape parameter is equal to two, the Subbotin distribution reduces to the Gaussian (normal) distribution, and if it is equal to unity, the Subbotin distribution reduces to the Laplace (double-exponential) distribution.

To demonstrate the empirical relevance of our model, we investigate the empirical density of profit rates, which is indeed reasonably described by a Laplace distribution. This prompts us to ask why the empirical shape parameter is close to unity, what this implies about the competitive environment that firms are facing, and whether variations in the shape parameter correspond to qualitative changes in the competitive environment. Since the maximum entropy principle only informs us of the stationary distribution, it does not shed light on the *dynamics* that lead to the stationary distribution. In order to extend the model in a dynamic direction, we utilize a particular class of stochastic processes known as *diffusion processes*, and construct a diffusion process that has the Subbotin as its stationary density. The rationale for resorting to diffusion processes is twofold. First, the process is parsimoniously described by only two functions, the so-called *drift* and *diffusion function* and, second, a considerable analytical apparatus relating to diffusion processes is already in place. This diffusion process will be introduced heuristically at first, starting from the assumption that the Subbotin distribution is the stationary distribution. Since the arising drift function has a singularity at m, we also provide a rather careful mathematical treatment of this process in Appendix A.

Examination of the diffusion process extends the maximum entropy results in two important ways. First, it provides additional insights into variations of the shape parameter of the stationary distribution. We show that the benchmark Laplace case, where the shape parameter equals unity, corresponds to a drift term that is independent of a firm's current state of profitability, implying that competition is a 'global' mechanism that acts with equal force on all companies, irrespective of a firm's particular deviation from the average rate of profit. Second, the diffusion process shows that the complex mechanisms of competition *simultaneously* generate (i) the fluctuations in the destinies of individual companies and (ii) the drift towards an average profit rate. Thus competition cannot be described by a deterministic skeleton with superimposed noise, because the drift function depends on the scale of fluctuations in the diffusion function. Put differently, switching off the noise in the diffusion process also eliminates the systematic drift towards the average rate of profit. Viewed from this perspective, classical competition becomes a truly stochastic phenomenon, where the fluctuations of individual destinies and the dissipation of profitable business opportunities are two sides of the same coin.

Finally, our findings also have a direct bearing on the *persistence of profits* (PP) literature that started with Mueller (1977). Studies of PP (see, e.g. Geroski and Jacquemin, 1988; Glen et al., 2001, 2003; Goddard and

Wilson, 1999; Gschwandtner, 2005; Kambhampati, 1995; Maruyama and Odagiri, 2002; Mueller, 1990) employ a first-order auto-regressive process as its standard workhorse, which in light of our results is clearly misspecified because the (stationary) process is the discrete-time analogue of the Ornstein-Uhlenbeck process, whose stationary distribution is Gaussian and therefore counter-factual. What PP methodology essentially lacks is the realization that the variance of the 'error term' (noise) is crucially intertwined with the 'speed of convergence' (drift function) towards the 'norm' (average profit rate), which only becomes apparent once we focus our attention on the distribution of profit rates.

2 Maximum Entropy Distribution of Profit Rates

We view profit rates as an inherently stochastic phenomenon, and take the position that competition among firms disperses their profit rates, denoted x, around an exogenously given measure m of central tendency. More formally, we assume that dispersion is measured by the *standardized α-th moment*, $\sigma^\alpha = E|x-m|^\alpha$, with $x, m \in \mathbb{R}$ and $\alpha, \sigma > 0$. At first, the assumption that the complexities of economic competition disperse profit rates around some average rate does not seem to get us anywhere. But by further assuming that in the absence of further information all profit rate outcomes around m are most evenly distributed, MEP establishes a correspondence between the moment constraint and a statistical distribution (see Jaynes, 1978).

Formally, MEP under a standardized α-th moment constraint defines a variational problem that maximizes the entropy $H[f(x)]$ of the profit rate density $f(x)$, defined as

$$H[f(x)] \equiv -\int_{-\infty}^{+\infty} f(x) \log f(x)\ dx, \tag{1}$$

subject to the constraint on the standardized α-th moment,

$$\int_{-\infty}^{+\infty} f(x) \left| \frac{x-m}{\sigma} \right|^\alpha\ dx = 1\,. \tag{2}$$

and subject to the natural constraint that normalizes the density,

$$\int_{-\infty}^{+\infty} f(x)\ dx = 1\,. \tag{3}$$

Proposition 1. *The maximum entropy distribution of profit rates under the standardized α-th moment constraint* (2) *is a* Subbotin distribution*,*

$$f(x; m, \sigma, \alpha) = \frac{1}{2\sigma\alpha^{1/\alpha}\Gamma(1+1/\alpha)} \exp\left(-\frac{1}{\alpha}\left|\frac{x-m}{\sigma}\right|^\alpha\right). \tag{4}$$

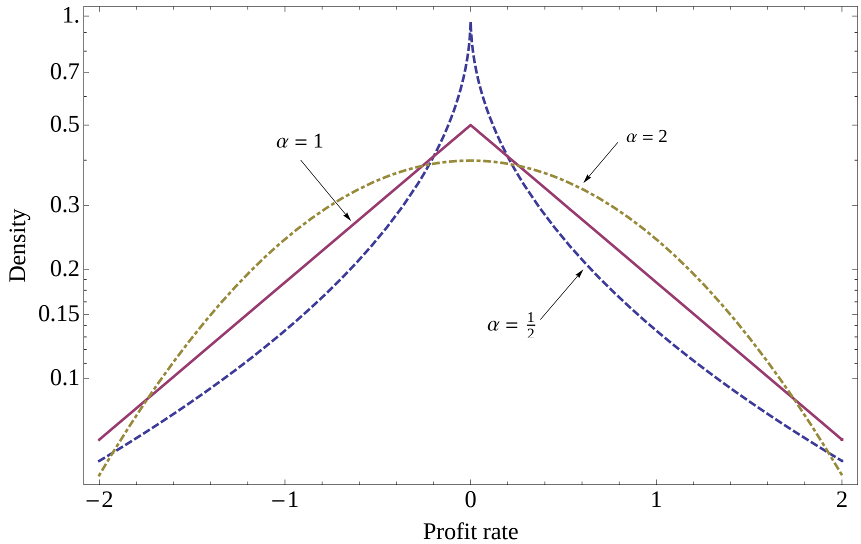

Figure 1: Subbotin distribution with $m = 0$, $\sigma = 1$, and varying shape parameter α. On semi-log scale, the Laplace distribution ($\alpha = 1$, solid curve) has linear slope while the Gaussian ($\alpha = 2$, dash-dotted curve) becomes a parabola.

Proof. The Lagrangian associated with the variational program (1)–(3) is

$$\mathcal{L} = H[f(x)] - \mu \left[\int_{-\infty}^{+\infty} f(x)\, dx - 1 \right] - \lambda \left[\int_{-\infty}^{+\infty} \left| \frac{x-m}{\sigma} \right|^{\alpha} f(x)\, dx - 1 \right],$$

where μ and λ denote the multipliers. Letting $\xi \equiv 1 + \mu$, the first order condition implies that the solution will have the functional form

$$f(x) = \exp(-\xi) \cdot \exp\left(-\lambda \left| \frac{x-m}{\sigma} \right|^{\alpha} \right). \tag{5}$$

Integrating by substitution in order to invert the constraints, and using the definition of the gamma function, Eq. (3) yields the normalizing constant, or *partition function*,

$$\exp(-\xi) = \frac{1}{2\sigma} \frac{1}{\alpha^{1/\alpha}\Gamma(1 + 1/\alpha)}, \tag{6}$$

and consequently Eq. (2) yields

$$\lambda = \frac{1}{\alpha}. \tag{7}$$

Since $f(x)$ is a positive function, $\partial^2\mathcal{L}/\partial f(x)^2 = -1/f(x) < 0$, and the solution is a maximum. □

The Subbotin distribution (4), illustrated in Figure 1, is characterized by a location parameter m, a scale parameter $\sigma > 0$, and a shape parameter $\alpha > 0$. If α is smaller (greater) than two, the distribution is leptokurtic (platykurtic). If $\alpha = 1$ the Subbotin reduces to the Laplace distribution, if $\alpha = 2$ it reduces to the Gaussian, and if $\alpha \to \infty$ it tends to a uniform. If $\alpha \to 0$, the statistical equilibrium distribution turns into Dirac's δ-distribution at m, including the more conventional competitive equilibrium concept of a situation in which each firm 'faces' an identical profit rate as a special case.[2]

From a methodological viewpoint, statistical equilibrium modeling is less ambitious than conventional Walrasian theory because it does not seek, nor is it able, to predict the actual profit rate outcome for each individual business firm. On the other hand, the statistical equilibrium approach is capable of translating a parsimonious description of the system, given by the dispersion constraint (2), into the distributional outcome (4). The distribution is a *stationary* or *statistical equilibrium outcome* in the sense that it measures the competitive tendency for profit rate equalization on a characteristic time scale that is large enough to accommodate the time scale of idiosyncratic shocks.

Paraphrasing Foley's economic interpretation of MEP, the outcome of the particular maximum entropy program (1)–(3) corresponds to the profit rate distribution that arises from the *most decentralized activity of competitive firms*. Business firms typically engage in a plethora of competitive strategies that aim more or less directly at the maximization of profit, for instance by seeking increases in market share or revenues through product differentiation, price undercutting, advertising, customer relationship management, etc. In addition, firms might simultaneously or separately seek to reduce costs by downsizing operations, by exploiting increasing returns to scale, or by adopting or inventing cost-cutting technologies. It is exactly in the presence of such complex and multi-dimensional environments that MEP comes into its own. While MEP cannot identify the impact of particular competitive strategies, all such strategies, along with the ensuing complex feed-back mechanisms, are in principle included in the statistical equilibrium outcome of Proposition 1. The only prerequisite for interpreting the MEP distribution as the outcome of the most decentralized economic activity under the dispersion constraint (2), or as the outcome that can be achieved in the most evenly distributed number of ways under the dispersion constraint, is that the number of firms in the economy is large (see Foley, 1994). Statistical equilibrium modeling thus excludes situations of system-wide collusion, which in any case should become increasingly difficult to realize as the number of firms increases.

[2]From the viewpoint of entropy maximization, this particular case is the most improbable of all feasible results because it has a multiplicity of unity, which generally applies to unique competitive Walrasian equilibria (see Foley, 1994).

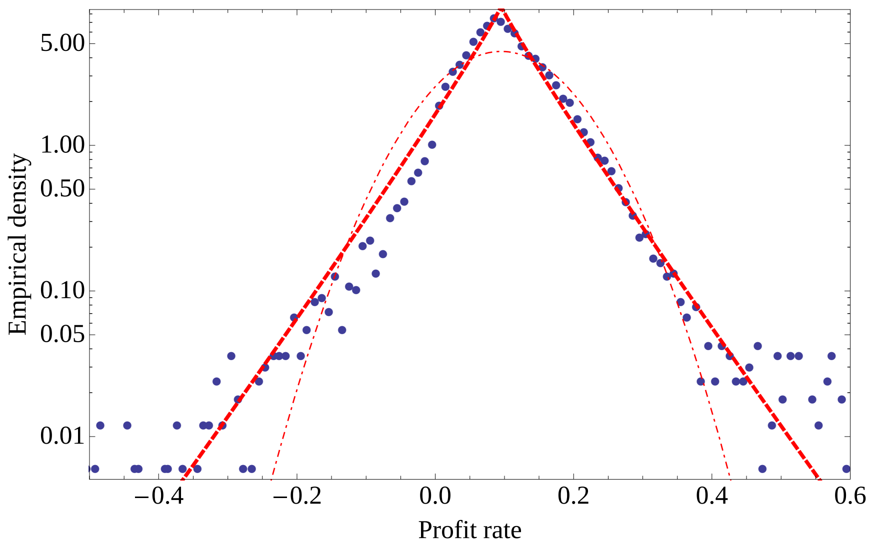

Figure 2: Pooled empirical density of annual profit rates (16 821 observations for the period 1980-2006), measured by the ratio of operating income over total assets, for 623 publicly traded non-bank companies in the United States. The median profit rate is $m = 9.5\%$ over this time span, and coincides with the mode of the empirical density. Maximum likelihood estimation of the Subbotin shape parameter yields $\alpha = 0.94 \pm 0.01$ with a scale parameter $\sigma = 0.0577 \pm 0.0006$. The (thick) dashed curve illustrates the corresponding fit, while the (thin) dash-dotted curve shows a Gaussian fit using the sample mean and standard deviation.

3 Empirical Distribution of Profit Rates

MEP cannot provide information about the individual destinies of companies, yet it manages to associate a distributional outcome with the dispersion constraint that presumably reflects the behavioral process of competition. The usefulness of such a theoretical prediction naturally depends on how well it describes the empirical profit rate distribution. Our data are from *Thomson Datastream* and consist of annual observations for 623 US publicly traded non-bank companies (operating in 36 different sectors on a two-digit classification level) that are present in every year during the period 1980-2006.[3] We calculate annual profit rates as the ratio of operating income to

[3]The entire sample of companies at our disposal consists of 9025 firms, yet our focus on long-lived companies and the corresponding sample size is consistent with most studies in the PP literature. Notice that the sum of total assets of long-lived companies accounts for well over half of the overall sum of total assets in each year. Even so, we also briefly discuss our preliminary results regarding the entire sample in the last section. We excluded banks because their balance sheets differ structurally from those of non-banks. It is noteworthy,

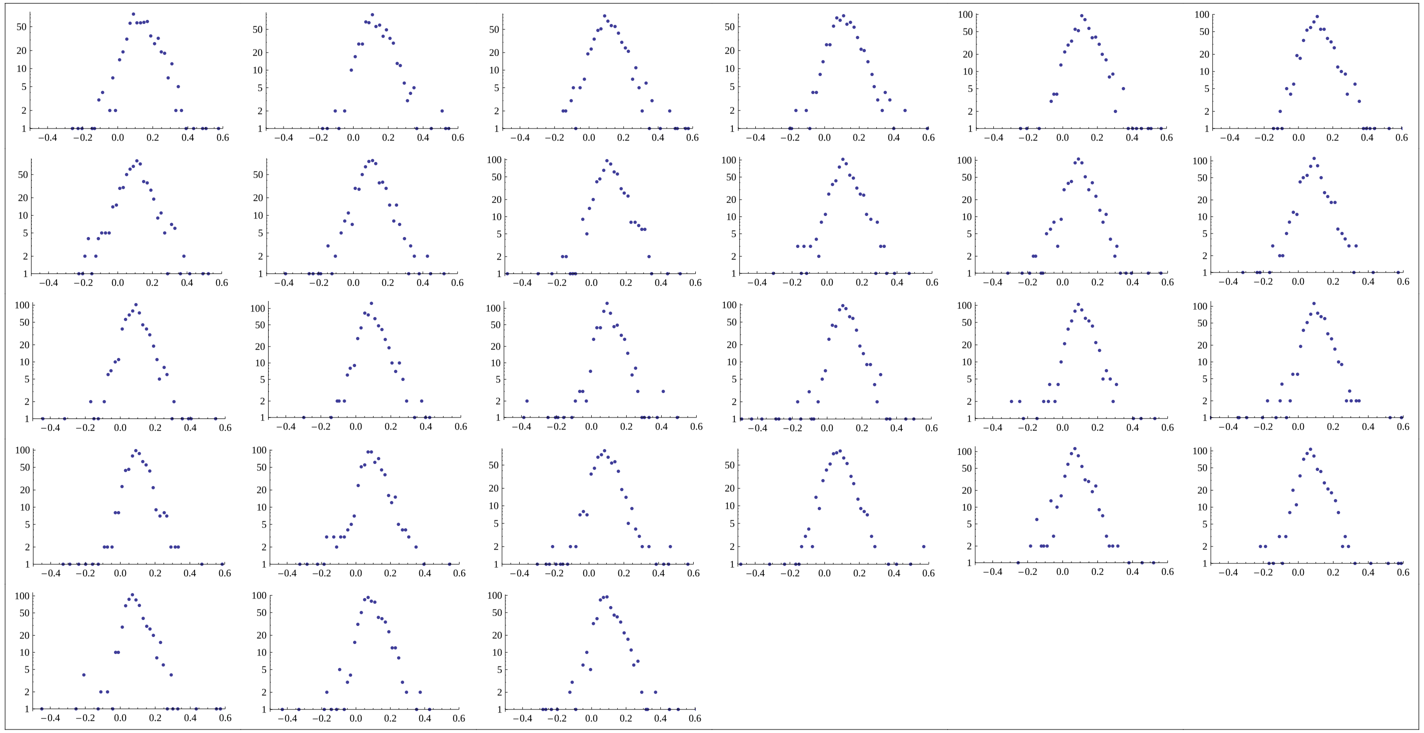

Figure 3: Year-by-year distribution of profit rates, 1980–2006 (top left to bottom right). Casual inspection reveals the dominance of tent-shaped distributions on semi-log scale around a constant mode, illustrating that Figure 2 is not an artifact of aggregation.

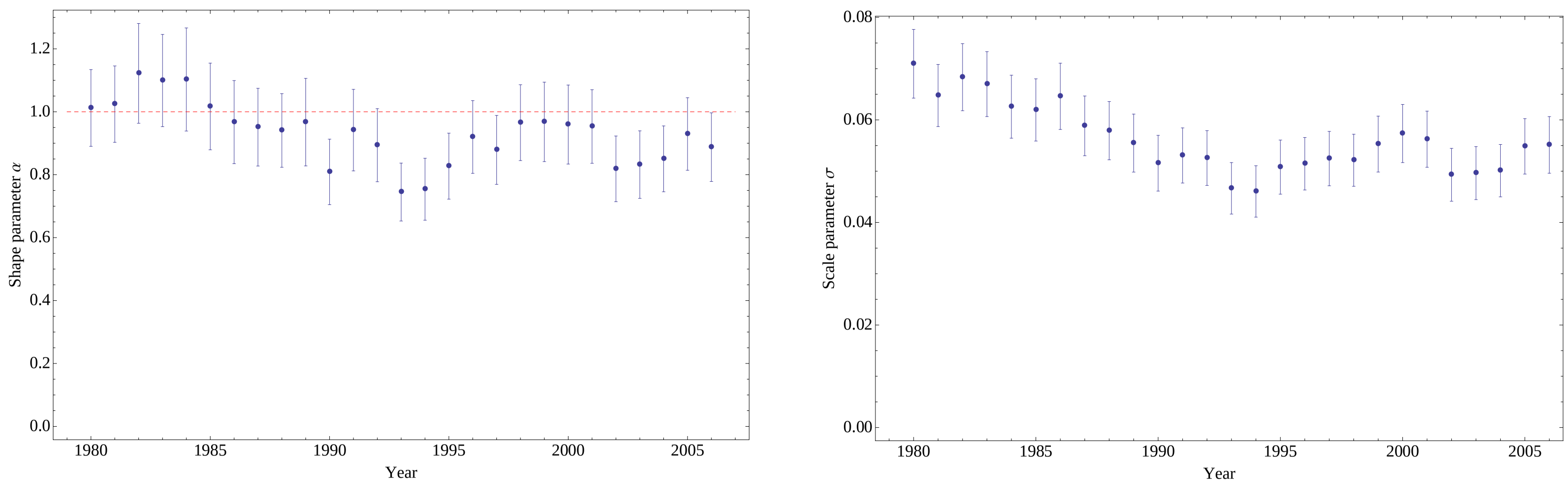

Figure 4: Year-by-year maximum likelihood estimates of the shape parameter α (left panel) and scale parameter σ (right panel). The left panel illustrates that in twenty out of twenty-seven years the estimated shape parameter is consistent with a Laplace distribution ($\alpha = 1$) within a 95% confidence interval. Notice that both parameter estimates do not fluctuate erratically but rather exhibit slightly cyclical behavior on a characteristic scale, further testifying to the usefulness of the Laplace distribution as a benchmark case.

total assets, and start out by plotting the empirical density of pooled profit rates in Figure 2. It is a rather remarkable feature of Figure 2 that the pooling of profit rates across all years results in a sharply peaked uni-modal distribution despite the fact that these are *raw* data, which have not been normalized or standardized in any way. The maximum likelihood estimate of the shape parameter is close to unity for the pooled data and points towards a Laplace distribution of profit rates.

To ensure that these findings are not an artifact of aggregation, we consider the year-by-year distribution of profit rates as well, shown in Figure 3, and we also estimate the corresponding year-by-year Subbotin parameters with maximum likelihood, whose time evolution is shown in Figure 4. Both figures rebut potential concerns that the pooled Laplace benchmark in Figure 2 is due to aggregation. On one hand the year-by-year distributions of profit rates are tent-shaped on semi-log scale, as in the pooled case, and on the other hand both the shape and scale parameters do not fluctuate erratically over time. In addition, the shape parameter is not significantly different from a Laplacian at the five percent level in almost seventy-five percent of the cases.

In light of the preceding evidence, the Laplace distribution would appear to represent a reasonable benchmark case for the empirical density of profit rates, begging the question what a shape parameter close to unity implies about the competitive environment in which firms are operating. More generally, what kind of qualitative changes in the competitive environment could be reflected in significant deviations of α from unity? Such questions, however, are hard to answer with MEP because the principle offers little in the direction of an economic interpretation of the parameters α and σ. Hence we extend the statistical equilibrium model into a dynamical setting by considering a diffusion process whose stationary distribution will be given by the Subbotin density.

4 The Dynamic Evolution of Profit Rates

There are essentially three reasons why we take recourse to diffusion processes among the much broader class of stochastic processes to describe the dynamic evolution of profit rates $\{X_t, t \geq 0\}$. First, a diffusion is parsimoniously described by two functions, the *drift* and the *diffusion function.* Second, an analytical apparatus relating for instance to existence and uniqueness theorems is available for diffusions, and third, a simple closed-form solution for the stationary distribution turns out to exist in our case of interest.

though, that the pooled distribution of bank profit rates is also close to a Laplacian, with $\alpha = 0.91 \pm 0.03$ and $\sigma = 0.0052 \pm 0.0001$, and with a mode of $m = 1.5\%$. The material is available upon request.

We consider a *time-homogeneous diffusion* on the real line, which takes the general form

$$dX_t = A(X_t)\,dt + \sqrt{D(X_t)}\,dW_t \ , \tag{8}$$

where $A(x)$ and $D(x) > 0$ denote the drift and diffusion function, and dW_t denotes Wiener increments. A diffusion thus decomposes the profit rate increment dX_t into two factors: a random term governed by the diffusion function, and a systematic effect captured by the drift term, both of which are due to the complex and continually evolving environment that business firms create, as we will argue shortly. Finally, from an economic point of view, the assumption of a time-homogeneous diffusion implies that the nature of the underlying competitive mechanism is time invariant.

Our strategy is to heuristically construct a diffusion that has the Subbotin density as its stationary distribution, and to demonstrate subsequently with mathematical rigour that this indeed yields a regular diffusion on the real line. Regularity here means that from any starting point x any other real y is reached in finite time with positive probability. If a stationary distribution with density $p_e(x)$ to the diffusion process (8) exists,[4] it obeys (in most cases of interest) the textbook formula

$$p_e(x) = \frac{\kappa}{D(x)} \exp\left(2\int_{x_0}^{x} \frac{A(y)}{D(y)} dy\right) \ , \tag{9}$$

where κ is the normalizing constant. We will subsequently show that this is indeed the case in our situation. Here x_0 may be chosen freely and κ of course depends on x_0. Therefore, Eq. (9) serves to establish a relationship between our stationary distribution of interest, and the drift and diffusion function that we want to identify. Knowledge of the functional form of the stationary distribution is, however, not sufficient to uniquely characterize the diffusion process since there is still a degree of freedom. Following the principle of parsimony, we opt to exploit this degree of freedom in a simple manner by assuming a constant diffusion function $D(x) = D$, meaning that idiosyncratic shocks are independent of the current state of a firm's profit rate.[5] Then, straightforward manipulation of Eq. (9) uniquely expresses the drift $A(x)$ as a function of the stationary distribution and its derivative $p_e'(x)$,

$$A(x) = \frac{D}{2}\frac{p_e'(x)}{p_e(x)} \ . \tag{10}$$

[4]The existence of a stationary distribution, in fact, implies additional conditions on the drift and diffusion, and a full characterization of the process at the boundaries for the different values of the underlying parameters, which we consider in Appendix A.

[5]Alternatively one could, for instance, prescribe a linear drift term and then construct diffusion functions that yield particular stationary distributions of interest Bibby et al. (2005).

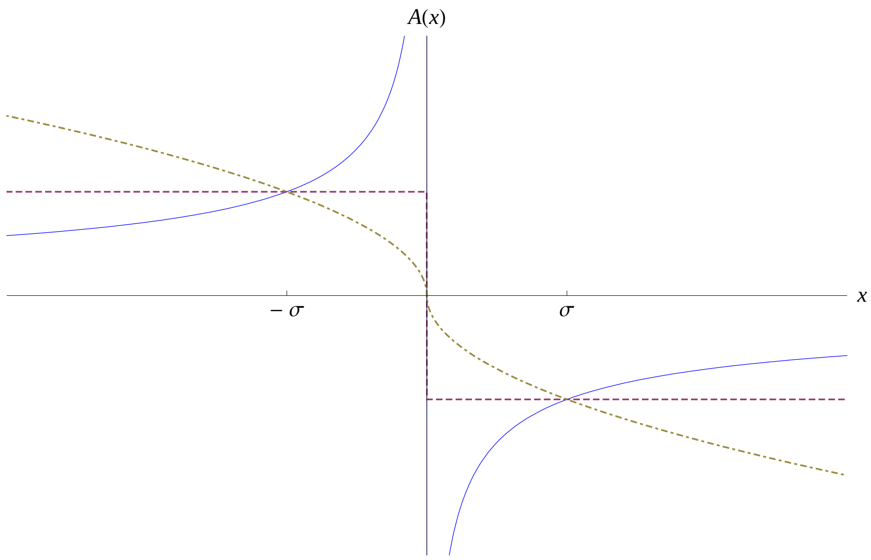

Figure 5: The qualitative behavior of the drift function $A(x)$, given by Eq. (11), depends on the parameter α. The axes originate in $(m, 0)$, and the curves are plotted for $\alpha = 0.5$ (solid curve), $\alpha = 1$ (dashed), and $\alpha = 1.5$ (dash-dotted). The strength with which profit rates are pulled back towards m is equal for all parametrizations of $\alpha > 0$ when $x = \pm\sigma$.

Hence, utilizing the functional form of the Subbotin distribution (4) in Eq. (10), we obtain the drift function

$$A(x) = -\frac{D}{2\sigma}\,\mathrm{sgn}(x-m)\left|\frac{x-m}{\sigma}\right|^{\alpha-1}\,, \tag{11}$$

where $\mathrm{sgn}(\cdot)$ denotes the signum function, and $A(m) = 0$. This result motivates the following proposition:

Proposition 2. *The stochastic differential equation*

$$dX_t = -\frac{D}{2\sigma}\,\mathrm{sgn}(X_t-m)\left|\frac{X_t-m}{\sigma}\right|^{\alpha-1} dt + \sqrt{D}\,dW_t \tag{12}$$

defines a regular diffusion on the real line for all $\alpha, \sigma > 0$ and $m \in \mathbb{R}$, with a Subbotin stationary distribution given by (4).

Proof. See Appendix A. □

Our economic interpretation of the dynamic evolution of profit rates rests on the assumption that all firms are subject to the same process (12), possibly with different diffusion constants, since $p_e(x)$ is independent of D, but with identical parameters α, σ and m. Then each firm's destiny corresponds to a different realization of (12), such that the stationary distribution represents the cross-sectional statistical equilibrium outcome (4) arising from the interactions of competitive firms. Put differently, the diffusion process decomposes the complexities of a competitive environment into a drift and diffusion function, whereby the latter captures idiosyncratic factors, while the former describes the systematic tendency for competition to equalize profit rates. Figure 5 illustrates that this mean-reverting drift towards m is generally non-linear, and depends qualitatively on the value of α.

5 Diffusion, Dispersion, and the Process of Competition

Viewed from the perspective of the diffusion process, deviations of the empirical shape parameter from unity measure qualitative changes in the economic environment created by competitive firms. If $\alpha > 1$, the systematic force towards profit rate equalization becomes stronger the further profit rates deviate from m, and symmetrically, if $\alpha < 1$, this force becomes weaker the further profit rates deviate from m. In a more applied setting, it would probably pay off to study the defining characteristics of sectors that show deviations of α in either direction in order to understand why certain industries are more or less prone to large deviations of profit rates from the average. A firm that operates in an environment where $\alpha < 1$, and succeeds in being very profitable at a given point in time, should look more optimistically into the future than a firm whose profitability is equally far from the average, but which operates in an environment where $\alpha > 1$. Looking at profitability from this angle suggests that α is an aggregate measure of competitive pressures within and across industries.

Notably, an equilibrium Laplace distribution ($\alpha = 1$) is obtained from the diffusion

$$dX_t = -\frac{D}{2\sigma} \ \mathrm{sgn}(X_t - m) \ dt + \sqrt{D} \ dW_t \ , \tag{13}$$

showing that the empirical benchmark case corresponds to a scenario in which the drift is constant, and therefore independent of a firm's current profit rate. To be more precise, the magnitude of the drift is constant while the dependence on the profit rate is incorporated into the signum function, thus yielding the only case where the Subbotin diffusion is independent of the current *level* of the profit rate. Consequently, the extension of MEP into a diffusion model provides a further step in the direction of understanding the peculiarity of the empirically observed Laplace case, because it reveals that competitive pressures act with equal force on all companies irrespective of their current profitability.

Furthermore, if $\alpha = 0$ the diffusion turns into a particular case of a *Bessel process*, with an equilibrium δ-distribution at m. Actually, Karlin and Taylor

(1981, Example 6, pp. 238–9) show that the point m then behaves as an *exit boundary* with total absorption in finite time. Here, the case $\alpha = 0$ leads to a change in the nature of the diffusion's boundary condition, whereas MEP relates this case to an outcome with minimal multiplicity. None the less both, the diffusion and MEP, highlight the peculiarity of a situation in which all firms are equally profitable.

The most salient point of our model is that the level of idiosyncratic noise D turns up in the drift $A(x)$, given by (11). Hence, our diffusion model decomposes the metaphor of competition into the *contemporaneous* presence of individual fluctuations and a systematic tendency towards profit rate equalization. Redefining the coefficients of the drift and diffusion as

$$\mu = \frac{D}{2\sigma^\alpha} \qquad \text{and} \qquad \lambda = \sqrt{D} \, , \tag{14}$$

we obtain the fundamental relationship

$$\frac{\lambda^2}{2\mu} = \sigma^\alpha \, , \tag{15}$$

which adeptly ties up the diffusion model with the entropy formalism, since the Subbotin distribution arises from MEP if we prescribe the dispersion

$$\sigma^\alpha = E\,|x - m|^\alpha \, . \tag{16}$$

It is the simultaneous and inseparable presence of individual fluctuations and a mean-reverting drift towards m that ultimately leads to the emergence of an equilibrium distribution. Strikingly, Eq. (15) reveals that the dispersion of profit rates measures the relative strength of one effect over the other.

Our pre-analytical vision of competition as a complex feed-back mechanism results in the diffusion (12), and as a consequence methodologically rules out a deterministic skeleton with some added noise on top of it. The introductory quote from Smith already illuminates the intrinsically random and inter-connected nature of competition among economic agents, highlighting that the success of one firm cannot be attributed to its effort alone, but crucially depends on what other agents are doing as well. Therefore Eq. (12) does not represent the fate an atomistic firm might desire for itself, but rather demonstrates the impossibility of such an endeavor in a competitive environment.

It is precisely the intertwined presence of centrifugal and centripetal forces in the process of competition, whereby the competitive practices of individual firms lead to the dissipation of profitable business opportunities, that is absent from PP methodology. Typically, the econometric workhorse of PP consists of an AR(1) process of profit rates of the form

$$x_{i,t} = \lambda_i x_{i,t-1} + \beta_i + \varepsilon_{i,t} \, , \tag{17}$$

where β_i represents the 'permanent' profit of firm i, possibly being zero, and $\varepsilon_{i,t}$ designates random shocks to firm i's profitability (see, e.g., Cable and Mueller, 2008, for a recent review of the subject). Rewriting (17) in first-difference form yields

$$\Delta x = x_{i,t} - x_{i,t-1} = -(1 - \lambda_i)x_{i,t-1} + \beta_i + \varepsilon_{i,t}\,, \tag{18}$$

which corresponds to the discrete-time analogue of the well-known Ornstein-Uhlenbeck process if $0 \leq \lambda_i < 1$ such that the process is stationary.[6] Since the stationary distribution of the Ornstein-Uhlenbeck process is Gaussian, an AR(1) process is clearly misspecified because the distribution of profit rates is markedly non-Gaussian, as we have shown in Section 3. The fundamental difference between PP and statistical equilibrium modeling is that the former focuses on the time-series behavior of individual firms, addressing the question whether there is individual convergence towards some profit rate 'norm,' while the latter starts from the distributional properties of a cross-section of firms and stresses the concept of a convergence in distribution.

6 Discussion and Conclusion

To capture the stochastic and intertwined aspects of competition, we have proposed a statistical equilibrium model that starts from a dispersion constraint, motivated by the notion of classical competition, which MEP translates into a Subbotin distribution of profit rates. Extending the statistical equilibrium model to a diffusion that has an equilibrium Subbotin distribution, we are then able to decompose the process of competition into two interdependent terms, the drift and diffusion function, which respectively capture the systematic tendency towards profit rate equalization on the one hand, and idiosyncratic factors on the other. As it turns out, dispersion measures the relative strength of these two effects.

Essentially, our model considers the distribution of profit rates as a statistical equilibrium outcome arising from the decentralized complex interactions of competitive firms, and the corresponding diffusion suggests that the empirical benchmark of a Laplace distribution represents a collection of firms whose interactions create a 'competitive field' that influences individual firms independently of their current profit rate.

From a methodological viewpoint, our diffusion model reveals that the process of competition is an inherently stochastic phenomenon, because the

[6] The studies of Goddard and Wilson (1999) or Kambhampati (1995) cannot even reject the unit root hypothesis in a substantial number of cases, yet a simple inspection of profit rate time series already reveals the mean-reverting nature of the process. More formally, the absence of a mean-reverting drift would imply a linear increase of the variance in the *distribution* of profit rates, which is rebutted by our results in Figures 2 and 3.

level of idiosyncratic fluctuations enters the systematic tendency for profit rate equalization. Thus it is not possible to switch off the idiosyncratic noise without eliminating the systematic drift towards an average profit rate, casting some doubt on models that approach competition from a purely deterministic perspective, as well as on those that fail to capture the interaction effects of a single firm with the competitive environment created by the economy-wide ensemble of firms.

Needless to say that our model is far from being complete. By treating the average profit rate as an exogenous variable, we have effectively eliminated its determining factors from consideration. Nevertheless it is quite remarkable in our opinion that the average profit rate, measured by the mode of the distribution, is constant over a period of almost three decades at about ten percent. Another important aspect that needs to be addressed in future work concerns the systematic influence of firm entry and exit. The empirical results we have presented here refer to the subset of long-lived companies, which provides an important yet partial description of the macroscopic properties of competition. We can consider long-lived firms as a kind of measurement device that is 'dipped' into the competitive environment as a whole, and therefore captures the effects of entry and exit dynamics at least indirectly. At this point we have conducted a preliminary analysis that includes all publicly traded US companies in the Datastream database, which reveals that the pooled profit rate distribution across all firms and years is highly asymmetric around the mode of ten percent. It is straightforward to generalize the Subbotin distribution with respect to asymmetries by adding another shape and scale parameter such that one set of shape and scale parameters respectively describes the distribution to the left and to the right of the mode, say α_r, σ_r and α_l, σ_l. While the shape parameter for observations to the right of the mode remains Laplacian ($\alpha_r \approx 1$), we find that it is much smaller for observations that are to the left of the mode ($\alpha_l < 1$), also implying that the distribution has considerably more mass to the left of the mode. Intuitively, the asymmetry tells us that it is much easier for firms to fail than to be successful over long periods of time. These findings certainly call for a theoretical extension of our statistical equilibrium model to explicitly include firm entry and exit dynamics, and to explain the observed asymmetries in the competitive process.

Last but not least, there is also evidence that the cross-sectoral distribution of firm growth rates is Laplacian (see Alfarano and Milaković, 2008; Stanley et al., 1996), as are many distributions on the sectoral level, though some deviations from the Laplace distribution do show up on the sectoral level as well (see Bottazzi and Secchi, 2006). Notice, however, that growth and profit rates are dimensionally very different quantities. Growth rates are time differences in (logarithmic) firm size, while profit rates are measured by the ratio of operating income to total assets during a time period. Firms sometimes have to shrink in order to restore profitability, while at other times they might become unprofitable by expanding the size of operations in an effort to increase their market power, which possibly restores profitability in future periods. These simple examples illustrate that both the direction of causality and the level of correlation between growth and profit rates are far from trivial, and we are currently investigating this relationship in more detail.

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A Proof of Proposition 2

A.1 General considerations

We recall the definition of

$$A(x) = -\frac{D}{2\sigma}\operatorname{sgn}(x-m)\left|\frac{x-m}{\sigma}\right|^{\alpha-1}$$

with $A(m) = 0$. If $\alpha = 2$, then $A(x)$ is Lipschitz continuous, which is the usual condition for the existence of a regular diffusion on the real line as a solution to the stochastic differential equation (12). The arising diffusion is the well-known Ornstein-Uhlenbeck process. Lipschitz continuity no longer holds for $\alpha \neq 2$. The case $0 < \alpha < 2$ is the more intricate one because there is a singularity at m for $\alpha < 1$, and we will consider it in detail in sections A.2 to A.6. We shortly remark on the case $\alpha > 2$ in section A.7. For easier notation we henceforth use $m = 0, \sigma = 1$.

A.2 The diffusion on $(0, \infty)$

First we obtain a solution to (12) on the positive half-line. For $\alpha = 1$, $A(x)$ is constant on $(0, \infty)$. Hence, according to A.1, we obtain a diffusion on $(0, \infty)$ that solves (12). For $0 < \alpha < 2, \alpha \neq 1$, $A(x)$ is not Lipschitz continuous on $(0, \infty)$ due to the behaviour in $x = 0$. To obtain a diffusion on $(0, \infty)$ we apply the usual localization argument. For each $n \in \mathbb{N}$ we choose a bounded Lipschitz continuous function $A_n(x)$ on $(0, \infty)$ such that $A_n(x) = A(x)$ for $x \in (\frac{1}{n}, \infty)$. Then we solve (12) with $A_n(x)$ instead of $A(x)$. This yields a diffusion Y_t^n. Here Y_t^{n+1} extends Y_t^n in the way that they are equal (with probability one) up to the random time when one of them leaves the state space interval $(\frac{1}{n}, \infty)$. Hence they may be glued together to define a regular diffusion Y_t on $(0, \infty)$ that solves (12). To extend this to a diffusion on the entire real line, it is necessary to investigate the boundary behaviour at 0 utilizing the *scale* and *speed measure*.

A.3 Scale and speed measure

In general, the scale function and scale measure are given by

$$S(x) = \int_{x_0}^{x} s(y)dy \text{ with } s(y) = \exp\left(-\int_1^y \frac{2A(z)}{D(z)}dz\right), \ x \in (0, \infty),$$

$$S[a, b] = S(b) - S(a), \ 0 < a < b < \infty.$$

Here any $x_0 \in (0, \infty)$ may be inserted, and subsequently we will use $x_0 = 1$. The speed density and speed measure are given by

$$m(x) = \frac{1}{D(x)s(x)}, \ x \in (0, \infty), \ M[a, b] = \int_a^b m(y)dy, \ 0 < a < b < \infty.$$

For the boundary 0, we define

$$S(0, b] = \lim_{a \downarrow 0} S[a, b], \ M(0, b] = \lim_{a \downarrow 0} M[a, b].$$

Obviously $0 < S[a, b] < \infty$, $0 < M[a, b] < \infty$ for all $0 < a < b < \infty$, and we compute

$$s(y) = \exp\left(-\int_1^y (-1)z^{\alpha-1}dz\right) = \exp\left(\frac{y^\alpha}{\alpha} - \frac{1}{\alpha}\right),$$

$$S(0, 1] = \int_0^1 s(y)dy = \int_0^1 \exp\left(\frac{y^\alpha}{\alpha} - \frac{1}{\alpha}\right) dy < \infty,$$

thus $S(0, b] < \infty$ for all $0 < b < \infty$. Similarly,

$$m(y) = \frac{1}{D} \exp\left(-\frac{y^\alpha}{\alpha} + \frac{1}{\alpha}\right),$$

$$M(0, 1] = \frac{1}{D} \int_0^1 \exp\left(-\frac{y^\alpha}{\alpha} + \frac{1}{\alpha}\right) dy < \infty,$$

hence $M(0, b] < \infty$ for all $0 < b < \infty$. Furthermore,

$$S[a, \infty) = \lim_{b \uparrow \infty} S[a, b] = \infty, \ M[a, \infty) < \infty \text{ for all } 0 < a < \infty.$$

A.4 Boundary behaviour

For any arbitrarily chosen $a > 0$, let

$$\Sigma(0) = \int_0^a M[y,a]dS(y),\ N(0) = \int_0^a S[y,a]dM(y),$$

$$\Sigma(\infty) = \int_a^\infty M[a,y]dS(y),\ N(\infty) = \int_a^\infty S[a,y]dM(y).$$

Using Karlin and Taylor (1981, Lemma 6.3, Chapter 15), we obtain from A.3

$$\Sigma(0) < \infty,\ N(0) < \infty \text{ and } \Sigma(\infty) = \infty,$$

and an easy argument shows that

$$N(\infty) = \int_a^\infty \int_a^y \frac{1}{D} \exp\left(\frac{z^\alpha}{\alpha} - \frac{1}{\alpha}\right) dz \exp\left(-\frac{y^\alpha}{\alpha} + \frac{1}{\alpha}\right) dy = \infty.$$

In the terminology of Karlin and Taylor (1981, p. 234), ∞ is a natural boundary (as ∞ is for Brownian motion) and can be omitted from the state space, whereas 0 is a regular boundary. A regular boundary can be added to the state space. To specify the behavior in 0, we set $M(\{0\}) = 0$ which stands for instant reflection. So we have defined a diffusion Y_t with state space $[0,\infty)$ that is a solution of (12) and is immediately reflected when it reaches 0. Using Karlin and Taylor (1981, pp. 192–197), one can show that Y_t reaches 0 with probability one in finite expected time from any starting point x.

A.5 The diffusion on $(-\infty, \infty)$

Having defined Y_t as a diffusion on $[0,\infty)$ that satisfies (12) with instant reflection, we have, with $Y'_t = -Y_t$, a diffusion on $(-\infty, 0]$ which again satisfies (12) with instant reflection in 0, and is characterized by $s'(y) = s(-y)$, $S'[c,d] = S[-c,-d]$, $m'(y) = m(-y)$, $M'[c,d] = M[-c,-d]$. These two may be glued together to define a diffusion X_t on $(-\infty,\infty)$. An informal way to describe this is the following: We use an independent randomization each time the process reaches the boundary zero. Using this randomization we let $X_t = Y_t$ or $X_t = Y'_t$, each with probability $\frac{1}{2}$, up to the next time point when the process reaches the boundary zero. Starting the process with a symmetric distribution on $(-\infty,\infty)$, this defines a symmetric distribution on $(-\infty,\infty)$. More formally, we consider the functions $\bar{s}, \bar{m}$ with $\bar{s} = s(x), x > 0$, $\bar{s} = -s(x), x < 0$, $\bar{m} = m(x), x > 0$, $\bar{m} = m(-x), x < 0$, and define the corresponding scale measure $\bar{S}$ and speed measure $\bar{M}$ on $(-\infty,\infty)$. Then there exists a diffusion X_t on $(-\infty,\infty)$ that has scale and speed measures corresponding to our informal construction (see, e.g., Stummer, 1993).

A.6 The stationary distribution

Since the restoring force $-\frac{D}{2}\,\mathrm{sgn}(x)|x|^{\alpha-1}$ is directed towards zero, the diffusion X_t is positive-recurrent because from any starting point x the point zero is reached in finite expected time according to A.4. Hence there exists a unique stationary distribution. Utilizing the scale function and speed density, this stationary distribution can be expressed as

$$p_e(x) = m(x)[K_1 S(x) + K_2],$$

where $K_1, K_2 > 0$ are normalizing constants, see Karlin and Taylor (1981, p. 221). Recalling $N(\infty) = \infty$ in A.4, we have in our case $\int m(x)S(x)dx = \infty$, and hence $K_1 = 0$. This yields

$$p_e(x) = \frac{\kappa}{D(x)} \exp\left(2\int_{x_0}^{x} \frac{A(y)}{D(y)}dy\right),$$

which is formula (9). So for any $m \in \mathbb{R}, \sigma > 0$ we obtain

$$p_e(x) = \frac{\kappa}{D} \exp\left(\int_0^x -\frac{1}{\sigma}\,\mathrm{sgn}(y-m)\left|\frac{y-m}{\sigma}\right|^{\alpha-1} dy\right) = \frac{\kappa}{D}\exp\left(-\frac{1}{\alpha}\left|\frac{x-m}{\sigma}\right|^{\alpha}\right),$$

which proves formula (4).

A.7 The case $\alpha > 2$

For $\alpha > 2$, the function $A(x)$ is not Lipschitz continuous on the entire real line, but on any interval $[-n, n]$. For any $n \in \mathbb{N}$ we use a bounded Lipschitz continuous function A_n which is equal to A on $[-n, n]$. Similarly to (A.2), we obtain a diffusion on the real line which satisfies (12). As in the previous case, this diffusion has a unique stationary distribution with density given by (4).

B Simulation

We simulate the processes Y_t and X_t with the Euler-Maruyama method. Let X_0 be normally distributed and $Y_0 = |X_0|$. Let $\Delta t > 0$. For all $n \in \mathbb{N}$ we compute

$$\tilde{Y}_{n\Delta t} = Y_{(n-1)\Delta t} + A(Y_{(n-1)\Delta t})\Delta t + \sigma Z_n$$

where Z_n is normal distributed with mean 0 and variance Δt. Since $\tilde{Y}_{n\Delta t}$ can become negative, we define $Y_{n\Delta t} = \max\left(\tilde{Y}_{n\Delta t}, 0\right)$. If $X_{(n-1)\Delta t} \neq 0$ let $X_{n\Delta t} = \mathrm{sgn}(X_{(n-1)\Delta t})Y_{n\Delta t}$. If $X_{(n-1)\Delta t} = 0$, we set $X_{n\Delta t}$ equal to $Y_{n\Delta t}$ or $-Y_{n\Delta t}$, both with probability $\frac{1}{2}$. The results are shown in Figure 6.

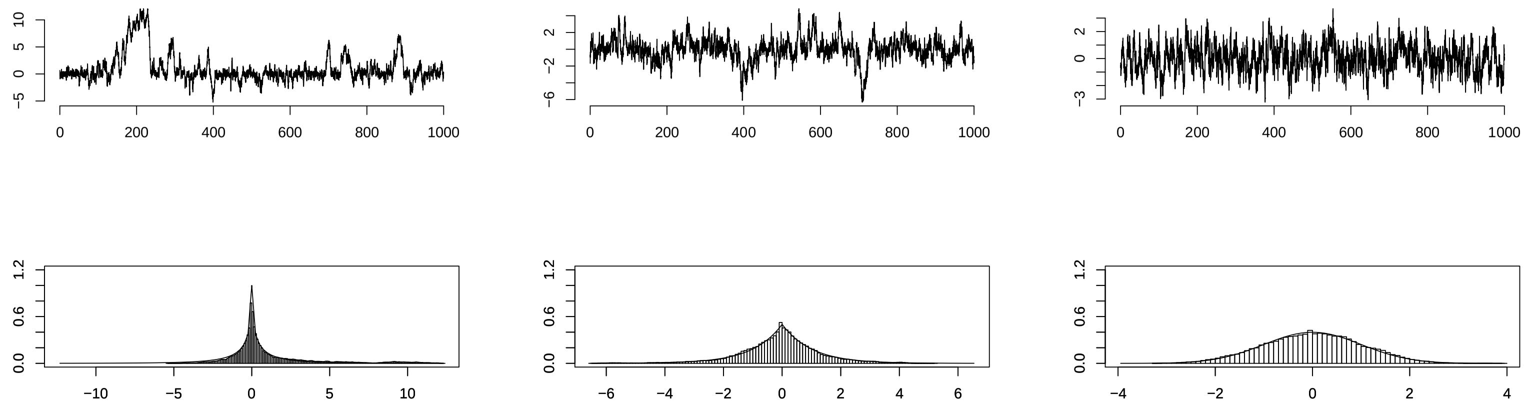

Figure 6: The simulated process X_t (top) and the stationary distribution, compared with the Subbotin distribution (bottom) for $m = 0$, $\sigma = 1$, and $\alpha = 0.5$ (left panel), $\alpha = 1$ (center), and $\alpha = 2$ (right).