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RESEARCH ARTICLE

Endowments, patience types, and uniqueness in two-good HARA utility economies

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Abstract
This paper establishes a link between endowments, patience types, and the parameters of the HARA Bernoulli utility function that ensure equilibrium uniqueness in an economy with two goods and two impatience types with additive separable preferences. We provide sufficient conditions that guarantee uniqueness of equilibrium for any possible value of γ in the HARA utility function $\frac{\gamma}{1-\gamma}\left(b+\frac{a}{\gamma}x\right)^{1-\gamma}$. The analysis contributes to the literature on uniqueness in pure exchange economies with two-goods and two agent types and extends the result in Loi and Matta (2022).

Keywords Uniqueness · Excess demand function · Additive separable preferences · HARA utility · Polynomial approximation

JEL Classifications C62 · D51 · D58

1 Introduction

The relationship between risk aversion, the number of consumer types I and the uniqueness of the equilibrium price has been analysed in a recent article (Loi and Matta 2022). More precisely, in an economy with two goods and an arbitrary number I of impatience types, where each type has additive separable preferences with a

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HARA Bernoulli utility function $u_H(x) := \frac{\gamma}{1-\gamma}\left(b + \frac{a}{\gamma}x\right)^{1-\gamma}$, it has been shown that the equilibrium is unique if the parameter γ is in the range $\left(1, \frac{I}{I-1}\right]$.

While it is well known (Hens and Loeffler 1995; Mas-Colell 1991; Mas-Colell et al. 1995) the effect on uniqueness when γ takes a value between 0 and 1 (in the case $\gamma = 1$ the function becomes logarithmic), it is not known what conditions ensure uniqueness when γ is greater than 2. In this perspective, Loi and Matta (2022) analysed only the particular case when $\gamma = 3$ and $I = 2$, and found sufficient conditions that ensure uniqueness.

It is a natural question to ask whether a similar result can be found outside the above interval. More specifically, this would mean finding sufficient conditions that guarantee uniqueness of the equilibrium for any value of the γ parameter. This is related to Geanakoplos and Walsh (2018)'s remarks on the difficulty of finding a sufficient condition, expressible in closed form, that would allow, for DARA-type utilities, the introduction of more heterogeneity among agents in order to overcome the restrictive assumption of identical endowments to ensure uniqueness [see Geanakoplos and Walsh (2018, Proposition 2)].

This paper provides a positive answer to the question above for HARA utilities, an important subclass of the DARA type. More precisely, our main result, Theorem 2, shows the connection between endowments, patience types and the parameters of the HARA utility function that ensure the uniqueness of the equilibrium. To obtain this result, we will follow the approach of Loi and Matta (2022), where the excess demand function is approximated by a polynomial whose variable, the price, is raised to a power dependent on γ. An algebraic result, Lemma 4, which links the existence of a double root of a polynomial to an inequality involving its coefficients, allows us to prove our main result.

At the end of the paper we also give a more direct analytic proof of Theorem 2 suggested by an anonymous referee.

For an overview of the literature on uniqueness, in addition to the well-known contributions by Kehoe (1998), Mas-Colell (1991) and of Geanakoplos and Walsh (2018) and Toda and Walsh (2017) for two-good, two-agent pure exchange economies, we also refer the reader to the following recent contributions. Giménez (2022), using an offer curve approach, establishes sufficient conditions to guarantee the uniqueness of competitive equilibrium by properly restricting the distribution of endowments and preference profiles in a two-commodity, two-agent exchange economy.

We refer the reader to Won (2023) and Won (2024) for a microfoundation for equilibrium uniqueness in multi-good, multi-agent economies by characterising the shape of individual demand functions. In particular, Won (2024) discusses the transformation of HARA utility functions into CRRA utility functions. Finally, Toda and Walsh (2024) provides a state-of-art review of the literature on the uniqueness and multiplicity of equilibria in general equilibrium models in finance, macroeconomics, and trade.

This short note is organised as follows. Section 2 analyses the economic setting using the polynomial approach. Section 3 proves our main result.

2 Preliminaries

Consider an economy with two goods and $I = 2$ impatience types, where type i has preferences represented by the utility function

$$u_i(x, y) = u_H(x) + \beta_i u_H(y), \tag{1}$$

where u_H is HARA, i.e.

$$u_H(x) := \frac{\gamma}{1-\gamma}\left(b + \frac{a}{\gamma}x\right)^{1-\gamma}, \ \gamma > 0, \gamma \neq 1, a > 0, b \geqslant 0. \tag{2}$$

Let ε be a rational number $\frac{m}{n}$, $m, n \in \mathbb{N}$ sufficiently close to $\frac{1}{\gamma}$. Suppose that $\gamma > 2$ and, hence, $n > 2m$. Denoting by (e_i, f_i) consumer i's endowments, the standard maximisation problem over the budget constraint $px_i + y_i \leqslant pe_i + f_i$ gives (see Loi and Matta (2022, formula (14))) the aggregate excess demand function for good x:

$$\sum_{i=1}^{2} \frac{b - bp^{\varepsilon}\sigma_i + a\varepsilon\,(pe_i + f_i)}{a\varepsilon\,(p + \sigma_i p^{\varepsilon})} - (e_1 + e_2), \tag{3}$$

where

$$\varepsilon \approx \frac{1}{\gamma}, \ \sigma_i := \beta_i^{\varepsilon}, \ i = 1, 2.$$

Following Loi and Matta (2022), we combine terms over a common denominator and take the numerator, then we collect terms in p, divide by p^{ε}, and we get:

$$\begin{aligned}&p(-ae_1\sigma_1\varepsilon - ae_2\sigma_2\varepsilon - b\sigma_1 - b\sigma_2) + p^{1-\varepsilon}(af_1\varepsilon + af_2\varepsilon + 2b)+\\&p^{\varepsilon}(-ae_1\sigma_1\sigma_2\varepsilon - ae_2\sigma_1\sigma_2\varepsilon - 2b\sigma_1\sigma_2) + af_1\sigma_2\varepsilon + af_2\sigma_1\varepsilon + b\sigma_1 + b\sigma_2\end{aligned}$$

Recalling that $\varepsilon = \frac{m}{n}$ and by letting, with a slight abuse of notation, $x := p^{1/n}$, we rewrite the previous expression in decreasing order as follows:

$$A(e, \sigma, a, b)x^n + B(f, a, b)x^{n-m} + C(e, \sigma, a, b)x^m + D(f, \sigma, a, b), \tag{4}$$

where

$$\begin{aligned}A(e, \sigma, a, b) &:= -(e_1\sigma_1 + e_2\sigma_2) - \frac{b}{a\varepsilon}(\sigma_1 + \sigma_2) < 0,\\B(f, a, b) &:= (f_1 + f_2) + \frac{2b}{a\varepsilon} > 0,\\C(e, \sigma, a, b) &:= -(e_1 + e_2)\sigma_1\sigma_2 - \frac{2b}{a\varepsilon}\sigma_1\sigma_2 < 0,\\D(f, \sigma, a, b) &:= (f_1\sigma_2 + f_2\sigma_1) + \frac{b}{a\varepsilon}(\sigma_1 + \sigma_2) > 0.\end{aligned} \tag{5}$$

For notational convenience, we introduce the following terms:

$$A_1 = -\sigma_1\bar{e}_1 - \sigma_2\bar{e}_2,\, A_2 = \bar{f}_1 + \bar{f}_2,\, A_3 = -\sigma_1\sigma_2\left(\bar{e}_1 + \bar{e}_2\right),\, A_4 = \sigma_2\bar{f}_1 + \sigma_1\bar{f}_2,$$

where

$$\bar{e}_i = b + \frac{a}{\gamma}e_i,\ \bar{f}_i = b + \frac{a}{\gamma}f_i.$$

Then, one immediately gets

$$A(e,\sigma,a,b) = \frac{\gamma}{a}A_1,\ B(f,a,b) = \frac{\gamma}{a}A_2,\ C(e,\sigma,a,b) = \frac{\gamma}{a}A_3,\ D(e,\sigma,a,b) = \frac{\gamma}{a}A_4. \tag{6}$$

Lemma 1 *If either*

$$\sigma_1 < \sigma_2,\ e_1 \leqslant e_2 \ \text{and}\ f_2 \leqslant f_1, \tag{7}$$

or

$$\sigma_2 < \sigma_1,\ e_2 \leqslant e_1 \ \text{and}\ f_1 \leqslant f_2, \tag{8}$$

holds, then the coefficients (5) satisfy the inequalities

$$A_1A_4 - A_2A_3 < 0 \tag{9}$$

or, equivalently,

$$A(e,\sigma,a,b)D(e,\sigma,a,b) - B(e,\sigma,a,b)C(e,\sigma,a,b) < 0. \tag{10}$$

Proof We see that

$$\begin{aligned} A_1A_4 - A_2A_3 &= -\left(\sigma_1 - \sigma_2\right)\left(\sigma_1\bar{e}_1\bar{f}_2 - \sigma_2\bar{e}_2\bar{f}_1\right) \\ &= -\gamma^{-2}\left(\sigma_1 - \sigma_2\right)\left(\left(b\gamma + ae_1\right)\left(b\gamma + af_2\right)\sigma_1\right. \\ &\quad \left. - \left(b\gamma + ae_2\right)\left(b\gamma + af_1\right)\sigma_2\right). \end{aligned}$$

Then $A_1A_4 - A_2A_3 < 0$ if and only if

$$\left(\sigma_1 - \sigma_2\right)\left(\frac{\left(b\gamma + ae_1\right)\left(b\gamma + af_2\right)}{\left(b\gamma + ae_2\right)\left(b\gamma + af_1\right)}\frac{\sigma_1}{\sigma_2} - 1\right) > 0 \tag{*}$$

If (7) holds, then we see that

$$\frac{\left(b\gamma + ae_1\right)\left(b\gamma + af_2\right)}{\left(b\gamma + ae_2\right)\left(b\gamma + af_1\right)} < 1 < \frac{\sigma_2}{\sigma_1},$$

which implies (*).

If (8) holds, then we have

$$\frac{(b\gamma + ae_1)\,(b\gamma + af_2)}{(b\gamma + ae_2)\,(b\gamma + af_1)} > 1 > \frac{\sigma_2}{\sigma_1}$$

which leads to (*). The equivalence between (9) and (10) follows by (6). □

3 Main result

In this section we present our main result, Theorem 2. As far as uniqueness is concerned, we will assume an arbitrary $\gamma > 2$. In fact, the case $\gamma \in (1, 2]$ is a particular case of Loi and Matta (2022, Theorem 1), while the case $\gamma \leqslant 1$ is a well known result in the literature (Hens and Loeffler 1995; Mas-Colell 1991; Mas-Colell et al. 1995).

Observe that the zero set of aggregate demand function amounts to studying the zeros of polynomial (4). In fact, according to Loi and Matta (2022)'s approach it is possible to approximate γ with a rational number, since $\mathbb{Q}$ is dense in $\mathbb{R}$, in such a way that the cardinality of the set of regular equilibria does not decrease Loi and Matta (2022, Lemma 9). To provide a geometric insight, this corresponds to small perturbations of the aggregate demand function that do not allow a decrease in the number of the equilibria.

Theorem 2 *In an economy with two goods and two impatient types with HARA preferences* (1)*, if the conditions* (7) *and* (8) *hold, then the equilibrium price is unique.*

Remark 3 For the general type of DARA, Geanakoplos and Walsh (2018) observe there is not a closed-form expression that ensures uniqueness, but conditions (7) and (8) represent a closed-form expression for HARA utilities, an important subclass of utilities of type DARA. They are a generalization of the sufficient condition of Geanakoplos and Walsh (2018, Prop. 5) in the two-agent framework. Notice also that they are the same as those presented in Loi and Matta (2022), here suitably generalised.

Proof By Loi and Matta (2022, Theorem 1) the equilibrium is unique if and only if the polynomial (4), $P(x)$, has a unique positive root. We will prove that the inequality (10), which holds by Lemma 1, implies that $P(x)$ has a unique positive root. Assume by contradiction[1] that $P(x) = Ax^n + Bx^{n-m} + Cx^m + D$, with $n > 2m$, has more than one positive root. We claim that $P(x)$ has a double positive root and hence one can achieve the conclusion of Theorem 2 by the following algebraic Lemma 4. In order to prove the claim notice that $P'(x) = x^{m-1}Q(x)$, where

$$Q(x) = nAx^{n-m} + (n-m)Bx^{n-2m} + mC.$$

[1] We thank an anonymous referee for suggesting a more direct and explicit proof. The previous one used the topological argument of continuous dependence of the roots of a polynomial on its coefficients in a path-connectedness space.

Note also that

$$\begin{aligned}Q'(x) &= n(n-m)Ax^{n-m-1} + (n-m)(n-2m)Bx^{n-2m-1}\\ &= (n-m)x^{n-2m-1}\left(nAx^m + (n-2m)B\right).\end{aligned}$$

Since $n \geqslant 2m+1$, $A < 0$, and $B > 0$, it follows that Q' has a single sign change from positive to negative as x ranges from 0 to ∞. Therefore Q is first increasing, and then decreasing. Thus $P(x)$ is a polynomial that is first decreasing, then (potentially) increasing, and then decreasing. If P is strictly decreasing, then of course there is a unique root, so we are done. So suppose P is not strictly decreasing and hence has a local minimum as well as a local maximum. If the signs of the local minimum and the maximum differ, then P has three roots. Suppose that is the case. Then by increasing D, we can shift the graph upward, and the local minimum eventually becomes a double root. At this point the inequality $AD - BC < 0$ must still hold because initially we had $AD - BC < 0$ with $A < 0 < D$, so increasing D makes this inequality even stronger. Thus P has a double root proving the claim. □

Lemma 4 *If the polynomial* (4)*,* $P(x) = Ax^n + Bx^{n-m} + Cx^m + D$*,* $ABCD \neq 0$*, has a double positive root, then* $AD - BC \geqslant 0$*.*

Proof Let $\alpha > 0$. By dividing the polynomial $P(x)$ by $(x-\alpha)^2$ we can write[2]

$$P(x) = Q(x)(x-\alpha)^2 + R(x),$$

where $R(x) = R_1x + R_2$, is the remainder, i.e. a degree one polynomial. If α has a double root for $P(x)$, then $P(\alpha) = R(\alpha)$, and by differentiation we also have $P'(\alpha) = R'(\alpha)$. These two conditions uniquely determine the coefficients of $R(x)$, namely R_1 and R_2 satisfy the following equations

$$\begin{cases} R_1\alpha + R_2 = A\alpha^n + B\alpha^{n-m} + C\alpha^m + D\\ R_1 = nA\alpha^{n-1} + (n-m)B\alpha^{n-m-1} + mC\alpha^{m-1}.\end{cases}$$

Thus

$$\begin{cases} m\alpha^{m-1}C = -(n-m)\alpha^{n-m-1}B - n\alpha^{n-1}A\\ D = (m-1)\alpha^m C + (n-m-1)\alpha^{n-m}B + (n-1)\alpha^n A.\end{cases}$$

Multiplying the second equation by $m\alpha^{m-1}$, we get

$$m\alpha^{m-1}D = (m-1)\alpha^m m\alpha^{m-1}C + m(n-m-1)\alpha^{n-1}B + m(n-1)\alpha^{n+m-1}A,$$

where, substituting $m\alpha^{m-1}C$ with the RHS of the first equation and multiplying by A, we obtain

$$m\alpha^{m-1}AD = (n-2m)\alpha^{n-1}AB + (n-m)\alpha^{n+m-1}A^2.$$

[2] We thank an anonymous referee for suggesting a quicker way to describe the remainder $R(x)$.

Moreover, we observe that

$$m\alpha^{m-1}BC = -(n-m)\alpha^{n-m-1}B^2 - n\alpha^{n-1}AB.$$

We can then write

$$m\alpha^{m-1}(AD - BC) = (n-m)\alpha^{n-m-1}(\alpha^{2m}A^2 + B^2) + 2(n-m)\alpha^{n-1}AB,$$

as

$$(n-m)\alpha^{n-m-1}(\alpha^{2m}A^2 + B^2 + 2\alpha^m AB),$$

or, equivalently,

$$(n-m)\alpha^{n-m-1}(\alpha^m A + B)^2.$$

Hence, we have

$$AD - BC = \frac{n-m}{m}\alpha^n(\alpha^n A + B)^2 \geqslant 0,$$

and we are done. □

An alternative proof of Theorem 2

We end the paper with an alternative proof of Theorem 2 by using an analytic approach suggested by an anonymous referee. The idea is to tackle the problem using a simplified setting, obtained by transforming HARA utility functions into CRRA type functions. By introducing $\bar{x} = b + \frac{a}{\gamma}x$ and $\bar{y} = b + \frac{a}{\gamma}y$ and using $\bar{e}_i$ and $\bar{f}_i$, the maximisation problem becomes $\max_{\bar{x},\bar{y}} \bar{u}_i(\bar{x},\bar{y}) = \frac{\gamma}{1-\gamma}\bar{x} + \beta_i \frac{\gamma}{1-\gamma}\bar{y} \quad \text{s.t.} \quad p\bar{x} + \bar{y} = p\bar{e}_i + \bar{f}_i$.

Proof We are going to show that if $A_1A_4 - A_2A_3 < 0$, then the numerator of the excess aggregate demand function for the first good

$$N(p) := A_1 p + A_2 p^{1-\frac{1}{\gamma}} + A_3 p^{\frac{1}{\gamma}} + A_4$$

has a unique positive solution. Then the proof of Theorem 2 will follow by combining (4) with $x = p^{1/n}$, (6) and Lemma 1. We introduce a function

$$K(p) = A_1\left(p + \frac{A_2}{A_1}p^{1-\frac{1}{\gamma}} + \frac{A_3}{A_1}p^{\frac{1}{\gamma}} + \frac{A_2}{A_1}\frac{A_3}{A_1}\right) = A_1\left(p^{1-\frac{1}{\gamma}} + \frac{A_3}{A_1}\right)\left(p^{\frac{1}{\gamma}} + \frac{A_2}{A_1}\right)$$

Since $p^{1-\frac{1}{\gamma}} + A_3/A_1 > 0$, $K(p) = 0$ has a unique solution $p_K \equiv (-A_2/A_1)^\gamma > 0$.

The function $K(p)$ differs from $N(p)$ in the constant terms: for all $p > 0$,

$$N(p) - K(p) = \frac{A_1 A_4 - A_2 A_3}{A_1} > 0$$

This property is exploited to verify that $N(p) = 0$ has a unique positive solution. We differentiate $N(p)$.

$$N'(p) = \frac{A_3 p^{-1+\frac{1}{\gamma}} - A_2 p^{-\frac{1}{\gamma}} + A_1 \gamma p^{-\frac{1}{\gamma}} \left(p^{\frac{1}{\gamma}} + A_2/A_1\right)}{\gamma}$$

For all $p \in (p_K, \infty)$, $p^{\frac{1}{\gamma}} + A_2/A_1 > 0$ implies $N'(p) < 0$. Consequently, N has the following properties:

(1) $N(p) > K(p) \geqslant 0$ for all $p \in (0, p_K]$,
(2) N is strictly decreasing in (p_K, ∞) with $\lim_{p \to \infty} N(p) = -\infty$, and
(3) $N(p_K) = K(p_K) + \frac{A_1 A_4 - A_2 A_3}{A_1} = \frac{A_1 A_4 - A_2 A_3}{A_1} > 0$.

These three properties ensure that $N(p)$ has a unique solution in (p_K, ∞), and the proof is complete. □

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