On the Impact of Agents With Influenced Opinions in the Swarm Social Behavior

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***Abstract*—We consider a simplified version of the Taylor model, typically used in the collective dynamics of continuous exchange of opinions, to describe the properties of swarm formation in the presence of external sources of influence or prejudices affecting a number of agents in the network. Such external sources are responsible for the breakdown of the consensus equilibrium and directly influence certain other individuals in the network, which we denote as quasi-stubborn agents. These quasi-stubborn agents participate in consensus with other individuals, but are able to indirectly influence the opinions of the entire system. In particular, we show that the swarm in steady-state moves towards the convex hull of the opinions of the quasi-stubborn agents. This is an interesting result that allows a more accurate estimation of the final opinions in a social network. In the case of two prejudiced agents, an explicit expression of the stationary opinions is provided in terms of the Moore-Penrose inverse of the Laplacian of the graph. Numerical simulations are presented to illustrate the properties of the considered model.**

***Index Terms*—Opinion formation, social networks, Laplacian matrix, stubborn agents.**

I. Introduction

Opinion formation represents an interesting dynamic process that can be used to explain most real-world situations: influence maximisation, link prediction, discovery of influential nodes, community detection, trend detection, to name a few [1], [2]. For these reasons, much attention is paid to the analysis of the emergence, evolution and diffusion of opinions in society. In particular, opinion dynamics leads to an interdisciplinary field of research that combines elements of economics, control theory, applied mathematics and computer science to study how opinions evolve starting from the interactions of agents [3], [4]. The influence of neighbours on users' opinions is described, for example, in [5] by using evolutionary game theory and introducing a mechanism to change or maintain opinions in order to maximise the gain for the user. Various models have been developed, inspired by those used in physics, to incorporate many of these elements and identify the mechanisms involved in the opinion formation process, with the practical aim of simulating the formation and propagation of opinions under different conditions [6]. The basic idea of all models of opinion dynamics is that nodes or actors in a social network have a variable representing their opinion, which is updated according to some predefined rules. Of course, they represent a simplification of real opinion dynamics, but they are useful to illustrate aspects of real opinion formation such as agreement, fragmentation, formation of clusters of people with the same opinion [7]. The various models proposed in the literature fall mainly into two categories: macroscopic and microscopic. Macroscopic models are usually based on high-dimensional stochastic models and consider individuals as a continuous mass rather than discrete particles. In this framework, mean-field theory is used to approximate the behaviour of the original model by averaging [8], [9]. Microscopic models, on the other hand, mainly use agent modelling and define a set of rules by which individuals interact [10], [11]. Sociologist John R. P. French and statistician M. H. DeGroot introduced one of the earliest and most well-known models. They proposed a simple process that allows agents to reach consensus by repeatedly integrating their opinions. The model assumes that each member of a population has an opinion, encoded with a real scalar, which is updated synchronously as a weighted average of his or her opinion and that of his or her neighbors (see [12], [13]). While the French-DeGroot model is formulated in discrete-time, its continuos-time counterpart was proposed by Abelson [14] and an extension in presence of stubborn and prejudiced agents was developed by Taylor [15]. Since the work of [14], many contributions have focused on the dynamics of opinions in continuous-time. For example, the work in [16] examines two continuous-time opinion dynamics models where individuals discuss opinions on several logically interdependent topics. In [17], a continuous-time mean-preserving opinion model is proposed in which each agent considers another agent as a neighbour if their opinions differ by less than 1, and the opinions of the agents are continuously attracted to the opinions of their neighbours. Reference [18] presented sufficient conditions to achieve modulus consensus over time-varying signed networks. In [19], control of a continuous-time opinion

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dynamics model with a leader is proposed, where interactions between individuals can be both state- and time-dependent.

In the field of opinion dynamics, particular attention has traditionally been paid to consensus formation and the emergence of heterogeneous (non-consensual) states [20]. Indeed, there can be persistent fluctuation of opinions and disagreement if there are obstinate agents in a society with conflicting views who never update their opinions. These obstinate agents may represent leaders, political parties or media sources who try to influence the beliefs of the rest of society [21]. The difficulty of identifying the properties of social networks is highlighted in [22], where a model of online opinion dynamics is used to numerically simulate and predict the extent to which people hold their own opinions or accept other people's opinions in discussions about a hot topic.

In this letter, we use the properties of the Taylor model to study how individuals' opinions are altered by the presence of external biases. More specifically, we consider a society of interacting agents (or individuals) who communicate and exchange information with each other, where some individuals participate in the joint formation of opinions but are additionally influenced by biases (initial prejudices) or external sources of influence. While it is known that the final opinions tend to be in a convex combination of the initial biases, less research has been done on the opinions of the agents who are directly influenced and whether it is possible to set a less restrictive boundary on the final opinions of the individuals. With this in mind, this letter aims to show that the final opinions of individuals tend to be a convex hull of the final opinions of agents who are directly influenced by external opinions (these agents are denoted as quasi-stubborn agents). In the case of two quasi-stubborn agents, the final opinion is explicitly expressed by the Moore-Penrose inverse of the Laplacian of the graph defining the connections between the individuals. For the general case with m quasi-stubborn agents, an expression for the final state of the entire social network is given.

This letter is organized as follows. We start with the main features of the simplified Taylor model in Section II. In Section III, we examine the influence of quasi-stubborn agents to the social network, discussing some properties about their stationary opinions. An explicit expression of the final opinions of the agents is given in Section IV. Some simulated examples are given in Section V. The last section is devoted to concluding remarks and future developments.

Notations: We use $e_p \in \mathbb{R}^n$ to denote a standard basis vector, where the pth element is 1 while the rest are zeros. The vector of ones in $\mathbb{R}^n$ is denoted by $\mathbb{1}_n$ while I_p denotes the identity matrix of order p and $0_{p\times q}$ the $p \times q$ matrix with zero entries. The j-th element of the vector v_i is indicated as v_i^j.

II. The Taylor Social Behavior Model

We consider a social network with n individuals, described by a graph $\mathcal{G} = (V, E)$ where $V = \{1, \dots, n\}$ is the set of nodes and $E \subseteq \{(i,j) : i,j \in V, i \neq j\}$ (no self-loops are allowed) is the set of unordered pairs of vertices defining the edges of the graph. Agents are the vertices of the graph while edges indicate the pair of agents that have interactions. The set of neighbors of agent i is indicated by $\mathcal{N}_i = \{j \mid (i,j) \in E\}$ and represents the set of agents that individual i interacts with. $\mathcal{G}$ is encoded by the Laplacian symmetric matrix L. Since it is assumed that $\mathcal{G}$ is connected, L has the following properties [23]: L has one simple zero eigenvalue, i.e., $\lambda_n = 0$, with the associated eigenvector $\mathbb{1}_n$; the second smallest eigenvalue $\lambda_{n-1} > 0$.

In this section we consider a simplified version of the model proposed in [15] and show some properties about the stationary positions of agents. For the reader's convenience, we briefly describe the Taylor model in the case of a social network encoded by a graph with unit-weighted edges. The Taylor model involves n agents with opinions $x_1, \dots, x_n \in \mathbb{R}$ and $m \geq 1$ communication sources providing static opinions $s_1, \dots, s_m \in \mathbb{R}$. The opinions of the agents obey to the model

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} (x_j(t) - x_i(t)) + \sum_{k=1}^{m} b_{ik}(s_k - x_i(t)) \tag{1}$$

with $b_{ik} \geq 0,\ i = 1, \dots, n,\ k = 1, \dots, m$. Some agents are free of the external influence, in this case $b_{i1} = \cdots = b_{im} = 0$, while others with $\sum_{k=1}^{m} b_{ik} > 0$ can be influenced by one or more sources. Unlike the Abelson model [14], the system (1) is usually asymptotically stable and converges to the unique equilibrium determined by $s_1, \dots, s_m$. An equivalent representation of the Taylor model can be found in [24]

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i} (x_j(t) - x_i(t)) + \gamma_i(u_i - x_i(t)) \tag{2}$$

where $\gamma_i \geq 0$. If $\gamma_i = \sum_k b_{ik},\ u_i = \frac{1}{\gamma_i}\sum_k b_{ik}s_k$, the model (2) reduces to (1). The quantity u_i is called the *prejudice*, or *bias*, of the agent i; moreover the agent i is prejudiced if $\gamma_i > 0$, otherwise $u_i = 0$. The stability properties of the Taylor model were discussed in [24] where it is shown that the final opinion of any agent is in a convex hull of the external opinions.

The importance of the Taylor model (1) generalised to the multidimensional case is also related to the problems of containment, where multiple leaders interacting with the other agents (followers) aim at driving and holding the followers into the convex hull imposed by their states (opinions). The convex hull then represents the domain that contains the opinions of the followers [25]. In this frame, the opinions of agents $x_i \in \mathbb{R}^d,\ i = 1, \dots, n$ represent, for example, the positions of mobile robots or vehicles, while $s_1, \dots, s_k \in \mathbb{R}^d$ represent the positions of k static leaders. The containment problem is to control the agents to reach the convex hull spanned by the leaders, i.e., the region $\mathcal{S} \subset \mathbb{R}^d$, which is defined as $\mathcal{S} := \{\sum_{p=1}^{k} \alpha_p s_p,\ \alpha_p \geq 0,\ \sum_{p=1}^{k} \alpha_p = 1\}$.

III. The Influence of the Quasi-Stubborn Agents

We consider the model (2) where the agents influenced by external opinions are referred as quasi-stubborn agents and, without loss of generality it is assumed that the subset of quasi-stubborn agents is $\mathcal{Q} = \{i : i \leq m\}$. As a consequence, $\gamma_i = 1$ if $i \in \mathcal{Q}$ and $\gamma_i = 0$ otherwise.

From (2) it can be seen that the last $n - m$ agents strive to minimise the differences of opinion with their connected neighbours, while the opinions of the first m quasi-stubborn agents are also influenced towards their bias u_i.

Eq. (2) can be rewritten in matrix form as

$$\dot{x}(t) = -(L + W)x(t) + Bu \tag{3}$$

where $x = [x_1, \ \cdots, \ x_n]^T \in \mathbb{R}^n$ is the vector containing the opinions of all agents, $W \in \mathbb{R}^{n\times n}$ is the diagonal matrix $W = \sum_{k=1}^m e_k e_k^T$, $B \in \mathbb{R}^{n\times m}$ is the input matrix $B = [e_1 \ e_2 \ \cdots e_m]$ and $u \in \mathbb{R}^m$ with $u = [u_1 \ u_2 \ \cdots \ u_m]^T$ contains the bias opinions of quasi-stubborn agents. Note that the matrix $L_W = L + W$ is obtained from the perturbation of the graph Laplacian by adding a diagonal matrix W with m elements of the diagonal equal to one in position corresponding to the indices of quasi-stubborn agents. As shown in the next result, the matrix L_W has positive eigenvalues.

Lemma 1: The smallest eigenvalue of L_W is positive, i.e., $\lambda_n(L_W) > 0$.

Proof: Because L_W is positive semi-definite, $\lambda_i(L_W) \geq 0$. If an eigenvalue $\lambda = 0$ exists for L_W with a related eigenvector v, we have $L_W v = Lv + Wv = \lambda v = 0$. When the above equality is multiplied by v^T from left, the result is $v^T Lv + v^T Wv = 0$. Considering that L and W are positive semi-definite matrices, i.e., $v^T Lv \geq 0$ and $v^T Wv \geq 0$, it must be $v^T Lv = 0$ and $v^T Wv = 0$. Then we have $v = \mathbb{1}_n$, but $\mathbb{1}_n^T W \mathbb{1}_n = m$ resulting in a contradiction; this simply means that $\lambda > 0$. ■

Remark 1: Lemma 1 allows to infer that the matrix L_W is invertible and thus the opinions of the agents in steady-state are given by $x_\infty = lim_{t\to\infty} x(t) = L_W^{-1} Bu$.

It is straightforward to note that the steady-state opinions of the agents can be rewritten as

$$x_\infty = \sum_{k=1}^m \pi_k u_k \tag{4}$$

with $\pi_k \in \mathbb{R}^n$ such that

$$L_W \pi_k = e_k, \quad k = 1, \ldots, m. \tag{5}$$

The following three results provide some useful properties of the matrix L_W and vectors $\pi_k, \ k = 1, \ldots, m$ propaedeutic to the main contribution of this letter.

Lemma 2: All the entries of vectors π_k with $k = 1, \ldots, m$ in (4) are positive.

Proof: Since the matrix L_W is a Z-matrix with positive eigenvalues then it is an invertible M-matrix. Moreover it is additionally irreducible because the graph is assumed connected, then it has positive inverse [26]. Consequently $\pi_k, \ k = 1, \ldots, m$ are vectors of positive elements. ■

Lemma 3: The solution of (5) satisfies the property $\sum_{k=1}^m \pi_k = \mathbb{1}_n$.

Proof: Summing the two sides of (5) for k from 1 to m yields to $\sum_{k=1}^m L_W \pi_k = L_W \sum_{k=1}^m \pi_k = \sum_{k=1}^m e_k$. Since L_W is invertible and $L_W \mathbb{1}_n = L\mathbb{1}_n + W\mathbb{1}_n = \sum_{k=1}^m e_k$ the thesis follows. ■

Lemma 4: Each vector π_k with $k = 1, \ldots, m$, satisfies $\sum_{i=1}^m \pi_k^i = 1$.

Proof: Rewrite (5) as $L\pi_k = e_k - W\pi_k$. Multiplying by $\mathbb{1}_n^T$ from left and considering that $\mathbb{1}_n^T L = 0$ yields

$$0 = 1 - \mathbb{1}_n^T W\pi_k = 1 - \mathbb{1}_n^T \sum_{i=1}^m e_i e_i^T \pi_k = 1 - \sum_{i=1}^m \pi_k^i.$$

■

As a further result about the final opinions in the whole network, the next proposition gives an insight into the final average opinion of the agents.

Proposition 1: Agents evolving according to the model (2) converge to a constant opinion in the convex hull of $u_1, \ldots, u_m$. Moreover, the mean of the final opinions of the quasi-stubborn agents coincides with the mean of the biased opinions.

Proof: From the steady-state opinions x_∞ and from Lemmas 2 and 3 the first part is proved. Note also that from $L_W x_\infty = Bu$ and multiplying on the left by $\mathbb{1}_n^T$, one has $\mathbb{1}_n^T W x_\infty = \mathbb{1}_n^T Bu$ and $\sum_{k=1}^m x_\infty^k = \sum_{k=1}^m u_k$. ■

The above result is common in various stubbornness-based models in which agents' opinions tend towards the convex hull of the external sources of influence or prejudices affecting the quasi-stubborn agents' opinions (see, for example, the containment problem briefly discussed in the previous section). The following theorem represents one of the main results of this letter and shows that the agents reach a constant opinion within the convex hull of the opinions of the quasi-stubborn agents in the steady-state. This property makes it possible to have more details and to determine the final opinions of the individuals more accurately and less conservatively w.r.t. the case of the convex hull of the prejudices.

Theorem 1: The steady-state opinions of the agents belong to the convex hull of the quasi-stubborn agents' opinions, namely

$$x_\infty^k = \sum_{i=1}^m \alpha_k^i x_\infty^i \tag{6}$$

with $\alpha_k \in \mathbb{R}^m$ such that $\sum_{i=1}^m \alpha_k^i = 1, \ \alpha_k^i \geq 0, \ \forall k = 1, \ldots, n, \ \forall i = 1, \ldots, m$.

Proof: By setting $\Lambda = [\alpha_1 \ \alpha_2 \ \ldots \ \alpha_n] \in \mathbb{R}^{m\times n}$ the thesis in (6) can be expressed in matrix form as

$$x_\infty = \Lambda^T B^T x_\infty.$$

By using (4) and (5), this turns out to be equivalent to

$$L_W^{-1} Bu = \Lambda^T B^T L_W^{-1} Bu.$$

Since this equation holds for every u, this means that

$$L_W^{-1} B = \Lambda^T B^T L_W^{-1} B. \tag{7}$$

Using Lemma 3, we obtain $\Lambda^T \mathbb{1}_m = \mathbb{1}_m$. Consider the following block partitioning of L_W and L_W^{-1}:

$$L_W = \begin{bmatrix} H & R^T \\ R & C \end{bmatrix}, \quad L_W^{-1} = \begin{bmatrix} \Pi & V^T \\ V & Z \end{bmatrix}$$

where $H = H^T$, $\Pi = \Pi^T \in \mathbb{R}^{m\times m}$, $Z = Z^T$ and $C = C^T \in \mathbb{R}^{(n-m)\times(n-m)}$, R and $V \in \mathbb{R}^{(n-m)\times m}$. Observe that, setting $\Gamma = B^T L_W^{-1}$, since $\Pi = \Gamma B$, the equality (7) becomes

$$\Pi\Lambda = \Gamma. \tag{8}$$

Since $\Gamma = B^T L_W^{-1} = [\,\Pi \;\; V^T\,]$, by substituting in (8) one has $\Lambda = [I_m \;\; \Pi^{-1} V^T\,]$.

The aim is to show that the matrix $\Pi^{-1} V^T$ has all entries greater or equal than zero. Note that $H\Pi + R^T V = I_m$ and this implies $H\Pi = I_m - R^T V$. H and Π are nonsingular being principal submatrices of the positive definite matrices L_W and L_W^{-1} so that $I_m - R^T V$ is nonsingular leading to

$$\Pi^{-1} = (I_m - R^T V)^{-1} H.$$

Analogously $I_{n-m} - VR^T$ is nonsingular. The equality $V(I_m - R^T V)^{-1} = (I_{n-m} - VR^T)^{-1} V$ implies

$$V\Pi^{-1} = V(I_m - R^T V)^{-1} H = (I_{n-m} - VR^T)^{-1} VH.$$

Since $VH + ZR = 0_{(n-m)\times m}$ then $VH = -ZR$. Moreover $VR^T + ZC = I_{n-m}$ implies $C^{-1} Z^{-1} = (I_{n-m} - VR^T)^{-1}$. It follows that

$$V\Pi^{-1} = -C^{-1} Z^{-1} ZR = -C^{-1} R.$$

Since C^{-1} and $-R$ are non negative matrices, $V\Pi^{-1}$ is also non negative and the elements of the matrix Λ are all non negative. Therefore Eq. (8) represents a set of n decoupled linear systems, each of them consisting in m equations on m unknowns. As a consequence the system (2) admits a unique convex combination of the opinions as defined in (6). ■

Remark 2: Note that in the standard Taylor model [15], [24], the opinions of the agents are known to tend towards the convex hull of static biases, while the proposed framework provides a more accurate description of the region containing the final state of the social network. Indeed, the opinions in the social network are now directed towards the convex hull of the final opinions of the quasi-stubborn agents.

IV. Explicit Formula for Steady-State Opinions: A Constructive Proof

In this section we consider Eq. (5), whose solution provides the weight vectors that give the stationary opinions of the individuals as a convex combination of the biases, and we give an explicit expression for the weight vectors. As expected, individuals' final opinions depend strongly on the topology of the influence network. In particular, we show that the final opinions depend on the inverse of a matrix whose dimension is equal to the number m of quasi-stubborn agents and the Moore-Penrose inverse of L, i.e., $L^\dagger$ [27], where $L^\dagger L = LL^\dagger = I_n - \frac{\mathbb{1}_n \mathbb{1}_n^T}{n}$.

We obtain a direct relationship between the vectors π_k, $k = 1, \dots, m$ defined in (5) and the columns of $L^\dagger$. This makes it possible to exploit the structure of the solution of (5) to obtain explicit formulas for the stationary opinions in the case of two quasi-stubborn agents.

Firstly, note that $L_W = L + BB^T = J + QQ^T$ where $J = L - \frac{1}{n^2}\mathbb{1}_n \mathbb{1}_n^T$ and $Q = [B \;\; \frac{1}{n}\mathbb{1}_n]$, $Q \in \mathbb{R}^{n\times(m+1)}$. By considering that the inverse of J is $J^{-1} = L^\dagger - \mathbb{1}_n \mathbb{1}_n^T$ and by using the Woodbury matrix identity, one has

$$\begin{aligned} L_W^{-1} &= (J + QQ^T)^{-1} \\ &= J^{-1} - J^{-1} Q \Big(I_{m+1} + Q^T J^{-1} Q\Big)^{-1} Q^T J^{-1}. \end{aligned}$$

The above formula requires the inversion of a matrix of order $m+1$, i.e., $(I_{m+1} + Q^T J^{-1} Q)^{-1} = M - \mathbb{1}_{m+1}\mathbb{1}_{m+1}^T$ with

$$M = \begin{bmatrix} B^T L^\dagger B + I_m & 0_{m\times 1} \\ 0_{1\times m} & 1 \end{bmatrix}.$$

By using the Sherman–Morrison formula, it follows that

$$\Big(I_{m+1} + Q^T J^{-1} Q\Big)^{-1} = M^{-1} + \frac{M^{-1}\mathbb{1}_{m+1}\mathbb{1}_{m+1}^T M^{-1}}{1 - \mathbb{1}_{m+1}^T M^{-1} \mathbb{1}_{m+1}}$$

from which it results that the inverse of L_W requires the inversion of the matrix $B^T L^\dagger B + I_m$ with dimension m and the pseudo-inverse of L.

It is worth noting that

$$\begin{aligned} L_W^{-1} Q &= (J + QQ^T)^{-1} Q \\ &= J^{-1} Q \Big(I_{m+1} + Q^T J^{-1} Q\Big) \\ &= \big[\,(L^\dagger - \mathbb{1}_n \mathbb{1}_n^T) B \quad -\mathbb{1}_n\,\big] \left(M^{-1} + \frac{M^{-1}\mathbb{1}_{m+1}\mathbb{1}_{m+1}^T M^{-1}}{1 - \mathbb{1}_{m+1}^T M^{-1}\mathbb{1}_{m+1}}\right). \end{aligned}$$

According to Eq. (5) the first m columns are of interest, since they are related to the vectors u_k, $k = 1, \dots, m$. Then, by defining $S = (B^T L^\dagger B + I_m)^{-1}$ it follows that

$$L_W^{-1} B = L^\dagger BS - \frac{L^\dagger BS \mathbb{1}_m \mathbb{1}_m^T S}{\mathbb{1}_m^T S \mathbb{1}_m} + \frac{\mathbb{1}_n \mathbb{1}_m^T S}{\mathbb{1}_m^T S \mathbb{1}_m}. \tag{9}$$

The above formula can be exploited to compute the mean opinion of the network as $\bar{x}_\infty = \sum_{k=1}^m \bar{\pi}_k u_k$ where $\bar{\pi}_k$ is the mean of π_k. From (9), the vector of means of π_k, $k = 1, \dots, m$ can be computed as $[\,\bar{\pi}_1 \;\; \cdots \;\; \bar{\pi}_m\,] = \frac{\mathbb{1}_m^T S^{-1}}{\mathbb{1}_m^T S^{-1} \mathbb{1}_m}$.

A. Steady-State Opinions in the Case of Two Quasi-Stubborn Agents

In this subsection we consider the case of two quasi-stubborn agents. The importance of an explicit formula for the final opinions in a social network with two quasi-stubborn agents is highlighted in [20], where this high-level description of opinion associations allows a closer look at emerging non-consensus states by also determining the maximum spread of opinions and relating it to the structure of the social network. If we consider two quasi-stubborn agents, the following result, starting from (9), gives further explicit expressions for π_1 and π_2 in a simpler formulation.

Theorem 2: In the case of two external biases, i.e., u_1 and u_2, an explicit expression of π_1 and π_2 is

$$\pi_1 = \frac{L^\dagger(e_1 - e_2) + \big(1 - e_2^T L^\dagger (e_1 - e_2)\big)\mathbb{1}_n}{2 + (e_1 - e_2)^T L^\dagger (e_1 - e_2)}, \tag{10}$$

$$\pi_2 = \frac{L^\dagger(e_2 - e_1) + \big(1 - e_1^T L^\dagger (e_2 - e_1)\big)\mathbb{1}_n}{2 + (e_1 - e_2)^T L^\dagger (e_1 - e_2)}. \tag{11}$$

Proof: For the sake of simplicity, the proof consists of verifying that (10) and (11) satisfy (5). Let us consider the case of u_1, then it must be true that

$$(L + e_1 e_1^T + e_2 e_2^T)\pi_1 = e_1.$$

Note that

$$\begin{aligned} L\pi_1 &= \frac{LL^\dagger(e_1-e_2)+\left(1-e_2^TL^\dagger(e_1-e_2)\right)L\mathbb{1}_n}{2+(e_1-e_2)^TL^\dagger(e_1-e_2)} \\ &= \frac{(e_1-e_2)}{2+(e_1-e_2)^TL^\dagger(e_1-e_2)}, \\ e_1^T\pi_1 &= \frac{1+(e_1-e_2)^TL^\dagger(e_1-e_2)}{2+(e_1-e_2)^TL^\dagger(e_1-e_2)}, \\ e_2^T\pi_1 &= \frac{1}{2+(e_1-e_2)^TL^\dagger(e_1-e_2)} \end{aligned} \tag{12}$$

resulting in

$$(e_1e_1^T+e_2e_2^T)\pi_1 = \frac{e_1+e_2+(e_1-e_2)^TL^\dagger(e_1-e_2)e_1}{2+(e_1-e_2)^TL^\dagger(e_1-e_2)}. \tag{13}$$

By summing (12) and (13) the proof follows. Using the same reasoning, we obtain the proof for π_2. ■

When analysing social networks, one of the most interesting questions is how to evaluate and possibly maximise the influence of certain opinions. For example, m potential individuals are selected to be convinced to adopt a product and use the word-of-mouth effect to spread the information widely and successfully create further adoptions in the network [28]. From the Theorem 1 we know that the two most extreme opinions are held by quasi-stubborn agents. Thus, to find the maximum spread of opinions it is sufficient to consider the final opinion distance between x_∞^1 and x_∞^2. From Eq. (4), which is specialised for the case of two persistent opinions, it follows that $x_\infty^1 = \pi_1^1u_1+\pi_2^1u_2$, $x_\infty^2 = \pi_1^2u_1+\pi_2^2u_2$. The maximum spread can consequently be defined as $D_{max} = x_\infty^1 - x_\infty^2$, i.e.,

$$D_{max} = \left|(\pi_1^1-\pi_1^2)u_1+(\pi_2^1-\pi_2^2)u_2\right|$$

and by using (10) and (11), it follows that

$$D_{max} = \frac{(e_1-e_2)^TL^\dagger(e_1-e_2)}{2+(e_1-e_2)^TL^\dagger(e_1-e_2)}|u_1-u_2|. \tag{14}$$

It is also worth noting that the mean opinion can be easily computed as $\bar{x}_\infty = \frac{\mathbb{1}_n^T\pi_1}{n}u_1 + \frac{\mathbb{1}_n^T\pi_2}{n}u_2$ and considering that $\mathbb{1}_n^TL^\dagger(e_1-e_2)=0$, it follows that

$$\bar{x}_\infty = \frac{(1-e_2^TL^\dagger(e_1-e_2))u_1+(1-e_1^TL^\dagger(e_2-e_1))u_2}{2+(e_1-e_2)^TL^\dagger(e_1-e_2)}.$$

V. Numerical Results

In this section, numerical simulations are presented to illustrate the effectiveness of theoretical results in the previous sections. We consider a network of $n = 20$ nodes that are connected via 60 undirected edges (see Fig. 1).

A. Example 1

The opinions of each agents is considered to be a vector in $\mathbb{R}^2$. In particular the biases of the first $m = 4$ quasi-stubborn agents are set as $u_1 = [-1 \quad 0]^T$, $u_2 = [1 \quad 0]^T$, $u_3 = [1 \quad 1]^T$, $u_4 = [-1 \quad 1]^T$. Note that all the provided results can be extended to the multidimensional case by considering a model (3) for each element of the opinion vector. Therefore, as expected, the final opinions will be in the convex hull of the biases and more precisely in the convex hull (polytope) of the final opinions of the quasi-stubborn agents.

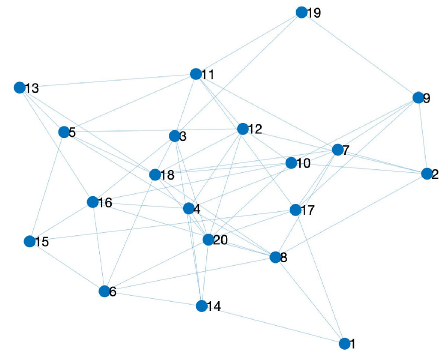

Fig. 1. Graph of the considered multi-agent system, $n = 20$.

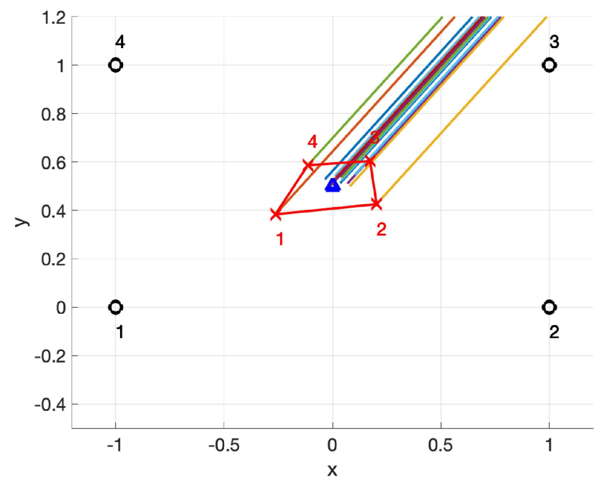

Fig. 2. Example 1. Agents' final opinions. Biases of the quasi-stubborn agents (black circles). Steady-state opinions of the quasi-stubborn agents (red crosses). The red polytope contains the opinions of the other agents. Mean of the final opinions of the quasi-stubborn agents (blue triangle).

Fig. 2 depicts the trend of the opinions close to the steady-state. Specifically, the biases of the quasi-stubborn agents are depicted by black circles and labelled. The more accurate containment area can be identified by considering the final opinions of the quasi-stubborn agents (red crosses). Indeed, the red polytope contains the final opinions of the individuals. Moreover, according to Proposition 1, the mean of the quasi-stubborn opinions at steady-state is equal to the mean of the initial prejudices (see the blue triangle in Fig. 2).

It is important to emphasise that the analysis can be used for design purposes, for example in a containment problem. More precisely, given a target polytopic region in $\mathbb{R}^2$ with m vertices, m quasi-stubborn agents have to be used with steady-state opinions x_∞^k, $k = 1,\ldots,m$ chosen as the vertices of the polytope. Then the external influences $u_k, k = 1,\ldots,m$ can be chosen as

$$\left[u_1^T,\ldots,u_m^T\right]^T = \left(B^TL_W^{-1}B\otimes I_2\right)^{-1}\left[x_\infty^{1^T},\ldots,x_\infty^{m^T}\right]^T$$

where $\otimes$ represents the Kronecker product, so that to drive the network in the prescribed region.

B. Example 2

This example focuses on scalar opinions with two quasi-stubborn agents. Starting from the topology in Fig. 1 and considering as biases $u_1 = 3$ and $u_2 = 50$, the overall opinions

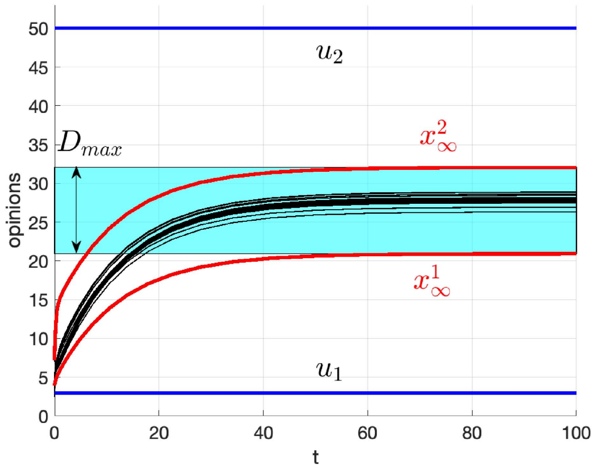

Fig. 3. Example 2. Agents' opinions (black lines). Quasi-stubborn agents' opinions (red lines). Biases (blue lines). Maximum spread between the final opinions (cyan region).

evolutions are depicted in Fig. 3 where the maximum spread D_{max} obtained by Eq. (14) is shown and it coincides with the distance between the two final quasi-stubborn opinions (x_∞^1, x_∞^2).

VI. Conclusion

A symmetric social network, encoded by a graph with unit weighted edges and with the presence of external biases, has been analysed by exploiting the Taylor model. In addition to the well-known property that the final opinions of the agents lie in the convex hull of the external opinions, it has been shown that the quasi-stubborn agents, i.e., the individuals who are directly influenced by these opinions, move the entire network in the convex hull of their final states. This result improves the understanding of the model of opinion diffusion under consideration. An explicit expression for the final opinions in terms of the Moore-Penrose inverses of the Laplacian matrix was also provided. Future studies will go in two directions. First, an attempt will be made to extend the obtained results to the case of a social network encoded by a directed weighted graph. Then, an attempt will be made to extend the proposed analysis to the case with time-varying persistent opinions.

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