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ASYMPTOTICS OF Z-CONVEX POLYOMINOES

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Abstract. The degree of convexity of a convex polyomino P is the smallest integer k such that any two cells of P can be joined by a monotone path inside P with at most k changes of direction. In this paper we show that one can compute in polynomial time the number of polyominoes of area n and degree of convexity at most 2 (the so-called Z-convex polyominoes). The integer sequence that we have computed allows us to conjecture the asymptotic number a_n of Z-convex polyominoes of area n, $a_n \sim \frac{C\cdot\exp(\pi\sqrt{11n/4})}{n^{3/2}}$.

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1. Introduction

A polyomino is a geometrical figure consisting of a finite set of connected unitary squares (called cells) in the plane $\mathbb{Z}\times\mathbb{Z}$, considered up to translations. Polyominoes gained popularity after the paper of S. Golomb [1]. Nowadays they are widely studied by physicists, mathematicians, computer scientists and also by biologists.

The problem of counting the number c_n of polyominoes with n cells (*i.e.* of area n) is probably one of the fundamental open problems in combinatorial geometry (see problem (37) in [2]). The problem has been solved up to $n \leq 56$ [3] and no closed-form expression for c_n is known. Due to the difficulty of the problem, simpler classes of polyominoes have been introduced and widely studied. In particular, the class of convex polyominoes (polyominoes where the intersection with an infinite horizontal or vertical stripe is a finite segment) and some of its subclasses have been thoroughly investigated [4–7].

For some classes C of polyominoes the generating function $\phi_C(x) = \sum_{n\geq 0} c_n x^n$ is known, either explicitly (by means of a closed-form expression) or implicitly (by means of a non-closed-form expression, or a functional equation satisfied by $\phi_C(x)$), see for instance [8]. This usually allows one to get an estimate of the asymptotic growth of c_n (the number of polyominoes of area n in C) using standard analytical methods.

Unfortunately, there are classes of polyominoes for which no information about $\phi_C(x)$ is known. In these cases one can exploit an efficient algorithm for the exhaustive generation of C in order to compute c_n for small (but still significant) values of n. For example, Constant Amortized Time (CAT) algorithms for generating several classes of polyominoes have been recently developed, where the exhaustive generation is done by semiperimeter [9, 10] or by area [11, 12].

Keywords and phrases: Convex polyominoes, counting problem, integer sequences.

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In this paper we consider a particular class containing all convex polyominoes P with the property that any two cells of P can be joined by a path in P with at most two changes of direction. This is the class of Z-convex polyominoes introduced in [13] (Z resembles the shape of the path connecting two cells) and studied in [14]. Its generating function with respect to the area is still unknown and the only way to enumerate it (up to now) is to use the CAT algorithm presented in [15]. We recall that in [13] the generating function of Z-convex polyominoes with respect to the semiperimeter has been computed.

We show how to decompose Z-convex polyominoes in order to obtain a set of formulas that can be used for computing the number of Z-convex polyominoes of area n in polynomial time (under the uniform cost model). We have also developed a C++ program that produces the series of coefficients. This series is analysed in order to obtain a conjecture on the number of Z-convex polyominoes of area n. More precisely, we conjecture that $c_n \sim \frac{C\cdot\exp(\pi\sqrt{11n/4})}{n^{3/2}}$ with $C = 0.095 \pm 0.003$.

2. Notation and preliminaries

Let P be a polyomino with an $r \times c$ minimal bounding rectangle. The r rows (resp., c columns) of P are numbered from bottom to top (resp., from left to right). The *area* of P is the number of its cells, denoted by $\mathrm{A}(P)$. We say that P is null (that is, $P = \epsilon$) if $\mathrm{A}(P) = 0$. A cell of P is identified by a pair of integers (i, j), where i (resp., j) is the row (resp., column) index. Two cells $a = (i, j)$ and $a' = (i', j')$ are *adjacent* if $|i - i'| + |j - j'| = 1$. Given two cells a and b of P, a *path* in P from a to b is a sequence $q_1, q_2, \ldots, q_k$ of cells of P, with $q_1 = a$ and $q_k = b$, such that q_i and q_{i+1} are adjacent for all i with $1 \le i < k$. A *step* is a sequence of two adjacent cells $(i, j), (i', j')$. More precisely, a step is called a

North step if $j' = j$ and $i' = i + 1$;
West step if $i' = i$ and $j' = j - 1$;
South step if $i' = i - 1$ and $j' = j$;
East step if $j' = j + 1$ and $i' = i$.

A path in P is uniquely identified by a pair (q, β), where q is the starting cell and β is a string in $\{\mathsf{N}, \mathsf{W}, \mathsf{S}, \mathsf{E}\}^\star$. The number of *changes of direction* in a path $\beta = \beta_1\beta_2 \cdots \beta_r$ is defined as the number of indices i such that $\beta_i \neq \beta_{i+1}$, with $1 \le i < r$. A path is *monotone* if $\beta \in \{\mathsf{N}, \mathsf{W}\}^+$ (NW-path) or $\beta \in \{\mathsf{N}, \mathsf{E}\}^+$ (NE-path) or $\beta \in \{\mathsf{S}, \mathsf{E}\}^+$ (SE-path) or $\beta \in \{\mathsf{S}, \mathsf{W}\}^+$ (SW-path).

A polyomino P is *horizontally convex* (resp., *vertically convex*) if any row (resp. column) of P consists of exactly one segment. The class of *convex* polyominoes contains all polyominoes that are horizontally and vertically convex. It has been proved [16], Proposition 1, that a polyomino P is convex if and only if any two cells of P are joined by a monotone path in P.

The *degree of convexity* of a convex polyomino P, denoted by $\deg_c(P)$, is defined as the least integer k such that any two cells of P can be joined by a monotone path with at most k changes of direction. A convex polyomino is called k-convex if its degree of convexity is at most k. When $k = 2$ we have the class of *Z-convex* polyominoes, denoted by ZConv and introduced in [13] (Z resembles the shape of a monotone path with two changes of directions).

In the following, we consider a Z-convex polyomino as the result of the concatenation of polyominoes belonging to well-known subclasses of convex polyominoes. Given a convex polyomino P and its bounding rectangle B, we say that P is a *stack* (resp., *Ferrers diagram*, *parallelogram*, *rectangle*) if it shares exactly two adjacent (resp., three, two opposite, four) vertices with B. A stack P is a *left* (resp., *right*) stack if the column with the largest area is the last (resp., first) one. Analogously, in a left (resp., right) Ferrers diagram the largest column is the last (resp., first) one. We denote by L (resp., R) the set of left (resp., right) stacks. The set of left (resp., right) Ferrers diagrams is F_L (resp., F_R). Furthermore, we indicate by C (resp., T) the set of parallelograms (resp., rectangles). For a class A of polyominoes, $\mathsf{A}(n)$ indicates the set of $P \in A$ of area n. Lastly, the *height* of a polyomino P in $\mathsf{L} \cup \mathsf{R} \cup \mathsf{F}_L \cup \mathsf{F}_R \cup \mathsf{T}$, denoted by $\textsc{height}(P)$, is the area of its largest column.

Let j be a column of P, by $\textsc{low}(j)$ (resp., $\textsc{high}(j)$) we denote the row index of the bottom cell (resp., top cell) of j. Lastly, $\textsc{first}(P)$ (resp., $\textsc{last}(P)$) indicates the first (resp., last) column of P. The following definition

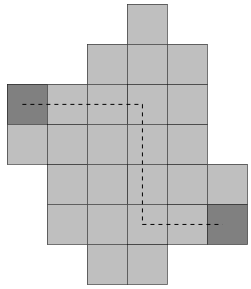

FIGURE 1. A polyomino in LR.

introduces some binary relations on the set of columns of a convex polyomino. These relations play a special role in the decomposition of a Z-convex polyomino.

Definition 2.1. Let i and j be two columns of a convex polyomino P. We say that

- i *includes* j, denoted by $j \subseteq i$, if and only if $\textsc{low}(i) \leq \textsc{low}(j)$ and $\textsc{high}(i) \geq \textsc{high}(j)$, see Figure 2a;
- i and j are *overlapping*, denoted by $i \upharpoonleft\!\downharpoonright j$, if and only if $\textsc{low}(j) < \textsc{low}(i) \leq \textsc{high}(j) < \textsc{high}(i)$ or $\textsc{low}(i) < \textsc{low}(j) \leq \textsc{high}(i) < \textsc{high}(j)$, see Figure 2b;
- i and j are *disjoint*, denoted by $i \asymp j$, if and only if $\textsc{low}(i) > \textsc{high}(j)$ or $\textsc{low}(j) > \textsc{high}(i)$, see Figure 2c.

We also write $j \subsetneq i$ if $j \subseteq i$ and $j \neq i$. Given a convex polyomino P, let e be the rightmost column of P such that $c \subseteq e$ for all columns c to the left of e. Then, P is called *descending* (resp., *ascending*) if the leftmost column j of P such that $j \upharpoonleft\!\downharpoonright e$ satisfies $\textsc{low}(e) > \textsc{low}(j)$ (resp., $\textsc{low}(e) < \textsc{low}(j)$), see Figure 2d (resp., e).

The set of descending convex polyominoes is indicated by DConv. If P is neither descending nor ascending (that is, there is not a column j such that $j \upharpoonleft\!\downharpoonright e$, hence $j \subseteq e$ for all j) then P is in $\mathsf{T} \cup \mathsf{F}_L \cup \mathsf{F}_R \cup \mathsf{L} \cup \mathsf{R}$ or belongs to the class LR containing all convex polyominoes that are the concatenation of two polyominoes, $P = P_1 \cdot P_2$, where $P_1 \in \mathsf{L} \cup \mathsf{F}_L$, $P_2 \in \mathsf{T} \cup \mathsf{R} \cup \mathsf{F}_R$ and $\textsc{first}(P_2) \subsetneq \textsc{last}(P_1)$. Since any P in LR contains a column $\bar{\jmath}$ such that $j \subseteq \bar{\jmath}$ for all columns j, one has $\deg_c(P) \leq 2$. Indeed, for any two cells $a \in P_1$ and $b \in P_2$ there is always a path from a to b with at most two changes of direction occurring on $\bar{\jmath}$, see Figure 1. The set of Z-convex polyominoes can be characterized in terms of inclusion between columns. This characterization is the basis of the decompositions that we introduce in the sequel.

Theorem 2.2. *A convex polyomino P is in ZConv if and only if for any two disjoint columns i and j of P there exist a column k, with $1 \leq i < k < j \leq \textsc{last}(P)$, such that $i \subsetneq k$ and $j \subsetneq k$.*

Proof. See [15], Theorem 1. □

We denote by DConv_2 (resp., AConv_2) the set of descending (resp., ascending) polyominoes of degree of convexity 2, see Figure 2d (resp., e). Clearly, one has

$$\mathsf{ZConv} = \mathsf{T} \cup \mathsf{F}_L \cup \mathsf{F}_R \cup \mathsf{L} \cup \mathsf{R} \cup \mathsf{LR} \cup \mathsf{AConv}_2 \cup \mathsf{DConv}_2,$$

where the unions are disjoint. Because of symmetry one has $|\mathsf{DConv}_2(n)| = |\mathsf{AConv}_2(n)|$, hence from here on we consider only descending polyominoes. Thus, for any $n \geq 0$ one has

$$|\mathsf{ZConv}(n)| = |\mathsf{T}(n)| + 2 \cdot |\mathsf{F}_L(n)| + 2 \cdot |\mathsf{L}(n)| + |\mathsf{LR}(n)| + 2 \cdot |\mathsf{DConv}_2(n)|, \tag{2.1}$$

and the counting problem for ZConv is reduced to the counting problem for DConv_2 and to some other simpler counting problems (easily solved in polynomial time, see Sect. 5).

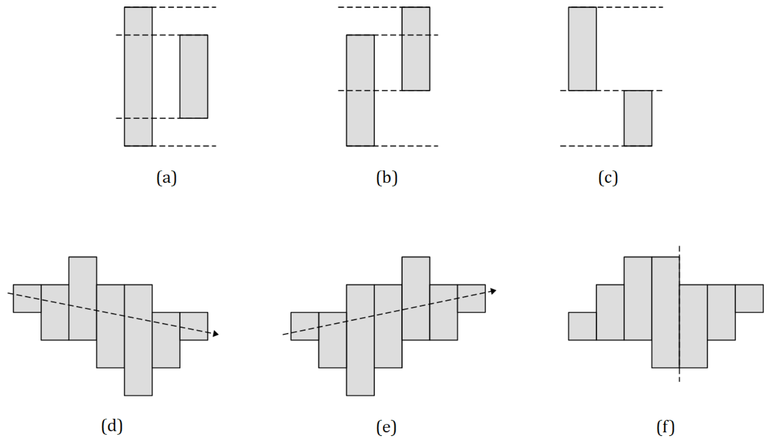

FIGURE 2. An included column (a), two overlapping columns (b), two disjoint columns (c), a polyomino in DConv_2 (d), a polyomino in AConv_2 (e) and a polyomino in LR (f).

3. Polyominoes decomposition

Computing $|\mathsf{DConv}_2(n)|$ is not immediate. Our approach is based on breaking a polyomino in DConv down into simpler polyominoes. As a matter of fact, a descending convex polyomino is the concatenation of at most four simple polyominoes. We introduce a decomposition that is the first of many steps leading to a set of formulas for computing $|\mathsf{DConv}(n)|$.

Definition 3.1. [standard decomposition] Let $P \in \mathsf{DConv}$. Then, we can decompose P as $P = L \cdot F \cdot C \cdot R$ for suitable polyominoes $L \in \mathsf{L} \cup \mathsf{T} \cup \mathsf{F}_L$, $F \in \mathsf{F}_R \cup \mathsf{T} \cup \{\epsilon\}$, $C \in \mathsf{C} \cup \mathsf{T} \cup \mathsf{F}_R$, and $R \in \mathsf{R} \cup \mathsf{T} \cup \mathsf{F}_R \cup \{\epsilon\}$ such that

- if $F \neq \epsilon$ then $\textsc{first}(F) \subsetneq \textsc{last}(L)$, $\textsc{low}(\textsc{last}(L)) = \textsc{low}(\textsc{last}(F))$ and $\textsc{last}(F) \upharpoonleft\!\downharpoonright \textsc{first}(C)$;
- $\textsc{last}(L) \upharpoonleft\!\downharpoonright \textsc{first}(C)$ and $\textsc{low}(\textsc{last}(L)) > \textsc{low}(\textsc{first}(C))$;
- if $R \neq \epsilon$ then $\textsc{first}(R) \subsetneq \textsc{last}(C)$ and $\textsc{low}(\textsc{last}(C)) < \textsc{low}(\textsc{first}(R))$.

The standard decomposition of a polyomino $P \in \mathsf{DConv}$ is unique. Indeed:

- $\textsc{last}(L)$ is the rightmost column $\bar{\jmath}$ of P such that $j \subseteq \bar{\jmath}$ for all columns j the the left of $\bar{\jmath}$;
- $\textsc{first}(C)$ is the first column e to the right of $\textsc{last}(L)$ such that $\textsc{last}(L) \upharpoonleft\!\downharpoonright e$;
- all columns between $\textsc{last}(L)$ and $\textsc{first}(C)$ belong to F;
- $\textsc{first}(R)$ is the first column j to the right of $\textsc{first}(C)$ such that $\textsc{low}(j) > \textsc{low}(j-1)$.

Figure 3 illustrates the standard decomposition of some descending convex polyominoes. Given $P \in \mathsf{DConv}_2$, we point out that by Theorem 2.2, each column c of P to the right of $\textsc{last}(L)$ satisfies the relation $c \upharpoonleft\!\downharpoonright \textsc{last}(L)$ or $c \subseteq \textsc{last}(L)$.

The following Lemma provides a property used to define some subclasses of DConv_2. These classes appear in the refinements of the standard decomposition that we do to compute $|\mathsf{DConv}_2(n)|$.

Lemma 3.2. *Let $P \in \mathsf{DConv}$ and consider its standard decomposition, $P = L \cdot F \cdot C \cdot R$. Then, P belongs to DConv_2 if and only if for any two disjoint columns j_1, j_2 of P (with $j_1 < j_2$) one has $j_1 \in L \wedge (j_2 \subseteq \textsc{last}(L) \vee j_2 \subseteq \bar{\jmath})$, where $\bar{\jmath}$ is the rightmost column of C that includes j_1.*

Proof. ($\Rightarrow$) Let $P \in \mathsf{DConv}_2$ and suppose that there exist two disjoint columns j_1 and j_2 with $j_1 \notin L$ and $j_1 < j_2$. Obviously, j_1 cannot belong to R, since this implies $j_2 \subseteq j_1$. If j_1 is a column of F or C, then any column j_2 to its right such that $j_1 \asymp j_2$ also satisfies $j_2 \asymp j'$ for all $j' < j_1$ (since P is descending). In particular, one has $j_2 \asymp \textsc{last}(L)$, which implies $\deg_c(P) > 2$, since no column in P includes both j_2 and $\textsc{last}(L)$.

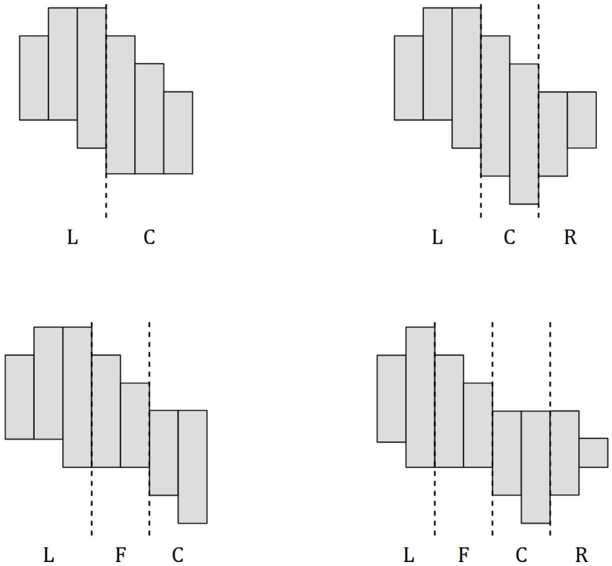

FIGURE 3. The standard decomposition of some polyominoes in DConv_2.

Thus, j_1 must be in L. By Definition 3.1, j_2 can only belong to C or R. If j_2 belongs to C there exists (by Thm. 2.2) at least one column j, with $j_1 < j < j_2$, that includes both j_1 and j_2. It is immediate that column j is in C since for all columns c in L or F one has $\textsc{low}(c) > \textsc{low}(j_2)$. Thus, we simply consider the rightmost column $\bar{\jmath}$ with that property. Otherwise, j_2 is in R. If $\textsc{low}(j_2) < \textsc{low}(\textsc{last}(L))$ the column including j_2 must be in C and we apply the above reasoning to get $\bar{\jmath}$. Lastly, if $\textsc{low}(j_2) \geq \textsc{low}(\textsc{last}(L))$ both j_1 and j_2 are included in $\textsc{last}(L)$.

($\Leftarrow$) The hypotheses of Theorem 2.2 are satisfied hence $P \in \mathsf{DConv}_2$. $\square$

We indicate by LCR_2 (resp., LC_2, LFCR_2, LFC_2) the subset of DConv_2 containing polyominoes whose standard decomposition is $L \cdot C \cdot R$ (resp., $L \cdot C$, $L \cdot F \cdot C \cdot R$, $L \cdot F \cdot C$). Furthermore, we introduce a subset of DConv_2 called Z_2. This subset is one of the components we need when we think to a polyomino in DConv as the result of some combinatorial operations applied to polyominoes that are easier to count.

Definition 3.3 (Z_2)**.** Z_2 is the set of all P in $\mathsf{LCR}_2 \cup \mathsf{LC}_2$ such that:

1. $l \subsetneq \textsc{first}(C)$ for all columns l of L, with $\mathrm{A}(l) < \mathrm{A}(\textsc{last}(L))$;
2. $c \upharpoonleft\!\downharpoonright \textsc{last}(L)$ for all columns c of R (if $R \neq \epsilon$).

In the sequel, polyominoes with disjoint columns will be recursively decomposed into simpler polyominoes. Thus, given a class A of polyominoes we consider the partition $\mathsf{A} = \mathsf{A}^\bullet \cup \mathsf{A}^\circ$, where $\mathsf{A}^\bullet$ (resp., A°) contains those polyominoes in A that have (resp., do not have) disjoint columns. In particular, there is a subset of $\mathsf{Z}_2^\bullet$ which plays a special role in the decomposition of a polyomino in $\mathsf{DConv}_2^\bullet$.

Definition 3.4 (s-$\mathsf{Z}_2^\bullet$)**.** The set s-$\mathsf{Z}_2^\bullet$ contains all $P \in \mathsf{Z}_2^\bullet$ that can be written as $P = L \cdot C \cdot R$ with $L \in \mathsf{L} \cup \mathsf{T} \cup \mathsf{F}_L$, $C \in \mathsf{C} \cup \mathsf{T} \cup \mathsf{F}_R$, $R \in \mathsf{R} \cup \mathsf{T} \cup \mathsf{F}_R$ and:

- $l \subsetneq \textsc{last}(C)$ for all columns l of L, with $\mathrm{A}(l) < \mathrm{A}(\textsc{last}(L))$;
- $\textsc{low}(\textsc{last}(C)) \leq \textsc{low}(\textsc{first}(R))$;
- $\textsc{first}(R) \subsetneq \textsc{last}(C)$;
- $\textsc{first}(L) \asymp \textsc{first}(R)$.

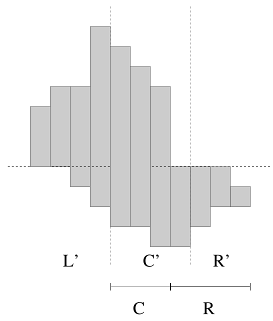

FIGURE 4. Standard decomposition vs. decomposition in Definition 3.4.

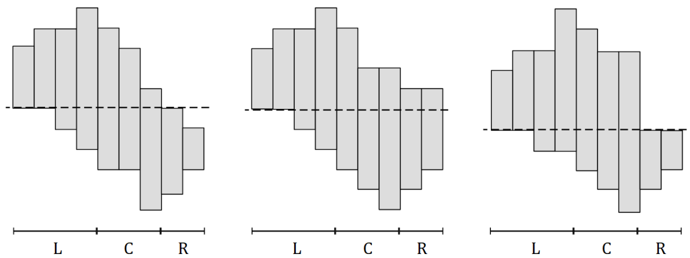

FIGURE 5. From left to right: a polyomino in $\mathsf{Z}_2^\bullet \setminus$ s-$\mathsf{Z}_2^\bullet$, a polyomino in Z_2° and a polyomino in s-$\mathsf{Z}_2^\bullet$.

We point out that polyominoes C and R in Definition 3.4 are not necessarily the same C and R that appear in the standard decomposition of P. More precisely, if we consider the standard decomposition $P = L' \cdot C' \cdot R'$ compared to $P = L \cdot C \cdot R$ given by Definition 3.4, one might have $C \neq C'$, $R \neq R'$, with $L = L'$, and $C' \cdot R' = C \cdot R$, see Figure 4. Figure 5 shows examples of polyominoes in the above defined sets.

4. Operations on polyominoes

In this section we introduce two operations used to obtain polyominoes in DConv_2 by appropriately combining polyominoes belonging to classes that are easier to count.

We start by refining the standard decomposition $P = L \cdot F \cdot C \cdot R$ of a polyomino P in DConv_2. Write L as $L = L' \cdot D$, with $D \in \mathsf{T}$ and $\textsc{last}(L') \subsetneq \textsc{first}(D)$, and consider the leftmost column c of L such that $c \upharpoonleft\!\downharpoonright \textsc{first}(C)$ (possibly $c = \textsc{first}(D)$). Notice that L cannot contain a column c such that $c \asymp \textsc{first}(C)$, and P cannot contain a column e such that $e \asymp \textsc{last}(L)$. Indeed, in both cases the degree of convexity of P would be at least 3 (by Thm. 2.2). Now, let e be the leftmost column of R such that $e \subsetneq \textsc{last}(D)$ (remark: $P \in \mathsf{Z}_2$ if $e = \epsilon$ and $c = \textsc{first}(D)$). Columns c and e lead to the decomposition

$$P = L' \cdot D \cdot F \cdot C \cdot R \quad (\text{if } c = \textsc{first}(D) \wedge e = \epsilon) \text{ or} \tag{4.1}$$
$$P = L_1 \cdot c \cdot L_2 \cdot D \cdot F \cdot C \cdot R_1 \cdot e \cdot R_2 \quad (\text{if } c \neq \textsc{first}(D) \wedge e \neq \epsilon) \text{ or} \tag{4.2}$$
$$P = L_1 \cdot c \cdot L_2 \cdot D \cdot F \cdot C \cdot R \quad (\text{if } c \neq \textsc{first}(D) \wedge e = \epsilon) \text{ or} \tag{4.3}$$
$$P = L' \cdot D \cdot F \cdot C \cdot R_1 \cdot e \cdot R_2 \quad (\text{if } c = \textsc{first}(D) \wedge e \neq \epsilon) \tag{4.4}$$

(remark: L_1, L_2, F, R, R_1, R_2 might be null). See Figure 6 for an example, where c, e, R_2 are red, D is black, L_1, C, R_1 are yellow and L_2 is null.

It is immediate that the polyominoes $c \cdot L_2 \cdot D \cdot e \cdot R_2$ (case 4.2), $c \cdot L_2 \cdot D$ (case 4.3) and $D \cdot e \cdot R_2$ (case 4.4) belong to $\mathsf{T} \cup \mathsf{L} \cup \mathsf{R} \cup \mathsf{F}_L \cup \mathsf{LR}$, whereas the polyominoes $L' \cdot D \cdot C \cdot R$ (case 4.1), $L_1 \cdot D \cdot C \cdot R_1$ (case 4.2),

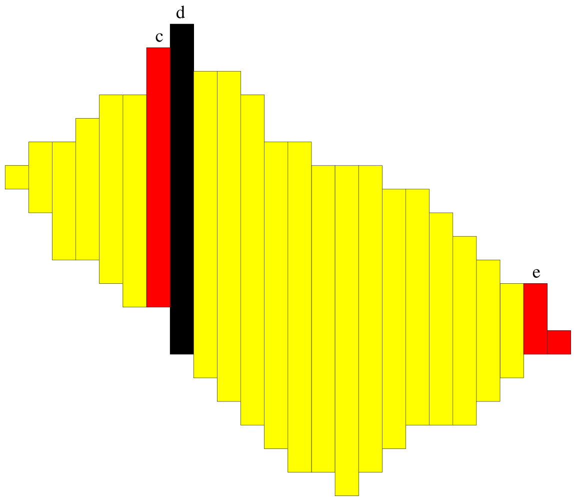

FIGURE 6. The refinement of the standard decomposition of a polyomino in $\mathsf{DConv}_2 \setminus \mathsf{Z}_2$.

$L_1 \cdot D \cdot C \cdot R$ (case 4.3) and $L' \cdot D \cdot C \cdot R_1$ (case 4.4) are in Z_2 (concatenation is done by keeping the position $\textsc{low}(f)$ of each column f fixed).

This refinement of the standard decomposition suggests us to define a partial function $\oplus : (\mathsf{F}_R \cup \{\epsilon\}) \times (\mathsf{T} \cup \mathsf{L} \cup \mathsf{R} \cup \mathsf{F}_L \cup \mathsf{LR}) \times \mathsf{Z}_2 \mapsto \mathsf{DConv}_2 \cup \{\perp\}$ (see Fig. 7) that is used to write the main equation for counting DConv_2. This definition is rather technical as it comprises a lot of conditions, used to ensure that the particular concatenation of polyominoes provides a polyomino in DConv_2.

Definition 4.1 ($\oplus$)**.** Let $F \in \mathsf{F}_R \cup \{\epsilon\}$, $P \in \mathsf{T} \cup \mathsf{L} \cup \mathsf{R} \cup \mathsf{F}_L \cup \mathsf{F}_R \cup \mathsf{LR}$ and $Q \in \mathsf{Z}_2$. Write P as $P = L \cdot D \cdot R$ where $D \in \mathsf{T}$ and $g \subsetneq \textsc{first}(D)$ for g in L or R (L and R possibly null). Furthermore, write Q as $Q = L' \cdot D' \cdot C' \cdot R'$, with $D' \in \mathsf{T}$, $\textsc{first}(C') \upharpoonleft\!\downharpoonright \textsc{last}(D')$ and $\textsc{last}(L') \subsetneq \textsc{first}(D')$ ($L' \cdot D' \in T \cup \mathsf{F}_L \cup \mathsf{L}$, $C' \in \mathsf{C} \cup \mathsf{T} \cup \mathsf{F}_R$, $R' \in \mathsf{R} \cup \mathsf{F}_R \cup \mathsf{T} \cup \{\epsilon\}$). Then, $\oplus(F, P, Q)$ is a polyomino W in DConv_2, with

$$W = L' \cdot L \cdot D' \cdot F \cdot C' \cdot R' \cdot R,$$

if and only if $D = D'$ and for $d = \textsc{last}(D)$, $d' = \textsc{last}(D')$, all the following conditions are satisfied:

(if $L, L' \neq \epsilon$)

1. $\textsc{high}(d) - \textsc{high}(\textsc{first}(L)) < \textsc{high}(d') - \textsc{high}(\textsc{last}(L'))$;
2. $\textsc{high}(d) - \textsc{low}(\textsc{first}(L)) \geq \textsc{high}(d') - \textsc{low}(\textsc{last}(L'))$;

(if $F \neq \epsilon$)

3. $\mathrm{A}(\textsc{first}(F)) < \mathrm{A}(d')$;
4. $\mathrm{A}(\textsc{last}(F)) \geq \mathrm{A}(d') - (\textsc{high}(d') - \textsc{high}(\textsc{first}(C')))$;

(if $R, R' \neq \epsilon$)

5. $\textsc{high}(d) - \textsc{high}(\textsc{first}(R)) \geq \textsc{high}(d') - \textsc{high}(\textsc{last}(R'))$.

If F, P and Q do not satisfy conditions 1–5 we set $\oplus(F, P, Q) = \perp$ (undefined).

Notice that $\oplus(F, P, Q) = Q$ if and only if $F = \epsilon$, $P = D$ and $Q = L \cdot D \cdot C \cdot R$. Moreover, $\oplus$ is immediately extended to sets of polyominoes by setting

$$\oplus(\mathsf{A}, \mathsf{B}, \mathsf{D}) = \{\oplus(F, P, Q) \mid F \in \mathsf{A}, P \in \mathsf{B}, Q \in \mathsf{D}\}.$$

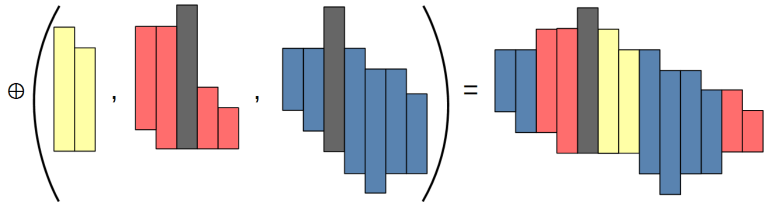

FIGURE 7. The function $\oplus$.

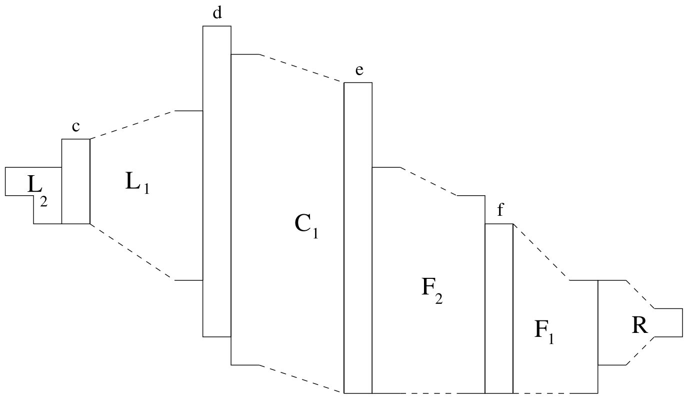

FIGURE 8. The decomposition of a polyomino in $\mathsf{Z}_2^\bullet$, case (4.6).

By decompositions (4.2), (4.3), (4.4) and Definition 4.1 it follows that

$$\mathsf{DConv}_2 = \bigcup_{\substack{F \in \mathsf{F}_R \cup \{\epsilon\} \\ P \in \mathsf{T} \cup \mathsf{L} \cup \mathsf{R} \cup \mathsf{F}_L \cup \mathsf{F}_R \cup \mathsf{LR},\, P \neq \epsilon \\ Q \in \mathsf{Z}_2,\, Q \neq \epsilon}} \oplus(F, P, Q). \tag{4.5}$$

We point out that all unions in (4.5) are disjoint. Indeed, if both $\oplus(F, P, Q)$ and $\oplus(F', P', Q')$ are defined, the polyomino $\oplus(F, P, Q)$ is equal to $\oplus(F', P', Q')$ if and only if $F = F'$, $P = P'$ and $Q = Q'$. In the next section we proceed further, and we show how to obtain a formula for computing $|\mathsf{DConv}_2(n)|$, from (4.5).

From (4.5) it is clear that the main subproblem is computing $|\mathsf{Z}_2(n)|$, since $|\mathsf{F}_L(n)|$ and $|\mathsf{T}(n) \cup \mathsf{L}(n) \cup \mathsf{R}(n) \cup \mathsf{F}_L(n) \cup \mathsf{F}_R(n) \cup \mathsf{LR}(n)|$ are easily computed in polynomial time (see Sect. 5). So, we focus on the most difficult problem, that is, the computation of $|\mathsf{Z}_2^\bullet(n)|$ (a formula for $|\mathsf{Z}_2^\circ(n)|$ can be obtained quite easily, see Section 5). To this aim, we introduce a particular refinement (see Fig. 8) of the standard decomposition $L \cdot C \cdot R$ (R possibly null) of a given $P \in \mathsf{Z}_2^\bullet$.

Write L as $L = L' \cdot D$, where D is a rectangle such that $\mathrm{A}(\textsc{last}(L')) < \mathrm{A}(\textsc{first}(D))$. Let e be the rightmost column of C such that $\textsc{last}(L') \subsetneq e$ (e exists by Def. 3.3). If $e = \textsc{last}(C)$ set $c = \textsc{first}(L)$, otherwise let c be the leftmost column in L' such that c is included in e but not in column $e+1$. Lastly, consider the leftmost column f in $C \cdot R$ such that $f \asymp c$.

We stress that if f belongs to C one has $\textsc{low}(f') = \textsc{low}(e)$ for any column f' of C to the right of e. Indeed, if $\textsc{low}(f') < \textsc{low}(e)$ no column of P includes both f' and c, and so $\deg_c(P) > 2$ by Theorem 2.2. So, there exist two right Ferrers diagrams F_1, F_2, two parallelograms (or rectangles) C_1, C_2, and two right stacks (or Ferrers diagrams) R_1, R_2 such that

$$P = L_2 \cdot c \cdot L_1 \cdot D \cdot C_1 \cdot e \cdot F_2 \cdot f \cdot F_1 \cdot R \quad (\text{if } f \in C) \tag{4.6}$$

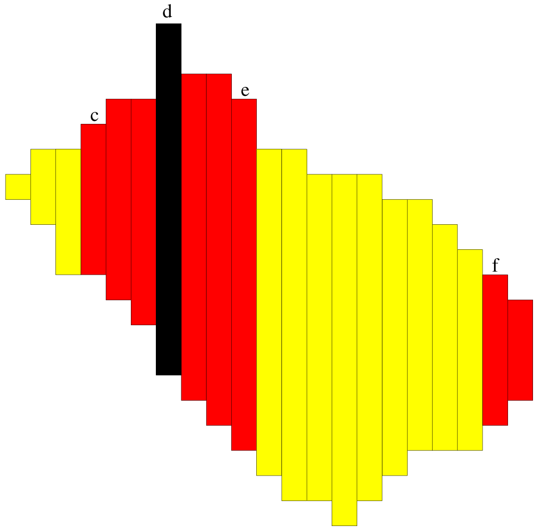

FIGURE 9. The decomposition of P in $\mathsf{Z}_2^\bullet$ gives rise to $P'' \in$ s-$\mathsf{Z}_2^\bullet$ (red) and $Q'' \in \mathsf{Z}_2$ (yellow) such that $P = P'' \parallel Q''$.

or

$$P = L_2 \cdot c \cdot L_1 \cdot D \cdot C_1 \cdot e \cdot C_2 \cdot R_2 \cdot f \cdot R_1 \quad (\text{if } f \in R), \tag{4.7}$$

where $g \upharpoonleft\downharpoonright c$ for any column g in F_2 and $\textsc{low}(h) = \textsc{low}(e)$ for all h in F_2 or in F_1 (in case (4.6)), and where no column of $C_2 \cdot R_2$ is disjoint from c (in case (4.7)). In particular, in case (4.7) e could be the last column of C and then L_2 and C_2 would be null. Moreover, $C_2 = \epsilon$ implies $L_2 = \epsilon$, since one has $\textsc{last}(L_2) \asymp f$ and so there must be a column to the right of e that includes both $\textsc{last}(L_2)$ and f. Similarly, in case (4.6) one has that $L_2 \neq \epsilon$ implies $F_2 \neq \epsilon$, since $\textsc{last}(L_2) \asymp f$ implies the existence of a column including both $\textsc{last}(L_2)$ and f. See Figure 8 for an example of the decomposition of a polyomino in $\mathsf{Z}_2^\bullet$.

We associate with case (4.6) the two polyominoes $P' = c \cdot L_1 \cdot D \cdot C_1 \cdot e \cdot f \cdot F_1 \cdot R$ and $Q' = L_2 \cdot D \cdot F_2$, see Figure 8 (we recall that concatenation is done by keeping the row index of the bottom cell of each column). Similarly, in case (4.7) we consider $P'' = c \cdot L_1 \cdot D \cdot C_1 \cdot e \cdot f \cdot R_1$ and $Q'' = L_2 \cdot D \cdot C_2 \cdot R_2$. By construction it follows that P' and P'' belong to s-$\mathsf{Z}_2^\bullet$, whereas Q' and Q'' are in $\in \mathsf{Z}_2 \cup \mathsf{T}$. Indeed, by definition of c, $\textsc{last}(L_2)$ is included in $\textsc{first}(F_2)$ (in case (4.6)) or in $\textsc{first}(C_2)$ (in case (4.7)). Moreover, Q' and Q'' are in T if and only if $P \in$ s-Z_2.

Figure 9 illustrates case (4.7). Here, P'' consists of 8 red columns and 1 black column (joined to form a single polyomino) whereas Q'' consists of 12 yellow columns and 1 black column.

The idea we exploit to count $\mathsf{Z}_2^\bullet$ is to obtain a polyomino P in $\mathsf{Z}_2^\bullet \setminus$ s-$\mathsf{Z}_2^\bullet$ by somehow combining two polyominoes that are uniquely determined by the decomposition of P seen above. In other words, we define an operator $\parallel$: $\mathsf{Z}_2 \times$ s-$\mathsf{Z}_2^\bullet \mapsto \mathsf{Z}_2^\bullet$, that we call a *pseudo-shuffle*, see Figure 9.

Definition 4.2 (pseudo-shuffle $\parallel$)**.** Let $P \in \mathsf{Z}_2$ and $P' \in$ s-$\mathsf{Z}_2^\bullet$. Consider the standard decomposition of P, $P = L \cdot C \cdot R$ or $P = L \cdot C$, and write P' as in Definition 3.4, $P' = L' \cdot C' \cdot R'$. Lastly, let $L = L_1 \cdot D$ and $L' = L_2 \cdot D'$, with $D, D' \in \mathsf{T}$ and $\textsc{last}(L_1) \subsetneq \textsc{last}(D)$, $\textsc{last}(L_2) \subsetneq \textsc{last}(D')$.

Let $d = \textsc{last}(D)$ and $d' = \textsc{last}(D')$, then

$$P \parallel P' = L_1 \cdot L_2 \cdot D' \cdot C' \cdot C \cdot R \cdot R'$$

is a polyomino in $\mathsf{Z}_2^\bullet$ if and only if $D = D'$ and (if $L_2 \neq \epsilon$)

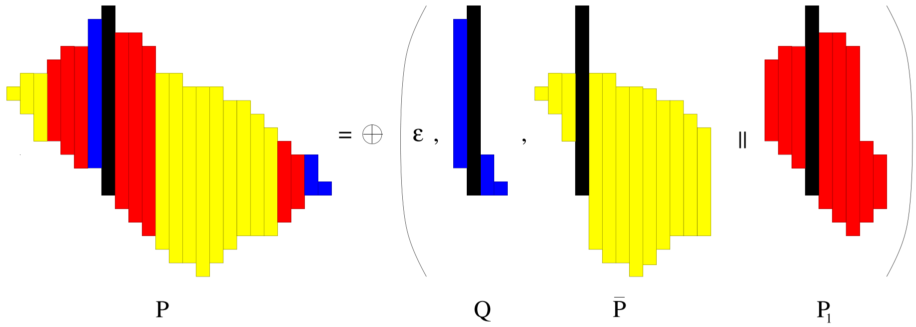

FIGURE 10. The first two steps in the decomposition of a polyomino $P \in \mathsf{DConv}_2^\bullet$, $P = \oplus(\epsilon, Q, \bar{P} \parallel P_1)$, with $Q \in \mathsf{LR}, \bar{P} \in \mathsf{Z}_2^\bullet, P_1 \in$ s-$\mathsf{Z}_2^\bullet$.

1. $\textsc{high}(d) - \textsc{high}(\textsc{last}(L_1)) > \textsc{high}(d') - \textsc{high}(\textsc{first}(L_2))$;
2. $\textsc{high}(d) - \textsc{low}(\textsc{last}(L_1)) \leq \textsc{high}(d') - \textsc{low}(\textsc{first}(L_2))$;

and (if $C \neq \epsilon$)

3. $\textsc{high}(d') - \textsc{high}(\textsc{last}(C')) < \textsc{high}(d) - \textsc{high}(\textsc{first}(C))$;
4. $\textsc{high}(d) - \textsc{low}(\textsc{last}(C')) \leq \textsc{high}(d') - \textsc{low}(\textsc{first}(C))$;

and (if $R \neq \epsilon$)

4. $\textsc{high}(d) - \textsc{high}(\textsc{last}(R)) < \textsc{high}(d') - \textsc{high}(\textsc{first}(R'))$;
5. $\textsc{high}(d) - \textsc{low}(\textsc{last}(R)) \geq \textsc{high}(d') - \textsc{low}(\textsc{first}(R'))$;
6. $\textsc{high}(d) - \textsc{high}(\textsc{last}(R)) < \textsc{high}(d') - \textsc{low}(\textsc{first}(L_2))$;

or (if $R = \epsilon$ and $C \neq \epsilon$)

7. $\textsc{high}(d) - \textsc{high}(\textsc{first}(C)) < \textsc{high}(d') - \textsc{high}(\textsc{first}(L_2))$.

We set $P \parallel Q = \perp$ if P and Q do not satisfy all conditions of Definition 4.2. By Definition 4.2 it follows that a polyomino $P \in \mathsf{Z}_2^\bullet$ is in s-$\mathsf{Z}_2^\bullet$ or can be uniquely written as the pseudo-shuffle of finitely many polyominoes,

$$P = \underbrace{(\cdots(}_{k-1} P_k \parallel P_{k-1}) \parallel P_{k-2}) \parallel \cdots) \parallel P_2) \parallel P_1, \tag{4.8}$$

with $P_i \in$ s-$\mathsf{Z}_2^\bullet$ for $1 \leq i < k$, and $P_k \in \mathsf{Z}_2^\circ \cup$ s-$\mathsf{Z}_2^\bullet$. Figure 10 illustrates the first two steps in the decomposition of a polyomino in $\mathsf{DConv}_2^\bullet$ (then, we consider the decomposition $\bar{P} = \bar{\bar{P}} \parallel P_2$, and so on, see Figure 11 for the full decomposition of $\bar{P}$).

The pseudo-shuffle is immediately extended to sets of polyominoes by setting

$$\mathsf{A} \parallel \mathsf{B} = \{P \parallel Q \mid P \in \mathsf{A}, Q \in \mathsf{B}\}.$$

From here on we denote by $\mathsf{F}_R(n,h,e)$ (resp., $\mathsf{LR}(n,h,e)$, $\mathsf{T}(n,h,e)$, $\mathsf{L}(n,h,e)$) the set of right Ferrers diagrams (resp., polyominoes in LR, rectangles, left stacks) of area n, height h and with e columns of area h. We also consider the set $\mathsf{Z}_2(n,h,e)$ of all $P \in \mathsf{Z}_2(n)$ where the polyomino L in the standard decomposition of P, $P = L \cdot C \cdot R$, can be written as $L = L_1 \cdot D$ where D is a rectangle with e columns of height h such that $\textsc{height}(L_1) < h$. From the previous definitions and (4.5) it follows that

$$\mathsf{DConv}_2^\bullet(n) = \bigcup_{m,d,e,a,c} \oplus(\mathsf{F}_R(a,c), \mathsf{G}(n-m-a+d\cdot e, d, e), \mathsf{Z}_2^\bullet(m,d,e)) \tag{4.9}$$

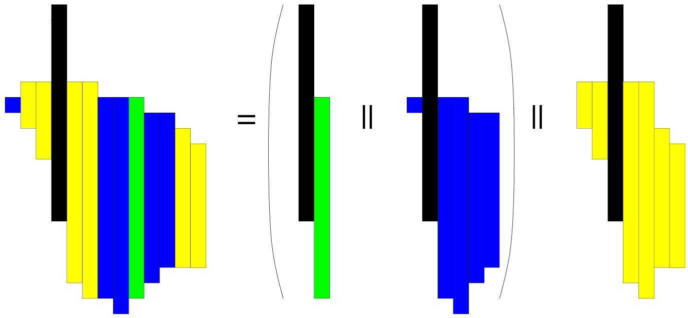

FIGURE 11. The pseudo-shuffle decomposition of the polyomino $\bar{P}$ in Figure 10.

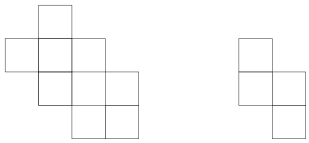

FIGURE 12. The polyominoes of smallest area in $\mathsf{Z}_2^{\bullet}$ (left) and Z_2° (right).

where $\mathsf{G}(b,d,e) = \mathsf{L}(b,d,e) \cup \mathsf{F}_L(b,d,e) \cup \mathsf{F}_R(b,d,e) \cup \mathsf{R}(b,d,e) \cup \mathsf{T}(b,d,e) \cup \mathsf{LR}(b,d,e)$, $\mathsf{F}_R(a,c) = \sum_{i=1}^{\lfloor a/c \rfloor} \mathsf{F}_R(a,c,i)$ and the union is taken on all a, c, d, e such that:

- $9 \leq m \leq n$ ($\mathsf{Z}_2^{\bullet}(m,d,e) = \emptyset$ for $m < 9$, see Fig. 12);
- $3 \leq d \leq m-6$ (for $d < 3$ there can be no disjoint columns, the sum of the areas of the two disjoint columns with the area of the column including them is at least 6, see Fig. 12);
- $1 \leq e \leq \lfloor (m-6)/d \rfloor$;
- $0 \leq a \leq n-m$;
- $0 \leq c \leq a$ and $c < d$.

Analogously, one has

$$\mathsf{DConv}^{\circ}(n) = \bigcup_{m,d,e,a,c} \oplus \left(\mathsf{F}_R(a,c), \mathsf{G}(n-m-a+d\cdot e, d, e), \mathsf{Z}_2^{\circ}(m,d,e)\right), \tag{4.10}$$

with

- $4 \leq m \leq n$;
- $2 \leq d \leq m-2$;
- $1 \leq e \leq \lfloor (m-2)/d \rfloor$ (at least two cells for $\textsc{first}(C)$);
- $0 \leq a \leq n-m$;
- $0 \leq c \leq a$ and $c < d$.

Lastly, from (4.8) one has

$$\mathsf{Z}_2^{\bullet}(n,d,e) = \text{s-}\mathsf{Z}_2^{\bullet}(n,d,e) \cup \bigcup_{m,d,e} \mathsf{Z}_2(n-m+d\cdot e, d, e) \parallel \text{s-}\mathsf{Z}_2^{\bullet}(m,d,e), \tag{4.11}$$

where $9 \leq m \leq n-2$, $3 \leq d \leq m-6$ and $1 \leq e \leq \lfloor (m-6)/d \rfloor$. Notice that in (4.11) m is at most $n-2$ since the area of A in $A \parallel B$ is at least $d \cdot e + 2$ (the area of $\textsc{first}(C)$ is at least 2).

5. Computing $|\mathsf{ZConv}(n)|$

We have seen in the previous section that a polyomino P in DConv_2 can be obtained by exploiting the functions $\oplus$ and $\|$, see (4.9), (4.10) and (4.11). Definitions 4.1 and 4.2 state some conditions that are closely related to the decomposition of P. More precisely, these conditions refer to the area and to the (relative) position of some suitable columns of each polyomino used in the decomposition. So, in order to compute $|\mathsf{DConv}_2(n)|$ from (4.9), (4.10) and (4.11) we need to introduce some functions that count (with respect some suitable parameters) the polyominoes that appear in the decomposition.

Let $S_L(n,h)$ (resp., $S_R(n,h)$) be the number of polyominoes of height h in $\mathsf{L}(n) \cup \mathsf{F}_L(n) \cup \mathsf{T}(n)$ (resp., $\mathsf{R}(n) \cup \mathsf{F}_R(n) \cup \mathsf{T}(n)$). Obviously one has $S_L(n,h) = S_R(n,h)$. It is immediate that

$$S_L(n,h) = \sum_{1 \le i \le h} (h-i+1) \cdot S_L(n-h,i),$$

with $S_L(i,i) = 1$ and $S_L(j,i) = 0$ for $i > j$. Notice that $S_L(n,h)$ is also the of number of unimodal sequences of weight n with largest term h. This is the coefficient of q^n in $\frac{q^h}{(1-q)^2(1-q^2)^2 \cdots (1-q^{h-1})^2(1-q^h)^2}$, see [17], Section 2.5.

In the sequel, we indicate by $S(n,h)$ the number of polyominoes of height h in $\mathsf{L}(n) \cup \mathsf{F}_L(n)$. Obviously, one has $S(n,h) = S_L(n,h) - \max(1-(n \mod h), 0)$.

We also denote by $S_L(m,p,q,y)$ (resp., $S_R(m,p,q,y)$) the number of polyominoes P counted by $S_L(m,p)$ (resp., $S_R(m,p)$) with smallest column of area q and $|\textsc{low}(\textsc{first}(P)) - \textsc{low}(\textsc{last}(P))| = y$ (see Fig. 13). We point out that the number of right Ferrers diagrams of area m, height p and smallest column of area q is $S_L(m,p,q,0)$. This number is easily computed by means of the following equation,

$$S_L(m,p,q,0) = \sum_{e=q}^{p} S_L(m-p,e,q,0) \quad (\text{if } m > p),$$

with $S_L(m,p,q,0) = 0$ if $p \ne q$ and $p+q > m$, and $S_L(m,p,q,0) = 1$ if $m = p = q$ or $m = p+q$.

Figure 13 suggests us how to compute $S_L(m,p,q,y)$ when $y > 0$. Indeed, if a polyomino P counted by $S_L(m,p,q,y)$ has at least three columns one necessarily has $m-p-q > 0$, and then

$$S_L(m,p,q,y) = \sum_{i=0}^{\gamma_1} \sum_{z=0}^{\gamma_2} S_L(m-p, q+i+z, q, i),$$

where $\gamma_1 = \min(y, m-p-2q)$ and $\gamma_2 = \min(p-q-y, m-p-2q-i)$.

Otherwise, the polyomino P has at most two columns, which means $m-p-q \le 0$. Thus, we have $S_L(m,p,q,y) = 1$ if $m = p+q \wedge y \le p-q$ or $p = q = m \wedge y = 0$, and $S_L(m,p,q,y) = 0$ if $m \ne p+q \wedge p \ne q$ or $y > p-q$.

Lastly, we indicate by $S_L^{\square}(m,p,q,y,e)$ the number of polyominoes in $\mathsf{R}(m,p,e) \cup \mathsf{F}_R(m,p,e) \cup \mathsf{T}(m,p,e)$ that are counted by $S_L(m,p,q,y)$. It is immediate that for $e > 0$ one has

$$S_L^{\square}(m,p,q,y,e) = \sum_{r=q}^{p-1} \sum_{i=\delta_1}^{\delta_2} S_L(m-p \cdot e, r, q, i),$$

where $\delta_1 = \max(0, r-q-(p-q-y))$ and $\delta_2 = \min(y, r-q)$ (see Fig. 13 and consider the second to last column as the column of area r). The function $S_R^{\square}(m,p,q,y,e)$ is defined similarly.

Functions S_L and S allow us to easily compute $|\mathsf{LR}(n)|$, one of the values that appear in the formula for $|\mathsf{ZConv}(n)|$, see (2.1). Indeed, since any polyomino $P \in \mathsf{LR}$ is (uniquely) decomposed as the concatenation

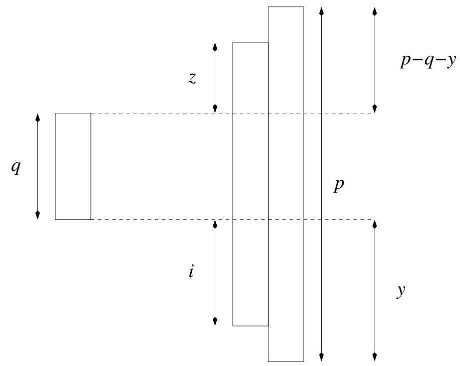

FIGURE 13. The recursive decomposition of a polyomino counted by $S_L(m,p,q,y)$.

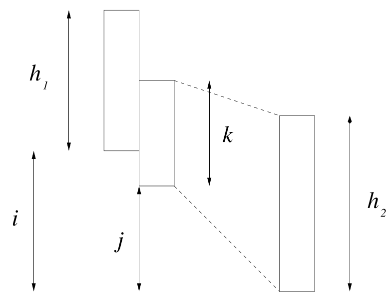

FIGURE 14. The recursive decomposition of a polyomino counted by $C(n,h_1,h_2,i)$.

$P = P_1 \cdot P_2$ of two polyominoes $P_1 \in \mathsf{L} \cup \mathsf{F}_L$ and $P_2 \in \mathsf{R} \cup \mathsf{F}_R \cup \mathsf{T}$ with $\textsc{first}(P_2) \subseteq \textsc{last}(P_1)$ and $\textsc{height}(P_1) > \textsc{height}(P_2)$, it follows that for $n > 3$ one has:

$$|\mathsf{LR}(n)| = \sum_{h=2}^{n-2} \sum_{a=h+1}^{n-1} \sum_{k=1}^{\min(h-1,n-a)} (h-k+1) \cdot S_L(a,h) \cdot S(n-a,k).$$

In the sequel we need to know the number $C(n,h_1,h_2,i)$ of descending parallelograms P of area n with $\mathrm{A}(\textsc{first}(P)) = h_1$, $\mathrm{A}(\textsc{last}(P)) = h_2$ and $\textsc{low}(\textsc{first}(P)) - \textsc{low}(\textsc{last}(P)) = i$. From Figure 14 it follows that

$$C(n,h_1,h_2,i) = \sum_{j=0}^{i} \sum_{k=i-j+1}^{h_1+i-j} C(n-h_1,k,h_2,j),$$

with $C(n,h_1,h_2,i) = 1$ if $h_1 = h_2 = n \wedge i = 0$ or $h_1 + h_2 = n \wedge \max(0, h_2 - h_1) \leq i \leq h_2 - 1$, and $C(n,h_1,h_2,i) = 0$ if $n < h_1$ or $h_1 \neq h_2 \wedge h_1 + h_2 > n$ or $h_1 \neq h_2 \wedge h_1 + h_2 = n \wedge (i < \max(0,h_2-h_1) \vee i > h_2 - 1)$ or $n = h_1 = h_2 \wedge i > 0$.

The functions introduced so far refer to the components that are easier to count. Now we focus on the most difficult part and let $Z_2^\bullet(n,h_1,h_2,h_3,h_4,\delta_1,\delta_2,\delta_3,e)$ (resp., $Z_2^\circ(n,h_1,h_2,h_3,h_4,\delta_1,\delta_2,\delta_3,e)$) be the number of polyominoes P in $\mathsf{Z}_2^\bullet(n,h_1,e)$ (resp., $P \in \mathsf{Z}_2^\circ(n,h_1,e)$) whose standard decomposition $P = L \cdot C$ or $P = L \cdot C \cdot R$ satisfies the following conditions (see Fig. 15):

- $L = L_1 \cdot D$ where D is a rectangle of height h_1 with e columns;
- $\mathrm{A}(\textsc{last}(L_1)) = h_2$, $h_2 < h_1$, $\mathrm{A}(\textsc{first}(C)) = h_3$, $\mathrm{A}(\textsc{last}(P)) = h_4$;

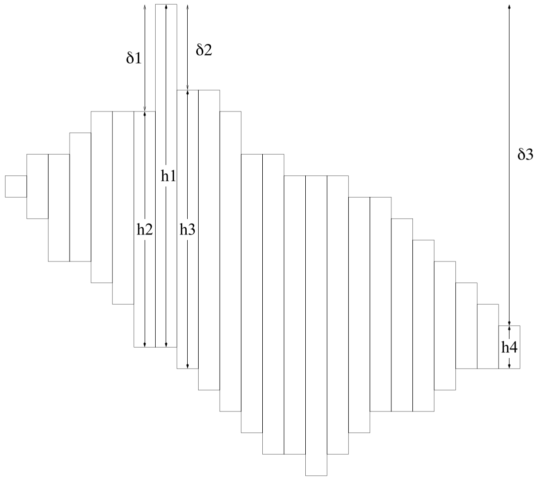

FIGURE 15. A polyomino counted by $\mathsf{Z}_2^\bullet(n, h_1, h_2, h_3, h_4, i_1, \delta_1, \delta_2, \delta_3, 1)$.

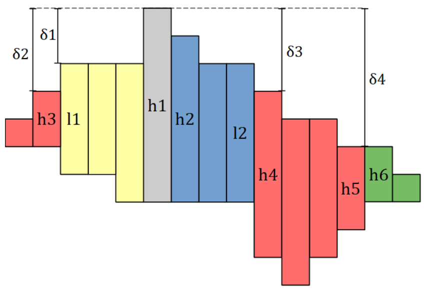

FIGURE 16. Computing $|\mathsf{DConv}_2^\bullet(n)|$: see (5.1). The grey column and the red columns form the polyomino counted by $Z_2^\bullet(a_1, h_1, h_3, h_4, h_5, \delta_2, \delta_3, \delta_4, e)$, blue columns form a polyomino in $\mathsf{F}_R \cup \mathsf{T}$ (counted by $S_R(a_3, h_2, l_2, 0)$), green columns form the polyomino in $\mathsf{R} \cup \mathsf{F}_R \cup \mathsf{T}$ (counted by $S_R(n - a_1 - a_2 - a_3, h_6)$). Lastly, if we rotate clockwise by 180 degrees the polyomino consisting of the grey column and the yellow columns we obtain a polyomino counted by $S_R^\square(a_2 + h_1, h_1, l_1, \delta_1, 1)$.

- $\textsc{high}(\textsc{last}(L)) - \textsc{high}(\textsc{last}(L_1)) = \delta_1$, $\textsc{high}(\textsc{last}(L)) - \textsc{high}(\textsc{first}(C)) = \delta_2$, $\textsc{high}(\textsc{last}(L)) - \textsc{high}(\textsc{last}(P)) = \delta_3$.

From (4.9) and Definition 4.1 we get the following formula (see Fig. 16 for the meaning of indices of summations):

$$\begin{aligned}|\mathsf{DConv}_2^\bullet(n)| = &\sum_{\substack{h_1, e, a_1, h_3, \delta_2\\ h_4, \delta_3, h_5, \delta_4}} Z_2^\bullet(a_1, h_1, h_3, h_4, h_5, \delta_2, \delta_3, \delta_4, e)\cdot\\ &\sum_{a_2, l_1, \delta_1} S_R^\square(a_2 + h_1, h_1, l_1, \delta_1, 1) \cdot \sum_{a_3, h_2, l_2} S_R(a_3, h_2, l_2, 0)\cdot\\ &\sum_{h_6} (h_1 - \delta_4 - h_6 + 1) \cdot S_R(n - a_1 - a_2 - a_3, h_6), \end{aligned} \tag{5.1}$$

where the sums are taken on all $a_1, a_2, a_3, h_1, h_2, h_3, h_4, h_5, h_6, l_1, l_2, \delta_1, \delta_2, \delta_3, \delta_4, e$ such that (we refer to decompositions (4.1)–(4.4) of $P \in \mathsf{DConv}_2^\bullet$ and to the standard decomposition $L' \cdot C'$ or $L' \cdot C' \cdot R'$ (with $L' = L_1 \cdot D'$, $D' \in \mathsf{T}$) of a polyomino P' counted by $Z_2^\bullet(a_1, h_1, h_3, h_4, h_5, \delta_2, \delta_3, \delta_4, e)$, see Fig. 16):

- $3 \leq h_1 \leq n-6$ (for $h_1 < 3$ there can be no disjoint columns, and at least 6 cells are needed for two disjoint columns and the column including them, see Fig. 12);
- $1 \leq e \leq \lfloor (n-6)/h_1 \rfloor$;
- $a_1 \geq h_1 \cdot e + 6$;
- $1 \leq h_3 < h_1$;
- $1 \leq \delta_2 \leq h_1 - h_3$;
- $h_1 - \delta_2 < h_4 \leq a_1 - h_3 - 2$ ($\textsc{first}(C')$ must include $\textsc{last}(L_1)$, and the area of any column c in $C' \cdot R'$ is at least 2 since $c \updownarrows \textsc{last}(L)$ and $\textsc{low}(c) < \textsc{low}(\textsc{last}(L))$);
- $\max(h_1 - h_4 - 1, 1) \leq \delta_3 \leq \delta_2$ (since $\textsc{first}(C') \updownarrows \textsc{last}(L')$ and $\textsc{last}(L_1) \subseteq \textsc{first}(C')$);
- $2 \leq h_5 \leq a_1 - h_3 - h_4$ (remark: $C' \cdot R'$ has at least two columns);
- $\delta_2 < \delta_4 < h_1$;
- $a_2 = 0$ or $h_3 + \delta_2 - \delta_3 < a_2 \leq n - a_1$;
- $h_3 + \delta_2 - \delta_3 < l_1 \leq a_2$;
- $\delta_2 + h_3 - l_1 \leq \delta_1 < \delta_3$;
- $a_3 = 0$ or $h_1 - \delta_3 \leq a_3 \leq n - a_1 - a_2$;
- $h_1 - \delta_3 \leq h_2 \leq \min(a_3, h_1 - 1)$;
- $h_1 - \delta_3 \leq l_2 \leq h_2$;
- $h_6 \leq h_1 - \delta_4$ (the first column c of the polyomino counted by $S_R(n - a_1 - a_2 - a_3, h_6)$ is included in the segment of $\textsc{last}(P')$ of area $h_1 - \delta_4$ overlapping $\textsc{last}(L')$. Thus, the number of ways of placing c is $(h_1 - \delta_4 - h_6 + 1)$).

Now, the problem becomes computing $Z_2^\bullet(n, h_1, h_2, h_3, h_4, i_1, i_2, i_3, e)$. Without loss of generality we suppose $e = 1$. Indeed, for $e > 1$ one has

$$Z_2^\bullet(n, h_1, h_2, h_3, h_4, i_1, i_2, i_3, e) = Z_2^\bullet(n - h_1 \cdot (e-1), h_1, h_2, h_3, h_4, i_1, i_2, i_3, 1).$$

Let $\boldsymbol{\alpha} = n, h_1, h_2, h_3, h_4, i_1, i_2, i_3, 1$ and $\boldsymbol{\beta} = n - a_1 - a_2 - a_3, h_1, k_2, k_3, k_4, e_1, e_2, e_3, 1$. From (4.11) and Definition 4.2 we immediately obtain a recurrence equation. Indeed (see Fig. 17),

$$Z_2^\bullet(\boldsymbol{\alpha}) = \text{s-}Z_2^\bullet(\boldsymbol{\alpha}) + \sum_{a_1, x_1, j_1} S_R(a_1, h_2, x_1, j_1) \cdot \sum_{a_2, j_2, x_2} C(a_2, x_2, h_3, j_2) \cdot \sum_{a_3, j_3, x_3} S_L(a_3, x_3, h_4, j_3) \cdot \sum_{\substack{e_1, k_2, e_2 \\ k_3, e_3, k_4}} Z_2^\bullet(\boldsymbol{\beta}) + Z_2^\circ(\boldsymbol{\beta}). \tag{5.2}$$

Indeed, a polyomino P counted by $Z_2^\bullet(\boldsymbol{\alpha})$ is either in s-$\mathsf{Z}_2^\bullet$ (and then counted by s-$Z_2^\bullet(\boldsymbol{\alpha})$) or is the pseudo-shuffle of two polyominoes, $P = P' \parallel P''$, with $P'' \in$ s-$\mathsf{Z}_2^\bullet$ and $P' \in \mathsf{Z}_2$. Thus, the first three sums in equation (5.2) count the number of $P'' \in$ s-$\mathsf{Z}_2^\bullet(a_1 + a_2 + a_3 + h_1, h_1, 1)$ (for specific values of the variables used to identify the components L, C and R in the decomposition of P''), whereas the fourth sum counts the number of $P' \in \mathsf{Z}_2(n - a_1 - a_2 - a_3, h_1, 1)$ such that $P' \parallel P''$ is defined (and then belongs to $Z_2^\bullet(\boldsymbol{\alpha})$). To ensure this, we refer to the decompositions $L' \cdot D' \cdot C' \cdot R'$ or $L' \cdot D' \cdot C'$ of $P' \in \mathsf{Z}_2$, and to $L'' \cdot D'' \cdot C'' \cdot R''$ (remark: $D' = D''$) of $P'' \in$ s-$\mathsf{Z}_2^\bullet$ (see Fig. 17), and we take the sums on all nonnegative integers $a_1, a_2, a_3, x_1, x_2, x_3, k_2, k_3, k_4, j_1, j_2, j_3, e_1, e_2, e_3$ such that (remark: the order of nested sums in (5.2) follows the order of the conditions below):

- $h_2 \leq a_1 < n - h_1 - h_3 - h_4$;
- $x_1 \leq h_2$;
- $0 \leq j_1 \leq h_2 - x_1$;
- $h_3 \leq a_2 < n - a_1 - h_1 - h_4$;
- $0 \leq j_2 \leq i_1 - i_2$ (since $\textsc{last}(L'') \subsetneq \textsc{last}(C'')$);
- $h_3 - j_2 \leq x_2 \leq a_2 - h_3$ (if $a_2 \neq h_3$) or $x_2 = h_3$ (if $a_2 = h_3$ and $j_2 = 0$);
- $h_4 \leq a_3 < n - h_1 - a_1 - a_2$ ($a_1 + a_2 + a_3 + h_1 < n$ since $P' \neq \epsilon$);

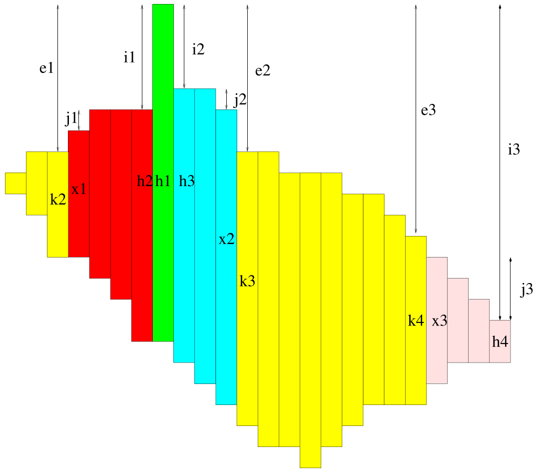

FIGURE 17. A polyomino P counted by $Z_2^{\bullet}(\boldsymbol{\alpha})$. The green column and the yellow columns form the polyomino $P' \in \mathsf{Z}_2$ (counted by $Z_2^{\bullet}(\boldsymbol{\beta})$), red columns form the polyomino $P_1 \in \mathsf{L} \cup \mathsf{F}_L \cup \mathsf{T}$ (counted by $S_L(a_1, h_2, x_1, j_1)$), blue columns form the polyomino $P_2 \in \mathsf{C}$ (counted by $C(a_2, h_3, x_2, j_2)$), whereas pink columns form the polyomino $P_3 \in \mathsf{R} \cup \mathsf{F}_R \cup \mathsf{T}$ (counted by $S_R(a_3, x_3, h_4, j_3)$) . One has $P'' = P_1 \cdot P_2 \cdot P_3 \in$ s-$Z_2^{\bullet}$ and $P = P' \parallel P''$.

- $0 \le j_3 \le i_3 - i_1 - j_1 - x_1$ (since $\text{FIRST}(R'') \asymp \text{FIRST}(L'')$);
- $x_3 = h_4$ (if $j_3 = 0$) or $j_3 + h_4 \le x_3 \le \min(n - h_1 \cdot e - a_1 - a_2 - h_4 - 1, i_2 + j_2 + x_2 - i_3 + j_3)$ (since $\text{FIRST}(R'') \subsetneq \text{LAST}(C'')$);
- $i_1 + j_1 < e_1 < i_1 + j_1 + x_1$;
- $0 \le k_2 \le i_1 + j_1 + x_1 - e_1$;
- $i_1 + j_1 < e_2 \le e_1$(since $\text{LAST}(L') \subseteq \text{FIRST}(C')$);
- $i_3 - j_3 + x_3 - e_2 \le k_3 \le n - k_1 - a_1 - a_2 - a_3$ (since $\text{FIRST}(R'') \subsetneq \text{FIRST}(C')$);
- $e_2 \le e_3 < i_1 + j_1 + x_1$ (since $\text{FIRST}(R'') \upharpoonleft\!\downharpoonright \text{LAST}(C')$);
- $i_3 - j_3 + x_3 - e_3 \le k_4 \le n - k_1 - a_1 - a_2 - a_3$.

We point out that all the previous conditions derive from definitions 3.3, 3.4 and 4.2. Furthermore, Definition 3.4 immediately leads to the following formula for s-$Z_2^{\bullet}(\boldsymbol{\alpha})$ (that holds only for $i_1 \ge i_2$, $i_1 + h_2 \le h_1$, $0 < i_2 < h_1$, $i_2 + h_3 > h_1$, and $i_3 < h_1$, otherwise s-$Z_2^{\bullet}(n, h_1, h_2, h_3, h_4, i_1, i_2, i_3, 1)$ is equal to 0),

$$\text{s-}Z_2^{\bullet}(n, h_1, h_2, h_3, h_4, i_1, i_2, i_3, 1) = \sum_{a_1, j_1, x_1} S_R(a_1, h_2, x_1, j_1) \cdot \sum_{a_2, j_2, x_2} C(a_2, x_2, h_3, j_2) \cdot \sum_{j_3, x_3} S_L(n - h_1 - a_1 - a_2, x_3, h_4, j_3). \tag{5.3}$$

Here, we refer to the decomposition $P = L \cdot C \cdot R$ (with $L = L_1 \cdot D$) given in Definition 3.4, and we take the sums on all nonnegative integers $a_1, a_2, x_1, x_2, x_3, j_1, j_2, j_3$ such that (see Fig. 18):

- $h_2 \le a_1 \le n - h_1 - h_3 - h_4$;
- $0 \le j_1 < i_3 - i_1$ (since $\text{FIRST}(R) \asymp \text{FIRST}(L_1)$);
- $1 \le x_1 \le \min(i_3 - i_1 - j_1, h_2)$;
- $h_3 \le a_2 \le n - h_1 - a_1 - h_4$;
- $0 \le j_2 \le i_1 - i_2$ (since $\text{LAST}(L_1) \subsetneq \text{LAST}(C)$);
- $x_2 = h_3$ (if $j_2 = 0$) or $h_3 - j_2 \le x_2 \le n - h_1 - a_1 - h_3 - h_4$;
- $0 \le j_3 \le i_3 - i_1 - j_1 - x_1$ (since $\text{FIRST}(R) \asymp \text{FIRST}(L)$);

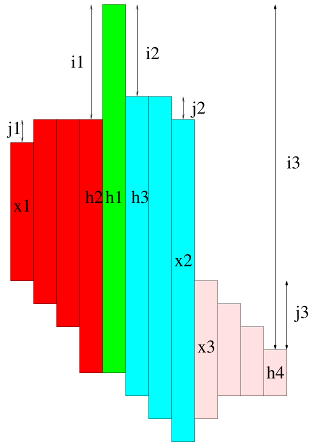

FIGURE 18. A polyomino counted by s-$Z_2^\bullet(n, h_1, h_2, h_3, h_4, i_1, i_2, i_3)$.

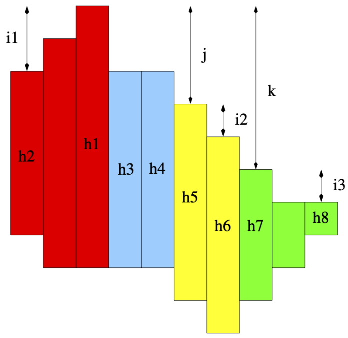

FIGURE 19. A polyomino in DConv°.

- $j_3 + h_4 \leq x_3 \leq i_2 + j_2 + x_2 - i_3 + j_3$ (since $\textsc{first}(R) \subsetneq \textsc{last}(C)$).

Notice that the previous conditions ensure that P_1 counted by $S_L(a_1, h_2, x_1, j_1)$ (rotated 180 degrees clockwise), P_2 counted by $C(a_2, x_2, h_3, j_2)$ (rotated 180 degrees clockwise) and P_3 counted by $S_R(n - h_1 - a_1 - a_2, x_3, h_4, j_3)$ are such that $P_1 \cdot P_2 \cdot P_3$ is in s-$Z_2^\bullet(n, h_1, h_2, h_3, h_4, i_1, i_2, i_3, e)$.

So far we have dealt with counting polyominoes with disjoint columns. Now, let's face the problem of computing the number of descending convex polyominoes without disjoint columns. By considering the standard decomposition $L \cdot C \cdot F \cdot R$ (possibly $F, R = \epsilon$) one has: (see Fig. 19):

$$|\mathsf{DConv}_2^\circ(n)| = \sum_{h_1, e, a_1, i_1, h_2} S_R^\square(a_1, h_1, h_2, i_1, e) \cdot \Bigg(1 + \sum_{a_2, h_3, h_4} S_R(a_2, h_3, h_4, 0) \cdot \sum_{a_3, h_5, h_6, i_2, j} C(a_3, h_6, h_5, i_2) \cdot (1 + \sum_{k, h_7, h_8, i_3} S_L(n - a_1 - a_2 - a_3, h_7, h_8, i_3))\Bigg) \tag{5.4}$$

where the sums are taken on all nonnegative integers $a_1, a_2, a_3, h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, i_1, i_2, i_3, j, k, e$ such that:

- $2 \leq h_1 \leq n - 2$, since $\mathrm{A}(C) \geq 2$;
- $1 \leq e \leq \lfloor (n-2)/h_1 \rfloor$;

- $h_1 \cdot e \leq a_1 \leq n-2$;
- $0 \leq i_1 < h_1$;
- $h_2 \leq h_1 - i_1$ and $h_1 \geq 1$ (if $i_1 > 0$) or $h_2 \geq 2$ (if $i_1 = 0$), since $h_1 = 1$ and $i_1 = 0$ implies $\textsc{first}(L) \asymp \textsc{first}(C)$;
- $h_1 - i_1 - h_2 < a_2 < n - a_1 - (h_1 - i_1 - h_2)$, since $C \neq \epsilon$ and $\textsc{first}(C) \upharpoonleft\downharpoonright \textsc{first}(L)$ (remark: the case $a_2 = 0$ corresponds to the first addend 1 in (5.4)) ;
- $h_1 - i_1 - h_2 < h_3 < h_1$, since $\textsc{first}(F) \upharpoonleft\downharpoonright \textsc{first}(L)$;
- $h_1 - i_1 - h_2 < h_4 \leq h_3$, since $\textsc{last}(F) \upharpoonleft\downharpoonright \textsc{first}(L)$;
- $1 \leq j < i_1 + h_2$ (if $a_2 = 0$) or $h_1 - h_4 \leq j < i_1 + h_2$ (if $a_2 > 0$);
- $h_1 - j < a_3 \leq n - a_1 - a_2$;
- $h_1 - j < h_5 \leq a_3$;
- $0 \leq i_2 < i_1 + h_2 - j$ since $\textsc{last}(C) \upharpoonleft\downharpoonright \textsc{first}(L)$;
- $h_6 = h_5$ (if $a_3 = h_5$) or $h_5 - i_2 \leq h_6 \leq a_3 - h_5$ (if $a_3 > h_5$);
- $j + i_2 \leq k < i_1 + h_2$;
- $1 \leq h_7 \leq \min(n - a_1 - a_2 - a_3, j + i_2 + h_6 - k)$, since $\textsc{first}(R) \subsetneq \textsc{last}(C)$;
- $0 \leq i_3 < i_1 + h_2 - k$;
- $1 \leq h_8 \leq h_7 - i_3$.

The last step consists of computing $Z_2^\circ(\boldsymbol{\alpha})$ ($\boldsymbol{\alpha} = n, h_1, h_2, h_3, h_4, i_1, i_2, i_3, 1$), that is used in equation (5.2). By Definition 3.3 and Figure 20 it follows that

$$Z_2^\circ(\boldsymbol{\alpha}) = \sum_{a_1, j_1, x_1} S_L(a_1, h_2, x_1, j_1) \cdot \Big(C(n - h_1 - a_1, h_4, h_3, i_3 - i_2) + \sum_{a_2, j_2, x_2} C(a_2, x_2, h_3, j_2) \cdot \sum_{j_3, x_3} S_L(n - h_1 \cdot e - a_1 - a_2, x_3, h_4, j_3))), \tag{5.5}$$

where the sums are taken over all non-negative integers $a_1, a_2, x_1, x_2, x_3, j_1, j_2, j_3$ (see Fig. 20) such that (we refer to the standard decomposition of $P \in Z_2^\circ$, $P = L \cdot C \cdot R$ or $P = L \cdot C$, with $L = L_1 \cdot D$):

- $h_2 \leq a_1 \leq n - h_1 - h_3 - h_4$ (if $h_3 \neq h_4$ or $i_2 \neq i_3$) or $h_2 \leq a_1 \leq n - h_1 - h_3$ (if $h_3 = h_4$ and $i_2 = i_3$);
- $0 \leq j_1 < h_2$;
- $i_3 - i_1 - j_1 < x_1 \leq h_2$, since $\textsc{last}(P) \upharpoonleft\downharpoonright \textsc{first}(L)$;
- $a_2 \leq n - h_1 - a_1 - h_4$ (remark: $\mathrm{A}(R) \geq h_4$) and $a_2 \geq h_3$ (if $i_2 + h_3 \geq i_3 + h_4$) or $a_2 > h_3 + h_4$ (if $i_2 + h_3 < i_3 + h_4$);
- $0 \leq j_2 < i_1 + j_1 + x_1 - i_2$, since $\textsc{last}(C) \upharpoonleft\downharpoonright \textsc{first}(L)$;
- $x_2 = h_3$ (if $a_2 = h_3$) or $\max(h_3 - j_2, i_3 + h_4 - i_2 - j_2) \leq x_2 \leq a_2 - h_3$, since $\textsc{last}(R) \subsetneq \textsc{last}(C)$;
- $0 \leq j_3 \leq i_3 - i_2 - j_2$;
- $j_3 + h_4 \leq x_3 < j_3 - i_3 + i_2 + j_2 + x_2$ (to ensure that $\textsc{low}(\textsc{last}(C)) < \textsc{low}(\textsc{first}(R))$);

Finally, we exploit the previous formulas to compute the number of Z-convex polyominoes,

$$|\mathsf{ZConv}(n)| = |\mathsf{T}(n)| + 2 \cdot |\mathsf{F}_L(n)| + 2 \cdot |\mathsf{L}(n)| + |\mathsf{LR}(n)| + 2 \cdot (|\mathsf{DConv}_2^\bullet(n)| + |\mathsf{DConv}_2^\circ(n)|).$$

We point out that the value $|\mathsf{ZConv}(n)|$ can be computed in polynomial time. Indeed, by applying dynamic programming one can develop a program that uses $O(1)$ tables of size $O(n^8)$ to store all intermediate results for the sets of values defined above.

The previous formulas are the basis of a C++ program under development. The goal is to compute the counting sequence $\{c_n\}$ of Z-convex polyominoes for $n \leq N$, where N is an integer that is large enough to obtain the most accurate estimation of the asymptotic form of the coefficients, as has been done for L-convex polyominoes [18]. At the moment, we have a β-version of the program (that uses a non-optimized data structure) that produced the coefficients in Table 1. We point out that this integer sequence does not currently appear in

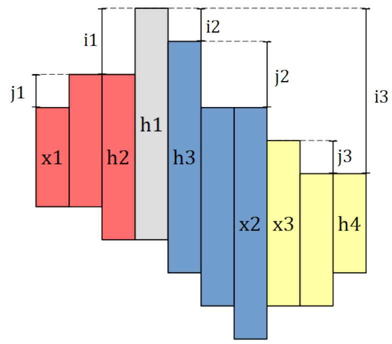

FIGURE 20. A polyomino counted by $Z_2^\circ(n, h_1, h_2, h_3, h_4, i_1, i_2, i_3, 1)$.

TABLE 1. $|\mathsf{ZConv}(n)|$ for $0 \leq n \leq 70$.

0, 1, 2, 6, 19, 55, 148, 370, 874, 1966, 4240, 8816, 17773, 34858, 66734, 125014, 229647, 414412
735762, 1286908, 2220035, 3781065, 6363460, 10591124, 17444763, 28453652, 45984090,
73671398, 117061785, 184562194, 288836144, 448846754, 692828996, 1062596751,
1619750728, 2454592300, 3698861168, 5543870866, 8266217558, 12264097608, 18108408216,
26614409924,38941858286,56734472110,82313536326, 118945843908, 171213356406,
245521741732, 350797907519, 499444170806, 708639882712, 1002112444338, 1412540714209,
1984808599052, 2780398734144, 3883311845028, 5408022969255, 7510151515584, 10400739110270,
14365314313088, 19789295317410, 27191768575390, 37270314602040, 50960377670716,
69513746774069, 94602105582945, 128453407239846, 174031156719558,235269684178159,
317382363101408,427264704189028

OEIS and is long enough to obtain an estimate of the number of Z-convex polyominoes of area n, as shown in Section 6.

6. Analysis of series

In [18] it was conjectured, with compelling evidence, that the asymptotics of 1-convex (otherwise called L-convex) polyominoes, enumerated by area is

$$[q^n]A(q) \sim \frac{13\sqrt{2}}{768 \cdot n^{3/2}} \exp(\pi\sqrt{13n/6}),$$

where $A(q)$ is the area generating function. We expect similar behaviour for k-convex polyominoes, enumerated by area. That is to say, if $A_k(q)$ is the area generating function of k-convex polyominoes, then

$$[q^n]A_k(q) \sim \frac{C_k}{n^{\alpha_k}} \exp(\pi\sqrt{\beta_k n}),$$

where C_k is expected to be an algebraic number and α_k and β_k are expected to be k-dependent rational constants.

The analysis of series with asymptotics of this type is described in detail in [19], and demonstrated in the case of L-convex polyominoes in [18], so we will not repeat the discussion here, but simply apply the methods described there.

We currently have 70 exact terms of the generating function $A_2(q)$. Using the method of series extension [20] we have obtained a further 155 *approximate* terms.

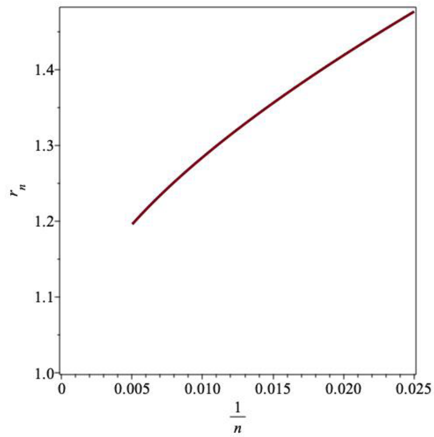

FIGURE 21. Ratios vs. $1/n$.

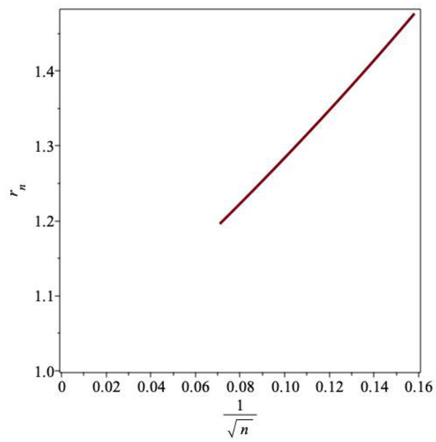

FIGURE 22. Ratios vs. $1/\sqrt{n}$.

As we are only considering 2-convex polyominoes in the following analysis, we will drop the subscripts, and write $a_n = [q^n]A_2(q) \sim \frac{C}{n^\alpha} \exp(\pi\sqrt{\beta n})$.

First, we consider the ratios of successive coefficients, $r_n = a_n/a_{n-1}$. For a power-law singularity, one expects the sequence of ratios to approach the growth constant linearly when plotted against $1/n$. In our case the growth constant is 1. That is to say, there is no exponential growth.

For a singularity of the assumed type, which is called a *stretched exponential,* the ratio of coefficients behaves as

$$r_n = \frac{a_n}{a_{n-1}} = 1 + \frac{\pi\sqrt{\beta}}{2\sqrt{n}} + O\left(\frac{1}{n}\right), \tag{6.1}$$

so we expect the ratios to approach a limit of 1 linearly when plotted against $1/\sqrt{n}$, and to display curvature when plotted against $1/n$.

We show the ratios plotted against $1/n$ and $1/\sqrt{n}$ in Figures 21 and 22 respectively. These plots are behaving as expected, with the plot against $1/n$ displaying considerable curvature, while the plot against $1/\sqrt{n}$ is closer to linear, but still displays some curvature, presumably due to the presence of an $O(1/n)$ term. We can eliminate

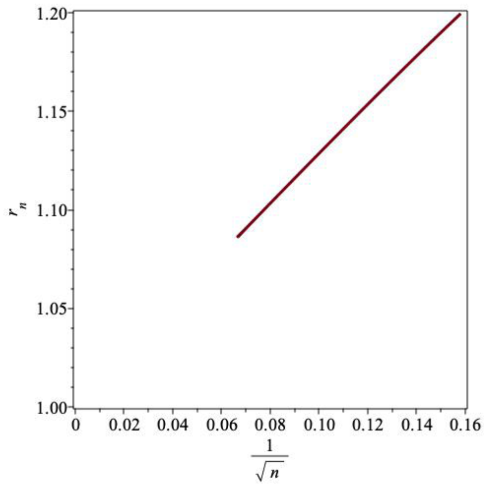

FIGURE 23. Linear intercepts vs. $1/\sqrt{n}$.

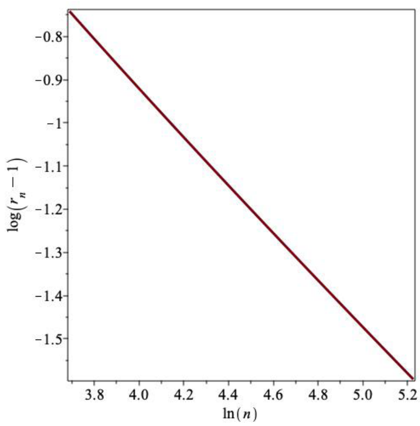

FIGURE 24. Log-log plot of $r_n - 1$ against n.

this term by considering the linear intercepts,

$$l_n = n \cdot r_n - (n-1) \cdot r_{n-1},$$

which eliminates that term. These are shown in Figure 23, which plot is seen to be convincingly linear, and going to the expected value of 1.

We can readily obtain further evidence that the power inside the exponential term is indeed a square root. From (6.1), one sees that

$$r_n - 1 = \frac{\pi\sqrt{\beta}}{2\sqrt{n}} + O\left(\frac{1}{n}\right). \tag{6.2}$$

Accordingly, a plot of $\log(r_n - 1)$ versus $\log n$ should be linear, with gradient $-1/2$. In Figure 24, we show the log-log plot, and in Figure 25 we show the local gradient plotted against $1/\sqrt{n}$. The linearity of the first plot is obvious, while the second is convincingly going to a limit of -0.5 as $n \to \infty$.

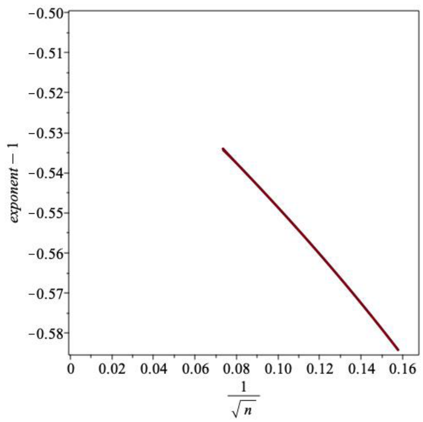

FIGURE 25. Gradient of log-log plot.

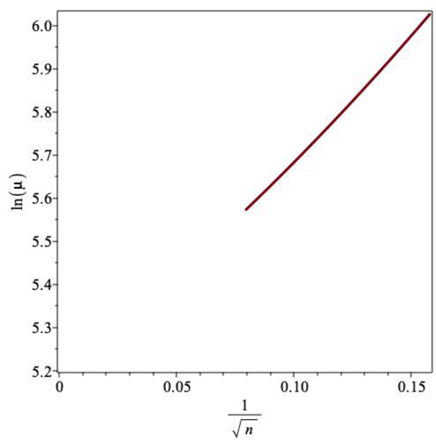

FIGURE 26. $\log(\mu_1)$ vs. $1/\sqrt{n}$.

Having convincingly established that the relevant exponent is a square-root, just as for stack polyominoes, it remains to determine the other parameters. There are several ways one might proceed, but here is one that works quite well.

Recall that $(r_n - 1) \sim \frac{\log \mu_1}{2\sqrt{n}}$, where $\mu_1 = \exp(\pi\sqrt{\beta})$, so $2\sqrt{n} \cdot (r_n - 1) \sim \log \mu_1$. We show the relevant plot in Figure 26. The second plot, Figure 27, eliminates the presumed sub-dominant $O(1/n)$ term.

The plot suggests $\log \mu_1 \approx 5.20$. Linear extrapolation gives 5.210 ± 0.005. Recall that L-convex polyominoes grow as $\exp(2\pi\sqrt{13n/6})$, so if we guess that these grow as $\exp(\pi\sqrt{\beta n})$, then $\pi\sqrt{\beta} \approx 5.210$, implying $\beta \approx 2.750$. It seems reasonable to conjecture that $\beta = 11/4$ exactly.

So at this stage we have

$$a_n \sim C \cdot \frac{\exp(\pi\sqrt{\beta n})}{n^{\alpha}},$$

with the conjecture that $\beta = 11/4$.

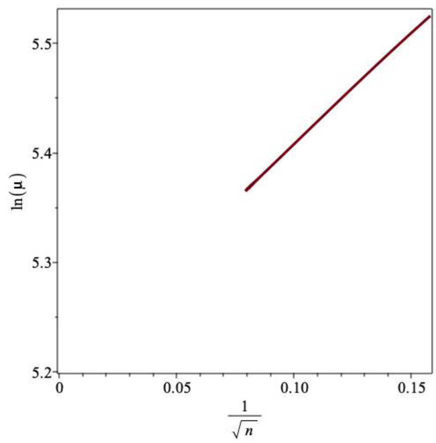

FIGURE 27. $\log(\mu_1)$ vs. $1/\sqrt{n}$, eliminating $O(1/n)$ term.

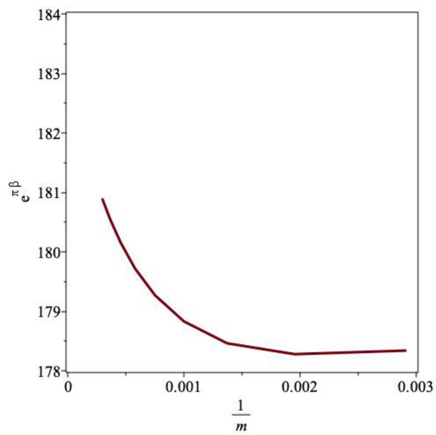

FIGURE 28. Ratios a_m/a_{m-1} vs. $1/m$.

To estimate α, we set $m = n^2$, so

$$a_m \sim C \cdot \frac{\exp(n\pi\sqrt{\beta})}{n^{2\alpha}}.$$

This is of the form $C\mu^n n^g$, so we can use the usual ratio method. In particular,

$$r_m \equiv \frac{a_m}{a_{m-1}} \sim \exp(\pi\sqrt{\beta})\left(1 - \frac{2\alpha}{m} + o(1/m)\right)$$

In Figure 28, we show the ratios plotted against $1/m$, and estimate that $\exp(\pi\beta) \approx 183$, implying $\beta = 2.7497$, in excellent agreement with our previous conjecture $\beta = 11/4$.

In Figure 29, we show the exponent estimates, obtained by rearranging the above equation to give

$$2\alpha \sim (1 - r_m \cdot \exp(-\pi\sqrt{\beta}))m.$$

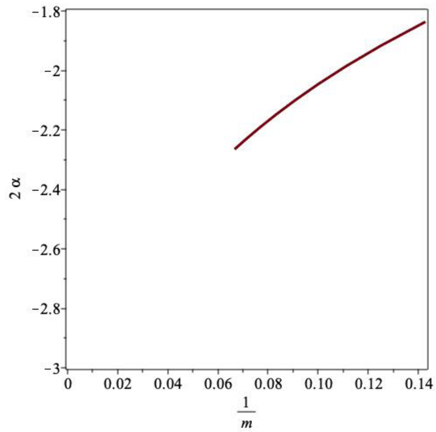

FIGURE 29. Estimators of 2α against $1/m$.

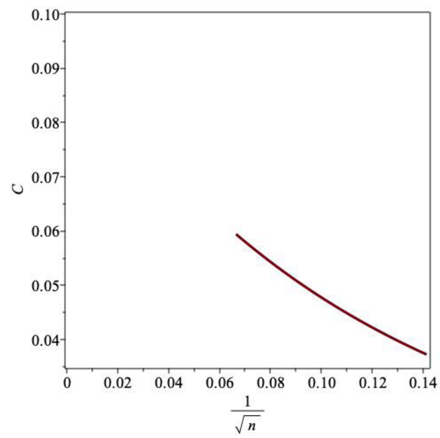

FIGURE 30. Estimates of C vs. $1/\sqrt{n}$.

We have plotted estimators of 2α against $1/m$, and 2α is estimated to be -3, or $\alpha = -3/2$, exactly the same value as for L-convex polyominoes.

So at this stage we can reasonably conjecture that

$$a_n \sim C \cdot \frac{\exp(\pi\sqrt{11n/4})}{n^{3/2}}.$$

In order to calculate the constant C, we form the sequence

$$C_n \equiv \frac{a_n \cdot n^{3/2}}{\exp(\pi\sqrt{11n/4})},$$

and extrapolate the sequence C_n using any of a variety of standard methods. In Figure 30, we show the sequence C_n plotted against $1/\sqrt{n}$, after having eliminated the $\mathrm{O}(1/n)$ term. In Figure 31, we show the sequence C_n plotted against $1/\sqrt{n}$, after having eliminated the $\mathrm{O}(1/n)$ term and the $\mathrm{O}(1/n^{3/2})$ term. From the latter plot we estimate $C = 0.095 \pm 0.003$.

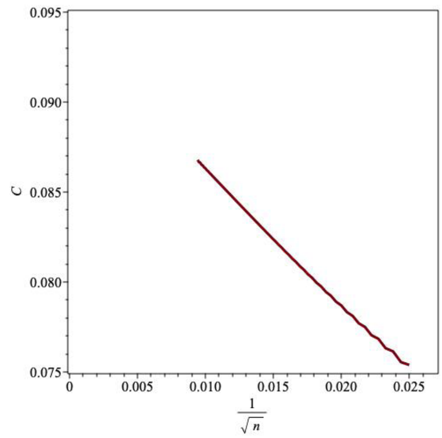

FIGURE 31. Refined estimates of C vs. $1/\sqrt{n}$.

We conclude with the conjecture that the asymptotic form of the coefficients of Z-convex polyominoes is

$$a_n \sim \frac{C \cdot \exp(\pi\sqrt{11n/4})}{n^{3/2}}, \tag{6.3}$$

with $C = 0.095 \pm 0.003$. Unfortunately, this estimate is insufficiently precise to conjecture an exact value for the constant C, though $C = \frac{11\sqrt{3}}{200} = 0.09526\cdots$ is a useful mnemonic.

7. Conclusion and perspectives

We have shown that the counting problem for Z-convex polyominoes is in P (the class of problem solved in polynomial time under the uniform cost model). The sequence of coefficients provided by the C++ program has been analyzed in order to obtain a conjecture on the asymptotic form of coefficients. We plan to develop an optimized program to produce a longer sequence, with the goal of obtaining a conjecture for the exact value of the constant C in (6.3).

Furthermore, we think that the idea of decomposing a convex polyomino into simpler pieces could be applied also to efficiently enumerate other classes of convex polyominoes. In particular, in [21] the standard decomposition is the basis of a different technique used to count convex polyominoes of degree of convexity at most k, for $k > 2$.

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