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A note on isomorphic generalized Petersen graphs with an application to the crossing number of $GP[3k-1,k]$ and $GP[3k+1,k]$

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Abstract

For positive integers n and k, the generalized Petersen graph $GP[n,k]$ is the graph on the $2n$ vertices $\{u_0, u_1, \dots, u_{n-1}, x_0, x_1, \dots, x_{n-1}\}$ and the edges $\{\{u_i, x_i\}, \{u_i, u_{i+1}\}, \{x_i, x_{i+k}\}\}$, where $i = 0, 1, \dots, n-1$, addition modulo n. The crossing number of a graph G is defined as the least number of crossings of edges of G when G is drawn in a surface, which in our case will be the Euclidean plane. We prove a conjecture presented by Zhou and Wang [*Int. J. Math. Comb.* **4** (2012) 116–123] on the crossing number of $GP[3k-1,k]$ and derive a quick way to check if a result by Watkins can be used to establish whether two generalized Petersen graphs on different parameters are isomorphic.

Keywords: drawing; crossing number; generalized Petersen graph.

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1. Introduction

The study of crossing numbers originated in Turán's Brick Factory Problem. In this problem, Turán [7] asked for a factory plan that minimized the number of crossings between tracks connecting brick kilns to storage sites. Mathematically this problem can be formalized as asking for the crossing number of a complete bipartite graph. Zarankiewicz [10] claimed that he had solved this problem in 1953, but a flaw in his argument that unvalidated his result was later discovered independently by Ringel and Kainen (see [5]).

A *graph* $G = G(V,E)$ of order n is composed of a set V of n vertices and a set E of m edges consisting of unordered pairs of vertices. A *drawing* of a graph G is a function f that maps G into a surface, which in our case will be the Euclidean plane, such that each vertex $v \in V(G)$ is mapped to a point $f(v)$ in the plane, and each edge $\{u,v\} \in E(G)$ is mapped to a polygonal path $[f(u), f(v)]$ in the plane. For simplicity's sake, we shall "abuse" terminology by referring to the points and curves in a drawing of G as the vertices and the edges of the graph G. A *good drawing* is one in which no edge crosses itself, an edge contains no vertex except its end points, no pair of edges touch each other without intersecting, any two edges have at most one point in common (which could either be a vertex or a crossing) and no three edges have a common point other than a vertex. A good drawing D of a graph G exhibiting the least possible number of crossings is said to be an *optimal drawing* of G. The number of crossings in such a drawing D is defined as the *crossing number* of G and is denoted by $\nu_D(G)$, or simply by $\nu(G)$. A graph G is *planar* if G can be drawn in the plane without any crossings. Such a drawing of G is called a *plane drawing* or a *planar embedding* of G.

2. Main results

Generalized Petersen graphs were first studied by Coxeter [2] and later named by Watkins [8]. The generalized Petersen graph $GP[n,k]$ consists of an n-circuit, n "spokes" incident to the vertices of the n-circuit, and another circuit or circuits (according to the values of n and k) joining the vertices of every k^{th} spoke. More formally, we have the following.

Definition 2.1. *The generalized Petersen graph $GP[n,k]$ is the graph with vertices $\{u_0, u_1, \dots, u_{n-1}\}$ and $\{x_0, x_1, \dots, x_{n-1}\}$ and edges $\{u_i, u_{i+1}\}$ (called outer edges), $\{u_i, x_i\}$ (called spokes) and $\{x_i, x_{i+k}\}$ (called inner edges), where $i = 0, 1, \dots, n-1$, addition is taken modulo n.*

A useful observation is that the crossing number of isomorphic graphs is identical, that is, if $G_1 \simeq G_2$, then $\nu(G_1) = \nu(G_2)$ This is very often used when dealing with generalized Petersen graphs, since two graphs on different parameters,

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which may at first seem to be totally different, may turn out to be exactly the same. In particular, Watkins [8] proved the following results.

Lemma 2.1. (Watkins' Isomorphism Results [8])

1. *$GP[n,k]$ and $GP[n,n-k]$ are isomorphic.*
2. *$GP[n,k_1]$ and $GP[n,k_2]$ are isomorphic if $k_1k_2 = 1 \bmod n$.*

A direct corollary of the above lemma is that if $k_1k_2 \in \{1,-1\}$ (mod n) then $GP[n,k_1]$ is isomorphic to $GP[n,k_2]$.

The crossing number of various families of generalized Petersen graphs has been determined by different authors (see for example, [1,3,9]). Of importance to us are the following results by Fiorini, Richter and Salazar.

Theorem 2.1. [4,6] *Consider $GP[n,3]$ and set $n = 3m+h$ where $h \in \{0,1,2\}$. Then we have $\nu(GP[7,3]) = 3$, $\nu(GP[9,3]) = 2$, and*

$$\nu(GP[3m+h,3]) = \begin{cases} m & \text{if } h = 0, m \geq 4. \\ m+3 & \text{if } h = 1, m \geq 3. \\ m+2 & \text{if } h = 2, m \geq 2. \end{cases}$$

Zhou and Wang in [11] used a detailed case-by-case argument to obtain an upper bound and a lower bound for the crossing number of $GP[3k-1,k]$. Their result, quoted also in the survey [1], is presented in Theorem 2.2 below.

Theorem 2.2. [11] *Consider $GP[3k-1,k]$ for $k \geq 3$. Then,*

$$k \leq \nu(GP[3k-1,k]) \leq k+1.$$

In [11], the authors also conjectured that $\nu(GP[3k-1,k]) = k+1$, for $k \geq 3$. In the next theorem, we prove their conjecture and a further result by using Lemma 2.1.

Theorem 2.3. *$\nu(GP[3k-1,k]) = k+1$ and $\nu(GP[3k+1,k]) = k+3$, where $k \geq 3$.*

Proof. Consider

$$\begin{aligned} \nu(GP[3k-1,3]) &= \nu(GP[3m+2,3]) \quad (\text{for } k = m+1) \\ &= m+2 \quad (\text{by Theorem 2.1}) \\ &= k+1. \end{aligned}$$

However $GP[3k-1,k] \simeq GP[3k-1,3]$ since $3k = 1 \bmod (3k-1)$, and thus $\nu(GP[3k-1,k]) = k+1$.

Similarly, $GP[3k+1,k] \simeq GP[3k+1,3]$ since $3k = (-1) \bmod (3k+1)$ and thus $\nu(GP[3k+1,k]) = k+3$. □

In the rest of this work, we derive a quick way how to check if two generalized Petersen graphs on different parameters are isomorphic by applying Lemma 2.1. More precisely, given two integers k and ℓ, we determine which values of n satisfy Watkins' conditions so as to establish if $GP[n,k]$ and $GP[n,\ell]$ are isomorphic.

Theorem 2.4. *Given k and ℓ, let $\alpha = \gcd(k-1,\ell-1)$ such that $k = \alpha p+1$ and $\ell = \alpha t+1$, for positive integers p and t. Then $GP[n,k]$ is isomorphic to $GP[n,\ell]$ if $n = \frac{\alpha}{\gamma}(\alpha pt+p+t)$ and $n \geq 2\max\{k,\ell\}$, for any positive integer γ.*

Proof. We remark that since $k-1 = \alpha p$ and $\ell - 1 = \alpha t$, where $\alpha = \gcd(k-1,\ell-1)$, then p and t are relatively prime positive integers. As $GP[n,k] \simeq GP[n,n-k]$, then we can assume that $n \geq 2k$ and similarly $n \geq 2\ell$, implying that $n \geq 2\max\{k,\ell\}$. By Lemma 2.1, if $k\ell = 1 \bmod n$ then $GP[n,k] \simeq GP[n,\ell]$. Thus, for some positive integer γ, we have $\gamma n + 1 = k\ell = \alpha^2 pt + \alpha p + \alpha t + 1$. Hence $\gamma n = \alpha^2 pt + \alpha p + \alpha t$, implying that $n = \frac{\alpha}{\gamma}(\alpha pt+p+t)$. □

A congruent argument is used to prove Theorem 2.5 when $k\ell = -1 \bmod n$. The proof is omitted here as it follows the same lines as that of Theorem 2.4.

Theorem 2.5. *Given k and ℓ, let $\alpha = \gcd(k-1,\ell-1)$ such that $k = \alpha p+1$ and $\ell = \alpha t+1$ for positive integers p and t. Then $GP[n,k]$ is isomorphic to $GP[n,\ell]$ if $n = \frac{\alpha(\alpha pt+p+t)+2}{\gamma}$ and $n \geq 2\max\{k,\ell\}$, for any positive integer γ.*

3. Applications

We remark that applying Theorem 2.4 for the case when $\ell = 3$ and any k, we get that

$$n \geq 2\max\{k,3\} = \begin{cases} 6, & \text{if } k \leq 2 \\ 2k, & \text{if } k \geq 3. \end{cases}$$

which implies that

$$\gamma = \frac{3k-1}{n} \leq \begin{cases} \frac{1}{3}, & \text{if } k = 1 \\ \frac{5}{6}, & \text{if } k = 2 \\ \frac{3k-1}{2k}, & \text{if } k \geq 3. \end{cases}$$

Thus, the cases when $k \leq 2$ do not arise since γ is a positive integer. In all other cases $\gamma = 1$. Also,

$$\alpha = \gcd(k-1,2) = \begin{cases} 1, & \text{if } k \text{ is even.} \\ 2, & \text{if } k \text{ is odd.} \end{cases}$$

Thus, when k is even and $k \geq 4$, $p = k-1$, $t = 2$ and $\gamma = 1$, implying that $n = 3k-1$. On the other hand, when k is odd and $k \geq 3$, $p = \frac{k-1}{2}$, $t = 1$ and $\gamma = 1$, implying that $n = 3k-1$ as well. Hence the first result of Theorem 2.3 follows immediately. Similarly, it can be shown that the second result of Theorem 2.3 follows by Theorem 2.5.

Given particular values of k and ℓ, an easy way how to find all n such that $k\ell = \pm 1 \bmod n$ is illustrated in the following example. For instance, if $k = 31$ and $\ell = 21$, we can determine all the values of n such that $GP[n,31] \simeq GP[n,21]$. We note that α is $\gcd(20,30) = 10$. Thus $p = 3$, $t = 2$ and $n \geq 2\max\{k,\ell\}$, which implies that $n \geq 62$. By first applying Theorem 2.4, $n = \frac{650}{\gamma}$ and thus the possible values of γ are 1, 2, 5 and 10 implying that n is 650, 325, 130 and 65, respectively. Then, applying Theorem 2.5, $n = \frac{652}{\gamma}$ resulting in the possible values of γ being 1, 2 and 4 implying that n is 652, 326 and 163, respectively. Therefore $GP[n,31] \simeq GP[n,21]$ provided that $n \in \{65,130,163,325,326,650,652\}$.

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