Activate students? Let them fold! Mathematical paper folding in secondary education in France and Germany

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Though the potential of mathematical paper folding for learning mathematics is known, this practice is still rare in mathematics classes. To inform the design of mathematics lessons enriched with folding, we investigated via a literature review what mathematical paper folding is done in secondary education and why. The research question that we explore in this paper is: What reasons do teachers from France and Germany report for implementing mathematical folding activities in authentic classroom situations? Through a grounded theory approach, we found reasons like: 'to activate students by letting them manipulate paper' and 'to visualise mathematics'. Teachers state that folding allows for dynamic representations that support the transition from informal to formal mathematics and the practice of skills. We summarise these findings in two major categories of reasons: 'to activate students by providing folding tasks' and 'to grow mathematical understanding by folding'.

Keywords: Haptic motion, mathematical paper folding, origami, real classroom experience, representations.

Introduction

In the Netherlands, the word for tinkering – *freubelen* – refers directly to the last name of the German pedagogue Friedrich Fröbel (1785-1852). He advocated the use of paper folding to teach the first principles of mathematics to young children. To explain and explore mathematics, paper folding—or origami—is still under development today. There are many ideas and examples of mathematical folding and origami activities for the mathematics lessons (Haga, 2008; Hull, 2013; Meyer & Mukoda, 2021; Row, 1901) from primary school (Golan, 2011) to university courses (Demaine & O'Rourke, 2007). Research on origami tasks in mathematics education concerns topics like geometry (Boakes, 2009; Golan, 2011), axiomatizing (Nedrenco, 2018) and logic. Serre and Spreafico state, in the context of their research on teaching mathematical logic with origami, that it can "…motivate students to learn, and actively participate during the lessons, and make it easier for them to grasp complex abstract concepts" (2018, p. 225).

Mathematical folding is known to have great potential, but if we look around, we see that only few teachers use it in secondary school mathematics classes. To understand how one could argue for the scaling up of folding in classroom, we research the motivations of those who already successfully implement it. This study is part of a long-term research project, aiming to explore how mathematical folding activities can be used effectively in mathematics lessons, and what support teachers need. As we want to connect with the teaching culture of secondary education, we investigate *what* mathematical paper folding is done in secondary education and *why*. We did a first survey in the Netherlands, going through the digital archive of 95 volumes of Euclides, the magazine of the Dutch association of mathematics teachers (NVvW) (van Wijk, 2022, 2023). Analysing this preliminary survey, we introduced eight categories: geometry, symmetry and ratios, optimisation problems, folding for understanding functions, folding for convincing, folding logically, didactic considerations, and remaining topics (e.g., 'fun'). These categories are still quite diverse. In this next, expanded survey, including other European countries, we bring a specific focus: the reasons for folding in classroom mentioned by teachers. In this way we have a starting point for developing mathematics lessons enriched with mathematical folding, in line with current teaching practices.

Theoretical framework

The word origami stems from the Japanese words *ori* and *kami*, meaning *folding* and *paper*, and at present, origami is a worldwide phenomenon. It covers activities from folding as a hobby to scientific research on protein folding, from folding as an astonishing artform to an efficient means of (un)folding solar panels for satellites. Mathematical paper folding is part of origami, and the emphasis is on the haptic process of (un)folding, in relation to a mathematical problem, to enable reflection while creating crease lines, areas and shapes, and to analyse the result mathematically (based on Friedman & Rittberg, 2019; Haga, 2008; Nedrenco, 2018). Mathematics educators use origami, or mathematical folding, in a variety of ways, but there is a lack of research investigating if folding is effective as a mathematics teaching tool (Boakes, 2009).

For the coding of reasons mentioned by teachers we used a grounded theory approach (Boeije, 2010). As far as we know, research into professional reports by teachers from secondary education on mathematical folding has not been done before. The grounded theory approach allowed us to explore their reports with an open mind. Besides that, we want to use the results from this study to design classroom interventions. Grounded theory and staying ‘close to the data’ can make this possible. Before we elaborate on our method, we summarize available literature from experts in the field of teaching mathematics with origami (Table 1). The outcomes of our study are compared with this summary. The known examples of the use of mathematical folding stem from expert origami teachers at university and primary school. For example, successful applications of origami and mathematics in university are lecture series by Demaine, Hull, and Meyer. Demaine uses origami as a basis to teach mathematics and computer science (Demaine & O’Rourke, 2007). Meyer has a very different audience. She teaches the course ‘general education mathematics’, which includes many students who fear mathematics. According to Meyer, a course based on origami “is accessible (…) and students get a peek at more complex math” (2020, min. 9–11). A successful application of origami to learn geometry and skills in preschool is done by (Golan, 2011), via the project Origametria. A special detail to mention here is that Golan started using origami without a mathematical goal in a special school (grade 1-10), as a remedial teaching tool for children from Israel and Palestine. Vardi and Golan (2009) state that origami is “enriching emotionally, expanding mentally and developing motorically” (p. 4). These examples show the application of mathematical folding for students with various learning requirements: from students with special needs, to students with math anxiety, and mathematically gifted university students. If we look at several books on origami for mathematics education, only the book of Tubis and Mills (2006) contains a chapter on “Why teach origami?”. They start this chapter with the philosophy of education of Fröbel. He propagates paper folding for mental development, and to informally introduce geometry. Tubis and Mills mention the positive mathematical experience, interaction and cooperation, patience and perseverance, manual dexterity, spatial visualization, mathematical thinking and language and communication skills. Next to that, they list 17 mathematical concepts related to origami and provide tips for the teacher. This book is one of the few that is aimed at 7th to 10th grade. Like in all books, most space is reserved for describing folding tasks. From the books for college and university level, Haga mentions the great interest, enthusiasm, and eagerness of the students for paper folding (2008, p. 1). Hull (2013) states that paper folding enables active and discovery-based learning, with hands-on participation. Meyer and Mukoda (2021, p. vii) state on abstraction: “Mathematics can be abstract, but here our fingers are doing the work: for example, creating a triangle that is congruent to another triangle.”, and they mention involvement of the student. We can summarize the remarks from the expert origami and mathematics teachers related to “why teach origami” in Table 1, consisting of four categories:

Affective reasons	Skills	Manual skills	Learning mathematics
Positive mathematical experience (TM), Interest, enthusiasm, eagerness (H), Involvement (M), Students with special needs (M, G)	Interaction, cooperation, patience, perseverance, language, communication skills (TM)	Manual dexterity (TM), Fingers do the work (M) Hands-on participation (H)	Spatial visualization, mathematical thinking, mental development, Informal introduction of geometry. (TM), Active and discovery-based learning (H).

Table 1: “Why teach origami” based on Tubis and Mills (TM), Haga (H), Meyer (M) and Golan (G)

Most experts give affective reasons, name the use of hands for folding, and write about learning mathematics. Tubis and Mills stand out with their list of skills. This first summary in Table 1 provides some answers to “why teach origami” from the expert teachers. These teachers have in common that the instructors could dedicate a series of full lessons to mathematical origami topics of their choice. This situation would not stand for secondary schools, where teachers are bound to a certain number of teaching hours, a prescribed curriculum, a prescribed textbook, and so on. As practice shows that secondary school teachers cannot teach full courses based on mathematical folding, we want to investigate via a literature review *what* mathematical paper folding is done in secondary education and *why,* despite the limits of school practice. To create a more international perspective, we have chosen to include two other European countries in our review. The research question that we explore in this paper is: *What reasons do teachers from France and Germany report for implementing mathematical folding activities in authentical classroom situations in secondary school?*

Method

We reviewed articles on mathematical paper folding written by teachers from Germany and France, using a grounded theory approach. We will elaborate on the coding process in the next paragraph. In the sample we included contributions from the French and German mathematics teachers’ societies, written by teachers. The French teachers’ society, founded in 1910, is called APMEP and is aimed at teachers from kindergarten to university. The German teachers’ society, founded in 1819, is called MNU and is aimed at secondary education teachers for mathematics, biology, chemistry, physics, computer science and technology. Following referrals from these contributions, we included some commercial journals for mathematics teachers, aimed at secondary education. The articles from the MNU and Mathematik Lehren were available by membership. We checked if the authors teach, or have taught, in secondary education by googling the authors. We only included articles written in the country's language. We list a summary of the inclusion criteria of the associations of mathematics teachers and journals:

France: Search words: plier, origami. Teachers' society: APMEP. Journals: PLOT (Until 2017), Bulletin Vert (Until 2017), Au fil des Maths (from 2018), Tangente Éducation. Websites: www.apmep.fr, www.tangente-mag.com.

Germany: Search words: papierfalten, falten, origami. Teachers' society: MNU. Journals: MNU-Journals, ML: Mathematik Lehren, PM: Praxis der Mathematik in der Schule[1] (Until 2017). Websites: www.mnu.de, www.friedrich-verlag.de.

Process of coding

After we made the summary (Table 1), we coded again from scratch, first for France, and then Germany. We started with 130 French articles. After a first check, like filtering duplicates and articles with the word "multiplier", 72 articles remained. For the first round of coding for France we uploaded these articles from the sample into MAXQDA. We read the articles, and filtered again many articles for several reasons, e.g., not aimed at secondary education, not written by secondary education teachers, or off-topic, like tinkering. Eleven articles remained. The full coding process, with categories in italics, is summarized in Figure 1. The first step, the open coding procedure, means getting a first impression of the data, break it up data into smaller parts, and make first categories.

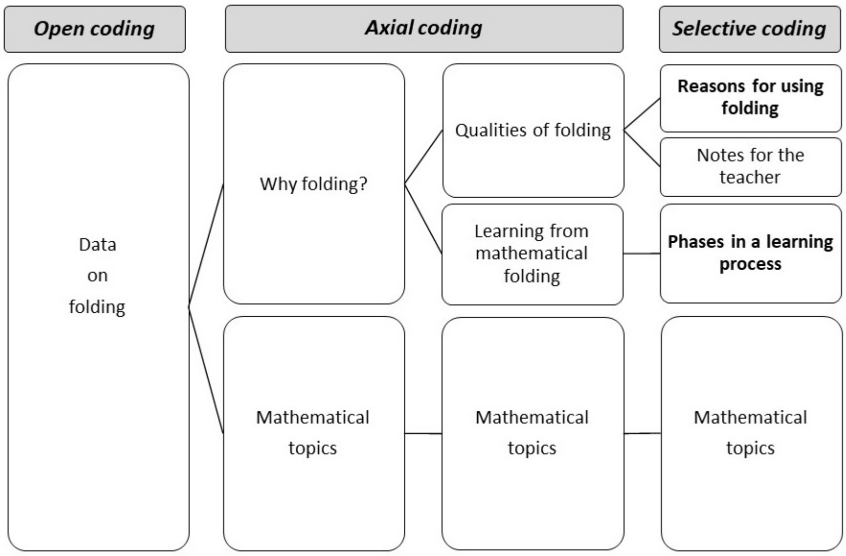

Figure 1: Summary of the coding procedure

We started off coding texts about *mathematical topics* (what is folded) and texts with reasons, tips, and stories of folding in the classroom: the category *why folding*. The category *mathematical topics* remained throughout the coding process, and many subcodes, from "proof" to "fractions" and "hexaflexagons" were added. In the axial coding step, we made connections between categories and related categories to subcategories. The *why folding* topic had to be split up. A part became *learning from mathematical folding*. This topic contained codes like "cooperation" and "recollect prior knowledge". The other part was *qualities of folding*, where specific qualities were addressed like "visualizing symmetry", and "activate the students". At the selective coding procedure, we

[1] Only following referrals (from MNU reviews)

determined important categories to answer the research question. Whe made a distinction between the *reasons for using mathematical folding* in the classroom and the more practical *tips for teachers*, like "use a flat surface and colored paper". We renamed *learning from mathematical folding*, as all categories are about learning, into *phases in a learning process*. For triangulation, we discussed the categories at each phase and co-coded two articles in phase 1 and 2. When we collectively agreed on the categorisation, we ran the different coding cycle steps for the German articles (180 from the sample, 55 in MAXQDA and 12 remaining articles for axial coding). After selective coding of the German and French articles we jointly discussed the various categories with all four authors and agreed on the classification, names, and descriptions of the categories.

Results

As we are doing explorative research from which we want to use the results to design classroom interventions, the result of our study is a completed codebook on the data of France and Germany. We aggregated all information that explained teachers' choices for folding activities, resulting in four categories: 1. Reasons for using folding mentioned by mathematics teachers, 2. Phases in a learning process supported by mathematical folding, 3. Mathematical topics covered with mathematical folding, and 4. Notes for the teacher. To answer the research question, we will only present data from category 1 and 2 (made bold in Figure 1) in Table 2 and 3.

Reasons for using folding	**Description of reason**	FRA # frag.	**GER** # frag.
1.1 To activate students by letting them manipulate paper	Students manipulating the paper during a task, using their hands, and/or being actively engaged.	16	10
1.2 To visualise mathematics	Students being able to see and view an (intermediate) shape, from different angles.	7	10
1.3 To involve and motivate the student	Students' involvement, attitude, and commitment in the folding activities.	7	7
1.4 To ground understanding	Students grasp the mathematical topic better by folding.	5	7
1.5 To provide an alternative perspective on mathematics	Folding activities differ from the standard way of learning mathematics.	5	4
1.6 To implement low floor, high ceiling tasks	The possible profundity of mathematical topics that begin with and/or follow from simple folding tasks.	4	5
1.7 To solve problems beyond ruler and compass	Problems (like the trisection of an angle) that cannot be done by straight edge and compass.	6	0
1.8 To encourage reflection	Folding encourages students to (mathematically) reflect on the topic at hand while folding.	3	2

Table 2: Reasons to use folding mentioned by mathematics teachers

An example of 1.3: "Au-delà de l'aspect manipulatoire qui permet d'impliquer plus facilement les élèves, des propriétés au programme peuvent être visualisées et travaillées grâce aux pliages." *Beyond* *the manipulative aspect, which makes it easier to involve the pupils, properties in the curriculum can be visualized and worked on thanks to the folding*.(Aoustin & Brilleaud, 2021, p. 6).

Phase in a learning process	**Description of phase**	**FRA** # frag.	**GER** # frag.
2.1 Transition from informal to formal	Folding allows students to take a step in the learning process, and/or to ground it, towards more formal mathematics.	6	29
2.2 Practice skills (mathematical, 21st century)	Skills that students learn or practice during mathematical folding: like cooperation, attention, and concentration.	14	15
2.3 Recollection of prior knowledge	Application of a folding activity involving (en passant) retrieval of prior knowledge.	4	6

Table 3: Phases in a learning process supported by mathematical folding

An example of 2.1: "Im Anschluss daran können diese Entdeckungen formuliert, überprüft und begründet sowie eventuell verallgemeinert werden." *Subsequently, these discoveries can be formulated, verified, and justified, and possibly generalised. (*Kaufmann, 2016, p. 12)

Discussion

Using the presented results, we can provide more clarity on the question about reasons for implementing mathematical folding activities in authentical classroom situations in secondary school reported by teacher from France and Germany. There is an overlap in Table 1 of the experts and Table 2 and 3 of the secondary school teachers, in (sub)categories like skills, visualisation, manipulation and involvement of the students. Based on this overlap, we conjecture that experience with teaching mathematical folding tasks of secondary school teachers is in line with what 'professionals' in the field of mathematical paper folding indicate. We can also see some interesting extra categories from France and Germany. The articles from France and Germany are written to inform other teachers, and maybe, next to that, to convince other teachers to use folding in the mathematics lesson. The subcategory 1.5 'To provide an alternative perspective on mathematics' shows the reader that there are alternative representations of mathematics besides paper and pencil tasks. Subcategory 1.7 'To solve problems beyond to ruler and compass' points at an alternative way of doing tasks. It is only coded in the French articles, and the references to this feature of folding were mostly used at the start of an article. This seems to justify the advantages of mathematical paper folding in the classroom compared to 'formal mathematics'. Some more pedagogical reasons from the secondary education teachers are the categories like 'To ground understanding' 1.4, and 1.8 'To encourage reflection'. In some categories the growth in learning mathematics is emphasised, like in 2.1 'Transition from informal to formal' and 1.6 'To implement low floor, high ceiling tasks'. A quote from Hull points to seeing the folded paper as the 'experimental laboratory' of students:

> The math ceases to be an abstract entity, only existing in their [students] mind. It becomes tangible, something they can hold in their hand and count or use to compute data from which patterns, conjectures, and theorems flow. (Hull, 2013, p. xiv)

This argument by Hull might indicate that 'switching' between the different representations can add educational value, even for students who already have considerable abstract knowledge.

This research is based on articles on authentic classroom experiences from teachers. It was hard to find articles from teachers who described real life situations in the classroom with mathematical folding. Many articles were discarded because they were not written by teachers in secondary education, or the lessons were not taught there. Based on the number of articles used of this research, we have some remarks. Though we have used multiple sources, we cannot claim, for example, that the activation of students is only the result of the folding activity, but that it (partly) might be a result of 'doing something new' during the lessons. We are aware that in other (European) countries, e.g., Italy and Belgium, mathematical folding is applied in the classroom. For our purposes, the present study is sufficient.

Conclusion

Though several reasons for using paper folding were mentioned in the earlier literature, our research on authentic situations allowed us to uncover several additional ones. We divided the group of reasons found into possible categories. This allows us to align them with the theoretical grounding of learning mathematics. Trying to summarize all reasons in a condensed way, we suggest considering the first category of reasons as '*To activate students by providing folding tasks*', as all subcategories describe actions of the students or features of the folding tasks. The second emerging category was the phase in the learning process where mathematical folding is implemented. We suggest characterizing these reasons as '*To grow mathematical understanding by folding'*, as teachers indicate that folding supports students from informal, tangible, mathematics to more formal mathematics, while also working on their skills.

In the reviewed articles, teachers show great enthusiasm for folding in the classroom. From this research, we have a good impression of how teachers perceive and use mathematical folding. Further research should more clearly highlight how students experience mathematical folding, and how the potential of folding can be embedded in classroom practice to engage all students in mathematical activities and support mathematics learning.

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