PARTITION COMPLEXES AND TREES

GIJS HEUTS AND IEKE MOERDIJK

(Communicated by Julie Bergner)

ABSTRACT. We construct a functor from the partition complex of a finite set A to a category of trees with leaves labelled by A and prove that it is homotopy initial. This construction and our proof are elementary and require very few preliminaries, but imply an equivalence between different bar constructions of an operad in great generality.

1. INTRODUCTION

Let A be a finite set. The partitions of A form a partially ordered set under refinement. This poset has an initial object (the indiscrete partition) and a final object (the discrete partition); removing these two extreme partitions leaves a poset for which we write $\mathbf{P}(A)$. The nerve $N\mathbf{P}(A)$ is a simplicial set called the *partition complex* of A. These partition complexes play a central role in the study of barcobar (or Koszul) duality for operads [2, 4], as we will review in Section 4.

In this short note we give an elementary relation between the partition complex $N\mathbf{P}(A)$ and a certain category $\mathbf{T}(A)$ of rooted trees with set of leaves A. These two objects play an important role in the literature on operads. However, the precise and direct comparison between the two that we provide here seems not to have been observed before.

The category $\mathbf{T}(A)$ has a morphism $f\colon T \to S$ if the tree T can be obtained from S by contracting a set of inner edges of S. This category has an initial object, namely the 'corolla' C_A with leaves A and no internal edges. Removing this object from $\mathbf{T}(A)$ gives a subcategory that we denote $\mathbf{T}(A)^+$.

The category of simplices of the simplicial set $N\mathbf{P}(A)$ will be denoted $\boldsymbol{\Delta}/N\mathbf{P}(A)$. This can be thought of as a category of layered trees. It has the same weak homotopy type as $N\mathbf{P}(A)$ itself (see Section 2), and our result is as follows:

Theorem 1.1. *There is a functor*

$$\varphi\colon \boldsymbol{\Delta}/N\mathbf{P}(A) \to \mathbf{T}^+(A)$$

with the property that for every object T of $\mathbf{T}^+(A)$, the slice category φ/T is weakly contractible. In particular, φ induces a weak homotopy equivalence between the classifying spaces of these two categories. In fact, φ even induces homology isomorphisms with coefficients in an arbitrary presheaf of abelian groups.

The second statement of the theorem is a consequence of the first by Quillen's Theorem A [14]. However, the first statement is significantly sharper than just φ inducing a homotopy equivalence of classifying spaces, as illustrated by the last sentence of the theorem. As an application this immediately implies a comparison of different versions of the bar construction of an operad, namely the simplicial one and the bar construction of Ginzburg–Kapranov [6] and Getzler–Jones [5]. Such a comparison has only been available in the context of differential graded operads by work of Fresse [4, Theorem 4.1.8] (see also [12, 17]) through a much more elaborate argument. By contrast, our proof is short, requires only the most basic concepts of the theory of simplicial sets, and applies to operads in any homotopical context (e.g. spaces or spectra). It would be interesting to see if our argument can be adapted to more general settings, such as different bar constructions for PROPs and properads (cf. [18, Theorem 6.7] and [7]).

The essential content of this paper consists of Sections 2 and 3. In Section 2 we review the partition complex. In Section 3 we discuss the category $\mathbf{T}(A)$ of trees and prove Theorem 1.1. Section 4 stands apart from the rest of the paper: its sole purpose is to *illustrate* the various occurrences of the category $\mathbf{T}(A)$ in the literature. A functor $\varphi \colon \mathbf{D} \to \mathbf{C}$ with the property that φ/c is a weakly contractible category for each object c of $\mathbf{C}$ is called *homotopy initial*. For readers unfamiliar with homotopy initial functors, we include a brief review of their elementary properties in the appendix.

2. Partition complexes and layered trees

Fix a finite set A. We introduced its partition complex $N\mathbf{P}(A)$ above. The p-simplices of this simplicial set are chains of length p of nontrivial partitions, which can be thought of as *layered trees* with p internal layers. For example, the partitions

$$(abcde)(f) \leq (ab)(cde)(f)$$

of the set $A = \{a, b, c, d, e, f\}$ define a 1-simplex in $N\mathbf{P}(A)$ represented by the layered tree
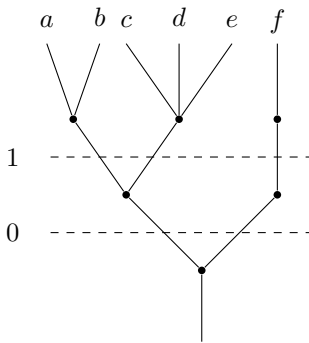
Here the layer labelled 1 represents the partition $(ab)(cde)(f)$, whereas the layer labelled 0 represents $(abcde)(f)$. The numbering of the dotted lines refers to the simplicial structure, so that the ith face map corresponds to removing (or rather, contracting) all of the internal edges on the dotted line. Let us fix some terminology.

(1) Observe that a simplex in $N\mathbf{P}(A)$ is nondegenerate if there is no horizontal level at which all vertices are unary (meaning they have only one incoming edge, if we orient our trees towards the root). A simplex is called *elementary* if it has exactly

one non-unary vertex at every horizontal level. As an example, observe that the picture above does not represent an elementary simplex, but the two trees pictured below do represent elementary simplices, namely the two chains of partitions

$$(abcde)(f) \leq (ab)(cde)(f) \leq (a)(b)(cde)(f),$$
$$(abcde)(f) \leq (ab)(cde)(f) \leq (ab)(c)(d)(e)(f).$$
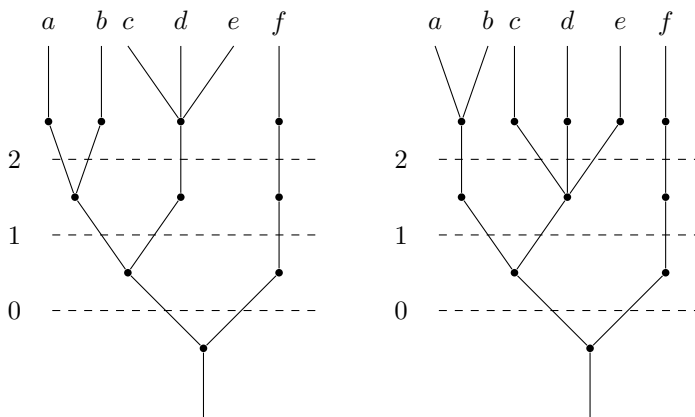
As this example illustrates, any nondegenerate and non-elementary simplex in $N\mathbf{P}(A)$ is a face of an elementary one, but the latter need not be unique.

(2) The category of simplices of $N\mathbf{P}(A)$ is denoted $\mathbf{\Delta}/N\mathbf{P}(A)$. Objects of this category are *layered trees* (with A as set of leaves), morphisms are opposite to compositions of face operations deleting a layer and contracting all edges in that layer, and degeneracy operations inserting a new layer consisting entirely of unary vertices. There is a functor

$$\zeta \colon \mathbf{\Delta}/N\mathbf{P}(A) \longrightarrow \mathbf{P}(A)$$

sending a chain of partitions to its final element. This functor induces a homotopy equivalence of classifying spaces. Indeed, as is well known, this is the case for any small category $\mathbf{C}$ in place of $\mathbf{P}(A)$, since the projection

$$\zeta \colon \mathbf{\Delta}/N\mathbf{C} \longrightarrow \mathbf{C}$$

sending $(c_0 \to \cdots \to c_p)$ to c_p satisfies the hypothesis of Quillen's Theorem A again, see [16].

3. Trees without layers and the main theorem

There is a more basic category of trees $\mathbf{T}(A)$ whose objects are isomorphism classes of finite rooted trees with A as set of leaves, in which every vertex has at least two incoming edges. Morphisms in this category are compositions of 'face

maps' creating an inner edge, as in
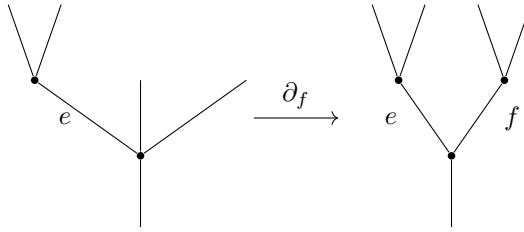
Each such morphism $S \to T$ maps edges to edges, while it preserves the root and induces the identity on the set of leaves. Note that a morphism $S \to T$ exists only if S can be obtained from T by contracting a sequence of inner edges and is unique in that case. Thus $\mathbf{T}(A)$ is in fact a poset. This category of trees is the same as the one considered by Ching [2] and by Hoffbeck and the second author [11].

The category $\mathbf{T}(A)$ has an initial object, namely the minimal tree with leaves A connected by a single vertex to its root: we label this tree C_A and refer to it as the corolla with A leaves. We shall write $\mathbf{T}^+(A)$ for the full subcategory of $\mathbf{T}(A)$ obtained by omitting this initial object; in other words, the full subcategory of trees with at least one inner edge.

Forgetting the layers and deleting unary vertices now defines a functor from layered trees to trees,

$$\varphi \colon \mathbf{\Delta}/N\mathbf{P}(A) \to \mathbf{T}^+(A),$$

which is the functor of Theorem 1.1. The remainder of this section is devoted to its proof and a couple of examples.

Proof of Theorem 1.1. For an object T of $\mathbf{T}^+(A)$, a *layering* of T is a p-simplex σ of $N\mathbf{P}(A)$ for which $\varphi(\sigma) = T$. An *elementary* such layering corresponds to a linear ordering of the vertices of T compatible with the partial ordering defined by the tree structure. In particular, $p = |V(T)| - 2$ in this case, where $V(T)$ is the set of vertices of T. If τ is a face of a layering of T, then there exists a map $\varphi(\tau) \to T$ in $\mathbf{T}^+(A)$. Vice versa, if σ is a level tree for which $\varphi(\sigma)$ maps to T, then σ is a face of a layering of T.

Now fix a tree T in $\mathbf{T}^+(A)$. The slice φ/T is the category of simplices of a simplicial set $L(T)$ generated by the $(|V(T)|-2)$-simplices representing elementary layerings of T. The corolla C_A is not an object of $\mathbf{T}^+(A)$, but we introduce the convention that $L(C_A) = \varnothing$. Also, in the course of the argument below, we will consider trees in $\mathbf{T}^+(B)$ for sets B different from A.

We call a vertex v of T a *leaf vertex* if all its incoming edges are leaves of the tree T. For such a v, let $L^v(T) \subseteq L(T)$ be the simplicial subset generated by simplices which represent elementary layerings of T with v on top. Then

$$L(T) = \bigcup_v L^v(T)$$

with v ranging over leaf vertices of T. Moreover, for any such v the simplicial set $L^v(T)$ is clearly a cone on the simplicial set $L(\partial_v T)$, with $\partial_v T$ the tree obtained from T by removing the vertex v and its incoming edges. To be precise, $L^v(T)$ may be identified with the join $L(\partial_v T) \star \Delta[0]$. (This identification results from the observation that any layering of $\partial_v T$, or a face of it, may be naturally upgraded to

an object of $L^v(T)$ by attaching the corolla with vertex v at the top.) Similarly, if $v_1, \ldots, v_k$ is a finite set of leaf vertices of T, then the intersection

$$L^{v_1}(T) \cap \cdots \cap L^{v_n}(T)$$

is a cone on $L(\partial_{v_1, \ldots, v_n} T)$, where $\partial_{v_1, \ldots, v_n} T$ is obtained from T by removing all of the vertices $v_1, \ldots, v_n$ and their incoming edges. Thus, the $L^v(T)$ together form a cover of $L(T)$ by weakly contractible simplicial sets, all of whose finite intersections are again contractible. So $L(T)$ is itself weakly contractible. Hence its category of simplices $\boldsymbol{\Delta}/NL(T)$ is weakly contractible as well, cf. item (2) of Section 2. This proves the first part of the theorem. The statement about homology isomorphisms now follows from item (d) of the appendix. $\qquad\square$

Example 3.1. An operad $\mathbf{O}$ in the category of sets defines a presheaf $N\mathbf{O}$ on $\mathbf{T}^+(A)$ called its *nerve*. This presheaf assigns to any tree T the set of labellings of the vertices of T by operations in $\mathbf{O}$ of arity corresponding to the number of inputs of the vertex. The functoriality with respect to edge contractions uses the composition of operations in $\mathbf{O}$. The category of elements of $N\mathbf{O}$ is denoted

$$\mathbf{T}^+(A, \mathbf{O}) := \mathbf{T}^+(A)/N\mathbf{O}.$$

The presheaf $N\mathbf{O}$ pulls back to a presheaf $\varphi^*(N\mathbf{O})$ on $\boldsymbol{\Delta}/N\mathbf{P}(A)$ whose category of elements we denote $\boldsymbol{\Delta}/N\mathbf{P}(A)_{\mathbf{O}}$. It is the category of elements of a simplicial set $N\mathbf{P}(A)_{\mathbf{O}}$. This simplicial set is a *labelled partition complex* of the kind considered in [4] and [17]. As explained in item (a) of the appendix, our theorem implies that the functor

$$\boldsymbol{\Delta}/N\mathbf{P}(A)_{\mathbf{O}} \longrightarrow \mathbf{T}^+(A, \mathbf{O})$$

is again homotopy initial, so in particular a weak homotopy equivalence.

Example 3.2. The previous example can be generalized to operads in simplicial sets, or in chain complexes, for example. (As explained in the appendix, really any symmetric monoidal model category or ∞-category could be used instead.) Concretely, let us consider the case of an operad $\mathbf{O}$ in chain complexes. Again it provides a presheaf $N\mathbf{O}$ on the category $\mathbf{T}^+(A)$. Item (d) of the appendix implies that φ induces an isomorphism between the homology of the presheaf $N\mathbf{O}$ and the homology of the presheaf $\varphi^*(N\mathbf{O})$ on the category $\boldsymbol{\Delta}/N\mathbf{P}(A)$. This isomorphism gives a different and much more direct proof of the Fresse's comparison between different bar constructions for an operad (cf. [4, Theorem 4.1.8]) mentioned in the introduction, see also the discussion in the next section. Indeed, the homology of $\varphi^*(N\mathbf{O})$ is the homology of the simplicial bar construction of $\mathbf{O}$, whereas the homology of $N\mathbf{O}$ itself is that of the operadic bar construction of Ginzburg–Kapranov [6] and Getzler–Jones [5]. Note that Fresse's proof goes via a construction in the opposite direction, from trees to layered trees, which he calls 'levelization'.

Remark 3.3. The category $\mathbf{T}^+(A, \mathbf{O})$ we described above only concerns trees without unary or nullary vertices and is therefore insensitive to the terms $\mathbf{O}(1)$ and $\mathbf{O}(0)$ of the operad. It would be interesting to see if our methods can be extended to cover the case where one includes potentially nontrivial unary operations. A comparison of bar constructions in this more general setting is given by Vallette [18, Theorem 6.7].

4. More on the category of trees

In this concluding section we collect some remarks on the category of trees $\mathbf{T}(A)$ described above, which should illustrate its relevance and some of its basic properties.

(1) As A ranges over finite sets, the categories $\mathbf{T}(A)$ (as well as $\mathbf{T}^+(A)$) form an operad in the category of small categories. Indeed, for $a \in A$, a composition map of the form

$$\mathbf{T}(B) \circ_a \mathbf{T}(A) \to \mathbf{T}(A \circ_a B)$$

is given by grafting trees with leaves B onto the leaf a of a tree with leaves A. Here $A \circ_a B$ denotes the set $(A - \{a\}) \amalg B$. An algebra for this operad is a category with a colax symmetric monoidal structure, as has been emphasized by Hinich [8].

(2) The homotopy type of the classifying space of $\mathbf{T}^+(A)$ (or equivalently of the partition complex $N\mathbf{P}(A)$) is a wedge of $(|A|-1)!$ spheres of dimension $|A|-3$ (cf. [15]).

(3) Let $\mathcal{E}$ be a symmetric monoidal simplicial model category (or symmetric monoidal ∞-category) with a zero object. Then there is a 'trivial' operad $\mathbf{1}$ having the monoidal unit in arity 1 and the zero object in all others. If $\mathbf{O}$ is an operad with $\mathbf{O}(1)$ the monoidal unit, then there is a unique augmentation $\mathbf{O} \to \mathbf{1}$ and one may form its simplicial bar construction with object of p-simplices

$$B\mathbf{O}_p = \mathbf{O} \circ \cdots \circ \mathbf{O}.$$

Here there are p copies of $\mathbf{O}$ on the right, the symbol $\circ$ denotes the usual composition product of symmetric sequences, the inner face maps use the operad structure of $\mathbf{O}$, while the outer face maps use the augmentation. The realization of this simplicial object in $\mathcal{E}$ is (one possible version of) the bar construction $B\mathbf{O}$ of the operad $\mathbf{O}$.

It is not difficult to see (cf. [4, Section 4.3]) that this simplicial bar construction $B\mathbf{O}$ can be described explicitly in terms of the labelled partition complexes introduced above. Indeed, the term $\mathbf{O}^{\circ p}(A) = B\mathbf{O}_p(A)$ consists of layered trees of height p with leaves A, and each internal vertex labelled by an operation of $\mathbf{O}$. Combining this with our result, one deduces fairly easily that $B\mathbf{O}(A)$ is a suspension of the cofiber of the map

$$\mathrm{hocolim}_{T \in \mathbf{T}^+(A)} N\mathbf{O}(T) \to \mathrm{hocolim}_{T \in \mathbf{T}(A)} N\mathbf{O}(T) \simeq \mathbf{O}(A).$$

This gives an alternative description for the simplicial bar construction in terms of trees. In the case where $\mathcal{E}$ is the category of chain complexes over a commutative ring, this reproduces the comparison of bar constructions given by Fresse [4]. If $\mathcal{E}$ is the category of pointed topological spaces or spectra, Ching [2] observes that the simplicial bar construction is even *homeomorphic* to the one defined in terms of trees; our result is a bit weaker, but applies more generally.

(4) If again $\mathbf{O}$ is an operad in a suitable simplicial model category, then the term $\mathrm{hocolim}_{T \in \mathbf{T}(A)} N\mathbf{O}$ is essentially the Boardman–Vogt W-construction $W\mathbf{O}(A)$ (as described in [1]). Indeed, the latter can be represented by 'the space' of trees with leaves A, vertices labelled by operations of $\mathbf{O}$ and inner edges labelled by a length t in the 1-simplex Δ^1. If such a length is 0, then the resulting tree is identified with the one where that inner edge is contracted and the operations corresponding to its endpoints are composed. Viewed in this way, the subobject $\mathrm{hocolim}_{T \in \mathbf{T}^+(A)} N\mathbf{O}(T)$ can be thought of as the *decomposable* operations in $W\mathbf{O}(A)$, i.e., the ones arising as a composition of two or more operations in the operad $W\mathbf{O}$. Writing $\mathrm{Indec}(W\mathbf{O}(A))$

for the quotient of $W\mathbf{O}(A)$ by the decomposable operations, the previous remark implies an equivalence of symmetric sequences

$$B\mathbf{O} \simeq \Sigma\mathrm{Indec}(W\mathbf{O}).$$

In the context of topological operads, this observation goes back to Salvatore [2, Lemma 2.4].

(5) The Fulton–MacPherson operad $\mathbf{FM}_d$ has as its terms $\mathbf{FM}_d(A)$ certain manifolds with corners that compactify the configuration spaces $\mathrm{Conf}_d(A)$ of $|A|$ points in $\mathbb{R}^d$, or rather their quotients by translation and dilation (cf. the discussion in [9, Section 1.8]). These manifolds are naturally stratified, with strata indexed by the objects of the category $\mathbf{T}(A)$. The stratum corresponding to a tree is homeomorphic to a product over the vertices of T of the configuration spaces corresponding to the inputs of each vertex; the corolla C_A corresponds to the maximal open stratum. The operad $\mathbf{FM}_d$ is weakly equivalent to the operad $\mathbf{E}_d$ of little d-cubes but has more favourable cofibrancy properties.

Using the description of $\mathbf{FM}_d(A)$ as a stratified space, it is not difficult to see that the homotopy colimit

$$\mathrm{hocolim}_{\mathbf{T}^+(A)} N\mathbf{FM}_d(-)$$

is equivalent to the boundary of the manifold $\mathbf{FM}_d(A)$. Thus, one deduces from (3) above that the terms of the bar construction (and hence of the bar construction of the little d-disks operad $\mathbf{E}_d$) are equivalent to the spaces

$$\Sigma\big(\mathbf{FM}_d(A)/\partial\mathbf{FM}_d(A)\big).$$

The reader might compare this to the remark immediately following [3, Lemma 9.5]. This reference contains a much more detailed analysis of the bar construction of the Fulton–MacPherson operad and includes a proof of the 'self-duality' of the $\mathbf{E}_d$-operad.

Appendix

This appendix is a service to the reader not familiar with homotopy initial functors, illustrating some useful properties of this notion. None of this material is original. A standard account in the context of model categories is [10, Section 19.6] or [13, Section 4.1] in the context of ∞-categories.

If $\varphi\colon \mathbf{D} \to \mathbf{C}$ is a functor and c an object of $\mathbf{C}$, then φ/c (or simply $\mathbf{D}/c$, leaving φ implicit) denotes the category with objects the pairs consisting of an object d of $\mathbf{D}$ and a morphism $\alpha\colon \varphi(d) \to c$, and with morphisms from α to another $\alpha'\colon \varphi(d') \to c$ the morphisms $\beta\colon d \to d'$ in $\mathbf{D}$ making the resulting triangle

$$\varphi(d) \xrightarrow{\varphi(\beta)} \varphi(d')$$
$$\alpha \searrow \qquad \swarrow \alpha'$$
$$c$$

commute. The functor φ is called *homotopy initial* (or *homotopy left cofinal* by some) if for each object c of $\mathbf{C}$, the category φ/c has contractible classifying space.

Quillen's Theorem A [14] states that if φ is homotopy initial, then it induces a homotopy equivalence of classifying spaces $B\mathbf{D} \simeq B\mathbf{C}$. However, the properties of a homotopy initial functor are much stronger than this. As suggested by the terminology, restriction along φ preserves arbitrary homotopy limits, as explained

in (e) below. (Alternatively, restriction along $\varphi \colon \mathbf{D}^{\mathrm{op}} \to \mathbf{C}^{\mathrm{op}}$ preserves homotopy colimits.) Let us list several useful properties of such functors, most of which can be seen as special instances of the last example (e).

(a) If X is a presheaf of sets on $\mathbf{C}$, with its restriction $\varphi^* X$ a presheaf on $\mathbf{D}$, then the induced functor

$$\mathbf{D}/\varphi^* X \to \mathbf{C}/X$$

between the respective categories of elements is again homotopy initial (since the slice of $\mathbf{D}/\varphi^* X$ over (c, x) is isomorphic to $\mathbf{D}/c$), hence induces a homotopy equivalence

$$B(\mathbf{D}/\varphi^* X) \xrightarrow{\simeq} B(\mathbf{C}/X).$$

Note that this map is a pullback of the map $B\mathbf{D} \to B\mathbf{C}$, but that the projection of the codomain to $B\mathbf{C}$ is rarely a fibration.

(b) The same applies more generally for a fibered category $p \colon \mathbf{E} \to \mathbf{C}$ and its pullback $\varphi^* \mathbf{E} \to \mathbf{D}$. To prove this, one shows similarly that for e an object of $\mathbf{E}$, the classifying space of $\varphi^* \mathbf{E}/e$ is weakly equivalent to that of $\mathbf{D}/p(e)$ by another application of Quillen's Theorem A. (The directions are important here; the result would not generally hold for opfibered categories.)

(c) Generalizing (a), one could take $X_\bullet$ to be a simplicial presheaf on $\mathbf{C}$. Then

$$\mathbf{D}/\varphi^* X_\bullet \to \mathbf{C}/X_\bullet$$

is a map of bisimplicial sets that is a weak equivalence in each fixed simplicial degree (of $X_\bullet$), hence induces a weak equivalence on diagonals.

(d) If A is a presheaf of abelian groups on $\mathbf{C}$, then φ induces an isomorphism $H_*(\mathbf{C}; A) \to H_*(\mathbf{D}; \varphi^* A)$ in homology. (Here homology is defined as the collection of left derived functors of the colimit functor from abelian presheaves on $\mathbf{C}$ to the category of abelian groups.) For this result it is not necessary to require that A is locally constant, as for ordinary weak equivalences. *Proof:* If $P_\bullet \to A$ is a resolution of A where $P_\bullet = \mathbb{Z}[X_\bullet]$ is free on a simplicial presheaf $X_\bullet$, then the presheaf $\varphi^*(P_\bullet) = \mathbb{Z}[\varphi^* X_\bullet]$ is a similar resolution of $X_\bullet$. The result then follows because for each simplicial degree p, the map $H_*(\mathbf{C}; \mathbb{Z}[X_p]) \to H_*(\mathbf{D}; \mathbb{Z}[\varphi^* X_p])$ can be identified with

$$H_*(\mathbf{C}/X_p; \mathbb{Z}) \xrightarrow{\simeq} H_*(\mathbf{D}/\varphi^*(X_p); \mathbb{Z}),$$

which is an isomorphism by case (a).

(e) Consider the category $\mathcal{E}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves with values in a simplicial model category $\mathcal{E}$, and similarly $\mathcal{E}^{\mathbf{D}^{\mathrm{op}}}$. These presheaf categories can be equipped with the projective model structures and φ induces a Quillen adjunction

$$\mathcal{E}^{\mathbf{D}^{\mathrm{op}}} \underset{\varphi^*}{\overset{\varphi_!}{\rightleftarrows}} \mathcal{E}^{\mathbf{C}^{\mathrm{op}}}.$$

Constructing a similar adjunction for the trivial functor $\mathbf{C} \to 1$ gives the Quillen adjunction

$$\mathcal{E}^{\mathbf{C}^{\mathrm{op}}} \underset{\mathrm{const}}{\overset{\mathrm{colim}_{\mathbf{C}}}{\rightleftarrows}} \mathcal{E}$$

and similarly for $\mathbf{D}$. If φ is homotopy initial, then there is a natural isomorphism of derived functors (cf. [10, Section 19.6])

$$\mathbf{L}\mathrm{colim}_{\mathbf{D}} \circ \mathbf{R}\varphi^* \cong \mathbf{L}\mathrm{colim}_{\mathbf{C}}.$$

In other words, the restriction functor $\mathbf{R}\varphi^*$ preserves homotopy colimits.

Proof: This is rather straightforward from the usual Bousfield–Kan formula for homotopy colimits, which states that for an $\mathcal{E}$-valued presheaf X on $\mathbf{C}$ taking values in cofibrant objects, its homotopy colimit may be computed as a tensor product

$$X(-) \otimes_{\mathbf{C}} N(\mathbf{C}/-).$$

Similarly, one computes the homotopy colimit of $\varphi^* X$ as

$$X(\varphi(-)) \otimes_{\mathbf{D}} N(\mathbf{D}/-).$$

The latter is isomorphic to

$$X(-) \otimes_{\mathbf{C}} N(\mathbf{C}/\varphi(-)),$$

and the natural comparison map $N(\mathbf{C}/\varphi(-)) \to N(\mathbf{C}/-)$ is a weak homotopy equivalence of simplicial diagrams on $\mathbf{C}$ by the assumption that φ is homotopy initial. This implies the result.

Examples (a) and (d) can be seen as special cases of this preservation of homotopy colimits: for example (a), this relies on the fact that the classifying space of the category of elements $\mathbf{C}/X$ is a model for the homotopy colimit of the functor

$$\mathbf{C}^{\mathrm{op}} \to \mathbf{Sets} \xrightarrow{\mathrm{const}} \mathbf{sSets}.$$

(f) There is an ∞-categorical version of (e) above, stating that precomposition by φ preserves colimits with values in a fixed ∞-category $\mathcal{E}$ (cf. [13, Theorem 4.1.3.1]). Also, example (b) can be seen as a special instance of the fact the pullback of a homotopy initial functor along a cartesian fibration is again homotopy initial [13, Proposition 4.1.2.15].

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