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Embodied instrumentation in a dynamic geometry environment: eleven-year-old students' dragging schemes

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Abstract

Digital technologies for mathematics education are continuously developing. Still, much remains unknown about how students use these tools and how this affects learning. For example, tablets nowadays come with multi-touch options that allow for a more embodied approach to geometry education, compared to mouse interactions. However, little is known about how students use these opportunities to develop bodily-based conceptualizations of geometric concepts in a touch-based dynamic geometry environment (DGE). The aim of this study was to investigate students' dragging schemes from an embodied instrumentation perspective and to identify the types of embodied-dragging schemes that the students use, while transforming one type of parallelogram into another. Fifty-seven 11-year-old students worked on a task on transforming a given parallelogram into a rectangle and next into a square, using a tablet-enabled DGE. Results showed that students used three types of embodied dragging schemes: (a) action-perception dragging guided by perceived prototypical images of shapes, (b) sequentially-coordinated dragging based on initial perception and then utilizing the affordances of the artefacts, and (c) adaptive dragging, effectively integrating action-perception loops and geometrical properties. In schemes of types (b) and (c), geometric properties of the constructed shapes emerged and guided students' action-perception loops. As a conclusion, this description informs teachers, textbook authors, and designers of digital assessment on how to design student activities. From a theoretical perspective, the embodied instrumentation lens provided a fruitful approach to study student–tool interactions in geometry that does justice to the bodily foundations of mathematical cognition.

Keywords Dragging · Dynamic geometry · Embodied instrumentation · Embodiment · Instrumental genesis · Mathematics education · Primary education

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1 Introduction

It is widely known that the use of digital technology in math education is complex, not self-evident, and does not automatically lead to learning. The mathematics education community is still struggling to get to grips on that. To orchestrate student learning, it is important that researchers, teachers, textbook authors, and designers of technology-enhanced assessment have a detailed understanding of the learning processes involved in tool use. This also holds for the domain of geometry and the use of digital geometry environments (DGE). Portable and handheld digital technologies offer opportunities for enacting embodied learning in mathematics education (Abrahamson & Bakker, 2016; Georgiou et al., 2021). The use of DGE on multi-touch tablet devices seems to have high potential for primary school age students, because it facilitates direct embodied interaction with geometric objects through dragging (Dubé et al., 2015; Xie et al., 2018). It is important to know, however, how exactly this interplay between embodied interaction and geometric conceptualizations takes place. The goal of this study, therefore, is to empirically investigate in detail how students use dragging to explore invariant properties of geometrical objects. To address this question, we explore 11-year-old students' activity investigating relations between different types of parallelograms while using a tablet-enabled DGE.

We describe and analyze the identified dragging schemes through an embodied instrumentation (EI) lens (Alberto et al., 2019; Drijvers, 2019; Shvarts et al., 2021) because it does justice to the embodied nature of students' actions. Thus, we set up an inventory of what we call embodied-instrumented schemes for the case of hand dragging to report on 11-year-old student's embodied experiences in a DGE. By the term embodied-instrumented scheme, we refer to the amalgam of techniques, in terms of action and perception, and student's conceptualizations, as to emphasize the role of bodily experience. As a more theoretical "hidden agenda", the study explores the value of the relatively new embodied instrumentation framework for researching these matters in the given research context.

2 Theoretical framework

The study's theoretical framework includes notions from embodied instrumentation to theorize students' embodied-instrumented schemes, and insights on dragging from mathematics education research.

2.1 Embodied instrumentation

Embodied instrumentation highlights the role of the body in learning mathematics with technology through revisiting the instrumental approach from an embodied cognitive perspective (Alberto et al., 2019). Shvarts et al. (2021) posit that instrumental and embodied cognition theories can be coordinated and aligned in a meaningful way. The networking of the two theories has resulted in a comprehensive theoretical framework that underlines the complexity of user-tool interaction and reconciles the embodied nature of instrumentation schemes and the instrumental nature of sensorimotor schemes (Drijvers, 2019). An embodied instrumentation approach to learning mathematics with digital tools (artefacts) may facilitate exploring the co-emergence of sensorimotor schemes, tool techniques, and mathematical cognition. It also highlights the role of body in regulating instrumented actions and acknowledges new developments in digital technology. Thus, we use the embodied instrumentation framework to integrate an instrumental perspective, in which we describe schemes, and an embodied perspective, in which we feature the role of bodily action and perception in conceptualizing mathematics concepts.

2.1.1 Instrumentation schemes

Based on the instrumental approach, a mathematical tool (artefact) becomes an integral part of an instrument through instrumental genesis that involves the construction of schemes (Artigue, 2002). The notion of scheme is inspired by Vergnaud's (2009) definition as "the invariant organization of activity for a certain class of situations" (p. 88) that comprises an intentional, a generative, an epistemic, and a computational aspect. The intentional aspect involves goals that include sub-goals and anticipations. The generative aspect includes sequences of actions. The epistemic aspect involves operational invariants that consist of theorems in actions, propositions considered true, and concepts in action, concepts that are considered relevant. Finally, the computational aspect includes possibilities of inference. Furthermore, two features of schemes are worth highlighting. First, the invariance in this definition should be understood as relative, as schemes are adaptive and dynamically developing. Second, a scheme is founded on a sequential organization of activity for a certain situation, which opens the horizon for the embodiment perspective. In the present study, the touchscreen-based dragging tool is the main artefact inviting scheme development and gives birth to dragging schemes (Lopez-Real & Leung, 2006).

2.1.2 Sensorimotor schemes in tablet technologies

The theory of embodied cognition suggests that cognition is shaped by bodily activity (Barsalou, 2010). Research findings indicate that various aspects of mathematical understanding are embodied (Flood et al., 2020; Radford 2009). Knowledge is thought of as embodied action that is part of a complex dynamic system of behaviour. Sensorimotor schemes appear through multiple action-perception loops and include attentional anchors, describing dynamical patterns of eye movements, while their hands move, gestures, verbal explanations, and conceptualizations (Duijzer et al., 2019).

Action-perception-loops are the product of the interaction of recurrent sensorimotor patterns and actions that are coordinated to direct the acquisition of data (Little & Sommer, 2013). This interaction between actions and sensation is crucial in respect to the sensory inputs we receive and results to a perceptually guided action. For instance, when students explore geometric concepts in a tablet-enabled DGE, their perspective input changes based on how they shift their gaze on the screen and how they use the dragging tool. Conceptualizing the examined concepts depends on the effective coordination of dragging actions and perception in terms of implied geometric properties.

An important aspect of action-perception loops is where students' gaze focusses, in terms of attentional anchors (Abrahamson & Sánchez-García, 2016). Attentional anchors are described as imaginary perceptual structures or routines for orienting toward the digital environment. They function as self-imposed motor constraints while students coordinate their motor actions and perception. In the case of tablet-enabled DGE, students' attentional anchors could be points, lines, angles, shapes, measures, or foci areas on the screen that could guide or impose their further actions. Besides attentional anchors, students' gestures are also significant in providing information regarding their manifest embodied knowledge (Alibali & Nathan, 2012). Three types of gestures that students use to produce explanations are pointing gestures, reflecting the grounding of cognition in the physical environment, representational gestures, manifesting mental simulations of action and perception, and metaphoric gestures that reflect body-based conceptual metaphors of mathematical ideas. In this study, these types of gestures are used to illustrate dragging actions and attributes of the shapes.

Tablet technologies, through their touch-based interactions, provide a kinaesthetic orientation of learning, while multiple senses are incorporated and offer opportunities to do justice to the embodied character of mathematical cognition (Abrahamson & Bakker, 2016). Touchscreen tablets incorporate and enable students' emerging sensorimotor enactments and visualizations of mathematical concepts (Price et al., 2020). In the present study, we describe students' sensorimotor schemes through analyzing their action-perception loops, attentional anchors, gestures, and verbal explanations.

2.1.3 Embodied-instrumented schemes and body-artefact functional systems

In line with work done by others (Abrahamson & Bakker, 2016), we adopt the position that the development of sensorimotor schemes goes hand in hand with instrumental genesis. This alignment might lead to schemes in which embodied experiences still form the basis, and through a process of reflective abstraction gives birth to embodied-instrumented schemes.

The emergence of embodied-instrumented schemes could also be described in terms of the genesis of body-artefact functional systems that regulate instrumented actions, as proposed by Shvarts et al. (2021). A functional system emerges as a synergy of many action-perception loops. Artefacts (such as the dragging tool) are included in the action-perception loops of instrumented actions and body potentialities (including the brain) and affordances of the environment (possibilities for action) frame perception and action that trigger the interaction between them. In this way, an artefact gets incorporated into a body-artefact functional system. Multiple levels of action are traced in the functional systems, such as verbal expressions (pragmatic and epistemic) and sensory-motor co-ordinations. In the present study, the coupling of students' body potentialities and the affordances of DGE facilitate the emergence of embodied-instrumented schemes that reflect body-artefact functional systems.

2.2 Dragging in a DGE

2.2.1 Dragging modalities and dragging schemes

Dragging is considered a dynamic and powerful tool to acquire mathematical knowledge, through continuous, real-time transformations in a way that the properties of geometric objects can be kept invariant or approximately invariant (Leung, 2008). Dragging in DGE may give the potential for conceptualizing mathematical concepts by using a variety of dragging modalities (strategies) (Baccaglini-Frank, 2019; Leung, et al., 2013). From the perspective of instrumental approach, dragging modalities can be seen as artefacts supporting conjecturing. A particular way of dragging may become an instrument with a scheme developed by a learner. These instrumentation schemes are referred to as dragging schemes (Leung, 2015), as they constitute the "reasoning" that accompanies particular uses of dragging.

Two broad categories of dragging modalities are dragging for testing, to check the presence of desired properties, and dragging for searching/discovering that consists of dragging to look for new properties of the figure (Holzl, 2001). Arzarello et al. (2002) described the following dragging modalities: wandering dragging (randomly dragging a point on the screen), maintaining dragging (induce a particular property to become invariant), dragging with trace activated and dragging test (verify whether a figure has been properly constructed). Furthermore, Leung (2008) proposed four dragging modalities to study the variation of a DGE figure: contrast by seeing differences; separation by separating out hidden geometrical properties; generalization by discerning or verifying invariants; and fusion by experiencing different features at the same time.

Students' work in a DGE could facilitate passing from iconic visualization to non-iconic visualization, as defined by Duval (1995). This means conceptualizing that a drawing is a representation of a geometrical object (Mithalal & Balacheff, 2019), and for the case of DGE, that the dynamic images on the device's screen are representations of a theoretical geometrical figure (Sinclair & Yurita, 2008). In the meantime, students that adopt an iconic visualization recognize an object because its shape is similar to an already known object and are guided by the perceptual recognition of similarities and differences among shapes and prototypical images (Duval, 2017).

2.2.2 Dragging in haptic devices

As DGE were initially designed without considering touch-based mobile devices, most of the research findings described in Sections 2.2.1 and 2.2.2 involve dragging mediated by the mouse. However, dragging in haptic devices gives new technical potentialities, such as multi-dragging, and it affords new modes of thinking through hand movements that are directly tied to the actions on screen (Ng, 2019). For instance, haptic dragging provides a new perspective to direct and indirect motion (Mariotti, 2014). Therefore, research is needed to investigate in depth students' dragging schemes in hand-held DGE to grasp the haptic nature of dragging in tablets. The embodied instrumentation framework, as discussed in Section 2.1.3, seems to be a valuable theoretical framework for this; indeed, this study also explores the applicability of the framework to this research context.

2.3 Research question

In line with the study's goal, and the lenses of embodied instrumentation and geometric dragging, the research question is: What types of embodied-instrumented dragging schemes emerge while 11-year-old students transform one type of parallelogram into another using a handheld/multitouch DGE?

3 Methods

3.1 Subjects

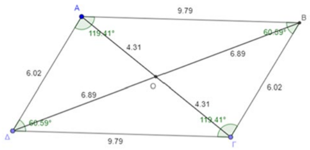
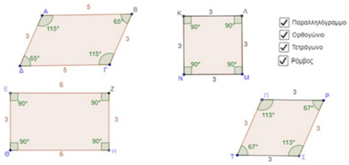
Subjects in the study were fifty-seven 11-year-old students, 29 girls and 28 boys from three Grade 5 classes. None of the students were identified as having learning disabilities or other cognitive or sensory incapacities. They reflected a broad spectrum of academic achievement levels. The school's population had a middle to high socioeconomic status. The students had used tablets in mathematics several times before. Their previous geometry lessons in Grade 5 involved tablet-enabled DGE to explore types of angles and triangles and the area formula of rectangles, parallelograms, and triangles. Overall, students had used DGE eight times in previous lessons. Consent was given by the students and their parents.

3.2 Procedure and task

3.2.1 The intervention context

In collaboration with the three teachers, the research team designed a geometry module consisting of three 80-minute lessons. The lessons were delivered by one of the members of the research team. The module was part of a Geometry Unit in Grade 5 and the relevant attainment targets included exploring and understanding the properties of parallelograms

Table 1 Description of the lessons

Lesson	Learning goal	Tool – Activity
1	Recognize parallelograms, describe their general features and explore their properties using DGE	Ready-made construction of a parallelogram with dynamic measures of side length and angle size. Students were asked to drag the vertices of the parallelogram to different positions and fill in a table regarding the length of its sides and the angles size. Based on the accumulated data, students made conjectures regarding the equity of the opposite sides and angles. 
2	Recognize rectangles, square and rhombi, describe their characteristics and explore their properties.	Ready-made construction of a rectangle, square and rhombi with dynamic measures of side length and angle size. Students were asked to drag the vertices of the shapes and fill in a table regarding the length of sides and angles size. Based on the accumulated data, students made conjectures regarding the properties of each shape. 
3	Transform a parallelogram into a special type of parallelogram and explore the inclusion relations of parallelograms, rectangles and squares.	Ready-made construction of an arbitrary parallelogram with dynamic measures of side length and angle size based on the following construction protocol: Point A, Point B, Segment AB, Point C, Segment CA, Line through C parallel to AB, Line through B parallel to CA, Point D (intersection of parallel lines). Dragging of any of the three free vertices A, B and C changes (1) the length of two sides, or (2) the length of all four sides, or (3) angles measure, (4) or side length and angles measure simultaneously, based on the direction of the dragging movement. In addition, students can drag directly segment AB or AC, which changes only the length of the two sides. Dragging of a point or a side maintains the parallelogram properties. Simultaneous dragging of the vertices is feasible and dynamic measures of side length and angle-size are available Students were asked to transform the parallelogram to a rectangle and then to a square.

and the inclusion relations of quadrilaterals. Students were familiar with angle measurement in degrees and the Babylonian history of angle measurement, using GeoGebra. Table 1 presents the learning goals of each lesson, the provided tool, and a short description of the main activity. In lessons 1 and 2, students explored the properties of parallelograms, rectangles, squares, and rhombi by manipulating ready-made constructions.

3.2.2 The task and its delivery

This paper reports on the results of one task in lesson 3. In this task, students were asked to transform an arbitrary parallelogram into a rectangle in a DGE, and then into a square. Afterwards, students were asked to write down their procedure and to provide a definition of a rectangle and a square based on their work. Next, the researchers raised clarifying questions in informal mini interviews.

The dynamic figure provided in the task was presented as an applet. Table 1 presents the GeoGebra construction protocol. The dynamic changes of the shape on the screen through the continuous reconstruction provided by the DGE constitute the base of the perception of the "draggable figure" and invite the identification of invariants and add-on properties. This identification is extremely important for noticing the properties of parallelograms. Besides grasping the dynamic changes, students are expected to conceptualize how discerning invariants and add-on properties results in identifying the common geometric properties of the initial construction and the transformed one. In addition, students hopefully recognize whether the dynamic images on the screen are examples of the targeted shape or not.

3.3 Data collection and analysis

This exploratory study used qualitative methods for data collection. In each class, four PhD students observed four students each to capture their actions while working (in total, 48 out of the 57 students were observed). Each researcher used an observation protocol, consisting of a set of questions examining the way in which each student utilized the tablet-based DGE. We videotaped the screens of the tablets to capture students' actions while working and audio-recorded their oral explanations. Thus, the data consisted of researchers' observations and field notes, students' written explanations and provided definitions, audio-recorded explanations from mini interviews, and screen-recording videos. Of course, it would have been interesting to include student language in the data and data analysis. However, for the sake of setting up an inventory of dragging schemes, we wanted to include a somewhat larger number of students to have a wide variety of possible schemes emerging. This scale did not allow us to include language.

A qualitative interpretive framework was used in the analysis of the data (Miles & Huberman, 1994). The data analysis consisted of three phases (see Table 2): (1) identifying common embodied-instrumentation schemes, (2) examining and interpreting scheme development during the work, and (3) elaborating scheme development.

In phase 1, we used a combined theory-driven and data-driven approach to identify emerging embodied-instrumentation schemes in four steps. In step 0, before the analysis, a coding scheme was formulated to evaluate students' dragging schemes, including distinctions among perceptual and geometrical conceptualizations, and incorporating hand-held dragging actions, action-perception loops, attentional anchors, gestures, and explanations. This initial framework was based on the theories on dragging schemes (Baccaglini-Frank & Mariotti, 2010), prototypical figure reasoning (Duval, 2017; Hershkowitz, 1989), dragging

Table 2 Overview of data analysis

Phase and objective	Outline	Data-driven/theory driven
Phase 1: Qualitative data analysis Identification of students' embodied instrumentation schemes	Step 0: Preformulating schemes based on theories on dragging in DGE and embodied instrumentation	Theory-driven
	Step 1: Observing each student's enactments and fragments from explanations	Data-driven
	Step 2: Categorizing data from step 1, by using preformulated schemes and refinement of initial coding	Data and theory-driven
	Step 3: Identifying global patterns of schemes for more students, by using the categorized data of step 2	Data and theory-driven
Phase 2: Interpretive analysis of embodied-instrumentation schemes Examination of students' scheme development by identifying dragging techniques, sensorial perception and geometrical understanding during the work	Defining, refining and specifying the integration of hand-held dragging techniques, sensorimotor schemes and levels of geometrical reasoning based on students' observed and latent enactments and reasoning	Data and theory-driven
Phase 3: Case study and comparison More detailed examination of students' scheme development	Further examining how instrumentation and sensorimotor schemes integrate and how the embodied-instrumentation schemes support geometrical understanding of inclusion relations in parallelograms, by zooming in on developing personal schemes of students and comparing students that adopted different schemes	Data and theory-driven

Table 3 Coding framework

Action/perception enactments	Attentional anchor	Gestures when providing explanations	Verbalization	Dragging modalities
Sensorial perception	Vertices-angles of the shape	Pointing gestures to the shape on the screen	Perception-iconic visualization explanations	Random dragging/wandering
Action-perception loops	Sides	Representational gestures simulating actions and perception	Prototypical images	Dragging around the given position of vertices
Action-perception loops driven mainly by students' prototypical images	Entire shape	Metaphoric gestures reflecting mathematical ideas	Non-iconic visualization explanations	Dragging of only one point/utilization of available free points
Enriched action-perception loops guided by integrating student's perception, feedback of the learning environment and geometrical properties	Measurement tools		Euclidean-geometric properties explanation	Explorative dragging
	Combination of the above			Maintaining dragging (making properties to become invariant)

based on sensorial perception and dragging consistent with Euclidean axioms (Leung, 2015), sensorimotor schemes (Alibali & Nathan, 2012; Duijzer et al., 2019), instrumental genesis (Artigue, 2002), and embodied instrumentation (Drijvers, 2019).

In step 1, we analyzed student's dragging actions for all students of two classes in the sample (36 out of 57 students), in terms of action-perception loops, attentional anchors, gestures, and oral and written explanations and definitions. In step 2, we categorized the data from step 1 by using the a priori framework. A constant comparative method was used, which allowed students' enactments to be compared, creating categories of schemes (Tesch, 2013). In the meantime, the initial theory-driven coding scheme was further elaborated and refined to include the observed data. In step 3, we used the modified categorized data of step 2 to identify patterns for the remaining students. Through iteratively going through the categorized data, both within one student and across students, we identified global patterns. In this step, we adapted again the a priori framework to the global patterns and resulted to the description of three schemes. To establish interrater reliability of the data analysis, a second researcher coded a 50% sample of data. Agreement among coders reached 80%. Any discrepancies were discussed until full agreement was reached.

To interpret the identified schemes, in phase 2, we elaborated on defining, refining, and specifying the integration of hand-held dragging techniques, and sensorimotor schemes based on students' observed and latent enactments and explanations. Table 3 presents the main categories of the theory-data driven codes. This analysis allowed us to exemplify the bodily foundation of the embodied-instrumentation schemes, in terms of Shvarts et al.'s (2021) body-artefact functional system levels and describe the different aspects of the schemes.

In phase 3, we validated the description of the schemes derived in phase 2 through zooming in on case studies and comparing students who adopted different schemes. We focused on conceiving how the embodied-instrumentation schemes support a geometrical understanding of inclusion relations in parallelograms. This zooming and comparison completed the information about the characteristics of each embodied-dragging scheme, the nuances, and the additional details that delineate boundaries between them.

4 Results

The aim of the study was to investigate the types of embodied-dragging schemes exhibited by the students. As a result, we identified three global embodied-dragging schemes:

I. *Action-perception dragging* (n=23)

The intentional aspect of this scheme was to transform the provided draggable construction, so it looks like the targeted shape, by establishing resemblance to typical examples or prototypes, such as horizontally aligned rectangles, without giving attention to measurement accuracy. To do so, a sequence of enactments took place that were initiated by iterative action-perception loops that relied on interpreting the dynamic images of the learning environment and integrating it with students' perceived salient characteristics of the involved shapes.

II. *Sequentially-coordinated dragging* (n=15)

The intentional aspect of this scheme was to transform the provided construction to the targeted shape in steps, by making first a rough dynamic image on the screen, following a series of intentional dragging actions, and then to establish measurement accuracy. This scheme includes two types of iterative action-perception loops: the first one relies on inter-

Table 4 Comprehensive description of the embodied-instrumentation schemes

Embodied-instrumentation scheme	Dragging modalities	Sensorimotor enactments	Operational invariants
Action-perception dragging: Transform the parallelogram to the targeted shape based on visual-perceptual characteristics of shapes	Explorative dragging to chase/establish right-angles in a convenient way (multitouch-dragging, easy-position dragging)	Action-perception iterative loops initiated by coordinating the dynamic images on screen with students' perceived typical examples of shapes and prototypes. Representational gestures to illustrate their intention to make vertical and horizontal lines	**Concepts in action**: right angles, description of a rectangle as the shape with four right angles **Theorems in action**: a parallelogram might give birth to a rectangle and vice-versa
Sequentially-coordinated dragging: Transform the parallelogram to the targeted shape in steps; establishing a rough dynamic image on screen and then fine-tuning adaptations to meet geometric properties	Drag for separation/maintaining to emerge invariant properties	Action-perception-artefact loops that integrated the DGE feedback, the artefact affordances and geometric properties Representational gestures to illustrate how a purposeful dragging enactment varies angle and size measurements	**Concepts in action**: moving parallelogram, right-angled parallelogram **Theorems in action**: rectangle as the right-angled parallelogram, rectangle is constructed by fixing the angles of a parallelogram
Adaptive Dragging: Transform the parallelogram to the targeted shapes in an intentional way by discerning invariant and variant properties	Discerning dragging; check the presence of desired properties and look for new properties	Enriched body-artefact functional system initiated by the effective integration of body potentialities, artefact affordances and geometrical properties, including flexible attentional anchors, representational gestures and verbal explanations indicating that each new shape is the result of an action on the dragged shape	**Concepts in action**: size of angles, side length, vertex, opposite sides, opposite angles, equality **Theorems in action**: a parallelogram is transformed to a rectangle if you make one its angles a right one, a rectangle is a parallelogram

preting the dynamic images of the learning environment to meet their perceived exemplar of the requested shape, and the second one integrates students' knowledge of the geometric properties in the action-perception loop, by exploiting the affordances of the artefact.

III. *Adaptive dragging* (*n*=7)

The intentional aspect of this scheme was to make the necessary transformations, by discerning invariant and variant geometrical properties through well-considered and flexible dragging modalities that take full advantage of the artefact affordances. Thus, it initiates a body-artefact functional system that effectively integrates action-perception with knowledge about geometrical properties and facilitates the development of a haptic conceptualization of the inclusion relations of parallelograms through discerning dragging that adds new properties to the draggable figure.

In the following, we provide a detailed description of these schemes by analyzing their intentional, generative (in terms of dragging modalities and sensorimotor enactments), and the epistemic (in terms of operational invariants) aspects (see Table 4). We do not report on the computational aspect because students did not really engage in this type of activity.

4.1 Action-perception embodied-instrumented dragging scheme

The main intention of this scheme was to drag the provided construction in an appropriate way, so it looks like a familiar example of a rectangle. This led to series of iterative dragging actions that could be characterized as right-angle chasing by dragging vertices or sides in convenient positions on the screen. We observed students' exploring procedure to transform the angles of the parallelogram to right-angled ones. Student's actions included one-finger and multiple-finger dragging of vertices and sides of the given figure to make a rough construction of the targeted shape, without considering the available measures.

Students dragged the vertex of an angle to make it a right one through quick finger movements and observed that the image on the screen changed too quickly and was too difficult to establish perpendicularity of sides. They gradually slowed down finger movement and interchanged the selected draggable point. Their last movements were more careful and slow compared to the initial ones since they grasped that a random dragging movement changes the size of all the angles of the shape. Perception consisted of watching and interpreting the dynamic images on the screen to establish the image of right angles and vertical and horizontal sides. Students' explorative dragging actions and the interpretation of the feedback provided by the DGE initiated and supported an action-perception iterative cycle and sensory-motor coordination until transforming the given figure to the targeted one. Students used pointing and representational gestures to explain their work. For instance, they used pointing gestures to show the points or sides they dragged and representational gestures to illustrate their rationale of creating vertical/horizontal sides.

Students' perception established right angles by applying a variety of dragging actions. A further analysis of student's dragging modalities' characteristics and attentional anchors provided three main types of perceptual dragging behaviour: *vertical/horizontal aligning-dragging*, *multitouch-dragging*, and *inflexible-dragging*. In the following, we provide examples.

Students who applied *vertical/horizontal aligning-dragging* (*n*=16) dragged the construction to change the orientation, by aligning the "right" angles with a virtual horizontal or vertical line or dragged the vertices of the given shape in appropriate positions to make the opposite angles right in a convenient, easily perceptual position on the screen. The main attentional anchor was the four angles of the dynamic image on the screen. For instance, Fig. 1a presents the starting position of the parallelogram and Fig. 1b the trace of Joseph's big and fast movements to create right angles,

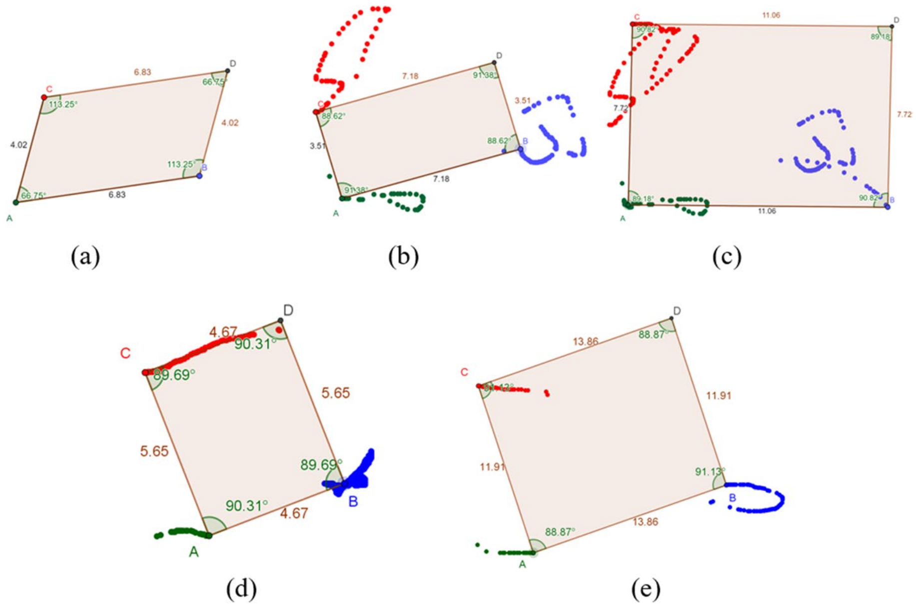


Fig. 1 Action-perception embodied-dragging using vertical/horizontal aligning and multitouch dragging

by switching the dragging point between the three available ones and without significantly changing the initial orientation of the construction. After making several movements, he slowed down and dragged point B to the right side of the screen. He explained that "it is difficult to work like this … I must put it down (indicating with a representational gesture a horizontal line)". Then, he dragged alternatively points A and C, so the side AB fits the virtual horizontal line. That was done more easily by enlarging the sides of the construction (see Fig. 1c), as a slight movement did not alter the dynamic image to a great extent and made angle manipulation more convenient.

The second type of perceptual-dragging (n=3), *multitouch-dragging*, was characterized by the use of multiple fingers to drag simultaneously all the possible vertices to form the perceived rectangle. Students that applied this type of dragging did not change the initial position of the construction but tried to instantly drag its vertices to unveil the hidden rectangle. Their explanations indicated that their actions were guided by a strong perceived example of a rectangle and their attentional anchor jumped between the four angles of the construction, based on the activated ones, while dragging each of the three available points.

Figure 1d presents the trace of the dragging procedure of Costas, who dragged points A and C with two fingers of the left hand and point B with the right hand, coordinating perceptually the three movements. When asked to describe his strategy, he supported that "I tried to fix the angles, all of them were skewed" and used a representational gesture to form a right-angle with his hands (∟). He thought that the provided construction did not fit his perceived rectangle and consequently needed appropriate corrections. Each correction was implemented by dragging at the same time one or more of the free vertices of the given construction and yielded a new action-perception loop, until the dynamic image on the screen matched the perceived example. The sequence of iterative loops resulted in enlarging the construction (see Fig. 1e), as hand dragging became smoother when the figure was enlarged.

The third type of perceptual dragging (n=4), *inflexible-dragging*, intended first to transform the provided construction so it does not look like a parallelogram and then establish resemblance to a prototype. This resulted in inflexible dragging modalities that followed the constraints imposed by salient-prototypical features. For instance, Ann first moved the entire construction by dragging an internal point (see Fig. 2a). Then, she dragged side AC to several positions (see Fig. 2b). When she realized that the image on the screen still looked like a parallelogram, she decided to drag freely points A and B (see Fig. 2c). She explained that:

before making a rectangle, the shape should stop being a parallelogram, so I dragged vertices *(pointing vertices A and B)* to make it become straight (*making a representational gesture with the two hands, indicating from up to down two parallel lines*, | |).

When asked if the construction is no longer a parallelogram, she answered that "it does not look like a parallelogram anymore…so now I can try to make a rectangle." Then, she dragged point B to the right (see Fig. 2d), explaining that "a rectangle has two big and two small sides, so I had to make the two of them bigger."

Integrating analysis of students' dragging and sensorimotor behaviour with students' verbal explanations shed light onto students' conceptualizations and the involved operational invariants. Students exhibited an iconic visualization, as their description of the involved shapes was founded on the perceived features of the dynamic images on the screen. They considered a rectangle as the shape with four right angles that is represented on the screen by dynamic images that resemble typical examples of rectangles or prototypes. That explained their persistence to simultaneously drag the free vertices of the parallelogram, to transform them into right-angled ones. Explanations such as "a rectangle has four right angles",

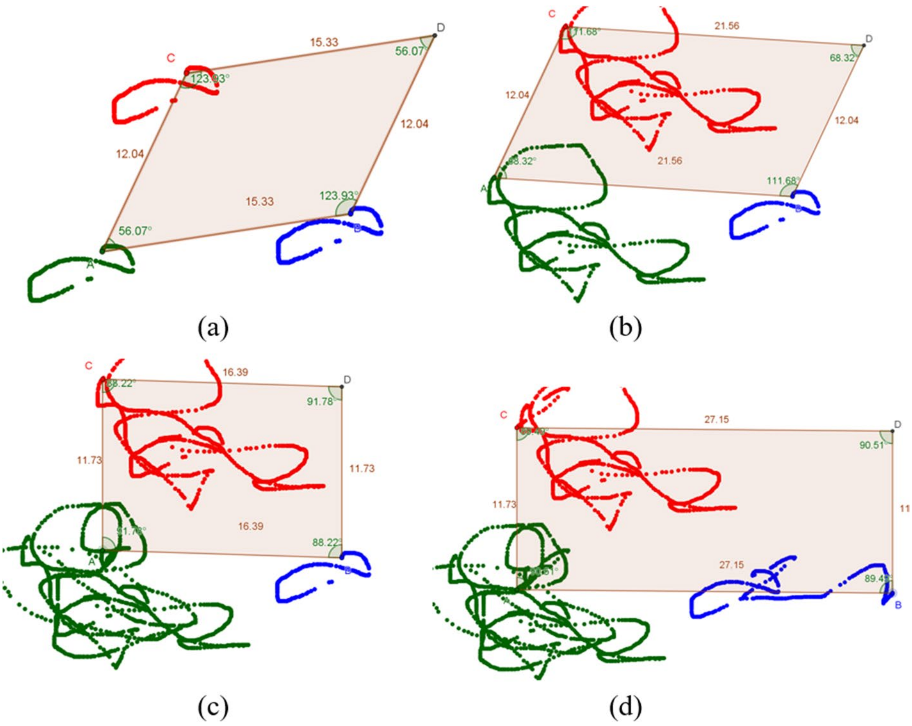


Fig. 2 Ann's action-perception inflexible dragging

"parallelogram is the shape that has parallel sides, while rectangle is the shape that has only right-angles", and "parallelogram is slanted" indicated the epistemic aspect.

The dynamic transformation facilitated understanding that the involved shapes are related, without conceiving the exact relation, while the transformation of the provided construction into the targeted shape relied on perceptual interpretation. During the mini interviews, students suggested that "when you make straight a parallelogram, you get a rectangle", and "rectangle is the parallelogram that you change its sides from slanted to vertical ones". These theorems in actions helped them to infer the goals of their dragging. Thus, the invariant of the scheme was that "a rectangle is a straight parallelogram" and one or more of following actions were critical requirements; change the image on the screen to stop looking like a parallelogram, adjust the angles to look like right ones, and establish that the two sides are substantially longer than the other two. When we asked them to explain whether a parallelogram is always a rectangle or a rectangle is always a parallelogram, they recalled the way they had worked and responded that their working procedure might indicate that a parallelogram gives birth to a rectangle; thus, a rectangle is the "child" of a parallelogram. However, their understanding was fragile. When asked if it could be also supported that a parallelogram is always a rectangle, they suggested that this argument could also be true, because as you change a parallelogram to a rectangle, you can do the inverse procedure.

4.2 Sequentially-coordinated dragging

The second main type of dragging scheme, the sequentially-coordinated scheme, intended to transform the provided construction to the targeted shape in steps, by exploiting at each stage different affordances of the DGE and conceptualizations of the examined geometric concepts. A sequence of intentional dragging actions attempted to transform the construction to roughly resemble a rectangle and then to establish measurement accuracy by fixing angle size to 90 degrees. To do so, students coordinated action-perception and the affordances provided by the artefact (angle and length measurements).

Students made a rough sketch of the requested shape by dragging mainly one of the free vertices. Their moves were slow and mainly circular around the starting position on the screen, indicating a sense of control and anticipation of unveiling a right angle through the dragging procedure. The perceived image of a right angle guided the procedure, by switching the attentional anchor between the angles of the construction and the adjacent sides. Students opted to focus their attention on the opposite angle of the vertex that they dragged, as they conceived that a slight movement changed the size of the four angles. When they considered that the dynamic image on the screen met the perceived characteristics of a rectangle, they modified their dragging behaviour. They made fine-tuning adaptations based on the available measures to improve the accuracy of the construction. They made careful movements and switched the dragging point between the available ones, based on the finger used, and the position of the finger with respect to the screen. During this process, different types of attentional anchors were observed, including the sides of the construction, and angle and side length measures. To maintain attention on a specific element, students utilized indirect motion of the construction's elements. For instance, they focused on one angle and moved its opposite vertex instead of the vertex of the angle in focus or its adjacent sides.

Two types of action-perception loops could explain the aforementioned dragging actions. The first loop was driven by students' dynamic interaction of dragging modalities and perceptual apprehension of the targeted shape, while the second loop effectively integrated the affordances of the DGE and the geometrical properties involved. The coordination of the two types of loops triggered students to utilize the haptic capabilities of the DGE and the measurement tools. Students' perception filtered the dynamic image on the screen based on the available measures to meet the geometrical property of four equal right angles. Students explained that it is not sufficient to look like a rectangle, but the angles of the construction should be exactly 90 degrees. Each successive dragging action intended to make the size of the four angles exactly 90 degrees. In this sense, an emerging body-artefact functional system was observed, as action-perception loops integrated tool affordances, geometric properties, and verbal explanations.

In the following, we provide an example of a sequentially-coordinated dragging. Figure 3a shows the dragging sequence of Sylvia that first dragged vertices A and C (using different fingers) and then vertex B to make the corresponding angles 90 degrees. She easily observed that by dragging one vertex, at the same time, all the measures changed. Thus, she decided to drag only point B, and observed how changing the position of the one vertex affects the size of the other ones. After establishing a rough sketch of a rectangle, she concluded that it is easier to drag vertex C to make A right-angled, by making fine-tuning adjustments and observing the effect on the measure of A (having angle A and its corresponding measure at the focus of her attentional anchor). Sylvia followed

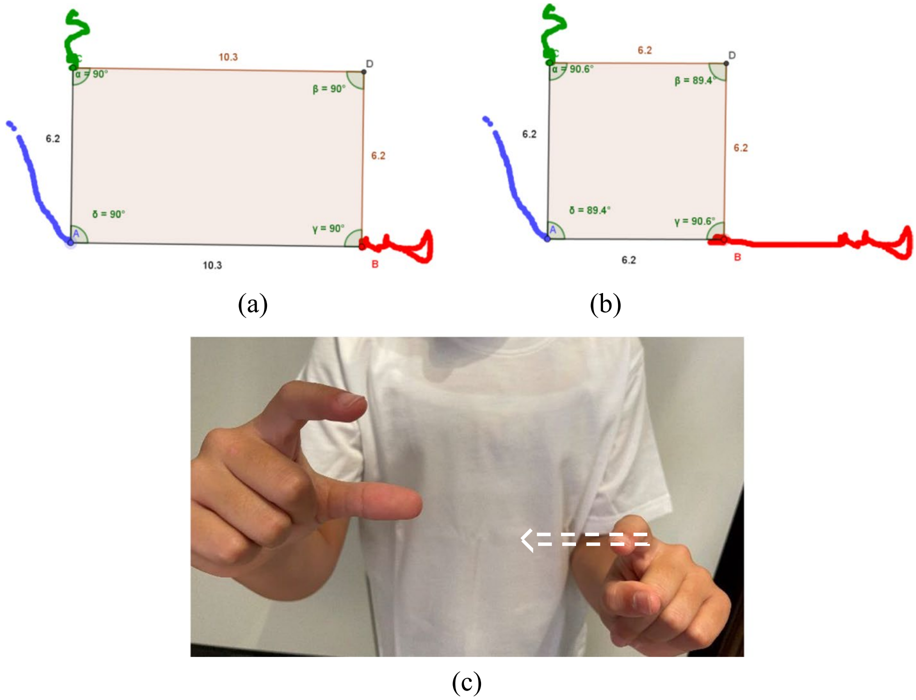


Fig. 3 Sequentially-coordinated dragging to convert the parallelogram into a rectangle and then into a square

the same dragging scheme to convert her rectangle to a square (see Fig. 3b). She instantly dragged point B to the left to make the side AB smaller and then, based on the provided measurement feedback, she made slow and careful movements to make the four sides of the construction equal and kept at the same time the size of the four angles constant (90 degrees). During this procedure, she concluded that A and C should remain at the same place, dragged B and then fine-tuned B to co-ordinate equality of successive sides and angles. She illustrated with her hands the procedure, by showing with two fingers of her right hand that the points A and C should remain stable and point B being dragged slowly to the right (see Fig. 3c). She explained that "to make a perfect square, you need to be very careful, when I drag B, I look at all the measures, because if I make a wrong move everything will collapse".

Students exhibited an emergent non-iconic geometric visualization, as they conceived that the dynamic images on the screen represented the concept of a parallelogram. They conceptualized, through the dragging procedure, that a rectangle is an example of a parallelogram. The main concepts in action were right-angled parallelogram and transformation of the parallelogram. The theorems in action involved the propositions "a rectangle is a special case of a parallelogram, as the product of the transformation of a parallelogram", and "a rectangle is the parallelogram that has four right angles." These theorems in actions revealed the decisive role of haptic dragging, as they explained the transformation based on actions controlled by their hands: "a rectangle is the parallelogram that you move its corners to make right angles" and "rectangle is what you get when you drag the corners of a parallelogram until you fix them at 90 degrees." Thus, the invariant of the scheme involved the requirement of transforming a parallelogram to a rectangle by adding four right angles.

Analysis of students' explanations during the mini interviews showed how students conceptualized the inclusion relations. They concluded that a parallelogram and a rectangle are related in terms of an intentional dynamic transformation to establish that the new shape (rectangle) has an additional variant attribute (right angles). Two indicative assertions were "every time you move (his finger) you get a new parallelogram, if you manage to make the right move you get a rectangle" and "when I roughly made a rectangle, I looked at the angles, the opposite ones were equal, then I softly dragged to make all of them equal (90 degrees)."

4.3 Adaptive embodied-dragging

The intention of the third main type of dragging scheme, the adaptive embodied-dragging scheme, was to drag flexibly the provided construction and transform it to the targeted shape, by utilizing the affordances of the DGE and the geometrical properties of the shapes involved. The scheme included an intentional and self-regulated sequence of dragging actions to coordinate at the same time students' perceived characteristics of the targeted shape, the feedback provided by the DGE (dynamic images on the screen and measures), and geometrical properties. Students' initial action was to experiment by dragging the vertices of the construction and to watch how each movement changes the dynamic image on the screen. Then, they applied a fine-maintaining dragging to induce a right angle, in terms of establishing accurately the angle size in steps. The action-perception interaction was guided by the angle size measurements of the DGE, as students focused on the angle size of one vertex and opted to drag the opposite one, without significantly changing the orientation of the construction on the screen. In case of difficulty, they flexibly experimented dragging alternatively the opposite or the adjacent vertex, based on the fine-tuning accuracy of the finger used or changed the targeted angle. Students commented that it is easier to establish accuracy by making diagonal moves. That is the reason they mainly preferred to drag the opposite vertex of the targeted angle, by making movements along a virtual diagonal line. Their attentional anchor switched between the dynamic measures of the two opposite angles and the draggable vertex to trace more easily the virtual diagonal line. Students' flexibility to adjust their dragging plan based on finger fine-movement accuracy underlies the importance of body potentialities. Students expressed a sense of strong power and ownership of the construction, making clear that the movements were not random. Some indicative utterances were "I am grasping the rectangle …" and "I can make this *(pointing to a side)* as long as I want, by dragging at the same time to the right and left…" (see Fig. 4).

The adaptive-dragging scheme included dynamic action-perception loops that progressively evolved. An enriched body-artefact functional system emerged that effectively integrated body potentialities, affordances of the artefacts, and geometrical properties of the shapes. This was evident in sophisticated dragging behaviour, when students were asked to explain what is changed by dragging. They responded that the opposite sides and angles are kept equal and the only thing they could do was to drag appropriately so one angle is set exactly 90 degrees. For instance, Nick explained: "A rectangle has four right angles, I will try to make one of the parallelograms angles a right one, the opposite will also be a right one and see what happens to the other two…". In addition, this system included self-regulating mechanisms, and verbal explanations of the conceptualized geometrical relations that unfolded while students were working.

Figure 5 presents an indicative adaptive-dragging enactment. Sophie observed that angle A was bigger than a right angle (Fig. 5a). She dragged A to make it smaller (Fig. 5b). When she realized that there was a need to fine-tune the dragging procedure to make it exactly 90 degrees, she decided to drag the opposite angle (D). When she realized that point D was fixated, she decided to change strategy and dragged point C and then the opposite angle (point B). She used one hand for point C and the other hand for point B, trying to coordinate the movement of the two points to achieve her goal (see Fig. 5c).

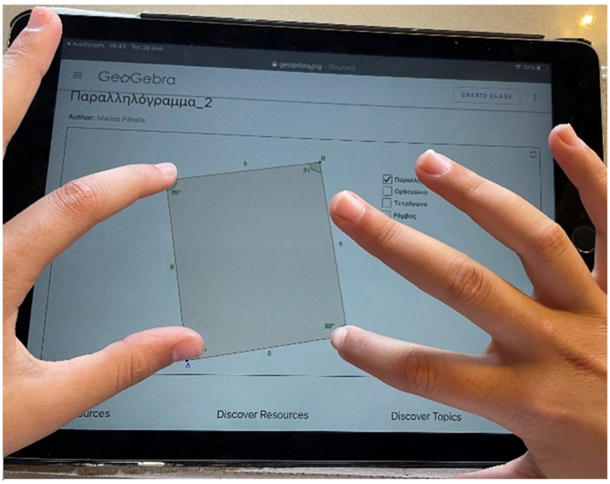


Fig. 4 Adaptive multi-finger dragging

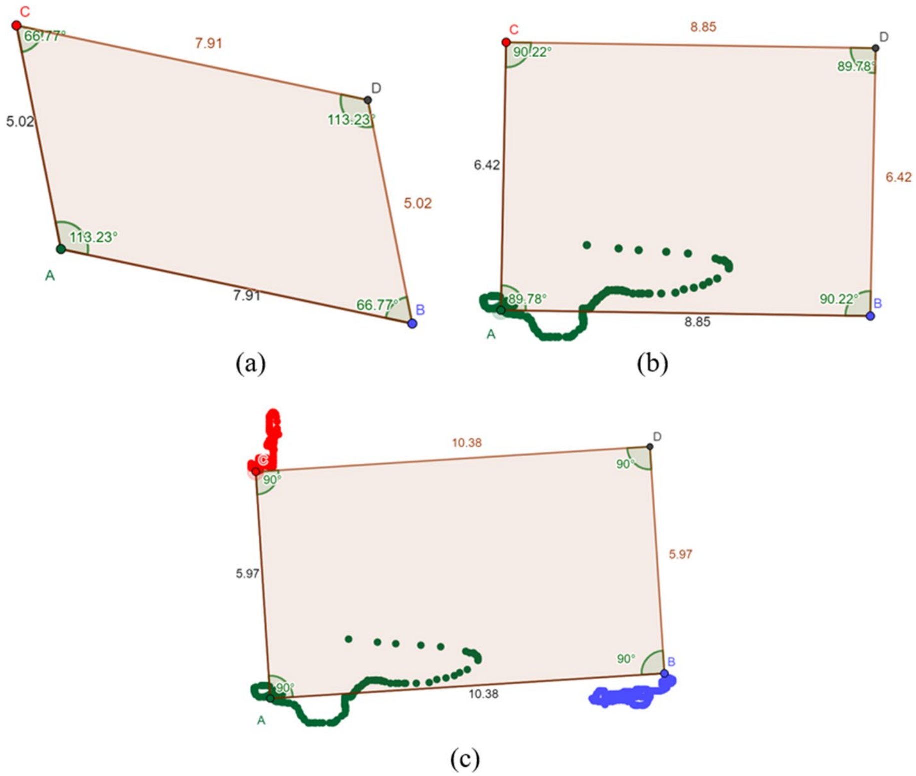


Fig. 5 Sophie's adaptive-dragging modalities

The adaptive scheme relates to a deep non-iconic geometric visualization, as students' formation of the geometric concept was mainly related to critical attributes. For instance, in the case of the parallelogram, students identified the equality of opposite sides as an invariant property and supported that "dragging one vertex of a parallelogram changes the size of the angles and the sides, but the opposite ones keep being equal." That also facilitated comparing the shapes involved not only by identifying common attributes, but by discerning variant and invariant properties during dragging. The main theorem in action was that a parallelogram is transformed to a rectangle if you make one of its angles a right one. Students integrated the feedback provided by the artefact and the property of equal opposite angles of a parallelogram to infer that making one angle 90 degrees makes at the same time the opposite one a right-angled and so on. This proposition explains the fact that they focused on setting only one angle of the parallelogram and sophisticated definitions, such as "you can get a rectangle when you make an angle of a parallelogram 90 degrees" and "rectangle is a parallelogram with at least one right angle." They used phrases such as "you change", "you drag", and "you make." Students used representational gestures to animate their dragging modalities, such as illustrating with one finger the word "you drag" (see Fig. 6a) and multiple finger moving-illustrations to display "you change" and "you make" (see Fig. 6b).

Students provided definitions and explanations indicating that they conceptualized that a rectangle is a special type of parallelogram as it was formed by a dragging procedure that

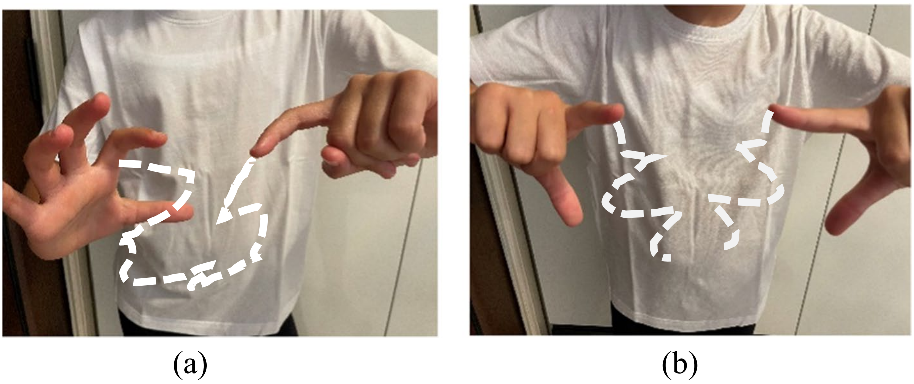


Fig. 6 Representational gestures to express the sense of ownership and illustrate their dragging actions

did not change the critical features of the parallelogram (invariant properties) but added an additional one. For instance, Jason described that "dragging one vertex of a parallelogram changes the size of the angles and the sides, but the opposite ones keep being equal", while Chris further explained that "you can easily switch a shape between a parallelogram and a rectangle, by just one move... if your shape is a parallelogram you just make one angle right one (variant property), if your shape is a rectangle you do not have to do anything, it is already a parallelogram (invariant properties)."

5 Conclusion and discussion

The central question addressed in this paper is what types of embodied-instrumented dragging schemes emerge while 11-year-old students transform one type of parallelogram into another using a DGE on a touchscreen device. In answering this question, we identified three types of embodied-instrumented dragging schemes: (I) the *action-perception dragging* that involved mainly students' perception, (II) the *sequentially-coordinated dragging*, that first activated students' perception and, at a second stage, utilized the artefact's measurement affordances, and (III) the *adaptive-dragging* that integrated students' perception, artefact affordances, and geometrical properties. We conclude that particular embodied dragging actions and experiences foster students' instrumental genesis through the conceptualization of the shape's critical features and properties. The study illustrates their potential for conceptualizing the inclusion relations of parallelograms.

Before discussing these conclusions, we should mention four key limitations of the design and data collection. First, we recognize that students' attentional anchors could have been examined more accurately by using eye-tracking technology. Our data includes only systematic observation of students when working with respect to embodied enactments, gestures, and attentional anchors. A second limitation concerns data regarding students' gestures. Our design aimed to capture student's enactments in the setting of an authentic lesson that made impossible video recording of the gestures and explanations of each student. Third, the design of the study focused on examining students' embodied-dragging schemes in a single lesson; thus, we could not trace any possible progressions. Fourth and final, although we acknowledge the importance of language and social interaction, we did not analyze extensively these two parameters, but we incorporated only students' explanations regarding the inclusion relations of shapes. This is because of the sample size and our wish to emphasize specific embodied modalities, such as finger action and perception.

In discussing this study's findings, we now reflect on the characteristics of the three schemes in relation to findings from the literature. In line with Shvarts et al. (2021), the embodied *action-perception dragging scheme* entails action-perception loops driven by a dynamic interaction of the dynamic images provided by the DGE and students' perceived characteristics of the targeted shape and strong prototypical image stereotypes (Hershkowitz, 1989). This interaction initiated specific multi-finger dragging that facilitated students visualizing and manipulating geometric shapes. Through this haptic exploration, students experience that from moving a given shape, another shape may emerge, thus confirming Arzarello et al.'s (2012) findings. However, students' understanding reflected an iconic visualization, based on what shapes look like (Duval, 2017). In line with Mithalal and Balacheff (2019), they ignored the shared and different geometrical properties. The perceptual limitations guided dragging behaviour to search for non-critical attributes, such as orientation.

The *sequentially-coordinated dragging scheme* can be described in terms of a body-artefact functional system because the provided tools become incorporated into action-perception loops to fulfil the functional request of searching specific geometric properties of the shape (Shvarts et al., 2021). Geometric properties, perceived shape characteristics, and definitions were artefacts that became part of the functional system. In addition, the system coupled body-artefact potentialities and environment through quick transformations. The transformations could explain the sequential work of students and specific enactments, such as dragging based on attentional anchor that extends Mariotti's findings (2014) about direct and indirect movements, as switching between attentional anchors facilitates students conceiving how indirect movements emerge.

The *adaptive-dragging scheme* reflects to a great extent Shvarts et al.'s (2021) body-artefact functional system with multiple levels of action. It included sensory-motor co-ordinations of the feedback provided by the learning environment, the measurement tools of the DGE, the geometric properties of parallelograms, and the conceptualization of variant and invariant properties of shapes (Leung, 2008). The transformative nature of the adaptive-dragging scheme could also be supported by the fact that students dragged to check the presence of desired properties and look for new properties of the shape (Holzl, 2001). Together, the dragging enactment procedure and the intentionality-driven transformations of the action-perception loops helped students to discern the variant and invariant properties that were essential to classify the parallelograms (Duval, 2017).

The identified embodied schemes provide significant information on student learning by analyzing the operational invariants of each scheme. It seems that conceptualizing the inclusion relations of the involved parallelograms relates to specific dragging modalities and geometric interpretations of each scheme. The action-perception dragging scheme did not include grasping the exact relation of the involved shapes, while the sequentially-coordinated dragging scheme entailed conceptualizing the rectangle as a special case of a parallelogram by executing a specific dragging action. Finally, the adaptive-dragging scheme made possible a conceptual understanding of the inclusion relations of parallelograms in terms of variant and invariant properties based on embodied experiences, as expressed by students' definitions of a rectangle and a square (Mariotti & Fischbein, 1997). It seems that as students coordinate action and perception, they move from informal goal-directed motions to more formal mathematics because they conceptualize how dragging actions variate critical features and properties of the draggable figures. Therefore, in line with Ng (2019), the study shows that haptic dragging affords new modes of thinking, such as haptic conceptualization of the inclusion relations of shapes through sequential transformations.

In the introduction, we mentioned the study's secondary aim of exploring the value of the relatively new embodied instrumentation framework in the given research context. As a result of applying the EI framework, the study's main theoretical contribution is the enrichment of the description of students' dragging scheme development, taking into consideration the bodily foundations of geometry knowledge. We put into practice the idea of embodied instrumentation that proved to be helpful indeed in characterizing and understanding students' actions because of the joint focus on two crucial aspects: the user-tool interaction central in instrumentation theory and the importance of physical action and perception prominent in embodied cognition theory. As such, the study informed us on how to combine the lenses of instrumentation, embodiment, and, more specifically, haptic dragging to provide an encompassing global view on the processes that are taking place when students use multitouch technology in geometry.

The study has implications for geometry teaching. The identified embodied schemes can help mathematics teachers to better understand the role of perception, students' actions and haptic dragging in grasping important geometrical concepts in a DGE. In addition, the description of the schemes in terms of actions, techniques, and theorems in action may inform teachers about students' potential difficulties in conceptualizing inclusion relations in geometry. Our results exemplify the importance of the types of questions a teacher may ask students, while they are struggling with a DGE situation. It is important to pose questions that help students reflect on their way of working while manipulating geometric shapes through haptic dragging in terms of what they see, what their plan is, and what actions are needed to carry out their plan. A teacher raising these types of questions may help students improve their dragging techniques and understand the consequences of their actions on the screen as well as the geometric properties involved. Keeping in mind that students explored and manipulated the shapes actively as a physical extension of their fingers and gained ownership of shapes, teachers can ask students to gesture with their hands the transformation of one shape to another and to describe verbally what is being changed (Alibali & Nathan, 2012). The findings show the importance of controlling the perceptual recognition of geometric figures by geometric properties. Therefore, teachers may insist on explaining the effect of a particular action in respect to variant and non-variant geometric properties. Reflecting of geometrical properties based on actions provides frames of reference to reason about relations between shapes based on their own experience (Triadafillides, 1995). As a final recommendation for practice, our results may encourage curriculum designers to design activities that activate embodied experiences in tablet-enabled DGE to explore the properties of geometric shapes and the relations among them, with an emphasis on linking perception, DGE affordances, and geometric properties. In addition, based on the embodied schemes, curriculum and software developers may foster the connections between action-perception loops and geometrical properties through providing structured feedback in terms of incorporating specific features, such as automatic recolouring of a construction when it turns into another shape while being dragged.

Availability of data and material Anonymized interview transcripts from participants who consented to data sharing

Declarations

Conflict of interest There authors declare no competing interests.

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