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ARCHIMEDEAN SPECTRAL DENSITIES FOR NONSTATIONARY SPACE-TIME GEOSTATISTICS

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Abstract: In this work we investigate methods for modelling space-time nonstationary data. We combine the approaches based on spatial adapting and spectral tempering of spectral densities with the desirable features of Archimedean functionals. We propose a class of spectral densities whose elements are shown to have a natural ordering with respect to the set of functionals, and some interesting closed forms for the covariance functions obtained as Fourier transform of the elements of this new class are shown. Finally, we show a new class of nonstationary space-time covariance functions, that are additionally asymmetric in time, and related to this methodology.

Key words and phrases: Archimedean functionals, orderings, space-time asymmetry, space-time geostatistics, spatially adaptive, spectral tempering.

1. Introduction and Setup

Nonstationarity is one of the most challenging problems in fields dealing with the analysis of spatio-temporal phenomena: the environment, oceanography, petroleum engineering, and ecology, for example. In the Geostatistical framework where one considers the data as realisations of a space-time continuous random field, popular assumptions on the process under study are those of Gaussianity, stationarity and isotropy.

On the other hand, part of the recent literature (Christakos and Hristopulos (1998), Christakos (2000) and Fuentes (2002, 2004)) reckons that stationarity can be an unrealistic assumption with respect to the great majority of geostatistical applications. Thus, it would be desirable to have covariance models that do not depend exclusively on the separation vector between two points of the spatio-temporal domain.

Unfortunately, few models for nonstationary spatial data have been proposed. It should be stressed that nonstationarity can be modelled with respect to either the deterministic trend component, or to the stochastic one, or even to both. In particular, one can focus on (a) an appropriate representation of the underlying process; (b) direct construction of the covariance function in the spatial domain; (c) construction of a class of spectral densities and computation of the Fourier transform in order to (possibly) obtain a closed form for the resulting covariance.

Examples of (a) can be found in Higdon, Swall and Kern (1999), Fuentes (2001) and Fuentes and Smith (2001). As far as (b) is concerned, crucial contributions can be found in Christakos and Hristopulos (1998), Christakos (2000) and, more recently, in Paciorek and Schervish (2004, 2006) and Stein (2005b). Possible drawbacks of this approach have been emphasised in several cases and in simulation studies, as for example in Cressie (1985) and Zimmerman and Zimmerman (1991). As pointed out in Gneiting (2002) and Furrer, Genton and Nychka (2006), the use of compactly supported correlation functions could provide a good solution to these problems.

The work of Pintore and Holmes (2007) represents a novel and ingenious approach to model nonstationarity by working in the spectral domain. The authors refer to a *spatially adaptive* spectral density to denote the fact that a parametric spectral density can be *adapted* by imposing the parameter vector to be a smooth function of the spatial locations. Thus, they achieve nonstationary spectral densities through factorisation of two spectral densities that are spatially adapted at different locations. Another approach proposed by the same authors regards the *tempering* of a parametric or nonparametric spectral density with a latent, strictly positive, and continuous spatial process that possibly varies smoothly over location.

In this paper we define a new class of functionals, dubbed Archimedean, show some of its theoretical properties, and build a new class of spectral densities that are spatially adaptive or tempered, and that can be used for nonstationary space-time data sets. The elements of this class may admit a natural ordering with respect to the parameter of their *generators*, which are completely monotone functions defined on the positive real line. After some fundamentals presented in Section 2, we explain, in Section 3, our methodology as well as the theoretical properties of the Archimedean functionals. In Section 4 we present the Archimedean class of stationary and nonstationary spectral densities, with particular emphasis on the latter. Then, we discuss some properties of its elements and present some examples of nonstationary covariance functions obtained with this method. In Section 5, we focus on the construction of Archimedean space-time covariance functions that are nonstationary, adaptive in space, and both adaptive and asymmetric in time. The paper ends with some discussion.

2. Fundamentals

We consider real-valued continuous Gaussian space-time random fields $\{Z(\mathbf{s},t) : (\mathbf{s},t) \in \mathbb{R}^d \times \mathbb{R}\}$, with covariance function $\mathrm{C}(\mathbf{s}_1, t_1, \mathbf{s}_2, t_2) = \mathrm{cov}(Z(\mathbf{s}_1, t_1), Z(\mathbf{s}_2, t_2))$.

Under the assumption of weak stationarity of the associated space-time random field, *i.e.* $\mathrm{C}(\mathbf{s}_1, t_1, \mathbf{s}_2, t_2) = \mathrm{C}(\mathbf{s}_1 - \mathbf{s}_2, t_1 - t_2, \mathbf{0}, 0)$ for any $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^d$, and any $t_1, t_2 \in \mathbb{R}$, Bochner's theorem (1933) established a one-to-one correspondence between continuous covariance functions and Fourier transforms of positive bounded measures F defined on $\mathbb{R}^d \times \mathbb{R}$ by

$$\mathrm{C}(\mathbf{s}_1, t_1, \mathbf{s}_2, t_2) = \mathrm{C}(\mathbf{h}, u, \mathbf{0}, 0) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} \mathrm{e}^{i\boldsymbol{\omega}'\mathbf{h} + i\tau u} \mathrm{d}F(\boldsymbol{\omega}, \tau),$$

where $(\mathbf{h}, u) = (\mathbf{s}_1 - \mathbf{s}_2, t_1 - t_2)$ is the so-called space-time lag vector. If, in addition, F is absolutely continuous with respect to the Lebesgue measure, then the previous expression can be rewritten as

$$\mathrm{C}(\mathbf{h}, u, \mathbf{0}, 0) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} \mathrm{e}^{i\boldsymbol{\omega}'\mathbf{h} + i\tau u} f(\boldsymbol{\omega}, \tau) \mathrm{d}\boldsymbol{\omega} \mathrm{d}\tau,$$

where f is called the spectral density associated to the weakly stationary space-time random field. Through Fourier inversion, assuming C absolutely integrable on its domain, one can show that

$$f(\boldsymbol{\omega}, \tau) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \mathrm{e}^{-i\boldsymbol{\omega}'\mathbf{h} - i\tau u} \mathrm{C}(\mathbf{h}, u, \mathbf{0}, 0) \mathrm{d}\mathbf{h} \mathrm{d}u, \qquad (\boldsymbol{\omega}, \tau) \in \mathbb{R}^d \times \mathbb{R}.$$

Note that it is common to write $\mathrm{C}(\mathbf{h}, u)$ instead of $\mathrm{C}(\mathbf{h}, u, \mathbf{0}, 0)$ (Cressie and Huang (1999)).

The specialisation of Bochner's representation to the spatial case is straightforward.

Completely monotone functions are infinitely differentiable nonnegative functions φ defined in $[0, \infty)$ such that $(-1)^n \varphi^{(n)}$ is nonnegative for any natural number n (in particular they are convex functions). Bernstein's Theorem (see Berg and Forst (1975)) states that completely monotone functions are Laplace transforms of positive and bounded measures F, *i.e.*,

$$\varphi(t) := \mathcal{L}[F](t) = \int_0^\infty \mathrm{e}^{-rt} \mathrm{d}F(r). \qquad (2.1)$$

For an extensive review on completely monotone functions we refer the reader to the book of Widder (1941) as a general reference, and to Berg, Christensen and Ressel (1984), and Berg and Forst (1975) for the use of completely monotone functions in a more abstract setting.

3. Methodology: Archimedean Composition of Two Real Functions

Let Φ be the class of real functions φ defined on some domain $D(\varphi) \subset \mathbb{R}$, admitting a proper inverse φ^{-1}, defined in $D(\varphi^{-1}) \subset \mathbb{R}$, and such that $\varphi(\varphi^{-1}(t)) = t$

Table 3.1. Examples of Archimedean compositions for some possible choices of the generating function $\varphi \in \Phi$.

$\varphi(t)$	$\varphi^{-1}(t)$	$\mathcal{A}_\psi(f_1, f_2)(\boldsymbol{\omega})$	Remarks
$\exp(-t)$	$-\log t$	$f_1(\boldsymbol{\omega})^{1/2} f_2(\boldsymbol{\omega})^{1/2}$	$f_1, f_2 : \mathbb{R}^d \to [0, \infty)$ $\log 0 := -\infty$ $\exp(-\infty) := 0$
$1/t$	$1/t$	$2/[1/f_1(\boldsymbol{\omega}) + 1/f_2(\boldsymbol{\omega})]$	$f_1, f_2 : \mathbb{R}^d \to [0, \infty)$ $1/0 := \infty,\ 1/\infty := 0$ $0/0 := 0$
$M(1-t/M)_+$	$M(1-t/M)_+$	$f_1/2 + f_2/2$	$f_1, f_2 : \mathbb{R}^d \to [0, M]$ for some $M > 0$ $(u)_+ = \max(u, 0)$
$-\log t$	$\exp(-t)$	$-\log([\exp(-f_1(\boldsymbol{\omega}))+\exp(-f_2(\boldsymbol{\omega}))]/2)$	$f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$

for all $t \in D(\varphi^{-1})$. Also, let $\Phi_{\rm c}$ and $\Phi_{\rm cm}$ be the subclasses of Φ obtained by restricting φ to be, respectively, convex or completely monotone on the positive real line.

Now, let us take the *Archimedean class of functionals* to be the class

$$\mathfrak{A} := \left\{\psi : D(\varphi^{-1}) \times D(\varphi^{-1}) \to \mathbb{R} : \psi(u, v) = \varphi\Big(\frac{1}{2}\varphi^{-1}(u) + \frac{1}{2}\varphi^{-1}(v)\Big),\ \varphi \in \Phi\right\},$$

with $\mathfrak{A}_{\rm c}$ and $\mathfrak{A}_{\rm cm}$ the corresponding subclasses of $\mathfrak{A}$ when restricting φ to belong, respectively, to $\Phi_{\rm c}$ and $\Phi_{\rm cm}$.

If $\psi \in \mathfrak{A}$, then we should write φ_ψ as the function such that, for u, v nonnegative, $\psi(u, v) = \varphi_\psi(\varphi_\psi^{-1}(u)/2 + \varphi_\psi^{-1}(v)/2)$. For ease of notation, we write φ instead of φ_ψ, whenever no confusion can arise.

For $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}_+$ such that $f_1(\mathbb{R}^d) \cup f_2(\mathbb{R}^d) \subset D(\varphi^{-1})$ for some $\varphi \in \Phi$, take the *Archimedean composition of f_1 and f_2 with generating function ψ*, to be

$$\mathcal{A}_\psi(f_1, f_2)(\boldsymbol{\omega}) = \psi\left(f_1(\boldsymbol{\omega}), f_2(\boldsymbol{\omega})\right), \qquad \psi \in \mathfrak{A}, \tag{3.1}$$

for all $\boldsymbol{\omega} \in \mathbb{R}^d$. Throughout the paper, we refer to $\psi \in \mathfrak{A}$ or to $\varphi \in \Phi$ as the generating functions of $\mathcal{A}_\psi$. Observe that $\mathcal{A}_\psi(f, f) = f$ for any function f and generating function ψ.

Remark 1. Even when $f_1(\mathbb{R}^d) \cup f_2(\mathbb{R}^d) \subset D(\varphi^{-1})$ is violated, it is sometimes possible, through conventions, such as are shown in Table 3.1, to define the appropriate Archimedean composition of f_1, f_2.

3.1. Ordering relations for Archimedean compositions

Ordering relations can be found among the set of Archimedean compositions of two fixed functions indexed by convex generating functions (a maximal element can also be found when both fixed functions are upper bounded). From now on, we set $\mathcal{A}_\Pi(f_1, f_2) = f_1^{1/2} f_2^{1/2}$, which is the Archimedean composition associated to $\varphi(t) = \exp(-t)$. Further, let $\mathcal{A}_\Sigma(f_1, f_2) = (1/2)(f_1 + f_2)$, the Archimedean composition associated to $\varphi(t) = M(1 - t/M)_+$ when $f_1, f_2 : \mathbb{R}^d \to [0, M]$, and let $\mathcal{A}_H(f_1, f_2) = 2/(1/f_1 + 1/f_2)$, the Archimedean composition associated to $\varphi(t) = 1/t$.

We write $h \le g$ whenever $h(\boldsymbol{\omega}) \le g(\boldsymbol{\omega})$ for all $\boldsymbol{\omega} \in \mathbb{R}^d$. We have the following results.

Proposition 1. *For any pair of functions f_1, f_2 and arbitrary generating functions $\varphi, \varphi_1, \varphi_2 \in \Phi_{\mathrm{cm}}$, corresponding to $\psi, \psi_1, \psi_2 \in \mathfrak{A}_{\mathrm{cm}}$, we have the following pointwise order relations.*

(i) *If $\varphi_1^{-1} \circ \varphi_2$ is convex, then $\mathcal{A}_{\psi_1}(f_1, f_2) \le \mathcal{A}_{\psi_2}(f_1, f_2)$.*

(ii) *If $\varphi_1^{-1} \circ \varphi_2$ is concave, then $\mathcal{A}_{\psi_1}(f_1, f_2) \ge \mathcal{A}_{\psi_2}(f_1, f_2)$.*

(iii) $\mathcal{A}_\psi(f_1, f_2) \le \mathcal{A}_\Pi(f_1, f_2) \le \frac{f_1+f_2}{2}$ ($= \mathcal{A}_\Sigma(f_1, f_2)$ *whenever f_1 and f_2 are bounded).*

(iv) $\mathcal{A}_H(f_1, f_2) \le \mathcal{A}_\Pi(f_1, f_2) \le \frac{f_1+f_2}{2}$ ($= \mathcal{A}_\Sigma(f_1, f_2)$ *whenever f_1 and f_2 are bounded).*

Proof. Recall that completely monotone functions are strictly decreasing, continuous and have a proper inverse function. Then

$$\begin{gathered}\mathcal{A}_{\psi_1}(f_1, f_2)(\boldsymbol{\omega}) \le \mathcal{A}_{\psi_2}(f_1, f_2)(\boldsymbol{\omega}) \text{ iff} \\ \varphi_1\left(\frac{1}{2}\varphi_1^{-1}(f_1(\boldsymbol{\omega})) + \frac{1}{2}\varphi_1^{-1}(f_2(\boldsymbol{\omega}))\right) \le \varphi_2\left(\frac{1}{2}\varphi_2^{-1}(f_1(\boldsymbol{\omega})) + \frac{1}{2}\varphi_2^{-1}(f_2(\boldsymbol{\omega}))\right) \text{ iff} \\ \varphi_1\left(\frac{1}{2}g(h_1(\boldsymbol{\omega})) + \frac{1}{2}g(h_2(\boldsymbol{\omega}))\right) \le \varphi_2\left(\frac{1}{2}h_1(\boldsymbol{\omega}) + \frac{1}{2}h_2(\boldsymbol{\omega})\right) \text{ iff} \\ \frac{1}{2}g(h_1(\boldsymbol{\omega})) + \frac{1}{2}g(h_2(\boldsymbol{\omega})) \ge g\left(\frac{1}{2}h_1(\boldsymbol{\omega}) + \frac{1}{2}h_2(\boldsymbol{\omega})\right),\end{gathered} \tag{3.2}$$

where we take, with abuse of notation, $h_1 := \varphi_2^{-1} \circ f_1$, $h_2 := \varphi_2^{-1} \circ f_1$ and $g := \varphi_1^{-1} \circ \varphi_2$.

For (i), when $\varphi_1^{-1} \circ \varphi_2$ is convex, the last inequality of (3.2) holds, hence the first inequality. For (ii), when $\varphi_1^{-1} \circ \varphi_2$ is concave, reverse the last inequality of (3.2), then reverse the first one.

Since $-\log \varphi$ is a concave function for any completely monotone function φ (see Widder (1941)), we get the first inequality of (iii). The second part of (iii) and the inequalities of (iv) are the classical inequalities of the harmonic, geometric and arithmetic means, as well as particular cases of (i) and the first part of (iii).

Table 3.2. Examples of ordering relations for Archimedean compositions for some choices of the generating functions.

Generating functions	Set	Order relation
$\Phi_1 = \{\varphi_\theta(t) = t^{-\theta} : \theta > 0\}$ $\mathfrak{A}_1 = \{\psi_\theta : \varphi_\theta \in \Phi_1\}$	$\{\mathcal{A}_{\psi_\theta}(f_1, f_2)\}_{\theta>0}$	$\mathcal{A}_{\psi_{\theta_1}}(f_1, f_2) \le \mathcal{A}_{\psi_{\theta_2}}(f_1, f_2)$ iff $\theta_1 \le \theta_2$
$\Phi_2 = \{\varphi_\theta(t) = \exp(t^{-\theta}) : 0 < \theta \le 1\}$ $\mathfrak{A}_2 = \{\psi_\theta : \varphi_\theta \in \Phi_2\}$	$\{\mathcal{A}_{\psi_\theta}(f_1, f_2)\}_{0<\theta\le1}$	$\mathcal{A}_{\psi_{\theta_1}}(f_1, f_2) \le \mathcal{A}_{\psi_{\theta_2}}(f_1, f_2)$ iff $\theta_1 \le \theta_2$
$\Phi_3 = \{\varphi_\theta(t) = 1-(1-\mathrm{e}^{-t})^{-\theta} : 0<\theta\le1\}$ $\mathfrak{A}_3 = \{\psi_\theta : \varphi_\theta \in \Phi_3\}$	$\{\mathcal{A}_{\psi_\theta}(f_1, f_2)\}_{0<\theta\le1}$	$\mathcal{A}_{\psi_{\theta_1}}(f_1, f_2) \le \mathcal{A}_{\psi_{\theta_2}}(f_1, f_2)$ iff $\theta_1 \le \theta_2$

When restricting to parametric families of generating functions, the pointwise order between Archimedean compositions can coincide with the natural order between parameters. In Table 2 we show three cases where this identification can be done.

4. Nonstationary Covariance Functions via Archimedean Spatial Adaptation of Parametric Spectra

4.1. Archimedean stationary spectral densities

As a direct consequence of Proposition 1 and (2.1), we can get the following result.

Corollary 1. *For any pair of spectral densities $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}$, and for any generating function $\varphi \in \Phi_{\mathrm{cm}}$, the composition $\boldsymbol{\omega} \mapsto \mathcal{A}_\psi(f_1, f_2)(\boldsymbol{\omega})$ defines a valid class of stationary spectral densities defined on $\mathbb{R}^d$.*

This result can be rephrased by saying that for any choice of the generator $\varphi \in \Phi_{\mathrm{cm}}$ and for any pair of spectral densities $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}_+$, there exists a real-valued weakly stationary Gaussian random field whose covariance function is exactly the Fourier transform of the Archimedean spectral density defined by (3.1). Now, the variance of such random field is $\mathrm{C}(\mathbf{0}) = \int \mathcal{A}_\psi(f_1(\boldsymbol{\omega}), f_2(\boldsymbol{\omega}))\mathrm{d}\boldsymbol{\omega}$. This implies that the natural ordering shown in Proposition 1 induces an ordering into the variance of the associated weakly stationary Gaussian random fields.

The Matérn class of stationary and isotropic covariance functions (Matérn (1960)) has been widely used in spatial statistics. Its equation, following the Stein (1999, p.31) parametrisation, has the form

$$\mathrm{C}_\theta(\mathbf{s}_1, \mathbf{s}_2) = \frac{\pi^{d/2}}{2^{\nu-1}\Gamma(\nu + d/2)\alpha^{2\nu}} \mathcal{M}_\nu(\alpha\|\mathbf{s}_1 - \mathbf{s}_2\|), \tag{4.1}$$

with $\mathbf{s}_1, \mathbf{s}_2$ points of $\mathbb{R}^d$, $\theta = (\alpha, \nu)' \in \mathbb{R}^2_+$, where α is a scaling parameter, ν governs the level of smoothness of the associated process, and $\mathcal{M}_\nu(t) = |t|^\nu \mathcal{K}_\nu(|t|)$, with t real and $\mathcal{K}_\nu$ the modified Bessel function of the second kind of order ν

(Abramowitz and Stegun (1965)). The Matérn class is particularly important in spatial modelling as it allows associated processes for any level of smoothness. Its related spectral density has a rather simple form,

$$f_\theta(\boldsymbol{\omega}) \propto \left(\alpha^2 + \|\boldsymbol{\omega}\|^2\right)^{-\nu-\frac{d}{2}}, \quad \boldsymbol{\omega} \in \mathbb{R}^d. \tag{4.2}$$

Here we show that (4.2) can be effectively used to build Archimedean spectral densities that constitute a very general and rich class. In particular, if $f_i(\cdot;\alpha_i,\nu_i)$, $i = 1, 2$, are Matérn spectral densities, the completely monotone generator $\varphi(t) = t^{-\beta}$, $\beta > 0$, leads to the Archimedean composition

$$\mathcal{A}_\psi(f_1, f_2)(\boldsymbol{\omega}) \propto \left(\frac{1}{2}\left(\alpha_1^2 + \|\boldsymbol{\omega}\|^2\right)^{\beta_1} + \frac{1}{2}\left(\alpha_2^2 + \|\boldsymbol{\omega}\|^2\right)^{\beta_2}\right)^{-\beta}, \tag{4.3}$$

with $\beta_i = (\nu_i + d/2)/\beta$. This class of spectral densities has an apparent analogy with the important Stein class (2005a), that possesses desirable features in terms of mean square differentiability of the associated Gaussian random field. This new class will be reprised in detail when dealing with the nonstationary case.

4.2. Nonstationary Archimedean spectral densities for spatial data

Pintore and Holmes (2007) describe two methods aiming at the obtention of nonstationary covariance functions, working with nonparametric or parametric spectral densities. The first method involves parametric families of spectral densities, say $\{f_\theta : \theta \in \Theta\}$, and smooth functions $\theta : \mathbb{R}^d \to \Theta$ so that $\mathbf{s} \mapsto \theta(\mathbf{s}) \in \Theta$. They define respectively a spatially adaptive spectrum and the resulting nonstationary spectral density as

$$f_{NS}^{\mathbf{s}}(\boldsymbol{\omega}) \propto f_{\theta(\mathbf{s})}(\boldsymbol{\omega}), \tag{4.4}$$

$$\begin{aligned} f_{NS}^{\mathbf{s}_1,\mathbf{s}_2}(\boldsymbol{\omega}) &= f_{NS}^{\mathbf{s}_1}(\boldsymbol{\omega})^{\frac{1}{2}} f_{NS}^{\mathbf{s}_2}(\boldsymbol{\omega})^{\frac{1}{2}} \\ &\propto f_{\theta(\mathbf{s}_1)}(\boldsymbol{\omega})^{\frac{1}{2}} f_{\theta(\mathbf{s}_2)}(\boldsymbol{\omega})^{\frac{1}{2}}. \end{aligned} \tag{4.5}$$

In their second proposal, a parametric or nonparametric spectral density f is *tempered* by either a strictly positive stochastic process or a strictly positive spatial deterministic function $\eta : \mathbb{R}^d \to \mathbb{R}_+$, in the sense that

$$f_{NS}^{\mathbf{s}}(\boldsymbol{\omega}) \propto [f(\boldsymbol{\omega})]^{\eta(\mathbf{s})}. \tag{4.6}$$

Then the resulting nonstationary tempered spectral density is obtained as

$$f_{NS}^{\mathbf{s}_1,\mathbf{s}_2}(\boldsymbol{\omega}) = f_{NS}^{\mathbf{s}_1}(\boldsymbol{\omega})^{\frac{1}{2}} f_{NS}^{\mathbf{s}_2}(\boldsymbol{\omega})^{\frac{1}{2}} \propto [f(\boldsymbol{\omega})]^{\frac{\eta(\mathbf{s}_1)+\eta(\mathbf{s}_2)}{2}}. \tag{4.7}$$

Finally, they prove that the function

$$(\mathbf{s}_1, \mathbf{s}_2) \mapsto \int_{\mathbb{R}^d} \mathrm{e}^{i\boldsymbol{\omega}'(\mathbf{s}_1-\mathbf{s}_2)} f_{NS}^{\mathbf{s}_1,\mathbf{s}_2}(\boldsymbol{\omega}) \mathrm{d}\boldsymbol{\omega}, \tag{4.8}$$

where $f_{NS}^{\mathbf{s}_1,\mathbf{s}_2}$ can be either defined by (4.5) or (4.7), is a (nonstationary) covariance function if and only if, respectively, $f_{\theta(\mathbf{s})}$ or $|f(\cdot)|^{\eta(\mathbf{s})}$ are absolutely integrable for almost all $\mathbf{s} \in \mathbb{R}^d$. The extension to the spatio-temporal setting is valid and straightforward.

For the remainder of the paper, we deal only with compositions of parametric spectral densities, whereas the tempering in (4.6) will be considered only with respect to deterministic functions, thus being a special case of (4.4).

We are interested in finding a class of link functions that, whenever applied to a spectral density adapted at two different locations, allow one to build valid spectral densities.

Theorem 1. *For any* $\psi \in \mathfrak{A}_{\rm cm}$, *and for* $f_{NS}^{\mathbf{s}}$ *defined as at* (4.4), *the function*

$$(\mathbf{s}_1, \mathbf{s}_2) \mapsto \int_{\mathbb{R}^d} \mathrm{e}^{i\boldsymbol{\omega}'(\mathbf{s}_1-\mathbf{s}_2)} \mathcal{A}_\psi \left(f_{NS}^{\mathbf{s}_1}, f_{NS}^{\mathbf{s}_2} \right)(\boldsymbol{\omega}) \mathrm{d}\boldsymbol{\omega} \tag{4.9}$$

for $(\mathbf{s}_1, \mathbf{s}_2) \in \mathbb{R}^d \times \mathbb{R}^d$, *is positive definite if and only if* $f_{\theta(\mathbf{s})}$ *is absolutely integrable for almost all* $\mathbf{s} \in \mathbb{R}^d$.

Proof. Let $\{c_i\}_{i=1}^n \subset \mathbb{R}$ and $\{\mathbf{s}_i\}_{i=1}^n \subset \mathbb{R}^d$ be, respectively, arbitrary families of scalars and spatial locations, $n \in \mathbb{N}$. Define $\mathrm{C}_{\mathcal{A}_\psi}(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ as at (4.9). We have that

$$\begin{aligned}
&\sum_{i,j=1}^n c_i c_j \mathrm{C}_{\mathcal{A}_\psi}(\mathbf{s}_i, \mathbf{s}_j) \\
&\propto \int_\Omega \sum_{i,j=1}^n c_i c_j \mathrm{e}^{i\boldsymbol{\omega}'(\mathbf{s}_i-\mathbf{s}_j)} \varphi\left(\frac{1}{2}\varphi^{-1}\left(f_{\theta(\mathbf{s}_i)}(\boldsymbol{\omega})\right) + \frac{1}{2}\varphi^{-1}\left(f_{\theta(\mathbf{s}_j)}(\boldsymbol{\omega})\right) \right) \mathrm{d}\boldsymbol{\omega} \\
&= \int_\Omega \sum_{i,j=1}^n c_i c_j \mathrm{e}^{i\boldsymbol{\omega}'\mathbf{s}_i} \mathrm{e}^{-i\boldsymbol{\omega}'\mathbf{s}_j} \int_0^\infty \mathrm{e}^{-r\left[\frac{1}{2}\varphi^{-1}\left(f_{\theta(\mathbf{s}_i)}(\boldsymbol{\omega})\right)+\frac{1}{2}\varphi^{-1}\left(f_{\theta(\mathbf{s}_j)}(\boldsymbol{\omega})\right)\right]} \mathrm{d}F(r) \mathrm{d}\boldsymbol{\omega} \\
&= \int_\Omega \int_0^\infty \left[\sum_{i=1}^n c_i \mathrm{e}^{i\boldsymbol{\omega}'\mathbf{s}_i} \mathrm{e}^{-r\frac{1}{2}\varphi^{-1}\left(f_{\theta(\mathbf{s}_i)}(\boldsymbol{\omega})\right)} \right] \left[\sum_{j=1}^n c_j \mathrm{e}^{-i\boldsymbol{\omega}'\mathbf{s}_j} \mathrm{e}^{-r\frac{1}{2}\varphi^{-1}\left(f_{\theta(\mathbf{s}_j)}(\boldsymbol{\omega})\right)} \right] \mathrm{d}F(r) \mathrm{d}\boldsymbol{\omega} \\
&= \int_\Omega \int_0^\infty \left| \sum_{i=1}^n c_i \mathrm{e}^{i\boldsymbol{\omega}'\mathbf{s}_i} \mathrm{e}^{-r\frac{1}{2}\varphi^{-1}\left(f_{\theta(\mathbf{s}_i)}(\boldsymbol{\omega})\right)} \right|^2 \mathrm{d}F(r) \mathrm{d}\boldsymbol{\omega} \geq 0,
\end{aligned}$$

where we have used the definition of $\mathrm{C}_{\mathcal{A}_\psi}$, the representation theorem for completely monotone functions, (2.1) (where $\mathrm{d}F$ is a nonnegative measure), the relation $z\bar{z} = |z|^2$ for complex z, and the assumption that the integral is finite. To ensure the latter, we use the integral Minkowsky inequality and again the representation theorem for completely monotone functions, followed by the integral

Cauchy-Schwartz inequality, to get

$$|\mathrm{C}_{\mathcal{A}_\psi}(\mathbf{s}_1,\mathbf{s}_2)| \le \int_\Omega \int_0^\infty \mathrm{e}^{-\frac{r}{2}\varphi^{-1}\left(f_{\theta(\mathbf{s}_i)}(\boldsymbol{\omega})\right)} \mathrm{e}^{-\frac{r}{2}\varphi^{-1}\left(f_{\theta(\mathbf{s}_j)}(\boldsymbol{\omega})\right)} \mathrm{d}F(r)\mathrm{d}\boldsymbol{\omega}$$

$$= \left[\int_\Omega \int_0^\infty \mathrm{e}^{-r\varphi^{-1}\left(f_{\theta(\mathbf{s}_i)}(\boldsymbol{\omega})\right)} \mathrm{d}F(r)\mathrm{d}\boldsymbol{\omega}\right]^{\frac{1}{2}} \left[\int_\Omega \int_0^\infty \mathrm{e}^{-r\varphi^{-1}\left(f_{\theta(\mathbf{s}_j)}(\boldsymbol{\omega})\right)} \mathrm{d}F(r)\mathrm{d}\boldsymbol{\omega}\right]^{\frac{1}{2}}$$

$$= \left[\int_\Omega f_{\theta(\mathbf{s}_i)}(\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega}\right]^{\frac{1}{2}} \left[\int_\Omega f_{\theta(\mathbf{s}_j)}(\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega}\right]^{\frac{1}{2}} < \infty.$$

Now, for the converse, if $\mathrm{C}_{\mathcal{A}_\psi}$ is a covariance function, the simple evaluation

$$\infty > \mathrm{C}_{\mathcal{A}_\psi}(\mathbf{s},\mathbf{s}) = \int_\Omega f_{\theta(\mathbf{s})}(\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega}$$

completes the proof.

Some comments are in order. It can be easily seen that the approaches in Pintore and Holmes (2007), as in equations (4.5) and (4.7), are particular cases of Theorem 1 under the choice $\mathcal{A}_\Pi$. This implies that Lemma 4.1 and 5.1 in Pintore and Holmes (2007) are special cases of Theorem 1. Furthermore, note that the construction in Theorem 1 preserves the margins of the spatially adaptive spectral densities, so that $\mathcal{A}_\psi(f_{\theta(\mathbf{s})}, f_{\theta(\mathbf{s})}) = f_{\theta(\mathbf{s})}$ for all $\mathbf{s} \in \mathbb{R}^d$.

The variance of the Gaussian random fields with Archimedean spectra at a point $\mathbf{s} \in \mathbb{R}^d$ is $\mathbb{V}\mathrm{ar}(Z(\mathbf{s})) = \mathrm{C}(\mathbf{s},\mathbf{s}) = \int \mathcal{A}_\psi(f^{\mathbf{s}}_{NS}(\boldsymbol{\omega}), f^{\mathbf{s}}_{NS}(\boldsymbol{\omega}))\mathrm{d}\boldsymbol{\omega} = \int f^{\mathbf{s}}_{NS}(\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega}$, which is independent of the generator. Thus, nonstationary Gaussian random fields with Archimedean spectral densities that are obtained with the same margins, but with different generators, have the same variance.

4.2.1 An example

The Matérn spectral density at (4.2) and the Archimedean class at (4.3) can be effectively used in order to create classes of nonstationary Archimedean spectral densities. The procedure can be summarised as follows: take $\alpha(\cdot), \nu(\cdot)$ to be smooth functions of the spatial location, $\alpha(\mathbf{s})$ to be bounded away from zero, and $\nu(\mathbf{s})$ positive for all $\mathbf{s}$. Use the completely monotone function $\varphi(t) = t^{-\beta}$, $\beta > 0$, and apply (4.9) to get

$$\mathcal{A}_\psi(f_{\theta(\mathbf{s}_1)}, f_{\theta(\mathbf{s}_2)})(\boldsymbol{\omega}) \propto \left(\frac{1}{2}\left(\alpha(\mathbf{s}_1)^2 + \|\boldsymbol{\omega}\|^2\right)^{\frac{\nu(\mathbf{s}_1)+\frac{d}{2}}{\beta}} + \frac{1}{2}\left(\alpha(\mathbf{s}_2)^2 + \|\boldsymbol{\omega}\|^2\right)^{\frac{\nu(\mathbf{s}_2)+\frac{d}{2}}{\beta}}\right)^{-\beta}. \tag{4.10}$$

This adaptive class of spectral densities is analogous to well-known constructions, such as those of Paciorek and Schervish (2004), Pintore and Holmes (2007) and Stein (2005b), who obtain forms of localised Matérn covariance functions.

Thus, consider a spatially adapted spectrum with respect to the parameter α and impose the following setting: starting from (4.10) and letting $\nu(\mathbf{s}) := \nu = \beta - d/2$ for all $\mathbf{s} \in \mathbb{R}^d$, one can show that

$$\mathcal{A}_\psi(f_{\theta(\mathbf{s}_1)}, f_{\theta(\mathbf{s}_2)})(\boldsymbol{\omega}) = \left(\alpha^2_{\mathbf{s}_1,\mathbf{s}_2} + \|\boldsymbol{\omega}\|^2\right)^{-\nu-\frac{d}{2}},$$

with $\alpha_{\mathbf{s}_1,\mathbf{s}_2} = [(\alpha(\mathbf{s}_1)^2 + \alpha(\mathbf{s}_2)^2)/2]^{1/2}$. The resulting spatially adapted Matérn covariance admits the equation

$$C_{\mathcal{A}_\psi}(\mathbf{s}_1, \mathbf{s}_2) = \frac{\pi^{\frac{d}{2}}}{2^{\nu-1}\Gamma(\nu + \frac{d}{2})\alpha^{2\nu}_{\mathbf{s}_1,\mathbf{s}_2}} \mathcal{M}_\nu(\alpha_{\mathbf{s}_1,\mathbf{s}_2}\|\mathbf{s}_1 - \mathbf{s}_2\|), \tag{4.11}$$

where $\mathcal{M}_\nu$ was defined at (4.1). Three adapted versions of well-known covariance functions can be derived from (4.11): (a) the spatially adaptive covariance function associated to the continuous spatial autoregressive process of the first order or, as it is also known, the spatial isotropic Ornstein-Uhlembeck process (Yaglom (1987)), for $\nu = 1/2$; (b) the spatially adaptive Gaussian covariance for $\nu \to \infty$; and (c) when $\nu = 3/2$, one obtains

$$\mathrm{C}_{\mathcal{A}_\psi}(\mathbf{s}_1, \mathbf{s}_2) = \frac{d}{2}\pi^d \alpha^{-3}_{\mathbf{s}_1,\mathbf{s}_2} \mathrm{e}^{-\alpha_{\mathbf{s}_1,\mathbf{s}_2}\|\mathbf{s}_1-\mathbf{s}_2\|} \left(1 + \alpha_{\mathbf{s}_1,\mathbf{s}_2}\|\mathbf{s}_1 - \mathbf{s}_2\|\right).$$

The covariance function at (4.11) is the same as the one proposed by Paciorek and Schervish (2004) under a different setting, and which is a particular case of the Stein (2005b) class; a similar example is proposed by Pintore and Holmes (2007). It should be stressed that, while the connection with Pintore and Holmes is immediate, since their approach is a special case of Theorem 1, no similar considerations can be made for Paciorek and Schervish (2004) and Stein (2005b). In particular, it is easy to see that Stein's (2005b) construction is obtained by working directly on the spatial domain, and that it is very difficult to obtain a general form for the Fourier pair associated to his class.

5. Archimedean Spatially Adaptive-Temporally Adaptive and Asymmetric Covariances

Recent literature has been focused on the construction of nonstationary covariances in the spatial setting. The extension to the spatio-temporal setting should be through methods based on simple procedures yielding closed forms with straightforward computations. Such methods could be based on direct construction in the spatio-temporal domain, or on the specification of a parametric structure for a spectrum for which the computation of the Fourier transform is feasible. It is worth remarking that the construction proposed for the spatial case can be directly extended to the spatio-temporal one, but the obtained classes are analytically intractable. For the space-time setting, we propose an alternative strategy, leading to the adaptive version of an important class of spectral densities proposed by Stein (2005a).

Let $f_\theta : \mathbb{R}^d \to \mathbb{R}_+$ be a parametric spectral density, $\beta : \mathbb{R}^d \to \mathbb{R}$ an even nonnegative Borel-measurable function, and $\phi : \mathbb{R}^d \to \mathbb{R}$ an odd Borel-measurable function, where $t \in \mathbb{R}$ denotes the temporal index. By Proposition 5 in Stein (2005a), the $\mathbb{R}^d$-Fourier transform of the function

$$\widetilde{f}_{(\theta;t)}(\boldsymbol{\omega}) = \exp\left(-|t|\beta(\boldsymbol{\omega}) - it\phi(\boldsymbol{\omega})\right) f_\theta(\boldsymbol{\omega}) \tag{5.1}$$

gives a space-time covariance function that is asymmetric in time.

In this section we show that this function can be used in order to build covariance functions that are adaptive in space, and both adaptive and asymmetric in time. To do this, consider the composition

$$\begin{aligned} g(\boldsymbol{\omega}; \mathbf{s}_1, \mathbf{s}_2, t_1, t_2) = \mathcal{A}_\Pi\left(f_{\theta(\mathbf{s}_1)}, f_{\theta(\mathbf{s}_2)}\right)(\boldsymbol{\omega}) & \\ \times \exp\Big(- \frac{1}{2}(\gamma(t_1)\beta(\boldsymbol{\omega}; \xi(\mathbf{s}_1)) + \gamma(t_2)\beta(\boldsymbol{\omega}; \xi(\mathbf{s}_2))) & \\ - \frac{i}{2}\left(t_1\phi(\boldsymbol{\omega}; \epsilon(t_1)) - t_2\phi(\boldsymbol{\omega}; \epsilon(t_2))\right)\Big), & \end{aligned} \tag{5.2}$$

where the deterministic functions $\xi(\cdot)$ and $\epsilon(\cdot)$ are nonnegative on their domains. We assume that the function $\beta(\cdot; \xi(\mathbf{s}))$ is nonnegative and Borel-measurable for almost every $\mathbf{s} \in \mathbb{R}^d$; $\phi(\cdot; \epsilon(t))$ is Borel-measurable for almost every $t \in \mathbb{R}$. Finally, we assume that $\gamma(\cdot)$ is a univariate nonnegative intrinsically stationary variogram.

Following the arguments of Theorem 1, one can easily show that

$$\mathrm{C}(\mathbf{s}_1, t_1, \mathbf{s}_2, t_2) = \int_{\mathbb{R}^d} \mathrm{e}^{i\boldsymbol{\omega}'(\mathbf{s}_1 - \mathbf{s}_2)} g(\boldsymbol{\omega}; \mathbf{s}_1, \mathbf{s}_2, t_1, t_2) \mathrm{d}\boldsymbol{\omega},$$

g as defined at (5.2), is a covariance function on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$.

Thus, we can derive interesting examples by calculating the Fourier transform of (5.2). For instance, let $\beta(\boldsymbol{\omega}; \xi(\mathbf{s})) = \xi(\mathbf{s}) \log(1 + \alpha^{-2}\|\boldsymbol{\omega}\|^2)$, with ξ a nonnegative parameter that depends on the spatial location, $\alpha > 0$, $\phi(\boldsymbol{\omega}; \epsilon(t)) = \epsilon(t)\boldsymbol{\omega}'\mathbf{z}$, with $\mathbf{z}$ a unit vector of $\mathbb{R}^d$ and ϵ a positive function of time, and the Matérn-type form for the parametric spectral density $f_{\theta(\mathbf{s})}(\boldsymbol{\omega}) = (1 + \alpha^{-2}\|\boldsymbol{\omega}\|^2)^{-\nu(\mathbf{s}) - d/2}$, $\mathbf{s} \in \mathbb{R}^d$. Using (5.2), one can show that the spatial spectral density adaptive on both space and time, can be written as

$$\mathrm{e}^{-i\epsilon_t \boldsymbol{\omega}'\mathbf{z}} \left(1 + \alpha^{-2}\|\boldsymbol{\omega}\|^2\right)^{-\nu_{\mathbf{s}}\xi_{(\mathbf{s},t)} - \frac{d}{2}},$$

where $(\epsilon_t, \nu_{\mathbf{s}}\xi_{(\mathbf{s},t)})'$ has $\epsilon_t = (1/2)(\epsilon(t_1)t_1 - \epsilon(t_2)t_2)$, $\nu_{\mathbf{s}} = (1/2)(\nu(\mathbf{s}_1) + \nu(\mathbf{s}_2))$, and $\xi_{(\mathbf{s},t)} = (1/2)(\xi(\mathbf{s}_1)\gamma(t_1) + \xi(\mathbf{s}_2)\gamma(t_2))$, and for which the associated adapted Fourier transform is

$$\mathrm{C}(\mathbf{s}_1, t_1, \mathbf{s}_2, t_2) = \frac{\pi^{d/2}\alpha^d}{2^{\nu_{\mathbf{s}}\xi_{(\mathbf{s},t)}-1}\Gamma(\nu_{\mathbf{s}}\xi_{(\mathbf{s},t)} + d/2)}\mathcal{M}_{(\nu_{\mathbf{s}}\xi_{(\mathbf{s},t)})}(\alpha\|(\mathbf{s}_1-\mathbf{s}_2)-\epsilon_t\mathbf{z}\|), \quad (5.3)$$

where $\mathcal{M}_\nu$ was defined at (4.1).

Observe that the spatial margin $\mathrm{C}(\mathbf{s}_1, 0, \mathbf{s}_2, 0)$ is of Matérn type and that nonstationarity is induced via the adaptive smoothing parameter, as in Stein (2005b). The adaptive parameters $\epsilon_t, \nu_{\mathbf{s}}$ and $\xi_{(\mathbf{s},t)}$ act, respectively, in time, space, and both space and time. In particular, ϵ_t expresses a weighted difference of instants t_1, t_2 with weights proportional to $\epsilon(t_1), \epsilon(t_2)$. This parameter defines the asymmetric structure of (5.3). The parameter $\nu_{\mathbf{s}}$ governs the smoothness of the associated spatial covariance, which is infinitely differentiable away from the origin. Unlike ϵ_t, it is an unweighted average of the adaptive smoothing parameters $\nu(\mathbf{s}_1), \nu(\mathbf{s}_2)$. Finally, the space-time parameter $\xi_{(\mathbf{s},t)}$ can be seen as a weighted average of the temporal variogram γ evaluated at points t_1, t_2 with weights, respectively, proportional to $\xi(\mathbf{s}_1)$ and $\xi(\mathbf{s}_2)$. Observe that if the chosen parametric variogram model does not have a nugget effect, then it is continuous at the origin, where its value is zero. As far as the temporal margin is concerned, it can be easily shown that it depends on the choice of the parametric temporal variogram.

6. Discussion

This work has been devoted to methods of construction of a class of spatially adaptive spectral densities that constitutes a nontrivial generalisation of the class presented in Pintore and Holmes (2007). We have shown that this class presents mathematical features having some implications on the variance of the associated stationary or nonstationary Gaussian random field. Also, we have presented some examples of families of adaptive spectra including, as a special case, the spatially adaptive Matérn class. Finally, for space-time, we have built an extension of the Stein's (2005a) class to the spatially adaptive case, and discussed some interesting closed forms that can be obtained through this procedure.

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