$\mathcal{H}_\infty$ observer design for a class of nonlinear discrete systems

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Abstract

In this paper, the problem of observing the state of a class of discrete nonlinear system is addressed. The design of the observer is dealt with using $\mathcal{H}_\infty$ performance techniques, taking into account disturbance and noise attenuation. The result is an LMI optimization problem that can be solved by standard optimization techniques. A design strategy is proposed based on the available disturbances information.

1 Introduction

The design of observers for non linear discrete systems has been studied by several authors in the literature. The idea behind is to obtain information about internal variables which are not directly available at the output, they are corrupted by noise or, in any practical situation, they are not accessible any time they are required. As usual in the nonlinear setting, there is no general solution for any nonlinearity. Observability conditions, as reviewed in [2], must be assumed. The simplest general assumption is to consider that the state and measurement functions satisfy some conic condition, [14]. Also, model uncertainties, noise and disturbances are assumed to be generally bounded. Most of the published works deal with a class of linear systems with additive nonlinearity characterized by a non linear term in the state and output equation, that are assumed to fulfill a Lipschitz condition. The use of linear matrix inequalities has made possible to address the design of observers for that class of systems

surpassing the drawbacks of previous approaches, where a high gain was needed to compensate for the non linear term, as initially proposed in [3]. Another alternative is the use of proportional/integral observers [1], that is, observers where the corrective action is proportional to the observation error and its integral, leading to a more complex observer dynamics. This approach has been applied for single output continuous time uniformly observed systems [4].

There are three ways of considering the Lipschitz condition in the additive nonlinearity in order to incorporate it in the LMI. The simplest one is the scalar form, consisting of

$$\|f(x_1) - f(x_2)\| \leq \gamma \|x_1 - x_2\|. \tag{1}$$

This condition is taken into account, for example, in [15]. The drawback of this approach is that it can lead to very conservative results, or even to the non feasibility of the LMI for large γ. A more complex form of the Lipschitz condition consists of incorporating a matrix in the form

$$\|f(x_1) - f(x_2)\| \leq \|F(x_1 - x_2)\|. \tag{2}$$

The advantage of this approach is that if the matrix F is adequately chosen, based on the form of the function $f(x)$, the resulting LMI is less conservative and more likely to have a feasible solution. This approach is used, for example, in [10], [11] or [13].

The most complex form of the Lipschitz condition (proposed in [16]) assumes that there are known upper and lower bounds on the elements of the jacobian matrix of $f(x)$, as

$$a_{ij} \leq \frac{\partial f_i}{\partial x_j} \leq b_{ij}. \tag{3}$$

This idea, and the use of the differential mean value theorem (DMVT), allows to express the condition as

$$f(x_1) - f(x_2) = \left(\sum_{i,j=1}^{n} h_{ij} M_{ij} \right) (x_1 - x_2) \tag{4}$$

where M_{ij} are empty matrices, except the element i, j that is 1, and where the terms h_{ij} are time varying but bounded by $a_{ij} \leq h_{ij} \leq b_{ij}$. Based on this condition, the error dynamics can be expressed as a Linear Parameter Varying system, with bounded parameters, that can be taken into account easily in the LMI's by considering the convex hull. The advantage of this approach is that less conservative results can be obtained. The important drawback is that the number of LMI to be solved simultaneously can grow exponentially with the system order (for a system of order n with p outputs the number of LMI can be up to 2^{n^2+np}).

In the present paper, the design of observers for non linear discrete systems is addressed. The Lipschitz condition in the form of matrix F is considered, and the disturbance and noise attenuation are taken into account in order to minimize the norm of the estimation error. The use of Lipschitz matrix conditions is similar to the one developed in [10] and extended in [12] (where time delays and uncertainties are also considered) for continuous systems. In [13], the discrete observer design based on the matrix Lipschitz condition is studied, but the disturbances are not taken into account. In [15], on the other hand,

the discrete case is also studied, but with the scalar Lipschitz condition and no disturbances consideration. With respect to the approach developed in [16], the presented work has the advantage of reaching a simple to solve LMI optimization problem, that consists of one single LMI, independently of the system order (compared to the up to 2^{n^2+np} simultaneous LMI that must be solved in [16]). The drawback is that the result of the proposed approach is more conservative than the one presented in [16], and therefore, some problems solved by that approach could lead to an unfeasible LMI if the approach of the present paper is used. On the other hand, the proposed approach can be extended to the case when the outputs are measured scarcely and irregularly in time, increasing the LMI's to be solved to a number equal to the possible measuring scenarios, while extending the approach of [16] to this case could lead to a really huge and almost unsolvable number of LMI's. In summary, the contribution of the paper is the design of a discrete observer for $\mathcal{H}_\infty$ disturbance attenuation, based on a matrix Lipschitz condition in the nonlinear terms, that has the advantage of a less conservative result if compared to previous works that use scalar Lipschitz conditions, and the advantage of a much simpler (lower computer cost) optimization problem to be solved compared to the approach in [16].

The outline of the paper is as follows: first, the problem is introduced, including the plant and observer equations, then, the prediction error dynamics is obtained, and the main result (the $\mathcal{H}_\infty$ design of the observer) is developed. Some examples illustrate the applicability of the proposed approach, compared with other works, and finally the main conclusions are summarized.

2 Problem statement

2.1 Plant and observer

Consider a discrete nonlinear time-invariant MIMO system described by the equations

$$\boldsymbol{x}[t+1] = \boldsymbol{A}\,\boldsymbol{x}[t] + \boldsymbol{f}(\boldsymbol{x}[t], \boldsymbol{u}[t]) + \boldsymbol{w}[t], \tag{5a}$$

$$\boldsymbol{y}[t] = \boldsymbol{C}\,\boldsymbol{x}[t] + \boldsymbol{h}(\boldsymbol{x}[t]) + \boldsymbol{v}[t]. \tag{5b}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $\boldsymbol{u} \in \mathbb{R}^{n_u}$ is the control input vector, $\boldsymbol{y} \in \mathbb{R}^{n_y}$ is the measured output variables, $\boldsymbol{w}[t] \in \mathbb{R}^n$ is the state disturbance and $\boldsymbol{v}[t] \in \mathbb{R}^{n_y}$ is the measurement noise. The pair $(\boldsymbol{A}, \boldsymbol{C})$ is assumed to be observable. The functions $\boldsymbol{f}(\cdot) : \mathbb{R}^{n+n_u} \to \mathbb{R}^n$ y $\boldsymbol{h}(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n_y}$ are known nonlinear functions that are assumed to fulfill the Lipschitz condition, i.e.,

$$\|\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{x}_2, \boldsymbol{u})\|_p \leq \|\boldsymbol{F} \cdot (\boldsymbol{x}_1 - \boldsymbol{x}_2)\|_p, \tag{6}$$

$$\|\boldsymbol{h}(\boldsymbol{x}_1) - \boldsymbol{h}(\boldsymbol{x}_2)\|_p \leq \|\boldsymbol{H} \cdot (\boldsymbol{x}_1 - \boldsymbol{x}_2)\|_p, \tag{7}$$

for any vectorial norm p.

In order to estimate the state from the output measurements, a model based observer is proposed. The state is initially estimated in open loop, leading to

$$\hat{\boldsymbol{x}}[t^-] = \boldsymbol{A}\,\hat{\boldsymbol{x}}[t-1] + \boldsymbol{f}(\hat{\boldsymbol{x}}[t-1], \boldsymbol{u}[t-1]), \tag{8a}$$

This estimation is updated with the measurement as

$$\hat{\boldsymbol{x}}[t] = \hat{\boldsymbol{x}}[t^-] + \boldsymbol{L}\,(y[t] - \boldsymbol{C}\,\hat{\boldsymbol{x}}[t^-] - \boldsymbol{h}(\hat{\boldsymbol{x}}[t^-])). \tag{8b}$$

where $\boldsymbol{L}$ is the gain matrix to be designed.

The dynamic of the state observer depends on the gain matrix, L, that must be designed to assure predictor stability and a proper attenuation of the disturbances and sensor noises.

2.2 Prediction error

In order to design a predictor, that is, the predictor gain L, with these properties, the prediction error dynamic equation must be obtained, that is, an explicit relationship between prediction error at measurement instants t and previous one $t-1$ must be obtained. If the process equations are introduced in the state estimation equations, (8), the estimation error can be expressed as

$$\begin{aligned}\tilde{\boldsymbol{x}}[t] =& \boldsymbol{A}\tilde{\boldsymbol{x}}[t-1] + \boldsymbol{f}(\boldsymbol{x}[t-1], \boldsymbol{u}[t-1]) - \boldsymbol{f}(\hat{\boldsymbol{x}}[t-1], \boldsymbol{u}[t-1]) - \\ & - \boldsymbol{L}\left(\boldsymbol{C}\tilde{\boldsymbol{x}}[t^-] + \boldsymbol{h}(\boldsymbol{x}[t]) - \boldsymbol{h}(\hat{\boldsymbol{x}}[t^-])\right) + \boldsymbol{w}[t-1] - \boldsymbol{L}\boldsymbol{v}[t]\end{aligned}$$

where $\tilde{\boldsymbol{x}}[t] = \boldsymbol{x}[t] - \hat{\boldsymbol{x}}[t]$. As it is observed, due to the presence of $\boldsymbol{f}$ and $\boldsymbol{h}$, it is not possible to explicitly write $\tilde{\boldsymbol{x}}[t]$ as a function of $\tilde{\boldsymbol{x}}[t-1]$. In order to simplify the next mathematical developments, the following notation is introduced

$$\begin{aligned}\tilde{\boldsymbol{x}}[t^-] &= \boldsymbol{x}[t] - \hat{\boldsymbol{x}}[t^-], \\ \tilde{\boldsymbol{f}}[t] &= \boldsymbol{f}(\boldsymbol{x}[t], \boldsymbol{u}[t]) - \boldsymbol{f}(\hat{\boldsymbol{x}}[t], \boldsymbol{u}[t]), \\ \tilde{\boldsymbol{h}}[t^-] &= \boldsymbol{h}(\boldsymbol{x}[t]) - \boldsymbol{h}(\hat{\boldsymbol{x}}[t^-]).\end{aligned}$$

The predictor error dynamics at sampling instants can then be written as

$$\tilde{\boldsymbol{x}}[t] = \tilde{\boldsymbol{x}}[t^-] - \boldsymbol{L}\left(\boldsymbol{C}\tilde{\boldsymbol{x}}[t^-] + \tilde{\boldsymbol{h}}[t^-] + \boldsymbol{v}[t]\right), \tag{9}$$

where the open loop estimation error ($\tilde{\boldsymbol{x}}[t^-]$) can be written as a function of the information at the previous control period as

$$\tilde{\boldsymbol{x}}[t^-] = \boldsymbol{A}\tilde{\boldsymbol{x}}[t-1] + \tilde{\boldsymbol{f}}[t-1] + \boldsymbol{w}[t-1], \tag{10}$$

and the functions $\tilde{\boldsymbol{f}}[t]$ and $\tilde{\boldsymbol{h}}[t^-]$ fulfill

$$\|\tilde{\boldsymbol{f}}[t]\| \leq \|\boldsymbol{F}\,\tilde{\boldsymbol{x}}[t]\|, \tag{11}$$

$$\|\tilde{\boldsymbol{h}}[t^-]\| \leq \|\boldsymbol{H}\,\tilde{\boldsymbol{x}}[t^-]\|. \tag{12}$$

The design objective of the predictor is to find a gain $\boldsymbol{L}$ that stabilizes the observer and assures a proper attenuation of state disturbance and measurement noise. For the next section some previous results must be obtained.

Lemma 1 *[9] For any pair of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and any positive definite matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$, the following condition holds*

$$2\boldsymbol{x}^\intercal \boldsymbol{y} \leq \boldsymbol{x}^\intercal \boldsymbol{P}\boldsymbol{x} + \boldsymbol{y}^\intercal \boldsymbol{P}^{-1}\boldsymbol{y}.$$

Lemma 2 *Assume that $\boldsymbol{x}$ is a vector and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{P}$ are matrices of proper dimensions, such that $\boldsymbol{P}$ is symmetric and positive definite ($P = P^\intercal > 0$). Assume that $\boldsymbol{y}$ is a vector that satisfies*

$$\|\boldsymbol{y}\| \leq \|\boldsymbol{F}\,\boldsymbol{x}\|, \tag{13}$$

being $\boldsymbol{F}$ *a matrix of proper dimensions. Then, for any* $\boldsymbol{\varepsilon} > 0$

$$(\boldsymbol{Ax}+\boldsymbol{By})^{\intercal}\,\boldsymbol{P}\,(\boldsymbol{Ax}+\boldsymbol{By}) \leq \boldsymbol{x}^{\intercal}\boldsymbol{Wx}, \tag{14}$$

with

$$\boldsymbol{W} = \boldsymbol{A}^{\intercal}\left(\boldsymbol{P}+\boldsymbol{PB}\left(\boldsymbol{\varepsilon I}-\boldsymbol{B}^{\intercal}\boldsymbol{PB}\right)^{-1}\boldsymbol{B}^{\intercal}\boldsymbol{P}\right)\boldsymbol{A}+\boldsymbol{\varepsilon F}^{\intercal}\boldsymbol{F}. \tag{15}$$

Proof 1 *Expanding the left expression in* (14) *one obtains*

$$(\boldsymbol{Ax}+\boldsymbol{By})^{\intercal}\,\boldsymbol{P}\,(\boldsymbol{Ax}+\boldsymbol{By}) = \boldsymbol{x}^{\intercal}\boldsymbol{A}^{\intercal}\boldsymbol{PAx}+2\boldsymbol{x}^{\intercal}\boldsymbol{A}^{\intercal}\boldsymbol{PBy}+\boldsymbol{y}^{\intercal}\boldsymbol{B}^{\intercal}\boldsymbol{PBy}.$$

Adding and subtracting $\boldsymbol{\varepsilon y}^{\intercal}\boldsymbol{y}$ *on the right term it yields*

$$(\boldsymbol{Ax}+\boldsymbol{By})^{\intercal}\,\boldsymbol{P}\,(\boldsymbol{Ax}+\boldsymbol{By}) = \boldsymbol{x}^{\intercal}\boldsymbol{A}^{\intercal}\boldsymbol{PAx}+2\boldsymbol{x}^{\intercal}\boldsymbol{A}^{\intercal}\boldsymbol{PBy}-\boldsymbol{y}^{\intercal}\left(\boldsymbol{\varepsilon I}-\boldsymbol{B}^{\intercal}\boldsymbol{PB}\right)\boldsymbol{y}+\boldsymbol{\varepsilon y}^{\intercal}\boldsymbol{y}.$$

Applying lemma 1, it leads to

$$(\boldsymbol{Ax}+\boldsymbol{By})^{\intercal}\boldsymbol{P}(\boldsymbol{Ax}+\boldsymbol{By}) \leq \boldsymbol{x}^{\intercal}\boldsymbol{A}^{\intercal}\boldsymbol{PAx}+\boldsymbol{x}^{\intercal}\boldsymbol{A}^{\intercal}\boldsymbol{PB}\left(\boldsymbol{\varepsilon I}-\boldsymbol{B}^{\intercal}\boldsymbol{PB}\right)^{-1}\boldsymbol{B}^{\intercal}\boldsymbol{PAx}+\boldsymbol{\varepsilon y}^{\intercal}\boldsymbol{y}.$$

Taking into account (13) *it is easy to obtain*

$$(\boldsymbol{Ax}+\boldsymbol{By})^{\intercal}\boldsymbol{P}(\boldsymbol{Ax}+\boldsymbol{By}) \leq \boldsymbol{x}^{\intercal}\left(\boldsymbol{A}^{\intercal}\boldsymbol{PA}+\boldsymbol{A}^{\intercal}\boldsymbol{PB}\left(\boldsymbol{\varepsilon I}-\boldsymbol{B}^{\intercal}\boldsymbol{PB}\right)^{-1}\boldsymbol{B}^{\intercal}\boldsymbol{PA}+\boldsymbol{\varepsilon F}^{\intercal}\boldsymbol{F}\right)\boldsymbol{x}.$$

Lemma 3 *Assume* $\boldsymbol{x}$ *and* $\boldsymbol{u}$ *are vectors and* $\boldsymbol{B}$, $\boldsymbol{P}$ *matrices of proper dimensions (such that* $P = P^{\intercal} > 0$*). Then, for any* $\boldsymbol{\Gamma} \succ \boldsymbol{0}$

$$(\boldsymbol{x}+\boldsymbol{Bu})^{\intercal}\,\boldsymbol{P}\,(\boldsymbol{x}+\boldsymbol{Bu})-\boldsymbol{u}^{\intercal}\boldsymbol{\Gamma}^2\boldsymbol{u} \leq \boldsymbol{x}^{\intercal}\boldsymbol{Wx}, \tag{16}$$

with

$$\boldsymbol{W} = \boldsymbol{P}+\boldsymbol{PB}\left(\boldsymbol{\Gamma}^2-\boldsymbol{B}^{\intercal}\boldsymbol{PB}\right)^{-1}\boldsymbol{B}^{\intercal}\boldsymbol{P}. \tag{17}$$

Proof 2 *Expanding the left expression in* (16) *it is easy to obtain*

$$(\boldsymbol{x}+\boldsymbol{Bu})^{\intercal}\,\boldsymbol{P}\,(\boldsymbol{x}+\boldsymbol{Bu})-\boldsymbol{u}^{\intercal}\boldsymbol{\Gamma}^2\boldsymbol{u} = \boldsymbol{x}^{\intercal}\boldsymbol{Px}+2\boldsymbol{x}^{\intercal}\boldsymbol{PBu}-\boldsymbol{u}^{\intercal}\left(\boldsymbol{\Gamma}^2-\boldsymbol{B}^{\intercal}\boldsymbol{PB}\right)\boldsymbol{u}.$$

Applying lemma 1, it leads to

$$(\boldsymbol{x}+\boldsymbol{Bu})^{\intercal}\,\boldsymbol{P}\,(\boldsymbol{x}+\boldsymbol{Bu})-\boldsymbol{u}^{\intercal}\boldsymbol{\Gamma}^2\boldsymbol{u} \leq \boldsymbol{x}^{\intercal}\left(\boldsymbol{P}+\boldsymbol{PB}\left(\boldsymbol{\Gamma}^2-\boldsymbol{B}^{\intercal}\boldsymbol{PB}\right)^{-1}\boldsymbol{B}^{\intercal}\boldsymbol{P}\right)\boldsymbol{x}.$$

3 $\mathcal{H}_\infty$ design

Theorem 1 *Consider the predictor algorithm* (8) *applied to system* (5)*. For some given* $\boldsymbol{\Gamma}_w, \boldsymbol{\Gamma}_v > 0$*, assume that there exist some matrices* $\boldsymbol{P} = \boldsymbol{P}^{\intercal} \in \mathbb{R}^{n\times n}$*,* $\boldsymbol{X} \in \mathbb{R}^{n\times n_y}$ *and some scalars* $\boldsymbol{\varepsilon}_f, \boldsymbol{\varepsilon}_h > 0$ *such that the next inequality fulfills*

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{PA}-\boldsymbol{XCA} & \boldsymbol{P}-\boldsymbol{XC} & \boldsymbol{P}-\boldsymbol{XC} & \boldsymbol{X} & \boldsymbol{X} \\ \star & \begin{pmatrix} \boldsymbol{P}-\boldsymbol{I}-\epsilon_f\boldsymbol{F}^{\intercal}\boldsymbol{F} \\ -\epsilon_h\boldsymbol{A}^{\intercal}\boldsymbol{H}^{\intercal}\boldsymbol{HA} \end{pmatrix} & -\epsilon_h\boldsymbol{A}^{\intercal}\boldsymbol{H}^{\intercal}\boldsymbol{H} & -\epsilon_h\boldsymbol{A}^{\intercal}\boldsymbol{H}^{\intercal}\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & \epsilon_f\boldsymbol{I}-\epsilon_h\boldsymbol{H}^{\intercal}\boldsymbol{H} & -\epsilon_h\boldsymbol{H}^{\intercal}\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & \star & \boldsymbol{\Gamma}_w^2-\epsilon_h\boldsymbol{H}^{\intercal}\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} \\ \star & \star & \star & \star & \epsilon_h\boldsymbol{I} & \boldsymbol{0} \\ \star & \star & \star & \star & \star & \boldsymbol{\Gamma}_v^2 \end{bmatrix} \succ \boldsymbol{0}, \tag{18}$$

with

$$\boldsymbol{\Gamma}_v = \mathrm{diag}\{\gamma_{v_1}, \ldots, \gamma_{v_{n_y}}\}$$
$$\boldsymbol{\Gamma}_w = \mathrm{diag}\{\gamma_{w_1}, \ldots, \gamma_{w_n}\}.$$

Then, defining the predictor gain matrix as $\boldsymbol{L} = \boldsymbol{P}^{-1}\boldsymbol{X}$*, under null disturbances, the prediction error converges to zero asymptotically, and, under null initial conditions, the following condition holds*

$$\|\tilde{\boldsymbol{x}}[t]\|_2^2 \leq \|\boldsymbol{\Gamma}_v \boldsymbol{v}[t]\|_2^2 + \|\boldsymbol{\Gamma}_w \boldsymbol{w}[t]\|_2^2. \tag{19}$$

Proof 3 *In order to prove the theorem, a cost index including estimation error and disturbances is created. That index is bounded using the Lyapunov function of the state estimation error. Introducing the state estimation error dynamics in the index bound it is demonstrated that if LMI* (18) *holds, then the cost index is negative and therefore, condition* (19) *holds. It is also demonstrated that if* (18) *holds, then the Lyapunov function of the state estimation error decreases, proving the convergence of the state estimation algorithm.*

Consider the index

$$J = \sum_{t=0}^{\infty} \left(\tilde{\boldsymbol{x}}[t]^{\intercal} \tilde{\boldsymbol{x}}[t] - \boldsymbol{v}[t]^{\intercal} \boldsymbol{\Gamma}_v^2 \boldsymbol{v}[t] - \boldsymbol{w}[t]^{\intercal} \boldsymbol{\Gamma}_w^2 \boldsymbol{w}[t] \right).$$

Taking the Lyapunov function $\mathcal{V}[t] = \mathcal{V}(\tilde{\boldsymbol{x}}[t]) = \tilde{\boldsymbol{x}}[t]^{\intercal} \boldsymbol{P} \tilde{\boldsymbol{x}}[t]$ *and assuming null initial conditions, one can write*

$$\begin{aligned} J &\leq \sum_{t=1}^{\infty} \left(\tilde{\boldsymbol{x}}[t-1]^{\intercal} \tilde{\boldsymbol{x}}[t-1] - \boldsymbol{v}[t-1]^{\intercal} \boldsymbol{\Gamma}_v^2 \boldsymbol{v}[t-1] - \boldsymbol{w}[t]^{\intercal} \boldsymbol{\Gamma}_w^2 \boldsymbol{w}[t] \right) + \mathcal{V}[t]|_{t=\infty} - \mathcal{V}[t]|_{t=0} \\ &= \sum_{t=1}^{\infty} \left(\tilde{\boldsymbol{x}}[t-1]^{\intercal} \tilde{\boldsymbol{x}}[t-1] - \boldsymbol{v}[t-1]^{\intercal} \boldsymbol{\Gamma}_v^2 \boldsymbol{v}[t-1] - \boldsymbol{w}[t]^{\intercal} \boldsymbol{\Gamma}_w^2 \boldsymbol{w}[t] + \Delta\mathcal{V}[t] \right), \end{aligned}$$

where $\Delta\mathcal{V}[t] = \mathcal{V}[t] - \mathcal{V}[t-1]$*. Substituting* $\Delta\mathcal{V}[t]$ *by*

$$\begin{aligned} \Delta\mathcal{V}[t] &= \tilde{\boldsymbol{x}}[t]^{\intercal} \boldsymbol{P} \tilde{\boldsymbol{x}}[t] - \tilde{\boldsymbol{x}}[t-1]^{\intercal} \boldsymbol{P} \tilde{\boldsymbol{x}}[t-1] \\ &= \underbrace{\left((\boldsymbol{I} - \boldsymbol{L}\boldsymbol{C}) \tilde{\boldsymbol{x}}[t^-] - \boldsymbol{L}\tilde{\boldsymbol{h}}[t^-] - \boldsymbol{L}\boldsymbol{w}[t] \right)}_{\star}^{\intercal} \boldsymbol{P} \, (\star) - \tilde{\boldsymbol{x}}[t-1]^{\intercal} \boldsymbol{P} \tilde{\boldsymbol{x}}[t-1], \end{aligned}$$

lemma 3 can be applied to eliminate the term $\boldsymbol{v}[t]^{\intercal} \boldsymbol{\Gamma}_v^2 \boldsymbol{v}[t]$*, leading to*

$$\begin{aligned} J \leq \sum_{t=1}^{\infty} \Bigg(& \tilde{\boldsymbol{x}}[t-1]^{\intercal} (\boldsymbol{I} - \boldsymbol{P}) \tilde{\boldsymbol{x}}[t-1] - \boldsymbol{v}[t-1]^{\intercal} \boldsymbol{\Gamma}_v^2 \boldsymbol{v}[t-1] \\ & + \underbrace{\left((\boldsymbol{I} - \boldsymbol{L}\boldsymbol{C}) \tilde{\boldsymbol{x}}[t^-] - \boldsymbol{L}\tilde{\boldsymbol{h}}[t^-] \right)}_{\star}^{\intercal} \boldsymbol{P}_v \, (\star) \Bigg). \end{aligned}$$

with

$$\boldsymbol{P}_v = \boldsymbol{P} + \boldsymbol{P}\boldsymbol{L} \left(\boldsymbol{\Gamma}_v^2 - \boldsymbol{L}^{\intercal} \boldsymbol{P} \boldsymbol{L} \right)^{-1} \boldsymbol{L}^{\intercal} \boldsymbol{P}. \tag{20}$$

Applying lemma 2 to eliminate the term $\tilde{\boldsymbol{h}}[t^-]$ *it yields*

$$J \leq \sum_{t=1}^{\infty} \left(\tilde{\boldsymbol{x}}[t-1]^{\intercal} (\boldsymbol{I} - \boldsymbol{P}) \tilde{\boldsymbol{x}}[t-1] - \boldsymbol{w}[t-1]^{\intercal} \boldsymbol{\Gamma}_w^2 \boldsymbol{w}[t-1] + \tilde{\boldsymbol{x}}[t^-]^{\intercal} \boldsymbol{P}_h \tilde{\boldsymbol{x}}[t^-] \right).$$

with

$$\boldsymbol{P}_h = (\boldsymbol{I} - \boldsymbol{L}\boldsymbol{C})^{\intercal} \left(\boldsymbol{P}_v + \boldsymbol{P}_v \boldsymbol{L} \left(\epsilon_h \boldsymbol{I} - \boldsymbol{L}^{\intercal} \boldsymbol{P}_v \boldsymbol{L} \right)^{-1} \boldsymbol{L}^{\intercal} \boldsymbol{P}_v \right) (\boldsymbol{I} - \boldsymbol{L}\boldsymbol{C}) + \epsilon_h \boldsymbol{H}^{\intercal} \boldsymbol{H}. \tag{21}$$

Substituting the open loop prediction error by $\tilde{\boldsymbol{x}}[t^-] = \boldsymbol{A}\tilde{\boldsymbol{x}}[t-1] + \tilde{\boldsymbol{f}}[t-1] + \boldsymbol{w}[t-1]$ *and applying lemma 3 to eliminate the term* $\boldsymbol{w}[t-1]^{\intercal} \boldsymbol{\Gamma}_w^2 \boldsymbol{w}[t-1]$ *it yields*

$$J \leq \sum_{t=1}^{\infty} \left(\tilde{\boldsymbol{x}}[t-1]^{\intercal} (\boldsymbol{I} - \boldsymbol{P}) \tilde{\boldsymbol{x}}[t-1] + \underbrace{\left(\boldsymbol{A}\tilde{\boldsymbol{x}}[t-1] + \tilde{\boldsymbol{f}}[t-1] \right)}_{\star}^{\intercal} \boldsymbol{P}_w (\star) \right),$$

with

$$\boldsymbol{P}_w = \boldsymbol{P}_h + \boldsymbol{P}_h \left(\boldsymbol{\Gamma}_w^2 - \boldsymbol{P}_h \right)^{-1} \boldsymbol{P}_h. \tag{22}$$

Applying now lemma 2 it yields that

$$J \leq \sum_{t=1}^{\infty} \left(\tilde{\boldsymbol{x}}[t-1]^{\intercal} (\boldsymbol{I} + \boldsymbol{P}_f - \boldsymbol{P}) \tilde{\boldsymbol{x}}[t-1] \right), \tag{23}$$

with

$$\boldsymbol{P}_f = \boldsymbol{A}^{\intercal} \left(\boldsymbol{P}_w + \boldsymbol{P}_w \left(\epsilon_f \boldsymbol{I} - \boldsymbol{P}_w \right)^{-1} \boldsymbol{P}_w \right) \boldsymbol{A} + \epsilon_f \boldsymbol{F}^{\intercal} \boldsymbol{F}. \tag{24}$$

Condition (19) *holds if* $J < 0$*, but this will always be true if*

$$\boldsymbol{I} + \boldsymbol{P}_f - \boldsymbol{P} \prec \boldsymbol{0}.$$

Substituting $\boldsymbol{P}_f$ *as a function of* $\boldsymbol{P}_w$ *(using* (24)*) and applying Schur complements, the previous condition is equivalent to condition*

$$\begin{bmatrix} \boldsymbol{A}^{\intercal} \boldsymbol{P}_w \boldsymbol{A} - \boldsymbol{P} + \boldsymbol{I} + \epsilon_f \boldsymbol{F}^{\intercal} \boldsymbol{F} & \boldsymbol{A}^{\intercal} \boldsymbol{P}_w \\ \boldsymbol{P}_w \boldsymbol{A} & \boldsymbol{P}_w - \epsilon_f \boldsymbol{I} \end{bmatrix} \prec \boldsymbol{0}.$$

sustituting $\boldsymbol{P}_w$ *as a function of* $\boldsymbol{P}_h$ *(using* (22)*) and applying Schur complements leads to*

$$\begin{bmatrix} \boldsymbol{A}^{\intercal} \boldsymbol{P}_h \boldsymbol{A} + \epsilon_f \boldsymbol{F}^{\intercal} \boldsymbol{F} - \boldsymbol{P} + \boldsymbol{I} & \boldsymbol{A}^{\intercal} \boldsymbol{P}_h & \boldsymbol{A}^{\intercal} \boldsymbol{P}_h \\ \boldsymbol{P}_h \boldsymbol{A} & \boldsymbol{P}_h - \epsilon_f \boldsymbol{I} & \boldsymbol{P}_h \\ \boldsymbol{P}_h \boldsymbol{A} & \boldsymbol{P}_h & \boldsymbol{P}_h - \boldsymbol{\Gamma}_w^2 \end{bmatrix} \prec \boldsymbol{0}.$$

Substituting $\boldsymbol{P}_h$ *as a function of* $\boldsymbol{P}_v$ *(using* (21)*) and applying Schur comple-*

ments it leads to

$$\begin{bmatrix} \begin{pmatrix} \boldsymbol{P} - \epsilon_f \boldsymbol{F}^\intercal \boldsymbol{F} - \boldsymbol{I} \\ -\epsilon_h \boldsymbol{A}^\intercal \boldsymbol{H}^\intercal \boldsymbol{H} \boldsymbol{A} \end{pmatrix} & -\epsilon_h \boldsymbol{A}^\intercal \boldsymbol{H}^\intercal \boldsymbol{H} & -\epsilon_h \boldsymbol{A}^\intercal \boldsymbol{H}^\intercal \boldsymbol{H} & \boldsymbol{0} \\ -\epsilon_h \boldsymbol{H}^\intercal \boldsymbol{H} \boldsymbol{A} & \epsilon_f \boldsymbol{I} - \epsilon_h \boldsymbol{H}^\intercal \boldsymbol{H} & -\epsilon_h \boldsymbol{H}^\intercal \boldsymbol{H} & \boldsymbol{0} \\ -\epsilon_h \boldsymbol{H}^\intercal \boldsymbol{H} \boldsymbol{A} & -\epsilon_h \boldsymbol{H}^\intercal \boldsymbol{H} & \boldsymbol{\Gamma}_w^2 - \epsilon_h \boldsymbol{H}^\intercal \boldsymbol{H} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \epsilon_h \boldsymbol{I} \end{bmatrix} -$$

$$- \begin{bmatrix} \boldsymbol{A}^\intercal (\boldsymbol{I} - \boldsymbol{L}\boldsymbol{C})^\intercal \\ (\boldsymbol{I} - \boldsymbol{L}\boldsymbol{C})^\intercal \\ (\boldsymbol{I} - \boldsymbol{L}\boldsymbol{C})^\intercal \\ \boldsymbol{L}^\intercal \end{bmatrix} \boldsymbol{P}_v \begin{bmatrix} \boldsymbol{A}^\intercal (\boldsymbol{I} - \boldsymbol{L}\boldsymbol{C})^\intercal \\ (\boldsymbol{I} - \boldsymbol{L}\boldsymbol{C})^\intercal \\ (\boldsymbol{I} - \boldsymbol{L}\boldsymbol{C})^\intercal \\ \boldsymbol{L}^\intercal \end{bmatrix}^\intercal \succ \boldsymbol{0}.$$

Substituting $\boldsymbol{P}_v$ *as a function of* $\boldsymbol{P}$ *using* (20)*, applying Schur complements twice, and taking into account that* $\boldsymbol{P}\boldsymbol{L} = \boldsymbol{X}$ *it finally leads to* (18)*.*

Applying the same mathematical manipulations as before, the increment of the Lyapunov function, i.e.,

$$\Delta \mathcal{V}[t] = \tilde{\boldsymbol{x}}[t]^\intercal \boldsymbol{P} \tilde{\boldsymbol{x}}[t] - \tilde{\boldsymbol{x}}[t-1]^\intercal \boldsymbol{P} \tilde{\boldsymbol{x}}[t-1],$$

will be negative if

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{P}(\boldsymbol{I} - \boldsymbol{L}\boldsymbol{C})\boldsymbol{A} & \boldsymbol{P}(\boldsymbol{I} - \boldsymbol{L}\boldsymbol{C}) & \boldsymbol{P}\boldsymbol{L} \\ \star & \begin{pmatrix} \boldsymbol{P} - \epsilon_f \boldsymbol{F}^\intercal \boldsymbol{F} \\ -\epsilon_h \boldsymbol{A}^\intercal \boldsymbol{H}^\intercal \boldsymbol{H} \boldsymbol{A} \end{pmatrix} & -\epsilon_h \boldsymbol{A}^\intercal \boldsymbol{H}^\intercal \boldsymbol{H} & \boldsymbol{0} \\ \star & \star & \epsilon_f \boldsymbol{I} - \epsilon_h \boldsymbol{H}^\intercal \boldsymbol{H} & \boldsymbol{0} \\ \star & \star & \star & \epsilon_h \boldsymbol{I} \end{bmatrix} \succ \boldsymbol{0}, \tag{25}$$

holds. It must be noted that the matrix in inequality (25) *can be formed taking the first, second, third and fifth blocks of rows and columns of matrix in LMI* (18) *with* $\boldsymbol{X} = \boldsymbol{P}\boldsymbol{L}$ *and adding matrix* $\mathrm{diag}\{\boldsymbol{0}, \boldsymbol{I}, \boldsymbol{0}, \boldsymbol{0}\}$*, that is a semidefinite matrix. This implies that if* (18) *holds,* (25) *holds, and then, the Lyapunov function of the state observed error decrease and the estimation error of algorithm (under null disturbances and noises) decreases exponentially to zero.*

Remark 1 (Design procedure) *If the* ℓ_2 *norm of disturbance and noise measurement signals are considered to be known, the upper bound on* $\|\tilde{\boldsymbol{x}}[t]\|_2$ *can be reduced to a minimum value by minimizing*

$$\sum_{i=1}^{n_y} \gamma_{v_i}^2 \|v_i[t]\|_2^2 + \sum_{i=1}^{n} \gamma_{w_i}^2 \|w_i[t]\|_2^2$$

along the variables γ_{v_i}*,* γ_{w_i}*,* $\boldsymbol{P}$ *and* $\boldsymbol{X}$ *that satisfy the LMI* (18)*. This convex minimization problem can be easily addressed using standard LMI solvers (such as Matlab LMI toolbox) that solve problems of the form*

$$\text{Minimize} \quad \boldsymbol{h}^\intercal \boldsymbol{x} \quad \text{subject to} \quad \boldsymbol{M}(\boldsymbol{X}) \prec \boldsymbol{0}, \tag{26}$$

where $\boldsymbol{h}$ *is a constant vector,* $\boldsymbol{X}$ *denotes the matrix variables,* $\boldsymbol{x}$ *is a vector with all the components of* $\boldsymbol{X}$*, and* $\boldsymbol{M}(\boldsymbol{X})$ *represents the matrices of the LMI problem. The computation cost of this minimization problem for one simple LMI, such as (18), is relatively low. First,* $\boldsymbol{\Gamma}_v^2$ *and* $\boldsymbol{\Gamma}_w^2$ *must be expressed as*

matricial variables and $\gamma_{v_1}^2, \ldots, \gamma_{v_{n_v}}^2, \gamma_{w_1}^2, \ldots, \gamma_{w_{n_m}}^2$ *must be written as the last components of vector* $\boldsymbol{x}$ *in* (26)*. Then, the vector* $\boldsymbol{h}^\intercal$ *is defined as*

$$\boldsymbol{h}^\intercal = [0 \ \ldots \ 0 \ \|v_1[t]\|_2^2 \ \ldots \ \|v_{n_y}[t]\|_2^2 \ \|w_1[t]\|_2^2 \ \ldots \ \|w_n[t]\|_2^2].$$

The previous remark also applies if the RMS norms of the disturbances and measurement noise are known. In that case, the upper bound on $\|\tilde{x}[t]\|_{RMS}$ can be minimized if

$$\gamma_v^2 \|v[t]\|_{RMS}^2 + \gamma_w^2 \|w[t]\|_{RMS}^2$$

is minimized along all variables γ_v, γ_w, $\boldsymbol{P}$ and $\boldsymbol{X}$ that satisfy the LMI (18).

Remark 2 *With respect the practical computation of matrices* F *and* H*, a simple general procedure may be to calculate the jacobian of* f *and* h*, and then to obtain the maximum of the absolute values of each one of their elements, in the domain of validity of the involved variables for the specific problem. Those maximums would then form the elements of matrices* F *and* H*. For a specific problem, however, tighter matrix bounds* F *and* H *could perhaps be obtained through the exploitation of the structure of the particular functions* f *and* h*.*

Remark 3 *The proposed approach can be extended to the case when the measurements are taken irregularly and scarcely in time (see [7] for details in the linear case). Assume that the output* $y[t]$ *is measured only every* N_k *periods, where* N_k *can take values in a finite integer set of size* s*, and define* t_k *as the instant when the* k*-th measurement is taken (hence* $N_k = t_k - t_{k-1}$*). Then the prediction error at instant* t_k *can be expressed as a function of the error at instant* t_{k-1}*, using a variant observer matrix gain* L_k *that is a function of* N_k*. As a result, if there are* s *possible values of* N_k*, one can obtain* s *LMI's to be solved to obtain* s *matrix gains,* $L(N_k)$ *that are precalculated off line and applied as a function of the measurement characteristics,* N_k*. The authors are finishing the detailed development of the irregular measurement case.*

On the other hand the extension of the work presented in [16] to the scarce measurement case would lead to a huge and almost unsolvable number of LMI's.

4 Examples

In this section some examples will illustrate the applicability of the proposed observer design technique, comparing it with other approaches.

4.1 Example 1

Consider the example of the flexible robot modeled in [5] and [8], being studied [6] and [16] (example 1). If an Euler approximation is applied, the dynamics of the robot can be described by equations

$$\boldsymbol{x}[t] = (\boldsymbol{I} + T\boldsymbol{A})\boldsymbol{x}[t-1] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.33T\sin(x_3) \end{bmatrix} + \boldsymbol{w}[t-1]$$

$$y[t] = \boldsymbol{C}\boldsymbol{x}[t] + v[t]$$

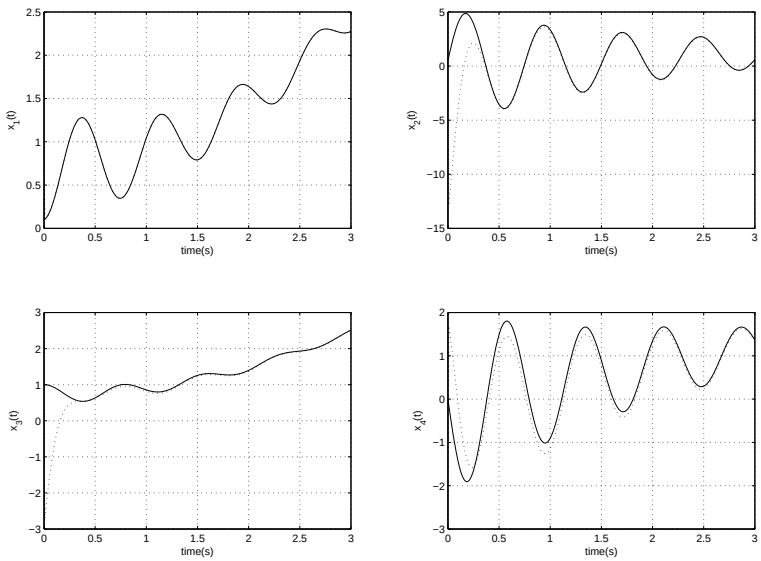

Figure 1: States (solid lines) and estimations (dotted lines)

where T is the sampling period and

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

The components of the state represent the angular position of the motor (x_1), its angular velocity (x_2), the angular position of the link (x_3) and its velocity (x_4). Vector $\boldsymbol{w}$ is the state disturbance whose components are null except the third one with an assumed norm of $\|w_3\|_2 = 0.1T$, and v is the measurement noise with an assumed norm $\|v\|_2 = 0.1$. The matrix $\boldsymbol{F}$ that fits the Lipschitz condition is in this case

$$\boldsymbol{F} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3.33T & 0 \end{bmatrix}.$$

If the method proposed in this work is applied, an observer gain

$$\boldsymbol{L} = \begin{bmatrix} 0.9996 & 16.5691 & 1.9360 & -5.1833 \end{bmatrix}^{\mathsf{T}}$$

is obtained. This gain cannot be compared with the one obtained in [16] because here a discrete-time observer is considered, but in figure 1 the evolution of the state and its estimate is shown to be similar to the behavior reached in [16].

If the scalar Lipschitz condition is applied, that is:

$$\|\boldsymbol{f}(\boldsymbol{x}[t]) - \boldsymbol{f}(\hat{\boldsymbol{x}}[t])\| \leq 0.33T\|\tilde{\boldsymbol{x}}[t]\|$$

the solution of the LMI problem is unfeasible (applying LMIs on this work with matrix $\boldsymbol{F} = 0.33T\boldsymbol{I}$), showing that the matrix Lipschitz condition is less conservative than the scalar one.

The proposed method calculates the gain as a function of the available information of disturbances and measurement noises. To show this idea, assume now that the system has a smaller measurement noise of $\|v\|_2 = 0.01$. The resulting observer gain is then

$$\boldsymbol{L} = \begin{bmatrix} 0.9998 & 24.0856 & 3.3106 & -6.5132 \end{bmatrix}^{\intercal}.$$

On the other hand, if the input disturbance is assumed to have an smaller value of $\|w_3\|_2 = 0.01T$, the resulting observer gain is

$$\boldsymbol{L} = \begin{bmatrix} 0.9996 & 12.5120 & 1.3297 & -4.1024 \end{bmatrix}^{\intercal}.$$

This shows that the proposed design strategy fits the observer gain to minimize the observation error taking into account the available information of the disturbances and noises.

4.2 Example 2

Consider the MIMO discrete time non linear system defined by

$$A = \begin{bmatrix} 0.15 & 0.2 & -0.01 \\ 0.1 & 0.9 & -0.1 \\ 0.02 & 0.26 & 0.8 \end{bmatrix} \quad B = \begin{bmatrix} 0.6 & 0.4 \\ 1 & 0 \\ 0.15 & 0.9 \end{bmatrix} \quad C = \begin{bmatrix} 0.5 & 1 & 0.5 \\ 0 & -1.5 & 1 \end{bmatrix}$$

$$f(x,u) = B\,u + \begin{bmatrix} 0.05\,sin(x_1)cos(2x_2) - 0.01\,sin(x_3) \\ 0.01\,sin(x_2)cos(2x_1)sin(x_3) \\ 0.15\,sin^2(3x_3) + 0.01\,sin(x_2)cos(3x_1) \end{bmatrix}$$

$$h(x) = \begin{bmatrix} 0.05\,sin(2x_1)cos(x_2)sin(2x_3) \\ 0.05\,cos^2(x_3) + 0.05\,sin(2x_2)cos(x_1) \end{bmatrix}$$

and assume that the disturbances w and v are vectors of independent white noises of variances 0.2, 0.3, 0.1 and 0.1, 0.1 respectively.

In this case the matrices that define the bounds on the Lipschitz conditions of f and h are easy to obtain

$$F = \begin{bmatrix} 0.05 & 0.1 & 0.01 \\ 0.02 & 0.01 & 0.01 \\ 0.03 & 0.01 & 0.9 \end{bmatrix}$$

$$H = \begin{bmatrix} 0.1 & 0.05 & 0.1 \\ 0.05 & 0.1 & 0.1 \end{bmatrix}$$

Applying the proposed design method, the following observer matrix gain is obtained:

$$L = \begin{bmatrix} 0.1048 & 0.01135 \\ 0.5409 & -0.2762 \\ 0.8134 & 0.5439 \end{bmatrix},$$

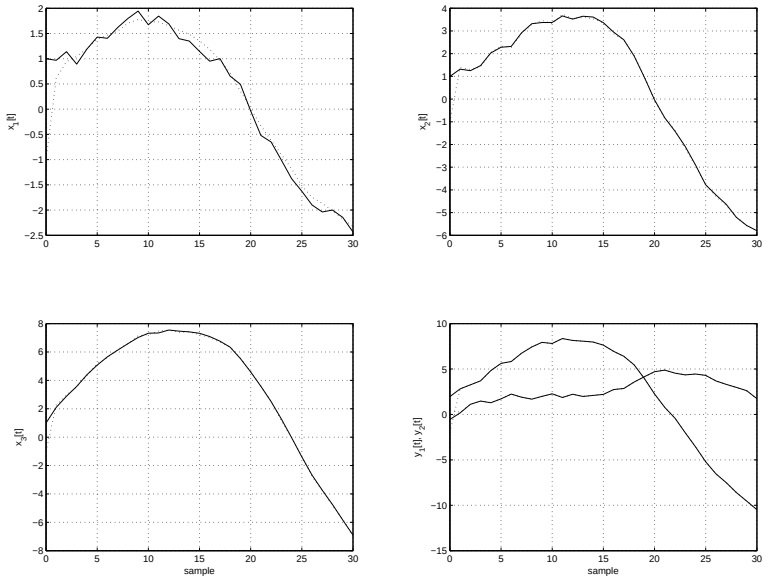

Figure 2: States, outputs (solid lines) and estimations (dotted lines)

and its implementation leads to the state estimations shown in figure 2. On the other hand, if a scalar Lipschitz condition is taken into account instead of the matrix one, the value of the bounding constant would be $\gamma = 0.9$. In this case, the resulting LMI is unfeasible, and hence, no solution can be found. Finally, in order to apply the method described in [16], a huge number of $2^{15} = 32768$ LMI's should be solved simultaneously, because in this case $n = 3, p = 3, q = 2$, implying a huge computational effort.

5 Conclusions

In this paper, the design of observers for non linear discrete systems has been addressed. The non linear terms in the state and output equation has been assumed to fulfill a matrix Lipschitz condition, leading to a less conservative result than the assumption of a scalar Lipschitz condition.

The proposed design strategy takes into account the attenuation of disturbances and measurement noise.

The final design procedure is based on the solution of a convex minimization problem subject to a linear matrix inequality, that can be solved by means of standard LMI solvers.

The problem is formulated in terms of the available knowledge about the norms of the disturbances, leading to a solution that minimizes the bound on the state estimation error norm.

The proposed approach is suitable to be extended to the case when the measurements are taken scarcely and irregularly in time. Those alternative

approaches already discussed would also lead to a huge number of LMI's to be solved.

References

[1] S. Beale and B. Shafai. Robust control design with a proportional integral observer. *International Journal of Control*, 50:97–111, 1989.

[2] G. Besançon. An overview on observer tools for nonlinear systems. *In G. Besançon (Ed.) Nonlinear Observers and Applications. Springer-Verlag. LNCIS*, 363:1–33, 2007.

[3] G. Bornard and H. Hammouri. A high gain observer for a class of uniformly observable systems. *Proc. of the 30th IEEE CDC*, pages 1494–1496, 1991.

[4] K.K. Busawon and P. Kabore. Disturbance attenuation using proportional integral observers. *INT. J. CONTROL*, 74(6):618–627, 2001.

[5] R. Marino and P. Tomei. *Nonlinear Control Design*. Prentice-Hall, 1995.

[6] P.R. Pagilla and Y. Zhu. Controller and observer design for lipschitz nonlinear systems. *American Control Conference ACC'04*, 2004.

[7] R. Sanchis, I. Peñarrocha, and P. Albertos. Design of robust output predictors under scarce measurements with time-varying delays. *Automatica*, 43:281–289, 2007.

[8] M. Spong. Modeling and control of elastic joint robots. *ASME Journal of Dynamic Systems, Measurements and Control*, 109:310–319, 1987.

[9] Y. Wang, L. Xie, and C.E. de Souza. Robust control of a class of uncertain nonlinear systems. *System Control Letters*, 19:139–149, 1992.

[10] Z. Wang and J. Burnham. Robust filtering for a class of stochastic uncertain nonlinear time-delay systems via exponential state estimation. *IEEE transactions on signal processing*, 49(4):794–804, 2001.

[11] Z. Wang, D. P. Goodall, and K. Burnham. On designing observers for time-delay systems with nonlinear disturbances. *International Journal of Control*, 75(11):803–811, 2002.

[12] Z. Wang, Y. Liu, and X. Liu. H-infinity filtering for uncertain stochastic time-delay systems with sector-bounded nonlinearities. *Automatica*, 44(5):1268–1277, 2008.

[13] S. Xu, J. Lu, S. Zhou, and C. Yang. Design of observers for a class of discrete-time uncertain nonlinear systems with time delay. *Journal of the Franklin Institute*, 341:295–308, 2004.

[14] E.E. Yaz, C.S. Jeonga, A. Bahakeemb, and Y.I. Yaz. Discrete-time nonlinear observer design with general criteria. *Journal of the Franklin Institute*, 344:918–928, 2007.

[15] A. Zemouche and M. Boutayeb. Observer design for lipschitz nonlinear systems: The discrete-time case. *IEEE Transactions on Circuits and Systems*, 53(8):777–781, 2006.

[16] A. Zemouche, M. Boutayeb, and G. Iulia Bara. Observers for a class of lipschitz system swith extension to h_∞ performance analysis. *Systems and Control Letters*, 57:18–27, 2008.