The Projectivity of Bridgeland Moduli Spaces of del Pezzo Surface of Picard Rank Three

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Main Theorem

S :del-Pezzo Surf of $\rho = 3$

$v \in K_0(S)$,

σ : Bridgeland stab condi,

$\mathcal{M}_\sigma(v)$: Moduli sp of σ-ss obj

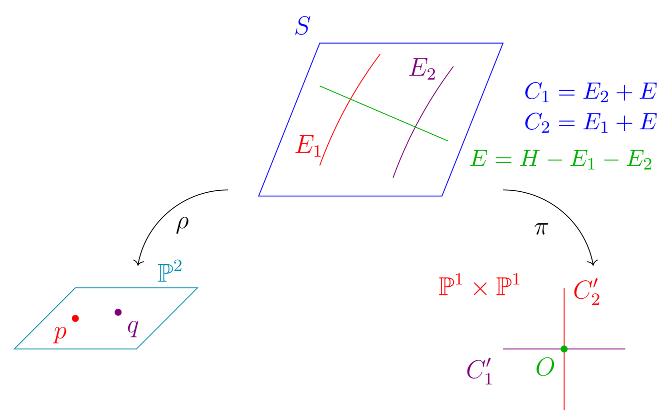

Figure 1: Notations of Divisors

Main Theorem

If $\mathcal{O}_S$ is semi-stable for $\sigma_{D,A}$,

$\mathcal{M}_{\sigma_{D,A}}(v)$ **is a Projective Scheme**

BSC from Divisors (Geometrical Setting)

$$N^1(S) \times \mathrm{Amp}(S) \xrightarrow{\sim} \mathbf{Stab}_{div}(S) \subset \mathbf{Stab}(S)$$
$$(D, A) \longmapsto \sigma_{D,A} = (Z_{D,A}, \mathcal{A}_{D,A})$$
$$\left(Z_{D,A} = -\int_S e^{-(D+iH)}\mathrm{ch},\ \mathcal{A}_{D,A} = \langle \mathcal{F}_{D,H}[1], \mathcal{Q}_{D,H}\rangle_{\mathrm{ex}}\right)$$

Rmk In general, $\mathcal{M}_{\sigma_{D,A}}(v)$ **is only an Alg Stack.**

BSC from FEC and Quiver (Algebraic Setting)

$\mathbb{E} = (E_1, \ldots, E_n)$: Strong FEC on S

Then, we have

1. $\mathbb{F} := (F_n, \ldots, F_1)$: **Ext-EC** $(F_i := \mathbb{L}_{E_{i-1}} \ldots \mathbb{L}_{E_1}(E_i))$
2. $^\exists(Q, \rho)$: Quiver w/ rel &

$$^\exists\Phi_{\mathbb{E}} : D^b(S) \xrightarrow{\sim} D^b(\mathrm{Rep}(Q,\rho))$$
$$\cup \qquad\qquad \cup$$
$$\mathcal{A}_{\mathbb{F}} := \langle F_n, \ldots, F_1\rangle_{\mathrm{ex}} \xrightarrow{\sim} \mathrm{Rep}(Q, \rho)$$

3. $^\exists$BSC w/ the heart $\mathcal{A}_{\mathbb{F}}$ via $\Phi_{\mathbb{E}}|_{\mathcal{A}_{\mathbb{F}}}$

Important Fact (ABCH,King) :

$^\forall\vec{h} \in \mathbb{H}^n_-$, $^\exists\sigma_{\mathbb{F},\vec{h}} = (\mathcal{A}_{\mathbb{F}}, Z_{\vec{h}}) \in \mathbf{Stab}(S)$ &

$\mathcal{M}_{\sigma_{\mathbb{F},\vec{h}}}(v)$ **is a Proj Sch.**

Rotations give isoms of Bridgeland moduli spaces

$^\exists$Action $\mathbb{R} \curvearrowright \mathbf{Stab}(S)$: called **Rotation** (Figure 2)

$$(\sigma, \varphi) \mapsto \sigma[\varphi] = (\mathcal{A}[\varphi], Z[\varphi]),$$
$$(Z[\varphi] = e^{-\varphi\pi i}Z,\ \mathcal{A}[\varphi] = \langle \mathcal{F}_\varphi[1], \mathcal{Q}_\varphi\rangle_{\mathrm{ex}})$$

Importance of the Rotation :

$$\mathcal{M}_\sigma(v) \simeq \mathcal{M}_{\sigma[\varphi]}(v)$$

Esp, $^\exists\varphi$ s.t. $\sigma_{D,A}[\varphi] = \sigma_{\mathbb{F},\vec{h}}$, then $\mathcal{M}_{\sigma_{D,A}}(v)$ is **Proj !**

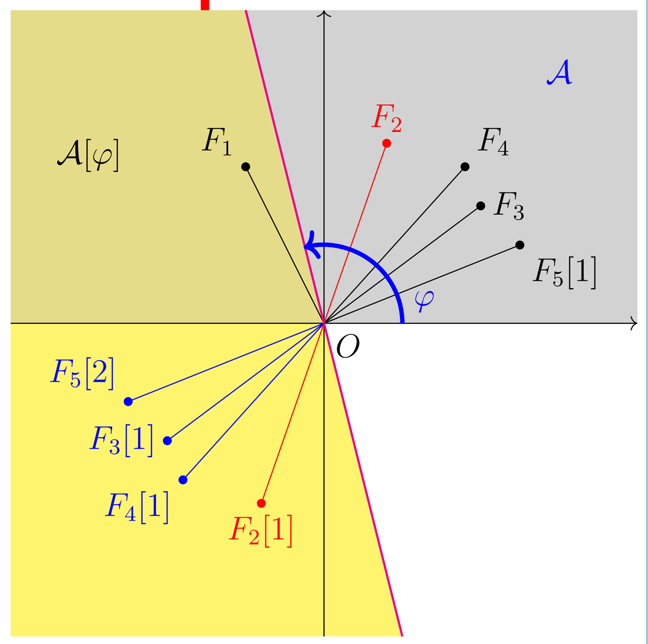

Figure 2: Rotation of BSC

All we need is quiver region

Def (Quiver Region)

$$QR_{\mathbb{F}} = \{\sigma \in \mathbf{Stab}_{div}(S) \mid {}^\exists\varphi \in \mathbb{R},\ \text{s.t. } \sigma[\varphi] = \sigma_{\mathbb{F},\vec{h}}\}$$

Want : $^\exists\mathbb{E}_1, \ldots, \mathbb{E}_r$: strong FECs s.t.

$$\mathbf{Stab}_{div}(S) \subset \bigcup_{i=1,\ldots,r} QR_{\mathbb{F}_i}$$

↓ By Wall & Chamber Structure of $\mathrm{Stab}(S)$
"Bertram's nested wall Theorem" (Figure 3)

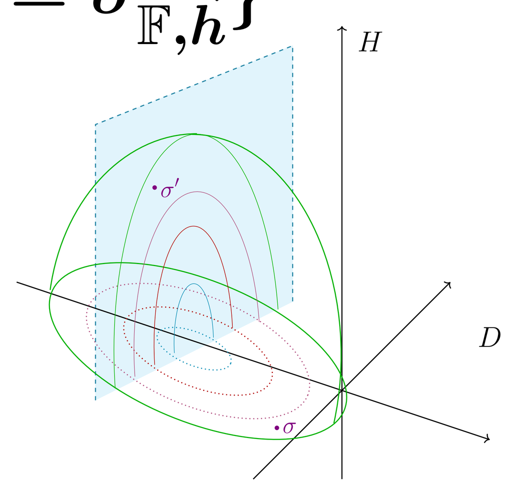

Figure 3: Bertram's Nested Wall Theorem
σ, σ' are in the same chamber $\mathcal{C}$, $\mathcal{M}_\sigma(v) \simeq \mathcal{M}_{\sigma'}(v)$

Enough To Show : $^\forall A \in \mathrm{Amp}(S)$

$$\lim_{\epsilon \to 0}\left((N^1(S) \times \epsilon A) \cap \bigcup_{i=0}^{r} QR_{\mathbb{F}_i}\right) \supset \textbf{An unit cube in } \mathbb{R}^3$$

Novelties of Research

- $\rho(S)$ is **Greater** than any other known cases.
- **More FECs are needed** (In Known cases, **at most two** FECs were sufficient).
  Length (or $\mathrm{rk}(K_0)$) up ⇝ Quiver Region small ⇝ more FECs are needed

Known Cases by the quiver region method

Projective Plane $\mathbb{P}^2$ (ABCH) $\quad$ $\mathbb{P}^1 \times \mathbb{P}^1$ (Arcara-Miles) $\quad$ $\mathrm{Bl}_p\,\mathbb{P}^2$ (Arcara-Miles)

Calculations

We need 14 strong FECs...

Strong Exceptional Collections	Dual Collection
$(\mathcal{O}, \mathcal{O}(C_1), \mathcal{O}(C_2), \mathcal{O}(H), \mathcal{O}(C_1+C_2))$	$(\mathcal{O}(-H)[2], \mathcal{O}(E)\|_E[1], \mathcal{O}(-C_2)[1], \mathcal{O}(-C_1)[1], \mathcal{O})$
$(\mathcal{O}, \mathrm{E}'_{02}, \mathcal{O}(C_1), \mathcal{O}(C_2), \mathcal{O}(-H))$	$(\mathcal{O}(-C_1-C_2)[2], \mathcal{O}_{E_2}(-1)[1], \mathcal{O}_{E_1}(-1)[1], \mathcal{O}(-H)[1], \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(E_1), \mathcal{O}(E_2), E_{2,1,1,2}(H), \mathcal{O}(H))$	$(\mathcal{O}(-C_1-C_2)[2], \mathcal{O}(-E)[1], \mathcal{O}_{E_2}(-1), \mathcal{O}_{E_1}(-1), \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(E), \mathcal{O}(C_2), \mathcal{O}(C_1), \mathcal{O}(C_1+C_2))$	$(\mathcal{O}(-E)[2], \mathcal{O}(-E_1)[1], \mathcal{O}(-E_2)[1], \mathcal{O}_E(-1), \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(E), E^1_3, \mathcal{O}(C_1), \mathcal{O}(C_2))$	$(\mathcal{O}(-C_1-H)[2], \mathcal{O}(-C_2-H)[2], \mathcal{O}(-H)[1], \mathcal{O}_E(-1), \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(E), E^1_3, \mathcal{O}(CC_1), \mathcal{O}(C_2))$	$(\mathcal{O}(-C_1-H)[2], \mathcal{O}(-C_1-H)[2], \mathcal{O}_E(-1)[1], \mathcal{O}(-C_1-C_2)[1], \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(C_2-C_1), G^1, \mathcal{O}(C_2-E), \mathcal{O}(C_2)))$	$(\mathcal{O}(-C_1-H)[2], \mathcal{O}_E(-1)[1], \mathcal{O}(-C_1)[1], \mathcal{O}(C_2-C_1), \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(C_1-C_2), G^2, \mathcal{O}(C_1-E), \mathcal{O}(C_1)))$	$(\mathcal{O}(-C_2-H)[2], \mathcal{O}_E(-1)[1], \mathcal{O}(-C_2)[1], \mathcal{O}(C_1-C_2), \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(C_2-C_1), G'^1, F'^1, \mathcal{O}(C_2)))$	$(\mathcal{O}(-C_1-H)[2], \mathcal{O}(-C_1+E)[1], \mathcal{O}_E(-1), \mathcal{O}(C_2-C_1), \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(C_1-C_2), G'^2, F'^2, \mathcal{O}(C_1)))$	$(\mathcal{O}(-C_2-H)[2], \mathcal{O}(-C_2+E)[1], \mathcal{O}_E(-1), \mathcal{O}(C_1-C_2), \mathcal{O})$
$(\mathcal{O}, G^3, F^3, I'^3, \mathcal{O}(C_1))$	$(\mathcal{O}(-C_2-H)[2], \mathcal{O}(-2C_1-C_2)[2], \mathcal{O}_{E_2}(-1)[1], \mathcal{O}(-C_1-H)[1], \mathcal{O})$
$(\mathcal{O}, G^4, F^4, I'^4, \mathcal{O}(C_2))$	$(\mathcal{O}(-C_1-H)[2], \mathcal{O}(-C_1-2C_2)[2], \mathcal{O}_{E_1}(-1)[1], \mathcal{O}(-C_2-H)[1], \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(E_2), G'^5, F''^5, \mathcal{O}(C_1))$	$(\mathcal{O}(-C_2-H)[2], \mathcal{O}(-2C_1-C_2)[2], \mathcal{O}(-C_1-C_2)[1], \mathcal{O}_{E_2}(-1), \mathcal{O})$
$(\mathcal{O}, \mathcal{O}(E_1), G'^6, F''^6, \mathcal{O}(C_2))$	$(\mathcal{O}(-C_1-H)[2], \mathcal{O}(-C_1-2C_2)[2], \mathcal{O}(-C_1-C_2)[1], \mathcal{O}_{E_1}(-1), \mathcal{O})$

Table 4: Strong exceptional collection and its dual collection.

Conditions determining the Quiver Region of $\mathbb{F}'$

$\mathbb{F}' = (\mathcal{O}(-C_1-C_2)[2], \mathcal{O}_{E_2}(-1)[1], \mathcal{O}_{E_1}(-1)[1], \mathcal{O}(-H)[1], \mathcal{O})$

1. $\mathcal{O}, \mathcal{O}(-C_1-C_2)[1] \in \mathcal{A}$
2. $\beta(\mathcal{O}(-H)[1]) > \beta(\mathcal{O}_{E_i}(-1)), \beta(\mathcal{O}(-C_1-C_2)[1])$
3. $\beta(\mathcal{O}) > \beta(\mathcal{O}(-H))$(if $\mathcal{O}(-H) \in \mathcal{A}$)
4. $\beta(\mathcal{O}) > \beta(\mathcal{O}_{E_i}(-1)), \beta(\mathcal{O}_{E_1}(-1)), \beta(\mathcal{O}(-C_1-C_2)[1])$(if $\mathcal{O}(-H)[1] \in \mathcal{A}$)

The longer the FEC, the greater the number of conditionals. As a result, the quiver region becomes smaller, and more FECs are needed.

Figures of Quiver Regions

$$\lim_{\epsilon\to 0}\left(\{\sigma_{xC_1+yC_2+zE,\epsilon H} \mid x, y, z \in \mathbb{R}\} \cap \bigcup_{i=0}^{r} QR_{\mathbb{F}_i}\right)$$

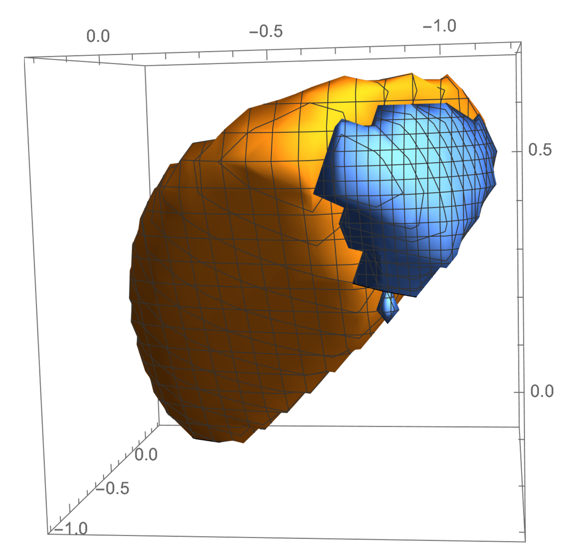

Figure 5: Quiver region of $\mathbb{F}'$ in (x, y, z)-space

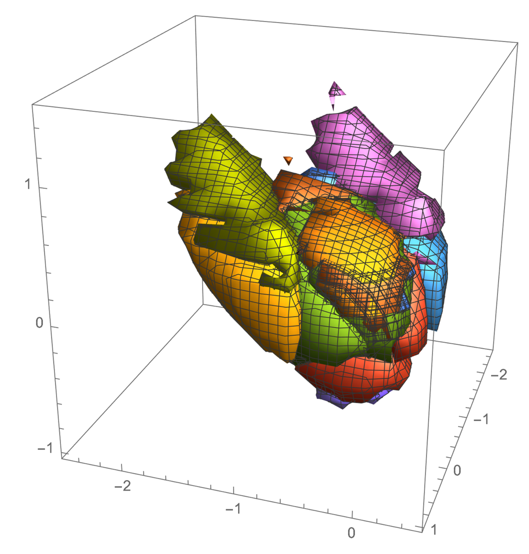

Figure 6: Quiver region that is combined of all in Table 4.

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