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Gelfand–Tsetlin polytopes and random contractions away from the limiting shape. (*)

Benoît Collins (1) and Anthony Metcalfe (2)

Abstract. — In this paper, we consider a sequence of selfadjoint matrices A_n having a limiting spectral distribution as $n \to \infty$, and we consider a sequence of full flags $\{0 \leqslant p_1^{(n)} \leqslant \ldots \leqslant p_i^{(n)} \leqslant \ldots \leqslant 1_n\}$ chosen at random according to the uniform measure on full flag manifolds. We are interested in the behaviour of the extremal eigenvalues of $p_i^{(n)} A_n p_i^{(n)}$. This problem is known to be equivalent to the study of uniform probability measures on Gelfand–Tsetlin polytopes. Our main results consist in explicit uniform estimates for extremal eigenvalues, and the fact that an outlier behavior has an exponentially small probability. This problem is of intrinsic interest in random matrix theory, but it was motivated from a problem in Quantum Information Theory, which we discuss. The proofs rely on a reinterpretation of the problem with the help of determinantal point processes and the techniques are based on steepest descent analysis.

Résumé. — Dans cet article, nous nous intéressons à une suite de matrices autoadjointes A_n possédant une distribution spectrale lorsque $n \to \infty$, et nous étudions une suite de drapeaux complets $\{0 \leqslant p_1^{(n)} \leqslant \ldots \leqslant p_i^{(n)} \leqslant \ldots \leqslant 1_n\}$ choisis au hasard selon la loi uniforme sur les varietes drapeaux complètes. Nous nous intéressons au comportement des valeurs propres extrêmes de $p_i^{(n)} A_n p_i^{(n)}$. Il est connu que ce problème est équivalent à l'étude de la mesure de probabilité uniforme sur des polytopes de Gelfand–Tsetlin. Notre résultat principal consiste en des estimées uniformes pour des valeurs propres extrémales, et le fait que les outliers sont de probabilité exponentiellement petite. Ce problème revêt un interêt intrinsèque en matrices aléatoires; par ailleurs, il trouve une motivation dans des questions d'information quantique que nous évoquons aussi. Les preuves se fonde sur une interpretation du problème a l'aide de processus de points déterminantaux, et les techniques reposent sur de l'analyse de type « steepest descent ».

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1. Introduction

1.1. Two facets of the same problem

A (weak) Gelfand–Tsetlin pattern is an n-tuple, $(y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in \mathbb{R} \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^n$, which satisfies the constraints

$$y_1^{(r+1)} \geqslant y_1^{(r)} \geqslant y_2^{(r+1)} \geqslant y_2^{(r)} \geqslant \cdots \geqslant y_r^{(r)} \geqslant y_{r+1}^{(r+1)},$$

for all $r \in \{1, \dots, n-1\}$. We refer to Subsection 2.1 for precise definitions and properties. The study of this subset of $\mathbb{R}^{n(n+1)/2}$ is very natural and has led to many deep results. For example, if $y^{(n)}$ is fixed, the collection of weak Gelfand–Tsetlin patterns form a polytope, and the study of the uniform probability measure on it is the object of many research results. We refer for example to [11, 12, 13] and references therein.

For the above uniform measure and under some assumptions on n, j and $y^{(n)}$ to be specified subsequently, it is known that some regions of $\mathbb{R}$ are highly unlikely to have elements $y_i^{(j)}$. While the description of these zones is well understood, quantifying the un-likelihood remained to be studied and it is one purpose of this paper to provide answers to this problem.

Let us now turn to the following random matrix problem. For selfadjoint matrices A_n, we consider a sequence of full flags $\{0 \leqslant p_1^{(n)} \leqslant \dots \leqslant p_i^{(n)} \leqslant \dots \leqslant 1_n\}$. Recall that a full flag is a maximal sequence of (selfadjoint) projections whose images are increasing for the inclusion order. In particular, in our setup, $rk(p_i^{(n)}) = i$, $\mathrm{Im}(p_i^{(n)}) \subset \mathrm{Im}(p_{i+1}^{(n)})$. The collection of full flags is a compact subset of n-tuples of matrices, on which unitary matrices act transitively by global conjugation, therefore there exists a unique invariant probability measure on full flags. We consider a random maximal flag according to this measure and we are interested in the joint set of eigenvalues of $p_i^{(n)} A_n p_i^{(n)}$. It is well-known ([3]) that this yields a Gelfand–Tsetlin pattern, provided that we denote by $y_1^{(i)} \geqslant \cdots \geqslant y_i^{(i)}$ the eigenvalues of the matrix $p_i^{(n)} A_n p_i^{(n)}$ corresponding to eigenvectors in $\mathrm{Im}(p_i^{(n)})$). In addition, its distribution is the uniform measure discussed above.

In this paper, we actually focus on the behaviour of the extremal eigenvalues of $p_i^n A_n p_i^n$. This unexpected connection allows to exploit properties from both facets to derive analytic estimates. For example, the fact that the uniform measure can be seen as the push forward of a measure on the unitary group implies some Gaussian concentration for each $y_i^{(j)}$ (see e.g. the book [1]) typically, there exists constants C, c such that for any n and for any $\varepsilon > 0$,

$$P(|y_i^{(j)} - E(y_i^{(j)})| \geqslant \varepsilon) \leqslant C \exp(-nc\varepsilon^2) \tag{1.1}$$

Such estimates are far from obvious from the study of uniform measure in polytopes in general (see for example partial results in the special case of random polytopes [21]) and they hint at the fact that the Gelfand–Tsetlin polytope has an exceptional behaviour.

1.2. Motivations from Quantum Information theory

Quantum Information Theory (often abbreviated by QIT in this paper) questions the information theoretic possibilities and limitations of using quantum protocols, e.g. quantum measurements and quantum channels. It has made very important progress in the last decades with a need for ever increasingly involved mathematics. In particular, random techniques have proven to be very useful for solving important problems, such as the problem of additivity of the Minimum Output Entropy.

Let us recall here briefly this problem. For further details, we refer to [9]. A *Quantum Channel* Φ is a map $M_n(\mathbb{C}) \to M_k(\mathbb{C})$ that is linear, preserves the trace, and such that for any l,

$$\Phi \otimes \mathrm{Id}_l : M_n(\mathbb{C}) \otimes M_l(\mathbb{C}) \to M_k(\mathbb{C}) \otimes M_l(\mathbb{C})$$

takes a positive matrix to a positive matrix (the map Φ is said to be *completely positive*). A *density matrix* is a positive matrix of trace 1, and for ρ a density matrix, its von Neumann entropy is $H(\rho) = -\sum \lambda_i(\rho)\log(\lambda_i(\rho))$, where $\lambda_1(\rho) \geqslant \lambda_2(\rho) \geqslant \dots$ are the eigenvalues of ρ. Here, the entropy function $x \log x : (0,1) \to \mathbb{R}_-$ is extended by continuity to $[0,1]$ and takes value 0 at 0 and 1. The *Miminum Output Entropy* (aka MOE) of a quantum channel Φ is

$$H_{\min}(\Phi) = \min_{\rho \text{ density matrix}} H(\Phi(\rho)),$$

and the problem of additivity asks whether it is true, for any Φ_1, Φ_2 quantum channels,

$$H_{\min}(\Phi_1 \otimes \Phi_2) = H_{\min}(\Phi_1) + H_{\min}(\Phi_2).$$

The importance of the question relies in the fact that a systematic equality implies the additivity of the classical capacity of quantum channels (i.e. the amount of classical information that can be sent through quantum channels is additive). This result has been proved to be false, i.e. there exist quantum channels Φ_1, Φ_2 such that $H_{\min}(\Phi_1 \otimes \Phi_2) < H_{\min}(\Phi_1) + H_{\min}(\Phi_2)$, see [16] and [17] for important preliminary results. However all constructions so far rely on the probabilistic method, i.e. on finding adequate sequences of random channels that satisfy the strict inequality with high probability. No non-random example is known at this point. Actually, it is very difficult, if not impossible, to estimate the size of matrices involved in creating a counterexample. While some strategies [2, 6, 15, 16] might in principle yield dimensions that can actually be described numerically, they yield extremely small violations. On the other hand, the strategy known to yield the best violation [4, 5], while giving an optimal estimate on the output (iff more than 183), makes it even more difficult to estimate the required dimension for the input.

Let us now outline why this dimension estimate is difficult. The results of [4, 5] rely on the fact that the largest eigenvalue of random matrix models converge almost surely. Typically, the matrix models involved are as follows:

$$p(A \otimes 1_n)p$$

where $A \in M_k$ is selfadjoint deterministic and p is a random uniform projection in $M_k \otimes M_n$ of rank approximately tkn (for a fixed $t \in (0,1]$). The spectrum of such an operator has been known since Voiculescu to converge almost surely to the free contraction of the spectral distribution of A by the relative dimension of p. The operator norm of this object is called $\|A\|_t$. Since this part is just a motivation, but not essential to the main results, we refer to [4, 5] for a thorough introduction and detail. In the core of this paper, we will not use the notation $\|A\|_t$ and rather study the Gelfand Tsetlin cone globally, so here, to link the topics, we will just note that

$$\|A\|_t = \sup\{x, (x,t) \in \mathcal{L}\}, \tag{1.2}$$

provided that the eigenvalues of A correspond to the top eigenvalues of the Gelfand Tsetlin cone. In the above equation, for the definition of $\mathcal{L}$, we refer to Definition 2.5 in the body of the manuscript. The papers of [4, 8] are the first ones that prove that the largest eigenvalue converges almost surely to $\|A\|_t$. However, nothing is known about the speed of convergence, except in the notable case where A itself is a projection, [7] but the techniques at hand in [4, 8] do not allow us to quantify the speed of convergence.

On the other hand, the set of eigenvalues of $p(A \otimes 1_n)p$ is known to be a determinantal point process. Such a determinantal point process is actually a particular case of a more general determinantal point process on Gelfand–Tsetlin patterns, as per Defosseux' results [10]. For us, this potential of applications to mathematical physics was a compelling motivation to undertake in this paper a systematic study of the top elements in the Gelfand Tsetlin cone.

The large dimension limit study of this determinantal point process has been initiated by the second author and his coauthors, with very fine asymptotic results inside the spectrum and at the boundary [11, 12, 13, 19]. In this respect, our paper is a continuation of the aforementioned papers. The main result is Theorem 2.16. Since it is quite technical, instead of stating it here, let us mention that it means that uniformly,

$$P(|y_1^{(j)} - f(j/n))| \geqslant \varepsilon) \leqslant C\exp(-nh(\varepsilon))$$

for some constants C and a strictly increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $h(0) = 0$. For a precise statement, we refer to Theorem 2.16.

A seemingly technical, yet necessary contribution of our work is to replace $E(y_1^{(j)})$ that appears for example in (1.1) by an explicit $f(j/n)$. Specifically, under reasonable assumptions, such as in the case of Subsection 2.2 (i.e. the case that motivated us in QIT, and that satisfies all technical assumptions of Theorem 2.16), one rules out the possibility of a tame fluctuation with respect to the mean or median, but misbehaved with respect to a limiting quantity (for example due to the mean or median converging too slowly towards a limit). As of today, all these computations are possible only thanks to the determinantal structures and the algebra and steepest descent analysis behind. Note also that in principle, our results allow us to systematically compute $h(\varepsilon)$. For the sake of keeping things within a reasonable pages number, we do not discuss this question systematically in this manuscript. This question will be discussed with completely different methods, highly specific to the largest eigenvalue, in a future work of F. Parraud.

To close this introduction, we would like to make the following remark. Many advanced analytical techniques have proved to be very useful towards solving problems in quantum information theory. This includes notably random matrix theory, but also large deviation theory, free probability theory, large dimensional convex analysis. We hope that this paper will also serve as an invitation to consider saddle point methods, determinantal point process, and possibly Riemann Hilbert techniques as possible additional mathematical techniques in the toolbox that can be used in quantum information theory

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1.3. Aims, structure and assumptions

In this section, to mitigate reader confusion, we give a brief description of the aims and structure of the paper, and the assumptions to be used in each section. The mathematical objects referenced in this section will be defined where appropriate.

The main result of this paper, Theorem 2.16, concerns explicit uniform estimates for extremal eigenvalues. The eigenvalues form Gelfand–Tsetlin patterns of particles, and we prove an exponentially small probability for the local asymptotic behaviour of the relevant eigenvalues/particles. We use steepest descent analysis to obtain the explicit bounds. This is highly technical: First we must understand the global asymptotic behaviour. We then use this understanding to identify the global asymptotic region which contains the extremal eigenvalues. Finally, we perform a steepest descent analysis within that region to understand the local asymptotic behaviour.

Steepest descent analysis is powerful but involved by nature, and by far the most complex part of such an analysis is proving the existence of appropriate contours of steepest descent/ascent. Additionally, steepest descent authors are normally only interested in proving convergence, and not in the explicit bounds we obtain in Theorem 2.16. These bounds, essential for our intended purpose, necessitates that we find exact contours of descent/ascent (see Definition 4.6), a problem greatly more complex than simply proving existence. We must also prove explicit bounds at each step of the calculation. The length of this paper reflects these unavoidable technical obstacles.

The assumptions at work throughout much of the paper are, in fact, weaker than those ultimately used in our main result, Theorem 2.16. Although the stronger assumptions of Theorem 2.16 are sufficient for our intended application to QIT, many of the steepest descent related results hold in greater generality, and we try to be as general as possible where we can in the hope that the steepest descent related results will trigger subsequent interest in random matrix theory.

Section 2 contains the mathematical preliminaries of the steepest descent problem, and a statement of the main result, Theorem 2.16. We give the minimum amount of information in order to state Theorem 2.16 without ambiguity, but leave the definitions of some quantities and statements of some results until later in the paper where they can more naturally be introduced. Section 2.4 applies Theorem 2.16 to an example relevant to QIT.

Section 3 examines the global asymptotic behaviour of the Gelfand–Tsetlin patterns. The assumptions here are quite broad, and stated at the beginning of the section, to be of greater interest to the random matrix theory community. We identify the *liquid region*, $\mathcal{L}$. Metcalfe, [19], proved universal bulk asymptotic behaviour in $\mathcal{L}$ using steepest descent analysis. We next identity the *edge*, $\mathcal{E}$, a natural subset of $\partial\mathcal{L}$ where steepest descent analysis suggests universal edge asymptotic behaviour, and perhaps other novel universal asymptotic behaviours (see Remarks 2.8 and 2.13). This is beyond the scope of this paper. Finally, we identify $\mathcal{O}$, the region in which we perform the steepest descent analysis in this paper. We then restrict our scope with the additional assumption that $\mu[\{b\}] > 0$ (see Lemma 3.10 and Section 3.3), since it is sufficient for our applications to QIT, and since it allows us to obtain a simpler global description of $\mathcal{O}$ (see Figure 2.2) and to simplify the setup of the steepest descent analysis.

Section 4 examines the local asymptotic behaviour around a fixed point $(\chi, \eta) \in \mathcal{O}$ using steepest descent techniques. The assumptions used are stated clearly at the beginning of the section. As stated above we assume that $\mu[\{b\}] > 0$, and we assume that u_n, r_n, v_n, s_n are defined as in (2.23). Note, Theorem 2.16 additionally assumes that $r_n = s_n$, which trivially gives $\phi_{r_n,s_n}(u_n, v_n) = 0$. This additional assumption is not otherwise used, and all asymptotic results of Section 4 hold without it. We use this condition as it is sufficient for our applications to QIT, and it avoids an involved asymptotic analysis of $\phi_{r_n,s_n}(u_n, v_n)$ for general r_n and s_n. Nevertheless, we conjecture that the steepest descent techniques of Section 4 are sufficient to examine the asymptotic behaviour of $\phi_{r_n,s_n}(u_n, v_n)$ for general r_n and s_n, and that Theorem 2.16 holds in this case also. The assumption $\mu[\{b\}] > 0$ can also be weakened. Indeed, whenever $\mu[\{b\}] = 0$ and $(\chi, \eta) \in \mathcal{O}$, $f'_{(\chi,\eta)}$ has either 2 distinct roots of multiplicity 1 in $(b, +\infty)$ and no other roots in $(b, +\infty)$, or simply 1 distinct root of multiplicity 1 in $(b, +\infty)$ and no other roots in $(b, +\infty)$. The geometric interpretation of $\mathcal{O}$ is therefore more complex than that shown in Figure 2.2. Regardless, the asymptotic techniques in Section 4 prove that Theorem 2.16 holds whenever 2 distinct roots of multiplicity 1 exist. The other case would require a more detailed analysis, and is beyond the scope of this paper.

Section 5 is a necessary technical examination of the behaviour of the roots of the relevant steepest descent functions. The assumptions needed are again weaker than those in Theorem 2.16. Indeed, only assumptions about the behaviour of the asymptotic measure, μ, are required, and these are stated clearly at the beginning of the section. Note, it is necessary to understand the behaviour of the roots in their entirety, and not just in the interval where the steepest descent analysis is carried out, as Theorem 5.2 employs a subtle counting argument. We exhaust all possible root behaviours, some of which are not directly related to the asymptotic situations in this paper, for the sake of completeness and general interest in the random matrix theory community.

We finish with Section 6 that contains applications to random geometry and QIT.

2. Mathematical preliminaries

2.1. The determinantal structure of Gelfand–Tsetlin patterns

A Gelfand–Tsetlin pattern of depth n is an n-tuple, $(y^{(1)}, y^{(2)}, \dots, y^{(n)}) \in \mathbb{R} \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^n$, which satisfies the interlacing constraint

$$y_1^{(r+1)} \geqslant y_1^{(r)} > y_2^{(r+1)} \geqslant y_2^{(r)} > \cdots \geqslant y_r^{(r)} > y_{r+1}^{(r+1)},$$

for all $r \in \{1, 2, \dots, n-1\}$, denoted $y^{(r+1)} \succ y^{(r)}$. Equivalently, this can be considered as an interlaced configuration of $\frac{1}{2}n(n+1)$ particles in $\mathbb{R} \times \{1, 2, \dots, n\}$ by placing a particle at position $(u, r) \in \mathbb{R} \times \{1, 2, \dots, n\}$ whenever u is an element of $y^{(r)}$. An example of such a configuration is given in Figure 2.1. Note, the particles obtained from $y^{(r)}$, for all $r \in \{1, 2, \dots, n\}$, are referred to as the particles on *row* r of the interlaced configuration.

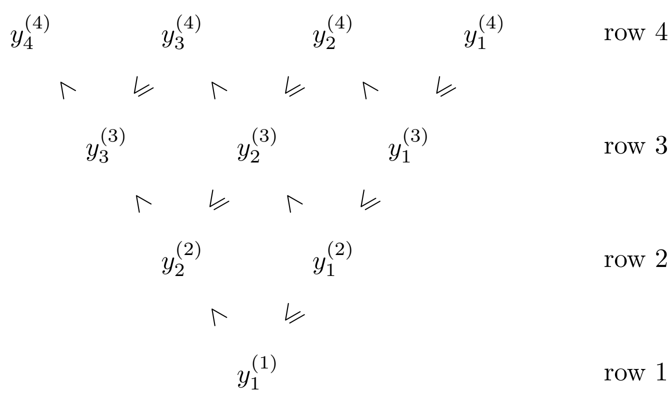

Figure 2.1. A visualisation of a Gelfand-Tsetlin pattern of depth 4.

For each $n \geqslant 1$, fix $x^{(n)} \in \mathbb{R}^n$ with $x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)}$. Let Ω_n represent the set of Gelfand–Tsetlin patterns of depth n with the particles on row n in the deterministic positions defined by $x^{(n)}$, and let ν_n represent the uniform probability measure on Ω_n:

$$\begin{aligned} &\mathrm{d}\nu_n[y^{(1)},\dots,y^{(n)}] \\ &= \frac{1}{Z_n}\cdot\begin{cases}\delta_{x^{(n)}}(y^{(n)})\mathrm{d}y^{(n)}\mathrm{d}y^{(n-1)}\dots\mathrm{d}y^{(1)} & \text{if } y^{(n)}\succ y^{(n-1)}\succ\cdots\succ y^{(1)},\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

where Z_n is a normalisation constant. Let $E_n := \mathbb{R}\times\{1,2,\dots,n\}$ and $N := \frac{1}{2}n(n+1)$, and recall the above equivalence of Ω_n as a set of configurations of N particles in E_n. (Ω_n,ν_n) is therefore equivalent to a probability space on configurations of N particles in E_n. Such probability spaces are commonly referred to as *random point fields.* Baryshnikov, [3], showed that this field arises naturally as an *eigenvalue minor process*: $y^{(n)} = x^{(n)}$ are the fixed eigenvalues of a random Hermitian matrix of size n with unitarily invariant distribution, and $y^{(r)}$ are the random eigenvalues of the principal minor of size r for all $r\in\{1,2,\dots,n-1\}$ (consisting of the first r rows and columns).

The above random point field was studied in Metcalfe, [19], and we now recall some important properties. First, for each $m\leqslant N$ define a measure, $\mathbb{M}_m$, on E_n^m by:

$$\mathbb{M}_m[B] := \mathbb{E}\left[\sum_{1\leqslant i_1\neq i_2\neq\cdots\neq i_m\leqslant N} 1_{\{\omega\in\Omega_n:(\omega_{i_1},\omega_{i_2},\dots,\omega_{i_m})\in B\}}\right],$$

for any Borel subset $B\subset E_n^m$, where the expectation is with respect to ν_n. Note, $\mathbb{M}_m[B]$ is the expected number of m-tuples of particles from Ω_n that are contained in B. In particular note that, when $m=1$ and $B=A\times\{r\}$ for any Borel $A\subset\mathbb{R}$ and $r\in\{1,2,\dots,n\}$, $\mathbb{M}_1[B]=\mathbb{M}_1[A\times\{r\}]$ is the expected number of particles on row r that are contained in A.

In [19] it is shown, for all $m\in\{1,2,\dots,n\}$ and Borel subsets $B\subset E_n^m$, that

$$\mathbb{M}_m[B] = \int_B \det[K_n((u_i,r_i),(u_j,r_j))]_{i,j=1}^m \mathrm{d}\lambda^m[(u,r)],$$

for some function $K_n : E_n^2\to\mathbb{C}$, where λ is the direct product of Lebesgue measure (on $\mathbb{R}$) with counting measure (on $\{1,2,\dots,n\}$). In words, the Radon–Nikodym derivative of $\mathbb{M}_m$ with respect to the reference measure λ^m exists, and is given by a determinant of a function of pairs of particle positions. Such random point fields are called *determinantal*, and the function $K_n : E_n^2\to\mathbb{C}$ is called the *correlation kernel.* In particular note that, when $m=1$ and $B=A\times\{r\}$ for any Borel $A\subset\mathbb{R}$ and $r\in\{1,2,\dots,n\}$,

$$\mathbb{M}_1[B] = \mathbb{M}_1[A\times\{r\}] = \int_A K_n((u,r),(u,r))\mathrm{d}u,$$

where integration is with respect to Lebesgue measure. Therefore, the expected number of particles on row r is a measure on $\mathbb{R}$ which is absolutely continuous with respect to Lebesgue measure, with density given by $u \mapsto K_n((u,r),(u,r))$ for all $u \in \mathbb{R}$.

Assumption 2.1. — *Let μ be a probability measure on $\mathbb{R}$ with compact support,* $\mathrm{Supp}(\mu) \subset [a,b]$ *with $b > a$ and $\{a,b\} \subset \mathrm{Supp}(\mu)$. Assume,*

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{x_i^{(n)}} \to \mu \text{ weakly.}$$

Then, rescaling the Gelfand–Tsetlin patterns vertically by $\frac{1}{n}$, the bulk of the rescaled particles asymptotically lie in $[a,b]\times[0,1]$ as $n \to \infty$. Indeed, as we shall see, the asymptotic bulk lies in a natural open subset of $[a,b]\times[0,1]$, which we denote below by $\mathcal{L}$. We provide global descriptions of $\mathcal{L}$ which arise naturally from steepest descent considerations (see Theorem 3.2), some examples of which are given in Figure 2.3. The local asymptotic behaviour of particles near a fixed point, $(\chi,\eta) \in [a,b]\times[0,1]$, is studied by considering $K_n((u_n,r_n),(v_n,s_n))$ as $n \to \infty$, where $\{(u_n,r_n)\}_{n\geqslant 1} \subset \mathbb{R}\times\{1,2,\dots,n-1\}$ and $\{(v_n,s_n)\}_{n\geqslant 1} \subset \mathbb{R}\times\{1,2,\dots,n-1\}$ satisfy:

$$\left(u_n, \frac{r_n}{n}\right) = (\chi,\eta) + o(1) \text{ and } \left(v_n, \frac{s_n}{n}\right) = (\chi,\eta) + o(1) \text{ as } n \to \infty. \quad (2.1)$$

The main result of this paper, Theorem 2.16, can then be stated at a high level as follows:

Theorem 2.2. — *Assume $\mu[\{b\}] > 0$. Then there exists an open subset ($\mathcal{O}$) in the lower right corner of $[a,b]\times[0,1]$, which lies outside the asymptotic bulk ($\mathcal{L}$). A global description of $\mathcal{O}$ arises naturally from steepest descent considerations. Moreover, the following is satisfied: Assume that $(u_n,\frac{r_n}{n})$ and $(v_n,\frac{s_n}{n})$ are contained in neighbourhoods of $\mathcal{O}$ whose description also arise naturally from steepest decent considerations. Take $r_n = s_n$ for all n so that the particles are on the same level of the Gelfand–Tsetlin patterns. Then $K_n((u_n,r_n),(v_n,s_n))$ decays exponentially as $n \to \infty$. Moreover, we obtain explicit bounds on the rates of decay, and explicit conditionals which describe how big we should take n (Definition 2.14 and Lemma 2.15).*

With the above result, and in particular the explicit bounds and description of n, we aimed to find explicit exponentially decaying bounds for the expected number of particles in subsets of $\mathcal{O}$, and over sets of the asymptotic measure μ. However the bounds we obtained were very complex, and this goal proved to be intractable. Instead, we consider an example calculation in Section 2.4 where $\mu := \frac{1}{4}\delta_1 + \frac{3}{4}\delta_{-1}$, and we take specific choices of $x^{(n)}, (v_n,s_n), (u_n,r_n)$. The bulk $\mathcal{L}$, and $\mathcal{O}$, for this example are depicted in

Figure 2.4. We obtain Corollary 2.17, which can be stated at a high level as follows:

Corollary 2.3. — *Take $\mu, x^{(n)}, (v_n, s_n), (u_n, r_n)$ as described in the previous paragraph. Fix $l \geqslant 2$. Then $[.5, .99] \times \{(1 - \frac{1}{l})\frac{1}{4}\}$ is a subset of $\mathcal{O}$, is the horizontal line depicted in Figure 2.4, and the closest vertical distance between this and the asymptotic bulk is $1/4l$. Moreover, we can find $C > 0$, and independent integers N, L for which the following is satisfied for all $l \geqslant L$ and $n \geqslant N$:*

$$\mathbb{M}_1[[.5, .99] \times \{n\eta\}] < Cl \ \exp\left(-n\frac{5}{12\sqrt{6}}\left(\frac{1}{l}\right)^{\frac{3}{2}}\right).$$

As stated above, we ultimately wished to find explicit values for C, N, L. However, while explicit values can in principle be obtained from Theorem 2.16, and the conditionals in Definition 2.14 and Lemma 2.15, providing these proved to be impractical giving the already considerable length of this paper.

Let us now continue with the analysis. Metcalfe, [19], gives the following expression for K_n:

$$K_n((u, r), (v, s)) = \widetilde{K}_n((u, r), (v, s)) - \phi_{r,s}(u, v), \tag{2.2}$$

for all $(u, r), (v, s) \in \mathbb{R} \times \{1, 2, \ldots, n-1\}$, where

$$\widetilde{K}_n((u, r), (v, s)) = \sum_{j=1}^{n} 1_{(x_j^{(n)} > u)} \frac{(x_j^{(n)} - u)^{n-r-1}}{(n-r-1)!} \frac{\partial^{n-s}}{\partial v^{n-s}} \prod_{i \neq j} \left(\frac{v - x_i^{(n)}}{x_j^{(n)} - x_i^{(n)}} \right),$$

and

$$\phi_{r,s}(u, v) := 1_{(v>u)} \cdot \begin{cases} 0 & \text{when } s \leqslant r, \\ 1 & \text{when } s = r+1, \\ (v-u)^{s-r-1}/(s-r-1)! & \text{when } s > r+1. \end{cases}$$

Next take the particle positions as in (2.1) for some fixed $(\chi, \eta) \in [a, b] \times [0, 1]$. Note, whenever $r_n \in \{1, 2, \ldots, n-2\}$, (2.2) and the Residue Theorem give,

$$K_n((u_n, r_n), (v_n, s_n)) = \frac{(n-s_n)!}{(n-r_n-1)!} J_n - \phi_{r_n, s_n}(u_n, v_n), \tag{2.3}$$

where

$$J_n := \frac{1}{(2\pi i)^2} \int_{c_n} \mathrm{d}w \int_{C_n} \mathrm{d}z \, \frac{1}{w-z} \frac{(z-u_n)^{n-r_n-1}}{(w-v_n)^{n-s_n+1}} \prod_{i=1}^{n} \left(\frac{w - x_i^{(n)}}{z - x_i^{(n)}} \right), \tag{2.4}$$

where c_n and C_n are any counter-clockwise simple closed contours which satisfy the following: C_n contains $\{x_j^{(n)} : x_j^{(n)} > u_n\}$ and does not contain any of $\{x_j^{(n)} : x_j^{(n)} < u_n\}$, and c_n contains v_n and C_n. Also note that for all $w, z \in \mathbb{C} \setminus \mathbb{R}$ the integrand can be written as:

$$\frac{\exp(nf_n(w) - n\widetilde{f}_n(z))}{w - z}, \tag{2.5}$$

where

$$f_n(w) := \frac{1}{n} \sum_{i=1}^{n} \log(w - x_i^{(n)}) - \frac{n - s_n + 1}{n} \log(w - v_n), \tag{2.6}$$

$$\widetilde{f}_n(z) := \frac{1}{n} \sum_{i=1}^{n} \log(z - x_i^{(n)}) - \frac{n - r_n - 1}{n} \log(z - u_n), \tag{2.7}$$

and log is the principal branch of the logarithm. Inspired by these, and by Assumption 2.1 and (2.1), define:

$$f_{(\chi,\eta)}(w) := \int_a^b \log(w - x)\mu[\mathrm{d}x] - (1 - \eta) \log(w - \chi), \tag{2.8}$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$.

Remark 2.4. — Let us point out that the above series of equations also appear frequently in free probability theory in the context of calculations on R-transforms and S-transforms. For a closely related example, we refer to [4].

Steepest descent analysis, and the above structure, intuitively suggests that the behaviour of $K_n((u_n, r_n), (v_n, s_n))$ as $n \to \infty$ depends on the roots of $f'_{(\chi,\eta)}$. In order to discuss this, we must first identify the largest possible domain of analytic extensions of $f'_{(\chi,\eta)}$. Recall that $\mathrm{Supp}(\mu) \subset [a, b]$ with $b > a$ and $\{a, b\} \subset \mathrm{Supp}(\mu)$, and $b \geqslant \chi \geqslant a$. Thus, for all $w \in \mathbb{C} \setminus \mathbb{R}$, it is natural to write,

$$\begin{aligned} f_{(\chi,\eta)}(w) = \int_{(\chi,b]} \log(w - x)\mu[\mathrm{d}x] - (1 - \eta - \mu[\{\chi\}]) \log(w - \chi) \\ + \int_{[a,\chi)} \log(w - x)\mu[\mathrm{d}x]. \end{aligned} \tag{2.9}$$

Also note, for all $w \in \mathbb{C} \setminus \mathbb{R}$, (2.8) gives,

$$f'_{(\chi,\eta)}(w) = C(w) - \frac{1 - \eta}{w - \chi}, \tag{2.10}$$

where $C : \mathbb{C} \setminus \mathrm{Supp}(\mu) \to \mathbb{C}$ denotes the *Cauchy* transform of μ:

$$C(w) := \int_a^b \frac{\mu[\mathrm{d}x]}{w - x}, \tag{2.11}$$

for all $w \in \mathbb{C}\backslash\mathrm{Supp}(\mu)$. Note that the above expression of $f'_{(\chi,\eta)}$ has a unique analytic extension to $\mathbb{C}\backslash(\mathrm{Supp}(\mu)\cup\{\chi\})$. Alternatively, for all $w \in \mathbb{C}\backslash\mathbb{R}$, (2.9) gives,

$$f'_{(\chi,\eta)}(w) = \int_{(\chi,b]} \frac{\mu[\mathrm{d}x]}{w-x} - \frac{1-\eta-\mu[\{\chi\}]}{w-\chi} + \int_{[a,\chi)} \frac{\mu[\mathrm{d}x]}{w-x}. \tag{2.12}$$

Finally note that the above expression has a unique analytic extension to the (possibly) larger set $\mathbb{C}\backslash(S_1 \cup S_2 \cup S_3)$, where $S_i := S_i(\chi,\eta)$ for all $i \in \{1,2,3\}$ are defined by:

$$\begin{aligned} S_1 &:= \mathrm{Supp}(\mu|_{(\chi,b]}), \\ S_2 &:= \begin{cases} \{\chi\} & \text{when } \mu[\{\chi\}] \neq 1-\eta, \\ \emptyset & \text{when } \mu[\{\chi\}] = 1-\eta, \end{cases} \\ S_3 &:= \mathrm{Supp}(\mu|_{[a,\chi)}). \end{aligned}$$

Note $S_1 = \emptyset$ when $b = \chi$, and $S_1 \neq \emptyset$ when $b > \chi$ (since $b \in \mathrm{Supp}(\mu)$). Similarly $S_3 = \emptyset$ when $\chi = a$, and $S_3 \neq \emptyset$ when $\chi > a$ (since $a \in \mathrm{Supp}(\mu)$). Theorem 5.2 characterises all possible behaviours of the roots of $f'_{(\chi,\eta)}$ in this domain. We now identify regions of $[a,b]\times[0,1]$ which can be defined by particular behaviours of the roots relevant to steepest descent considerations.

Definition 2.5. — *The liquid region, $\mathcal{L}$, is the set of all $(\chi,\eta) \in [a,b] \times [0,1]$ for which $f'_{(\chi,\eta)}$ has a root in $\mathbb{H} := \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$.*

Theorem 5.2 implies that $(\chi,\eta) \in \mathcal{L}$ if and only if $f'_{(\chi,\eta)}$ has exactly 1 root in $\mathbb{H}$, counting multiplicities. Steepest descent analysis then suggests, and Metcalfe [19] confirmed, universal bulk asymptotic behaviour whenever $(\chi,\eta) \in \mathcal{L}$: Fixing $(\chi,\eta) \in \mathcal{L}$, and choosing the parameters (u_n,r_n) and (v_n,s_n) of (2.1) appropriately, $K_n((u_n,r_n),(v_n,s_n))$ converges to the *Sine* kernel as $n \to \infty$. The Sine kernel is a so-called *universal* kernel as it has been observed asymptotically in the spectrum of other ensembles of random matrices and in related systems (see, for example, [14, 18, 20]).

Note, it is clear from the above observations that there is a natural map,

$$(\chi_{\mathcal{L}}(\cdot), \eta_{\mathcal{L}}(\cdot)) : \mathbb{H} \to \mathcal{L}.$$

Theorem 3.2 obtains an explicit expression for this and shows it is a homeomorphism, and so $\mathcal{L}$ is open. Lemma 3.4 examines $\partial\mathcal{L}$. Part (2) of that lemma shows that $(\chi_{\mathcal{L}}(\cdot), \eta_{\mathcal{L}}(\cdot)) : \mathbb{H} \to \mathcal{L}$ has a unique continuous extension to the following open subset of $\mathbb{R}$:

$$R := (\mathbb{R} \setminus \mathrm{Supp}(\mu)) \cup R_1, \tag{2.13}$$

where R_1 is the set of all *isolated atoms* of μ (see (2.14)). We also obtain an explicit expression for this extension (see (2.15)), denoted

$$(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \to \partial\mathcal{L} \subset [a,b] \times [0,1].$$

We now define:

Definition 2.6. — *The edge, $\mathcal{E} \subset \partial\mathcal{L}$, is the image of the curve $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \to \partial\mathcal{L} \subset [a,b] \times [0,1]$. The curve itself is called the edge curve.*

Theorem 3.7 give an alternative definition of $\mathcal{E}$ which is analogous to that of $\mathcal{L}$. Recall that $C : \mathbb{C} \setminus \mathrm{Supp}(\mu) \to \mathbb{C}$ denotes the *Cauchy* transform of μ (see (2.11)) and note that R is given by the disjoint union,

$$R := R^+ \cup R^- \cup R_0 \cup R_1, \tag{2.14}$$

where:

- $R^+ := \{t \in \mathbb{R} \setminus \mathrm{Supp}(\mu) : C(t) > 0\}$.
- $R^- := \{t \in \mathbb{R} \setminus \mathrm{Supp}(\mu) : C(t) < 0\}$.
- $R_0 := \{t \in \mathbb{R} \setminus \mathrm{Supp}(\mu) : C(t) = 0\}$.
- $R_1 := \{t \in \mathrm{Supp}(\mu) : \mu[\{t\}] > 0$ and there exists an open interval $I \subset \mathbb{R}$ with $t \in I$ and $I \setminus \{t\} \subset \mathbb{R} \setminus \mathrm{Supp}(\mu)\}$.

Lemma 3.4 then gives the following for $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \to \mathcal{E} \subset \partial\mathcal{L} \subset [a,b] \times [0,1]$:

$$\begin{aligned} &\chi_{\mathcal{E}}(t) = t + \frac{C(t)}{C'(t)} \text{ and } \eta_{\mathcal{E}}(t) = 1 + \frac{C(t)^2}{C'(t)} \\ &\qquad\qquad \text{when } t \in R^+ \cup R^- \cup R_0 = \mathbb{R} \setminus \mathrm{Supp}(\mu), \\ &\chi_{\mathcal{E}}(t) = t \text{ and } \eta_{\mathcal{E}}(t) = 1 \text{ when } t \in R_0, \\ &\chi_{\mathcal{E}}(t) = t \text{ and } \eta_{\mathcal{E}}(t) = 1 - \mu[\{t\}] \text{ when } t \in R_1. \end{aligned} \tag{2.15}$$

The above mapping is continuous in any open sub-interval of R. Next, define:

Definition 2.7. — *The edge, $\mathcal{E}$, is the disjoint union $\mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^- \cup \mathcal{E}_0 \cup \mathcal{E}_1$ where:*

- *$\mathcal{E}^+$ is the set of all $(\chi, \eta) \in [a,b] \times [0,1]$ for which $f'_{(\chi,\eta)}$ has a repeated root in $(\chi, +\infty) \setminus \mathrm{Supp}(\mu)$.*
- *$\mathcal{E}^-$ is the set of all $(\chi, \eta) \in [a,b] \times [0,1]$ for which $f'_{(\chi,\eta)}$ has a repeated root in $(-\infty, \chi) \setminus \mathrm{Supp}(\mu)$.*
- *$\mathcal{E}_0 := \{(\chi, \eta) \in [a,b] \times [0,1] : \chi \in R_0$ and $\eta = 1\}$.*
- *$\mathcal{E}_1 := \{(\chi, \eta) \in [a,b] \times [0,1] : \chi \in R_1$ and $\eta = 1 - \mu[\{\chi\}]\}$.*

In Section 3.2, we show that Definitions 2.6 and 2.7 are equivalent: First, starting with Definition 2.7, Corollary 5.3 (4) shows that $\{\mathcal{L}, \mathcal{E}^+, \mathcal{E}^-, \mathcal{E}_0, \mathcal{E}_1\}$

are pairwise disjoint. Moreover, Corollary 5.3 (1) shows that $f'_{(\chi,\eta)}$ has a unique real-valued repeated root in $\mathbb{R} \setminus \{\chi\}$ when $(\chi, \eta) \in \mathcal{E}^+ \cup \mathcal{E}^-$. Next, map each $(\chi, \eta) \in \mathcal{E}^+ \cup \mathcal{E}^-$ to the unique real-valued repeated root in $\mathbb{R} \setminus \{\chi\}$, and map each $(\chi, \eta) \in \mathcal{E}_0 \cup \mathcal{E}_1$ to χ. Then, Theorem 3.7 implies that this bijectively maps $\mathcal{E}$ to R, and the inverse of this map is the edge curve of Definition 2.6. Therefore the definitions are trivially equivalent, and (2.15) is a convenient parameterisation of the edge which is continuous in any open sub-interval of R, with the relevant root as parameter.

In Section 3.2, we also examine the geometric behaviour of the edge curve. First, fix $(\chi, \eta) \in \mathcal{E}$ and the corresponding $t \in R$. Next, let $m = m(t)$ denote the multiplicity of t as a root of $f'_{(\chi,\eta)}$. Then, Lemma 3.9 proves:

- The edge curve behaves like a parabola in neighbourhoods of (χ, η) when $t \in R^+ \cup R^-$ and $(\chi, \eta) \in \mathcal{E}^+ \cup \mathcal{E}^-$ and $m = 2$, when $t \in R_0$ and $(\chi, \eta) \in \mathcal{E}_0$ and $m = 1$, and when $t \in R_1$ and $(\chi, \eta) \in \mathcal{E}_1$ and $m = 0$.
- The edge curve behaves like an algebraic cusp of first order in neighbourhoods of (χ, η) when $t \in R^+ \cup R^-$ and $(\chi, \eta) \in \mathcal{E}^+ \cup \mathcal{E}^-$ and $m = 3$, and when $t \in R_1$ and $(\chi, \eta) \in \mathcal{E}_1$ and $m = 1$.

For clarity we state that the above exhaust all possibilities, and $m = 0$ means that $f'_{(\chi,\eta)}(t) \neq 0$.

Remark 2.8. — Steepest descent analysis, and the above root behaviour, suggests universal edge asymptotic behaviour whenever $(\chi, \eta) \in \mathcal{E}^+ \cup \mathcal{E}^-$:

- Fixing $(\chi, \eta) \in \mathcal{E}^+ \cup \mathcal{E}^-$, and choosing the parameters (u_n, r_n) and (v_n, s_n) appropriately, we conjecture that $K_n((u_n, r_n), (v_n, s_n))$ converges to the Airy kernel as $n \to \infty$ when $m = 2$.
- Fixing $(\chi, \eta) \in \mathcal{E}^+ \cup \mathcal{E}^-$, and choosing the parameters (u_n, r_n) and (v_n, s_n) appropriately, we conjecture that $K_n((u_n, r_n), (v_n, s_n))$ converges to the Pearcey kernel as $n \to \infty$ when $m = 3$.

Analogous behaviour when $m = 2$ is observed in Duse and Metcalfe, [13], for discrete interlaced Gelfand–Tsetlin patterns. Steepest descent analysis also suggests possible novel universal behaviours in the following situations:

- $(\chi, \eta) \in \mathcal{E}_0$ and $m = 0$.
- $(\chi, \eta) \in \mathcal{E}_1$ and $m = 0$.
- $(\chi, \eta) \in \mathcal{E}_1$ and $m = 1$.

See Duse and Johansson and Metcalfe, [11], for an analysis of edge points of discrete interlaced Gelfand–Tsetlin patterns which produced a previously

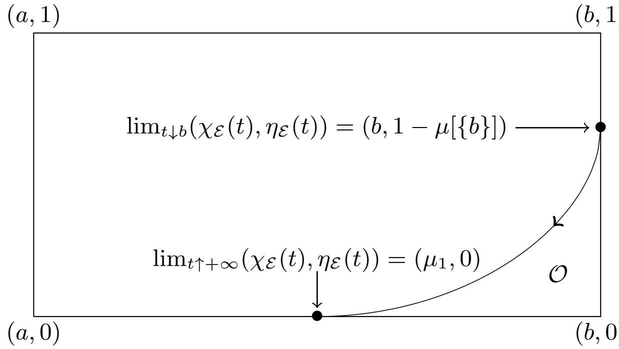

Figure 2.2. $(\chi_{\mathcal{E}}(\cdot),\eta_{\mathcal{E}}(\cdot)) : (b,+\infty) \to \mathcal{E}$ when $\mu[\{b\}] > 0$. The arrow represents the direction of the increasing parameter, and $\mu_1 := \int_a^b x\mu[\mathrm{d}x]$.

unobserved universal kernel, which we called the Cusp–Airy kernel. A further discussion on these points is beyond the scope of this paper.

Next define:

Definition 2.9. — *$\mathcal{O}$ is the set of all $(\chi,\eta) \in [a,b] \times [0,1]$ for which $\chi < b$, $\eta > 0$, and $f'_{(\chi,\eta)}$ has a root of multiplicity 1 in $(b,+\infty)$.*

Corollary 5.3 implies that $\{\mathcal{L},\mathcal{E},\mathcal{O}\}$ are pairwise disjoint.

The main result of this paper, Theorem 2.16, examines the local asymptotic behaviour in $\mathcal{O}$ using steepest descent techniques. The additional assumption that $\mu[\{b\}] > 0$ is used as it is sufficient for our application to QIT, and greatly simplifies the analysis. First, under this additional assumption, Lemma 3.10 shows that $(\chi_{\mathcal{E}}(\cdot),\eta_{\mathcal{E}}(\cdot)) : (b,+\infty) \to \mathcal{E}$ always behaves as in Figure 2.2. Next note, a simpler description of $\mathcal{O}$ exists when $\mu[\{b\}] > 0$ (see Definition 3.11): $\mathcal{O}$ is the set of all $(\chi,\eta) \in (a,b) \times (0,1)$ for which $1-\eta > \mu[\{\chi\}]$, $f'_{(\chi,\eta)}$ has 2 distinct roots of multiplicity 1 in $(b,+\infty)$, and $f'_{(\chi,\eta)}$ has no other roots in $(b,+\infty)$. This defines a map from $\mathcal{O}$ to

$$\angle := \{(t,s) \in (b,+\infty)^2 : t > s\}. \tag{2.16}$$

Theorem 3.12 shows that this map is a homeomorphism, and finds an explicit expression for the inverse of the homeomorphism, denoted,

$$(\chi_{\mathcal{O}}(\cdot),\eta_{\mathcal{O}}(\cdot)) : \angle \to \mathcal{O}.$$

Finally, Lemma 3.13 gives the following simple geometric interpretation of $\mathcal{O}$ in this case: $\mathcal{O}$ is that open subset of $(a,b)\times(0,1)$ in Figure 2.2 bounded by $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot))|_{(b,+\infty)}$ and the bounding box of $[a,b] \times [0,1]$. Steepest descent analysis, and the above root behaviour, suggest universal asymptotic behaviour whenever $\mu[\{b\}] > 0$ and $(\chi, \eta) \in \mathcal{O}$. Indeed, the correlation kernel should decay exponentially as $n \to \infty$. Theorem 2.16 confirms this intuition.

2.2. $\mathcal{L}$, $\mathcal{E}$ and $\mathcal{O}$ when μ is atomic

In this section we restrict to the case of purely atomic measures to illustrate the global geometric behaviours of $\mathcal{L}$, $\mathcal{E}$ and $\mathcal{O}$ as discussed in the previous section. Suppose,

$$\mu = \sum_{i=1}^{k} \alpha_i \delta_{b_i},$$

for some $k \geqslant 2$, $b_1, b_2, \dots, b_k \in \mathbb{R}$ with $b = b_1 > b_2 > \cdots > b_k = a$, and $\alpha_1, \alpha_2, \dots, \alpha_k > 0$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$. In this case, (2.13) easily gives $R = \mathbb{R}$. Definition 2.7 then implies that the edge curve is a map $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : \mathbb{R} \to \partial\mathcal{L} \subset [a,b] \times [0,1]$, and $\mathcal{E}$ is the image space of this map. Moreover, Theorem 3.7 implies that this map is bijective. Also, Definition 2.7 and Lemma 3.4 imply that $\partial\mathcal{L} = (\int_a^b x\mu[\mathrm{d}x], 0) \cup \mathcal{E}$. Finally, (2.11) and (2.15) imply that,

$$\begin{aligned} \chi_{\mathcal{E}}(t) &= t - \frac{\sum_{i=1}^k \alpha_i (t-b_i) \prod_{j \neq i} (t-b_j)^2}{\sum_{i=1}^k \alpha_i \prod_{j \neq i} (t-b_j)^2}, \\ \eta_{\mathcal{E}}(t) &= 1 - \frac{(\sum_{i=1}^k \alpha_i \prod_{j \neq i} (t-b_j))^2}{\sum_{i=1}^k \alpha_i \prod_{j \neq i} (t-b_j)^2}, \end{aligned} \tag{2.17}$$

for all $t \in \mathbb{R}$. In particular, note this gives $(\chi_{\mathcal{E}}(b_l), \eta_{\mathcal{E}}(b_l)) = (b_l, 1-\alpha_l) = (b_l, 1-\mu[\{b_l\}])$ for all atoms $b_l \in \{b_1, b_2, \dots, b_k\}$.

Suppose $k = 2$, $b = b_1 = 1$, $a = b_k = b_2 = -1$, and $\mu = \alpha\delta_1 + (1-\alpha)\delta_{-1}$ for some $\alpha \in (0,1)$. (2.17) then gives,

$$\begin{aligned} \chi_{\mathcal{E}}(t) &= t - \frac{\alpha(t-1)(t+1)^2 + (1-\alpha)(t+1)(t-1)^2}{\alpha(t+1)^2 + (1-\alpha)(t-1)^2}, \\ \eta_{\mathcal{E}}(t) &= 1 - \frac{(\alpha(t+1) + (1-\alpha)(t-1))^2}{\alpha(t+1)^2 + (1-\alpha)(t-1)^2}, \end{aligned} \tag{2.18}$$

for all $t \in \mathbb{R}$. The case where $\alpha = \frac{1}{4}$, i.e., $\mu = \frac{1}{4}\delta_1 + \frac{3}{4}\delta_{-1}$, is shown on the left of Figure 2.3. Note the atoms on the upper level at $(1,1)$ and $(-1,1)$, of size $\frac{1}{4}$ and $\frac{3}{4}$ respectively. In Metcalfe [19], it is shown that the sizes of these atoms decay linearly as η decreases. More exactly, since there is an the atom of size $\frac{1}{4}$ at the point $(1,1)$ on the upper level, then there is an atom

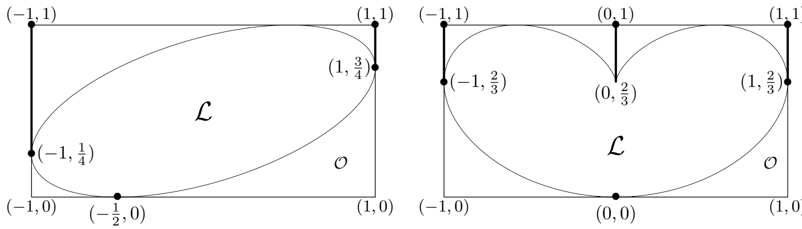

Figure 2.3. Left: $\mu = \frac{1}{4}\delta_1 + \frac{3}{4}\delta_{-1}$. $\mathcal{E}$ is composed of all points on $\partial\mathcal{L}$ except the lower tangent point $(\int_a^b x\mu[\mathrm{d}x], 0) = (-\frac{1}{2}, 0)$. Right: $\mu = \frac{1}{3}\delta_1 + \frac{1}{3}\delta_0 + \frac{1}{3}\delta_{-1}$. $\mathcal{E}$ is composed of all points on $\partial\mathcal{L}$ except the lower tangent point $(\int_a^b x\mu[\mathrm{d}x], 0) = (0, 0)$. In both cases $\lim_{t\to\pm\infty}(\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$ equals the lower tangent point, and parameter increases in the clockwise direction.

of size $\frac{1}{4} - (1 - \eta) = \eta - \frac{3}{4}$ at the point $(1, \eta)$ for all $1 \geqslant \eta > \frac{3}{4}$, and no atom at the point $(1, \eta)$ when $\frac{3}{4} \geqslant \eta \geqslant 0$. Similarly there is an atom of size $\frac{3}{4} - (1 - \eta) = \eta - \frac{1}{4}$ at the point $(-1, \eta)$ for all $1 \geqslant \eta > \frac{1}{4}$, and no atom at the point $(-1, \eta)$ when $\frac{1}{4} \geqslant \eta \geqslant 0$. The vertical solid lines in Figure 2.3 represent these atoms. Note that the edge curve is tangent to the boundary box at the points $(1, \frac{3}{4})$ and $(-1, \frac{1}{4})$ where the atoms "disappear".

The case where $\mu = \frac{1}{3}\delta_1 + \frac{1}{3}\delta_0 + \frac{1}{3}\delta_{-1}$ ($k = 3$, $b = b_1 = 1$, $b_2 = 0$, $a = b_3 = -1$) is shown on the right of Figure 2.3. Now we distinguish between the "outer" top level atoms at $(-1, 1)$ and $(1, 1)$, and the "inner" top level atom at $(0, 1)$. All top level atoms are of size $\frac{1}{3}$. Moreover, similar to before, the sizes of the "outer" atoms decay linearly as η decreases, and the edge curve is tangent to the boundary box at their points of disappearance ($(-1, \frac{2}{3})$ and $(1, \frac{2}{3})$). In contrast, though the size of the "inner" atom also decays linearly at η decreases, there is a cusp in the edge curve at the point of disappearance ($(0, \frac{2}{3})$).

For the general atomic measure, though the exact details are non-trivial and beyond the scope of this paper, a high level overview of analogous results is illuminating. We state the following without proof: We call the top level atoms at $(b_1, 1)$ and $(b_k, 1)$, of size α_1 and α_k respectively, the "outer" atoms. All other top level atoms when $k > 2$ are called the "inner" atoms. A similar decay in the size of all top level atoms occurs as η decreases. Moreover, the edge curve is tangent to the boundary box at the points of disappearance of the "outer" atoms ($(b_1, 1 - \alpha_1)$ and $(b_k, 1 - \alpha_k)$). Finally, each "inner" atom has an associated cusp, but the location of the cusp can be distinct from the point of disappearance of the atom.

Figure 2.3 also depicts $\mathcal{O}$ for these examples. $\mathcal{L}$, $\mathcal{E}$ and $\mathcal{O}$ for general atomic measures behaves similarly, irrespective of the number of cusps in the edge curve. For more general measures μ, however, $\mathcal{L}$, $\mathcal{E}$ and $\mathcal{O}$ are highly non-trivial to completely characterise. That is why, in this paper, we restrict to the case $\mu[\{b\}] > 0$. Then $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : (b, +\infty) \to \mathcal{E}$ and $\mathcal{O}$ always behave as described in the previous section, depicted in Figure 2.2.

2.3. Statement of main asymptotic result

In [19], Metcalfe examined the local asymptotic behaviour in $\mathcal{L}$ for the Gelfand–Tsetlin particle systems discussed in Section 2.1 as $n \to \infty$, and found universal bulk asymptotic behaviour. In [13], Duse and Metcalfe examined the local asymptotic behaviour of particles in $\mathcal{E}$ for analogous systems of discrete Gelfand–Tsetlin patterns and found universal edge asymptotic behaviour, and the authors have every expectation that analogous results hold in this case also (see Remark 2.13). The main asymptotic result of this paper, Theorem 2.16, concerns the local asymptotic behaviour of particles in neighbourhoods of $\mathcal{O}$ (see Definition 2.9), under the assumption that $\mu[\{b\}] > 0$.

As we discussed at the end of Section 2.1, the assumption that $\mu[\{b\}] > 0$ allows us to refine the definition of $\mathcal{O}$: $\mathcal{O}$ is the set of all $(\chi, \eta) \in (a, b) \times (0, 1)$ for which $1 - \eta > \mu[\{\chi\}]$, $f'_{(\chi,\eta)}$ has 2 distinct roots of multiplicity 1 in $(b, +\infty)$, and $f'_{(\chi,\eta)}$ has no other roots in $(b, +\infty)$. This defines a map from $\mathcal{O}$ to $\angle = \{(t, s) \in (b, +\infty)^2 : t > s\}$, a homeomorphism with inverse $(\chi_{\mathcal{O}}(\cdot), \eta_{\mathcal{O}}(\cdot)) : \angle \to \mathcal{O}$.

More specifically, Theorem 2.16 examines the asymptotic behaviour of $K_n((u_n, r_n), (v_n, s_n))$ (see (2.2)) under the following:

Assumption 2.10. — *Assume that $\mu[\{b\}] > 0$. Fix $(\chi, \eta) \in \mathcal{O} \subset (a, b) \times (0, 1)$ and the corresponding $(t, s) \in \angle$ with $(\chi, \eta) = (\chi_{\mathcal{O}}(t, s), \eta_{\mathcal{O}}(t, s))$. Also let $\{(u_n, r_n)\}_{n \geqslant 1} \subset \mathbb{R} \times \{1, 2, \ldots, n-1\}$ and $\{(v_n, s_n)\}_{n \geqslant 1} \subset \mathbb{R} \times \{1, 2, \ldots, n-1\}$ be sequences of particle positions which satisfy:*

$$\left(u_n, \frac{r_n}{n}\right) = (\chi, \eta) + o(1) \quad \text{and} \quad \left(v_n, \frac{s_n}{n}\right) = (\chi, \eta) + o(1) \quad \text{as } n \to \infty. \tag{2.19}$$

Consequently, for the remainder of this section it is natural to index $f'_{(\chi,\eta)}$ with $(t, s) \in \angle$ instead of (χ, η). In other words:

$$f_{(t,s)} := f_{(\chi,\eta)}.$$

Theorem 2.16 obtains the asymptotics of $K_n((u_n, r_n), (v_n, s_n))$ by performing a steepest descent analysis of the contour integral expression in (2.3). Note, Lemma 3.13 (4) implies that $f_{(t,s)}|_{(b,+\infty)}$ is real-valued, strictly increasing in (b, s), has a local maximum at s, is strictly decreasing in (s, t), has a local minimum at t, and is strictly increasing in $(t, +\infty)$. Also, (2.6), (2.7), (2.8), (2.19), and Assumption 2.1 imply that $f_n(t) - \widetilde{f}_n(s) \to f_{(t,s)}(t) - f_{(t,s)}(s) < 0$ as $n \to \infty$. (2.3), (2.4), (2.5), and intuition from steepest descent analysis then imply that $\exp(nf_n(t) - n\widetilde{f}_n(s)) \sim \exp(nf_{(t,s)}(t) - nf_{(t,s)}(s))$ will dominate the asymptotics as $n \to \infty$, i.e., exponential decay. The main result of this section, Theorem 2.16, proves this result, and gives exact bounds on the rate of convergence.

To state Theorem 2.16, we must motivate the choice of the $o(1)$ terms in (2.19) using steepest descent considerations. First recall that t and s are the unique roots of $f'_{(t,s)}$ in $(b, +\infty)$ and $t > s > b$ (see (2.16)). Next recall that $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)})$ are the deterministic positions of the particles on row n (see Section 2.1), $x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)}$, and $\mathrm{Supp}(\mu) \subset [a, b]$ (see Assumption 2.1). Note, an element of $x^{(n)}$ may act as a pole for the contour integral expression of (2.3), and so a problem may arise in the steepest descent analysis if these are not *eventually isolated* from the roots t and s. It is therefore convenient to assume:

Assumption 2.11. — *Assume that there exists an* $\xi = \xi(t, s) > 0$ *and* $N = N(t, s) \geqslant 1$ *for which* $t - 4\xi > s + 4\xi > s - 4\xi > b + 4\xi > x_1^{(n)} > b - 4\xi$ *and* $a + 4\xi > x_n^{(n)} > a - 4\xi$ *for all* $n > N$.

Next define,

$$
\begin{aligned}
P_n &:= \{x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}\}, \\
\mu_n &:= \frac{1}{n} \sum_{x \in P_n} \delta_x, \\
C_n(w) &:= \frac{1}{n} \sum_{x \in P_n} \frac{1}{w - x},
\end{aligned}
\tag{2.20}
$$

for all $n \geqslant 1$ and $w \in \mathbb{C} \setminus \mathrm{Supp}(\mu_n) = \mathbb{C} \setminus P_n$. Note, $\mu_n \to \mu$ weakly as $n \to \infty$ (see Assumption 2.1), $C_n : \mathbb{C} \setminus \mathrm{Supp}(\mu_n) \to \mathbb{C}$ is the Cauchy Transform of μ_n, and $\{t, s\} \subset (b + 4\xi, +\infty) \subset \mathbb{C} \setminus P_n$ for all $n > N$. Next, inspired by the explicit expression for $(\chi_{\mathcal{O}}(\cdot), \eta_{\mathcal{O}}(\cdot)) : \angle \to \mathcal{O}$ in Theorem 3.12, define:

Definition 2.12. — *Define* $\angle(\xi) := \{(T, S) \in (b + 4\xi, +\infty)^2 : T > S\}$. *Also, for all* $n > N$, *define*

$$
\chi_n(T, S) = \frac{TC_n(T) - SC_n(S)}{C_n(T) - C_n(S)} \quad \textit{and} \quad \eta_n(T, S) = 1 + \frac{C_n(T)C_n(S)(T - S)}{C_n(T) - C_n(S)},
$$

for all $(T,S) \in \angle(\xi)$. Note, Assumption 2.11 implies that $(t,s) \in \angle(\xi)$, and define $(\chi_n, \eta_n) := (\chi_n(t,s), \eta_n(t,s))$ for all $n > N$.

Note that, for any fixed $w \in (\mathbb{C} \setminus \mathbb{R}) \cup (b + 4\xi, +\infty)$ and $(T,S) \in \angle(\xi)$, Assumption 2.11 and Definition 2.12 imply that $C_n(w)$ and $\chi_n(T,S)$ and $\eta_n(T,S)$ are all well-defined for $n > N$. Moreover, since $\mu_n \to \mu$ weakly as $n \to \infty$:

$$C_n(w) \to C(w), \quad \chi_n(T,S) \to \chi_{\mathcal{O}}(T,S), \quad \eta_n(T,S) \to \eta_{\mathcal{O}}(T,S), \tag{2.21}$$

for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup (b + 4\xi, +\infty)$ and $(T,S) \in \angle(\xi)$.

Remark 2.13. — Given a fixed $(\chi, \eta) = (\chi_{\mathcal{O}}(t,s), \eta_{\mathcal{O}}(t,s)) \in \mathcal{O}$, (χ_n, η_n) can be considered as the equivalent non-asymptotic point. Since we have no control of the rate of convergence in Assumption 2.1, it is therefore natural to examine particles in neighborhoods of (χ_n, η_n) rather than neighborhoods of (χ, η), as we do in (2.23) below. Note, though beyond the scope of this paper, we stated at the beginning of this section that we expect universal edge asymptotic behaviour in neighborhoods of $\mathcal{E}$. Proceeding analogously, first define the equivalent non-asymptotic edge by simply replacing the asymptotic quantities in (2.15) with their non-asymptotic equivalents ($\chi_{n,\mathcal{E}}(t) := t + \frac{C_n(t)}{C_n'(t)}$ when $t \in R^+ \cup R^-$ etc), and then examine particles in neighborhoods of the non-asymptotic edge. See [13] for the analogous result for discrete Gelfand–Tsetlin patterns.

Next recall (see above) that $\exp(nf_n(t) - n\widetilde{f}_n(s))$ intuitively dominates the asymptotics as $n \to \infty$. Note, (2.6) and (2.7) imply that $f_n(t)$ depends on t, v_n, s_n, and $\widetilde{f}_n(s)$ depends on s, u_n, r_n. Intuition from steepest descent analysis then imply that v_n and s_n must depend on t, and u_n and r_n must depend on s. Moreover, (2.6), (2.7), (2.8), (2.19), and Lemma 3.13 (4) imply the following as $n \to \infty$:

- $f_n'(t) \to f_{(t,s)}'(t) = 0$ and $f_n''(t) \to f_{(t,s)}''(t) \neq 0$.
- $f_n'(s) \to f_{(t,s)}'(s) = 0$ and $f_n''(s) \to f_{(t,s)}''(s) \neq 0$.

(2.3), (2.4), (2.5), Definition 2.12, and intuition from steepest descent analysis then imply the following refinement of (2.19) for all $n > N$, a stronger assumption than Assumption 2.10:

$$\begin{aligned} \left(v_n, \frac{s_n}{n}\right) &= (\chi_n, \eta_n) + \mathbf{X}_n(t) n^{-\frac{1}{2}} \\ \left(u_n, \frac{r_n}{n}\right) &= (\chi_n, \eta_n) + \widetilde{\mathbf{X}}_n(s) n^{-\frac{1}{2}}, \end{aligned} \tag{2.22}$$

where $\mathbf{X}_n(t)$ is a vector depending on t, $\widetilde{\mathbf{X}}_n(s)$ is a vector depending on s, and $\|\mathbf{X}_n(t)\| = O(1)$ and $\|\widetilde{\mathbf{X}}_n(s)\| = O(1)$ for all n sufficiently large.

It remains to provide natural choices for $\mathbf{X}_n(t)$ and $\widetilde{\mathbf{X}}_n(s)$. Note, when $n > N$, we can proceed similarly to the proof of Lemma 3.14 to get:

$$(\chi_n(T,S),\eta_n(T,S)) = (\chi_n,\eta_n) + (T-t)\, c_{1,n}\, \mathbf{x_n}(s) + (S-s)\, c_{2,n}\, \mathbf{x_n}(t) \\ + O((|T-t| + |S-s|)^2),$$

for all $(T,S) \in \angle_\xi$ with $|T-t|$ and $|S-s|$ sufficiently small, where $\mathbf{x}_n(T) := (1, C_n(T))$ for all $T \in (b+2\xi, +\infty)$, and where $c_{1,n} = c_{1,n}(t,s) \to c_1(t,s) < 0$ and $c_{2,n} = c_{2,n}(t,s) \to c_2(t,s) < 0$ as $n \to \infty$ (see (3.18)). Note that the linear part of the above Taylor expansion is non-trivial, and is naturally decomposed in terms of the vectors $\mathbf{x}_n(t)$ and $\mathbf{x}_n(s)$. It therefore seems natural to assume the following stronger assumption than (2.22) for all $n > N$:

$$\begin{aligned} \left(v_n, \frac{s_n - 1}{n}\right) &= (\chi_n,\eta_n) + m_n \mathbf{x}_n(t) n^{-\frac12} + (y_{1,n}, y_{2,n}) n^{-1}, \\ \left(u_n, \frac{r_n + 1}{n}\right) &= (\chi_n,\eta_n) + \widetilde{m}_n \mathbf{x}_n(s) n^{-\frac12} + (\widetilde{y}_{1,n}, \widetilde{y}_{2,n}) n^{-1}, \end{aligned} \tag{2.23}$$

where $m_n, \widetilde{m}_n, y_{1,n}, y_{2,n}, \widetilde{y}_{1,n}, \widetilde{y}_{2,n} = O(1)$ for all n sufficiently large. Using $\frac{s_n-1}{n}$ and $\frac{r_n+1}{n}$ above, rather that simply $\frac{s_n}{n}$ and $\frac{r_n}{n}$, simplifies some expressions later.

Finally we give additional conditions on ξ and N (see Assumption 2.11) which are sufficient to obtain exact steepest descent bounds for $K_n((u_n,r_n),(v_n,s_n))$ for all $n > N$:

Definition 2.14. — *Assume that $\mu[\{b\}] > 0$, and fix $(\chi,\eta) \in \mathcal{O}$ and the corresponding $(t,s) \in \angle$ with $(\chi,\eta) = (\chi_{\mathcal{O}}(t,s), \eta_{\mathcal{O}}(t,s))$. Recall that $t > s > b > \chi > a$ and $1 > \eta > 0$ (see (2.16) and Definition 3.11). Also recall that $\chi_n = \chi_n(t,s)$, $\eta_n = \eta_n(t,s)$, $u_n = u_n(t,s)$, $r_n = r_n(t,s)$ $v_n = v_n(t,s)$ and $s_n = s_n(t,s)$ (see Definition 2.12 and (2.23)), and $\chi_n, u_n, v_n \to \chi$ and $\eta_n, \frac{r_n}{n}, \frac{s_n}{n} \to \eta$ as $n \to \infty$ (see (2.21), (2.23)). Finally recall (see Assumption 2.11) that exists an $\xi = \xi(t,s) > 0$ and $N = N(t,s) \geqslant 1$ for which $t-4\xi > s+4\xi > s-4\xi > b+4\xi > x_1^{(n)} > b-4\xi$ for all $n > N$. We first choose the above $\xi = \xi(t,s) > 0$ sufficiently small such that the following are also satisfied:*

- $t-4\xi > s+4\xi > s-4\xi > b+4\xi > b-4\xi > \chi+4\xi > \chi-4\xi > a+4\xi$.
- $1-2\xi > 1-\eta+2\xi > 1-\eta-2\xi > 2\xi$.

Next, given this new ξ, we choose the above $N = N(t,s) \geqslant 1$ sufficiently large such that the following are also satisfied for all $n > N$:

- $b+4\xi > x_1^{(n)} > b-4\xi$ *and* $a+4\xi > x_n^{(n)} > a-4\xi$,
- $\chi+4\xi > \{\chi_n, v_n, u_n\} > \chi-4\xi$,
- $1-\eta+2\xi > \{1-\eta_n, 1-\frac{s_n-1}{n}, 1-\frac{r_n+1}{n}\} > 1-\eta-2\xi$,

- $\{x \in P_n : x > v_n \vee u_n\} \neq \emptyset$ *and* $\{x \in P_n : x < v_n \wedge u_n\} \neq \emptyset$.

Above, $\alpha > \{x, y, z\} > \beta$ *denotes* $\alpha > x \vee y \vee z \geqslant x \wedge y \wedge z > \beta$. *Next, fix* $\theta \in (\frac{1}{3}, \frac{1}{2})$*, and choose the above* $N = N(t,s) \geqslant 1$ *sufficiently large such that the following are also satisfied for all* $n > N$*:*

- $n^{\frac{1}{3}-\theta} < \frac{1}{2}, n^{-\frac{1}{2}+\theta} < \frac{1}{2}, n^{-\theta} < \xi, n^{-\frac{1}{2}} < \frac{1}{2}\xi, |v_n - u_n| < \frac{1}{2}\xi$, $n^{-1} < \xi$.
- $n^{-\theta} < 2^{-6}(t-\chi)(t-b)^3(b-a)^{-1}|f''_{(t,s)}(t)|$.
- $n^{1-3\theta}(E_{2,n} + \widetilde{E}_{2,n}) < 1$*, where* $E_{2,n}, \widetilde{E}_{2,n}$ *are defined in Lemma 4.3(6) and (7).*

The above conditions on $N = N(t,s) \geqslant 1$ are still not yet sufficient. To obtain the remaining conditions we need to examine the root behaviour of $f'_{(t,s)}, f'_n, \widetilde{f}'_n$ more closely. We also consider the following "non-asymptotic" functions inspired by Definition 2.12 and by (2.10), (2.11), and (2.20):

$$f'_{(t,s),n}(w) := C_n(w) - \frac{1-\eta_n}{w-\chi_n} = \frac{1}{n}\sum_{x\in P_n}\frac{1}{w-x} - \frac{1-\eta_n}{w-\chi_n}, \tag{2.24}$$

for all $w \in \mathbb{C} \setminus (P_n \cup \{\chi_n\})$. The function $f_{(t,s),n}$ is unimportant and left undefined. Moreover, (2.6), (2.7), and (2.20) give,

$$f'_n(w) = C_n(w) - \frac{1-\frac{s_n-1}{n}}{w-v_n} = \frac{1}{n}\sum_{x\in P_n}\frac{1}{w-x} - \frac{1-\frac{s_n-1}{n}}{w-v_n}, \tag{2.25}$$

$$\widetilde{f}'_n(w) = C_n(w) - \frac{1-\frac{r_n+1}{n}}{w-v_n} = \frac{1}{n}\sum_{x\in P_n}\frac{1}{w-x} - \frac{1-\frac{r_n+1}{n}}{w-u_n}, \tag{2.26}$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$. We extend these functions analytically to $\mathbb{C} \setminus (P_n \cup \{v_n\})$ and $\mathbb{C} \setminus (P_n \cup \{u_n\})$ respectively. The following result will be proved in Section 4.1:

Lemma 2.15. — *Fix* $\xi = \xi(t,s) > 0$ *and* $N = N(t,s) \geqslant 1$ *as in Definition 2.14. Let* $B(t, 2\xi) \subset \mathbb{C}$ *represent the open ball of radius* 2ξ *centered on* t*, and similarly for* $B(s, 2\xi)$*. Then* $B(t, 2\xi)$ *and* $B(s, 2\xi)$ *are disjoint open subsets of* $(\mathbb{C} \setminus \mathbb{R}) \cup (b + 4\xi, +\infty)$*, and* $f'_{(t,s)}$ *is well-defined and analytic in* $B(t, 2\xi) \cup B(s, 2\xi)$*. Moreover:*

(1) $f'_{(t,s)}(t) = f'_{(t,s)}(s) = 0$.
(2) $f''_{(t,s)}(t) > 0$ *and* $f''_{(t,s)}(s) < 0$.
(3) t *and* s *are the unique roots of* $f'_{(t,s)}$ *in* $B(t, 2\xi)$ *and* $B(s, 2\xi)$ *respectively.*

Also, for all $n > N$, $f'_{(t,s),n}$ *is well-defined and analytic in* $B(t, 2\xi) \cup B(s, 2\xi)$*, and:*

(4) $f'_{(t,s),n}(t) = f'_{(t,s),n}(s) = 0$.

Moreover, we can choose the above $N = N(t,s) \geqslant 1$ *sufficiently large such that the following are also satisfied for all* $n > N$*:*

(5) $f''_{(t,s),n}(t) > \frac{1}{2} f''_{(t,s)}(t) > 0$ *and* $f''_{(t,s),n}(s) < \frac{1}{2} f''_{(t,s)}(s) < 0$.
(6) t *and* s *are the unique roots of* $f'_{(t,s),n}$ *in* $B(t,\xi)$ *and* $B(s,\xi)$ *respectively.*

Also, for all $n > N$, $B(t, 2n^{-\frac{1}{2}}) \subset B(t,\xi)$ *and* $B(s, 2n^{-\frac{1}{2}}) \subset B(s,\xi)$, f'_n *and* $\widetilde{f}'_n$ *are well-defined and analytic in* $B(t,2\xi) \cup B(s,2\xi)$*, and:*

(7) $|f'_n(t)|, |\widetilde{f}'_n(s)| = O(n^{-1})$, *and* $|f''_n(t) - f''_{(t,s),n}(t)|$, $|\widetilde{f}''_n(s) - f''_{(t,s),n}(s)| = O(n^{-\frac{1}{2}})$ *for all* n *sufficiently large (we give explicit bounds in the proof).*

Moreover, we can choose the above $N = N(t,s) \geqslant 1$ *sufficiently large such that the following are also satisfied for all* $n > N$*:*

(8) $f''_n(t) > \frac{1}{4} f''_{(t,s)}(t) > 0$ *and* $\widetilde{f}''_n(s) < \frac{1}{4} f''_{(t,s)}(s) < 0$.
(9) *Counting multiplicities,* f'_n *has exactly* 1 *root (denoted* t_n*) in* $B(t, n^{-\frac{1}{2}})$ *and exactly* 1 *root (denoted* s_n*) in* $B(s,\xi)$*. Also,* $t_n \in (t - n^{-\frac{1}{2}}, t + n^{-\frac{1}{2}}) \subset (t - \frac{\xi}{2}, t + \frac{\xi}{2})$ *and* $s_n \in (s - \xi, s + \xi)$.
(10) *Counting multiplicities,* $\widetilde{f}'_n$ *has exactly* 1 *root (denoted* $\widetilde{t}_n$*) in* $B(t,\xi)$ *and exactly* 1 *root (denoted* $\widetilde{s}_n$*) in* $B(s, n^{-\frac{1}{2}})$*. Also,* $\widetilde{t}_n \in (t-\xi, t+\xi)$ *and* $\widetilde{s}_n \in (s - n^{-\frac{1}{2}}, s + n^{-\frac{1}{2}}) \subset (s - \frac{\xi}{2}, s + \frac{\xi}{2})$.
(11) $f''_n(t_n) > \frac{1}{4} f''_{(t,s)}(t) > 0$ *and* $\widetilde{f}''_n(\widetilde{s}_n) < \frac{1}{4} f''_{(t,s)}(s) < 0$.

With the above lemma, we can finally state the main asymptotic result (proved in Section 4.3):

Theorem 2.16. — *Assume that* $\mu[\{b\}] > 0$*, and fix* $(\chi,\eta) \in \mathcal{O}$ *and the corresponding* $(t,s) \in \angle$ *with* $(\chi,\eta) = (\chi_{\mathcal{O}}(t,s), \eta_{\mathcal{O}}(t,s))$*. Define* u_n, r_n, v_n, s_n *as in* (2.23)*, fix* $\theta \in (\frac{1}{3}, \frac{1}{2})$*, and choose* $N = N(t,s) \geqslant 1$ *sufficiently large that the conditions of Definition 2.14 and the results Lemma 2.15 are both satisfied. Then, for all* $n > N$*,*

$$\left| nJ_n - \frac{\exp(nf_n(t) - n\widetilde{f}_n(s))}{4\pi(t-s)D_n\widetilde{D}_n} \right| < \frac{\exp(nf_n(t) - n\widetilde{f}_n(s))}{4\pi(t-s)D_n\widetilde{D}_n}\, n^{1-3\theta}\, F_n + \frac{\exp(nf_n(t) - n\widetilde{f}_n(s))}{t-s} \exp\Big(-\frac{1}{4} n^{1-2\theta} (D_n^2 \wedge \widetilde{D}_n^2)\Big) n^{1-\theta}\, G_n,$$

where J_n *is defined in* (2.4)*,* $F_n > 0$ *and* $G_n > 0$ *are defined in the proofs of Lemmas 4.8 and 4.9 (respectively) and satisfy* $F_n = O(1)$ *and* $G_n = O(1)$ *for* *all n sufficiently large, and $D_n := (\frac{1}{2}|f_n''(t)|)^{\frac{1}{2}} \geqslant 0$ and $\widetilde{D}_n := (\frac{1}{2}|\widetilde{f}_n''(s)|)^{\frac{1}{2}} \geqslant 0$. Finally, $\phi_{r_n,s_n}(u_n,v_n) = 0$ when $r_n = s_n$ for all $n > N$ (see (2.2)), and:*

$$K_n((u_n,r_n),(v_n,s_n)) = (1 - \frac{s_n}{n})\, nJ_n \quad \textit{when } r_n = s_n \textit{ for all } n > N.$$

Note that $f_n(t) - \widetilde{f}_n(s) \to f_{(t,s)}(t) - f_{(t,s)}(s) < 0$ as $n \to \infty$ (see (2.6), (2.7), (2.8), and Lemma 4.1(1)). Also note that $D_n^2 > \frac{1}{8}|f''_{(t,s)}(t)| > 0$ and $\widetilde{D}_n^2 > \frac{1}{8}|f''_{(t,s)}(s)| > 0$ for all $n > N$ (see Lemma 4.3(3)). Moreover, Definition 2.14 gives $n^{-1} < \xi$ and $1 - \frac{s_n - 1}{n} > 1 - \eta - 2\xi$ and $\xi < \frac{1}{4}(1-\eta)$ for all $n > N$, and so $1 > 1 - \frac{s_n}{n} > \frac{1}{4}(1-\eta) > 0$. Finally, recall that $\theta \in (\frac{1}{3}, \frac{1}{2})$. Theorem 2.16 thus shows that $|K_n((u_n,r_n),(v_n,s_n))|$ decays exponentially when $r_n = s_n$ as $n \to \infty$, and gives exact rates of decay.

Finally, write the denominator $(t-s)D_n\widetilde{D}_n$ of Theorem 2.16 as follows:

$$(t-s)D_n\widetilde{D}_n = (t-s)\frac{1}{2}(|f_n''(t)||\widetilde{f}_n''(s)|)^{\frac{1}{2}} = (t-s)^2\frac{1}{2}\left(\frac{|f_n''(t)|}{t-s}\frac{|\widetilde{f}_n''(s)|}{t-s}\right)^{\frac{1}{2}}.$$

Then, Lemma 4.1(4) shows that there exists natural bounds $c_1 = c_1(t,s) > 0$ and $c_2 = c_2(t,s) > 0$ for which the following is satisfied for all n sufficiently large:

$$c_1(t-s)^2 > (t-s)D_n\widetilde{D}_n > c_2(t-s)^2.$$

This demonstrates the natural dependence of the denominator on the term $t-s$. We will see this explicitly for an example in the next section.

2.4. Expected number of particles

Theorem 2.16 proves exponential decay for the correlation kernel in neighbourhoods of $\mathcal{O}$ as $n \to \infty$. Moreover, explicit bounds and rates of convergence have been obtained. However, it is clear that the bounds are very complex for the general case. In this section we consider an example calculation. We demonstrate how explicit bounds may be obtained in principle, but do not actually obtain these for brevity.

First we define the asymptotic measure of Assumption 2.1:

- $\mu := \frac{1}{4}\delta_1 + \frac{3}{4}\delta_{-1}$ (note, $1 = b > a = -1$).

With this μ, we will see below that $[.5, .99] \times (0, \frac{1}{4}) \subset \mathcal{O}$. Recall, Definition 3.11 and Theorem 3.12 imply that for each unique point in $\mathcal{O}$, there exists a corresponding unique point in $\angle = \{(t,s) : t > s > 1\}$ with $(\chi,\eta) = (\chi_{\mathcal{O}}(t,s), \eta_{\mathcal{O}}(t,s))$. We can therefore apply Theorem 2.16 to the following:

- Consider all $(\chi, \eta) \in [.5, .99] \times \{(1-\frac{1}{l})\frac{1}{4}\} \subset \mathcal{O}$ and all corresponding $(t, s) \in \angle$ with $(\chi, \eta) = (\chi_{\mathcal{O}}(t, s), \eta_{\mathcal{O}}(t, s))$, where $l \geqslant L \geqslant 2$ are integers.

Note that $\eta = (1 - \frac{1}{l})\frac{1}{4}$ for all above (χ, η), and recall that $t > s > 1$ for all above (t, s). The integer $L \geqslant 2$ will not be specified for brevity. However, we will see below that L can be fixed sufficiently large such that the relevant results hold for all $l \geqslant L$. We also adopt the following terminology for brevity: Whenever we say a statement holds for all l and corresponding pairs, we mean the statement holds for all $l \geqslant L$ and all $(\chi, \eta) \in [.5, .99] \times \{(1 - \frac{1}{l})\frac{1}{4}\}$ and all corresponding $(t, s) \in \angle$. Whenever we say a statement holds for any fixed l and all corresponding pairs, we mean if we fix any specific $l \geqslant L$, the statement holds for all $(\chi, \eta) \in [.5, .99] \times \{(1 - \frac{1}{l})\frac{1}{4}\}$ and all corresponding $(t, s) \in \angle$. Next choose parameters in Definition 2.14 for all l and all corresponding pairs:

- Fix $\theta := \frac{5}{12} \in (\frac{1}{3}, \frac{1}{2})$, $\xi > 0$, and integers $n \geqslant N \geqslant 1$.

Note, we do not specify explicit values for ξ and N for brevity. However, we show below that ξ can be fixed sufficiently small and N sufficiently large such that the requirements of Theorem 2.16 (Definition 2.14 and Lemma 2.15) are satisfied for any fixed $n \geqslant N$, and all l and corresponding pairs. We also demonstrate how, in principle, explicit values may be found. Next choose the remaining parameters of Theorem 2.16 for any fixed l and all corresponding pairs:

- Restrict the above n to integer multiples of $4l$.
- $x_1^{(n)} := 1$ and $x_i^{(n)} := 1 - (i-1)\frac{1}{n^2}$ for all $k \in \{2, 3, \dots, \frac{n}{4}\}$, and $x_n^{(n)} := -1$ and $x_i^{(n)} := -1 + (n-i)\frac{1}{n^2}$ for all $k \in \{\frac{n}{4} + 1, \frac{n}{4} + 2, \dots, n-1\}$: The particles on the top row of the Gelfand–Tsetlin pattern.
- $(u_n, r_n) := (\chi, n\eta)$ and $(v_n, s_n) := (\chi, n\eta)$: The parameters in (2.23). More exactly, in (2.23), we take $m_n = \widetilde{m}_n = 0$, $y_{1,n} = \widetilde{y}_{1,n} = n(\chi - \chi_n)$, $y_{2,n} = n(\eta - \eta_n) - 1$, and $\widetilde{y}_{2,n} = n(\eta - \eta_n) + 1$.

Note, the top level particles are distinct (a requirement of Section 2.1), and Assumption 2.1 is trivially satisfied. Note also, that r_n and s_n are integers as required, since $\eta = (1 - \frac{1}{l})\frac{1}{4}$ and n is a multiple of $4l$, and we will show below that the above choices of (u_n, r_n) and (v_n, s_n) satisfy the requirements of (2.23).

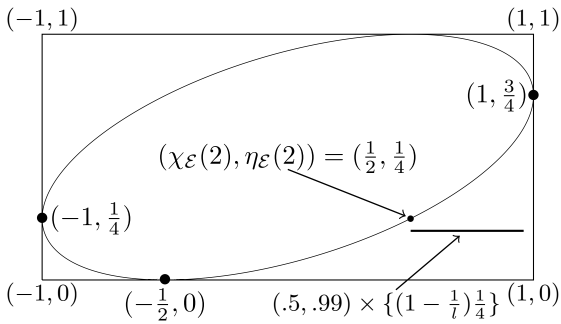

Figure 2.4. $\mathcal{L}$, $\mathcal{E}$ and $\mathcal{O}$ when $\mu = \frac{1}{4}\delta_1 + \frac{3}{4}\delta_{-1}$, and $[.5, .99] \times \{(1-\frac{1}{l})\frac{1}{4}\} \subset \mathcal{O}$. See Figure 2.3 for a more explicit description of $\mathcal{L}$, $\mathcal{E}$ and $\mathcal{O}$ in this case.

We then use Theorem 2.16 to estimate the following:

$$\begin{aligned} \mathbb{M}_1[[.5, .99] \times \{n\eta\}] &= \int_{.5}^{.99} K_n((\chi, n\eta), (\chi, n\eta)) \mathrm{d}\chi \\ &= \int_{.5}^{.99} K_n((u_n, r_n), (v_n, s_n)) \mathrm{d}\chi, \end{aligned} \tag{2.27}$$

where integration is with respect to Lebesgue measure. This is the expected number of particles on row $n\eta = n(1-\frac{1}{l})\frac{1}{4}$ that are contained in $[.5, .99]$ (see Section 2.1).

First recall (see Section 2.2) that $\mathcal{L}$, $\mathcal{E}$ and $\mathcal{O}$ for the above μ are shown in Figure 2.3 (reproduced in Figure 2.4), the edge curve $(\chi_\mathcal{E}(\cdot), \eta_\mathcal{E}(\cdot)) : \mathbb{R} \to \mathcal{E}$ is given by (2.18) with $\alpha = \frac{1}{4}$, the restriction $(\chi_\mathcal{E}(\cdot), \eta_\mathcal{E}(\cdot)) : (1, +\infty) \to \mathcal{E}$ is that lower right section of the edge curve in Figure 2.4 between $(1, \frac{3}{4})$ and $(-\frac{1}{2}, 0)$, and $\mathcal{O}$ is that open subset of $(-1, 1) \times (0, 1)$ bounded by $(\chi_\mathcal{E}(\cdot), \eta_\mathcal{E}(\cdot))|_{(1,+\infty)}$ and the bounding box of $[-1, 1] \times [0, 1]$. It follows that $(\frac{1}{2}, \frac{1}{4}) = (\chi_\mathcal{E}(2), \eta_\mathcal{E}(2))$ is a point of the lower right edge, and $[.5, .99] \times (0, \frac{1}{4}) \subset \mathcal{O}$. Then, $[.5, .99] \times \{(1-\frac{1}{l})\frac{1}{4}\}$ is a horizontal line in $\mathcal{O}$ for any $l \geqslant L$.

Consider the relevant asymptotic quantities in Theorem 2.16. Note, since $\mu = \frac{1}{4}\delta_1 + \frac{3}{4}\delta_{-1}$, (2.8) gives:

$$f_{(\chi,\eta)}(w) = \frac{1}{4}\log(w-1) + \frac{3}{4}\log(w+1) - (1-\eta)\log(w-\chi), \tag{2.28}$$

for all $(\chi, \eta) \in [-1, 1] \times [0, 1]$ and $w \in \mathbb{C} \setminus \{1, -1, \chi\}$, where log represents principal value. Note that $(.5, (1-\frac{1}{l})\frac{1}{4}) \in \mathcal{O}$ is the leftmost point of $[.5, .99] \times$

$\{(1-\frac{1}{l})\frac{1}{4}\}$ for any fixed l. Lemma 3.15 then gives the following for any fixed l and all corresponding pairs:

- $f_{(\chi,\eta)}(t) - f_{(\chi,\eta)}(s) < 0$ is maximised when $(\chi,\eta) = (.5,(1-\frac{1}{l})\frac{1}{4})$.
- $t > 1$ is minimised when $(\chi,\eta) = (.5,(1-\frac{1}{l})\frac{1}{4})$.
- $s > 1$ is maximised when $(\chi,\eta) = (.5,(1-\frac{1}{l})\frac{1}{4})$.

Let $(t_l, s_l) \in \angle$ denote the point in $\angle$ which corresponds $(.5,(1-\frac{1}{l})\frac{1}{4})$, i.e, $(.5,(1-\frac{1}{l})\frac{1}{4}) = (\chi_{\mathcal{O}}(t_l,s_l), \eta_{\mathcal{O}}(t_l,s_l))$. The above bounds thus imply the following for any fixed l and all corresponding pairs:

- $f_{(\chi,\eta)}(t) - f_{(\chi,\eta)}(s) < f_{(.5,(1-\frac{1}{l})\frac{1}{4})}(t_l) - f_{(.5,(1-\frac{1}{l})\frac{1}{4})}(s_l) < 0$.
- $t > t_l > 1$.
- $s_l > s > 1$.

We now apply Lemma 3.16 to analyse these further. Consider the corresponding points $2 \in (1,+\infty) = (b,+\infty)$ and $(\chi,\eta) = (\frac{1}{2},\frac{1}{4}) = (\chi_{\mathcal{E}}(2), \eta_{\mathcal{E}}(2)) \in \mathcal{E}$ (see Theorem 3.7). Note (2.28) gives $f'_{(\frac{1}{2},\frac{1}{4})}(2) = f''_{(\frac{1}{2},\frac{1}{4})}(2) = 0$ and $f'''_{(\frac{1}{2},\frac{1}{4})}(2) = \frac{1}{9}$. Then, since $l \geqslant L$, Lemma 3.16 implies that we can fix L sufficiently large such that the following is satisfied for any fixed $l \geqslant L$ and all corresponding pairs:

- $f_{(.5,(1-\frac{1}{l})\frac{1}{4})}(t_l) - f_{(.5,(1-\frac{1}{l})\frac{1}{4})}(s_l) < -\frac{5}{12\sqrt{6}}\,(\frac{1}{l})^{\frac{3}{2}}$.
- $2 + (\frac{6}{l})^{\frac{1}{2}} > t_l > 2 + \frac{1}{2}(\frac{6}{l})^{\frac{1}{2}} > 2$.
- $2 > 2 - \frac{1}{2}(\frac{6}{l})^{\frac{1}{2}} > s_l > 2 - (\frac{6}{l})^{\frac{1}{2}} > 1 + (\frac{6}{l})^{\frac{1}{2}}$.

The above then prove the following for any fixed l and all corresponding pairs:

- $f_{(\chi,\eta)}(t) - f_{(\chi,\eta)}(s) < -\frac{5}{12\sqrt{6}}\,(\frac{1}{l})^{\frac{3}{2}}$.
- $t - s > (\frac{6}{l})^{\frac{1}{2}}$.

The above also show the following for all l and corresponding pairs:

- $t > 2 > s$.

Next note that $(.99,(1-\frac{1}{l})\frac{1}{4}) \in \mathcal{O}$ is the rightmost point of $[.5,.99] \times \{(1-\frac{1}{l})\frac{1}{4}\}$ for any fixed l, and $(1-\frac{1}{l})\frac{1}{4} \geqslant (1-\frac{1}{L})\frac{1}{4}$ for all l. Lemma 3.15 then gives the following for all $l \geqslant L$ and corresponding pairs:

- $s > 1$ is minimised when $(\chi,\eta) = (.99,(1-\frac{1}{L})\frac{1}{4})$.
- $t > 1$ is maximised when $(\chi,\eta) = (.99,(1-\frac{1}{L})\frac{1}{4})$.

Note, (2.28) gives the following when $(\chi,\eta) = (.99, (1-\frac{1}{L})\frac{1}{4})$:

$$f_{(\chi,\eta)}(w) = \frac{1}{4}\log(w-1) + \frac{3}{4}\log(w+1) - \left(\frac{3}{4}+\frac{1}{4L}\right)\log(w-.99),$$

for all $w > 1$. Then, similar methods to those used in Lemma 3.15 easily give the following for all l and corresponding pairs for some constants $D, d > 0$:

- $D > t > s > 1 + d$.

In principle we can obtain explicit expressions for D and d, but we do not do so for brevity.

Next note, since $\mu := \frac{1}{4}\delta_1 + \frac{3}{4}\delta_{-1}$, (2.11) gives,

$$C(w) = \frac{1}{4}\frac{1}{w-1} + \frac{3}{4}\frac{1}{w+1},$$

for all $w > 1$. The above bounds then give the following for all l and corresponding pairs:

- $\frac{1}{4}\frac{1}{D-1} + \frac{3}{4}\frac{1}{D+1} < C(t) < \frac{1}{4}\frac{1}{2-1} + \frac{3}{4}\frac{1}{2+1} = \frac{1}{2}$.
- $\frac{1}{2} = \frac{1}{4}\frac{1}{2-1} + \frac{3}{4}\frac{1}{2+1} < C(s) < \frac{1}{4}\frac{1}{d} + \frac{3}{4}\frac{1}{d+2}$.

Moreover, for all l and corresponding pairs:

$$-\frac{C(t)-C(s)}{t-s} = \frac{1}{4}\frac{1}{(t-1)(s-1)} + \frac{3}{4}\frac{1}{(t+1)(s+1)}. \tag{2.29}$$

The above bounds then give the following for all l and all corresponding pairs:

- $-\frac{C(t)-C(s)}{t-s} > \frac{1}{4}\frac{1}{(D-1)(2-1)} + \frac{3}{4}\frac{1}{(D+1)(2+1)}$.
- $-\frac{C(t)-C(s)}{t-s} < \frac{1}{4}\frac{1}{(2-1)(d)} + \frac{3}{4}\frac{1}{(2+1)(d+2)}$.

Next consider $f''_{(\chi,\eta)}(t)$ and $f''_{(\chi,\eta)}(s)$ for all l and all corresponding pairs. Note, Lemma 4.1 (4) gives,

$$|f''_{(\chi,\eta)}(t)| = \int_{-1}^{1}\int_{-1}^{1} \frac{(t-s)(x-y)^2\mu[\mathrm{d}x]\mu[\mathrm{d}y]}{2C(s)(t-x)^2(t-y)^2(s-x)(s-y)},$$

$$|f''_{(\chi,\eta)}(s)| = \int_{-1}^{1}\int_{-1}^{1} \frac{(t-s)(x-y)^2\mu[\mathrm{d}x]\mu[\mathrm{d}y]}{2C(t)(s-x)^2(s-y)^2(t-x)(t-y)}.$$

The, since $\mu = \frac{1}{4}\delta_1 + \frac{3}{4}\delta_{-1}$,

$$\begin{aligned} |f''_{(\chi,\eta)}(t)| &= \frac{(t-s)\frac{3}{8}}{2C(s)(t-1)^2(t+1)^2(s-1)(s+1)}, \\ |f''_{(\chi,\eta)}(s)| &= \frac{(t-s)\frac{3}{8}}{2C(t)(s-1)^2(s+1)^2(t-1)(t+1)}. \end{aligned} \tag{2.30}$$

It is thus clear from the above bounds that we can find in principle explicit $D_1, D_2, d_1, d_2 > 0$ such that the following is satisfied for all l and corresponding pairs:

- $d_1 < |f''_{(\chi,\eta)}(t)|(t-s)^{-1} < D_1$.
- $d_2 < |f''_{(\chi,\eta)}(s)|(t-s)^{-1} < D_2$.

Next consider non-asymptotic quantities. Recall that $n \geqslant 1$ is a multiple of $4l$, and the above definition of $x^{(n)}$. (2.11) and (2.20) then give,

$$C(w) - C_n(w) = \frac{1}{n}\sum_{i=1}^{\frac{n}{4}}\left(\frac{1}{w-1} - \frac{1}{w-1+(i-1)\frac{1}{n^2}}\right) + \frac{1}{n}\sum_{i=\frac{n}{4}+1}^{n}\left(\frac{1}{w+1} - \frac{1}{w+1-(n-i)\frac{1}{n^2}}\right),$$

for all $w > 1$. It is thus clear from the above bounds that we can find in principle $B > 0$ such that the following is satisfied for all l and corresponding pairs:

- $|C(t) - C_n(t)| < \frac{B}{n}$.
- $|C(s) - C_n(s)| < \frac{B}{n}$.

We can similarly show that we can choose B such that the following is satisfied for all l and corresponding pairs:

- $|C'(t) - C'_n(t)| < \frac{B}{n}$ and $|C''(t) - C''_n(t)| < \frac{B}{n}$.
- $|C'(s) - C'_n(s)| < \frac{B}{n}$ and $|C''(s) - C'_n(s)| < \frac{B}{n}$.

Also, since $(v_n, s_n) = (\chi, n\eta)$, (2.6) and (2.28) give:

$$f_{(\chi,\eta)}(t) - f_n(t) = \frac{1}{n}\sum_{i=1}^{\frac{n}{4}}\Big(\log(t-1) - \log\Big(t-1+(i-1)\frac{1}{n^2}\Big)\Big) + \frac{1}{n}\sum_{i=\frac{n}{4}+1}^{n}\Big(\log(t+1) - \log\Big(t+1-(n-i)\frac{1}{n^2}\Big)\Big) + \frac{1}{n}\log(t-\chi).$$

The above bounds thus show that we can choose B and N (recall $n > N$) such that the following is satisfied for all l and corresponding pairs:

- $|f_{(\chi,\eta)}(t) - f_n(t)| < \frac{B}{n}$.

Similarly, since $(u_n, r_n) = (\chi, n\eta)$, (2.7), (2.28), and the above bounds show that we can choose B and N such that:

- $|f_{(\chi,\eta)}(s) - \widetilde{f}_n(s)| < \frac{B}{n}$.

Similarly, we can choose B and N such that:

- $|f'_{(\chi,\eta)}(t) - f'_n(t)| < \frac{B}{n}$ and $|f''_{(\chi,\eta)}(t) - f''_n(t)| < \frac{B}{n}$.
- $|f'_{(\chi,\eta)}(s) - \widetilde{f}'_n(s)| < \frac{B}{n}$ and $|f''_{(\chi,\eta)}(s) - \widetilde{f}''_n(s)| < \frac{B}{n}$.

Next note, for all l and all corresponding pairs, (2.20) gives:

$$\begin{aligned} - \frac{C_n(t) - C_n(s)}{t-s} &= \frac{1}{n} \sum_{i=1}^{\frac{n}{4}} \left(\frac{1}{(t - 1 + (i-1)\frac{1}{n^2})(s - 1 + (i-1)\frac{1}{n^2})} \right) \\ &\quad + \frac{1}{n} \sum_{i=\frac{n}{4}+1}^{n} \left(\frac{1}{(t + 1 - (n-i)\frac{1}{n^2})(s + 1 - (n-i)\frac{1}{n^2})} \right). \end{aligned}$$

Therefore, since $t > s > 1$ and $n \geqslant 1$,

$$-\frac{C_n(t) - C_n(s)}{t-s} > \frac{1}{4} \frac{1}{(t - 1 + \frac{1}{4})(s - 1 + \frac{1}{4})} + \frac{3}{4} \frac{1}{(t+1+1)(s+1+1)}.$$

The above bounds thus give the following for all l and corresponding pairs:

- $-\frac{C_n(t)-C_n(s)}{t-s} > \frac{1}{4} \frac{1}{(D-1+\frac{1}{4})(2-1+\frac{1}{4})} + \frac{3}{4} \frac{1}{(D+1+1)(2+1+1)}$.

Next recall (see Definition 2.12 and Theorem 3.12 that, $(\chi, \eta) = (\chi_{\mathcal{O}}(t,s), \eta_{\mathcal{O}}(t,s))$ and $(\chi_n, \eta_n) = (\chi_n(t,s), \eta_n(t,s))$. Definition 3.11 and Theorem 3.12 (replace μ by $\mu_n = \sum_i \delta_{x_i^{(n)}}$ etc) then give the following for all l and corresponding pairs:

- $1 = x_1^{(n)} > \chi_n > x_n^{(n)}$.
- $1 > \eta_n > 0$.

Moreover, the expressions for χ and χ_n give:

$$\begin{aligned} &(\chi_n - \chi)(t-s) \\ &= \frac{t-s}{C_n(t) - C_n(s)} \frac{t-s}{C(t) - C(s)} [-(C_n(t) - C(t))C(s) + (C_n(s) - C(s))C(t)]. \end{aligned}$$

It is thus clear from the above bounds that we can choose the $B > 0$ such that the following is satisfied for all l and all corresponding pairs:

- $|\chi_n - \chi|(t-s) < \frac{B}{n}$.

Thus, since $t-s > (\frac{6}{l})^{\frac{1}{2}}$ for any fixed l and all corresponding pairs (see above):

- $|\chi_n - \chi| < \frac{B\sqrt{l}}{n\sqrt{6}}$.

Similarly we can choose the $B > 0$ such that the following is satisfied for any fixed l and all corresponding pairs:

- $|\eta_n - \eta| < \frac{B\sqrt{l}}{n\sqrt{6}}$.

Next recall that $m_n = \widetilde{m}_n = 0$, $y_{1,n} = \widetilde{y}_{1,n} = n(\chi - \chi_n)$, $y_{2,n} = n(\eta - \eta_n) - 1$, and $\widetilde{y}_{2,n} = n(\eta - \eta_n) + 1$. The above bounds then give the following for any fixed l and all corresponding pairs, which we note trivially satisfy the requirements of (2.23):

- $|m_n| = |\widetilde{m}_n| = 0$.
- $|y_{1,n}| < B\sqrt{l/6}$ and $|\widetilde{y}_{1,n}| < B\sqrt{l/6}$.
- $|y_{2,n}| < 1 + B\sqrt{l/6}$ and $|\widetilde{y}_{2,n}| < 1 + B\sqrt{l/6}$.

Next consider the requirements of Definition 2.14 and Lemma 2.15. Recall that $\theta = \frac{5}{12} \in (\frac{1}{3}, \frac{1}{2})$, $v_n = u_n = \chi$, and $s_n = r_n = n\eta$. With these choices, and the above bounds, it is easy to see that $\xi > 0$ can be fixed sufficiently small, and $N \geqslant 1$ (recall $n > N$) can be fixed sufficiently large, such that all requirements are satisfied for all l and corresponding pairs. Moreover, we can in principle find explicit values but we do not do this for brevity.

Finally, we apply Theorem 2.16 to equation (2.27). Recall that $(u_n, r_n) = (v_n, s_n) = (\chi, n\eta)$, where $\eta = (1 - \frac{1}{l})\frac{1}{4}$, where $l \geqslant L$, $n \geqslant N$ is a multiple of $4l$, and $\theta = \frac{5}{12}$. We have shown above that the conditions of Theorem 2.16 are satisfied, and applying Theorem 2.16 for any fixed l we get:

$$\begin{aligned} &|K_n((\chi, n\eta), (\chi, n\eta))| \\ &\qquad < \frac{\exp(nf_n(t) - n\widetilde{f}_n(s))}{4\pi(t-s)D_n\widetilde{D}_n} + \frac{\exp(nf_n(t) - n\widetilde{f}_n(s))}{4\pi(t-s)D_n\widetilde{D}_n}\, n^{-\frac{1}{4}}\, F_n \\ &\qquad\quad + \frac{\exp(nf_n(t) - n\widetilde{f}_n(s))}{t-s}\ \exp\left(-\frac{1}{4}n^{\frac{1}{6}}(D_n^2 \wedge \widetilde{D}_n^2)\right) n^{\frac{7}{12}}\, G_n, \end{aligned} \tag{2.31}$$

where $F_n > 0$ and $G_n > 0$ are defined in the proofs of Lemmas 4.8 and 4.9 (respectively) and satisfy $F_n = O(1)$ and $G_n = O(1)$ for all n sufficiently large, and $D_n\widetilde{D}_n = \frac{1}{2}(|f_n''(t)||\widetilde{f}_n''(s)|)^{\frac{1}{2}} \geqslant 0$. Recall (see Lemma 2.15 (8)) that $(|f_n''(t)||\widetilde{f}_n''(s)|)^{\frac{1}{2}} > \frac{1}{4}(|f_{(\chi,\eta)}''(t)||f_{(\chi,\eta)}''(s)|)^{\frac{1}{2}}$, and (see above) $|f_{(\chi,\eta)}''(t)| > d_1(t-s)$ and $|f_{(\chi,\eta)}''(s)| > d_2(t-s)$ for all l and corresponding pairs, where in principle we can find explicit constants for $d_1, d_2 > 0$. Recall also that $t - s > (\frac{6}{l})^{\frac{1}{2}}$ for any fixed l and all corresponding pairs. Also, we have shown that $f_{(\chi,\eta)}(t) - f_{(\chi,\eta)}(s) < -\frac{5}{12\sqrt{6}}\ (\frac{1}{l})^{\frac{3}{2}}$, and $|f_n(t) - f_{(\chi,\eta)}(t)| < \frac{B}{n}$ and $|\widetilde{f}_n(s) - f_{(\chi,\eta)}(s)| < \frac{B}{n}$. Combined, the above show that we can choose N sufficiently large such that the following is satisfied for any fixed l and all corresponding $(\chi,\eta) \in [.5,.99] \times \{(1-\frac{1}{l})\frac{1}{4}\}$ and $(t,s) \in \angle$ with $(\chi,\eta) = (\chi_{\mathcal{O}}(t,s), \eta_{\mathcal{O}}(t,s))$:

- $f_n(t) - \widetilde{f}_n(s) < -\frac{5}{12\sqrt{6}}\,(\frac{1}{l})^{\frac{3}{2}} + \frac{2B}{n}$.
- $t - s > (\frac{6}{l})^{\frac{1}{2}}$.
- $D_n \widetilde{D}_n > \frac{1}{8}(t-s)\sqrt{d_1 d_2} > \frac{1}{8}(\frac{6}{l})^{\frac{1}{2}}\sqrt{d_1 d_2}$.

The first term on the RHS of (2.31) thus satisfies the following for any fixed l and all corresponding pairs:

$$\frac{\exp(nf_n(t) - n\widetilde{f}_n(s))}{4\pi(t-s)D_n\widetilde{D}_n} < \frac{\exp(-n\frac{5}{12\sqrt{6}}\,(\frac{1}{l})^{\frac{3}{2}} + 2B)}{3\pi(\frac{1}{l})\sqrt{d_1 d_2}}.$$

Finally, we state that we can find explicit bounds for $|F_n|$ and $|G_n|$ using similar methods to those discussed above. It thus follows that we can choose N sufficiently large such that the second and third terms on the RHS of (2.31) are also bounded by the above term for any fixed l and all corresponding pairs. Finally, (2.27) and (2.31) give the following corollary of Theorem 2.16

Corollary 2.17. — *Take $\mu := \frac{1}{4}\delta_1 + \frac{3}{4}\delta_{-1}$, and define $x^{(n)}$, (v_n, s_n), (u_n, r_n), N, B etc, as above. Fix $l \geqslant L$, and $n > N$ a multiple of $4l$. Then the expected number of particles on row $n\eta = n(1-\frac{1}{l})\frac{1}{4}$ that are contained in $[.5,.99]$ satisfies the following:*

$$\mathbb{M}_1[[.5,.99] \times \{n\eta\}] < Cl\ \exp\left(-n\frac{5}{12\sqrt{6}}\left(\frac{1}{l}\right)^{\frac{3}{2}}\right),$$

where $C := \frac{\exp(2B)}{2\pi\sqrt{d_1 d_2}}$ is a constant independent of l.

3. The global asymptotic behaviour

In this section we examine the global asymptotic behaviours of $\mathcal{L}$, $\mathcal{E}$ and $\mathcal{O}$, defined in Section 2.1. The analysis here is analogous to that given in Duse and Metcalfe, [12], for discrete interlaced Gelfand–Tsetlin patterns, and many of the methods and results are similar. However, it is still necessary to carry out the analysis in this context as understanding the global asymptotic behaviour is an essential first step to identifying natural regions in which universal local asymptotic behaviours can occur. Unless otherwise stated, only the following assumptions are required in this section:

- μ is a probability measure on $\mathbb{R}$ with compact support, $\mathrm{Supp}(\mu) \subset [a,b]$ with $\{a,b\} \subset \mathrm{Supp}(\mu)$, and $(\chi,\eta) \in [a,b] \times [0,1]$ is fixed.
- Assume that $b > a$ to avoid that degenerate case where μ is a single atom of mass 1. This implies that $\mu[\{\chi\}] \in [0,1)$.

3.1. The liquid region

Recall (see Definition 2.5) that the liquid region, $\mathcal{L}$, is the set of all $(\chi, \eta) \in [a, b] \times [0, 1]$ for which the following function has non-real roots (see (2.10)):

$$f'_{(\chi,\eta)}(w) = C(w) - \frac{1-\eta}{w-\chi}, \tag{3.1}$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$, where C is the Cauchy transform of μ (see (2.11)). We denote $f'_{(\chi,\eta)}$ simply by f' where no confusion is possible. Note, Definition 2.5 and Corollary 5.3 (1) imply the following, more refined, definition of $\mathcal{L}$:

Definition 3.1. — *The liquid region, $\mathcal{L}$, is the set of all $(\chi, \eta) \in (a, b) \times (0, 1)$ for which f' has a unique root in $\mathbb{H} := \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$. This root has multiplicity* 1.

Theorem 3.2. — *Let $W_{\mathcal{L}} : \mathcal{L} \to \mathbb{H}$ map $(\chi, \eta) \in \mathcal{L}$ to the corresponding root of f' in $\mathbb{H}$. This is a homeomorphism with inverse $(\chi_{\mathcal{L}}(\cdot), \eta_{\mathcal{L}}(\cdot)) : \mathbb{H} \to \mathcal{L}$ given by,*

$$\chi_{\mathcal{L}}(w) = w + \frac{C(\overline{w})(w-\overline{w})}{C(w) - C(\overline{w})} \quad \text{and} \quad \eta_{\mathcal{L}}(w) = 1 + \frac{C(w)C(\overline{w})(w-\overline{w})}{C(w) - C(\overline{w})}.$$

Proof. — We first show:

(i) $\mathcal{L}$ is non-empty.
(ii) $\mathcal{L}$ is open.
(iii) $W_{\mathcal{L}} : \mathcal{L} \to \mathbb{H}$ is continuous.
(iv) $W_{\mathcal{L}} : \mathcal{L} \to \mathbb{H}$ is injective.

The *invariance of domain theorem* then implies that $W_{\mathcal{L}}(\mathcal{L})$ is open and $W_{\mathcal{L}} : \mathcal{L} \to W_{\mathcal{L}}(\mathcal{L})$ is a homeomorphism. We complete the result by showing:

(v) $W_{\mathcal{L}} : \mathcal{L} \to W_{\mathcal{L}}(\mathcal{L})$ has inverse $w \mapsto (\chi_{\mathcal{L}}(w), \eta_{\mathcal{L}}(w))$ for all $w \in W_{\mathcal{L}}(\mathcal{L})$.
(vi) $W_{\mathcal{L}}(\mathcal{L}) = \mathbb{H}$.

Consider (i). Fix $w \in \mathbb{H}$ and define $(\chi, \eta) := (\chi_{\mathcal{L}}(w), \eta_{\mathcal{L}}(w))$, where $\chi_{\mathcal{L}}$ and $\eta_{\mathcal{L}}$ are defined in the statement of the theorem. We will show that:

(ia) $f'(w) = 0$.
(ib) $(\chi, \eta) \in (a, b) \times (0, 1)$ when $|w|$ is sufficiently large.

Definition 2.5 then implies that $(\chi, \eta) \in \mathcal{L}$ when $|w|$ is sufficiently large. This proves (i).

Consider (ia). First note, the definitions of $\chi = \chi_{\mathcal{L}}(w)$ and $\eta = \eta_{\mathcal{L}}(w)$ easily give $1 - \eta = (w - \chi)C(w)$. (3.1) then trivially gives $f'(w) = 0$. This proves (ia).

Consider (ib). First recall $\chi = \chi_{\mathcal{L}}(w)$ and $\eta = \eta_{\mathcal{L}}(w)$, and write χ and η as in (3.4) (below) to get $(\chi, \eta) \in \mathbb{R}^2$. Next note, Taylor expansions of (2.11) give:

$$\begin{aligned}
&C(w) = \frac{1}{w} + \frac{\mu_1}{w^2} + \frac{\mu_2}{w^3} + O\left(|w|^{-4}\right),\\
&C(w) - C(\overline{w})\\
&\quad = \left(\frac{1}{w} - \frac{1}{\overline{w}}\right)\left(1 + \mu_1\left(\frac{1}{w} + \frac{1}{\overline{w}}\right) + \mu_2\left(\frac{1}{w^2} + \frac{1}{|w|^2} + \frac{1}{\overline{w}^2}\right) + O\left(|w|^{-3}\right)\right),
\end{aligned}$$

where $\mu_1 := \int_a^b x\mu[\mathrm{d}x]$ and $\mu_2 := \int_a^b x^2\mu[\mathrm{d}x]$. Combine these with the expressions for $\chi = \chi_{\mathcal{L}}(w)$ and $\eta = \eta_{\mathcal{L}}(w)$ given in the statement of this lemma to get,

$$\chi = \mu_1 + O\left(|w|^{-1}\right) \quad \text{and} \quad \eta = \left(\mu_2 - \mu_1^2\right)\frac{1}{|w|^2} + O\left(|w|^{-3}\right).$$

Finally recall (see Assumption 2.1) that μ is a probability measure on $[a, b]$, $b > a$, and $\{a, b\} \in \mathrm{Supp}(\mu)$. Therefore,

$$\mu_1 = \int_a^b x\mu[\mathrm{d}x] < \int_{\{b\}} x\ \delta_b[\mathrm{d}x] = b.$$

Similarly $\mu_1 > a$, and

$$\mu_2 - \mu_1^2 = \frac{1}{2}\int_a^b \mu[\mathrm{d}x]\int_a^b \mu[\mathrm{d}y](x-y)^2 > \frac{1}{2}\int_{\{0\}} \delta_0[\mathrm{d}x]\int_{\{0\}} \delta_0[\mathrm{d}y]\ (x-y)^2 = 0.$$

Therefore $(\chi, \eta) \in (a, b)\times(0, 1)$ when $|w|$ is sufficiently large. This proves (ib).

Consider (ii). Fix $(\chi_1, \eta_1), (\chi_2, \eta_2) \in (a, b) \times (0, 1)$ with $(\chi_1, \eta_1) \in \mathcal{L}$. Define,

- $f_1'(w) := C(w) - (1 - \eta_1)/(w - \chi_1)$,
- $f_2'(w) := C(w) - (1 - \eta_2)/(w - \chi_2)$,

for all $w \in \mathbb{H}$. Note, since $(\chi_1, \eta_1) \in \mathcal{L}$, Definition 3.1 implies that f_1' has a unique root in $\mathbb{H}$. Denote this root by w_1, and fix $\epsilon > 0$ such that $B(w_1, 2\epsilon) \subset \mathbb{H}$. Next note, since w_1 is the unique root of f_1' in $\mathbb{H}$, the extreme value theorem gives,

$$\inf_{w \in \partial B(w_1, \epsilon)} |f_1'(w)| > 0.$$

Finally, $|f_1'(w) - f_2'(w)| \leqslant |\frac{1-\eta_1}{w-\chi_1} - \frac{1-\eta_2}{w-\chi_2}|$ for all $w \in \mathbb{H}$. Thus, whenever $|\chi_1 - \chi_2|$ and $|\eta_1 - \eta_2|$ are sufficiently small, $|f_1'(w)| > |f_1'(w) - f_2'(w)|$ for all $w \in \partial B(w_1, \epsilon)$. Rouché's Theorem thus implies that f_2' has a root in $B(w_1, \epsilon) \subset \mathbb{H}$. Definition 2.5 thus implies that $(\chi_2, \eta_2) \in \mathcal{L}$ whenever $|\chi_1 - \chi_2|$ and $|\eta_1 - \eta_2|$ are sufficiently small. This proves (ii).

Consider (iii). Fix $(\chi_1, \eta_1), (\chi_2, \eta_2) \in \mathcal{L}$, and define f_1', f_2' as in (ii). Also define w_1 and ϵ as in (ii), and let w_2 denote the unique root of f_2' in $\mathbb{H}$ (see Definition 3.1). Next, proceed as in (ii) to show that f_2' has a root in $B(w_1, \epsilon) \subset \mathbb{H}$ whenever $|\chi_1 - \chi_2|$ and $|\eta_1 - \eta_2|$ are sufficiently small. Thus we must have $w_2 \in B(w_1, \epsilon)$ whenever $|\chi_1 - \chi_2|$ and $|\eta_1 - \eta_2|$ are sufficiently small. Next recall that $w_1 = W_{\mathcal{L}}(\chi_1, \eta_1)$ and $w_2 = W_{\mathcal{L}}(\chi_2, \eta_2)$ (see statement of this lemma). Therefore $|W_{\mathcal{L}}(\chi_1, \eta_1) - W_{\mathcal{L}}(\chi_2, \eta_2)| < \epsilon$ whenever $|\chi_1 - \chi_2|$ and $|\eta_1 - \eta_2|$ are sufficiently small. Finally note that we can repeat the above analysis with ϵ replaced by any $\delta \in (0, \epsilon)$. This proves (iii).

Consider (iv). Fix $(\chi_1, \eta_1), (\chi_2, \eta_2) \in \mathcal{L}$ with $W_{\mathcal{L}}(\chi_1, \eta_1) = W_{\mathcal{L}}(\chi_2, \eta_2) = w \in \mathbb{H}$. (3.1) and the above definition of $W_{\mathcal{L}}$ then give,

$$C(w) = \frac{1 - \eta_1}{w - \chi_1} = \frac{1 - \eta_2}{w - \chi_2}.$$

Therefore $(\eta_2 - \eta_1)w = (1 - \eta_1)\chi_2 - (1 - \eta_2)\chi_1$. Then $w \in \mathbb{R}$ whenever $\eta_1 \neq \eta_2$, which contradicts $w \in \mathbb{H}$. Thus $\eta_1 = \eta_2$, and so $(1-\eta_1)(\chi_1-\chi_2) = 0$. Finally, $\eta_1 < 1$ since $(\chi_1, \eta_1) \in \mathcal{L}$ (see Definition 3.1), and so $\chi_1 = \chi_2$. This proves (iv).

Consider (v). Fix $(\chi, \eta) \in \mathcal{L}$ and let $w := W_{\mathcal{L}}(\chi, \eta) \in W_{\mathcal{L}}(\mathcal{L})$. (3.1) and the above definition of $W_{\mathcal{L}}$ then give $1 - \eta = (w - \chi)C(w)$. Complex conjugation then gives,

$$1 - \eta = (w - \chi)C(w) = (\overline{w} - \chi)C(\overline{w}).$$

Solving gives $(\chi, \eta) = (\chi_{\mathcal{L}}(w), \eta_{\mathcal{L}}(w))$. This proves (v).

Consider (vi). Recall that $W_{\mathcal{L}}(\mathcal{L})$ is open and that $W_{\mathcal{L}} : \mathcal{L} \to W_{\mathcal{L}}(\mathcal{L})$ is a homeomorphism with inverse $w \mapsto (\chi_{\mathcal{L}}(w), \eta_{\mathcal{L}}(w))$. Assume that $W_{\mathcal{L}}(\mathcal{L})$ is a proper subset of $\mathbb{H}$, i.e., that there exists a point $w \in \partial W_{\mathcal{L}}(\mathcal{L})$ with $w \in \mathbb{H} \setminus W_{\mathcal{L}}(\mathcal{L})$. Choose a sequence $\{w_k\}_{k \geqslant 1} \subset W_{\mathcal{L}}(\mathcal{L})$ with $w_k \to w$ as $k \to \infty$, and let $(\chi_k, \eta_k) := (\chi_{\mathcal{L}}(w_k), \eta_{\mathcal{L}}(w_k))$ for all $k \geqslant 1$. Note that we can always choose so that $\{(\chi_k, \eta_k)\}_{k \geqslant 1}$ is convergent as $k \to \infty$, $(\chi_k, \eta_k) \to (\chi, \eta)$ say. Also note (3.1) and the above definition of $W_{\mathcal{L}}$ gives $C(w_k) - (1-\eta_k)/(w_k - \chi_k) = 0$ for all $k \geqslant 1$. Letting $k \to \infty$ we get $C(w) - (1 - \eta)/(w - \chi) = 0$, and so $(\chi, \eta) \in \mathcal{L}$ and $w = W_{\mathcal{L}}(\chi, \eta)$. This contradicts the assumption that $w \in \mathbb{H} \setminus W_{\mathcal{L}}(\mathcal{L})$, and so $W_{\mathcal{L}}(\mathcal{L}) = \mathbb{H}$. This proves (vi). □

Note the following trivial corollary of Theorem 3.2:

Corollary 3.3. — *$\mathcal{L}$ is a non-empty, open, simply connected subset of $(a, b) \times (0, 1)$. Moreover, $\partial\mathcal{L}$ is the set of all $(\chi, \eta) \in [a, b] \times [0, 1]$ for which there exists a sequence, $\{w_k\}_{k \geqslant 1} \subset \mathbb{H}$, with $(\chi_{\mathcal{L}}(w_k), \eta_{\mathcal{L}}(w_k)) \to (\chi, \eta)$ as $k \to \infty$, and either $|w_k| \to \infty$ or $w_k \to t \in \mathbb{R}$ as $k \to \infty$.*

We end this section by using the above to examine $\partial\mathcal{L}$:

Lemma 3.4. — *First we consider those parts of $\partial\mathcal{L}$ which exist for any choice of μ:*

(1) $(\int_a^b x\mu[\mathrm{d}x], 0) \in \partial\mathcal{L}$. *Moreover* $(\chi_{\mathcal{L}}(w_k), \eta_{\mathcal{L}}(w_k)) \to (\int_a^b x\mu[\mathrm{d}x], 0)$ *as* $k \to \infty$ *for all* $\{w_k\}_{k\geqslant 1} \subset \mathbb{H}$ *with* $|w_k| \to \infty$.

(2) $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \partial\mathcal{L}$ *for all* $t \in R$, *where* $R \subset \mathbb{R}$ *is open and given by the disjoint union* $R = R^+ \cup R^- \cup R_0 \cup R_1$ *(see* (2.14)*), and where*

$$\begin{aligned} \chi_{\mathcal{E}}(t) &:= t + \frac{C(t)}{C'(t)} \quad \text{and} \quad \eta_{\mathcal{E}}(t) := 1 + \frac{C(t)^2}{C'(t)} && \text{when } t \in R^+ \cup R^-, \\ \chi_{\mathcal{E}}(t) &:= t \quad \text{and} \quad \eta_{\mathcal{E}}(t) := 1 && \text{when } t \in R_0, \\ \chi_{\mathcal{E}}(t) &:= t \quad \text{and} \quad \eta_{\mathcal{E}}(t) := 1 - \mu[\{t\}] && \text{when } t \in R_1. \end{aligned}$$

Moreover, whenever $t \in R$, $(\chi_{\mathcal{L}}(w_k), \eta_{\mathcal{L}}(w_k)) \to (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ *as* $k \to \infty$ *for all* $\{w_k\}_{k\geqslant 1} \subset \mathbb{H}$ *with* $w_k \to t$.

Next we impose restrictions on μ to examine other possible parts of $\partial\mathcal{L}$:

(3) $(t, 1) \in \partial\mathcal{L}$ *when there exists an interval* $I = (t_2, t_1)$ *with* $t \in I \subset \mathrm{Supp}(\mu)$, μ *is absolutely continuous on* I, *and the density of* μ *(denoted* φ*) satisfies one of the following:*
- $\sup_{x\in(t_2,t_1)} \varphi(x) < +\infty$ *and* $\inf_{x\in(t_2,t_1)} \varphi(x) > 0$.
- $\sup_{x\in(t_2,t)} \varphi(x) < +\infty$, $\inf_{x\in(t_2,t)} \varphi(x) > 0$, $\varphi(x) = 0$ *for all* $x \in (t, t_1)$.
- $\sup_{x\in(t,t_1)} \varphi(x) < +\infty$, $\inf_{x\in(t,t_1)} \varphi(x) > 0$, $\varphi(x) = 0$ *for all* $x \in (t_2, t)$.

Moreover, $(\chi_{\mathcal{L}}(w_k), \eta_{\mathcal{L}}(w_k)) \to (t, 1)$ *as* $k \to \infty$ *for all* $\{w_k\}_{k\geqslant 1} \subset \mathbb{H}$ *with* $w_k \to t$.

Proof. — Consider (1). Fix $\{w_k\}_{k\geqslant 1} \subset \mathbb{H}$ with $|w_k| \to \infty$ as $k \to \infty$. The proof of step (ib) in Theorem 3.2 then gives $(\chi_{\mathcal{L}}(w_k), \eta_{\mathcal{L}}(w_k)) \to (\int_a^b x\mu[\mathrm{d}x], 0)$. Corollary 3.3 then gives $(\int_a^b x\mu[\mathrm{d}x], 0) \in \partial\mathcal{L}$. This proves (1).

Consider (2) when $t \in R^+ \cup R^- \cup R_0 = \mathbb{R} \setminus \mathrm{Supp}(\mu)$ (see (2.14)). Fix $\{w_k\}_{k\geqslant 1} \subset \mathbb{H}$ with $w_k \to t$ as $k \to \infty$. First write (see Theorem 3.2),

$$\begin{aligned} \chi_{\mathcal{L}}(w_k) &= w_k + C(\overline{w_k}) \frac{w_k - \overline{w_k}}{C(w_k) - C(\overline{w_k})}, \\ \eta_{\mathcal{L}}(w_k) &= 1 + C(w_k) C(\overline{w_k}) \frac{w_k - \overline{w_k}}{C(w_k) - C(\overline{w_k})}. \end{aligned} \tag{3.2}$$

Thus, since $w_k \to t$ and $\overline{w_k} \to t$ as $k \to \infty$, where $w_k \in \mathbb{H}$ and $t \in \mathbb{R} \setminus \mathrm{Supp}(\mu)$, and since $C : \mathbb{C} \setminus \mathrm{Supp}(\mu) \to \mathbb{C}$ is analytic (see (2.11)),

$$\chi_{\mathcal{L}}(w_k) \to t + C(t) \frac{1}{C'(t)} \quad \text{and} \quad \eta_{\mathcal{L}}(w_k) \to 1 + C(t)C(t) \frac{1}{C'(t)} \quad \text{as } k \to \infty.$$

Therefore $(\chi_{\mathcal{L}}(w_k), \eta_{\mathcal{L}}(w_k)) \to (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ when $t \in R^+ \cup R^-$. Similarly when $t \in R_0$ (recall that $C(t) = 0$ in this case by (2.14)). Corollary 3.3 then gives $(\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t)) \in \partial\mathcal{L}$ when $t \in R^+ \cup R^- \cup R_0$. This proves (2) when $t \in R^+ \cup R^- \cup R_0$.

Consider (2) when $t \in R_1$. Fix $\{w_k\}_{k\geqslant 1} \subset \mathbb{H}$ with $w_k \to t$ as $k \to \infty$. Recall (see (2.14)) that $\mu[\{t\}] > 0$, and there exists an open interval $I \subset \mathbb{R}$ with $t \in I$ and $I \setminus \{t\} \subset \mathbb{R} \setminus \operatorname{Supp}(\mu)$. (2.11) thus gives,

$$C(w) = \frac{\mu[\{t\}]}{w - t} + C_I(w),$$

for all $w \in \mathbb{C} \setminus \operatorname{Supp}(\mu)$, where $C_I(w) := \int_{[a,b]\setminus I} \frac{\mu[\mathrm{d}x]}{w-x}$. Therefore,

$$\frac{C(w) - C(\overline{w})}{w - \overline{w}} = -\frac{\mu[\{t\}]}{(w-t)(\overline{w}-t)} + \frac{C_I(w) - C_I(\overline{w})}{w - \overline{w}},$$

Recall that $w_k, \overline{w_k} \to t \in I$ as $k \to \infty$, and note that C_I has a unique analytic extension to I. Thus, combined, the above give the following as $k \to \infty$:

$$C(w_k) = \frac{\mu[\{t\}]}{w_k - t} + C_I(t) + o(1), \qquad C(\overline{w_k}) = \frac{\mu[\{t\}]}{\overline{w_k} - t} + C_I(t) + o(1),$$
$$\frac{C(w_k) - C(\overline{w_k})}{w_k - \overline{w_k}} = -\frac{\mu[\{t\}]}{(w_k - t)(\overline{w_k} - t)} + C_I'(t) + o(1).$$

(3.2) thus gives the following for all k sufficiently large:

$$\chi_{\mathcal{L}}(w_k) = w_k + \left(\frac{\mu[\{t\}]}{\overline{w_k} - t} + O(1)\right)\left(-\frac{\mu[\{t\}]}{(w_k - t)(\overline{w_k} - t)} + O(1)\right)^{-1},$$

$$\eta_{\mathcal{L}}(w_k)$$
$$= 1 + \left(\frac{\mu[\{t\}]}{w_k - t} + O(1)\right)\left(\frac{\mu[\{t\}]}{\overline{w_k} - t} + O(1)\right)\left(-\frac{\mu[\{t\}]}{(w_k - t)(\overline{w_k} - t)} + O(1)\right)^{-1}.$$

Therefore, since $w_k, \overline{w_k} \to t$ as $k \to \infty$, $(\chi_{\mathcal{L}}(w_k), \eta_{\mathcal{L}}(w_k)) \to (t, 1 - \mu[\{t\}])$ when $t \in R_1$. Corollary 3.3 then gives $(t, 1 - \mu[\{t\}]) \in \partial\mathcal{L}$ when $t \in R_1$. This proves (2) when $t \in R_1$.

Consider (3) when $\sup_{x\in(t_2,t_1)} \varphi(x) < +\infty$ and $\inf_{x\in(t_2,t_1)} \varphi(x) > 0$. Fix $\{w_k\}_{k\geqslant 1} \subset \mathbb{H}$ with $w_k \to t$ as $k \to \infty$. Denote $u_k := \operatorname{Re}(w_k)$, $v_k := \operatorname{Im}(w_k)$, $R_k := \operatorname{Re}(C(w_k))$, and $I_k := -\operatorname{Im}(C(w_k))$, where C is the Cauchy transform of μ (see (2.11)). Then $u_k \to t$ and $v_k \searrow 0$ as $k \to \infty$, and

$$R_k = \int_a^b \frac{(u_k - x)\mu[\mathrm{d}x]}{(u_k - x)^2 + v_k^2} \quad \text{and} \quad I_k = \int_a^b \frac{v_k \mu[\mathrm{d}x]}{(u_k - x)^2 + v_k^2} \quad \text{for all } k. \tag{3.3}$$

Letting $\varphi^+ := \sup_{x\in(t_2,t_1)} \varphi(x)$ and $\varphi^- := \inf_{x\in(t_2,t_1)} \varphi(x)$, we will show:

(3a) $\pi\varphi^+ + O(v_k) \geqslant I_k \geqslant \pi\varphi^- + O(v_k)$ for all k sufficiently large.
(3b) $|R_k| \leqslant (\varphi^+ - \varphi^-)|\log(v_k)| + O(1)$ for all k sufficiently large.

Next write (see Theorem 3.2),

$$\chi_{\mathcal{L}}(w_k) = u_k - \frac{v_k R_k}{I_k} \qquad \text{and} \qquad \eta_{\mathcal{L}}(w_k) = 1 - \frac{v_k(R_k^2 + I_k^2)}{I_k}, \tag{3.4}$$

for all k. Then, since $u_k \to t$ and $v_k \searrow 0$ as $k \to \infty$, and since $+\infty > \varphi^+ \geqslant \varphi^- > 0$, (3a), (3b), and (3.4) give $(\chi_{\mathcal{L}}(w_k), \eta_{\mathcal{L}}(w_k)) \to (t, 1)$ as $k \to \infty$. Corollary 3.3 then gives $(t, 1) \in \partial\mathcal{L}$. This proves (3) when $\sup_{x\in(t_2,t_1)} \varphi(x) < +\infty$ and $\inf_{x\in(t_2,t_1)} \varphi(x) > 0$. Part (3) for the other cases follows similarly.

Consider (3a). Recall that $t \in (t_2, t_1) \subset \mathrm{Supp}(\mu)$, and μ is absolutely continuous on (t_2, t_1) with density φ. (3.3) then gives,

$$I_k = \int_a^{t_2} \frac{v_k \mu[\mathrm{d}x]}{(u_k - x)^2 + v_k^2} + \int_{t_2}^{t_1} \frac{v_k \varphi(x)\mathrm{d}x}{(u_k - x)^2 + v_k^2} + \int_{t_1}^{b} \frac{v_k \mu[\mathrm{d}x]}{(u_k - x)^2 + v_k^2}.$$

Recall that $u_k \to t \in (t_2, t_1)$ and $v_k \searrow 0$ as $k \to \infty$. Therefore,

$$I_k = \int_{t_2}^{t_1} \frac{v_k \varphi(x)\mathrm{d}x}{(u_k - x)^2 + v_k^2} + O(v_k),$$

for all k sufficiently large. Recall also that $\varphi^+ = \sup_{x\in(t_2,t_1)} \varphi(x) < +\infty$. Therefore,

$$\begin{aligned} I_k &\leqslant \int_{t_2}^{t_1} \frac{v_k (\varphi^+)\mathrm{d}x}{(u_k - x)^2 + v_k^2} + O(v_k) \\ &= -\varphi^+ \arctan\left(\frac{u_k - t_1}{v_k}\right) + \varphi^+ \arctan\left(\frac{u_k - t_2}{v_k}\right) + O(v_k). \end{aligned}$$

Thus, since $u_k \to t \in (t_2, t_1)$ and $v_k \searrow 0$ as $k \to \infty$, $I_k \leqslant -\varphi^+(-\frac{\pi}{2} + O(v_k)) + \varphi^+(\frac{\pi}{2} + O(v_k)) + O(v_k) = \pi\varphi^+ + O(v_k)$ for all k sufficiently large. Similarly, since $\varphi^- = \inf_{x\in(t_2,t_1)} \varphi(x) > 0$,

$$I_k \geqslant \int_{t_2}^{t_1} \frac{v_k (\varphi^-)\mathrm{d}x}{(u_k - x)^2 + v_k^2} + O(v_k),$$

for all k sufficiently large. Proceed as before to get $I_k \geqslant \pi\varphi^- + O(v_k)$. Combining both inequalities proves (3a).

Consider (3b). Recall that $t \in (t_2, t_1) \subset \mathrm{Supp}(\mu)$, and μ is absolutely continuous on (t_2, t_1) with density φ. (3.3) then gives,

$$R_k = \int_a^{t_2} \frac{(u_k - x)\mu[\mathrm{d}x]}{(u_k - x)^2 + v_k^2} + \int_{t_2}^{t_1} \frac{(u_k - x)\varphi(x)\mathrm{d}x}{(u_k - x)^2 + v_k^2} + \int_{t_1}^{b} \frac{(u_k - x)\mu[\mathrm{d}x]}{(u_k - x)^2 + v_k^2}.$$

Recall that $u_k \to t \in (t_2, t_1)$ and $v_k \searrow 0$ as $k \to \infty$. Therefore,

$$R_k = \int_{t_2}^{u_k} \frac{(u_k - x)\varphi(x)\mathrm{d}x}{(u_k - x)^2 + v_k^2} + \int_{u_k}^{t_1} \frac{(u_k - x)\varphi(x)\mathrm{d}x}{(u_k - x)^2 + v_k^2} + O(1),$$

for all k sufficiently large. Recall also that $\varphi^+ = \sup_{x\in(t_2,t_1)} \varphi(x) < +\infty$ and $\varphi^- = \inf_{x\in(t_2,t_1)} \varphi(x) > 0$. Therefore,

$$\begin{aligned} R_k &\leqslant \int_{t_2}^{u_k} \frac{(u_k - x)(\varphi^+)\mathrm{d}x}{(u_k - x)^2 + v_k^2} + \int_{u_k}^{t_1} \frac{(u_k - x)(\varphi^-)\mathrm{d}x}{(u_k - x)^2 + v_n^2} + O(1) \\ &= -\frac{\varphi^+}{2} \log((u_k - x)^2 + v_k^2)\Big|_{t_2}^{u_k} - \frac{\varphi^-}{2} \log((u_k - x)^2 + v_k^2)\Big|_{u_k}^{t_1} + O(1), \end{aligned}$$

for all k sufficiently large. Thus, since $u_k \to t \in (t_2, t_1)$ and $v_k \searrow 0$ as $k \to \infty$, $R_n \leqslant -(\varphi^+ - \varphi^-)\log(v_k) + O(1)$. Similarly,

$$R_k \geqslant \int_{t_2}^{u_k} \frac{(u_k - x)(\varphi^-)\mathrm{d}x}{(u_k - x)^2 + v_k^2} + \int_{u_k}^{t_1} \frac{(u_k - x)(\varphi^+)\mathrm{d}x}{(u_k - x)^2 + v_n^2} + O(1),$$

for all k sufficiently large. Proceed as before to get $R_k \geqslant (\varphi^+ - \varphi^-)\log(v_k) + O(1)$. Combining both inequalities proves (3b). □

Remark 3.5. — Note, Lemma 3.4 (1) and (2) finds parts of $\partial\mathcal{L}$ which exist for any choice of μ. Also note, (2.14) implies that $R = \mathbb{R}$ when μ is purely atomic. In that case, Corollary 3.3 implies that Lemma 3.4 (1) and (2) give a complete description of $\partial\mathcal{L}$. Finally note, Lemma 3.4 (3) imposes restrictions on μ to examine other possible parts of $\partial\mathcal{L}$. It is beyond the scope of this paper to ease these restrictions since the resulting technicalities are highly non-trivial.

3.2. The Edge, $\mathcal{E}$

In this section we define $\mathcal{E}$ as in Definition 2.7, and prove an analogous result for $\mathcal{E}$ to Theorem 3.2 for $\mathcal{L}$. As in Section 3.1, we denote $f'_{(\chi,\eta)}$ simply by f'. Recall in Theorem 3.2, $W_{\mathcal{L}} : \mathcal{L} \to \mathbb{H}$ maps each $(\chi, \eta) \in \mathcal{L}$ to the corresponding unique root of f' in $\mathbb{H}$, and $W_{\mathcal{L}}$ is a homeomorphism with inverse $(\chi_{\mathcal{L}}(\cdot), \eta_{\mathcal{L}}(\cdot)) : \mathbb{H} \to \mathcal{L}$. Recall also, Lemma 3.4 implies that $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \to \partial\mathcal{L}$ is the curve which is the unique continuous extension to $R = R^+ \cup R^- \cup R_0 \cup R_1 \subset \mathbb{R}$ (see (2.13), (2.14)) of $(\chi_{\mathcal{L}}(\cdot), \eta_{\mathcal{L}}(\cdot)) : \mathbb{H} \to \mathcal{L}$, and which is continuous in any open sub-interval of R. Finally note, Definition 2.7 and Corollary 5.3 imply the following, more refined, definition of $\mathcal{E}$:

Definition 3.6. — *The edge, $\mathcal{E}$, is the disjoint union $\mathcal{E} = \mathcal{E}^+ \cup \mathcal{E}^- \cup \mathcal{E}_0 \cup \mathcal{E}_1$ where:*

- *$\mathcal{E}^+$ is the set of all $(\chi, \eta) \in (a, b) \times (0, 1)$ for which $1 - \eta > \mu[\{\chi\}]$, and f' has a unique repeated root in $(\chi, +\infty) \setminus \mathrm{Supp}(\mu)$. This root has multiplicity 2 or 3.*

- $\mathcal{E}^-$ *is the set of all* $(\chi, \eta) \in (a,b) \times (0,1)$ *for which* $1-\eta > \mu[\{\chi\}]$, *and* f' *has a unique repeated root in* $(-\infty, \chi) \setminus \mathrm{Supp}(\mu)$. *This root has multiplicity* 2 *or* 3.
- $\mathcal{E}_0 := \{(\chi,\eta) : \chi \in R_0$ *and* $\eta = 1\}$. *Moreover, when* $(\chi,\eta) \in \mathcal{E}_0$, χ *is a root of* f' *of multiplicity* 1.
- $\mathcal{E}_1 := \{(\chi,\eta) : \chi \in R_1$ *and* $\eta = 1-\mu[\{\chi\}]\}$. *Moreover, when* $(\chi,\eta) \in \mathcal{E}_1$, *either* $f'(\chi) \neq 0$ *or* χ *is a root of* f' *of multiplicity* 1.

Theorem 3.7. — *Let* $W_{\mathcal{E}} : \mathcal{E} \to \mathbb{R}$ *map each* $(\chi,\eta) \in \mathcal{E}^+ \cup \mathcal{E}^-$ *to the corresponding real-valued repeated root, and map each* $(\chi,\eta) \in \mathcal{E}_0 \cup \mathcal{E}_1$ *to* χ. *Then* $W_{\mathcal{E}}(\mathcal{E}) = R$ *and* $W_{\mathcal{E}} : \mathcal{E} \to R$ *is a bijection with inverse* $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$. *Moreover, the image spaces of* $\mathcal{E}^+, \mathcal{E}^-, \mathcal{E}_0, \mathcal{E}_1$ *are (respectively)* R^+, R^-, R_0, R_1.

Proof. — We will show:

(1) $W_{\mathcal{E}}(\mathcal{E}^+) = R^+$ and $W_{\mathcal{E}} : \mathcal{E}^+ \to R^+$ is a bijection with inverse $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.
(2) $W_{\mathcal{E}}(\mathcal{E}^-) = R^-$ and $W_{\mathcal{E}} : \mathcal{E}^- \to R^-$ is a bijection with inverse $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.
(3) $W_{\mathcal{E}}(\mathcal{E}_0) = R_0$ and $W_{\mathcal{E}} : \mathcal{E}_0 \to R_0$ is a bijection with inverse $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.
(4) $W_{\mathcal{E}}(\mathcal{E}_1) = R_1$ and $W_{\mathcal{E}} : \mathcal{E}_1 \to R_1$ is a bijection with inverse $t \mapsto (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.

Note, (2.14) implies that R is the disjoint union, $R = R^+ \cup R^- \cup R_0 \cup R_1$. We will prove (1). Part (2) follows from similar considerations. Parts (3) and (4) trivially follow from (2.14), Definition 3.6, and Lemma 3.4 (2). Parts (1)–(4) give the required result.

Consider (1). We prove this by showing:

(1a) Fix $(\chi,\eta) \in \mathcal{E}^+$ and let $t := W_{\mathcal{E}}(\chi,\eta)$. Then $t \in R^+$ and $(\chi,\eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$.
(1b) Fix $t \in R^+$ and let $(\chi,\eta) := (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$. Then $(\chi,\eta) \in \mathcal{E}^+$ and $W_{\mathcal{E}}(\chi,\eta) = t$.

Consider (1a). First note, Definition 3.6, and the definition of $W_{\mathcal{E}}$ given in the statement of this theorem, imply that $(\chi,\eta) \in (a,b) \times (0,1)$, $1-\eta > \mu[\{\chi\}]$, and $t \in (\chi, +\infty) \setminus \mathrm{Supp}(\mu)$ is a repeated root of f'. Also, (2.10) gives,

$$f'(w) = C(w) - \frac{1-\eta}{w-\chi} \qquad \text{and} \qquad f''(w) = C'(w) + \frac{1-\eta}{(w-\chi)^2}, \tag{3.5}$$

for all $w \in \mathbb{C} \setminus (\mathrm{Supp}(\mu) \cup \{\chi\})$. Then, since $t \in (\chi, +\infty) \setminus \mathrm{Supp}(\mu)$ and $f'(t) = f''(t) = 0$, this gives

$$C(t) = \frac{1-\eta}{t-\chi} \qquad \text{and} \qquad C'(t) = -\frac{1-\eta}{(t-\chi)^2}.$$

The first part gives $C(t) > 0$, since $t > \chi$ and $1 > \eta$, and so $t \in R^+$ (see (2.14)). Also, solving the above equations gives $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ (see Lemma 3.4 (2)). This proves (1a).

Consider (1b). First note, Lemma 3.4 (2) implies that $(\chi, \eta) \in \partial\mathcal{L} \subset [a, b] \times [0, 1]$, and

$$\chi = t + \frac{C(t)}{C'(t)} \qquad \text{and} \qquad \eta = 1 + \frac{C(t)^2}{C'(t)}.$$

Next note, since $t \in R^+$, (2.14) implies that $t \in \mathbb{R} \setminus \mathrm{Supp}(\mu)$ and $C(t) > 0$. Also, since $t \in \mathbb{R} \setminus \mathrm{Supp}(\mu)$, (2.11) implies that $C'(t) < 0$. The first part of the above equation thus implies that $t > \chi$. Also, (3.5) holds as above, for all $w \in \mathbb{C} \setminus (\mathrm{Supp}(\mu) \cup \{\chi\})$. Substitute the above expressions for χ and η into (3.5) to get $f'(t) = f''(t) = 0$. Therefore, $(\chi, \eta) \in [a, b] \times [0, 1]$, and f' has a repeated root in $t \in (\chi, +\infty) \setminus \mathrm{Supp}(\mu)$. Definition 2.7 thus implies that $(\chi, \eta) \in \mathcal{E}^+$, and the definition of $W_{\mathcal{E}}$ given in the statement of this theorem gives $W_{\mathcal{E}}(\chi, \eta) = t$. This proves (1b). □

Note (see Lemma 3.4 (2)) that $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \to \partial\mathcal{L}$ is continuous in any open sub-interval of R. Also recall that Definitions 2.7 and 3.6 for $\mathcal{E}$ are equivalent. Note, Theorem 3.7 uses these definitions, and shows that $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \to \partial\mathcal{L}$ bijectively maps R to $\mathcal{E}$. Finally recall that Definition 2.6 defines $\mathcal{E}$ as the image of $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \to \partial\mathcal{L}$. Therefore:

Corollary 3.8. — *Definitions 2.6, 2.7, and 3.6 of $\mathcal{E}$ are equivalent.*

The curve, $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \to \mathcal{E}$, is called the *edge curve*. We now consider the geometric behaviour of the edge curve. Fix $(\chi, \eta) \in \mathcal{E}$ and the corresponding $t \in R$ with $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$, and again denote $f'_{(\chi,\eta)}$ simply by f'. Recall that $t = W_{\mathcal{E}}(\chi, \eta)$ (see Theorem 3.7), and let $m = m(t)$ denote the multiplicity of t as a root of f' (see Definition 3.6). Note, Definition 3.6 and Theorem 3.7 imply that the following exhaust all possibilities:

- $t \in R^+ \cup R^-$, $(\chi, \eta) \in \mathcal{E}^+ \cup \mathcal{E}^-$, and $m \in \{2, 3\}$.
- $t \in R_0$, $(\chi, \eta) \in \mathcal{E}_0$, and $m = 1$.
- $t \in R_1$, $(\chi, \eta) \in \mathcal{E}_1$, and $m \in \{0, 1\}$.

Above, $m = 0$ means that $f'(t) \neq 0$. We now show how the local geometric behaviour of the edge curve in a neighbourhood of (χ, η) depends on m:

Lemma 3.9. — *Define* $t, (\chi, \eta), m$, *as above. Also define the orthogonal vectors,* $\mathbf{x} = \mathbf{x}(t)$ *and* $\mathbf{y} = \mathbf{y}(t)$ *as,*

- $\mathbf{x} := (1, C(t))$ *and* $\mathbf{y} := (C(t), -1)$ *when* $t \in R^+ \cup R^-$.
- $\mathbf{x} := (1, 0)$ *and* $\mathbf{y} := (0, 1)$ *when* $t \in R_0$.
- $\mathbf{x} := (0, 1)$ *and* $\mathbf{y} := (1, 0)$ *when* $t \in R_1$.

Write,

$$(\chi_{\mathcal{E}}(s), \eta_{\mathcal{E}}(s)) - (\chi, \eta) = a(s)\mathbf{x} + b(s)\mathbf{y}, \tag{3.6}$$

for all $s \in R$ *sufficiently close to* t. *Then,*

$$a(s) = a_1(s-t) + a_2(s-t)^2 + O((s-t)^3), \tag{3.7}$$

$$b(s) = b_1(s-t)^2 + b_2(s-t)^3 + O((s-t)^4), \tag{3.8}$$

where $a_1 = a_1(t)$, $a_2 = a_2(t)$, $b_1 = b_1(t)$ *and* $b_2 = b_2(t)$ *satisfy the following:*

- $a_1 \neq 0$ *and* $b_1 \neq 0$ *when* $t \in R^+ \cup R^-$ *and* $m = 2$. *Similarly when* $t \in R_0$ *and* $m = 1$, *and when* $t \in R_1$ *and* $m = 0$. *Expressions for* a_1, b_1 *for these cases are given in* (3.11), (3.13), *and* (3.15).
- $a_1 = b_1 = 0$, $a_2 \neq 0$ *and* $b_2 \neq 0$ *when* $t \in R^+ \cup R^-$ *and* $m = 3$. *Similarly when* $t \in R_1$ *and* $m = 1$. *Expressions for* a_2, b_2 *for these cases are given in* (3.12) *and* (3.16).

Proof. — Consider (3.7) and (3.8) when $t \in R^+ \cup R^-$. Fix an interval, $I := (t_2, t_1)$, with $t \in I$, $I \subset R^+$ when $t \in R^+$, and $I \subset R^-$ when $t \in R^-$. Recall that $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$, and solve (3.6) to get,

$$(1 + C(t)^2)a(s) = (\chi_{\mathcal{E}}(s) - \chi_{\mathcal{E}}(t)) + (\eta_{\mathcal{E}}(s) - \eta_{\mathcal{E}}(t))C(t),$$

$$(1 + C(t)^2)b(s) = (\chi_{\mathcal{E}}(s) - \chi_{\mathcal{E}}(t))C(t) - (\eta_{\mathcal{E}}(s) - \eta_{\mathcal{E}}(t)),$$

for all $s \in I$. Also, Lemma 3.4 (2) gives,

$$\chi'_{\mathcal{E}}(t) = 2 - \frac{C''(t)C(t)}{C'(t)^2} \quad \text{and} \quad \eta'_{\mathcal{E}}(t) = \chi'_{\mathcal{E}}(t)C(t). \tag{3.9}$$

Note, the second part of this equation and Taylor expansions give (3.7) and (3.8) with:

- $a_1 = \chi'_{\mathcal{E}}(t)$.
- $2a_2 = \chi''_{\mathcal{E}}(t) + \chi'_{\mathcal{E}}(t)C'(t)C(t)(1 + C(t)^2)^{-1}$.
- $2b_1 = -\chi'_{\mathcal{E}}(t)C'(t)(1 + C(t)^2)^{-1}$.
- $6b_2 = -(2\chi''_{\mathcal{E}}(t)C'(t) + \chi'_{\mathcal{E}}(t)C''(t))(1 + C(t)^2)^{-1}$.

Consider (3.7) and (3.8) when $t \in R_0$. Recall (see (2.14)) that $C(t) = 0$, and there exists an interval, $I = (t_2, t_1)$, with $t \in I \subset \mathbb{R} \setminus \mathrm{Supp}(\mu)$. Note, solving (3.6) gives,

$$a(s) = \chi_{\mathcal{E}}(s) - \chi_{\mathcal{E}}(t) \qquad \text{and} \qquad b(s) = \eta_{\mathcal{E}}(s) - \eta_{\mathcal{E}}(t),$$

for all $s \in I$. Also note, similar to above, (3.9) holds. Moreover, since $C(t) = 0$, this equation gives $\chi'_{\mathcal{E}}(t) = 2$, $\eta'(t) = 0$, and $\eta''(t) = 2C'(t)$. Taylor expansions then give (3.7) and (3.8) with $a_1 = 2$ and $b_1 = C'(t)$. We ignore a_2 and b_2 here.

Consider (3.7) and (3.8) when $t \in R_1$. Recall (see (2.14)) that $\mu[\{t\}] > 0$ and there exists an open interval $I \subset \mathbb{R}$ with $t \in I$ and $I \setminus \{t\} \subset \mathbb{R} \setminus \mathrm{Supp}(\mu)$. Note, solving (3.6) gives,

$$a(s) = \eta_{\mathcal{E}}(s) - \eta_{\mathcal{E}}(t) \qquad \text{and} \qquad b(s) = \chi_{\mathcal{E}}(s) - \chi_{\mathcal{E}}(t),$$

for all $s \in I$. Next note, since $\mu[\{t\}] > 0$, (2.11) gives,

$$C(w) = \frac{\mu[\{t\}]}{w - t} + C_I(w),$$

for all $w \in \mathbb{C} \setminus \mathrm{Supp}(\mu)$ where $C_I(w) := \int_{[a,b]\setminus I} \frac{\mu[\mathrm{d}x]}{w-x}$. Lemma 3.4 (2) then gives,

$$\chi_{\mathcal{E}}(s) = s - (s-t)\frac{\mu[\{t\}] + (s-t)C_I(s)}{\mu[\{t\}] - (s-t)^2 C_I'(s)},$$
$$\eta_{\mathcal{E}}(s) = 1 - \frac{(\mu[\{t\}] + (s-t)C_I(s))^2}{\mu[\{t\}] - (s-t)^2 C_I'(s)},$$

for all $s \in I$. Taylor expansions then give (3.7) and (3.8) with:

- $a_1 = -2C_I(t)$.
- $a_2 = -3C_I'(t) - C_I(t)^2 \mu[\{t\}]^{-1}$.
- $b_1 = -C_I(t)\mu[\{t\}]^{-1}$.
- $b_2 = -2C_I'(t)\mu[\{t\}]^{-1}$.

Consider a_1, b_1 when $t \in R^+ \cup R^-$ and $m \in \{2,3\}$. First, proceed as in the proof of part (1a) in Theorem 3.7 to get $f'(w) = C(w) - (1-\eta)/(w-\chi)$ for all $w \in \mathbb{C} \setminus (\mathrm{Supp}(\mu) \cup \{\chi\})$. Differentiating and taking $w = t$ (recall $t \in \mathbb{R}\setminus(\mathrm{Supp}(\mu)\cup\{\chi\}$ by Definition 3.6) gives $f'''(t) = C''(t) - 2(1-\eta)/(t-\chi)^3$ and $f^{(4)}(t) = C'''(t) + 6(1-\eta)/(t-\chi)^4$. Next recall that $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ (see statement of this lemma). Lemma 3.4 (2) thus gives,

$$f'''(t) = C''(t) - 2\frac{C'(t)^2}{C(t)} \qquad \text{and} \qquad f^{(4)} = C'''(t) - 6\frac{C'(t)^3}{C(t)^2}. \tag{3.10}$$

Next note, since $t \in \mathbb{R} \setminus \mathrm{Supp}(\mu)$, (2.11) gives $C'(t) < 0$. (3.9) and (3.10) then give,

$$\chi'_{\mathcal{E}}(t) = -\frac{C(t)}{C'(t)^2} f'''(t) \qquad \text{and} \qquad \eta'_{\mathcal{E}}(t) = -\frac{C(t)^2}{C'(t)^2} f'''(t).$$

Moreover, the expressions for a_1, b_1 (see above) then give,

$$a_1 = -\frac{C(t)}{C'(t)^2} f'''(t) \qquad \text{and} \qquad b_1 = \frac{C(t)}{2C'(t)(1+C(t)^2)} f'''(t). \tag{3.11}$$

Finally recall $C(t) \neq 0$ since $t \in R^+ \cup R^-$ (see (2.14)), $C'(t) < 0$, and $m \in \{2,3\}$ is the multiplicity of t as a root of f' (see statement of this lemma). Therefore $a_1 \neq 0$ and $b_1 \neq 0$ when $t \in R^+ \cup R^-$ and $m = 2$, and $a_1 = b_1 = 0$ when $t \in R^+ \cup R^-$ and $m = 3$.

Consider a_2, b_2 when $t \in R^+ \cup R^-$ and $m = 3$. First note, (3.10) again holds. Therefore, since $m = 3$ and so $f'''(t) = 0$,

$$C''(t) = 2\frac{C'(t)^2}{C(t)} \qquad \text{and} \qquad C'''(t) = f^{(4)} + 6\frac{C'(t)^3}{C(t)^2}.$$

Substitute these into the expressions for a_2, b_2 (see above) to get,

$$a_2 = -\frac{C(t)}{2C'(t)^2} f^{(4)}(t) \qquad \text{and} \qquad b_2 = \frac{C(t)}{3C'(t)(1+C(t)^2)} f^{(4)}(t). \tag{3.12}$$

Finally recall $C(t) \neq 0$, $C'(t) < 0$, and $m = 3$ and so $f^{(4)}(t) \neq 0$. Therefore $a_2 \neq 0$ and $b_2 \neq 0$ when $t \in R^+ \cup R^-$ and $m = 3$.

Consider a_1, b_1 when $t \in R_0$. First recall (see above) that,

$$a_1 = 2 \qquad \text{and} \qquad b_1 = C'(t). \tag{3.13}$$

Next note, since $t \in \mathbb{R} \backslash \mathrm{Supp}(\mu)$ (see (2.14)), (2.11) gives $C'(t) < 0$. Therefore $a_1 \neq 0$ and $b_1 \neq 0$ when $t \in R_0$.

Consider a_1, b_1 when $t \in R_1$ and $m \in \{0,1\}$. Recall (see (2.14)) that $t \in \mathrm{Supp}(\mu)$, $\mu[\{t\}] > 0$, and there exists an open interval, $I \subset \mathbb{R}$ with $t \in I$ and $I \setminus \{t\} \subset \mathbb{R} \setminus \mathrm{Supp}(\mu)$. Moreover $(\chi, \eta) = (t, 1 - \mu[\{t\}])$ since $(\chi, \eta) \in \mathcal{E}_1$ (see Definition 3.6 and Theorem 3.7). (2.12) then gives $f'(w) = C_I(w)$ for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup I$, where $C_I(w) := \int_{[a,b]\setminus I} \frac{\mu[\mathrm{d}x]}{w-x}$. Therefore,

$$f'(t) = C_I(t) \qquad \text{and} \qquad f''(t) = C_I'(t). \tag{3.14}$$

The expressions for a_1, b_1 (see above) then give,

$$a_1 = -2f'(t) \qquad \text{and} \qquad b_1 = -\frac{f'(t)}{\mu[\{t\}]}. \tag{3.15}$$

Finally recall that $m \in \{0,1\}$ is the multiplicity of t as a root of f' (see statement of this lemma). Therefore $a_1 \neq 0$ and $b_1 \neq 0$ when $t \in R_1$ and $m = 0$, and $a_1 = b_1 = 0$ when $t \in R_1$ and $m = 1$.

Consider a_2, b_2 when $t \in R_1$ and $m = 1$. First note, (3.14) again holds. Therefore, since $m = 1$ and so $f'(t) = 0$,

$$C_I(t) = 0 \qquad \text{and} \qquad C_I'(t) = f''(t).$$

Substitute these into the expressions for a_2, b_2 (see above) to get,

$$a_2 = -3f''(t) \qquad \text{and} \qquad b_2 = -\frac{2f''(t)}{\mu[\{t\}]}. \tag{3.16}$$

Finally recall that $m = 1$ and so $f''(t) \neq 0$. Therefore $a_2 \neq 0$ and $b_2 \neq 0$ when $t \in R_1$ and $m = 1$. □

Note that (2.11) and (2.14) imply that $(b, +\infty) \subset R^+$. We end this section by considering the edge restricted to this interval:

Lemma 3.10. — *Recall $(b, +\infty) \subset R^+$ and consider $(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : (b, +\infty) \to \mathcal{E}^+ \subset \mathcal{E}$:*

1. *$\chi_{\mathcal{E}} : (b, +\infty) \to [a, b]$ is strictly decreasing with $\lim_{t\uparrow+\infty} \chi_{\mathcal{E}}(t) = \mu_1 := \int_a^b x\mu[\mathrm{d}x] \in (a, b)$. Moreover, when $\mu[\{b\}] > 0$, $\lim_{t\downarrow b} \chi_{\mathcal{E}}(t) = b$.*
2. *$\eta_{\mathcal{E}} : (b, +\infty) \to [0, 1]$ is strictly decreasing with $\lim_{t\uparrow+\infty} \eta_{\mathcal{E}}(t) = 0$. Moreover, when $\mu[\{b\}] > 0$, $\lim_{t\downarrow b} \eta_{\mathcal{E}}(t) = 1 - \mu[\{b\}]$.*
3. *$\chi'_{\mathcal{E}}(\cdot)/\eta'_{\mathcal{E}}(\cdot) : (b, +\infty) \to \mathbb{R}$ is positive and strictly decreasing with $\lim_{t\uparrow+\infty} \chi'_{\mathcal{E}}(t)/\eta'_{\mathcal{E}}(t) = 0$. Moreover, when $\mu[\{b\}] > 0$, $\lim_{t\downarrow b} \chi'_{\mathcal{E}}(t)/\eta'_{\mathcal{E}}(t) = +\infty$.*

$(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : (b, +\infty) \to \mathcal{E}$, when $\mu[\{b\}] > 0$ is depicted in Figure 2.2. Next:

4. *Fix $(\chi, \eta) \in \mathcal{E}$ and the corresponding $t \in (b, +\infty)$ with $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ (see Definition 3.6 and Theorem 3.7). Then $f'_{(\chi,\eta)}(s) > 0$ for all $s \in (b, t)$, $f'_{(\chi,\eta)}(t) = f''_{(\chi,\eta)}(t) = 0$ and $f'''_{(\chi,\eta)}(t) > 0$, and $f'_{(\chi,\eta)}(s) > 0$ for all $s \in (t, +\infty)$.*

Proof. — Consider (1). First recall that $(b, +\infty) \subset R^+$. Next recall (see (3.9), (3.10)) that $\chi'_{\mathcal{E}}(t) = -\frac{C(t)}{C'(t)^2} f'''(t)$ for all $t \in R^+$. Thus $\chi'_{\mathcal{E}}(t) < 0$ for all $t \in (b, +\infty)$ since $C(t) > 0$ ($t \in (b, +\infty) \subset R^+$), and since $f'''(t) > 0$ (see proof of part (4), below). Next note, Lemma 3.4 (2) and (2.11) give,

$$\chi_{\mathcal{E}}(t) = \frac{tC'(t) + C(t)}{C'(t)} = \frac{\int_a^b \mu[\mathrm{d}x] \frac{x}{(t-x)^2}}{\int_a^b \mu[\mathrm{d}x] \frac{1}{(t-x)^2}},$$

for all $t \in (b, +\infty)$. Therefore $\lim_{t\uparrow+\infty} \chi_{\mathcal{E}}(t) = \int_a^b \mu[\mathrm{d}x] x = \mu_1$. Moreover, when $\mu[\{b\}] > 0$,

$$\chi_{\mathcal{E}}(t) = \frac{\mu[\{b\}] \frac{b}{(t-b)^2} + \int_{[a,b)} \mu[\mathrm{d}x] \frac{x}{(t-x)^2}}{\mu[\{b\}] \frac{1}{(t-b)^2} + \int_{[a,b)} \mu[\mathrm{d}x] \frac{1}{(t-x)^2}},$$

for all $t \in (b, +\infty)$. Finally note, since μ is a probability measure, $\lim_{\epsilon\downarrow 0} \mu[(b-\epsilon, b)] = 0$, and so $\int_{[a,b)} \mu[\mathrm{d}x] \frac{x}{t-x} = o((t-b)^{-1})$ and $\int_{[a,b)} \mu[\mathrm{d}x] \frac{1}{(t-x)^2} =$

$o((t-b)^{-1})$ as $t \downarrow b$. Therefore $\lim_{t\downarrow b} \chi_{\mathcal{E}}(t) = b$ when $\mu[\{b\}] > 0$. This proves (1). Parts (2) and (3) follow similarly.

Consider (4). Recall that $t \in (b,+\infty)$, and $f'_{(\chi,\eta)}$ has a root of multiplicity 2 or 3 at t (see Definition 3.6). Indeed, since $(b,+\infty) = J_1$ (see (4.2)), part (a) of Theorem 5.2 implies that t is a root of $f'_{(\chi,\eta)}$ multiplicity 2, and $f'_{(\chi,\eta)}$ has no roots in $(b,+\infty)\setminus\{t\} = (b,t)\cup(t,+\infty)$. Therefore, it is sufficient to show that there exists an $s \in (t,+\infty)$ with $f'_{(\chi,\eta)}(s) > 0$. To see this, note (2.10) gives

$$f'_{(\chi,\eta)}(s) = \int_a^b \frac{\mu[\mathrm{d}x]}{s-x} - \frac{1-\eta}{s-\chi},$$

for all $s \in (b,+\infty)$. Thus, since $\mu[a,b] = 1$, $\chi \in (a,b)$ and $\eta \in (0,1)$ (see Definition 3.6), $\lim_{s\to+\infty} s f'_{(\chi,\eta)}(s) = \eta > 0$. This proves (4). □

3.3. Outside the liquid region, $\mathcal{O}$

In this section we additionally assume,

$$\mu[\{b\}] > 0.$$

We define $\mathcal{O}$ as in Definition 2.9, and we will prove an analogous result for $\mathcal{O}$ to Theorems 3.2 and 3.7. Again, we denote $f'_{(\chi,\eta)}$ simply by f'. First note, (2.10) gives

$$f'(w) = C(w) - \frac{1-\eta}{w-\chi}, \tag{3.17}$$

for all $w \in (\mathbb{C}\setminus\mathbb{R})\cup(b,+\infty)$. Next note, since $(b,+\infty) = J_1$ (see (4.2)), Definition 2.9 and Corollary 5.3 (1) imply the following, more refined, definition of $\mathcal{O}$: $\mathcal{O}$ is the set of all $(\chi,\eta) \in (a,b)\times(0,1)$ for which $1-\eta > \mu[\{\chi\}]$, f' has a root of multiplicity 1 in $(b,+\infty)$, and f' has at most 2 roots in $(b,+\infty)$ counting multiplicities. Also, since $\mu[\{b\}] > 0$, (2.11) and (3.17) give,

$$f'(w) = \frac{\mu[\{b\}]}{w-b} + \int_{[a,b)} \frac{\mu[\mathrm{d}x]}{w-x} - \frac{1-\eta}{w-\chi},$$

for all $w \in (\mathbb{C}\setminus\mathbb{R})\cup(b,+\infty)$. Then, since $\mu[a,b] = 1$, $\mu[\{b\}] > 0$, $\chi < b$, and $\eta > 0$,

$$\lim_{w\in(b,+\infty),w\downarrow b} f'(w) = +\infty \qquad \text{and} \qquad \lim_{w\in(b,+\infty),w\uparrow+\infty} wf'(w) = \eta > 0.$$

It easily follows that f' has an even number of roots in $(b,+\infty)$, counting multiplicities. Therefore, we can further refine the definition of $\mathcal{O}$:

Definition 3.11. — *When $\mu[\{b\}] > 0$, $\mathcal{O}$ is the set of all $(\chi,\eta) \in (a,b)\times(0,1)$ for which $1-\eta > \mu[\{\chi\}]$, f' has 2 distinct roots of multiplicity 1 in $(b,+\infty)$, and f' has no other roots in $(b,+\infty)$.*

Corollary 5.3 implies that $\{\mathcal{L}, \mathcal{E}, \mathcal{O}\}$ are pairwise disjoint. Next, as in Theorem 3.2, we prove that each point in $\mathcal{O}$ maps homeomorphically to its corresponding pair of roots:

Theorem 3.12. — *Define $\angle := \{(t,s) \in (b,+\infty)^2 : t > s\}$. Let $W_{\mathcal{O}} : \mathcal{O} \to \angle$ map each $(\chi,\eta) \in \mathcal{O}$ to the corresponding pair of roots of f' in $(b,+\infty)$. Then $W_{\mathcal{O}} : \mathcal{O} \to \mathbb{R}$ is a homeomorphism with inverse $(\chi_{\mathcal{O}}(\cdot,\cdot), \eta_{\mathcal{O}}(\cdot,\cdot)) : \angle \to \mathcal{O}$ given by,*

$$\chi_{\mathcal{O}}(t,s) = \frac{tC(t) - sC(s)}{C(t) - C(s)} \qquad \textit{and} \qquad \eta_{\mathcal{O}}(t,s) = 1 + \frac{C(t)C(s)(t-s)}{C(t) - C(s)}.$$

Proof. — We prove this result by proving the analogues of parts (i)–(vi) in the proof of Theorem 3.2. We will be more brief here, highlighting the differences only when necessary.

Consider (i). Fix $(t,s) \in \angle$ and define $(\chi,\eta) := (\chi_{\mathcal{O}}(t,s), \eta_{\mathcal{O}}(t,s))$. First note, the definitions of $\chi = \chi_{\mathcal{O}}(t,s)$ and $\eta = \eta_{\mathcal{L}}(t,s)$ and (3.17) trivially imply that $f'(t) = f'(s) = 0$. Next, proceed similarly to part (ib) in the proof of Theorem 3.2 to get $\chi = \mu_1 + O(t^{-1})$ and $\eta = (\mu_2 - \mu_1^2)/(ts) + O(t^{-3})$ whenever $t \in (b,+\infty)$ is sufficiently large and $s^{-1} = O(t^{-1})$, where $\mu_1 := \int_a^b x\mu[\mathrm{d}x]$ and $\mu_2 := \int_a^b x^2\mu[\mathrm{d}x]$. Finally recall that $b > \mu_1 > a$ and $\mu_2 - \mu_1^2 > 0$ (see part (ib) in the proof of Theorem 3.2). Therefore $f'(t) = f'(s) = 0$ and $(\chi,\eta) \in (a,b) \times (0,1)$ whenever $t \in (b,+\infty)$ is sufficiently large and $s^{-1} = O(t^{-1})$. Definition 2.9 then implies that $(\chi,\eta) \in \mathcal{O}$. This proves (i).

Consider (ii). Fix $(\chi_1,\eta_1),(\chi_2,\eta_2) \in (a,b) \times (0,1)$ with $(\chi_1,\eta_1) \in \mathcal{O}$. Define $f_1'(w) := C(w) - (1-\eta_1)/(w-\chi_1)$ and $f_2'(w) := C(w) - (1-\eta_2)/(w-\chi_2)$ for all $w \in (\mathbb{C}\setminus\mathbb{R}) \cup (b,+\infty)$. Let $(t_1,s_1) \in \angle$ denote the unique pair roots of f_1' in $(b,+\infty)$ (see Definition 3.11). Fix $\epsilon > 0$ such that t_1 is the unique root of f_1' in $B(t_1,2\epsilon)$, s_1 is the unique root of f_1' in $B(s_1,2\epsilon)$, $B(t_1,2\epsilon) \cap B(s_1,2\epsilon) = \emptyset$, and $B(t_1,2\epsilon) \cup B(s_1,2\epsilon) \subset (\mathbb{C}\setminus\mathbb{R}) \cup (b,+\infty)$. Then, whenever $|\chi_1 - \chi_2|$ and $|\eta_1 - \eta_2|$ are sufficiently small, proceed as in part (ii) in the proof of Theorem 3.2 to show that f_2' has exactly 1 root in $B(t_1,\epsilon)$, counting multiplicities, and exactly 1 root in $B(s_1,\epsilon)$. Denote these by t_2 and s_2 respectively, and note that $t_2 \neq s_2$ since $B(t_1,2\epsilon) \cap B(s_1,2\epsilon) = \emptyset$. Next note that roots of f_2' occur in complex conjugate pairs, and so we must have $t_2 \in (t_1-\epsilon, t_1+\epsilon) \subset (b,+\infty)$ and $s_2 \in (s_1-\epsilon, s_1+\epsilon) \subset (b,+\infty)$. Definition 2.5 thus implies that $(\chi_2,\eta_2) \in \mathcal{O}$ whenever $|\chi_1 - \chi_2|$ and $|\eta_1 - \eta_2|$ are sufficiently small. This proves (ii).

Consider (iii). This follows from similar arguments to those used to prove part (iii) in the proof of Theorem 3.2.

Consider (iv). Fix $(\chi_1,\eta_1),(\chi_2,\eta_2)\in\mathcal{O}$ with $W_{\mathcal{O}}(\chi_1,\eta_1)=W_{\mathcal{O}}(\chi_2,\eta_2)=(t,s)\in\angle$. Definition 3.11, the definition of $W_{\mathcal{O}}$ (see statement of this theorem), and (3.17), then give,

$$C(t)=\frac{1-\eta_1}{t-\chi_1}=\frac{1-\eta_2}{t-\chi_2}\qquad\text{and}\qquad C(s)=\frac{1-\eta_1}{s-\chi_1}=\frac{1-\eta_2}{s-\chi_2}.$$

Therefore $(\eta_2-\eta_1)t=(1-\eta_1)\chi_2-(1-\eta_2)\chi_1$ and $(\eta_2-\eta_1)s=(1-\eta_1)\chi_2-(1-\eta_2)\chi_1$. Then $t=s$ whenever $\eta_1\neq\eta_2$, which contradicts $(t,s)\in\angle$. Thus $\eta_1=\eta_2$, and so $(1-\eta_1)(\chi_1-\chi_2)=0$. Finally, $\eta_1<1$ since $(\chi_1,\eta_1)\in\mathcal{O}$ (see Definition 3.11), and so $\chi_1=\chi_2$. This proves (iv).

Consider (v). Fix $(\chi,\eta)\in\mathcal{O}$ and let $(t,s):=W_{\mathcal{O}}(\chi,\eta)$. Definition 3.11, the definition of $W_{\mathcal{O}}$, and (3.17) then give $1-\eta=(t-\chi)C(t)=(s-\chi)C(s)$. Solving gives $(\chi,\eta)=(\chi_{\mathcal{O}}(t,s),\eta_{\mathcal{O}}(t,s))$. This proves (v).

Consider (vi). This follows from similar arguments to those used to prove part (vi) in the proof of Theorem 3.2. □

Note, Theorem 3.12 implies that $\mathcal{O}$ is a non-empty, open, simply connected subset of $(a,b)\times(0,1)$. We end this section by proving analogous results to Lemma 3.4 and 3.10. We will be more brief here. As we will see, $\mathcal{O}$ is that open region bounded by $(\chi_{\mathcal{E}}(\cdot),\eta_{\mathcal{E}}(\cdot))|_{(b,+\infty)}$ and the bounding box of $[a,b]\times[0,1]$ in Figure 2.2:

Lemma 3.13. — *Consider $\partial\mathcal{O}$:*

(1) $(\chi_{\mathcal{E}}(t),\eta_{\mathcal{E}}(t))\in\partial\mathcal{O}$ *for all* $t\in(b,+\infty)$. *Moreover,* $(\chi_{\mathcal{O}}(t_k,s_k),\eta_{\mathcal{O}}(t_k,s_k))\to(\chi_{\mathcal{E}}(t),\eta_{\mathcal{E}}(t))$ *as* $k\to\infty$ *for all* $t\in(b,+\infty)$ *and* $\{(t_k,s_k)\}_{k\geqslant1}\subset\angle$ *with* $(t_k,s_k)\to(t,t)\in\partial\angle$.

(2) $(g(s),0)\in\partial\mathcal{O}$ *for all* $s\in(b,+\infty)$ *where* $g(s):=s-C(s)^{-1}$ *for all* $s\in(b,+\infty)$. *Moreover,* $g:(b,+\infty)\to\mathbb{R}$ *is strictly decreasing with* $\lim_{s\uparrow+\infty}g(s)=\mu_1:=\int_a^b x\mu[\mathrm{d}x]$ *and* $\lim_{s\downarrow b}g(s)=b$. *Finally,* $(\chi_{\mathcal{O}}(t_k,s_k),\eta_{\mathcal{O}}(t_k,s_k))\to(g(s),0)$ *as* $k\to\infty$ *for all* $s\in(b,+\infty)$ *and* $\{(t_k,s_k)\}_{k\geqslant1}\subset\angle$ *with* $(t_k,s_k)\to(+\infty,s)\in\partial\angle$.

(3) $(b,h(t))\in\partial\mathcal{O}$ *for all* $t\in(b,+\infty)$, *where* $h(t):=1-(t-b)C(t)$ *for all* $t\in(b,+\infty)$. *Moreover,* $h:(b,+\infty)\to\mathbb{R}$ *is strictly decreasing with* $\lim_{t\uparrow+\infty}h(t)=0$ *and* $\lim_{t\downarrow b}h(t)=1-\mu[\{b\}]$. *Finally,* $(\chi_{\mathcal{O}}(t_k,s_k),\eta_{\mathcal{O}}(t_k,s_k))\to(b,h(t))$ *as* $k\to\infty$ *for all* $t\in(b,+\infty)$ *and* $\{(t_k,s_k)\}_{k\geqslant1}\subset\angle$ *with* $(t_k,s_k)\to(t,b)\in\partial\angle$.

Moreover:

(4) *Fix* $(\chi,\eta)\in\mathcal{O}$ *and the corresponding* $(t,s)\in\angle$ *with* $(\chi,\eta)=(\chi_{\mathcal{O}}(t,s),\eta_{\mathcal{O}}(t,s))$ *(see Definition 3.6 and Theorem 3.12). Then* $f'_{(\chi,\eta)}(y)>0$ *for all* $y\in(b,s)$, $f'_{(\chi,\eta)}(s)=0$ *and* $f''_{(\chi,\eta)}(s)<0$, $f'_{(\chi,\eta)}(y)<0$ *for all* $y\in(s,t)$, $f'_{(\chi,\eta)}(t)=0$ *and* $f''_{(\chi,\eta)}(t)>0$, *and*

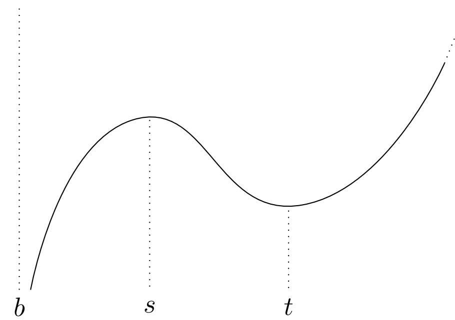

Figure 3.1. The behaviour of $y \mapsto f_{(\chi,\eta)}(y)$ for $y \in (b,\infty)$, for $(\chi,\eta) \in \mathcal{O}$ and $(t,s) \in \angle$ with $(\chi,\eta) = (\chi_{\mathcal{O}}(t,s), \eta_{\mathcal{O}}(t,s))$ when $\mu[\{b\}] > 0$. The function is strictly increasing in (b,s), strictly decreasing in (s,t), and strictly increasing in (t,∞).

$f'_{(\chi,\eta)}(y) > 0$ for all $y \in (t,+\infty)$. The resulting behaviour of the real-valued function $y \mapsto f_{(\chi,\eta)}(y)$ for all $y \in (b,\infty)$ is shown in Figure 3.1.

Proof. — Consider (1). Fix $t \in (b,+\infty)$ and $\{(t_k,s_k)\}_{k\geqslant 1} \subset \angle$ with $(t_k,s_k) \to (t,t) \in \partial\angle$. Write (see Theorem 3.12),

$$\chi_{\mathcal{O}}(t_k,s_k) = t_k + C(s_k)\frac{t_k - s_k}{C(t_k) - C(s_k)},$$
$$\eta_{\mathcal{O}}(t_k,s_k) = 1 + C(t_k)C(s_k)\frac{t_k - s_k}{C(t_k) - C(s_k)}.$$

Thus, since $t_k, s_k \to t \in (b,+\infty)$ as $k \to \infty$, and C is analytic in $(b,+\infty)$,

$$\chi_{\mathcal{O}}(t_k,s_k) \to t + C(t)\frac{1}{C'(t)} = \chi_{\mathcal{E}}(t) \qquad \text{as } k \to \infty,$$
$$\eta_{\mathcal{O}}(t_k,s_k) \to 1 + C(t)C(t)\frac{1}{C'(t)} = \eta_{\mathcal{E}}(t) \quad \text{as } k \to \infty.$$

This proves (1).

Consider (2). First note, $g'(s) = (C(s)^2 + C'(s))/C(s)^2$ for all $s \in (b,+\infty)$. Write as $C(s)^2 g'(s) = C(s)C(s) + \frac{1}{2}C'(s) + \frac{1}{2}C'(s)$, and use (2.11) to get,

$$\begin{aligned} C(s)^2 g'(s) &= \left(\int_a^b \frac{\mu[\mathrm{d}x]}{s-x}\right)\left(\int_a^b \frac{\mu[\mathrm{d}y]}{s-y}\right) - \frac{1}{2}\int_a^b \frac{\mu[\mathrm{d}x]}{(s-x)^2} - \frac{1}{2}\int_a^b \frac{\mu[\mathrm{d}y]}{(s-y)^2} \\ &= -\frac{1}{2}\int_a^b \mu[\mathrm{d}x]\int_a^b \mu[\mathrm{d}y]\left(\frac{1}{s-x} - \frac{1}{s-y}\right)^2. \end{aligned}$$

Thus $g'(s) < 0$ for all $s \in (b, +\infty)$, and so g is strictly decreasing. Next write (see (2.11)),

$$g(s) = \frac{sC(s) - 1}{C(s)} = \frac{\int_a^b \mu[\mathrm{d}x] \frac{x}{s-x}}{\int_a^b \mu[\mathrm{d}x] \frac{1}{s-x}}.$$

Therefore $\lim_{s\uparrow+\infty} g(s) = \int_a^b \mu[\mathrm{d}x]x = \mu_1$. Next note, since $\mu[\{b\}] > 0$, we can write:

$$g(s) = \frac{\mu[\{b\}]\frac{b}{s-b} + \int_{[a,b)} \mu[\mathrm{d}x]\frac{x}{s-x}}{\mu[\{b\}]\frac{1}{s-b} + \int_{[a,b)} \mu[\mathrm{d}x]\frac{1}{s-x}} = \frac{\mu[\{b\}]\frac{b}{s-b} + \int_{[a,b)} \mu[\mathrm{d}x]\frac{x}{s-x}}{\mu[\{b\}]\frac{1}{s-b} + \int_{[a,b)} \mu[\mathrm{d}x]\frac{1}{s-x}},$$

for all $s \in (b, +\infty)$. Also, since $\lim_{\epsilon\downarrow 0} \mu[(b-\epsilon, b)] = 0$, $\int_{[a,b)} \mu[\mathrm{d}x]\frac{1}{s-x} = o((s-b)^{-1})$ and $\int_{[a,b)} \mu[\mathrm{d}x]\frac{x}{s-x} = o((s-b)^{-1})$ as $s \downarrow b$. Therefore,

$$g(s) = \frac{\mu[\{b\}]\frac{b}{s-b} + o(\frac{1}{s-b})}{\mu[\{b\}]\frac{1}{s-b} + o(\frac{1}{s-b})} \to b \qquad \text{as } s \downarrow b.$$

Finally, fix $s \in (b, +\infty)$ and $\{(t_k, s_k)\}_{k\geqslant 1} \subset \angle$ with $(t_k, s_k) \to (+\infty, s) \in \partial\angle$. Recall (see Theorem 3.12),

$$\chi_{\mathcal{O}}(t_k, s_k) = \frac{t_k C(t_k) - s_k C(s_k)}{C(t_k) - C(s_k)}$$
$$\text{and} \quad \eta_{\mathcal{O}}(t_k, s_k) = 1 + \frac{C(t_k)C(s_k)(t_k - s_k)}{C(t_k) - C(s_k)}.$$

Therefore, since $t_k \to +\infty$ and $s_k \to s \in (b, +\infty)$ as $k \to \infty$, (2.11) gives the following for all k sufficiently large:

$$\chi_{\mathcal{O}}(t_k, s_k) = \frac{t_k(\frac{1}{t_k} + O(\frac{1}{t_k^2})) - (sC(s) + O(|s_k - s|))}{O(\frac{1}{t_k}) - (C(s) + O(|s_k - s|))},$$
$$\eta_{\mathcal{O}}(t_k, s_k) = 1 + \frac{(\frac{1}{t_k} + O(\frac{1}{t_k^2}))(C(s) + O(|s_k - s|))(t_k + O(1))}{O(\frac{1}{t_k}) - (C(s) + O(|s_k - s|))}.$$

Therefore $\chi_{\mathcal{O}}(t_k, s_k) \to (1 - sC(s))/(-C(s)) = g(s)$ and $\eta_{\mathcal{O}}(t_k, s_k) \to 1 + (C(s))/(-C(s)) = 0$ as $k \in \infty$. This proves (2).

Consider (3). First note $h'(t) = -C(t) - (t-b)C'(t)$ for all $t \in (b, +\infty)$. (2.11) then gives,

$$h'(t) = -\int_a^b \frac{\mu[\mathrm{d}x]}{t-x} + (t-b)\int_a^b \frac{\mu[\mathrm{d}x]}{(t-x)^2} = \int_a^b \mu[\mathrm{d}x]\frac{x-b}{(t-x)^2}.$$

Thus $h'(t) < 0$ for all $t \in (b, +\infty)$, and so h is strictly decreasing. Next, write (see (2.11)),

$$h(t) = 1 - (t-b)C(t) = 1 - \int_a^b \mu[\mathrm{d}x]\frac{t-b}{t-x}.$$

Therefore $\lim_{t\uparrow+\infty} h(t) = 1 - 1 = 0$. Next note, since $\mu[\{b\}] > 0$, we can write:

$$h(t) = 1 - \mu[\{b\}]\frac{t-b}{t-b} - \int_{[a,b)} \mu[\mathrm{d}x]\frac{t-b}{t-x} = 1 - \mu[\{b\}] - \int_{[a,b)} \mu[\mathrm{d}x]\frac{t-b}{t-x},$$

for all $t \in (b,+\infty)$. Also, since $\lim_{\epsilon\downarrow 0}\mu[(b-\epsilon,b)] = 0$, $\int_{[a,b)}\mu[\mathrm{d}x]\frac{1}{t-x} = o((t-b)^{-1})$ as $t \downarrow b$. Therefore, $h(t) = 1-\mu[\{b\}]+o(1) \to 1-\mu[\{b\}]$ as $t\downarrow b$. Finally, fix $t \in (b,+\infty)$ and $\{(t_k,s_k)\}_{k\geqslant 1} \subset \angle$ with $(t_k,s_k) \to (t,b) \in \partial\angle$. Recall (see Theorem 3.12),

$$\chi_{\mathcal{O}}(t_k,s_k) = \frac{t_k C(t_k) - s_k C(s_k)}{C(t_k) - C(s_k)}$$
$$\text{and}\quad \eta_{\mathcal{O}}(t_k,s_k) = 1 + \frac{C(t_k)C(s_k)(t_k-s_k)}{C(t_k)-C(s_k)}.$$

Then, since $\mu[\{b\}] > 0$, (2.11) gives,

$$\chi_{\mathcal{O}}(t_k,s_k) = \frac{t_k C(t_k) - s_k(\frac{\mu[\{b\}]}{s_k-b} + \int_{[a,b)}\frac{\mu[\mathrm{d}x]}{s_k-x})}{C(t_k) - (\frac{\mu[\{b\}]}{s_k-b} + \int_{[a,b)}\frac{\mu[\mathrm{d}x]}{s_k-x})},$$
$$\eta_{\mathcal{O}}(t_k,s_k) = 1 + \frac{C(t_k)(\frac{\mu[\{b\}]}{s_k-b} + \int_{[a,b)}\frac{\mu[\mathrm{d}x]}{s_k-x})(t_k-s_k)}{C(t_k) - (\frac{\mu[\{b\}]}{s_k-b} + \int_{[a,b)}\frac{\mu[\mathrm{d}x]}{s_k-x})}.$$

Therefore, since $t_k \to t \in (b,+\infty)$ and $s_k \to b$ as $k \to \infty$, and since $\lim_{\epsilon\downarrow 0}\mu[(b-\epsilon,b)] = 0$, the following are satisfied as $k \to \infty$:

$$\chi_{\mathcal{O}}(t_k,s_k) = \frac{(tC(t)+o(1)) - (b+o(1))(\frac{\mu[\{b\}]}{s_k-b} + o(\frac{1}{s_k-b}))}{(C(t)+o(1)) - (\frac{\mu[\{b\}]}{s_k-b} + o(\frac{1}{s_k-b}))},$$
$$\eta_{\mathcal{O}}(t_k,s_k) = 1 + \frac{(C(t)+o(1))(\frac{\mu[\{b\}]}{s_k-b} + o(\frac{1}{s_k-b}))(t-b+o(1))}{(C(t)+o(1)) - (\frac{\mu[\{b\}]}{s_k-b} + o(\frac{1}{s_k-b}))}.$$

Therefore, when $\mu[\{b\}] > 0$, $\chi_{\mathcal{O}}(t_k,s_k) \to b$ and $\eta_{\mathcal{O}}(t_k,s_k) \to 1-(t-b)C(t) = h(t)$ as $k \to \infty$. This proves (3).

Consider (4). Recall that $t,s \in (b,+\infty)$ with $t > s$, $f'_{(\chi,\eta)}$ has a root of multiplicity 1 at both t and s, and $f'_{(\chi,\eta)}$ has 0 roots in $(b,+\infty)\setminus\{t,s\}$ (see Definition 3.11 and Theorem 3.12). Therefore, it is sufficient to show that there exists an $y \in (t,+\infty)$ with $f'_{(\chi,\eta)}(y) > 0$. This follows similarly to the proof of Lemma 3.10 (4). ☐

Next we prove an analogous result for $\mathcal{O}$ to Lemma 3.9 for $\mathcal{E}$:

Lemma 3.14. — *Fix* $(\chi,\eta)\in\mathcal{O}$ *and the corresponding* $(t,s)\in\angle$ *with* $(\chi,\eta)=(\chi_{\mathcal{O}}(t,s),\eta_{\mathcal{O}}(t,s))$. *Define the vectors* $\mathbf{x}(T):=(1,C(T))$ *for all* $T\in(b,+\infty)$. *Then,*

$$(\chi_{\mathcal{O}}(T,S),\eta_{\mathcal{O}}(T,S))=(\chi,\eta)+(T-t)\,c_1\,\mathbf{x}(s)+(S-s)\,c_2\,\mathbf{x}(t)+O((|T-t|+|S-s|)^2),$$

for all $(T,S)\in\angle$ *with* $|T-t|$ *and* $|S-s|$ *sufficiently small, where* $c_1=c_1(t,s)$ *is negative, and* $c_2=c_2(t,s)$ *is negative. Expressions for* c_1 *and* c_2 *are given in* (3.18).

Proof. — This proof is similar to the proof of Lemma 3.9, and so we will be brief here. First recall (see Theorem 3.12),

$$\chi_{\mathcal{O}}(T,S)=\frac{TC(T)-SC(S)}{C(T)-C(S)},$$

for all $(T,S)\in\angle$. Next note, since $\chi=\chi_{\mathcal{O}}(t,s)$, Taylor expansions give,

$$\begin{aligned}\chi_{\mathcal{O}}(T,S)-\chi\\ &=-\frac{(T-t)(t-s)C(s)C'(t)}{(C(t)-C(s))^2}+\frac{(T-t)C(t)}{C(t)-C(s)}\\ &\quad+\frac{(S-s)(t-s)C(t)C'(s)}{(C(t)-C(s))^2}-\frac{(S-s)C(s)}{C(t)-C(s)}+O((|T-t|+|S-s|)^2),\end{aligned}$$

for all $(T,S)\in\angle$ with $|T-t|$ and $|S-s|$ sufficiently small. Next note, since $(t,s)\in\angle\subset(b,+\infty)^2$, (3.17) gives $f''(t)=C'(t)+(1-\eta)/(t-\chi)^2$ and $f''(s)=C'(s)+(1-\eta)/(s-\chi)^2$. Substitute for $(\chi,\eta)=(\chi_{\mathcal{O}}(t,s),\eta_{\mathcal{O}}(t,s))$ (see Theorem 3.12) to get,

$$f''(t)=C'(t)-\frac{C(t)(C(t)-C(s))}{C(s)(t-s)}\quad\text{and}\quad f''(s)=C'(s)-\frac{C(s)(C(t)-C(s))}{C(t)(t-s)}.$$

Substitute $C'(t)$ and $C'(s)$ from the above expressions into the Taylor expansion to get,

$$\chi_{\mathcal{O}}(T,S)-\chi=\frac{-(T-t)(t-s)C(s)f''(t)+(S-s)(t-s)C(t)f''(s)}{(C(t)-C(s))^2}+O((|T-t|+|S-s|)^2),$$

for all $(T,S) \in \angle$ with $|T-t|$ and $|S-s|$ sufficiently small. Similarly we can show that,

$$\eta_{\mathcal{O}}(T,S) - \eta = \frac{-(T-t)(t-s)C(s)^2 f''(t) + (S-s)(t-s)C(t)^2 f''(s)}{(C(t)-C(s))^2} + O((|T-t|+|S-s|)^2),$$

for all $(T,S) \in \angle$ with $|T-t|$ and $|S-s|$ sufficiently small. Finally recall that $f''(t) > 0$ and $f''(s) < 0$ (see Lemma 3.13 (4)). This proves the required result with,

$$c_1(t,s) := -\frac{(t-s)C(s)f''(t)}{(C(t)-C(s))^2} \quad \text{and} \quad c_2(t,s) := \frac{(t-s)C(t)f''(s)}{(C(t)-C(s))^2}. \qquad (3.18)$$

□

Next consider $(\chi,\eta) \in \mathcal{O}$ and the corresponding $(t,s) \in \angle$ with $(\chi,\eta) = (\chi_{\mathcal{O}}(t,s), \eta_{\mathcal{O}}(t,s))$. Recall that $\mathcal{O}$ is depicted in Figure 2.2, and is that region to the lower right of that sub-section of edge curve given by $\theta \mapsto (\chi_{\mathcal{E}}(\theta), \eta_{\mathcal{E}}(\theta))$ for all $T \in (b, +\infty)$. Recall also, Lemma 3.13 (4) proves that $f_{(\chi,\eta)}(s) - f_{(\chi,\eta)}(t) < 0$ (see also Figure 3.1). Moreover, Theorem 2.16 shows that correlation kernels of particles in neighborhoods of $(\chi,\eta) \in \mathcal{O}$ decay exponentially with approximate exponent of decay given by $f_{(\chi,\eta)}(s) - f_{(\chi,\eta)}(t) < 0$. We end this section by examining the behaviour of the exponent as $(\chi,\eta) \in \mathcal{O}$ changes. Lemma 3.15 examines what happens to the exponent as $(\chi,\eta) \in \mathcal{O}$ is moved closer to the edge curve along either horizontal or vertical paths (see Figure 3.3), and Lemma 3.16 examines the behaviour of the exponent in neighborhoods of $\mathcal{E}$.

Lemma 3.15. — *Fix $(\chi,\eta) \in \mathcal{O}$ and the corresponding $(t,s) \in \angle$ with $(\chi,\eta) = (\chi_{\mathcal{O}}(t,s), \eta_{\mathcal{O}}(t,s))$. Similarly fix $(X,Y) \in \mathcal{O}$ and the corresponding $(T,S) \in \angle$ with $(X,Y) = (\chi_{\mathcal{O}}(T,S), \eta_{\mathcal{O}}(T,S))$. Assume that one of the possibilities is satisfied:*

- $\chi < X$ *and* $\eta = Y$.
- $\chi = X$ *and* $\eta > Y$.

These possibilities are depicted on the left of Figure 3.3. Then the following are satisfied:

(1) $T > t > s > S$.
(2) $f_{(X,Y)}(T) - f_{(X,Y)}(S) < f_{(\chi,\eta)}(t) - f_{(\chi,\eta)}(s) < 0$.

Proof. — We will prove the result only when $\chi < X$ and $\eta = Y$. The other case follows from similar considerations.

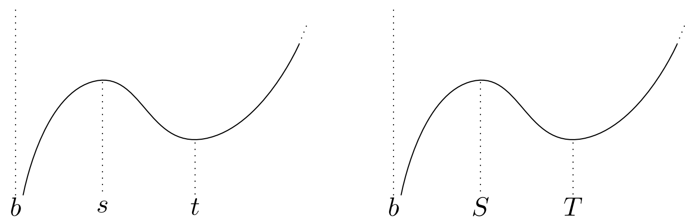

Figure 3.2. Left: The behaviour of $y \mapsto f_{(\chi,\eta)}(y)$ for $y \in (b,\infty)$. Right: The behaviour of $y \mapsto f_{(X,Y)}(y)$ for $y \in (b,\infty)$.

Take $\chi < X$ and $\eta = Y$. Consider (1). First note, similarly to Figure 3.1, Figure 3.2 depicts the behaviours of the real-valued functions $y \mapsto f_{(\chi,\eta)}(y)$ and $y \mapsto f_{(X,Y)}(y)$ for all $y \in (b,\infty)$. Note, (2.8) gives,

$$f_{(X,Y)}(w) = f_{(\chi,\eta)}(w) + (1-\eta)\log(w-\chi) - (1-Y)\log(w-X), \quad (3.19)$$

for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup (b,+\infty)$, where log represents principal value of the logarithm. Thus, since $t > s > b > \max\{\chi, X\}$ (see Definition 3.11 and Theorem 3.12), and $f'_{(\chi,\eta)}(t) = f'_{(\chi,\eta)}(s) = 0$ (see Lemma 3.13(4)),

$$f'_{(X,Y)}(t) = 0 + \frac{1-\eta}{t-\chi} - \frac{1-Y}{t-X}, \qquad f'_{(X,Y)}(s) = 0 + \frac{1-\eta}{s-\chi} - \frac{1-Y}{s-X}.$$

It follows that $f'_{(X,Y)}(t) < 0$ and $f'_{(X,Y)}(s) < 0$ since $\chi < X$ and $\eta = Y$, $t > s > b > X$, and $1 > Y > 0$ (see Definition 3.11 and Theorem 3.12). Finally note that $y \mapsto f_{(X,Y)}(y)$ for all $y \in (b,+\infty)$ is strictly decreasing only when $y \in (S,T)$ (see Figure 3.2). This proves (1).

Consider (2). Recall, Figure 3.2 depicts the behaviours of the real-valued functions $y \mapsto f_{(\chi,\eta)}(y)$ and $y \mapsto f_{(X,Y)}(y)$ for all $y \in (b,\infty)$. In particular note that $f_{(\chi,\eta)}(s) - f_{(\chi,\eta)}(t) < 0$ and $f_{(X,Y)}(S) - f_{(X,Y)}(T) < 0$. Recall also, part (1) gives $T > t > s > S$. Thus, since $y \mapsto f_{(X,Y)}(y)$ for all $y \in (S,T)$ is strictly decreasing in (S,T) (see Figure 3.2) $f_{(X,Y)}(T) - f_{(X,Y)}(S) < f_{(X,Y)}(t) - f_{(X,Y)}(s) < 0$. We can thus prove (2) by showing that $f_{(X,Y)}(t) - f_{(X,Y)}(s) < f_{(\chi,\eta)}(t) - f_{(\chi,\eta)}(s) < 0$.

To see the above, first recall that $t > s > b > \max\{\chi, X\}$. (3.19) then gives,

$$\begin{aligned} f_{(X,Y)}(t) - f_{(X,Y)}(s) &= f_{(\chi,\eta)}(t) - f_{(\chi,\eta)}(s) \\ &\qquad + (1-\eta)\log\left(\frac{t-\chi}{s-\chi}\right) - (1-Y)\log\left(\frac{t-X}{s-X}\right). \end{aligned}$$

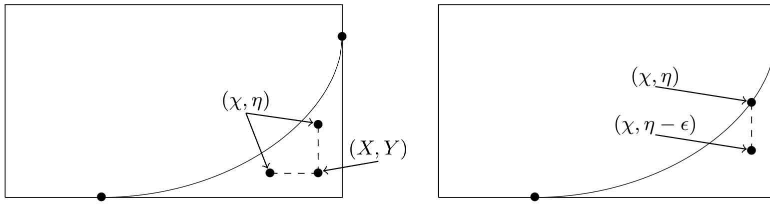

Figure 3.3. Left: The two possibilities of Lemma 3.15. Right: The situation in Lemma 3.16. In both the curve is that sub-section of the edge, $\mathcal{E}$, given by $\theta \mapsto (\chi_{\mathcal{E}}(\theta), \eta_{\mathcal{E}}(\theta))$ for all $\theta \in (b, +\infty)$.

Then, since $\eta = Y$,

$$\begin{aligned} f_{(X,Y)}(t) - f_{(X,Y)}(s) &= f_{(\chi,\eta)}(t) - f_{(\chi,\eta)}(s) + (1-Y)\log\left(\frac{t-\chi}{s-\chi}\frac{s-X}{t-X}\right) \\ &= f_{(\chi,\eta)}(t) - f_{(\chi,\eta)}(s) + (1-Y)\log\left(1 - \frac{(t-s)(X-\chi)}{(s-\chi)(t-X)}\right). \end{aligned}$$

Finally note that the logarithmic term on the right hand side is strictly negative since $\chi < X$ and $\eta = Y$, $t > s > b > X$, and $1 > Y > 0$. This proves that $f_{(X,Y)}(t) - f_{(X,Y)}(s) < f_{(\chi,\eta)}(t) - f_{(\chi,\eta)}(s) < 0$, which proves (2). □

Lemma 3.16. — *Fix $(\chi,\eta) \in \mathcal{E}$ and the corresponding $\theta \in (b, +\infty)$ with $(\chi,\eta) = (\chi_{\mathcal{E}}(\theta), \eta_{\mathcal{E}}(\theta))$ (see Definition 3.6 and Theorem 3.7. See also Figure 2.2). Recall that $\theta > b > \chi$, and $f'_{(\chi,\eta)}(\theta) = f''_{(\chi,\eta)}(\theta) = 0$, and $f'''_{(\chi,\eta)}(\theta) > 0$ (see Lemma 3.10(4)), and define $c = c(\theta) > 0$ by,*

$$c := (\theta-\chi)^{-1} f'''_{(\chi,\eta)}(\theta)^{-1}.$$

Next, fix $\epsilon > 0$ sufficiently small such that $\sqrt{c\epsilon} < \frac{1}{4}(\theta - b)$, $\eta - \epsilon > 0$, and such that (3.21) and (3.22) are satisfied. Finally note that $(\chi, \eta-\epsilon) \in \mathcal{O}$ since $\eta - \epsilon > 0$ (see right of Figure 3.3), and let $(t_\epsilon, s_\epsilon) \in \angle$ denote the point in $\angle$ which corresponds to $(\chi, \eta-\epsilon) \in \mathcal{O}$ (i.e., $(\chi,\eta-\epsilon) = (\chi_{\mathcal{O}}(t_\epsilon, s_\epsilon), \eta_{\mathcal{O}}(t_\epsilon, s_\epsilon))$. Then the following are satisfied:

(1) $\theta + 2\sqrt{c\epsilon} > t_\epsilon > \theta + \sqrt{c\epsilon} > \theta > \theta - \sqrt{c\epsilon} > s_\epsilon > \theta - 2\sqrt{c\epsilon} > b + 2\sqrt{c\epsilon}$.
(2) $f_{(\chi,\eta-\epsilon)}(t_\epsilon) - f_{(\chi,\eta-\epsilon)}(s_\epsilon) < -\frac{5}{6}\frac{\sqrt{c}}{\theta-\chi}\,\epsilon^{\frac{3}{2}}$.

Proof. — Consider (1). Recall that $(\chi, \eta-\epsilon) \in \mathcal{O}$, and (t_ϵ, s_ϵ) is the corresponding point in $\angle$. Recall also that the behaviour of the real-valued function $y \mapsto f_{(\chi,\eta-\epsilon)}(y)$ for all $y \in (b, +\infty)$ is described by Lemma 3.13 (4) and depicted in Figure 3.1 (replace t by t_ϵ and s by s_ϵ). Also note that

$\theta + 2\sqrt{c\epsilon} > \theta + \sqrt{c\epsilon} > \theta > \theta - \sqrt{c\epsilon} > \theta - 2\sqrt{c\epsilon} > b + 2\sqrt{c\epsilon}$ since $\theta > b$ and $\sqrt{c\epsilon} < \frac{1}{4}(\theta - b)$. (1) thus follows if we can prove the following:

(i) $f'_{(\chi,\eta-\epsilon)}(\theta) < 0$.
(ii) $f'_{(\chi,\eta-\epsilon)}(\theta + \sqrt{c\epsilon}) < 0$ and $f'_{(\chi,\eta-\epsilon)}(\theta - \sqrt{c\epsilon}) < 0$.
(iii) $f'_{(\chi,\eta-\epsilon)}(\theta + 2\sqrt{c\epsilon}) > 0$ and $f'_{(\chi,\eta-\epsilon)}(\theta - 2\sqrt{c\epsilon}) > 0$.

Consider (i). First note, (2.8) gives,

$$f_{(\chi,\eta-\epsilon)}(w) = f_{(\chi,\eta)}(w) - \epsilon \log(w - \chi), \tag{3.20}$$

for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup (b, +\infty)$, where log represents principal value of the logarithm. Thus, since $\theta > b > \chi$, and since $f'_{(\chi,\eta)}(\theta) = 0$ (see Definition 3.6 and Theorem 3.7),

$$f'_{(\chi,\eta-\epsilon)}(\theta) = 0 - \frac{\epsilon}{\theta - \chi}.$$

This proves (i).

Consider (ii). First note, since $f'_{(\chi,\eta)}(\theta) = f''_{(\chi,\eta)}(\theta) = 0$ (see Definition 3.6 and Theorem 3.7), Taylors theorem gives,

$$f'_{(\chi,\eta)}(\theta \pm \sqrt{c\epsilon}) = \frac{1}{2} f'''_{(\chi,\eta)}(\theta)(\pm\sqrt{c\epsilon})^2 + \frac{1}{2}\int_\theta^{\theta\pm\sqrt{c\epsilon}} f^{(4)}_{(\chi,\eta)}(y)(\theta \pm \sqrt{c\epsilon} - y)^2 \mathrm{d}y.$$

Recall $c = (\theta - \chi)^{-1} f'''_{(\chi,\eta)}(\theta)^{-1}$ (see statement of this lemma) and (3.20). Then,

$$\left| f'_{(\chi,\eta-\epsilon)}(\theta \pm \sqrt{c\epsilon}) + \frac{1}{2}\frac{\epsilon}{\theta - \chi} \right| \leqslant \frac{1}{2}\left| \int_\theta^{\theta\pm\sqrt{c\epsilon}} f^{(4)}_{(\chi,\eta)}(y)(\theta \pm \sqrt{c\epsilon} - y)^2 \mathrm{d}y \right| + \left| \frac{\epsilon}{\theta - \chi} - \frac{\epsilon}{\theta \pm \sqrt{c\epsilon} - \chi} \right|.$$

Next note, (2.8) gives,

$$\left| f^{(4)}_{(\chi,\eta)}(y) \right| \leqslant \int_a^b \frac{6}{|y - x|^4}\mu[\mathrm{d}x] + (1 - \eta)\frac{6}{|y - \chi|^4},$$

for all $y \in [\theta - \sqrt{c\epsilon}, \theta + \sqrt{c\epsilon}]$. Thus, since $\theta > b > \chi$ and $0 < \eta < 1$ (see Definition 3.6 and Theorem 3.7), and since $\theta - \sqrt{c\epsilon} - b > \frac{1}{2}(\theta - b) > 0$ (recall $4\sqrt{c\epsilon} < \theta - b$),

$$\frac{1}{2}\left| \int_\theta^{\theta\pm\sqrt{c\epsilon}} f^{(4)}_{(\chi,\eta)}(y)(\theta \pm \sqrt{c\epsilon} - y)^2 \mathrm{d}y \right| < \frac{1}{2}\frac{2^8}{(\theta - b)^4}(\sqrt{c\epsilon})^3.$$

Moreover, since $\theta > b > \chi$, and $\theta - \sqrt{c\epsilon} - b > \frac{1}{2}(\theta - b) > 0$,

$$\left| \frac{\epsilon}{\theta - \chi} - \frac{\epsilon}{\theta \pm \sqrt{c\epsilon} - \chi} \right| \leqslant \frac{2\sqrt{c}\epsilon^{\frac{3}{2}}}{(\theta - b)^2}.$$

Finally, choose $\epsilon > 0$ sufficiently small such that,

$$\frac{1}{4}\frac{\epsilon}{\theta-\chi} > \frac{2^7 c^{\frac{3}{2}}}{(\theta-b)^4}\,\epsilon^{\frac{3}{2}} + \frac{2c^{\frac{1}{2}}}{(\theta-b)^2}\,\epsilon^{\frac{3}{2}}. \tag{3.21}$$

Combined, the above give $f'_{(\chi,\eta-\epsilon)}(\theta \pm \sqrt{c\epsilon}) < 0$, which proves (ii). Part (iii) follows similarly.

Consider (2). Recall that the behaviour of the real-valued function $y \mapsto f_{(\chi,\eta-\epsilon)}(y)$ for all $y \in (b,+\infty)$ is described by Lemma 3.13 (4) and depicted in Figure 3.1 (replace t by t_ϵ and s by s_ϵ). Part (1) thus implies that $f_{(\chi,\eta-\epsilon)}(t_\epsilon) - f_{(\chi,\eta-\epsilon)}(s_\epsilon) < f_{(\chi,\eta-\epsilon)}(\theta+\sqrt{c\epsilon}) - f_{(\chi,\eta-\epsilon)}(\theta-\sqrt{c\epsilon})$. We will show:

(iv) $f_{(\chi,\eta-\epsilon)}(\theta+\sqrt{c\epsilon}) - f_{(\chi,\eta-\epsilon)}(\theta-\sqrt{c\epsilon}) < -\frac{5}{6}\frac{\sqrt{c}}{\theta-\chi}\,\epsilon^{\frac{3}{2}}$.

This proves (2).

Consider (iv). First note, (3.20) gives,

$$\begin{aligned} &f_{(\chi,\eta-\epsilon)}(\theta+\sqrt{c\epsilon}) - f_{(\chi,\eta-\epsilon)}(\theta-\sqrt{c\epsilon}) \\ &\qquad = f_{(\chi,\eta)}(\theta+\sqrt{c\epsilon}) - f_{(\chi,\eta)}(\theta-\sqrt{c\epsilon}) \\ &\qquad\qquad - \epsilon\log(\theta+\sqrt{c\epsilon}-\chi) + \epsilon\log(\theta-\sqrt{c\epsilon}-\chi). \end{aligned}$$

Thus, since $f'_{(\chi,\eta)}(\theta) = f''_{(\chi,\eta)}(\theta) = 0$ (see Definition 3.6 and Theorem 3.7), Taylors theorem applied to each term on the RHS gives,

$$\begin{aligned} &f_{(\chi,\eta-\epsilon)}(\theta+\sqrt{c\epsilon}) - f_{(\chi,\eta-\epsilon)}(\theta-\sqrt{c\epsilon}) \\ &= f_{(\chi,\eta)}(\theta) + \frac{1}{6}f'''_{(\chi,\eta)}(\theta)(\sqrt{c\epsilon})^3 + \frac{1}{6}\int_\theta^{\theta+\sqrt{c\epsilon}} f^{(4)}_{(\chi,\eta)}(y)(\theta+\sqrt{c\epsilon}-y)^3\mathrm{d}y \\ &\quad - f_{(\chi,\eta)}(\theta) - \frac{1}{6}f'''_{(\chi,\eta)}(\theta)(-\sqrt{c\epsilon})^3 - \frac{1}{6}\int_\theta^{\theta-\sqrt{c\epsilon}} f^{(4)}_{(\chi,\eta)}(y)(\theta-\sqrt{c\epsilon}-y)^3\mathrm{d}y \\ &\quad - \epsilon\log(\theta-\chi) - \epsilon\frac{\sqrt{c\epsilon}}{\theta-\chi} + \epsilon\int_\theta^{\theta+\sqrt{c\epsilon}} \frac{\theta+\sqrt{c\epsilon}-y}{(y-\chi)^2}\mathrm{d}y \\ &\quad + \epsilon\log(\theta-\chi) + \epsilon\frac{-\sqrt{c\epsilon}}{\theta-\chi} - \epsilon\int_\theta^{\theta-\sqrt{c\epsilon}} \frac{\theta-\sqrt{c\epsilon}-y}{(y-\chi)^2}\mathrm{d}y. \end{aligned}$$

Next recall (see statement of this lemma) that $c = (\theta-\chi)^{-1} f'''_{(\chi,\eta)}(\theta)^{-1}$. Therefore,

$$\begin{aligned}&\left| f_{(\chi,\eta-\epsilon)}(\theta+\sqrt{c\epsilon}) - f_{(\chi,\eta-\epsilon)}(\theta-\sqrt{c\epsilon}) + \frac{5}{3}\frac{\sqrt{c}}{\theta-\chi}\ \epsilon^{\frac{3}{2}} \right| \\ &\quad\leqslant \frac{1}{6}\left| \int_\theta^{\theta+\sqrt{c\epsilon}} f^{(4)}_{(\chi,\eta)}(y)(\theta+\sqrt{c\epsilon}-y)^3 \mathrm{d}y \right| \\ &\qquad + \frac{1}{6}\left| \int_\theta^{\theta-\sqrt{c\epsilon}} f^{(4)}_{(\chi,\eta)}(y)(\theta-\sqrt{c\epsilon}-y)^3 \mathrm{d}y \right| \\ &\qquad + \epsilon\left| \int_\theta^{\theta+\sqrt{c\epsilon}} \frac{\theta+\sqrt{c\epsilon}-y}{(y-\chi)^2}\mathrm{d}y \right| + \epsilon\left| \int_\theta^{\theta-\sqrt{c\epsilon}} \frac{\theta-\sqrt{c\epsilon}-y}{(y-\chi)^2}\mathrm{d}y \right|.\end{aligned}$$

Next proceed similarly to the proof of part (ii) above to get,

$$\frac{1}{6}\left| \int_\theta^{\theta\pm\sqrt{c\epsilon}} f^{(4)}_{(\chi,\eta)}(y)(\theta\pm\sqrt{c\epsilon}-y)^3 \mathrm{d}y \right| < \frac{2^5}{(\theta-b)^4}(\sqrt{c\epsilon})^4,$$

$$\left| \int_\theta^{\theta\pm\sqrt{c\epsilon}} \frac{\theta\pm\sqrt{c\epsilon}-y}{(y-\chi)^2}\mathrm{d}y \right| < \frac{2^2}{(\theta-b)^2}(\sqrt{c\epsilon})^2.$$

Finally, choose $\epsilon > 0$ sufficiently small such that,

$$\frac{5}{6}\frac{\sqrt{c}}{\theta-\chi}\ \epsilon^{\frac{3}{2}} > \frac{2^6}{(\theta-b)^4}\ c^2\ \epsilon^4 + \frac{2^3}{(\theta-b)^2}\ c\ \epsilon^2. \tag{3.22}$$

Combined, the above prove (iv). □

4. Steepest descent analysis

In this section we prove Theorem 2.16 via steepest descent analysis. Recall the following conditions from Theorem 2.16, which we assume throughout Section 4:

- Assume $\mu[\{b\}] > 0$.
- Fix $(\chi,\eta) \in \mathcal{O}$ and the corresponding $(t,s) \in \angle$ with $(\chi,\eta) = (\chi_{\mathcal{O}}(t,s), \eta_{\mathcal{O}}(t,s))$.
- Define u_n, r_n, v_n, s_n as in (2.23).
- Fix $\theta \in (\frac{1}{3}, \frac{1}{2})$.
- Define $\xi = \xi(t,s) > 0$ and $N = N(t,s) \geqslant 1$ as in Definition 2.14.

Using only the above, we will show that N can be chosen sufficiently large that Lemma 2.15 is satisfied. We will then prove Theorem 2.16 for this choice of N. Note, Theorem 2.16 assumes that $r_n = s_n$ for all $n > N$. As stated in

Section 1.3, this trivially gives $\phi_{r_n,s_n}(u_n, v_n) = 0$, but is not used elsewhere. All other asymptotic results in this section hold for general r_n and s_n.

4.1. The roots, and the local asymptotic behaviour, of the steepest descent functions

In this section we examine the roots of the steepest descent functions under the above conditions, and the local asymptotic behaviour of these functions in the neighbourhood of important roots. We begin with the roots of $f_{(\chi,\eta)}$. Note, it is now natural to index $f'_{(\chi,\eta)}$ with $(t,s) \in \angle$ instead of with $(\chi,\eta) \in \mathcal{O}$. (2.10) and (2.12) thus give,

$$\begin{aligned} f'_{(t,s)}(w) &= C(w) - \frac{1-\eta}{w-\chi}, \\ &= \int_{(\chi,b]} \frac{\mu[\mathrm{d}x]}{w-x} - \frac{1-\eta-\mu[\{\chi\}]}{w-\chi} + \int_{[a,\chi)} \frac{\mu[\mathrm{d}x]}{w-x} \end{aligned} \tag{4.1}$$

for all $w \in \mathbb{C} \setminus (S_1 \cup S_2 \cup S_3)$, where

$$\begin{aligned} S_1 &:= \mathrm{Supp}(\mu|_{(\chi,b]}), \\ S_2 &:= \begin{cases} \{\chi\} & \text{when } \mu[\{\chi\}] \neq 1-\eta, \\ \emptyset & \text{when } \mu[\{\chi\}] = 1-\eta, \end{cases} \\ S_3 &:= \mathrm{Supp}(\mu|_{[a,\chi)}). \end{aligned}$$

Note, Assumption 2.1 and Definition 3.11 give:

$$S_1 \neq \emptyset \text{ and } \mu[S_1] > 0, \quad 1-\eta-\mu[\{\chi\}] > 0, \quad S_3 \neq \emptyset \text{ and } \mu[S_3] > 0.$$
$$\mu[S_1] - (1-\eta-\mu[\{\chi\}]) + \mu[S_3] = \mu[a,b] - (1-\eta) = \eta \in (0,1).$$
$$b > \chi > a \text{ and } b = \sup S_1 \geqslant \inf S_1 \geqslant \chi \geqslant \sup S_3 \geqslant \inf S_3 = a.$$

Partition the domain of $f'_{(t,s)}$ as follows:

$$\mathbb{C} \setminus (S_1 \cup S_2 \cup S_3) = (\mathbb{C} \setminus \mathbb{R}) \cup J \cup K, \tag{4.2}$$

where $J := \bigcup_{i=1}^4 J_i$, $K := \bigcup_{i=1,3} K^{(i)}$, and

- $J_1 := (\sup S_1, +\infty) = (b, +\infty)$.
- $J_2 := (-\infty, \inf S_3) = (-\infty, a)$.
- $J_3 := (\sup S_2, \inf S_1) = (\chi, \inf S_1)$ (empty if $\inf S_1 = \chi$).
- $J_4 := (\sup S_3, \inf S_2) = (\sup S_3, \chi)$ (empty if $\sup S_3 = \chi$).
- $K^{(i)} := [\inf S_i, \sup S_i] \setminus S_i$ for all $i \in \{1,3\}$ (note, the indices are chosen to match those of S_i, and so there is no $K^{(2)}$).

This partition is depicted in Figure 4.1. Partition each $K^{(i)}$ as $\{K_1^{(i)}, K_2^{(i)}, \dots\}$, a set of pairwise disjoint open intervals, unique up to order, either empty or finite or countable, and which satisfy $\{\inf I, \sup I\} \subset S_i$ for any $I \in \{K_1^{(i)}, K_2^{(i)}, \dots\}$.

Lemma 4.1. — *We have:*

(1) $f'_{(t,s)}$ *has roots of multiplicity* 1 *at* $t, s \in J_1 = (b, +\infty)$ *where* $t > s$, *and has* 0 *roots in* $J_1 \setminus \{t, s\}$. *Moreover,* $f_{(t,s)}|_{(b,+\infty)}$ *is real-valued, is strictly increasing in* (b, s), *has a local maximum at* s ($f'_{(t,s)}(s) = 0$ *and* $f''_{(t,s)}(s) < 0$), *is strictly decreasing in* (s, t), *has a local minimum at* t ($f'_{(t,s)}(t) = 0$ *and* $f''_{(t,s)}(t) > 0$), *and is strictly increasing in* $(t, +\infty)$.

(2) $f'_{(t,s)}$ *has* 0 *roots in* $\mathbb{C} \setminus \mathbb{R}$, *and in each of* $\{J_2, J_3, J_4\}$.

(3) $f'_{(t,s)}$ *has at most* 1 *root, counting multiplicities, in each of* $\bigcup_{i=1,3}\{K_1^{(i)}, K_2^{(i)}, \dots\}$.

(4) *The following are expressions for* $f''_{(t,s)}(t) > 0$ *and* $f''_{(t,s)}(s) < 0$:

$$f''_{(t,s)}(t) = \int_a^b \frac{\mu[\mathrm{d}x](\chi - x)}{(t-x)^2(t-\chi)} = \int_a^b \int_a^b \frac{(t-s)(x-y)^2 \mu[\mathrm{d}x]\mu[\mathrm{d}y]}{2C(s)(t-x)^2(t-y)^2(s-x)(s-y)},$$

$$f''_{(t,s)}(s) = \int_a^b \frac{\mu[\mathrm{d}x](\chi - x)}{(s-x)^2(s-\chi)} = \int_a^b \int_a^b \frac{-(t-s)(x-y)^2 \mu[\mathrm{d}x]\mu[\mathrm{d}y]}{2C(t)(s-x)^2(s-y)^2(t-x)(t-y)}.$$

Proof. — Consider (1), (2), and (3). Note, since $(\chi, \eta) \in \mathcal{O}$, Lemma 3.13 (4), and possibility (a) of Theorem 5.2 trivially imply parts (1), (2), and (3).

Consider (4). First recall, part (1) gives $f'_{(t,s)}(t) = 0$. (4.1) then gives,

$$\frac{1-\eta}{t-\chi} = \int_a^b \frac{\mu[\mathrm{d}x]}{t-x}.$$

Moreover, (4.1) gives,

$$f''_{(t,s)}(t) = -\int_a^b \frac{\mu[\mathrm{d}x]}{(t-x)^2} + \frac{1-\eta}{(t-\chi)^2}.$$

Combined, the above give,

$$f''_{(t,s)}(t) = -\int_a^b \frac{\mu[\mathrm{d}x]}{(t-x)^2} + \int_a^b \frac{\mu[\mathrm{d}x]}{(t-x)(t-\chi)} = \int_a^b \frac{\mu[\mathrm{d}x](\chi - x)}{(t-x)^2(t-\chi)}.$$

This proves the first expression for $f''_{(t,s)}(t)$.

Consider the second expression for $f''_{(t,s)}(t)$. Recall that $\chi = \chi_{\mathcal{O}}(t, s)$, where an expression for $\chi_{\mathcal{O}}(t, s)$ is given in the statement of Theorem 3.12.

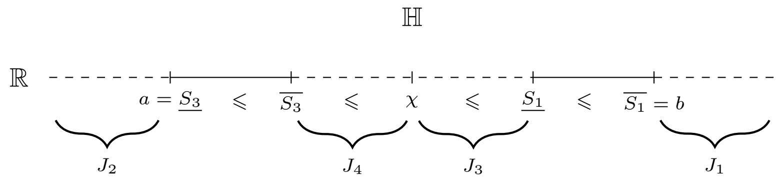

Figure 4.1. The sets of (4.2), with $b > \chi > a$, $K^{(i)} = [\inf S_i, \sup S_i]\backslash S_i$ for $i \in \{1, 3\}$, $\underline{S_1} := \inf S_1$, $\overline{S_1} := \sup S_1$, etc.

This gives,

$$f''_{(t,s)}(t) = \int_a^b \frac{\mu[\mathrm{d}x]}{(t-x)^2} \; \frac{\chi - x}{t - \chi} = \int_a^b \frac{\mu[\mathrm{d}x]}{(t-x)^2} \; \frac{(t-x)C(t) - (s-x)C(s)}{-(t-s)C(s)}.$$

Equation (2.11) then gives,

$$\begin{aligned} f''_{(t,s)}(t) &= \int_a^b \frac{\mu[\mathrm{d}x]}{(t-x)^2} \; \frac{1}{-(t-s)C(s)} \; \int_a^b \left(\frac{t-x}{t-y} - \frac{s-x}{s-y} \right) \mu[\mathrm{d}y] \\ &= -\int_a^b \int_a^b \frac{(x-y)}{C(s)(t-x)^2(t-y)(s-y)} \mu[\mathrm{d}x]\mu[\mathrm{d}y]. \end{aligned}$$

Thus, since x and y are dummy parameters,

$$\begin{aligned} &f''_{(t,s)}(t) \\ &= -\frac{1}{2}\int_a^b\int_a^b \left(\frac{(x-y)}{C(s)(t-x)^2(t-y)(s-y)} + \frac{(y-x)}{C(s)(t-y)^2(t-x)(s-x)} \right) \mu[\mathrm{d}x]\mu[\mathrm{d}y] \\ &= \frac{1}{2}\int_a^b\int_a^b \frac{(t-s)(x-y)^2\mu[\mathrm{d}x]\mu[\mathrm{d}y]}{C(s)(t-x)^2(t-y)^2(s-x)(s-y)}. \end{aligned}$$

This gives the second expression for $f''_{(t,s)}(t)$. We can similarly prove the first and second expression for $f''_{(t,s)}(s)$. This proves (4). ☐

We next prove Lemma 2.15 which examine the roots of the "non-asymptotic" functions, $f'_{(t,s),n}, f'_n, \widetilde{f}'_n$. Recall, (2.24), (2.25), and (2.26) give the following for all $n > N$:

$$f'_{(t,s),n}(w) = C_n(w) - \frac{1-\eta_n}{w - \chi_n} = \frac{1}{n} \sum_{x \in P_n} \frac{1}{w - x} - \frac{1 - \eta_n}{w - \chi_n}, \tag{4.3}$$

$$f'_n(w) = C_n(w) - \frac{1 - \frac{s_n - 1}{n}}{w - v_n} = \frac{1}{n} \sum_{x \in P_n} \frac{1}{w - x} - \frac{1 - \frac{s_n - 1}{n}}{w - v_n}, \tag{4.4}$$

$$\widetilde{f}'_n(w) = C_n(w) - \frac{1 - \frac{r_n + 1}{n}}{w - v_n} = \frac{1}{n} \sum_{x \in P_n} \frac{1}{w - x} - \frac{1 - \frac{r_n + 1}{n}}{w - u_n}. \tag{4.5}$$

where P_n and μ_n and C_n are defined in (2.20), and (χ_n, η_n), (u_n, r_n), (v_n, s_n) are defined in Definition 2.12 and (2.23). The above functions have domains $\mathbb{C} \setminus (P_n \cup \{\chi_n\}$ and $\mathbb{C} \setminus (P_n \cup \{v_n\})$ and $\mathbb{C} \setminus (P_n \cup \{u_n\})$ respectively. Also recall, Definition 2.14 gives the following for all $n > N$:

$$\begin{aligned} & t-4\xi > s+4\xi > s-4\xi > b+4\xi > b-4\xi > \chi+4\xi > \chi-4\xi > a+4\xi, \\ & b+4\xi > x_1^{(n)} > b-4\xi \text{ and } a+4\xi > x_n^{(n)} > a-4\xi, \\ & \chi+4\xi > \{\chi_n, v_n, u_n\} > \chi-4\xi, \\ & 1-2\xi > 1-\eta+2\xi > \left\{1-\eta_n, 1-\frac{s_n-1}{n}, 1-\frac{r_n+1}{n}\right\} > 1-\eta-2\xi > 2\xi, \end{aligned} \tag{4.6}$$

Note the above implies that $\xi < \frac{1}{8}(t-s), \frac{1}{16}(t-b), \frac{1}{24}(t-\chi), \frac{1}{8}(s-b), \frac{1}{4}(1-\eta)$, etc.

Proof of Lemma 2.15. — Fix $\xi = \xi(t,s) > 0$ and $N = N(t,s) \geqslant 1$ as in Definition 2.14. First note, (4.1) and (4.6) imply that $B(t, 2\xi)$ and $B(s, 2\xi)$ are disjoint open subsets of $(\mathbb{C} \setminus \mathbb{R}) \cup (b+4\xi, +\infty)$, and $f'_{(t,s)}$ is well-defined and analytic in $B(t,2\xi) \cup B(s,2\xi)$. Parts (1), (2), and (3) then follow trivially from Lemma 4.1 (1) and (2).

Next note (4.3) and (4.6) imply that $f'_{(t,s),n}$ is well-defined and analytic in $B(t,2\xi) \cup B(s,2\xi)$ for all $n > N$. Part (4) then follows trivially from (4.3) and Definition 2.12.

Consider (5). Note, (4.1) and (4.3) give the following for all $n > N$:

$$\begin{aligned} |f''_{(t,s),n}(t) - f''_{(t,s)}(t)| &\leqslant |C'_n(t) - C'(t)| + \left| \frac{1-\eta_n}{(t-\chi_n)^2} - \frac{1-\eta}{(t-\chi)^2} \right|, \\ |f''_{(t,s),n}(s) - f''_{(t,s)}(s)| &\leqslant |C'_n(s) - C'(s)| + \left| \frac{1-\eta_n}{(s-\chi_n)^2} - \frac{1-\eta}{(s-\chi)^2} \right|. \end{aligned}$$

Recall that $\chi_n = \chi_n(t,s)$ and $\chi = \chi_{\mathcal{O}}(t,s)$, and similarly for η_n and η. (2.21) and part (2) then give $f''_{(t,s),n}(t) \to f''_{(t,s)}(t) > 0$ and $f''_{(t,s),n}(s) \to f''_{(t,s)}(s) < 0$ as $n \to \infty$. This proves (5).

Consider (6). We will show the following:

$$\inf_{w \in \partial B(t,\xi)} |f'_{(t,s)}(w)| > 0 \text{ and } \lim_{n\to\infty} \sup_{w \in \mathrm{cl}(B(t,\xi))} |f'_{(t,s)}(w) - f'_{(t,s),n}(w)| = 0. \tag{i}$$

This implies that we can choose N such that the following is satisfied for all $n > N$:

$$\inf_{w \in \partial B(t,\xi)} |f'_{(t,s)}(w)| > \sup_{w \in \mathrm{cl}(B(t,\xi))} |f'_{(t,s)}(w) - f'_{(t,s),n}(w)|.$$

Then parts (1), (2), (3), and Rouché's theorem imply that $f'_{(t,s),n}$ has exactly 1 root in $B(t,\xi)$ for all $n > N$ and counting multiplicities. Similarly we can choose N such that $f'_{(t,s),n}$ has exactly 1 root in $B(s,\xi)$ for all $n > N$ and counting multiplicities. Moreover, part (4) implies that these roots are t and s respectively. This proves (6).

Consider (7). First note, for all $n > N$, Definition 2.14 and (4.3), (4.4), (4.5), and (4.6) give $B(t, 2n^{-\frac{1}{2}}) \subset B(t,\xi)$ and $B(s, 2n^{-\frac{1}{2}}) \subset B(s,\xi)$, and $f'_n, \widetilde{f}'_n$ are well-defined and analytic in $B(t,2\xi) \cup B(s,2\xi)$. Fix $n > N$. (4.3) and (4.4) then give the following:

$$|f'_n(t) - f'_{(t,s),n}(t)| = \left| \frac{1-\eta_n}{t-\chi_n} - \frac{1-\frac{s_n-1}{n}}{t-v_n} \right|,$$
$$|f''_n(t) - f''_{(t,s),n}(t)| = \left| \frac{1-\eta_n}{(t-\chi_n)^2} - \frac{1-\frac{s_n-1}{n}}{(t-v_n)^2} \right|.$$

Recall that $f'_{(t,s),n}(t) = 0$ (see part (4)), $\frac{C_n(t)}{1-\eta_n} = \frac{1}{t-\chi_n}$ (see Definition 2.12), and (2.23). Combined these give,

$$|f'_n(t)| = |f'_n(t) - f'_{(t,s),n}(t)| = \left| \frac{1-\eta_n}{t-\chi_n} - \frac{1-\eta_n}{t-\chi_n} \frac{1 - \frac{m_n}{t-\chi_n} n^{-\frac{1}{2}} - \frac{y_{2,n}}{1-\eta_n} n^{-1}}{1 - \frac{m_n}{t-\chi_n} n^{-\frac{1}{2}} - \frac{y_{1,n}}{t-\chi_n} n^{-1}} \right|,$$
$$|f''_n(t) - f''_{(t,s),n}(t)| = \left| \frac{1-\eta_n}{(t-\chi_n)^2} - \frac{1-\eta_n}{(t-\chi_n)^2} \frac{1 - \frac{m_n}{t-\chi_n} n^{-\frac{1}{2}} - \frac{y_{2,n}}{1-\eta_n} n^{-1}}{(1 - \frac{m_n}{t-\chi_n} n^{-\frac{1}{2}} - \frac{y_{1,n}}{t-\chi_n} n^{-1})^2} \right|.$$

Therefore,

$$|f'_n(t)| := \frac{|1-\eta_n|}{|t-\chi_n|} \frac{|-\frac{y_{1,n}}{t-\chi_n} n^{-1} + \frac{y_{2,n}}{1-\eta_n} n^{-1}|}{|1 - \frac{m_n}{t-\chi_n} n^{-\frac{1}{2}} - \frac{y_{1,n}}{t-\chi_n} n^{-1}|},$$
$$|f''_n(t) - f''_{(t,s),n}(t)|$$
$$:= \frac{|1-\eta_n|}{|t-\chi_n|^2} \frac{|(1 - \frac{m_n}{t-\chi_n} n^{-\frac{1}{2}} - \frac{y_{1,n}}{t-\chi_n} n^{-1})^2 - (1 - \frac{m_n}{t-\chi_n} n^{-\frac{1}{2}} - \frac{y_{2,n}}{1-\eta_n} n^{-1})|}{|1 - \frac{m_n}{t-\chi_n} n^{-\frac{1}{2}} - \frac{y_{1,n}}{t-\chi_n} n^{-1}|^2},$$

and so $|f'_n(t)| = B_{1,n} n^{-1}$ and $|f''_n(t) - f''_{(t,s),n}(t)| = B_{2,n} n^{-\frac{1}{2}}$ where,

$$B_{1,n} := \frac{|1-\eta_n|}{|t-\chi_n|} \frac{|-\frac{y_{1,n}}{t-\chi_n} + \frac{y_{2,n}}{1-\eta_n}|}{|1 - \frac{m_n}{t-\chi_n} n^{-\frac{1}{2}} - \frac{y_{1,n}}{t-\chi_n} n^{-1}|},$$
$$B_{2,n}$$
$$:= \frac{|1-\eta_n|}{|t-\chi_n|^2} \frac{|-\frac{m_n}{t-\chi_n} - 2\frac{y_{1,n}}{t-\chi_n} n^{-\frac{1}{2}} + \frac{y_{2,n}}{1-\eta_n} n^{-\frac{1}{2}} + (\frac{m_n}{t-\chi_n} n^{-\frac{1}{2}} + \frac{y_{1,n}}{t-\chi_n} n^{-1})^2 n^{\frac{1}{2}}|}{|1 - \frac{m_n}{t-\chi_n} n^{-\frac{1}{2}} - \frac{y_{1,n}}{t-\chi_n} n^{-1}|^2}.$$

Similarly we can show that $|\widetilde{f}'_n(s)| = \widetilde{B}_{1,n} n^{-1}$ and $|\widetilde{f}''_n(s) - f''_{(t,s),n}(s)| = \widetilde{B}_{2,n} n^{-\frac{1}{2}}$ where,

$$\widetilde{B}_{1,n} := \frac{|1-\eta_n|}{|s-\chi_n|} \, \frac{|-\frac{\tilde{y}_{1,n}}{s-\chi_n} + \frac{\tilde{y}_{2,n}}{1-\eta_n}|}{|1 - \frac{\tilde{m}_n}{s-\chi_n} n^{-\frac{1}{2}} - \frac{\tilde{y}_{1,n}}{s-\chi_n} n^{-1}|},$$

$$\widetilde{B}_{2,n}$$

$$:= \frac{|1-\eta_n|}{|s-\chi_n|^2} \, \frac{|-\frac{\tilde{m}_n}{s-\chi_n} - 2\frac{\tilde{y}_{1,n}}{s-\chi_n} n^{-\frac{1}{2}} + \frac{\tilde{y}_{2,n}}{1-\eta_n} n^{-\frac{1}{2}} + (\frac{\tilde{m}_n}{s-\chi_n} n^{-\frac{1}{2}} + \frac{\tilde{y}_{1,n}}{s-\chi_n} n^{-1})^2 n^{\frac{1}{2}}|}{|1 - \frac{\tilde{m}_n}{s-\chi_n} n^{-\frac{1}{2}} - \frac{\tilde{y}_{1,n}}{s-\chi_n} n^{-1}|^2}.$$

Finally recall (see (2.23)) that $m_n, \widetilde{m}_n, y_{1,n}, y_{2,n}, \widetilde{y}_{1,n}, \widetilde{y}_{2,n} = O(1)$ for all n sufficiently large, and (see (4.6)) that $\frac{3}{2}(1-\eta) > 1-\eta_n > \frac{1}{2}(1-\eta) > 0$ and $t-\chi_n > \frac{5}{6}(t-\chi) > 0$ and $s-\chi_n > \frac{3}{4}(s-\chi) > 0$. Combined with the above expressions this gives $B_{1,n}, B_{2,n}, \widetilde{B}_{1,n}, \widetilde{B}_{2,n} = O(1)$ for all n sufficiently large. This proves (7). Part (8) follows trivially from parts (5) and (7).

Consider (9). Recall that $f'_{(t,s)}, f'_n$ are well-defined and analytic in $B(s, 2\xi)$ for all $n > N$. We will show the following:

$$\inf_{w \in \partial B(s,\xi)} |f'_{(t,s)}(w)| > 0 \text{ and } \lim_{n\to\infty} \sup_{w \in \mathrm{cl}(B(s,\xi))} |f'_{(t,s)}(w) - f'_n(w)| = 0. \quad \text{(ii)}$$

This implies that we can choose N such that the following is satisfied for all $n > N$:

$$\inf_{w \in \partial B(s,\xi)} |f'_{(t,s)}(w)| > \sup_{w \in \mathrm{cl}(B(s,\xi))} |f'_{(t,s)}(w) - f'_n(w)|.$$

Then parts (1), (2), (3), and Rouché's theorem imply that f'_n has exactly 1 root in $B(s,\xi)$, for all $n > N$ and counting multiplicities. Denote this by s_n. Moreover, (4.4) implies that roots of f'_n occur in complex conjugate pairs, and so s_n must be real-valued. More exactly, $s_n \in (s-\xi, s+\xi)$ for all $n > N$.

Next, for all $n > N$, recall that $f'_{(t,s),n}, f'_n$ are well-defined and analytic in $B(t, 2\xi)$, and note that $B(t, 2n^{-\frac{1}{2}}) \subset B(t,\xi)$ (see Definition 2.14). Also recall (see part (5)) that $f'_{(t,s),n}(t) = 0$ for all $n > N$. Then, for all $w \in \mathrm{cl}(B(t, n^{-\frac{1}{2}}))$ and $n > N$, Taylors theorem gives,

$$f'_{(t,s),n}(w) = f''_{(t,s),n}(t)(w-t) + \int_t^w \mathrm{d}z \, f'''_{(t,s),n}(z)(w-z),$$

where the integral is along the straight line from t to w. It follows that,

$$\inf_{w \in \partial B(t, n^{-\frac{1}{2}})} |f'_{(t,s),n}(w)| \geqslant |f''_{(t,s),n}(t)|(n^{-\frac{1}{2}}) - \sup_{z \in \mathrm{cl}(B(t,n^{-\frac{1}{2}}))} |f'''_{(t,s),n}(z)|(n^{-\frac{1}{2}})^2,$$

for all $n > N$. Recall, part (5) gives $|f''_{(t,s),n}(t)| > \frac{1}{2}|f''_{(t,s)}(t)|$ for all $n > N$. Moreover, we will show that we can choose N such that the following is satisfied for all $n > N$:

$$\sup_{z\in \mathrm{cl}(B(t,n^{-\frac{1}{2}}))} |f'''_{(t,s),n}(z)|n^{-\frac{1}{2}} < \frac{1}{4}|f''_{(t,s)}(t)|, \tag{iii}$$

Combined the above give, for all $n > N$,

$$\inf_{w\in\partial B(t,n^{-\frac{1}{2}})} |f'_{(t,s),n}(w)| > \frac{1}{4}|f''_{(t,s)}(t)|(n^{-\frac{1}{2}}).$$

Finally, we will show that we can choose N such that for all $n > N$:

$$\frac{1}{4}|f''_{(t,s)}(t)|(n^{-\frac{1}{2}}) > \sup_{w\in \mathrm{cl}(B(t,n^{-\frac{1}{2}}))} |f'_{(t,s),n}(w) - f'_n(w)|, \tag{iv}$$

Therefore, for all $n > N$,

$$\inf_{w\in\partial B(t,n^{-\frac{1}{2}})} |f'_{(t,s),n}(w)| > \sup_{w\in \mathrm{cl}(B(t,n^{-\frac{1}{2}}))} |f'_{(t,s),n}(w) - f'_n(w)|.$$

Then parts (4), (5), (6), and Rouché's theorem imply that f'_n has exactly 1 root in $B(t, n^{-\frac{1}{2}})$, for all $n > N$ and counting multiplicities. Denote this by t_n. Moreover, (4.4) implies that roots of f'_n occur in complex conjugate pairs, and $n^{-\frac{1}{2}} < \frac{1}{2}\xi$ for all $n > N$ (see Definition 2.14), and so $t_n \in (t - n^{-\frac{1}{2}}, t - n^{-\frac{1}{2}}) \subset (t - \frac{1}{2}\xi, t + \frac{1}{2}\xi)$. This proves (9).

Consider (10). Recall that $f'_{(t,s)}, \widetilde{f}'_n$ are well-defined and analytic in $B(t, 2\xi)$ for all $n > N$. We will show the following:

$$\inf_{w\in\partial B(t,\xi)} |f'_{(t,s)}(w)| > 0 \text{ and } \lim_{n\to\infty} \sup_{w\in \mathrm{cl}(B(t,\xi))} |f'_{(t,s)}(w) - \widetilde{f}'_n(w)| = 0. \tag{v}$$

This implies that we can choose N such that the following is satisfied for all $n > N$:

$$\inf_{w\in\partial B(t,\xi)} |f'_{(t,s)}(w)| > \sup_{w\in \mathrm{cl}(B(t,\xi))} |f'_{(t,s)}(w) - \widetilde{f}'_n(w)|.$$

Then parts (1), (2), (3), and Rouché's theorem imply that $\widetilde{f}'_n$ has exactly 1 root in $B(t, \xi)$, for all $n > N$ and counting multiplicities. Denote it by $\widetilde{t}_n$. Moreover, (4.5) implies that roots of $\widetilde{f}'_n$ occur in complex conjugate pairs, and so $\widetilde{t}_n \in (t - \xi, t + \xi)$.

Next note, similar to part (9), $f'_{(t,s),n}(s) = 0$ for all $n > N$, and so Taylors theorem gives the following:

$$\inf_{w\in\partial B(s,n^{-\frac{1}{2}})} |f'_{(t,s),n}(w)| \geqslant |f''_{(t,s),n}(s)|(n^{-\frac{1}{2}}) - \sup_{z\in \mathrm{cl}(B(s,n^{-\frac{1}{2}}))} |f'''_{(t,s),n}(z)|(n^{-\frac{1}{2}})^2.$$

Recall that part (5) gives $|f''_{(t,s),n}(s)| > \frac{1}{2}|f''_{(t,s)}(s)|$ for all $n > N$. We will show that we can choose N such that the following are satisfied for all $n > N$:

$$\sup_{z\in \mathrm{cl}(B(s,n^{-\frac{1}{2}}))} |f'''_{(t,s),n}(z)|n^{-\frac{1}{2}} < \frac{1}{4}|f''_{(t,s)}(s)|, \tag{vi}$$

$$\frac{1}{4}|f''_{(t,s)}(s)|(n^{-\frac{1}{2}}) > \sup_{w\in \mathrm{cl}(B(s,n^{-\frac{1}{2}}))} |f'_{(t,s),n}(w) - \widetilde{f}'_n(w)|. \tag{vii}$$

Then parts (4), (5), (6), and Rouché's theorem imply that $\widetilde{f}'_n$ has exactly 1 root in $B(s, n^{-\frac{1}{2}})$, for all $n > N$ and counting multiplicities. Denote this by $\widetilde{s}_n$. Moreover, (4.5) implies that roots of $\widetilde{f}'_n$ occur in complex conjugate pairs, and $n^{-\frac{1}{2}} < \frac{1}{2}\xi$, and so $\widetilde{s}_n \in (s - n^{-\frac{1}{2}}, s - n^{-\frac{1}{2}}) \subset (s - \frac{1}{2}\xi, s + \frac{1}{2}\xi)$. This proves (10).

Consider (11). First recall, part (4) gives $f'_{(t,s),n}(t) = f'_{(t,s),n}(s) = 0$ for all $n > N$. We can then use (4.3), and proceed similarly to the proof of Lemma 4.1 (4) to get,

$$f''_{(t,s),n}(t) = \frac{1}{n}\sum_{x\in P_n} \frac{\chi_n - x}{(t-x)^2(t-\chi_n)},\ f''_{(t,s),n}(s) = \frac{1}{n}\sum_{x\in P_n} \frac{\chi_n - x}{(s-x)^2(s-\chi_n)},$$

for all $n > N$. Moreover, parts (9) and (10) give $f'_n(t_n) = \widetilde{f}'_n(\widetilde{s_n}) = 0$ for all $n > N$. We can then use (4.4) and (4.5) and proceed similarly to get,

$$f''_n(t_n) = \frac{1}{n}\sum_{x\in P_n} \frac{v_n - x}{(t_n-x)^2(t_n-v_n)},\ \widetilde{f}''_n(\widetilde{s}_n) = \frac{1}{n}\sum_{x\in P_n} \frac{u_n - x}{(\widetilde{s}_n-x)^2(\widetilde{s}_n-u_n)},$$

for all $n > N$. Next note, (2.23) and parts (9) and (10) give the following for all $n > N$: $|v_n - \chi_n| \leqslant |m_n|n^{-\frac{1}{2}} + |y_{1,n}|n^{-1}$, $|u_n - \chi_n| \leqslant |\widetilde{m}_n|n^{-\frac{1}{2}} + |\widetilde{y}_{1,n}|n^{-1}$, $|t_n - t| < n^{-\frac{1}{2}} < \frac{1}{2}\xi$, and $|\widetilde{s}_n - s| < n^{-\frac{1}{2}} < \frac{1}{2}\xi$. Finally recall that $x_1^{(n)} = \max P_n$ and $x_n^{(n)} = \min P_n$ (see (2.20)), and note (4.6) gives the following for all $n > N$ and $x \in P_n$: $\max\{|\chi_n - x|, |u_n - x|, |v_n - x|\} < \min\{2(b-\chi), 2(\chi - a)\}$, $|t - x| > \frac{3}{4}(t-b) > 0$, $|t - \chi_n| > \frac{5}{6}(t-\chi) > 0$, $|s - x| > \frac{1}{2}(s-b) > 0$, $|s - \chi_n| > \frac{3}{4}(s-\chi) > 0$, $|t_n - x| > \frac{23}{32}(t-b) > 0$, $|t_n - v_n| > \frac{39}{48}(t-\chi) > 0$, $|\widetilde{s}_n - x| > \frac{7}{16}(s-b) > 0$, and $|\widetilde{s} - u_n| > \frac{23}{32}(s-\chi) > 0$. Combined, the above imply that we can choose N sufficiently large such that the following are also satisfied for all $n > N$:

$$|f''_{(t,s),n}(t) - f''_n(t_n)| < \frac{1}{4}|f''_{(t,s)}(t)| \quad\text{and}\quad |f''_{(t,s),n}(s) - \widetilde{f}''_n(\widetilde{s}_n)| < \frac{1}{4}|f''_{(t,s)}(s)|.$$

Finally recall (see part (5)) that $f''_{(t,s),n}(t) > \frac{1}{2}f''_{(t,s)}(t) > 0$ and $f''_{(t,s),n}(s) < \frac{1}{2}f''_{(t,s)}(s) < 0$ for all $n > N$. This proves (11).

Consider (i). Note, the first part of (i) follows from the extreme value theorem, since $f'_{(t,s)}$ is analytic in $B(t, 2\xi)$ (see parts (1)–(3)). We prove the second part of (i) via contradiction: Assume that the second part does not hold. Then there exists a $\delta > 0$ for which, for all $n \geqslant 1$, there exists some $p_n \geqslant n$ and $z_n \in \mathrm{cl}(B(t,\xi))$ with $\delta < |f'_{(t,s),n}(z_n) - f'_{(t,s),p_n}(z_n)|$. Choosing $\{z_n\}_{n\geqslant 1}$ to be convergent, and denoting the limit by z, the triangle inequality gives

$$\delta < |f'_{(t,s)}(z_n) - f'_{(t,s)}(z)| + |f'_{(t,s)}(z) - f'_{(t,s),p_n}(z)| \\ + |f'_{(t,s),p_n}(z) - f'_{(t,s),p_n}(z_n)|. \quad (4.7)$$

Note, $|f'_{(t,s)}(z_n) - f'_{(t,s)}(z)| \to 0$ since $z_n \to z$, $\{z, z_1, z_2 \ldots\} \subset \mathrm{cl}(B(t,\xi))$, and $f'_{(t,s)}$ is analytic in $B(t,2\xi)$. Also, since $z \in \mathrm{cl}(B(t,\xi))$, (2.21), (4.1), and (4.3) imply that $|f'_{(t,s)}(z) - f'_{(t,s),p_n}(z)| \to 0$. Finally, (4.3) implies that,

$$|f'_{(t,s),p_n}(z) - f'_{(t,s),p_n}(z_n)| \leqslant \sup_{x\in P_n} \left| \frac{1}{z-x} - \frac{1}{z_n - x} \right| + \left| \frac{1-\eta_n}{z-\chi_n} - \frac{1-\eta_n}{z_n-\chi_n} \right|.$$

Then, since $z_n \to z$ and $\{z, z_1, z_2 \ldots\} \subset \mathrm{cl}(B(t,\xi))$, (4.6) implies that $|f'_{p_n}(z) - f'_{p_n}(z_n)| \to 0$. The above observations contradict (4.7), and so our assumption is false. This proves the second part of (i). Parts (ii) and (v) have similar proofs.

Consider (iii). First note, for all $n > N$, (4.3) gives,

$$f'''_{(t,s),n}(z) = \frac{1}{n} \sum_{x\in P_n} \frac{2}{(z-x)^3} - \frac{2(1-\eta_n)}{(z-\chi_n)^3},$$

for all $z \in \mathrm{cl}(B(t, n^{-\frac{1}{2}}))$. Next recall that $n^{-\frac{1}{2}} < \frac{1}{2}\xi$ for all $n > N$(see Definition 2.14), $x_1^{(n)} = \max P_n$ (see (2.20)), and note (4.6) gives the following for all $n > N$: $1 > 1 - \eta_n > 0$, $|z - \chi_n| > \frac{39}{48}(t-\chi) > 0$ for all $z \in \mathrm{cl}(B(t,n^{-\frac{1}{2}}))$, and $|z - x| > \frac{23}{32}(t-b) > 0$ for all $z \in \mathrm{cl}(B(t,n^{-\frac{1}{2}}))$ and $x \in P_n$. Thus, for all $n > N$,

$$\sup_{z\in \mathrm{cl}(B(t,n^{-\frac{1}{2}}))} |f'''_{(t,s),n}(z)| < \frac{1}{n} \sum_{x\in P_n} \frac{2}{(\frac{23}{32}(t-b))^3} + \frac{2}{(\frac{39}{48}(t-\chi))^3} \\ < \frac{2^3}{(t-b)^3} + \frac{2^2}{(t-\chi)^3}.$$

Part (iii) easily follows.

Consider (iv). Proceed similarly to the proof of part (7) above to get,

$$\begin{aligned}
&|f'_{(t,s),n}(w) - f'_n(w)| \\
&\qquad = \left|\frac{1-\eta_n}{w-\chi_n}\right| \left|1 - \frac{1 - \frac{m_n}{t-\chi_n}n^{-\frac{1}{2}} - \frac{y_{2,n}}{1-\eta_n}n^{-1}}{1 - \frac{m_n}{w-\chi_n}n^{-\frac{1}{2}} - \frac{y_{1,n}}{w-\chi_n}n^{-1}}\right| \\
&\qquad = \left|\frac{1-\eta_n}{w-\chi_n}\right| \left|\frac{(\frac{m_n}{t-\chi_n} - \frac{m_n}{w-\chi_n})n^{-\frac{1}{2}} + (\frac{y_{2,n}}{1-\eta_n} - \frac{y_{1,n}}{w-\chi_n})n^{-1}}{1 - \frac{m_n}{w-\chi_n}n^{-\frac{1}{2}} - \frac{y_{1,n}}{w-\chi_n}n^{-1}}\right|
\end{aligned}$$

for all $n > N$ and $w \in \mathrm{cl}(B(t, n^{-\frac{1}{2}}))$. In particular note that $|\frac{m_n}{t-\chi_n} - \frac{m_n}{w-\chi_n}|n^{-\frac{1}{2}} \leqslant \frac{|m_n|}{|t-\chi_n|\,|w-\chi_n|}n^{-1}$ for all $n > N$ and $w \in \mathrm{cl}(B(t, n^{-\frac{1}{2}}))$. Finally recall that $m_n, y_{1,n}, y_{2,n} = O(1)$ for all n sufficiently large (see (2.23)), $n^{-\frac{1}{2}} < \frac{1}{2}\xi$ for all $n > N$ (see Definition 2.14), and note (4.6) gives the following for all $n > N$: $\frac{3}{2}(1-\eta) > 1-\eta_n > \frac{1}{2}(1-\eta) > 0$, $t-\chi_n > \frac{5}{6}(t-\chi) > 0$, and $|w-\chi_n| > \frac{39}{48}(t-\chi) > 0$ for all $w \in \mathrm{cl}(B(t, n^{-\frac{1}{2}}))$. This proves (iv).

Consider (vi). First note, for all $n > N$, (4.3) gives,

$$f'''_{(t,s),n}(z) = \frac{1}{n}\sum_{x\in P_n} \frac{2}{(z-x)^3} - \frac{2(1-\eta_n)}{(z-\chi_n)^3},$$

for all $z \in \mathrm{cl}(B(s, n^{-\frac{1}{2}}))$. Next recall that $n^{-\frac{1}{2}} < \frac{1}{2}\xi$ for all $n > N$(see Definition 2.14), $x_1^{(n)} = \max P_n$ (see (2.20)), and note (4.6) gives the following for all $n > N$: $1 > 1-\eta_n > 0$, $|z-\chi_n| > \frac{23}{32}(s-\chi) > 0$ for all $z \in \mathrm{cl}(B(s, n^{-\frac{1}{2}}))$, and $|z-x| > \frac{7}{16}(s-b) > 0$ for all $z \in \mathrm{cl}(B(s, n^{-\frac{1}{2}}))$ and $x \in P_n$. Thus, for all $n > N$,

$$\begin{aligned}
\sup_{z\in\mathrm{cl}(B(t,n^{-\frac{1}{2}}))} |f'''_{(t,s),n}(z)| &< \frac{1}{n}\sum_{x\in P_n} \frac{2}{(\frac{7}{16}(s-b))^3} + \frac{2}{(\frac{23}{32}(s-\chi))^3} \\
&< \frac{2^4}{(s-b)^3} + \frac{2^3}{(s-\chi)^3}.
\end{aligned}$$

Part (vi) easily follows.

Consider (vii). Proceed similarly to the proof of part (7) above to get,

$$\begin{aligned}
&|f'_{(t,s),n}(w) - \widetilde{f}'_n(w)| \\
&\qquad = \left|\frac{1-\eta_n}{w-\chi_n}\right| \left|1 - \frac{1 - \frac{\tilde{m}_n}{s-\chi_n}n^{-\frac{1}{2}} - \frac{\tilde{y}_{2,n}}{1-\eta_n}n^{-1}}{1 - \frac{\tilde{m}_n}{w-\chi_n}n^{-\frac{1}{2}} - \frac{\tilde{y}_{1,n}}{w-\chi_n}n^{-1}}\right| \\
&\qquad = \left|\frac{1-\eta_n}{w-\chi_n}\right| \left|\frac{(\frac{\tilde{m}_n}{s-\chi_n} - \frac{\tilde{m}_n}{w-\chi_n})n^{-\frac{1}{2}} + (\frac{\tilde{y}_{2,n}}{1-\eta_n} - \frac{\tilde{y}_{1,n}}{w-\chi_n})n^{-1}}{1 - \frac{\tilde{m}_n}{w-\chi_n}n^{-\frac{1}{2}} - \frac{\tilde{y}_{1,n}}{w-\chi_n}n^{-1}}\right|
\end{aligned}$$

for all $n > N$ and $w \in \mathrm{cl}(B(s, n^{-\frac{1}{2}}))$. In particular note that $|\frac{\tilde{m}_n}{s-\chi_n} - \frac{\tilde{m}_n}{w-\chi_n}|n^{-\frac{1}{2}} \leqslant \frac{|\tilde{m}_n|}{|s-\chi_n|\,|w-\chi_n|}n^{-1}$ for all $n > N$ and $w \in \mathrm{cl}(B(s, n^{-\frac{1}{2}}))$. Finally recall that $\widetilde{m}_n, \widetilde{y}_{1,n}, \widetilde{y}_{2,n} = O(1)$ for all n sufficiently large (see (2.23)), $n^{-\frac{1}{2}} < \frac{1}{2}\xi$ for all $n > N$ (see Definition 2.14), and note (4.6) gives the following for all $n > N$: $\frac{3}{2}(1-\eta) > 1 - \eta_n > \frac{1}{2}(1-\eta) > 0$, $s - \chi_n > \frac{3}{4}(s-\chi) > 0$, and $|w - \chi_n| > \frac{23}{32}(s-\chi) > 0$ for all $w \in \mathrm{cl}(B(s, n^{-\frac{1}{2}}))$. This proves (vii). $\square$

Lemma 2.15 (8)–(11) examine the behaviour of the roots of f_n' and $\widetilde{f}_n'$ in neighbourhoods of t and s. Next we consider the remaining roots of f_n' and $\widetilde{f}_n'$ in their respective domains, outside of these neighbourhoods. Consider f_n'. First recall that $\{x \in P_n : x > v_n\} \neq \emptyset$ and $\{x \in P_n : x < v_n\} \neq \emptyset$ for all $n > N$ (see Definition 2.14). Next note that $x_1^{(n)} = \max\{x \in P_n : x > v_n\}$ and $x_n^{(n)} = \min\{x \in P_n : x < v_n\}$ (see (2.20)), and define,

$$X_n(v_n) := \min\{x \in P_n : x > v_n\} \quad \text{and} \quad x_n(v_n) := \max\{x \in P_n : x < v_n\}.$$

Then, for all $n > N$, partition the domain of f_n' as follows:

$$\mathbb{C} \setminus (P_n \cup \{v_n\}) = (\mathbb{C} \setminus \mathbb{R}) \cup J_n \cup K_n, \tag{4.8}$$

where $J_n := \bigcup_{i=1}^4 J_{i,n}$, $K_n := \bigcup_{i=1,3} K_n^{(i)}$, and

- $J_{1,n} := (x_1^{(n)}, +\infty)$.
- $J_{2,n} := (-\infty, x_n^{(n)})$.
- $J_{3,n} := (v_n, X_n(v_n))$.
- $J_{4,n} := (x_n(v_n), v_n)$.
- $K_n^{(1)} := [X_n(v_n), x_1^{(n)}] \setminus P_n$.
- $K_n^{(2)} := [x_n^{(n)}, x_n(v_n)] \setminus P_n$.

Partition each $K_n^{(i)}$ as $\{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$, a finite set of pairwise disjoint open intervals, unique up to order, which satisfy $\{\inf I, \sup I\} \subset P_n$ for any $I \in \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$. These sets are depicted in Figure 4.2. Note that $\sum_{i=1}^2 |\{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}| = |P_n| - 2 = n - 2$ when $v_n \notin P_n$, and $\sum_{i=1}^2 |\{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}| = |P_n| - 3 = n - 3$ when $v_n \in P_n$. Note, an analogous partition exists for $\mathbb{C} \setminus (P_n \cup \{u_n\})$, the domain of $\widetilde{f}_n'$, and we denote the analogous quantities by $\widetilde{J}_{1,n}, \widetilde{J}_{2,n}$, etc. Note, in particular, $J_{1,n} = \widetilde{J}_{1,n} = (x_1^{(n)}, +\infty)$.

Lemma 4.2. — *Fix ξ, N as above and $n > N$, and define $t_n, s_n, \widetilde{t}_n, \widetilde{s}_n$ as in Lemma 2.15 (9) and (10). Recall that $(t-\xi, t+\xi) \cup (s-\xi, s+\xi) \subset J_{1,n} = (x_1^{(n)}, +\infty)$ and $t - \xi > s + \xi > s - \xi > x_1^{(n)}$ (see (4.6)). Also recall that $t_n \in (t-\xi, t+\xi)$ and $s_n \in (s-\xi, s+\xi)$ are roots of f_n' of multiplicity 1 (see Lemma 2.15 (9)). Then the following are satisfied for all $n > N$:*

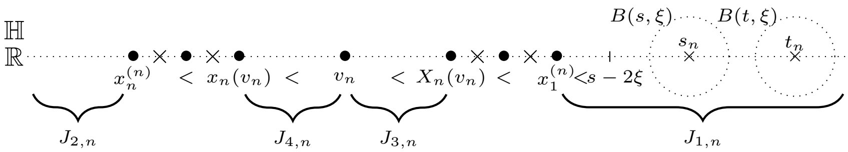

Figure 4.2. The roots of f_n' are represented by $\times$, and are each of multiplicity 1. Elements of $P_n \cup \{v_n\}$ are represented by $\bullet$. Above, $K_n^{(1)} = [X_n(v_n), x_1^{(n)}] \setminus P_n$, $K_n^{(3)} = [x_n^{(n)}, x_n(v_n)] \setminus P_n$.

(1) *f_n' has a root of multiplicity 1 at $t_n \in (t-\xi, t+\xi) \subset J_{1,n} = (x_1^{(n)}, +\infty)$, a root of multiplicity 1 at $s_n \in (s-\xi, s+\xi) \subset J_{1,n} = (x_1^{(n)}, +\infty)$, and 0 roots in $J_{1,n} \setminus \{t_n, s_n\}$. Moreover, $f_n|_{J_{1,n}}$ is real-valued, is strictly increasing in $(x_1^{(n)}, s_n)$, has a local maximum at s_n ($f_n'(s_n) = 0$ and $f_n''(s_n) < 0$), is strictly decreasing in (s_n, t_n), has a local minimum at t_n ($f_n'(t_n) = 0$ and $f_n''(t) > 0$), and is strictly increasing in $(t_n, +\infty)$.*
(2) *f_n' has 0 roots in $\mathbb{C} \setminus \mathbb{R}$, and in each of $\{J_{2,n}, J_{3,n}, J_{4,n}\}$.*
(3) *f_n' has exactly 1 root, counting multiplicities, in each of $\bigcup_{i=1}^{2} \{K_1^{(i)}, K_2^{(i)}, \dots\}$.*

Analogous results hold for $\widetilde{f}_n$ with the analogous roots of $\widetilde{f}_n'$, $\widetilde{t}_n \in (t-\xi, t+\xi)$ and $\widetilde{s}_n \in (s-\xi, s+\xi)$.

Proof. — Fix $n > N$. We will show the following:

(i) f_n' has $\sum_{i=1}^{2} |\{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \dots\}| + 2$ roots in $\mathbb{C} \setminus (P_n \cup \{v_n\}) = (\mathbb{C} \setminus \mathbb{R}) \cup J_n \cup K_n$.
(ii) f_n' an odd number of roots in each of $\bigcup_{i=1}^{2} \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \dots\}$.
(iii) $f_n|_{J_{1,n}}$ is real-valued, and $\lim_{w \to +\infty} w f_n'(w) > 2\xi > 0$.

Then, since $t_n \in (t-\xi, t+\xi) \subset J_{1,n}$ and $s_n \in (s-\xi, s+\xi) \subset J_{1,n}$ are roots of f_n' of multiplicity 1, parts (i), (ii), and a simple counting argument imply that the following: f_n' has a root of multiplicity 1 at $t_n \in (t-\xi, t+\xi) \subset J_{1,n}$, a root of multiplicity 1 at $s_n \in (s-\xi, s+\xi) \subset J_{1,n}$, 0 roots in each of $\{\mathbb{C} \setminus \mathbb{R}, J_{1,n} \setminus \{t_n, s_n\}, J_{2,n}, J_{3,n}, J_{4,n}\}$, and 1 root in each of $\bigcup_{i=1}^{2} \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \dots\}$. Moreover, since f_n' has a root of multiplicity 1 at both $t_n, s_n \in J_{1,n}$ with $t_n > s_n$, and 0 roots in $J_{1,n} \setminus \{t_n, s_n\} = (x_1^{(n)}, +\infty) \setminus \{t_n, s_n\}$, part (iii) implies that $f_n''(t_n) > 0$ and $f_n''(s_n) < 0$. The above prove parts (1), (2), and (3).

Consider (i). First note, for all $w \in \mathbb{C} \setminus (P_n \cup \{v_n\})$, (4.4) gives,

$$f_n'(w) = \frac{1}{n} \sum_{x \in P_n \setminus \{v_n\}} \frac{1}{w-x} - \frac{1 - \frac{s_n - 1}{n} - \frac{1}{n} 1_{(v_n \in P_n)}}{w - v_n},$$

Therefore, $f_n'(w) = \frac{1}{n} \frac{1}{w - v_n} (\prod_{y \in P_n \setminus \{v_n\}} \frac{1}{w-y}) Q_n(w)$, where Q_n is the polynomial,

$$Q_n(w) = (w - v_n) \sum_{x \in P_n \setminus \{v_n\}} \left(\prod_{y \in P_n \setminus \{v_n, x\}} (w-y) \right) - (n - (s_n - 1) - 1_{(v_n \in P_n)}) \left(\prod_{y \in P_n \setminus \{v_n\}} (w-y) \right).$$

Note that Q_n has no roots in $P_n \cup \{v_n\}$, and so the roots of Q_n and f_n' coincide. Also note that Q_n is a polynomial of degree $|P_n| = n$ when $v_n \notin P_n$, and of degree $|P_n| - 1 = n - 1$ when $v_n \in P_n$. Therefore, counting multiplicities, f_n' has n roots in $\mathbb{C} \setminus (P_n \cup \{v_n\})$ when $v_n \notin P_n$, and $n-1$ roots in $\mathbb{C} \setminus (P_n \cup \{v_n\})$ when $v_n \in P_n$. Finally recall (see (4.8) and the subsequent remarks) $\sum_{i=1}^2 |\{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \dots\}| = n-2$ when $v_n \notin P_n$, and $\sum_{i=1}^2 |\{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \dots\}| = n-3$ when $v_n \in P_n$. This proves (i).

Consider (ii). Fix $i \in \{1,2\}$, and any interval $I_n \in \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \dots\}$. Recall that $\inf I_n$ and $\sup I_n$ are either both consecutive elements of $\{x \in P_n : x > v_n\}$, or both consecutive elements of $\{x \in P_n : x < v_n\}$ (see (4.8)). In both cases, (4.4) gives,

$$\lim_{w \in \mathbb{R}, w \uparrow \sup I_n} f_n'(w) = -\infty \qquad \text{and} \qquad \lim_{w \in \mathbb{R}, w \downarrow \inf I_n} f_n'(w) = +\infty.$$

This proves (ii).

Consider (iii). Fix $n > N$. First note, (4.4) implies that $f_n|_{J_{1,n}}$ is real-valued, and $\lim_{w \to +\infty} w f_n'(w) = \frac{s_n - 1}{n}$. (4.6) then gives $\lim_{w \to +\infty} w f_n'(w) > 2\xi > 0$. This proves (iii). ☐

Finally, we examine Taylor expansions of f_n in neighbourhoods of t, and $\widetilde{f}_n$ in neighbourhoods of s:

Lemma 4.3. — *Fix ξ, N as above, and define $t_n, s_n, \widetilde{t}_n, \widetilde{s}_n$ as in Lemma 2.15(9) and (10). Fix $\theta \in (\frac{1}{3}, \frac{1}{2})$ as in Definition 2.14. For all $n > N$,* *define:*

$$b_n := |t_n + in^{-\theta} - t|n^{\theta} \qquad \text{and} \qquad \widetilde{b}_n := |\widetilde{s}_n + in^{-\theta} - s|n^{\theta},$$
$$\alpha_n := \operatorname{Arg}(t_n + in^{-\theta} - t) \qquad \text{and} \qquad \widetilde{\alpha}_n := \operatorname{Arg}(\widetilde{s}_n + in^{-\theta} - s),$$
$$D_n := \left(\frac{1}{2}|f_n''(t)|\right)^{\frac{1}{2}} \geqslant 0 \qquad \text{and} \qquad \widetilde{D}_n := \left(\frac{1}{2}|\widetilde{f}_n''(s)|\right)^{\frac{1}{2}} \geqslant 0.$$

Then the following is satisfied for all $n > N$:

(1) $1 \leqslant b_n < 2$ *and* $1 \leqslant \widetilde{b}_n < 2$, $|b_n - 1| < n^{-\frac{1}{2}+\theta}$ *and* $|\widetilde{b}_n - 1| \leqslant n^{-\frac{1}{2}+\theta}$.
(2) $|\alpha_n - \frac{\pi}{2}| < n^{-\frac{1}{2}+\theta}$ *and* $|\widetilde{\alpha}_n - \frac{\pi}{2}| < n^{-\frac{1}{2}+\theta}$.
(3) $D_n^2 > \frac{1}{8}|f''_{(t,s)}(t)| > 0$ *and* $\widetilde{D}_n^2 > \frac{1}{8}|f''_{(t,s)}(s)| > 0$.

Next note, for all $n > N$ (2.6), (2.7), (2.20), and (4.6) imply that $f_n, \widetilde{f}_n$ are well-defined and analytic in the disjoint sets $B(t, 2\xi)$ and $B(s, 2\xi)$. Recall, for all $n > N$, that $n^{-\frac{1}{2}} < \frac{1}{2}\xi$ (see Definition 2.14), and $n^{-\theta}b_n < 2\xi$ and $n^{-\theta}\widetilde{b}_n < 2\xi$ (see Definition 2.14 and part (1)). Finally, $E_{1,n}, \widetilde{E}_{1,n}, E_{2,n}, \widetilde{E}_{2,n} = O(1)$ for all n sufficiently large, where $E_{1,n}, \widetilde{E}_{1,n}, E_{2,n}, \widetilde{E}_{2,n} > 0$ are defined in the proof, and the following is satisfied for all $n > N$:

(4) $\sup_{w \in B(t, n^{-\frac{1}{2}})} |f_n(w) - f_n(t)| \leqslant E_{1,n} n^{-1}$.
(5) $\sup_{z \in B(s, n^{-\frac{1}{2}})} |\widetilde{f}_n(z) - \widetilde{f}_n(s)| \leqslant \widetilde{E}_{1,n} n^{-1}$.
(6) $\sup_{w \in \mathrm{cl}(B(0, n^{\frac{1}{2}-\theta} b_n D_n))} |nf_n(t + n^{-\frac{1}{2}} D_n^{-1} w) - nf_n(t) - w^2| \leqslant E_{2,n} n^{1-3\theta}$.
(7) $\sup_{w \in \mathrm{cl}(B(0, n^{\frac{1}{2}-\theta} \tilde{b}_n \tilde{D}_n))} |n\widetilde{f}_n(s + n^{-\frac{1}{2}} \widetilde{D}_n^{-1} w) - n\widetilde{f}_n(s) + w^2| \leqslant \widetilde{E}_{2,n} n^{1-3\theta}$.

Proof. — Fix $n > N$. Consider (1). First, since $b_n = \sqrt{1 + (t_n - t)^2 n^{2\theta}}$, it trivially follows that $b_n \geqslant 1$. Next recall that $b_n = \sqrt{1 + (t_n - t)^2 n^{2\theta}}$, $|t_n - t|n^{\theta} < n^{-\frac{1}{2}+\theta} < 1$ (see Definition 2.14 and Lemma 2.15(9)), and note that $\sqrt{1 + x^2} < \sqrt{2}$ for all $x \in [0, 1)$. This gives $b_n < \sqrt{2}$. Finally recall that $b_n = \sqrt{1 + (t_n - t)^2 n^{2\theta}}$, $|t_n - t|n^{\theta} < n^{-\frac{1}{2}+\theta} < 1$, and note that $|\sqrt{1 + x^2} - 1| \leqslant |x|$ for all $x \in [0, 1)$ with equality only when $x = 0$. This gives $|b_n - 1| < n^{-\frac{1}{2}+\theta}$. We can similarly show that $1 \leqslant \widetilde{b}_n < \sqrt{2}$, and $|\widetilde{b}_n - 1| < n^{-\frac{1}{2}+\theta}$. This proves (1).

Consider (2). First note, the definition of α_n gives,

$$\alpha_n = \begin{cases} \arctan(\frac{1}{(t_n - t)n^{\theta}}) & \text{when } t_n - t > 0, \\ \frac{\pi}{2} & \text{when } t_n - t = 0, \\ \arctan(\frac{1}{(t_n - t)n^{\theta}}) + \pi & \text{when } t_n - t < 0. \end{cases}$$

Recall that $|t_n - t|n^{\theta} < n^{-\frac{1}{2}+\theta} < 1$, and note that $|\arctan(\frac{1}{x}) - \frac{\pi}{2}| \leqslant |x|$ for all $x \in (0,1)$ with $\lim_{x\downarrow 0}|\arctan(\frac{1}{x}) - \frac{\pi}{2}| = 0$, and $|\arctan(\frac{1}{x}) + \frac{\pi}{2}| \leqslant |x|$ for all $x \in (-1,0)$ with $\lim_{x\uparrow 0}|\arctan(\frac{1}{x}) + \frac{\pi}{2}| = 0$. This gives $|\alpha_n - \frac{\pi}{2}| < n^{-\frac{1}{2}+\theta}$. Similarly we can show that $|\widetilde{\alpha}_n - \frac{\pi}{2}| < n^{-\frac{1}{2}+\theta}$. This proves (2).

Consider (3). Recall that $D_n^2 = \frac{1}{2}|f_n''(t)|$ and $\widetilde{D}_n^2 = \frac{1}{2}|\widetilde{f}_n''(s)|$. Also recall (see Lemma 2.15 (8)) that $f_n''(t) > \frac{1}{4} f_{(t,s)}''(t) > 0$ and $\widetilde{f}_n''(s) < \frac{1}{4} f_{(t,s)}''(s) < 0$. This proves (3).

Consider (4). First recall that f_n is well-defined and analytic in $B(t, n^{-\frac{1}{2}})$. Then, for all $w \in B(t, n^{-\frac{1}{2}})$, Taylors theorem gives,

$$f_n(w) = f_n(t) + f_n'(t)(w-t) + \int_t^w \mathrm{d}z \; f_n''(z)(w-z),$$

where the integral is along the straight line from t to w. Therefore,

$$|f_n(w) - f_n(t)| \leqslant |f_n'(t)|(n^{-\frac{1}{2}}) + \sup_{z \in B(t,n^{-\frac{1}{2}})} |f_n''(z)|(n^{-\frac{1}{2}})^2,$$

for all $w \in B(t, n^{-\frac{1}{2}})$. Next recall that $|f_n'(t)| = B_{1,n} n^{-1}$ where $B_{1,n} = O(1)$ for all n sufficiently large (see proof of Lemma 2.15 (7)). Therefore,

$$|f_n(w) - f_n(t)| \leqslant B_{1,n}(n^{-\frac{3}{2}}) + \sup_{z \in B(t,n^{-\frac{1}{2}})} |f_n''(z)|(n^{-\frac{1}{2}})^2,$$

for all $w \in B(t, n^{-\frac{1}{2}})$. Next note, (4.4) gives,

$$f_n''(z) = -\frac{1}{n} \sum_{x \in P_n} \frac{1}{(z-x)^2} + \frac{1 - \frac{s_n - 1}{n}}{(z - v_n)^2},$$

for all $z \in B(t, n^{-\frac{1}{2}})$. Next recall that $n^{-\frac{1}{2}} < \frac{1}{2}\xi$ (see Definition 2.14), $x_1^{(n)} = \max P_n$ (see (2.20)), and note (4.6) gives the following: $1 > 1 - \frac{s_n - 1}{n} > 0$, $|z - v_n| > \frac{39}{48}(t - \chi) > 0$ for all $z \in B(t, n^{-\frac{1}{2}})$, and $|z - x| > \frac{23}{32}(t - b) > 0$ for all $z \in B(t, n^{-\frac{1}{2}})$ and $x \in P_n$. Thus,

$$\sup_{z \in B(t,n^{-\frac{1}{2}})} |f_n''(z)| < \frac{1}{n} \sum_{x \in P_n} \frac{1}{(\frac{23}{32}(t-b))^2} + \frac{1}{(\frac{39}{48}(t-\chi))^2} < \frac{2}{(t-b)^2} + \frac{2}{(t-\chi)^2}.$$

Combined, the above give $|f_n(w) - f_n(t)| \leqslant E_{1,n} n^{-1}$ for all $w \in B(t, n^{-\frac{1}{2}})$, where

$$E_{1,n} := B_{1,n} n^{-\frac{1}{2}} + \frac{2}{(t-b)^2} + \frac{2}{(t-\chi)^2}.$$

Recall that $B_{1,n} = O(1)$ for all n sufficiently large. Thus $E_{1,n} = O(1)$ for all n sufficiently large. This proves (4).

Consider (5). Proceed similarly to the proof of part (4) to get,

$$|\widetilde{f}_n(w) - \widetilde{f}_n(s)| \leqslant \widetilde{B}_{1,n}(n^{-\frac{3}{2}}) + \sup_{z\in B(s,n^{-\frac{1}{2}})} |\widetilde{f}''_n(z)|(n^{-\frac{1}{2}})^2,$$

for all $w \in B(s, n^{-\frac{1}{2}})$. Next note, (4.5) gives,

$$\widetilde{f}''_n(z) = -\frac{1}{n}\sum_{x\in P_n} \frac{1}{(z-x)^2} + \frac{1-\frac{r_n+1}{n}}{(z-u_n)^2},$$

for all $z \in B(s, n^{-\frac{1}{2}})$. Recall that $n^{-\frac{1}{2}} < \frac{1}{2}\xi$, $x_1^{(n)} = \max P_n$, and note (4.6) gives the following: $1 > 1 - \frac{r_n+1}{n} > 0$, $|z-u_n| > \frac{23}{32}(s-\chi) > 0$ for all $z \in B(s, n^{-\frac{1}{2}})$, and $|z-x| > \frac{7}{16}(s-b) > 0$ for all $z \in B(s, n^{-\frac{1}{2}})$ and $x \in P_n$. Thus,

$$\begin{aligned}\sup_{z\in B(s,n^{-\frac{1}{2}})} |f''_n(z)| &< \frac{1}{n}\sum_{x\in P_n} \frac{1}{(\frac{7}{16}(t-b))^2} + \frac{1}{(\frac{23}{32}(s-\chi))^2}\\ &< \frac{2^3}{(t-b)^2} + \frac{2}{(s-\chi)^2}.\end{aligned}$$

Combined, the above give $|\widetilde{f}_n(w) - \widetilde{f}_n(t)| \leqslant \widetilde{E}_{1,n} n^{-1}$ for all $w \in B(s, n^{-\frac{1}{2}})$, where

$$\widetilde{E}_{1,n} := \widetilde{B}_{1,n} n^{-\frac{1}{2}} + \frac{2^3}{(s-b)^2} + \frac{2}{(s-\chi)^2}.$$

This proves (5).

Consider (6). First note Taylor's theorem gives,

$$\begin{aligned}f_n(t+n^{-\frac{1}{2}}D_n^{-1}w) = f_n(t) + f'_n(t)(n^{-\frac{1}{2}}D_n^{-1}w) + \frac{1}{2}f''_n(t)(n^{-\frac{1}{2}}D_n^{-1}w)^2\\ + \frac{1}{2}\int \mathrm{d}z\ f'''_n(z)(t+n^{-\frac{1}{2}}D_n^{-1}w - z)^2,\end{aligned}$$

for all $w \in \mathrm{cl}(B(0, n^{\frac{1}{2}-\theta}b_n D_n))$, where the integral is along the straight line from t to $t + n^{-\frac{1}{2}}D_n^{-1}w$. Recall that $f''_n(t) > 0$ (see Lemma 2.15 (8)), and $D_n^2 = \frac{1}{2}f''_n(t)$ (see statement of this lemma). Therefore,

$$\begin{aligned}f_n(t+n^{-\frac{1}{2}}D_n^{-1}w) = f_n(t) + f'_n(t)(n^{-\frac{1}{2}}D_n^{-1}w)\\ + n^{-1}w^2 + \frac{1}{2}\int \mathrm{d}z\ f'''_n(z)(t+n^{-\frac{1}{2}}D_n^{-1}w - z)^2.\end{aligned}$$

for all $w \in \mathrm{cl}(B(0, n^{\frac{1}{2}-\theta}b_n D_n))$. Next note, since $n^{-\frac{1}{2}}D_n^{-1}w \in B(0, n^{-\theta}b_n)$ for all $w \in B(0, n^{\frac{1}{2}-\theta}b_n D_n)$, and since the integral is along the straight line from t to $t + n^{-\frac{1}{2}} D_n^{-1} w$,

$$|f_n(t + n^{-\frac{1}{2}} D_n^{-1} w) - f_n(t) - n^{-1} w^2| \\ \leqslant |f_n'(t)| (n^{-\theta} b_n) + \frac{1}{2} \sup_{z \in \mathrm{cl}(B(t, n^{-\theta} b_n))} |f_n'''(z)| (n^{-\theta} b_n)^3,$$

for all $w \in \mathrm{cl}(B(0, n^{\frac{1}{2}-\theta} b_n D_n))$. Next recall that $|f_n'(t)| = B_{1,n} n^{-1}$ (see proof of Lemma 2.15 (7)), $b_n < 2$ (see part (1)), and $n^{-\theta} b_n < 2\xi$ (see Definition 2.14 and part (1)). Then,

$$|f_n(t + n^{-\frac{1}{2}} D_n^{-1} w) - f_n(t) - n^{-1} w^2| \\ < (B_{1,n} n^{-1})(n^{-\theta} 2) + \frac{1}{2} \sup_{z \in \mathrm{cl}(B(t, 2\xi))} |f_n'''(z)| (n^{-\theta} 2)^3,$$

for all $w \in \mathrm{cl}(B(0, n^{\frac{1}{2}-\theta} b_n D_n))$. Finally note, (4.4) gives,

$$f_n'''(z) = \frac{1}{n} \sum_{x \in P_n} \frac{2}{(z-x)^3} - \frac{2(1 - \frac{s_n - 1}{n})}{(z - v_n)^3},$$

for all $z \in \mathrm{cl}(B(t, 2\xi))$. Recall $x_1^{(n)} = \max P_n$, and note (4.6) gives the following: $1 > 1 - \frac{s_n - 1}{n} > 0$, $|z - v_n| > \frac{3}{4}(t - \chi) > 0$ for all $z \in \mathrm{cl}(B(t, 2\xi))$, and $|z - x| > \frac{5}{8}(t - b) > 0$ for all $z \in \mathrm{cl}(B(t, 2\xi))$ and $x \in P_n$. Therefore,

$$\sup_{z \in \mathrm{cl}(B(t, 2\xi))} |f_n'''(z)| < \frac{1}{n} \sum_{x \in P_n} \frac{2}{(\frac{5}{8}(t-b))^3} + \frac{2}{(\frac{3}{4}(t-\chi))^3} < \frac{2^4}{(t-b)^3} + \frac{2^3}{(t-\chi)^3}.$$

Combined, the above give $|f_n(t + n^{-\frac{1}{2}} D_n^{-1} w) - f_n(t) - n^{-1} w^2| < E_{2,n} n^{-3\theta}$ for all $w \in \mathrm{cl}(B(0, n^{\frac{1}{2}-\theta} b_n D_n))$, where

$$E_{2,n} := 2 B_{1,n} n^{2\theta - 1} + \frac{2^6}{(t-b)^3} + \frac{2^5}{(t-\chi)^3}.$$

Recall that $B_{1,n} = O(1)$ for all n sufficiently large and $2\theta - 1 < 0$. Thus $E_{2,n} = O(1)$ for all n sufficiently large. This proves (6).

Consider (7). First note Taylors theorem gives,

$$\widetilde{f}_n(s + n^{-\frac{1}{2}} \widetilde{D}_n^{-1} w) = \widetilde{f}_n(s) + \widetilde{f}_n'(s)(n^{-\frac{1}{2}} \widetilde{D}_n^{-1} w) + \frac{1}{2} \widetilde{f}_n''(s)(n^{-\frac{1}{2}} \widetilde{D}_n^{-1} w)^2 \\ + \frac{1}{2} \int \mathrm{d}z \, \widetilde{f}_n'''(z)(s + n^{-\frac{1}{2}} \widetilde{D}_n^{-1} w - z)^2,$$

for all $w \in \mathrm{cl}(B(0, n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n))$, where the integral is along the straight line from s to $s + n^{-\frac{1}{2}} \widetilde{D}_n^{-1} w$. Recall that $\widetilde{f}_n''(s) < 0$ (see Lemma 2.15 (8)), and

$\widetilde{D}_n^2 = -\frac{1}{2}\widetilde{f}_n''(s)$ (see statement of this lemma). Therefore,

$$\widetilde{f}_n(s+n^{-\frac{1}{2}}\widetilde{D}_n^{-1}w) = \widetilde{f}_n(s) + \widetilde{f}_n'(s)(n^{-\frac{1}{2}}\widetilde{D}_n^{-1}w) - n^{-1}w^2 + \frac{1}{2}\int \mathrm{d}z\, \widetilde{f}_n'''(z)(s+n^{-\frac{1}{2}}\widetilde{D}_n^{-1}w-z)^2,$$

for all $w \in \mathrm{cl}(B(0,n^{\frac{1}{2}-\theta}\widetilde{b}_n\widetilde{D}_n))$ Then, proceed similarly to part (6) to get,

$$|\widetilde{f}_n(s+n^{-\frac{1}{2}}\widetilde{D}_n^{-1}w) - \widetilde{f}_n(s) + n^{-1}w^2| < (\widetilde{B}_{1,n}n^{-1})(n^{-\theta}2) + \frac{1}{2}\sup_{z\in \mathrm{cl}(B(s,2\xi))}|\widetilde{f}_n'''(z)|(n^{-\theta}2)^3,$$

for all $w \in \mathrm{cl}(B(0,n^{\frac{1}{2}-\theta}\widetilde{b}_n\widetilde{D}_n))$. Next note, (4.5) gives,

$$\widetilde{f}_n'''(z) = \frac{1}{n}\sum_{x\in P_n}\frac{2}{(z-x)^3} - \frac{2(1-\frac{r_n+1}{n})}{(z-u_n)^3},$$

for all $z \in \mathrm{cl}(B(s,2\xi))$. Recall $x_1^{(n)} = \max P_n$, and note (4.6) gives the following: $1 > 1-\frac{r_n+1}{n} > 0$, $|z-u_n| > \frac{5}{8}(s-\chi) > 0$ for all $z \in \mathrm{cl}(B(s,2\xi))$, and $|z-x| > \frac{1}{4}(s-b) > 0$ for all $z \in \mathrm{cl}(B(s,2\xi))$ and $x \in P_n$. Therefore,

$$\sup_{z\in \mathrm{cl}(B(s,2\xi))}|f_n'''(z)| < \frac{1}{n}\sum_{x\in P_n}\frac{2}{(\frac{1}{4}(s-b))^3} + \frac{2}{(\frac{5}{8}(s-\chi))^3} < \frac{2^7}{(s-b)^3} + \frac{2^4}{(s-\chi)^3}.$$

Combined, the above give $|\widetilde{f}_n(s+n^{-\frac{1}{2}}\widetilde{D}_n^{-1}w) - \widetilde{f}_n(s) + n^{-1}w^2| < \widetilde{E}_{2,n}n^{-3\theta}$ for all $w \in \mathrm{cl}(B(0,n^{\frac{1}{2}-\theta}\widetilde{b}_n\widetilde{D}_n))$, where

$$\widetilde{E}_{2,n} := 2\widetilde{B}_{1,n}n^{2\theta-1} + \frac{2^9}{(s-b)^3} + \frac{2^6}{(s-\chi)^3}.$$

This proves (7). □

4.2. The contours of descent/ascent

In this section we define the contours to be used in the steepest descent analysis. First define:

Definition 4.4. — *As in the previous section, fix ξ and N and $\theta \in (\frac{1}{3},\frac{1}{2})$, and define $t_n, \widetilde{s}_n, u_n, r_n, v_n, s_n$ for all $n > N$. Recall that $t_n - v_n > 0$ and $\widetilde{s}_n - u_n > 0$ (see Lemma 2.15(9) and (10), and* (4.6)*). Define for all $n > N$:*

$$q_n := |t_n + in^{-\theta} - v_n|,$$
$$\widetilde{q}_n := |\widetilde{s}_n + in^{-\theta} - u_n|.$$

Also define $R_n : (0,1) \to \mathbb{R}$ *and* $I_n : (0,1) \to \mathbb{R}$ *as follows for all* $n > N$*:*

$$R_n(y) := (\widetilde{s}_n - u_n) \left(1 - \frac{1}{2} \frac{(\widetilde{q}_n)^2}{(\widetilde{s}_n - u_n)^2} \log(y)\right) y,$$

$$I_n(y) := \sqrt{(\widetilde{q}_n)^2 y - R_n(y)^2},$$

for all $y \in (0,1)$*.*

The next lemma shows that I_n is well-defined and other useful properties:

Lemma 4.5. — *For all* $n > N$*:*

(1) R_n *strictly increases in* $(0,1)$ *with* $\lim_{y \downarrow 0} R_n(y) = 0$ *and* $\lim_{y \uparrow 1} R_n(y) = \widetilde{s}_n - u_n$*.*
(2) $(\widetilde{q}_n)^2 y - R_n(y)^2 > 0$ *for all* $y \in (0,1)$*, and so* $I_n(y)$ *is well-defined and* $I_n(y) > 0$ *for all* $y \in (0,1)$*. Moreover,* $\lim_{y \downarrow 0} I_n(y) = 0$ *and* $\lim_{y \uparrow 1} I_n(y) = n^{-\theta}$*.*

Proof. — Fix $n > N$. Consider (1). First note, Definition 4.4 gives $\widetilde{s}_n - u_n > 0$ and,

$$R_n'(y) = (\widetilde{s}_n - u_n) \left(1 - \frac{1}{2} \frac{(\widetilde{q}_n)^2}{(\widetilde{s}_n - u_n)^2} - \frac{1}{2} \frac{(\widetilde{q}_n)^2}{(\widetilde{s}_n - u_n)^2} \log(y)\right),$$

for all $y \in (0,1)$. Next recall $\widetilde{q}_n = |\widetilde{s}_n + i n^{-\theta} - u_n|$ (see Definition 4.4), and so

$$1 - \frac{1}{2} \frac{\widetilde{q}_n^2}{(\widetilde{s}_n - u_n)^2} = \frac{1}{2} - \frac{1}{2} \frac{n^{-2\theta}}{(\widetilde{s}_n - u_n)^2}.$$

Thus $1 - \frac{1}{2} \frac{\tilde{q}_n^2}{(\tilde{s}_n - u_n)^2} > 0$ since $n^{-\theta} < \xi < \frac{1}{16}(s - \chi)$ (see Definition 2.14 and (4.6)) and $\widetilde{s}_n - u_n > \frac{23}{32}(s - \chi) > 0$ (see (4.6) and Lemma 2.15 (10)). Moreover, note that $\log(y) < 0$ for all $y \in (0,1)$. Combined, the above give $R_n'(y) > 0$ for all $y \in (0,1)$. Moreover, Definition 4.4 easily gives $\lim_{y \downarrow 0} R_n(y) = 0$ and $\lim_{y \uparrow 1} R_n(y) = \widetilde{s}_n - u_n$. This proves (1).

Consider (2). First note, part (1) gives $R_n(y) > 0$ for all $y \in (0,1)$. Thus, to prove (2), it is thus sufficient to show that $\widetilde{q}_n \sqrt{y} > R_n(y)$ for all $y \in (0,1)$, i.e. (see Definition 4.4) that,

$$\left(1 + \frac{n^{-2\theta}}{(\widetilde{s}_n - u_n)^2}\right)^{\frac{1}{2}} > \left(1 - \frac{1}{2} \left(1 + \frac{n^{-2\theta}}{(\widetilde{s}_n - u_n)^2}\right) \log(y)\right) \sqrt{y}.$$

We will show that the following are satisfied for all $y \in (0,1)$:

$$\left(1 + \frac{n^{-2\theta}}{(\widetilde{s}_n - u_n)^2}\right)^{\frac{1}{2}} > 1 - \frac{n^{-2\theta}}{(\widetilde{s}_n - u_n)^2} y \log(y). \tag{i}$$

$$1 - \frac{n^{-2\theta}}{(\widetilde{s}_n - u_n)^2} y \log(y) > \left(1 - \left(1 + \frac{n^{-2\theta}}{(\widetilde{s}_n - u_n)^2}\right) \log(y)\right) y. \tag{ii}$$

Replacing y in (i) and (ii) by $\sqrt{y}$ gives the required inequality. This proves (2).

Consider (i). Note that the RHS of this inequality is positive for all $y \in (0,1)$, since $0 > y\log(y)$. Thus, squaring both sides, it is sufficient to show that,

$$1 + \frac{n^{-2\theta}}{(\widetilde{s}_n - u_n)^2} > 1 - 2\frac{n^{-2\theta}}{(\widetilde{s}_n - u_n)^2} y\log(y) + \frac{n^{-4\theta}}{(\widetilde{s}_n - u_n)^4}(y\log(y))^2,$$

for all $y \in (0,1)$. Rewriting, it is sufficient to show that,

$$1 + 2y\log(y) > \frac{n^{-2\theta}}{(\widetilde{s}_n - u_n)^2}(y\log(y))^2,$$

for all $y \in (0,1)$. Next note that $0 > y\log(y) > -e^{-1}$ for all $y \in (0,1)$. Thus it is sufficient to show that,

$$1 - 2e^{-1} > \frac{n^{-2\theta}}{(\widetilde{s}_n - u_n)^2} e^{-2}.$$

Finally note that $n^{-\theta} < \xi < \frac{1}{16}(s-\chi)$ (see Definition 2.14 and (4.6)) and $\widetilde{s}_n - u_n > \frac{23}{32}(s-\chi) > 0$ (see (4.6) and Lemma 2.15 (10)). Thus it is sufficient to show that $1 - 2e^{-1} > (\frac{2}{23})^2 e^{-2}$ This is trivially true. This proves (i).

Consider (ii). Rewriting, it is sufficient to show that $1 > y(1-\log(y))$ for all $y \in (0,1)$. This is trivially true. This proves (ii). ☐

We now use the quantities in Definition 4.4 to define the contours to be used in the steepest descent analysis. Extend the definition of $R_n, I_n : (0,1) \to \mathbb{R}$ to the end-points $\{0,1\}$ using the well-defined limits shown Lemma 4.5, and define:

Definition 4.6. — *For all $n > N$, let γ_n^+ to be the contour which:*

- *starts at $t \in (s,+\infty)$,*
- *then traverses the straight line from t to $t_n + in^{-\theta}$,*
- *then traverses the counter-clockwise arc of $\partial B(v_n,q_n)$ from $t_n + in^{-\theta}$ to $v_n - q_n$,*
- *then ends at $v_n - q_n$.*

Note, γ_n^+ is trivially a continuous contour which begins and ends in $\mathbb{R}$, and is otherwise contained in $\mathbb{H}$. Let γ_n^- be the reflection of γ_n^+ in $\mathbb{R}$, let γ_n be the continuous closed contour given by $\gamma_n = \gamma_n^+ + \gamma_n^-$ with counter-clockwise orientation.

For all $n > N$, let Γ_n^+ to be the contour which:

- *starts at $s \in (b,t)$,*
- *then traverses the straight line from s to $\widetilde{s}_n + in^{-\theta}$,*

- *then traverses the contour* $y \mapsto u_n + R_n(1-y) + iI_n(1-y)$ *for* $y \in [0,1]$*,*
- *then ends at* u_n*.*

Note, Lemma 4.5 implies that Γ_n^+ *is a continuous contour which begins and ends in* $\mathbb{R}$*, and is otherwise contained in* $\mathbb{H}$*. Define* Γ_n^- *and* Γ_n *analogously to above.*

The next result prove properties of the contours which are useful for the steepest descent analysis, and Figure 4.3 depicts γ_n^+ and Γ_n^+:

Lemma 4.7. — *The following are satisfied for all* $n > N$*:*

(1) γ_n *contains* v_n *and* Γ_n*.*
(2) Γ_n *contains* $\{x \in P_n : x > u_n\}$ *and does not contain any of* $\{x \in P_n : x < u_n\}$*.*
(3) $\mathrm{Re}(f_n(w)) \leqslant \mathrm{Re}(f_n(t_n + in^{-\theta}))$ *for all* w *on that section of* γ_n^+ *given by the counter-clockwise arc of* $\partial B(v_n, q_n)$ *from* $t_n + in^{-\theta}$ *to* $v_n - q_n$*.*
(4) $\mathrm{Re}(\widetilde{f}_n(z)) \geqslant \mathrm{Re}(\widetilde{f}_n(\widetilde{s}_n + in^{-\theta})) = \mathrm{Re}(\widetilde{f}_n(u_n + R_n(1) + iI_n(1)))$ *for all* z *on that section of* Γ_n^+ *given by the contour* $y \mapsto u_n + R_n(1-y) + iI_n(1-y)$ *for* $y \in [0,1]$*.*
(5) $|w - z| \geqslant \frac{1}{2}(t-s)$ *for all* $w \in \gamma_n$ *and* $z \in \Gamma_n$*.*
(6) $|\gamma_n| \leqslant 8(t-\chi)$*, where* $|\cdot|$ *represents the length.*
(7) $|\Gamma_n| \leqslant 8(s-\chi)$*.*

Proof. — Fix $n > N$. Consider (1). First note, Definition 4.6 trivially implies that γ_n contains v_n. Next recall (see Definition 4.6), γ_n^+ starts at t and ends at $v_n - q_n = v_n - |t_n + in^{-\theta} - v_n|$, and Γ_n^+ starts at s and ends at u_n. Moreover, both contours are otherwise contained in $\mathbb{H}$, and $t > s > u_n > v_n - |t_n + in^{-\theta} - v_n|$ (this follows from (4.6), and since $n^{-\theta} < \xi$ by Definition 2.14, and since $|t_n - t| < \frac{1}{2}\xi$ by Lemma 2.15 (9)). Thus γ_n^+ contains the start and end points of Γ_n^+. Moreover, we will show in the proof of part (5), below, that $|w - z| \geqslant \frac{1}{2}(t-s)$ for all $w \in \gamma_n^+$ and $z \in \Gamma_n^+$. Combined, the above imply that γ_n^+ contains Γ_n^+. This proves (1). Consider (2). Note that Definition 4.6 and Lemma 4.5 imply that Γ_n^+ starts at s, ends at u_n, and is otherwise contained in $\mathbb{H}$. Also, (2.20) and (4.6) give $s > x_1^{(n)} = \max P_n$. Part (2) easily follows.

Consider (3). Define $g_n(y) := \mathrm{Re}(f_n(v_n + q_n e^{iy}))$ for all $y \in \mathbb{R}$. Note, to prove (3), it is sufficient to show that $g_n'(y) < 0$ for all $y \in (0, \pi)$. To prove

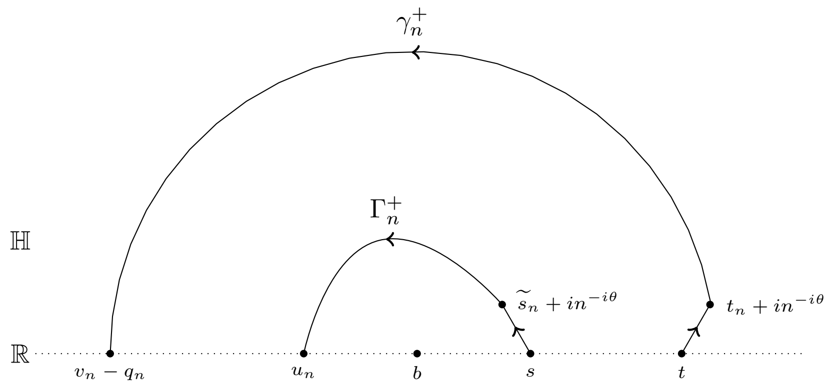

Figure 4.3. The contours γ_n^+ and Γ_n^+ defined in Definition 4.6, and whose properties are examined in Lemma 4.7. We remind the reader that $t_n \to t$ and $s_n \to s$ and $u_n \to \chi$ and $v_n - q_n \to \chi - (t-\chi)$ as $n \to \infty$, and that $t > s > b > \chi > \chi - (t-\chi)$.

this, first note, for all $y \in \mathbb{R}$, (2.6) and (2.20) give,

$$\begin{aligned} &g_n(y)\\ &= \frac{1}{2n}\sum_{x\in P_n} \log|v_n + q_n e^{iy} - x|^2 - \left(1 - \frac{s_n - 1}{n}\right)\log|v_n + q_n e^{iy} - v_n|\\ &= \frac{1}{2n}\sum_{x\in P_n} \log((v_n - x)^2 + 2(v_n - x)q_n\cos(y) + q_n^2) - \left(1 - \frac{s_n - 1}{n}\right)\log(q_n), \end{aligned}$$

where log is the natural logarithm. Differentiate to get $g_n'(y) = h_n(y)\sin(y)$ and $g_n''(y) = h_n'(y)\sin(y) + h_n(y)\cos(y)$ for all $y \in \mathbb{R}$, where

$$h_n(y) = -\frac{1}{n}\sum_{x\in P_n} \frac{(v_n - x)q_n}{(v_n - x)^2 + 2(v_n - x)q_n\cos(y) + q_n^2}.$$

We will show:

(i) $g_n'(0) = 0$ and $g_n''(0) = h_n(0)$ and $h_n(0) < 0$.
(ii) Assume that there exists a $Y \in (0,\pi)$ for which $g_n'(Y) = 0$. Then $g_n''(Y) = h_n'(Y)\sin(Y)$ and $h_n'(Y)\sin(Y) < 0$.

Part (i) implies that 0 is a local maximum of $g_n : \mathbb{R} \to \mathbb{R}$. Moreover, part (ii) implies that any extrema of g_n' in $(0,\pi)$ is also a local maximum. It follows that g_n has no extrema in $(0,\pi)$, and that $g_n'(y) < 0$ for all $y \in (0,\pi)$. This proves (3).

Consider (4). Define $G_n(y) := \mathrm{Re}(\widetilde{f}_n(u_n + R_n(y) + iI_n(y)))$ for all $y \in [0,1]$. To prove (4), it is sufficient to show that $G_n'(y) < 0$ for all $y \in (0,1)$. Remark that for all $y \in (0,1)$, (2.7) and (2.20) give,

$$G_n(y) = \frac{1}{2n} \sum_{x \in P_n} \log |u_n + R_n(y) + iI_n(y) - x|^2 \\ - \frac{1}{2}\left(1 - \frac{r_n+1}{n}\right) \log |R_n(y) + iI_n(y)|^2,$$

where log is the natural logarithm. Then, since $R_n(y)^2 + I_n(y)^2 = (\widetilde{q}_n)^2 y$ for all $y \in (0,1)$ (see Definition 4.4),

$$G_n(y) = \frac{1}{2n} \sum_{x \in P_n} \log((u_n - x)^2 + 2(u_n - x)R_n(y) + \widetilde{q}_n^2 y) \\ - \frac{1}{2}\left(1 - \frac{r_n+1}{n}\right) \log(\widetilde{q}_n^2 y).$$

Therefore, for all $y \in (0,1)$,

$$G_n'(y) = \frac{1}{2n} \sum_{x \in P_n} \frac{2(u_n - x)R_n'(y) + (\widetilde{q}_n)^2}{(u_n - x)^2 + 2(u_n - x)R_n(y) + (\widetilde{q}_n)^2 y} - \frac{1}{2}\left(1 - \frac{r_n+1}{n}\right)\frac{1}{y}.$$

Next recall that $\widetilde{f}_n'(\widetilde{s}_n) = 0$ (see Lemma 2.15(10)). (4.5) thus gives,

$$1 - \frac{r_n+1}{n} = \frac{1}{n} \sum_{x \in P_n} \frac{\widetilde{s}_n - u_n}{\widetilde{s}_n - x}.$$

Therefore, for all $y \in (0,1)$,

$$G_n'(y) = \frac{1}{2n} \sum_{x \in P_n} \frac{2(u_n - x)R_n'(y) + (\widetilde{q}_n)^2}{(u_n - x)^2 + 2(u_n - x)R_n(y) + (\widetilde{q}_n)^2 y} - \frac{1}{2n} \sum_{x \in P_n} \frac{\widetilde{s}_n - u_n}{(\widetilde{s}_n - x)y}.$$

Rewriting gives, for all $y \in (0,1)$,

$$G_n'(y) = \frac{1}{2n} \sum_{x \in P_n} \frac{(u_n - x)H_n(y) + (u_n - x)^2 M_n(y)}{((u_n - x)^2 + 2(u_n - x)R_n(y) + (\widetilde{q}_n)^2 y)(\widetilde{s}_n - x)y},$$

where,

$$H_n(y) := 2(\widetilde{s}_n - u_n) y R_n'(y) + (\widetilde{q}_n)^2 y - 2(\widetilde{s}_n - u_n) R_n(y),$$
$$M_n(y) := 2y R_n'(y) - (\widetilde{s}_n - u_n).$$

We will show:

(iii) $H_n(y) = 0$ for all $y \in (0,1)$.
(iv) $M_n(y) < 0$ for all $y \in (0,1)$.

Finally recall that $(u_n - x)^2 + 2(u_n - x)R_n(y) + (\widetilde{q}_n)^2 y = |u_n + R_n(y) + iI_n(y) - x|^2 > 0$ for all $y \in (0,1)$, and $\widetilde{s}_n - x > \frac{7}{16}(s-b) > 0$ for all $x \in P_n$ (see (4.6) and Lemma 2.15 (10)). Combined, the above give $G_n'(y) < 0$ for all $y \in (0,1)$. This proves (4).

Consider (5). First recall that $|t_n + in^{-\theta} - t| < 2n^{-\theta}$ (see Lemma 4.3 (1)), and that part of γ_n^+ outside $B(t, |t_n + in^{-\theta} - t|)$ is a subset of $\partial B(v_n, q_n)$ (see Definition 4.6). Also, $|\widetilde{s}_n + in^{-\theta} - s| < 2n^{-\theta}$ (see Lemma 4.3 (1)), and that part of Γ_n^+ outside $B(s, |\widetilde{s}_n + in^{-\theta} - s|)$ is a subset of the contour $y \mapsto u_n + R_n(y) + iI_n(y)$ for $y \in [0,1]$ (see Definition 4.6). We will show:

$$\inf_{w \in B(t, 2n^{-\theta})} \inf_{z \in B(s, 2n^{-\theta})} |w - z| > \frac{1}{2}(t-s). \tag{v}$$

$$\inf_{w \in B(t, 2n^{-\theta})} \inf_{y \in [0,1]} |w - (u_n + R_n(y) + iI_n(y))| > \frac{1}{2}(t-s). \tag{vi}$$

$$\inf_{w \in \partial B(v_n, q_n)} \inf_{z \in B(s, 2n^{-\theta})} |w - z| > \frac{1}{2}(t-s). \tag{vii}$$

$$\inf_{w \in \partial B(v_n, q_n)} \inf_{y \in [0,1]} |w - (u_n + R_n(y) + iI_n(y))| > \frac{1}{2}(t-s). \tag{viii}$$

Combined, the above give $|w - z| \geqslant \frac{1}{2}(t-s)$ for all $w \in \gamma_n^+$ and $z \in \Gamma_n^+$. Finally recall that γ_n^+ is a continuous contour which begins and ends in $\mathbb{R}$ and is otherwise in $\mathbb{H}$, γ_n^- is the reflection of γ_n^+ in $\mathbb{R}$, and $\gamma_n = \gamma_n^+ + \gamma_n^-$. Similarly for $\Gamma_n^+, \Gamma_n^-, \Gamma_n$. This proves (5).

Consider (6). Definitions 4.4 and 4.6 trivially give $|\gamma_n| \leqslant 2|t_n + in^{-\theta} - t| + 2\pi|t_n + in^{-\theta} - v_n|$. Therefore,

$$|\gamma_n| \leqslant 2|t_n - t| + 2\pi|t_n - v_n| + 2(1+\pi)n^{-\theta}.$$

Next, Definition 2.14, Lemma 2.15 (9), and (4.6) give the following: $|t_n - t| < \frac{1}{2}\xi < \frac{1}{48}(t-\chi)$, $|t_n - v_n| < \frac{57}{48}(t-\chi)$, and $n^{-\theta} < \xi < \frac{1}{24}(t-\chi)$. Combined, the above prove (6).

Consider (7). Note, Definition 4.6 gives,

$$|\Gamma_n| = 2|\widetilde{s}_n + in^{-\theta} - s| + 2\int_0^1 \mathrm{d}y \sqrt{(R_n'(y))^2 + (I_n'(y))^2}.$$

Denote, for simplicity, the constant $c_n = (\frac{n^{-\theta}}{\widetilde{s}_n - u_n})^2$. Note $c_n < (\frac{2}{23})^2$ since $n^{-\theta} < \xi < \frac{1}{16}(s-\chi)$ (see Definition 2.14 and (4.6)), and $\widetilde{s}_n - u_n > \frac{23}{32}(s-\chi)$

(see Lemma 2.15 (10) and (4.6)). Moreover, Definition 4.4 gives,

$$|\Gamma_n| = 2|\widetilde{s}_n + in^{-\theta} - s| + (1+c_n)(\widetilde{s}_n - u_n)\int_0^1 \frac{\mathrm{d}y}{\sqrt{y}}$$
$$\sqrt{\frac{1-y(1-\log(y)) + c_n y(1+\log(y))}{1-y(1-\frac{1}{2}\log(y))^2 + c_n(1+y\log(y) - \frac{1}{2}y\log(y)^2) + c_n^2(-\frac{1}{4}y\log(y)^2)}}.$$

Recall that $|\widetilde{s}_n + in^{-\theta} - s| < 2n^{-\theta}$ (see Lemma 4.3 (1)). Next, we state (without proof) the following inequalities, which hold for all $y \in (0,1)$:

- $1 - y(1-\log(y)) \leqslant 2(y-1)^2$.
- $y(1+\log(y)) \leqslant 1$.
- $1 - y(1-\frac{1}{2}\log(y))^2 \geqslant \frac{1}{4}(y-1)^2$.
- $1 + y\log(y) - \frac{1}{2}y\log(y)^2 \geqslant \frac{1}{4}$.
- $-\frac{1}{4}y\log(y)^2 \geqslant -\frac{1}{4}$.

Combined the above give:

$$|\Gamma_n| \leqslant 4n^{-\theta} + (1+c_n)(\widetilde{s}_n - u_n)\int_0^1 \frac{\mathrm{d}y}{\sqrt{y}}\sqrt{\frac{2(y-1)^2 + c_n(1)}{\frac{1}{4}(y-1)^2 + c_n(\frac{1}{4}) + c_n^2(-\frac{1}{4})}}.$$

Thus, since $c_n < (\frac{2}{23})^2 < \frac{1}{2}$,

$$\begin{aligned}|\Gamma_n| &< 4n^{-\theta} + (1+c_n)(\widetilde{s}_n - u_n)\int_0^1 \frac{\mathrm{d}y}{\sqrt{y}}\sqrt{\frac{2(y-1)^2 + c_n}{\frac{1}{4}(y-1)^2 + \frac{1}{8}c_n}} \\ &= 4n^{-\theta} + (1+c_n)(\widetilde{s}_n - u_n)\int_0^1 \frac{\mathrm{d}y}{\sqrt{y}}\sqrt{8} \\ &= 4n^{-\theta} + 4\sqrt{2}(1+c_n)(\widetilde{s}_n - u_n).\end{aligned}$$

Finally recall $c_n < (\frac{2}{23})^2$, and note Definition 2.14, Lemma 2.15 (10), and (4.6) give the following: $n^{-\theta} < \xi < \frac{1}{16}(s-\chi)$, and $|\widetilde{s}_n - u_n| < \frac{41}{32}(s-\chi)$. Combined, the above prove (7).

Consider (i). Recall that $g_n'(y) = h_n(y)\sin(y)$ and $g_n''(y) = h_n'(y)\sin(y) + h_n(y)\cos(y)$ for all $y \in \mathbb{R}$. It trivially follows that $g_n'(0) = 0$ and $g_n''(0) = h_n(0)$. It remains to show that $h_n(0) < 0$. To see this first note, the expression for $h_n : \mathbb{R} \to \mathbb{R}$ gives,

$$h_n(0) = -\frac{1}{n}\sum_{x\in P_n} \frac{(v_n - x)q_n}{(v_n - x + q_n)^2}.$$

Next recall (see proof of Lemma 2.15 (11)) that,

$$(t_n - v_n)q_n f_n''(t_n) = \frac{1}{n}\sum_{x\in P_n}\frac{(v_n - x)q_n}{(t_n - x)^2}.$$

Therefore,

$$\begin{aligned}h_n(0) &\leqslant -(t_n - v_n)q_n f_n''(t_n) + |h_n(0) + (t_n - v_n)q_n f_n''(t_n)| \\ &\leqslant -(t_n - v_n)q_n f_n''(t_n) \\ &\qquad + \max_{x\in P_n} \frac{|v_n - x|q_n|t_n - v_n - q_n|(|t_n - x| + |v_n - x + q_n|)}{|t_n - x|^2|v_n - x + q_n|^2}.\end{aligned}$$

Next note, Lemma 2.15 (10) and (4.6) give $t_n - v_n > 0$, and Lemma 2.15 (11) gives $f_n''(t_n) > \frac{1}{4} f_{(t,s)}''(t) > 0$, and so

$$\begin{aligned}h_n(0) &< -\frac{1}{4}(t_n - v_n)q_n|f_{(t,s)}''(t)| \\ &\qquad + \max_{x\in P_n} \frac{|v_n - x|q_n|t_n - v_n - q_n|(|t_n - x| + |v_n - x + q_n|)}{|t_n - x|^2|v_n - x + q_n|^2}.\end{aligned}$$

Finally recall $x_1^{(n)} = \max P_n$ and $x_n^{(n)} = \min P_n$ (see (2.20)), $|t_n - t| < \frac{1}{2}\xi$ (see Lemma 2.15 (9)), $|q_n - (t_n - v_n)| < n^{-\theta} < \xi$ (see Definitions 2.14 and 4.4), and note (4.6) gives the following for all $x \in P_n$: $t_n - v_n > \frac{39}{48}(t - \chi)$, $q_n > t_n - v_n - \xi > \frac{37}{48}(t - \chi)$, $|v_n - x| < \min\{2(b - \chi), 2(\chi - a)\} \leqslant b - a$, $q_n < t_n - v_n + \xi < \frac{59}{48}(t - \chi)$, $|t_n - v_n - q_n| < n^{-\theta}$, $|t_n - x| < \frac{41}{32}(t - b)$, $|v_n - x + q_n| = |(t_n - x) + q_n - (t_n - v_n)| < |t_n - x| + \xi < \frac{43}{32}(t - b)$, $|t_n - x| > \frac{23}{32}(t - b)$, $|v_n - x + q_n| = |(t_n - x) + q_n - (t_n - v_n)| > |t_n - x| - \xi > \frac{21}{32}(t - b)$. Combined, the above give,

$$\begin{aligned}h_n(0) &< -\frac{1}{4}\left(\frac{39}{48}(t - \chi)\right)\left(\frac{37}{48}(t - \chi)\right)|f_{(t,s)}''(t)| \\ &\qquad + \frac{(b - a)(\frac{59}{48}(t - \chi))n^{-\theta}\ (\frac{41}{32}(t - b) + \frac{43}{32}(t - b))}{(\frac{23}{32}(t - b))^2(\frac{21}{32}(t - b))^2} \\ &< -\frac{1}{8}(t - \chi)^2|f_{(t,s)}''(t)| + 8n^{-\theta}\frac{(b - a)(t - \chi)}{(t - b)^3}.\end{aligned}$$

Definition 2.14 finally gives $h_n(0) < 0$. This proves (i).

Consider (ii). Recall that $g_n'(y) = h_n(y)\sin(y)$ and $g_n''(y) = h_n'(y)\sin(y) + h_n(y)\cos(y)$ for all $y \in \mathbb{R}$. Thus, since $Y \in (0, \pi)$ and $g_n'(Y) = 0$, $\sin(Y) \neq 0$, $h_n(Y) = 0$, and $g_n''(Y) = h_n'(Y)\sin(Y)$. Finally note, the expression for $h_n : \mathbb{R} \to \mathbb{R}$ gives,

$$h_n'(Y)\sin(Y) = -\frac{1}{n}\sum_{x\in P_n} \frac{2(v_n - x)^2 q_n^2 \sin(Y)^2}{((v_n - x)^2 + 2(v_n - x)q_n\cos(Y) + q_n^2)^2}.$$

This proves (ii).

Part (iii) follows trivially from the expressions for R_n and R_n' in Definition 4.4 and Lemma 4.5.

Consider (iv). First note, for all $y \in (0,1)$, the expressions for R_n and R'_n give,

$$M_n(y) = (\widetilde{s}_n - u_n)\left(2y - \frac{(\widetilde{q}_n)^2}{(\widetilde{s}_n - u_n)^2} y - 1 - \frac{(\widetilde{q}_n)^2}{(\widetilde{s}_n - u_n)^2} y \log(y)\right).$$

Therefore, since $(\widetilde{q}_n)^2 = (\widetilde{s}_n - u_n)^2 + n^{-2\theta}$ (see Definition 4.4),

$$M_n(y) = (\widetilde{s}_n - u_n)\left(y - 1 - y\log(y) - \frac{n^{-2\theta}}{(\widetilde{s}_n - u_n)^2} y(1 + \log(y))\right),$$

for all $y \in (0,1)$. Note that $\widetilde{s}_n - u_n > 0$ (see Lemma 2.15 (10) and (4.6)). Also recall (see proof of part (7) above) that $\frac{n^{-2\theta}}{(\widetilde{s}_n - u_n)^2} < (\frac{2}{23})^2$. Finally, we state that $y - 1 - y\log(y) \leqslant 2e^{-1} - 1$ and $y(1+\log(y)) \geqslant -e^{-2}$ for all $y \in (0, e^{-1}]$, and $y - 1 - y\log(y) < 0$ and $y(1+\log(y)) > 0$ for all $y \in (e^{-1}, 1)$. Combined the above prove that $M_n(y) < 0$ for all $y \in (0,1)$. This proves (iv).

Consider (v). Note, Definition 2.14 and (4.6) gives $t > s$ and $n^{-\theta} < \xi < \frac{1}{8}(t-s)$. This proves (v). Consider (vi). Note,

$$\begin{aligned}
&\inf_{w \in B(t,2n^{-\theta})} \inf_{y \in [0,1]} |w - (u_n + R_n(y) + iI_n(y))| \\
&\qquad \geqslant \inf_{w \in B(t,2n^{-\theta})} |w - u_n| - \sup_{y \in [0,1]} |R_n(y) + iI_n(y)|.
\end{aligned}$$

Next note, for all $y \in [0,1]$, Definition 4.4 gives $|R_n(y) + iI_n(y)| = \widetilde{q}_n \sqrt{y} = |\widetilde{s}_n + in^{-\theta} - u_n|\sqrt{y}$. Therefore,

$$\begin{aligned}
&\inf_{w \in B(t,2n^{-\theta})} \inf_{y \in [0,1]} |w - (u_n + R_n(y) + iI_n(y))| \\
&\qquad\qquad \geqslant \inf_{w \in B(t,2n^{-\theta})} |w - u_n| - |\widetilde{s}_n + in^{-\theta} - u_n|.
\end{aligned}$$

Finally recall that $n^{-\theta} < \xi$, $|\widetilde{s}_n - s| < \frac{1}{2}\xi$ (see Lemma 2.15 (10)), and note (4.6) gives the following: $|w - u_n| \geqslant t - u_n - 2\xi$ for all $w \in B(t, 2n^{-\theta})$, $|\widetilde{s}_n + in^{-\theta} - u_n| < \widetilde{s}_n - u_n + \xi < s - u_n + \frac{3}{2}\xi$, and $\xi < \frac{1}{8}(t-s)$. Combined, the above prove (vi).

Consider (vii). Note that

$$\inf_{w \in \partial B(v_n,q_n)} \inf_{z \in B(s,2n^{-\theta})} |w - z| \geqslant \inf_{w \in \partial B(v_n,q_n)} |w - v_n| - \sup_{z \in B(s,2n^{-\theta})} |v_n - z|.$$

In addition, $\inf_{w \in \partial B(v_n,q_n)} |w - v_n| = q_n = |t_n + in^{-\theta} - v_n|$ (see Definition 4.4). Therefore,

$$\inf_{w \in \partial B(v_n,q_n)} \inf_{z \in B(s,2n^{-\theta})} |w - z| \geqslant |t_n + in^{-\theta} - v_n| - \sup_{z \in B(s,2n^{-\theta})} |v_n - z|.$$

Finally recall that $n^{-\theta} < \xi$, $|t_n - t| < \frac{1}{2}\xi$ (see Lemma 2.15 (9)), and note (4.6) gives the following: $|t_n + in^{-\theta} - v_n| > t_n - v_n - \xi > t - v_n - \frac{3}{2}\xi$, $|z - v_n| \leqslant$

$s - v_n + 2\xi$ for all $z \in B(s, 2n^{-\theta})$, and $\xi < \frac{1}{8}(t-s)$. Combined, the above prove (vii).

Consider (viii). Note,

$$\inf_{w \in \partial B(v_n, q_n)} \inf_{y \in [0,1]} |w - (u_n + R_n(y) + iI_n(y))| \geqslant \inf_{w \in \partial B(v_n, q_n)} |w - v_n| - |v_n - u_n| - \sup_{y \in [0,1]} |R_n(y) + iI_n(y)|.$$

Proceed as in the proofs of parts (vi,vii) above to get,

$$\inf_{w \in \partial B(v_n, q_n)} \inf_{y \in [0,1]} |w - (u_n + R_n(y) + iI_n(y))| \geqslant |t_n + in^{-\theta} - v_n| - |v_n - u_n| - |\widetilde{s}_n + in^{-\theta} - u_n|.$$

Next recall that $|t_n + in^{-\theta} - v_n| > t - v_n - \frac{3}{2}\xi$ (see proof of part (vii)), and $|\widetilde{s}_n + in^{-\theta} - u_n| < s - u_n + \frac{3}{2}\xi$ (see proof of part (vi)). Therefore,

$$\inf_{w \in \partial B(v_n, q_n)} \inf_{y \in [0,1]} |w - (u_n + R_n(y) + iI_n(y))| > t - s - 3\xi - 2|v_n - u_n|.$$

Finally recall that $|v_n - u_n| < \frac{1}{2}\xi$ (see Definition 2.14) and $\xi < \frac{1}{8}(t-s)$ (see (4.6)). Combined, the above prove (viii). □

4.3. Proof of Theorem 2.16 via steepest descent analysis

Fix ξ, N and $\theta \in (\frac{1}{3}, \frac{1}{2})$ as in the previous two sections. Fix $n > N$. Define $t_n, \widetilde{s}_n, u_n, r_n, v_n, s_n, \gamma_n, \Gamma_n$ as in the previous two sections. Recall (see Lemma 4.7 (1) and (2)) that Γ_n contains $\{x \in P_n : x > u_n\}$ and does not contain any of $\{x \in P_n : x < u_n\}$, and γ_n contains v_n and Γ_n. (2.3) and (2.20) thus gives,

$$K_n((u_n, r_n), (v_n, s_n)) = \frac{(n - s_n)!}{(n - r_n - 1)!} \, J_n - \phi_{r_n, s_n}(u_n, v_n),$$

where

$$J_n := \frac{1}{(2\pi i)^2} \int_{\gamma_n} \mathrm{d}w \int_{\Gamma_n} \mathrm{d}z \, \frac{1}{w - z} \frac{(z - u_n)^{n - r_n - 1}}{(w - v_n)^{n - s_n + 1}} \prod_{x \in P_n} \left(\frac{w - x}{z - x} \right).$$

Define $b_n, \widetilde{b}_n, \alpha_n, \widetilde{\alpha}_n$ as in Lemma 4.3, and so $n^{-\theta} b_n = |t_n + in^{-\theta} - t|$ and $\alpha_n = \mathrm{Arg}(t_n + in^{-\theta} - t)$, and $n^{-\theta}\widetilde{b}_n = |\widetilde{s}_n + in^{-\theta} - s|$ and $\widetilde{\alpha}_n = \mathrm{Arg}(\widetilde{s}_n + in^{-\theta} - s)$. Recall (see Lemma 4.3 (1) and (2)) that $1 \leqslant b_n < 2$ and $1 \leqslant \widetilde{b}_n < 2$, and $\max\{|b_n - 1|, |\widetilde{b}_n - 1|, |\alpha_n - \frac{\pi}{2}|, |\widetilde{\alpha}_n - \frac{\pi}{2}|\} < n^{-\frac{1}{2}+\theta}$. Also, Definition 4.6 implies that we can partition γ_n and Γ_n as follows:

$$\gamma_n = \gamma_n^{(l)} + \gamma_n^{(r)} \qquad \text{and} \qquad \Gamma_n = \Gamma_n^{(l)} + \Gamma_n^{(r)}, \tag{4.9}$$

where:

- $\gamma_n^{(l)}$ is that *local section* of γ_n given by the lines from $t_n - in^{-\theta} = t + n^{-\theta} b_n e^{-i\alpha_n}$ to t, and from t to $t_n + in^{-\theta} = t + n^{-\theta} b_n e^{i\alpha_n}$.
- $\Gamma_n^{(l)}$ is that *local section* of Γ_n given by the lines from $\widetilde{s}_n - in^{-\theta} = s + n^{-\theta} \widetilde{b}_n e^{-i\tilde{\alpha}_n}$ to s, and from s to $\widetilde{s}_n + in^{-\theta} = s + n^{-\theta} \widetilde{b}_n e^{i\tilde{\alpha}_n}$.
- $\gamma_n^{(r)}$ and $\Gamma_n^{(r)}$ are (respectively) the *remaining sections* of γ_n and Γ_n.

Then,

$$J_n = J_n^{(l,l)} + J_n^{(l,r)} + J_n^{(r,l)} + J_n^{(r,r)}, \tag{4.10}$$

where,

$$J_n^{(l,l)} := \frac{1}{(2\pi i)^2} \int_{\gamma_n^{(l)}} \mathrm{d}w \int_{\Gamma_n^{(l)}} \mathrm{d}z \, \frac{1}{w-z} \frac{(z-u_n)^{n-r_n-1}}{(w-v_n)^{n-s_n+1}} \prod_{x \in P_n} \left(\frac{w-x}{z-x} \right).$$

The other three terms on the RHS of (4.10) are defined analogously. As we shall see in the following lemmas, the asymptotic behaviour of $J_n^{(l,l)}$ dominates the other terms.

Consider first $J_n^{(l,l)}$. Define $D_n, \widetilde{D}_n$ as in Lemma 4.3, and recall that $D_n > (\frac{1}{8}|f''_{(t,s)}(t)|)^{\frac{1}{2}} > 0$ and $\widetilde{D}_n > (\frac{1}{8}|f''_{(t,s)}(s)|)^{\frac{1}{2}} > 0$ (see Lemma 4.3 (3)). Then:

Lemma 4.8. — *The following is satisfied:*

$$\left| J_n^{(l,l)} - \frac{\exp(nf_n(t) - n\widetilde{f}_n(s))}{4\pi(t-s) D_n \widetilde{D}_n} \, n^{-1} \right| < \frac{\exp(nf_n(t) - n\widetilde{f}_n(s))}{4\pi(t-s) D_n \widetilde{D}_n} \, n^{-3\theta} \, F_n,$$

where $F_n > 0$ is defined in the proof and satisfies $F_n = O(1)$ for all n sufficiently large.

Proof. — First, (2.6), (2.7), (2.20), and (4.10) give

$$J_n^{(l,l)} = \frac{1}{(2\pi i)^2} \int_{\gamma_n^{(l)}} \mathrm{d}w \int_{\Gamma_n^{(l)}} \mathrm{d}z \, \frac{\exp(nf_n(w) - n\widetilde{f}_n(z))}{w-z},$$

where (see (4.9)):

- $\gamma_n^{(l)}$ is the lines from $t+n^{-\theta} b_n e^{-i\alpha_n}$ to t, and from t to $t+n^{-\theta} b_n e^{i\alpha_n}$.
- $\Gamma_n^{(l)}$ is the lines from $s+n^{-\theta} \widetilde{b}_n e^{-i\tilde{\alpha}_n}$ to s, and from s to $s+n^{-\theta} \widetilde{b}_n e^{i\tilde{\alpha}_n}$.

A change of variables then gives,

$$J_n^{(l,l)} = \frac{n^{-1}}{(2\pi i)^2 D_n \widetilde{D}_n} \int_{h_n} \mathrm{d}w \int_{H_n} \mathrm{d}z \, \frac{\exp(nf_n(t + n^{-\frac{1}{2}} D_n^{-1} w) - n\widetilde{f}_n(s + n^{-\frac{1}{2}} \widetilde{D}_n^{-1} z))}{t - s + n^{-\frac{1}{2}} D_n^{-1} w - n^{-\frac{1}{2}} \widetilde{D}_n^{-1} z},$$

where:

- h_n is the lines from $n^{\frac{1}{2}-\theta} b_n D_n e^{-i\alpha_n}$ to 0, and from 0 to $n^{\frac{1}{2}-\theta} b_n D_n e^{i\alpha_n}$.
- H_n is the lines from $n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n e^{-i\tilde{\alpha}_n}$ to 0, and from 0 to $n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n e^{i\tilde{\alpha}_n}$.

h_n and H_n are shown in Figure 4.4. Note, letting cl denote closure in $\mathbb{C}$, $h_n \subset \mathrm{cl}(B(0, n^{\frac{1}{2}-\theta} b_n D_n))$ and $H_n \subset \mathrm{cl}(B(0, n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n))$. Lemma 4.3 (6) and (7) then give,

$$J_n^{(l,l)} = \frac{n^{-1}}{(2\pi i)^2 D_n \widetilde{D}_n} \int_{h_n} \mathrm{d}w \int_{H_n} \mathrm{d}z \frac{\exp(n f_n(t) - n\widetilde{f}_n(s) + w^2 + z^2 + n^{1-3\theta} g_n(w,z))}{t - s + n^{-\frac{1}{2}} D_n^{-1} w - n^{-\frac{1}{2}} \widetilde{D}_n^{-1} z},$$

where

$$n^{1-3\theta} g_n(w,z) := n f_n(t + n^{-\frac{1}{2}} D_n^{-1} w) - n f_n(t) - w^2 - n\widetilde{f}_n(s + n^{-\frac{1}{2}} \widetilde{D}_n^{-1} z)) + n\widetilde{f}_n(s) - z^2$$

satisfies,

$$\sup_{(w,z) \in \mathrm{cl}(B(0,n^{\frac{1}{2}-\theta} b_n D_n)) \times \mathrm{cl}(B(0,n^{\frac{1}{2}-\theta} \tilde{b}_n \tilde{D}_n))} |g_n(w,z)| \leqslant E_{2,n} + \widetilde{E}_{2,n}, \quad (4.11)$$

and $E_{2,n} + \widetilde{E}_{2,n} = O(1)$ for all n sufficiently large. Next recall that $|\alpha_n - \frac{\pi}{2}| \leqslant n^{-\frac{1}{2}+\theta}$ and $|\alpha_n - \frac{\pi}{2}| \leqslant n^{-\frac{1}{2}+\theta}$ (see Lemma 4.3 (2))), where $\theta \in (\frac{1}{3}, \frac{1}{2})$, and define:

- k_n is the line from $n^{\frac{1}{2}-\theta} b_n D_n e^{-i\frac{\pi}{2}}$ to $n^{\frac{1}{2}-\theta} b_n D_n e^{i\frac{\pi}{2}}$. c_n is the smallest arcs of $\partial B(0, n^{\frac{1}{2}-\theta} b_n D_n)$ from $n^{\frac{1}{2}-\theta} b_n D_n e^{-i\alpha_n}$ to $n^{\frac{1}{2}-\theta} b_n D_n e^{-i\frac{\pi}{2}}$, and from $n^{\frac{1}{2}-\theta} b_n D_n e^{i\frac{\pi}{2}}$ to $n^{\frac{1}{2}-\theta} b_n D_n e^{i\alpha_n}$.
- K_n is the line from $n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n e^{-i\frac{\pi}{2}}$ to $n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n e^{i\frac{\pi}{2}}$. C_n is the smallest arcs of $\partial B(0, n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n)$ from $n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n e^{-i\tilde{\alpha}_n}$ to $n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n e^{-i\frac{\pi}{2}}$, and from $n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n e^{i\frac{\pi}{2}}$ to $n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n e^{i\tilde{\alpha}_n}$.

These contours are also shown in Figure 4.4. Then, noting that h_n and $c_n + k_n$ have the same initial and final points, and similarly for H_n and $C_n + K_n$,

$$J_n^{(l,l)} = n^{-1} \frac{\exp(n f_n(t) - n\widetilde{f}_n(s))}{(2\pi)^2 D_n \widetilde{D}_n} (I_n^{(k,K)} + I_n^{(k,C)} + I_n^{(c,K)} + I_n^{(c,C)}), \quad (4.12)$$

where,

$$I_n^{(k,K)} := -\int_{k_n} \mathrm{d}w \int_{K_n} \mathrm{d}z \frac{\exp(w^2 + z^2 + n^{1-3\theta} g_n(w,z))}{t - s + n^{-\frac{1}{2}} D_n^{-1} w - n^{-\frac{1}{2}} \widetilde{D}_n^{-1} z},$$

and the other three terms on the RHS are defined analogously. Next write,

$$I_n^{(k,K)} = I_{1,n} + I_{2,n} + I_{3,n}, \quad (4.13)$$

where,

$$I_{1,n} := -\int_{k_n} \mathrm{d}w \int_{K_n} \mathrm{d}z \ \frac{\exp(w^2+z^2)}{t-s},$$
$$I_{2,n} := -\int_{k_n} \mathrm{d}w \int_{K_n} \mathrm{d}z \ \left(\frac{\exp(w^2+z^2+n^{1-3\theta}g_n(w,z))}{t-s} - \frac{\exp(w^2+z^2)}{t-s}\right),$$
$$I_{3,n} := -\int_{k_n} \mathrm{d}w \int_{K_n} \mathrm{d}z \ \left(\frac{\exp(w^2+z^2+n^{1-3\theta}g_n(w,z))}{t-s+n^{-\frac{1}{2}}D_n^{-1}w - n^{-\frac{1}{2}}\widetilde{D}_n^{-1}z} - \frac{\exp(w^2+z^2+n^{1-3\theta}g_n(w,z))}{t-s}\right).$$

We will show:

(i) $|I_{1,n} - \frac{\pi}{t-s}| < \exp(-n^{1-2\theta}(D_n^2 \wedge \widetilde{D}_n^2))\frac{\pi}{t-s}$.
(ii) $|I_{2,n}| \leqslant n^{1-3\theta}(E_{2,n}+\widetilde{E}_{2,n})\frac{2\pi}{t-s}$.
(iii) $|I_{3,n}| \leqslant n^{-\theta}\frac{2^5\pi}{(t-s)^2}$.
(iv) $|I_n^{(k,C)}| < \exp(-\frac{1}{4}n^{1-2\theta}\widetilde{D}_n^2)\frac{2^6\tilde{D}_n}{t-s}$.
(v) $|I_n^{(c,K)}| < \exp(-\frac{1}{4}n^{1-2\theta}D_n^2)\frac{2^6 D_n}{t-s}$.
(vi) $|I_n^{(c,C)}| < \exp(-\frac{1}{4}n^{1-2\theta}(D_n^2+\widetilde{D}_n^2))\frac{2^7 D_n\tilde{D}_n}{t-s}$.

Define,

$$F_n := n^{-1+3\theta}\ \frac{t-s}{\pi}\ (|I_{1,n} - \frac{\pi}{t-s}| + |I_{2,n}| + |I_{3,n}| + |I_n^{(k,C)}| + |I_n^{(c,K)}| + |I_n^{(c,C)}|).$$

Recall that $\theta \in (\frac{1}{3}, \frac{1}{2})$, $D_n > (\frac{1}{8}|f''_{(t,s)}(t)|)^{\frac{1}{2}} > 0$ and $\widetilde{D}_n > (\frac{1}{8}|f''_{(t,s)}(s)|)^{\frac{1}{2}} > 0$ (see Lemma 4.3 (3)), and $E_{2,n} + \widetilde{E}_{2,n} = O(1)$. It follows that $F_n = O(1)$ for all n sufficiently large. The required result then follows from (4.12), (4.13), and parts (i)–(vi).

Consider (i). First recall that $\theta \in (\frac{1}{3}, \frac{1}{2})$, $k_n(x) = ix$ for all $x \in [-n^{\frac{1}{2}-\theta}b_n D_n, n^{\frac{1}{2}-\theta}b_n D_n]$ and $K_n(y) = iy$ for all $y \in [-n^{\frac{1}{2}-\theta}\widetilde{b}_n \widetilde{D}_n, n^{\frac{1}{2}-\theta}\widetilde{b}_n \widetilde{D}_n]$. (4.13) thus gives,

$$I_{1,n} = \frac{1}{t-s}\int_{-n^{\frac{1}{2}-\theta}b_n D_n}^{n^{\frac{1}{2}-\theta}b_n D_n} \mathrm{d}x \int_{-n^{\frac{1}{2}-\theta}\tilde{b}_n \tilde{D}_n}^{n^{\frac{1}{2}-\theta}\tilde{b}_n \tilde{D}_n} \mathrm{d}y\ \exp(-x^2-y^2).$$

Therefore,

$$I_{1,n} < \frac{1}{t-s}\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\ \exp(-x^2-y^2) = \frac{\pi}{t-s}.$$

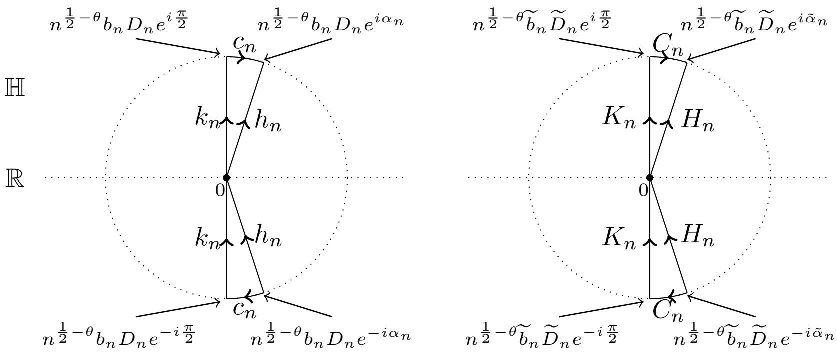

Figure 4.4. Left: The circle $B(0, n^{\frac{1}{2}-\theta} b_n D_n)$, and the contours h_n, k_n, c_n. Right: The circle $B(0, n^{\frac{1}{2}-\theta}\widetilde{b}_n \widetilde{D}_n)$, and the contours H_n, K_n, C_n. Recall, $\theta \in (\frac{1}{3}, \frac{1}{2})$, $\alpha_n = \frac{\pi}{2} + O(n^{-\frac{1}{2}+\theta})$, and $\widetilde{\alpha}_n = \frac{\pi}{2} + O(n^{-\frac{1}{2}+\theta})$.

Next recall that $b_n, \widetilde{b}_n \geqslant 1$ (see Lemma 4.3(1)). Therefore,

$$\begin{aligned} I_{1,n} &\geqslant \frac{1}{t-s} \int_0^{n^{\frac{1}{2}-\theta}(D_n \wedge \tilde{D}_n)} \mathrm{d}r \int_{-\pi}^{\pi} \mathrm{d}\phi \, r \exp(-r^2) \\ &= \frac{\pi}{t-s} \left(1 - \exp(-n^{1-2\theta}(D_n^2 \wedge \widetilde{D}_n^2))\right). \end{aligned}$$

Combined, the above prove (i).

Consider (ii). First recall that $\theta \in (\frac{1}{3}, \frac{1}{2})$, $k_n(x) = ix$ for all $x \in [-n^{\frac{1}{2}-\theta} b_n D_n, n^{\frac{1}{2}-\theta} b_n D_n]$ and $K_n(y) = iy$ for all $y \in [-n^{\frac{1}{2}-\theta}\widetilde{b}_n \widetilde{D}_n, n^{\frac{1}{2}-\theta}\widetilde{b}_n \widetilde{D}_n]$. (4.13) thus gives,

$$I_{2,n} = \int_{-n^{\frac{1}{2}-\theta} b_n D_n}^{n^{\frac{1}{2}-\theta} b_n D_n} \mathrm{d}x \int_{-n^{\frac{1}{2}-\theta}\tilde{b}_n \tilde{D}_n}^{n^{\frac{1}{2}-\theta}\tilde{b}_n \tilde{D}_n} \mathrm{d}y \\ \frac{\exp(-x^2-y^2)}{t-s} \left(\exp(n^{1-3\theta} g_n(ix, iy)) - 1\right).$$

Next recall that $n^{1-3\theta}(E_{2,n} + \widetilde{E}_{2,n}) < 1$ (see Definition 2.14), and note that $|\exp(x) - 1| \leqslant 2|x|$ when $|x| < 1$. (4.11) then gives,

$$\sup_{|x| \leqslant n^{\frac{1}{2}-\theta} b_n D_n, \, |y| \leqslant n^{\frac{1}{2}-\theta}\tilde{b}_n \tilde{D}_n} \left|\exp(n^{1-3\theta} g_n(ix, iy)) - 1\right| < 2n^{1-3\theta}(E_{2,n} + \widetilde{E}_{2,n}).$$

Therefore,

$$
\begin{aligned}
|I_{2,n}| &< \int_{-n^{\frac{1}{2}-\theta} b_n D_n}^{n^{\frac{1}{2}-\theta} b_n D_n} \mathrm{d}x \int_{-n^{\frac{1}{2}-\theta} \tilde{b}_n \tilde{D}_n}^{n^{\frac{1}{2}-\theta} \tilde{b}_n \tilde{D}_n} \mathrm{d}y \; \frac{\exp(-x^2-y^2)}{t-s} (2n^{1-3\theta}(E_{2,n}+\widetilde{E}_{2,n})) \\
&< \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \; \frac{\exp(-x^2-y^2)}{t-s} (2n^{1-3\theta}(E_{2,n}+\widetilde{E}_{2,n})).
\end{aligned}
$$

This proves (ii).

Consider (iii). First recall that $\theta \in (\frac{1}{3}, \frac{1}{2})$, $k_n(x) = ix$ for all $x \in [-n^{\frac{1}{2}-\theta} b_n D_n, n^{\frac{1}{2}-\theta} b_n D_n]$ and $K_n(y) = iy$ for all $y \in [-n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n, n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n]$. (4.13) thus gives,

$$
\begin{aligned}
I_{3,n} = \int_{-n^{\frac{1}{2}-\theta} b_n D_n}^{n^{\frac{1}{2}-\theta} b_n D_n} \mathrm{d}x \int_{-n^{\frac{1}{2}-\theta} \tilde{b}_n \tilde{D}_n}^{n^{\frac{1}{2}-\theta} \tilde{b}_n \tilde{D}_n} \mathrm{d}y & \frac{\exp(-x^2-y^2+n^{1-3\theta} g_n(ix,iy))}{t-s} \\
& \times \left(- \frac{n^{-\frac{1}{2}} D_n^{-1} ix - n^{-\frac{1}{2}} \widetilde{D}_n^{-1} iy}{t-s+n^{-\frac{1}{2}} D_n^{-1} ix - n^{-\frac{1}{2}} \widetilde{D}_n^{-1} iy} \right).
\end{aligned}
$$

Next recall that $n^{1-3\theta}(E_{2,n}+\widetilde{E}_{2,n}) < 1$ (see Definition 2.14), and note that $|\exp(x)| < 4$ when $|x| < 1$. (4.11) thus gives,

$$
\sup_{|x| \leqslant n^{\frac{1}{2}-\theta} b_n D_n, \, |y| \leqslant n^{\frac{1}{2}-\theta} \tilde{b}_n \tilde{D}_n} \exp(n^{1-3\theta} |g_n(ix,iy)|) < 4.
$$

Next recall that $b_n, \widetilde{b}_n < 2$ (see Lemma 4.3 (1)), and $n^{-\theta} < \xi$ (see Definition 2.14), and so $|n^{-\frac{1}{2}} D_n^{-1} ix| < 2n^{-\theta} < 2\xi$ for all $|x| \leqslant n^{\frac{1}{2}-\theta} b_n D_n$, and $|n^{-\frac{1}{2}} \widetilde{D}_n^{-1} iy| < 2n^{-\theta} < 2\xi$ for all $|y| \leqslant n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n$. Also, $\xi < \frac{1}{8}(t-s)$ (see (4.6)), and so

$$
\begin{aligned}
\sup_{|x| \leqslant n^{\frac{1}{2}-\theta} b_n D_n, \, |y| \leqslant n^{\frac{1}{2}-\theta} \tilde{b}_n \tilde{D}_n} & \left| - \frac{n^{-\frac{1}{2}} D_n^{-1} ix - n^{-\frac{1}{2}} \widetilde{D}_n^{-1} iy}{t-s+n^{-\frac{1}{2}} D_n^{-1} ix - n^{-\frac{1}{2}} \widetilde{D}_n^{-1} iy} \right| \\
& < \frac{2n^{-\theta}+2n^{-\theta}}{t-s-2\xi-2\xi} < \frac{4n^{-\theta}}{\frac{1}{2}(t-s)}.
\end{aligned}
$$

Combined the above give,

$$
\begin{aligned}
|I_{3,n}| &< \int_{-n^{\frac{1}{2}-\theta} b_n D_n}^{n^{\frac{1}{2}-\theta} b_n D_n} \mathrm{d}x \int_{-n^{\frac{1}{2}-\theta} \tilde{b}_n \tilde{D}_n}^{n^{\frac{1}{2}-\theta} \tilde{b}_n \tilde{D}_n} \mathrm{d}y \; \frac{\exp(-x^2-y^2)}{t-s} \frac{2^5 n^{-\theta}}{t-s} \\
&< \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \; \frac{\exp(-x^2-y^2)}{t-s} \frac{2^5 n^{-\theta}}{t-s}.
\end{aligned}
$$

This proves (iii).

Consider (iv). First recall (4.12):

$$I_n^{(k,C)} := -\int_{k_n} \mathrm{d}w \int_{C_n} \mathrm{d}z \, \frac{\exp(w^2 + z^2 + n^{1-3\theta} g_n(w,z))}{t - s + n^{-\frac{1}{2}} D_n^{-1} w - n^{-\frac{1}{2}} \widetilde{D}_n^{-1} z},$$

where k_n and C_n are given in Figure 4.4. Next recall that $\theta < \frac{1}{2}$ and $|\widetilde{\alpha}_n - \frac{\pi}{2}| < n^{-\frac{1}{2}+\theta}$ (see Lemma 4.3 (2)). It thus follows that $|\mathrm{Arg}(w)| = \frac{\pi}{2}$ for all w on k_n, and $\big||\mathrm{Arg}(z)| - \frac{\pi}{2}\big| < n^{-\frac{1}{2}+\theta}$ for all z on C_n. Therefore $\mathrm{Re}(w^2) = -|w|^2$ for all w on k_n. Moreover, since $n^{-\frac{1}{2}+\theta} < \frac{1}{2}$ (see Definition 2.14), $\mathrm{Re}(z^2) = |z|^2 \cos(2\,\mathrm{Arg}(z)) < -\frac{1}{4}|z|^2$ for all z on C_n. Therefore,

$$\big|\exp(w^2 + z^2)\big| = \exp(\mathrm{Re}(w^2 + z^2)) < \exp(-|w|^2 - \frac{1}{4}|z|^2),$$

for all w on k_n and z on C_n. Next, proceed similarly to part (iii) to get:

$$\left|\frac{\exp(n^{1-3\theta} g_n(w,z))}{t - s + n^{-\frac{1}{2}} D_n^{-1} w - n^{-\frac{1}{2}} \widetilde{D}_n^{-1} z}\right| < \frac{4}{\frac{1}{2}(t-s)} = \frac{8}{t-s},$$

for all w on k_n and z on C_n. Recall that $k_n(x) = ix$ for all $x \in [-n^{\frac{1}{2}-\theta} b_n D_n, n^{\frac{1}{2}-\theta} b_n D_n]$, and that C_n is composed of 2 arcs of $\partial B(0, n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n)$ with total length $2n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n |\widetilde{\alpha}_n - \frac{\pi}{2}| < 2n^{\frac{1}{2}-\theta} \widetilde{b}_n \widetilde{D}_n n^{-\frac{1}{2}+\theta} = 2\widetilde{b}_n \widetilde{D}_n$. Combined the above give,

$$\begin{aligned} |I_n^{(k,C)}| &< \int_{-\infty}^{\infty} \mathrm{d}x \, (2\widetilde{b}_n \widetilde{D}_n) \, \exp\left(-x^2 - \frac{1}{4} n^{1-2\theta} \widetilde{b}_n^2 \widetilde{D}_n^2\right) \frac{8}{t-s} \\ &= \sqrt{\pi} \, (2\widetilde{b}_n \widetilde{D}_n) \, \exp\left(-\frac{1}{4} n^{1-2\theta} \widetilde{b}_n^2 \widetilde{D}_n^2\right) \frac{8}{t-s}. \end{aligned}$$

Finally note that $\sqrt{\pi} < 2$, and $1 \leqslant \widetilde{b}_n < 2$ (see Lemma 4.3 (1)). This proves (iv).

Consider (v). Proceed similar to case (iv) to get the following for all w on c_n and z on K_n: $\big||\mathrm{Arg}(w)| - \frac{\pi}{2}\big| < |\alpha_n - \frac{\pi}{2}| < n^{-\frac{1}{2}+\theta} < \frac{1}{2}$, $|\mathrm{Arg}(z)| = \frac{\pi}{2}$,

$$\big|\exp(w^2 + z^2)\big| < \exp\left(-\frac{1}{4}|w|^2 - |z|^2\right),$$

$$\left|\frac{\exp(n^{1-3\theta} g_n(w,z))}{t - s + n^{-\frac{1}{2}} D_n^{-1} w - n^{-\frac{1}{2}} \widetilde{D}_n^{-1} z}\right| < \frac{8}{t-s}.$$

Next recall that c_n is composed of 2 arcs of $\partial B(0, n^{\frac{1}{2}-\theta} b_n D_n)$ with total length $2n^{\frac{1}{2}-\theta} b_n D_n |\alpha_n - \frac{\pi}{2}| < 2b_n D_n$, and $K_n(y) = iy$ for all $y \in$

$[-n^{\frac{1}{2}-\theta}\widetilde{b}_n\widetilde{D}_n, n^{\frac{1}{2}-\theta}\widetilde{b}_n\widetilde{D}_n]$. Combine the above with (4.12) to get,

$$\begin{aligned}|I_n^{(c,K)}| &< (2b_nD_n) \int_{-\infty}^{\infty} \mathrm{d}y \, \exp\left(-\frac{1}{4}n^{1-2\theta}b_n^2D_n^2 - y^2\right)\frac{8}{t-s} \\ &= (2b_nD_n)\sqrt{\pi}\exp\left(-\frac{1}{4}n^{1-2\theta}b_n^2D_n^2\right)\frac{8}{t-s}.\end{aligned}$$

Finally note that $\sqrt{\pi} < 2$, and $1 \leqslant b_n < 2$ (see Lemma 4.3(1)). This proves (v).

Consider (vi). Proceed similar to previous cases (iv) and (v) to get the following for all w on c_n and z on C_n: $\big||\mathrm{Arg}(w)| - \frac{\pi}{2}| < |\alpha_n - \frac{\pi}{2}| < n^{-\frac{1}{2}+\theta} < \frac{1}{2}$, $\big||\mathrm{Arg}(z)| - \frac{\pi}{2}| < |\widetilde{\alpha}_n - \frac{\pi}{2}| < n^{-\frac{1}{2}+\theta} < \frac{1}{2}$,

$$\begin{aligned}\big|\exp(w^2+z^2)\big| &< \exp\left(-\frac{1}{4}|w|^2 - \frac{1}{4}|z|^2\right), \\ \left|\frac{\exp(n^{1-3\theta}g_n(w,z))}{t-s+n^{-\frac{1}{2}}D_n^{-1}w - n^{-\frac{1}{2}}\widetilde{D}_n^{-1}z}\right| &< \frac{8}{t-s}.\end{aligned}$$

Next recall that c_n is composed of 2 arcs of $\partial B(0, n^{\frac{1}{2}-\theta}b_nD_n)$ with total length $2n^{\frac{1}{2}-\theta}b_nD_n|\alpha_n - \frac{\pi}{2}| < 2b_nD_n$, and C_n is composed of 2 arcs of $\partial B(0, n^{\frac{1}{2}-\theta}\widetilde{b}_n\widetilde{D}_n)$ with total length $2n^{\frac{1}{2}-\theta}\widetilde{b}_n\widetilde{D}_n|\widetilde{\alpha}_n - \frac{\pi}{2}| < 2\widetilde{b}_n\widetilde{D}_n$. Combine the above with (4.12) to get,

$$|I_n^{(c,C)}| < (2b_nD_n)\,(2\widetilde{b}_n\widetilde{D}_n)\,\exp\left(-\frac{1}{4}n^{1-2\theta}b_n^2D_n^2 - \frac{1}{4}n^{1-2\theta}\widetilde{b}_n^2\widetilde{D}_n^2\right)\frac{8}{t-s}.$$

Finally recall that $1 \leqslant b_n, \widetilde{b}_n < 2$. This proves (vi). □

Next we examine the asymptotic behaviour of the remaining terms of (4.10):

Lemma 4.9. — *The following is satisfied:*

$$\begin{aligned}&|J_n^{(l,r)} + J_n^{(r,l)} + J_n^{(r,r)}| \\ &\qquad < \frac{\exp(nf_n(t) - n\widetilde{f}_n(s))}{t-s}\,\exp\left(-\frac{1}{4}n^{1-2\theta}(D_n^2 \wedge \widetilde{D}_n^2)\right)\,n^{-\theta}\,G_n,\end{aligned}$$

where $G_n > 0$ is defined in the proof and satisfies $G_n = O(1)$ for all n sufficiently large.

Proof. — We will show:

$$|J_n^{(l,r)}| < \frac{2^3 \exp(nf_n(t) - n\widetilde{f}_n(s))}{t-s} \exp\left(-\frac{1}{4}n^{1-2\theta}\widetilde{D}_n^2\right) n^{-\theta} (s-\chi), \tag{i}$$

$$|J_n^{(r,l)}| < \frac{2^3 \exp(nf_n(t) - n\widetilde{f}_n(s))}{t-s} \exp\left(-\frac{1}{4}n^{1-2\theta}D_n^2\right) n^{-\theta} (t-\chi), \tag{ii}$$

$$|J_n^{(r,r)}| < \frac{2^4 \exp(nf_n(t) - n\widetilde{f}_n(s))}{t-s} \exp\left(-\frac{1}{4}n^{1-2\theta}(D_n^2 + \widetilde{D}_n^2)\right) (t-\chi)(s-\chi). \tag{iii}$$

Recall that $\theta \in (\frac{1}{3}, \frac{1}{2})$, and $D_n^2 > \frac{1}{8}|f''_{(t,s)}(t)| > 0$ and $\widetilde{D}_n^2 > \frac{1}{8}|f''_{(t,s)}(s)| > 0$ (see Lemma 4.3 (3)). The required result then easily follows from parts (i), (ii), and (iii) with,

$$G_n = 2^3(s-\chi) + 2^3(t-\chi) + 2^4 \exp\left(-\frac{1}{4}n^{1-2\theta}(D_n^2 \wedge \widetilde{D}_n^2)\right) n^{\theta} (t-\chi)(s-\chi).$$

Consider (i). Note, (2.6), (2.7), (2.20) and (4.10) give,

$$|J_n^{(l,r)}| \leqslant \frac{1}{(2\pi)^2}|\gamma_n^{(l)}||\Gamma_n^{(r)}| \sup_{(w,z)\in\gamma_n^{(l)}\times\Gamma_n^{(r)}} \left|\frac{\exp(nf_n(w) - n\widetilde{f}_n(z))}{w-z}\right|.$$

Recall (see (4.9)) that $\gamma_n^{(l)}$ is the lines from $t + n^{-\theta}b_n e^{-i\alpha_n}$ to t, and from t to $t + n^{-\theta}b_n e^{i\alpha_n}$. Therefore $|\gamma_n^{(l)}| = 2n^{-\theta}b_n < 4n^{-\theta}$ (see Lemma 4.3 (1)). Next recall (see Definition 4.6 and (4.9)) that $\Gamma_n^{(r)}$ traverses the contour $x \mapsto u_n + R_n(1-x) + iI_n(1-x)$ for $x \in [0,1]$, and its reflection in $\mathbb{R}$. Combine the above with Lemma 4.7 (4), (5), and (7) to get,

$$|J_n^{(l,r)}| < \frac{1}{(2\pi)^2}(4n^{-\theta})(8(s-\chi)) \sup_{w\in\gamma_n^{(l)}} \left|\frac{\exp(nf_n(w) - n\widetilde{f}_n(\widetilde{s}_n + in^{-\theta}))}{\frac{1}{2}(t-s)}\right|.$$

Thus, since $\frac{(4)(8)}{(2\pi)^2} < 1$, and $\widetilde{s}_n + in^{-\theta} = s + n^{-\theta}\widetilde{b}_n e^{i\tilde{\alpha}_n}$ (see Lemma 4.3),

$$|J_n^{(l,r)}| < \frac{2(s-\chi)}{t-s} n^{-\theta} \times \sup_{w\in h_n} \left|\exp(nf_n(t + n^{-\frac{1}{2}}D_n^{-1}w) - n\widetilde{f}_n(s + n^{-\frac{1}{2}}\widetilde{D}_n^{-1}z_n))\right|,$$

where $h_n \subset B(0, n^{\frac{1}{2}-\theta}b_n D_n)$ is defined in Figure 4.4, and $z_n \in \partial B(0, n^{\frac{1}{2}-\theta}\widetilde{b}_n\widetilde{D}_n)$ is defined by $z_n := n^{\frac{1}{2}-\theta}\widetilde{b}_n\widetilde{D}_n e^{i\tilde{\alpha}_n}$. Note that $h_n \subset \mathrm{cl}(B(0, n^{\frac{1}{2}-\theta}b_n D_n))$ and $z_n \in \partial B(0, n^{\frac{1}{2}-\theta}\widetilde{b}_n\widetilde{D}_n)$. Lemma 4.3 (6) and (7) then give:

$$|J_n^{(l,r)}| < \frac{2(s-\chi)}{t-s}\, n^{-\theta} \times \sup_{w\in h_n} \left|\exp(nf_n(t) - n\widetilde{f}_n(s) + w^2 + z_n^2 + n^{1-3\theta} g_n(w,z_n))\right|,$$

where

$$n^{1-3\theta} g_n(w,z) := nf_n(t + n^{-\frac{1}{2}} D_n^{-1} w) - nf_n(t) - w^2 - n\widetilde{f}_n(s + n^{-\frac{1}{2}} \widetilde{D}_n^{-1} z)) + n\widetilde{f}_n(s) - z^2.$$

We then proceed similarly to part (iii) of the previous lemma to get,

$$|J_n^{(l,r)}| < \frac{8(s-\chi)}{t-s}\, n^{-\theta} \sup_{w\in h_n} \left|\exp(nf_n(t) - n\widetilde{f}_n(s) + w^2 + z_n^2)\right|.$$

Recall that $|\mathrm{Arg}(w)| = \alpha_n$ for all w on h_n, $|\mathrm{Arg}(z_n)| = \widetilde{\alpha}_n$, $|\alpha_n - \frac{\pi}{2}| < n^{-\frac{1}{2}+\theta}$ and $|\widetilde{\alpha}_n - \frac{\pi}{2}| \leqslant n^{-\frac{1}{2}+\theta}$ (see Lemma 4.3(2)), and $|z_n| = n^{\frac{1}{2}-\theta}\widetilde{b}_n\widetilde{D}_n$. We then proceed similarly to parts (iv), (v) and (vi) of the previous lemma to get $\mathrm{Re}(w^2) < -\frac{1}{4}|w|^2 \leqslant 0$ for all w on h_n, and $\mathrm{Re}((z_n)^2) < -\frac{1}{4}|z_n|^2 = -\frac{1}{4}n^{1-2\theta}\widetilde{b}_n^2\widetilde{D}_n^2 \leqslant -\frac{1}{4}n^{1-2\theta}\widetilde{D}_n^2$. Combined, the above prove (i).

Consider (ii). First, proceed similarly to part (i) to get $|\Gamma_n^{(l)}| < 4n^{-\theta}$. Next recall (see Definition 4.6 and (4.9)) that $\gamma_n^{(r)}$ the counter-clockwise arc of $\partial B(v_n, q_n)$ from $t_n + in^{-\theta}$ to $v_n - q_n$, and its reflection in $\mathbb{R}$. (2.6), (2.7), (2.20), (4.10), and Lemma 4.7(3), (5), and (6) thus give,

$$|J_n^{(r,l)}| < \frac{1}{(2\pi)^2}(8(t-\chi))(4n^{-\theta}) \sup_{z\in\Gamma_n^{(l)}} \left|\frac{\exp(nf_n(t_n + in^{-\theta}) - n\widetilde{f}_n(z))}{\frac{1}{2}(t-s)}\right|.$$

Thus, since $t_n + in^{-\theta} = t + n^{-\theta} b_n e^{i\alpha_n}$,

$$|J_n^{(r,l)}| < \frac{2(t-\chi)}{t-s}\, n^{-\theta} \times \sup_{z\in H_n} \left|\exp(nf_n(t + n^{-\frac{1}{2}} D_n^{-1} w_n) - n\widetilde{f}_n(s + n^{-\frac{1}{2}} \widetilde{D}_n^{-1} z))\right|,$$

where $w_n \in \partial B(0, n^{\frac{1}{2}-\theta} b_n D_n)$ is defined by $w_n := n^{\frac{1}{2}-\theta} b_n D_n e^{i\alpha_n}$, and $H_n \subset B(0, n^{\frac{1}{2}-\theta}\widetilde{b}_n\widetilde{D}_n)$ is defined in Figure 4.4. Proceed similarly to part (i) to then get,

$$|J_n^{(r,l)}| < \frac{8(t-\chi)}{t-s}\, n^{-\theta} \sup_{z\in H_n} \left|\exp(nf_n(t) - n\widetilde{f}_n(s) + w_n^2 + z^2)\right|,$$
$$\mathrm{Re}((w_n)^2) < -\frac{1}{4}|w_n|^2 = -\frac{1}{4}n^{1-2\theta}b_n^2 D_n^2 \leqslant -\frac{1}{4}n^{1-2\theta}D_n^2,$$
$$\mathrm{Re}(z^2) < -\frac{1}{4}|z|^2 \leqslant 0$$

for all z on H_n. Combined, the above prove (ii).

Consider (iii). First, proceeding similarly to parts (i) and (ii),

$$|J_n^{(r,r)}| < \frac{1}{(2\pi)^2}(8(t-\chi))(8(s-\chi)) \left|\frac{\exp(nf_n(t_n+in^{-\theta}) - n\widetilde{f}_n(\widetilde{s}_n+in^{-\theta}))}{\frac{1}{2}(t-s)}\right|.$$

Then, define $w_n \in \partial B(0, n^{\frac{1}{2}-\theta}b_n D_n)$ and $z_n \in \partial B(0, n^{\frac{1}{2}-\theta}\widetilde{b}_n\widetilde{D}_n)$ as above, and proceed similarly to parts (i) and (ii) to get,

$$\begin{aligned}|J_n^{(r,r)}| &< \frac{4(t-\chi)(s-\chi)}{t-s} \left|\exp(nf_n(t+n^{-\frac{1}{2}}D_n^{-1}w_n) - n\widetilde{f}_n(s+n^{-\frac{1}{2}}\widetilde{D}_n^{-1}z_n))\right| \\ &< \frac{16(t-\chi)(s-\chi)}{t-s} \left|\exp(nf_n(t) - n\widetilde{f}_n(s) + w_n^2 + z_n^2)\right|,\end{aligned}$$

$\mathrm{Re}((w_n)^2) < -\frac{1}{4}n^{1-2\theta}D_n^2$, and $\mathrm{Re}((z_n)^2) < -\frac{1}{4}n^{1-2\theta}\widetilde{D}_n^2$. Combined, the above prove (iii). ☐

Finally we prove Theorem 2.16:

Proof of Theorem 2.16. — First recall (see (4.10)) that $J_n = J_n^{(l,l)} + J_n^{(l,r)} + J_n^{(r,l)} + J_n^{(r,r)}$. Lemmas 4.8 and 4.9 thus gives,

$$\begin{aligned}\left|nJ_n - \frac{\exp(nf_n(t) - n\widetilde{f}_n(s))}{4\pi(t-s)D_n\widetilde{D}_n}\right| &< \frac{\exp(nf_n(t) - n\widetilde{f}_n(s))}{4\pi(t-s)D_n\widetilde{D}_n}\, n^{1-3\theta}\, F_n \\ &+ \frac{\exp(nf_n(t) - n\widetilde{f}_n(s))}{t-s}\, \exp\left(-\frac{1}{4}n^{1-2\theta}(D_n^2 \wedge \widetilde{D}_n^2)\right) n^{1-\theta}\, G_n,\end{aligned}$$

where F_n and G_n are defined in the proof of Lemmas 4.8 and 4.9 (respectively). Moreover, (2.2) and (2.3) trivially give $\phi_{r_n,s_n}(u_n, v_n) = 0$ and $K_n((u_n, r_n), (v_n, s_n)) = (1 - \frac{s_n}{n})\, nJ_n$ when $r_n = s_n$ for all $n > N$, as required. ☐

5. The behaviour of the roots of $f'_{(\chi,\eta)}$

In this section we examine the behaviour of the roots of the function $f'_{(\chi,\eta)}$ given in (2.12). Only the following assumptions are required in this section:

- μ is a probability measure on $\mathbb{R}$ with compact support, $\mathrm{Supp}(\mu) \subset [a,b]$ with $\{a,b\} \subset \mathrm{Supp}(\mu)$, and $(\chi,\eta) \in [a,b] \times [0,1]$ is fixed.
- Assume that $b > a$ to avoid that degenerate case where μ is a single atom of mass 1. This implies that $\mu[\{\chi\}] \in [0,1)$.

Recall (see (2.12)),

$$f'_{(\chi,\eta)}(w) = \int_{(\chi,b]} \frac{\mu[\mathrm{d}x]}{w-x} - \frac{1-\eta-\mu[\{\chi\}]}{w-\chi} + \int_{[a,\chi)} \frac{\mu[\mathrm{d}x]}{w-x},$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$. The above expression has a unique analytic extension to the set $\mathbb{C} \setminus (S_1 \cup S_2 \cup S_3)$, where $S_i := S_i(\chi,\eta)$ for all $i \in \{1,2,3\}$ are defined by:

$$S_1 := \mathrm{Supp}(\mu|_{(\chi,b]}),$$
$$S_2 := \begin{cases} \{\chi\} & \text{when } \mu[\{\chi\}] \neq 1-\eta, \\ \emptyset & \text{when } \mu[\{\chi\}] = 1-\eta, \end{cases}$$
$$S_3 := \mathrm{Supp}(\mu|_{[a,\chi)}).$$

Note $S_1 = \emptyset$ when $b = \chi$, and $S_3 = \emptyset$ when $\chi = a$. Thus, since $b > a$, $\mathrm{Supp}(\mu) \subset [a,b]$ with $\{a,b\} \subset \mathrm{Supp}(\mu)$, $(\chi,\eta) \in [a,b] \times [0,1]$, and $\mu[\{\chi\}] \in [0,1)$, the following 12 cases exhaust all possibilities:

(a) $b > \chi > a$, $1 > \eta > 0$, $1-\eta > \mu[\{\chi\}]$, and $S_1 \neq \emptyset$, $S_2 = \{\chi\}$, $S_3 \neq \emptyset$.
(b) $b > \chi > a$, $1 > \eta = 0$, $1-\eta > \mu[\{\chi\}]$, and $S_1 \neq \emptyset$, $S_2 = \{\chi\}$, $S_3 \neq \emptyset$.
(c) $b > \chi > a$, $1 \geqslant \eta > 0$, $1-\eta < \mu[\{\chi\}]$, and $S_1 \neq \emptyset$, $S_2 = \{\chi\}$, $S_3 \neq \emptyset$.
(d) $b > \chi > a$, $1 \geqslant \eta > 0$, $1-\eta = \mu[\{\chi\}]$, and $S_1 \neq \emptyset$, $S_2 = \emptyset$, $S_3 \neq \emptyset$.
(e) $b > \chi = a$, $1 > \eta > 0$, $1-\eta > \mu[\{\chi\}]$, and $S_1 \neq \emptyset$, $S_2 = \{\chi\}$, $S_3 = \emptyset$.
(f) $b > \chi = a$, $1 > \eta = 0$, $1-\eta > \mu[\{\chi\}]$, and $S_1 \neq \emptyset$, $S_2 = \{\chi\}$, $S_3 = \emptyset$.
(g) $b > \chi = a$, $1 \geqslant \eta > 0$, $1-\eta < \mu[\{\chi\}]$, and $S_1 \neq \emptyset$, $S_2 = \{\chi\}$, $S_3 = \emptyset$.
(h) $b > \chi = a$, $1 \geqslant \eta > 0$, $1-\eta = \mu[\{\chi\}]$, and $S_1 \neq \emptyset$, $S_2 = \emptyset$, $S_3 = \emptyset$.
(i) $b = \chi > a$, $1 > \eta > 0$, $1-\eta > \mu[\{\chi\}]$, and $S_1 = \emptyset$, $S_2 = \{\chi\}$, $S_3 \neq \emptyset$.
(j) $b = \chi > a$, $1 > \eta = 0$, $1-\eta > \mu[\{\chi\}]$, and $S_1 = \emptyset$, $S_2 = \{\chi\}$, $S_3 \neq \emptyset$.
(k) $b = \chi > a$, $1 \geqslant \eta > 0$, $1-\eta < \mu[\{\chi\}]$, and $S_1 = \emptyset$, $S_2 = \{\chi\}$, $S_3 \neq \emptyset$.
(l) $b = \chi > a$, $1 \geqslant \eta > 0$, $1-\eta = \mu[\{\chi\}]$, and $S_1 = \emptyset$, $S_2 = \emptyset$, $S_3 \neq \emptyset$.

Moreover note:

- $b = \sup S_1 \geqslant \inf S_1 \geqslant \chi \geqslant \sup S_3 \geqslant \inf S_3 = a$ for possibilities (a-d).
- $b = \sup S_1 \geqslant \inf S_1 \geqslant \chi = a$ for possibilities (e-h).
- $b = \chi \geqslant \sup S_3 \geqslant \inf S_3 = a$ for possibilities (i-l).

The sets, S_1, S_2, S_3, for the above possibilities are depicted in Figure 5.1. Note, since $\mu[S_1] + \mu[\{\chi\}] + \mu[S_3] = 1$, we trivially have,

$$f'_{(\chi,\eta)}(w) = \int_{S_1} \frac{\mu[\mathrm{d}x]}{w-x} - \frac{\mu[S_1] + \mu[S_3] - \eta}{w - \chi} + \int_{S_3} \frac{\mu[\mathrm{d}x]}{w-x}, \tag{5.1}$$

for all $w \in \mathbb{C} \setminus (S_1 \cup S_2 \cup S_3)$.

Next write the domain of $f'_{(\chi,\eta)}$ as the disjoint union:

$$\mathbb{C} \setminus (S_1 \cup S_2 \cup S_3) = (\mathbb{C} \setminus \mathbb{R}) \cup J \cup K,$$

where $J := \bigcup_{i=1}^{4} J_i$, $K := \mathbb{R} \setminus (S \cup J)$, and

- $J_1 := (\sup S_1, +\infty)$.
- $J_2 := (-\infty, \inf S_3)$.
- $J_3 := (\chi, \inf S_1)$ when $S_1 \neq \emptyset$ and $S_2 = \{\chi\}$ and $\inf S_1 > \chi$. Otherwise, $J_3 := \emptyset$.
- $J_4 := (\sup S_3, \chi)$ when $S_3 \neq \emptyset$ and $S_2 = \{\chi\}$ and $\chi > \sup S_3$. Otherwise, $J_4 := \emptyset$.

Note that $K \subset \mathbb{R}$ is open, and so it can be partitioned as $K = \bigcup_{k=1}^{\infty} K_k$, where $\{K_1, K_2, \ldots\}$ is a set of pairwise disjoint open intervals. This partition is unique up to order, and is either empty, finite, or countable. The above sets for the different possibilities are also depicted in Figure 5.1.

Remark 5.1. — For the remainder of this section, whenever we say a number of roots, it should be implicitly understood that we mean that number of roots counting multiplicities.

The behaviour of the roots of $f'_{(\chi,\eta)}$ for the above possibilities is the following:

Theorem 5.2. — *For* (a), *$f'_{(\chi,\eta)}$ has at most* 2 *roots in each of $\{\mathbb{C} \setminus \mathbb{R}, J_1, J_2, J_3, J_4\}$, and at most* 3 *roots in each of $\{K_1, K_2, \ldots\}$. Moreover, when $f'_{(\chi,\eta)}$ has either* 1 *or* 2 *roots in some fixed $I \in \{\mathbb{C} \setminus \mathbb{R}, J_1, J_2, J_3, J_4\}$, then $f'_{(\chi,\eta)}$ has* 0 *roots in each of $\{\mathbb{C} \setminus \mathbb{R}, J_1, J_2, J_3, J_4\} \setminus \{I\}$, and at most* 1 *root in each of $\{K_1, K_2, \ldots\}$. Finally, when $f'_{(\chi,\eta)}$ has either* 2 *or* 3 *roots in some fixed $L \in \{K_1, K_2, \ldots\}$, then $f'_{(\chi,\eta)}$ has* 0 *roots in each of $\{\mathbb{C} \setminus \mathbb{R}, J_1, J_2, J_3, J_4\}$, and at most* 1 *root in each of $\{K_1, K_2, \ldots\} \setminus \{L\}$.*

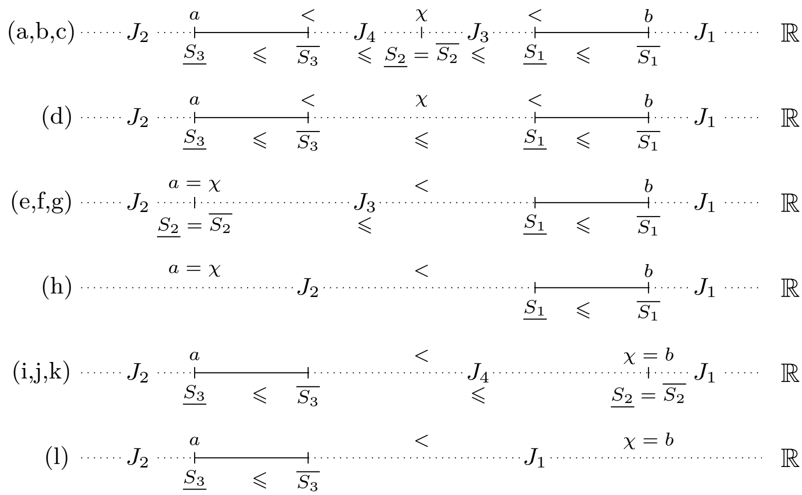

Figure 5.1. The sets $S_1, S_2, S_3, J_1, J_2, J_3, J_4$ for possibilities (a)–(l). When one of these sets is not depicted, it is understood to be empty. Also, J_3 is empty when $\inf S_1 = \chi$, and J_4 is empty when $\chi = \sup S_3$. Above, $\overline{S_i} := \sup S_i$ and $\underline{S_i} := \inf S_i$. Recall that $K = \mathbb{R} \setminus (S \cup J) = \bigcup_{k=1}^{\infty} K_k$, where $\{K_1, K_2, \ldots\}$ are disjoint open intervals. Finally, note that $[\inf S_i, \sup S_i] \setminus S_i$ is either empty or (finite or countable) union of intervals from $\{K_1, K_2, \ldots\}$.

For (b), $f'_{(\chi,\eta)}$ *has at most* 1 *root in each of* $\{J_1, J_2\} \cup \{K_1, K_2, \ldots\}$, *and* 0 *roots in each of* $\{\mathbb{C} \setminus \mathbb{R}, J_3, J_4\}$. *Moreover, when* $f'_{(\chi,\eta)}$ *has* 1 *root in some fixed* $I \in \{J_1, J_2\}$, *then* $f'_{(\chi,\eta)}$ *has* 0 *roots in* $\{J_1, J_2\} \setminus \{I\}$.

For (c), $f'_{(\chi,\eta)}$ *has* 0 *roots in each of* $\{\mathbb{C} \setminus \mathbb{R}, J_1, J_2\}$, *and at most* 1 *root in each of* $\{J_3, J_4\} \cup \{K_1, K_2, \ldots\}$.

For (d), $f'_{(\chi,\eta)}$ *has* 0 *roots in each of* $\{\mathbb{C} \setminus \mathbb{R}, J_1, J_2\}$, *and at most* 1 *root in each of* $\{K_1, K_2, \ldots\}$.

For (e), $f'_{(\chi,\eta)}$ *has* 0 *roots in each of* $\{\mathbb{C} \setminus \mathbb{R}, J_1, J_3\}$, *and at most* 1 *root in each of* $\{J_2\} \cup \{K_1, K_2, \ldots\}$.

For (f), $f'_{(\chi,\eta)}$ *has* 0 *roots in each of* $\{\mathbb{C} \setminus \mathbb{R}, J_1, J_2, J_3\}$, *and at most* 1 *root in each of* $\{K_1, K_2, \ldots\}$.

For (g), $f'_{(\chi,\eta)}$ *has* 0 *roots in each of* $\{\mathbb{C} \setminus \mathbb{R}, J_1, J_2\}$, *and at most* 1 *root in each of* $\{J_3\} \cup \{K_1, K_2, \ldots\}$.

For (h), $f'_{(\chi,\eta)}$ *has* 0 *roots in each of* $\{\mathbb{C}\setminus\mathbb{R}, J_1, J_2\}$, *and at most* 1 *root in each of* $\{K_1, K_2, \ldots\}$.

For (i), $f'_{(\chi,\eta)}$ *has* 0 *roots in each of* $\{\mathbb{C}\setminus\mathbb{R}, J_2, J_4\}$, *and at most* 1 *root in each of* $\{J_1\} \cup \{K_1, K_2, \ldots\}$.

For (j), $f'_{(\chi,\eta)}$ *has* 0 *roots in each of* $\{\mathbb{C}\setminus\mathbb{R}, J_1, J_2, J_4\}$, *and at most* 1 *root in each of* $\{K_1, K_2, \ldots\}$.

For (k), $f'_{(\chi,\eta)}$ *has* 0 *roots in each of* $\{\mathbb{C}\setminus\mathbb{R}, J_1, J_2\}$, *and at most* 1 *root in each of* $\{J_4\} \cup \{K_1, K_2, \ldots\}$.

For (l), $f'_{(\chi,\eta)}$ *has* 0 *roots in each of* $\{\mathbb{C}\setminus\mathbb{R}, J_1, J_2\}$, *and at most* 1 *root in each of* $\{K_1, K_2, \ldots\}$.

Proof. — We will prove the result only for possibilities (a) and (b) when the supports are as given on the top of Figure 5.2. The remaining results follow from similar considerations.

Consider (a) and (b) where the supports are given as on the top of Figure 5.2. First note, (5.1) trivially implies the following:

(i) Non-real roots of $f'_{(\chi,\eta)}$ occur in complex conjugate pairs.

Next, inspired by (5.1), define the following for all $n \geqslant 1$:

$$g_n(w) := \frac{1}{n} \sum_{x\in X_n} \frac{1}{w-x} - \frac{\frac{m+l}{n} - \eta}{w-\chi} + \frac{1}{n} \sum_{y\in Y_n} \frac{1}{w-y}, \tag{5.2}$$

for all $w \in \mathbb{C} \setminus (X_n \cup \{\chi\} \cup Y_n)$, where:

- $m := m(n)$ is a positive integer ($\geqslant 4$) with $\frac{m}{n} \to \mu[S_1] > 0$ as $n \to \infty$.
- $l := l(n)$ is a positive integer ($\geqslant 2$) with $\frac{l}{n} \to \mu[S_3] > 0$ as $n \to \infty$.
- X_n is a set of m distinct real-numbers with $\{a_2, a_1\} \subset X_n \subset [\underline{X_n}, a_2] \cup [a_1, \overline{X_n}]$ for all n, $\underline{X_n} \to \underline{S_1}$ and $\overline{X_n} \to \overline{S_1}$ as $n \to \infty$, and $\frac{1}{n}\sum_{x\in X_n} \delta_x \to \mu|_{(\chi,b]}$ weakly as $n \to \infty$.
- Y_n is a set of l distinct real-numbers with $\{\underline{S_3}\} \subset Y_n \subset [\underline{S_3}, \overline{S_3})$ for all n, $\overline{Y_n} \uparrow \overline{S_3} = \chi$ as $n \to \infty$, and $\frac{1}{n}\sum_{y\in Y_n} \delta_y \to \mu|_{[a,\chi)}$ weakly as $n \to \infty$.

These are depicted on the bottom of Figure 5.2. (5.1), (5.2), the above convergence as $n \to \infty$, and Rouché's theorem imply the following:

(ii) Suppose that $z \in \mathbb{C}\setminus(S_1 \cup S_2 \cup S_3) = (\mathbb{C}\setminus\mathbb{R}) \cup (J_1 \cup J_2 \cup J_3 \cup K_1)$ is a root of $f'_{(\chi,\eta)}$ of multiplicity $k \geqslant 1$. Fix $\epsilon > 0$ for which $B(z,\epsilon) \subset$

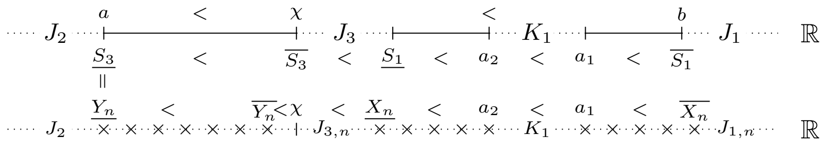

Figure 5.2. Top: An example support for possibilities (a) and (b). In words, S_1 is the union of two intervals, $S_1 = [\underline{S_1}, a_2] \cup [a_1, \overline{S_1}]$ with $\overline{S_1} > a_1 > a_2 > \underline{S_1}$. Also, S_3 is a single interval, $S_3 = [\underline{S_3}, \overline{S_3}]$ with $\overline{S_3} > \underline{S_3}$. Moreover, $\underline{S_1} > \chi = \overline{S_3}$, and so $J_3 = (\chi, \underline{S_1})$ and $J_4 = \emptyset$. Finally, $K = K_1 = (a_2, a_1)$. Bottom: Examples of the sets X_n and Y_n defined in (5.2). Elements of X_n and Y_n are denoted by $\times$.

$\mathbb{C} \setminus (S_1 \cup S_2 \cup S_3)$, and z is the unique root of $f'_{(\chi,\eta)}$ in $B(z, \epsilon)$. Then, for all n sufficiently large, g_n has k roots in $B(z, \epsilon)$.

Next, we will show, for all $n \geqslant 1$:

(iii) g_n has $m + l$ roots in $\mathbb{C} \setminus (X_n \cup \{\chi\} \cup Y_n\})$ for possibility (a), and at least $m + l - 1$ roots in $\mathbb{C} \setminus (X_n \cup \{\chi\} \cup Y_n\})$ for possibility (b).
(iv) g_n has an odd number of roots in any open interval bounded by any two consecutive elements of X_n for possibilities (a) and (b). Similarly for any two consecutive elements of Y_n.
(v) g_n has an even number of roots in each of $\{\mathbb{C} \setminus \mathbb{R}, J_{1,n}, J_2, J_{3,n}\}$ for possibility (a), where $J_{1,n} := (\overline{X_n}, +\infty)$ and $J_{3,n} := (\chi, \underline{X_n})$ (see Figure 5.2). g_n has an even number of roots in each of $\{\mathbb{C} \setminus \mathbb{R}, J_{3,n}\}$ for possibility (b).

We will then use parts (iii), (iv), and (v) to show:

(vi) For possibility (a) and all $n \geqslant 1$, g_n has either 0 or 2 roots in each of $\{\mathbb{C} \setminus \mathbb{R}, J_{1,n}, J_2, J_{3,n}\}$, and either 1 or 3 roots in K_1. Moreover, when g_n has 2 roots in some fixed $I \in \{\mathbb{C} \setminus \mathbb{R}, J_{1,n}, J_2, J_{3,n}\}$, then g_n has 0 roots in each of $\{\mathbb{C} \setminus \mathbb{R}, J_{1,n}, J_2, J_{3,n}\} \setminus \{I\}$, and 1 root in K_1. Finally, when g_n has 3 roots in K_1, then g_n has 0 roots in each of $\{\mathbb{C} \setminus \mathbb{R}, J_{1,n}, J_2, J_{3,n}\}$.
(vii) For possibility (b) and all $n \geqslant 1$, g_n has 0 roots in each of $\{\mathbb{C} \setminus \mathbb{R}, J_{3,n}\}$, 1 root in K_1, and at most 1 root in each of $\{J_{1,n}, J_2\}$. Moreover, when g_n has 1 root in some fixed $I \in \{J_{1,n}, J_2\}$, then g_n has 0 roots in $\{J_{1,n}, J_2\} \setminus \{I\}$.

Finally, we will use parts (i), (ii), (vi), (vii) to show:

(viii) For (a), $f'_{(\chi,\eta)}$ has at most 2 roots in each of $\{\mathbb{C}\setminus\mathbb{R}, J_1, J_2, J_3\}$, and at most 3 roots in K_1. Moreover, when $f'_{(\chi,\eta)}$ has either 1 or 2 roots in some fixed $I \in \{\mathbb{C}\setminus\mathbb{R}, J_1, J_2, J_3\}$, then $f'_{(\chi,\eta)}$ has 0 roots in each of $\{\mathbb{C}\setminus\mathbb{R}, J_1, J_2, J_3\} \setminus \{I\}$, and at most 1 root in K_1. Finally, when $f'_{(\chi,\eta)}$ has either 2 or 3 roots in K_1, then $f'_{(\chi,\eta)}$ has 0 roots in each of $\{\mathbb{C}\setminus\mathbb{R}, J_1, J_2, J_3\}$.

(ix) For (b), $f'_{(\chi,\eta)}$ has 0 roots in each of $\{\mathbb{C}\setminus\mathbb{R}, J_3\}$, at most 1 root in each of $\{J_1, J_2, K_1\}$. Moreover, when $f'_{(\chi,\eta)}$ has 1 root in some fixed $I \in \{J_1, J_2\}$, then $f'_{(\chi,\eta)}$ has 0 roots in each of $\{J_1, J_2\} \setminus \{I\}$.

Parts (viii) and (ix) prove the required results for possibilities (a) and (b) when the supports are as given on the top of Figure 5.2.

Consider (iii). Recall that the sets $\{X_n, \{\chi\}, Y_n\}$ are mutually disjoint, X_n consists of $m \geqslant 4$ distinct elements, and Y_n consists of $l \geqslant 2$ elements. Define the following polynomial:

$$p_n(w) := \frac{1}{n} \sum_{x\in X_n\cup Y_n} \left(\prod_{y\in X_n\cup Y_n, y\neq x} (w-y) \right)(w-\chi) \\ - \left(\frac{m+l}{n} - \eta\right)\left(\prod_{y\in X_n\cup Y_n} (w-y) \right),$$

for all $w \in \mathbb{C}$. Recall that $\eta > 0$ for possibility (a), and $\eta = 0$ for possibility (b). Therefore p_n has degree $m+l$ for (a), and degree at least $m+l-1$ for (b). Next note that p_n has 0 roots in $X_n \cup \{\chi\} \cup Y_n$, as can be seen by substitution. Also, (5.2) implies that the roots of p_n and g_n in $\mathbb{C}\setminus(X_n\cup\{\chi\}\cup Y_n)$ coincide, up to multiplicities. This proves (iii).

Consider (iv). Let x and y denote any two consecutive elements of X_n, or any two consecutive elements of Y_n, with $y > x$. Note, (5.2) implies that $g_n|_{(x,y)}$ is real-valued and continuous, and:

$$\lim_{w\in\mathbb{R}, w\downarrow x} g_n(w) = +\infty \qquad \text{and} \qquad \lim_{w\in\mathbb{R}, w\uparrow y} g_n(w) = -\infty.$$

Therefore g_n has an odd number of roots in (x, y). This proves (iv).

Consider (v). First note, (5.2) implies that non-real roots of g_n occur in complex conjugate pairs. Therefore g_n has an even number of roots in $\mathbb{C}\setminus\mathbb{R}$. Next note, (5.2) implies that $g_n|_{(\chi,\underline{X_n})}$ is real-valued and continuous, and:

$$\lim_{w\in\mathbb{R}, w\downarrow\chi} g_n(w) = -\infty \qquad \text{and} \qquad \lim_{w\in\mathbb{R}, w\uparrow\underline{X_n}} g_n(w) = -\infty.$$

Therefore g_n has an even number of roots in $J_{3,n} = (\chi, \underline{X_n})$. Finally note, (5.2) implies that $g_n|_{(\overline{X_n},+\infty)}$ is real-valued and continuous, and:

$$\lim_{w\in\mathbb{R}, w\downarrow\overline{X_n}} g_n(w) = +\infty \qquad \text{and} \qquad \lim_{w\in\mathbb{R}, w\uparrow+\infty} w g_n(w) = \eta.$$

Therefore, since $\eta > 0$ for possibility (a), g_n has an even number of roots in $J_{1,n} = (\overline{X_n}, +\infty)$. Similarly, for (a), g_n has an even number of roots in $J_2 = (-\infty, \underline{X_n})$.

Consider (vi). Note, part (iv) and Figure 5.2 imply that g_n has at least $m-1$ roots in $[\underline{X_n}, \overline{X_n}]$. More specifically, recalling that $\{a_2, a_1\} \subset X_n$, g_n has at least $m-2$ roots in $[\underline{X_n}, a_2] \cup [a_1, \overline{X_n}]$, and at least 1 root in $(a_2, a_1) = K_1$. Similarly, g_n has at least $l-1$ roots in $[\underline{Y_n}, \overline{Y_n}] = [\underline{S_3}, \overline{Y_n}]$. Part (iii) and Figure 5.2 thus imply that g_n has at most 2 roots in $(\mathbb{C}\setminus\mathbb{R}) \cup (J_{1,n} \cup J_2 \cup J_{3,n})$, and at most 3 roots in $(\mathbb{C}\setminus\mathbb{R}) \cup (J_{1,n} \cup J_2 \cup J_{3,n} \cup K_1)$. Part (vi) then follows from parts (iv) and (v). Part (vii) can be shown similarly.

Consider (viii). First suppose that $z \in \mathbb{C}\setminus\mathbb{R}$ is a root of $f'_{(\chi,\eta)}$ of multiplicity $k \geqslant 1$. Fix $\epsilon > 0$ such that $B(z,\epsilon) \subset \mathbb{C}\setminus\mathbb{R}$, and z is the unique root in $B(z,\epsilon)$. Note, part (i) implies that $\overline{z}$ is also a root of multiplicity k, and $\overline{z}$ is the unique root in $B(\overline{z},\epsilon)$. Then, for all n sufficiently large, part (ii) implies that g_n has k roots in both $B(z,\epsilon)$ and $B(\overline{z},\epsilon)$. Thus, since $B(z,\epsilon)$ and $B(\overline{z},\epsilon)$ are disjoint subsets of $\mathbb{C}\setminus\mathbb{R}$, g_n has at least $2k \geqslant 2$ roots in $\mathbb{C}\setminus\mathbb{R}$. Finally recall, part (vi) implies that g_n has either 0 or 2 roots in $\mathbb{C}\setminus\mathbb{R}$. Therefore $k = 1$, and so z and $\overline{z}$ are roots of $f'_{(\chi,\eta)}$ of multiplicity 1.

Next suppose that $z, w \in \mathbb{C}\setminus\mathbb{R}$ are roots of $f'_{(\chi,\eta)}$ of multiplicity $k = 1$ and $l \in \{0, 1\}$ respectively ($l = 0$ means w is not a root), and $w \notin \{z, \overline{z}\}$. Fix $\epsilon > 0$ such that $\{B(z,\epsilon), B(\overline{z},\epsilon), B(w,\epsilon), B(\overline{w},\epsilon)\}$ are disjoint subsets of $\mathbb{C}\setminus\mathbb{R}$, z is the unique root in $B(z,\epsilon)$, and w is the unique root in $B(w,\epsilon)$. Then we can proceed similarly to above to show, for all n sufficiently large, that g_n has k roots in each of $\{B(z,\epsilon), B(\overline{z},\epsilon)\}$, and l roots in each of $\{B(w,\epsilon), B(\overline{w},\epsilon)\}$. Thus g_n has at least $2k + 2l$ roots in $\mathbb{C}\setminus\mathbb{R}$. Thus, since $k = 1$, part (vi) implies $l = 0$. Combined, the above show, when $z \in \mathbb{C}\setminus\mathbb{R}$ is a root of $f'_{(\chi,\eta)}$, that z is a root of $f'_{(\chi,\eta)}$ of multiplicity 1, $\overline{z}$ is a root of $f'_{(\chi,\eta)}$ of multiplicity 1, and $f'_{(\chi,\eta)}$ has 0 roots in $(\mathbb{C}\setminus\mathbb{R}) \setminus \{z, \overline{z}\}$. Thus $f'_{(\chi,\eta)}$ has either 0 or 2 roots in $\mathbb{C}\setminus\mathbb{R}$.

Next suppose and $z \in J_1$ is a root of multiplicity $k \geqslant 1$. Fix $\epsilon > 0$ such that $B(z,\epsilon) \subset (\mathbb{C}\setminus\mathbb{R}) \cup J_1$, and z is the unique root in $B(z,\epsilon)$. Then, for all n sufficiently large, part (ii) implies that g_n has k roots in $B(z,\epsilon) \subset (\mathbb{C}\setminus\mathbb{R}) \cup J_1$. Recall, $J_1 = (\overline{S_1}, +\infty)$ and $J_{1,n} = (\overline{X_n}, +\infty)$ and $\overline{X_n} \to \overline{S_1}$ as $n \to \infty$. Therefore, for all n sufficiently large, $B(z,\epsilon) \subset (\mathbb{C}\setminus\mathbb{R}) \cup J_{1,n}$, and so g_n has at least $k \geqslant 1$ roots in $(\mathbb{C}\setminus\mathbb{R}) \cup J_{1,n}$. Finally recall, part (vi) implies that g_n

has either 0 or 2 roots in $(\mathbb{C} \setminus \mathbb{R}) \cup J_{1,n}$. Therefore $k = 1$ or $k = 2$, and so z is a root of $f'_{(\chi,\eta)}$ of multiplicity at most 2.

Next suppose that $z, w \in J_1$ are roots of $f'_{(\chi,\eta)}$ of multiplicity $k \in \{1, 2\}$ and $l \in \{0, 1, 2\}$ respectively, and $w \neq z$. Fix $\epsilon > 0$ such that $B(z, \epsilon)$ and $B(w, \epsilon)$ are disjoint subsets of $(\mathbb{C}\setminus\mathbb{R})\cup J_1$, z is the unique root in $B(z, \epsilon)$, and w is the unique root in $B(w, \epsilon)$. Then we can proceed similarly to above to show, for all n sufficiently large, that g_n has k roots in $B(z, \epsilon) \subset (\mathbb{C}\setminus\mathbb{R})\cup J_{1,n}$, and l roots in $B(w, \epsilon) \subset (\mathbb{C} \setminus \mathbb{R}) \cup J_{1,n}$. Thus g_n has at least $k + l$ roots in $(\mathbb{C} \setminus \mathbb{R}) \cup J_{1,n}$. Therefore, part (vi) implies that $l = 0$ when $k = 2$, and $l \in \{0, 1\}$ when $k = 1$. This implies, when $z \in J_1$ is a root of $f'_{(\chi,\eta)}$ of multiplicity 2, that $f'_{(\chi,\eta)}$ has 0 roots in $J_1 \setminus \{z\}$. Moreover, when $z \in J_1$ is a root of $f'_{(\chi,\eta)}$ of multiplicity 1, $f'_{(\chi,\eta)}$ has a root of multiplicity at most 1 in $J_1 \setminus \{z\}$. Similarly it can be shown, when $z, w \in J_1$ are distinct roots of $f'_{(\chi,\eta)}$ of multiplicity 1, that $f'_{(\chi,\eta)}$ has 0 roots in $J_1 \setminus \{z, w\}$. Therefore $f'_{(\chi,\eta)}$ has at most 2 roots in J_1.

Next suppose that $z \in J_1$ is a root of multiplicity $k \in \{1, 2\}$, and $w \in J_2$ is a root of multiplicity $l \geqslant 0$. Fix $\epsilon > 0$ such that $B(z, \epsilon) \subset (\mathbb{C} \setminus \mathbb{R}) \cup J_1$, $B(w, \epsilon) \subset (\mathbb{C} \setminus \mathbb{R}) \cup J_2$, z is the unique root in $B(z, \epsilon)$, and w is the unique root in $B(w, \epsilon)$. Then we can proceed similarly to above to show, for all n sufficiently large, that g_n has $k \in \{1, 2\}$ roots in $B(z, \epsilon) \subset (\mathbb{C} \setminus \mathbb{R}) \cup J_{1,n}$, and $l \geqslant 0$ roots in $B(w, \epsilon) \subset (\mathbb{C} \setminus \mathbb{R}) \cup J_2$. Note, since non-real roots of g_n occur in complex conjugate pairs, one of the following must be satisfied for the roots in $B(z, \epsilon)$:

- $k \in \{1, 2\}$, g_n has 0 roots in $B(z, \epsilon) \setminus (z - \epsilon, z + \epsilon) \subset \mathbb{C} \setminus \mathbb{R}$, and either 1 or 2 roots in $(z - \epsilon, z + \epsilon) \subset J_{1,n}$.
- $k = 2$, g_n has 2 roots in $B(z, \epsilon) \setminus (z - \epsilon, z + \epsilon) \subset \mathbb{C} \setminus \mathbb{R}$, and 0 roots in $(z - \epsilon, z + \epsilon) \subset J_{1,n}$.

In the first case, part (vi) implies that g_n has 2 roots in $J_{1,n}$, and 0 roots in $\mathbb{C} \setminus \mathbb{R}$ and J_2. Therefore, since $B(w, \epsilon) \subset (\mathbb{C} \setminus \mathbb{R}) \cup J_2$, g_n has 0 roots in $B(w, \epsilon)$. In the second case, part (vi) implies that g_n has 2 roots in $B(z, \epsilon) \setminus (z-\epsilon, z+\epsilon) \subset \mathbb{C}\setminus\mathbb{R}$, 0 roots in $(\mathbb{C}\setminus\mathbb{R}) \setminus B(z, \epsilon)$, and 0 roots in J_2. Therefore, since $B(w, \epsilon) \subset (\mathbb{C} \setminus \mathbb{R}) \cup J_2$, and since $B(z, \epsilon)$ and $B(w, \epsilon)$ are disjoint, g_n has 0 roots in $B(w, \epsilon)$. In both cases, this gives $l = 0$. Therefore, when $z \in J_1$ is a root of $f'_{(\chi,\eta)}$ of multiplicity 1 or 2, $f'_{(\chi,\eta)}$ has 0 roots in J_2.

We finally state that the rest of part (viii), and part (ix), can be shown using similar arguments. □

Recall the definitions of $\mathcal{L}$ and $\mathcal{E} = \mathcal{E}^+\cup\mathcal{E}^-\cup\mathcal{E}_0\cup\mathcal{E}_1$ given in Definitions 2.5 and 2.7. We end this section by using Theorem 5.2 to refine these definitions:

Corollary 5.3. — *We have:*

1. *Possibility* (a) *of Theorem 5.2 is satisfied whenever* $(\chi,\eta) \in \mathcal{L} \cup \mathcal{E}^+ \cup \mathcal{E}^- \cup \mathcal{O}$*, and so* $b > \chi > a$*,* $1 > \eta > 0$*,* $1-\eta > \mu[\{\chi\}]$*, and* $S_1 \neq \emptyset$*,* $S_2 = \{\chi\}$*,* $S_3 \neq \emptyset$*. When* $(\chi,\eta) \in \mathcal{L}$*,* $f'_{(\chi,\eta)}$ *has a unique root in* $\mathbb{H}$*, and this root is of multiplicity* 1*. When* $(\chi,\eta) \in \mathcal{E}^+$*,* $f'_{(\chi,\eta)}$ *has a unique repeated root in* $(\chi,+\infty)\setminus \mathrm{Supp}(\mu)$*, and this is of multiplicity either* 2 *or* 3*. When* $(\chi,\eta) \in \mathcal{E}^-$*,* $f'_{(\chi,\eta)}$ *has a unique repeated root in* $(-\infty,\chi)\setminus \mathrm{Supp}(\mu)$*, and this is of multiplicity either* 2 *or* 3*. When* $(\chi,\eta) \in \mathcal{O}$*,* $f'_{(\chi,\eta)}$ *has a root of multiplicity* 1 *in* $(b,+\infty)$*, and has at most* 2 *roots in* $(b,+\infty)$*.*
2. $\chi \in \mathbb{R} \setminus \mathrm{Supp}(\mu)$ *and* $\eta = 1$ *when* $(\chi,\eta) \in \mathcal{E}_0$*. Moreover, possibility* (d) *of Theorem 5.2 is satisfied, and so* $b > \chi > a$*,* $\eta = 1$*, and* $S_1 \neq \emptyset$*,* $S_2 = \emptyset$*,* $S_3 \neq \emptyset$*. Finally,* $f'_{(\chi,\eta)}$ *has a root of multiplicity* 1 *at* χ*.*
3. $\chi \in \mathrm{Supp}(\mu)$*,* $1 > \mu[\{\chi\}] > 0$*, and* $\eta = 1 - \mu[\{\chi\}]$ *when* $(\chi,\eta) \in \mathcal{E}_1$*. Moreover, one of possibilities* (d)*,* (h) *or* (l) *is satisfied. For* (d)*,* $b > \chi > a$*,* $S_1 \neq \emptyset$*,* $S_2 = \emptyset$*,* $S_3 \neq \emptyset$*, and* $f'_{(\chi,\eta)}$ *has either* 0 *or* 1 *root at* χ*. For* (h)*,* $\chi = a$*,* $S_1 \neq \emptyset$*,* $S_2 = S_3 = \emptyset$*, and* $f'_{(\chi,\eta)}$ *has* 0 *roots at* χ*. For* (l)*,* $\chi = b$*,* $S_1 = S_2 = \emptyset$*,* $S_3 \neq \emptyset$*, and* $f'_{(\chi,\eta)}$ *has* 0 *roots at* χ*.*
4. $\{\mathcal{L}, \mathcal{E}^+, \mathcal{E}^-, \mathcal{E}_0, \mathcal{E}_1, \mathcal{O}\}$ *is pairwise disjoint.*

Proof. — Consider (1) when $(\chi,\eta) \in \mathcal{L}$. First note, Definition 2.5 implies that $f'_{(\chi,\eta)}$ has roots in $\mathbb{C} \setminus \mathbb{R}$. Next note that this can only happen when possibility (a) of Theorem 5.2 is satisfied. Finally note, possibility (a) of Theorem 5.2 implies that $f'_{(\chi,\eta)}$ has at most 2 roots in $\mathbb{C} \setminus \mathbb{R}$. Thus, since non-real roots of $f'_{(\chi,\eta)}$ occur in complex conjugate pairs, $f'_{(\chi,\eta)}$ has exactly 1 roots in $\mathbb{H}$, and this is of multiplicity 1. This proves (1) when $(\chi,\eta) \in \mathcal{L}$.

Consider (1) when $(\chi,\eta) \in \mathcal{E}^+$. First note, Definition 2.7 implies that $f'_{(\chi,\eta)}$ has a repeated root in $(\chi,+\infty) \setminus \mathrm{Supp}(\mu)$. Next note that this can only happen when possibility (a) of Theorem 5.2 is satisfied. Finally note, possibility (a) of Theorem 5.2 implies that this root has multiplicity either 2 or 3. This proves (1) when $(\chi,\eta) \in \mathcal{E}^+$. We can similarly prove (1) when $(\chi,\eta) \in \mathcal{E}^-$.

Consider (1) when $(\chi,\eta) \in \mathcal{O}$. First note that Definition 2.9 implies that $\chi < b$, $\eta > 0$, and $f'_{(\chi,\eta)}$ has a root of multiplicity 1 in $J_1 = (b,+\infty)$. Next note that this can only happen when possibility (a) of Theorem 5.2 is satisfied. Finally note that possibility (a) of Theorem 5.2 implies that $f'_{(\chi,\eta)}$ has at most 2 roots in J_1. This proves (1) when $(\chi,\eta) \in \mathcal{O}$.

Consider (2). Recall that $(\chi, \eta) \in \mathcal{E}_0$. First note that (2.14) and Definition 2.7 imply that $\chi \in \mathbb{R} \setminus \mathrm{Supp}(\mu)$ (and so $\mu[\{\chi\}] = 0$), $C(\chi) = 0$, and $\eta = 1$. Next note that since $\eta = 1$ and $\mu[\{\chi\}] = 0$, (2.10) and (2.11) give $f'_{(\chi,\eta)}(w) = C(w)$ for all $w \in \mathbb{C} \setminus \mathrm{Supp}(\mu)$. Therefore $f'_{(\chi,\eta)}(\chi) = C(\chi) = 0$. Also, since $1 - \eta = \mu[\{\chi\}](= 0)$, one of possibilities (d), (h) or (l) of Theorem 5.2 is satisfied. Moreover, since $C(\chi) = 0$, (2.11) trivially implies that $\chi \neq a$ and $\chi \neq b$. Therefore possibility (d) must be satisfied. Finally note, possibility (d) of Theorem 5.2 implies that χ is a root of $f'_{(\chi,\eta)}$ of multiplicity 1.

Consider (3). Recall that $(\chi, \eta) \in \mathcal{E}_1$. First note that (2.14) and Definition 2.7 imply that $\chi \in \mathrm{Supp}(\mu)$, $\mu[\{\chi\}] > 0$, and $\eta = 1 - \mu[\{\chi\}]$. Thus, since $1 - \eta = \mu[\{\chi\}]$, one of possibilities (d), (h) or (l) of Theorem 5.2 is satisfied. For possibility (d), note that Theorem 5.2 implies that $b > \chi > a$, $S_1 \neq \emptyset$, $S_2 = \emptyset$, $S_3 \neq \emptyset$, and $f'_{(\chi,\eta)}$ has either 0 or 1 root at χ. Similarly, Theorem 5.2 gives the required results for those possibilities (h) and (l). This proves (3).

Consider (4). Suppose first that $(\chi, \eta) \in \mathcal{L}$. Part (1) of this result thus implies that possibility (a) of Theorem 5.2 is satisfied, and that $f'_{(\chi,\eta)}$ has a root in $\mathbb{C} \setminus \mathbb{R}$. Possibility (a) of Theorem 5.2 further implies that $f'_{(\chi,\eta)}$ has no real-valued repeated roots, and so $(\chi, \eta) \notin \mathcal{E}^+ \cup \mathcal{E}^-$ (see Definition 2.7). Moreover, possibility (a) implies that $f'_{(\chi,\eta)}$ has no roots in $J_1 = (b, +\infty)$, and so $(\chi, \eta) \notin \mathcal{O}$ (see Definition 2.9). Finally, none of possibilities (d), (h), (l) are satisfied, and so parts (2) and (3) of this lemma imply that $(\chi, \eta) \notin \mathcal{E}_0 \cup \mathcal{E}_1$.

Next suppose that $(\chi, \eta) \in \mathcal{E}^+$. Part (1) of this result thus implies that possibility (a) of Theorem 5.2 is satisfied, and that $f'_{(\chi,\eta)}$ has a unique repeated root in $(\chi, +\infty) \setminus \mathrm{Supp}(\mu)$. Possibility (a) of Theorem 5.2 further implies that $f'_{(\chi,\eta)}$ has no repeated roots in $(-\infty, \chi) \setminus \mathrm{Supp}(\mu)$, and so $(\chi, \eta) \notin \mathcal{E}^-$ (see Definition 2.7). Moreover, possibility (a) implies that $f'_{(\chi,\eta)}$ has no roots of multiplicity 1 in $J_1 = (b, +\infty)$, and so $(\chi, \eta) \notin \mathcal{O}$ (see Definition 2.9). Finally, none of possibilities (d), (h), (l) are satisfied, and so parts (2) and (3) of this lemma imply that $(\chi, \eta) \notin \mathcal{E}_0 \cup \mathcal{E}_1$.

Next suppose that $(\chi, \eta) \in \mathcal{E}^-$. Then, similar arguments to those used above show that $(\chi, \eta) \notin \mathcal{O} \cup \mathcal{E}_0 \cup \mathcal{E}_1$. Next suppose that $(\chi, \eta) \in \mathcal{O}$. Then, similar arguments to those used above show that $(\chi, \eta) \notin \mathcal{E}_0 \cup \mathcal{E}_1$. Finally suppose that $(\chi, \eta) \in \mathcal{E}_0$. Then $\eta = 1$, and Definition 2.7 trivially imples that $(\chi, \eta) \notin \mathcal{E}_1$. This proves (4). ☐

6. An application to a problem from Quantum Information Theory

Let us consider the following problem. We fix a parameter $t \in (0,1)$ and an integer $k \geqslant 1$, and take a sequence V_n of random subspaces of $\mathbb{C}^k \otimes \mathbb{C}^n$ of dimension $d = d_n \sim tkn$. Here, random means taken uniformly according to the uniform measure on the Grassmann manifold. For a given $x \in \mathbb{C}^k \otimes \mathbb{C}^n$, we recall that its *singular value decomposition* is

$$x = \sum_i \sqrt{\lambda_i(x)} e_i(x) \otimes f_i(x),$$

where $\lambda_1(x) \geqslant \lambda_2(x) \geqslant \ldots \geqslant 0$, and both (e_i) and (f_i) are families of orthonormal vectors. $\lambda_i(x)$ are always uniquely defined. As for $(e_i(x))$ and $(f_i(x))$ they are generically defined up to a phase (here, generically means that this statement holds true if all $\lambda_i(x)$ are distinct, and this is actually a necessary and sufficient condition).

It follows from Pythagoras' theorem that $\sum \lambda_i(x) = \|x\|_2^2$. We are interested in the subset $K_{k,t,n}$ of $\mathbb{R}^k$ of all possible singular values x for $x \in V_n$ of norm 1. The set $K_{k,t,n}$ is actually random, and it is a subset of the probability simplex $\Delta_k = \{\lambda_1, \ldots, \lambda_k, \lambda_i \geqslant 0, \sum \lambda_i = 1\}$. As per our definition of singular values, this set should consist of non-increasing eigenvalues, but for convenience we make an abuse of language we consider instead the symmetrized version of this set, i.e. any permutation of coordinates leaves the set $K_{k,t,n}$ invariant.

It was proved in [4] (Theorem 1.2) that this set actually converges in the Hausdorff distance to a set $K_{k,t}$ defined as $K_{k,t} = \{(a_1, \ldots, b_k) \in \Delta_k, \forall (a_1, \ldots, 1_k) \in \Delta_k, \sum a_i b_i \leqslant \|A\|_t\}$, where $\|A\|_t = \|(a_1, \ldots, a_k)\|_t$ is the free compressed t-norm, as introduced in the first section, cf. (1.2) (see also Definition 2.7). Recall that the *Hausdorff distance* between two compact subsets K, S of a complete metric space is the infimum over all $\varepsilon > 0$ such that $K \subset B(S, \varepsilon)$ and $S \subset B(K, \varepsilon)$, where $B(S, \varepsilon)$ is the ball of "center" S and radius ε, i.e. the collection of all elements that are ε-close to S. We also proved that it is true for the boundary of sets viewed as subsets of the affine space of real k-tuples that add up to 1 in the sense that the Hausdorff distance between $\partial K_{k,t,n}$ and $\partial K_{k,t}$ converges to zero almost surely. Thanks to the main result, we are able to upgrade the results mentioned earlier in this section as follows:

THEOREM 6.1. — *There exist constants C and a polynomial function $h(\varepsilon)$ such that for any $\varepsilon \in (0,1)$, $P(d(K_{k,t,n}, K_{k,t}) \geqslant \varepsilon) \leqslant Ce^{-nh(\varepsilon)}$.*

We do not give a complete proof of this result, as it is essentially contained in [4], however let us try to give a sense of the important ideas. It follows from linear algebra considerations that an element of V_n will satisfy $\sum_{i=1}^k \lambda_i(x)a_i|\langle e_i(x), h_i\rangle|^2 \geqslant \alpha$ if and only if, calling p the orthogonal projection onto V_n, $p(\sum a_i e_i e_i^*)p$, has operator norm at least α. Our main Theorem 2.16 allows us to estimate this quantity very precisely.

In order to prove the result, we need to be able to obtain such an estimate for all k-tuples (a_i), (h_i) simultaneously, where $(a_i) \in \mathbb{R}_+^k$, and (h_i) is a family of orthonormal vectors. Thanks to this estimate, we are able to estimate

$$P(|\langle (a_1, \dots, a_k), K_{k,t,n}\rangle - \langle (a_1, \dots, a_k), K_{k,t}\rangle| \geqslant \varepsilon),$$

and find it to be less than $Ce^{-nh(\varepsilon)}$.

In this problem, k is fixed, so we can take a finite η-net of $(a_i) \in \mathbb{R}_+^k, (h_i)$ for an appropriate metric, on the product of real eigenvalues and eigenvectors up to a phase, which, for this purpose, can be thought of as the convex set of trace one semidefinite selfadjoint matrices. By passing, let us note that this set is also known as the set of density matrices in QIT. Thanks to this net argument, and by a continuity argument, we can then take the sup over all probability vectors $(a_1, \dots, a_k)$ and estimate again

$$P\left(\sup_{(a_1,\dots,a_k)} |\langle (a_1, \dots, a_k), K_{k,t,n}\rangle - \langle (a_1, \dots, a_k), K_{k,t}\rangle| \geqslant \varepsilon\right),$$

and bound it alike by $Ce^{-nh(\varepsilon)}$, with constant worsened to take into account η and a union bound reasoning. This gives the desired result. Note that although we show the existence of actual constants and of an exponential speed of convergence, making the constants c, h is probably a difficult task, first because it requires to make every constant of Subsection 2.2 explicit, and secondly because it asks to understand in detail the procedure of optimizing the sup over all probability vectors. Partial work in this direction was completed [5], though the problem under consideration was simpler and yet required considerably involved developments in free probability theory.

In particular, given a continuous function, this result allows us to give estimates for

$$P(|\min\{f(x), x \in K_{k,t,n}\} - \min\{f(x), x \in K_{k,t}\}| > \varepsilon)$$

and we obtain similar upper bounds, of type $C\exp(-nh(\varepsilon))$. Thanks to the results of [5], it was known that the minimum output entropy for generic quantum channels can be generically violated if and only if the parameter $k \geqslant 183$, however, no estimate was available for the required dimension n of the input space, nor was any technique available to attack this problem. This paper contributes to solving this problem in the sense that combining the above result in the case where f is the entropy function H, together with the calculations of $\min\{H(x), x \in K_{k,t}\}$ of [5] yield a path towards answering this problem.

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