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Ultrafilters and non-Cantor minimal sets in linearly ordered dynamical systems

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Abstract It is well known that infinite minimal sets for continuous functions on the interval are Cantor sets; that is, compact zero dimensional metrizable sets without isolated points. On the other hand, it was proved in Alcaraz and Sanchis (Bifurcat Chaos 13:1665–1671, 2003) that infinite minimal sets for continuous functions on connected linearly ordered spaces enjoy the same properties as Cantor sets except that they can fail to be metrizable. However, no examples of such subsets have been known. In this note we construct, in ZFC, $2^{\mathfrak{c}}$ non-metrizable infinite pairwise non-homeomorphic minimal sets on compact connected linearly ordered spaces.

Keywords Dynamical system · Minimal set · Cantor set · Linearly ordered topological space

Mathematics Subject Classification (2000) 54H20 · 37E99

0 Introduction

The underlying object considered here is a (discrete) dynamical system; that is, a pair (X, φ) where $\varphi : X \longrightarrow X$ is a continuous function on a Tychonoff space X named

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the *phase space*. A subset $M \subseteq X$ is said to be *minimal* in (X, φ) (or *minimal* for φ) provided it is a minimal element in the partially ordereded by inclusion set of all nonempty closed sets $A \subseteq X$ such that $\varphi(A) \subseteq A$. If X is minimal in (X, φ), then (X, φ) is said to be a *minimal system*. Minimal subsets do not always exist, but Zorn's Lemma implies that every dynamical system with compact phase space has minimal sets. It is apparent that a minimal set is finite if and only if it is a periodic orbit but, in general, classifying infinite minimal sets is an arduous work (and in many occurrences, it is an open question). For instance, the unit circle is minimal for rotations of irrational index (see [6]) and there exist linearly ordered dynamical systems where every minimal set is finite (see [2]). For the unit interval, a widely known result characterizes infinite minimal sets (see for example [4]) as the zero-dimensional metrizable compact subsets without isolated points, that is, as the Cantor subsets of $[0, 1]$.

The foregoing results raise the question of studying minimal sets in linearly ordered dynamical systems; i.e., dynamical systems in which the topology of the phase space is induced by a linear order. From previous research on this kind of dynamical systems, carried out by Schirmer [7] and Baldwin [3], this problem was investigated in [2] where it was shown that infinite minimal sets in a linearly ordered dynamical system enjoy the same properties that characterize Cantor set except that they can fail to be metrizable. However, no examples of this type of non-metrizable minimal sets were known. The aim of this note is to exhibit $2^{\mathfrak{c}}$ non-homeomorphic, non-metrizable infinite minimal sets on compact connected linearly ordered dynamical systems. In Sect. 1 we give some results that will be useful to obtain our main objective. In Sect. 2, by means of free ultrafilters on ω, we construct $2^{\mathfrak{c}}$ dynamical systems containing, each of them, a minimal set, in such a way that they are pairwise non-homeomorphic. Finally, in Sect. 3 we construct a linearly ordered compact connected space X of size $2^{\mathfrak{c}}$ and a continuous function φ on X such that there are $2^{\mathfrak{c}}$ non-homeomorphic minimal sets for φ.

Our terminology and notation are standard. In particular, for topological spaces X and Y, the symbol $X \cong Y$ means that X and Y are homeomorphic. For a linearly ordered set $(X, \leq)$, if $x < y$, we will denote by $]x, y[$, $[x, y]$, $[x, y[$ and $]x, y]$ the open, closed and semiclosed intervals in X determined by x and y, respectively. The symbols $]x, \rightarrow[$, $]\leftarrow, x[$, $[x, \rightarrow[$, $]\leftarrow, x]$ will denote the open and closed, final and initial segments defined by $x \in X$. For a topological space X, its *weight*, $w(X)$, is the minimum infinite cardinal number τ such that X has a base for its topology with cardinality τ. We denote the set $\{0, 1, 2, \ldots, n, \ldots\}$ of all natural numbers by using the symbols ω or $\mathbb{N}$. The set of all the ultrafilters defined on ω is $\beta\omega$, and the set of all the free ultrafilters on ω is $\omega^* = \beta\omega \setminus \omega$.

For notions and concepts not defined here the reader can consult [1] and [5].

1 Basic results

Each linear order on a set X induces a Tychonoff topology (actually, a hereditarily normal topology) on X in such a way that the open intervals form a base for the open sets. A topological space X is said to be a *linearly ordered space* if its topology is induced by a linear order. The real line is a paradigmatic example of a linearly ordered space whereas there is no linear order inducing the usual topology of the complex plane.

Let I be the (closed) unit interval. Let J denote the product $I \times I$ endowed with the topology induced by the lexicographic order $<_l$ defined by letting $(x_1, y_1) <_l (x_2, y_2)$ whenever $x_1 < x_2$ or $x_1 = x_2$ and $y_1 < y_2$. It is well-known that J is a compact connected space. The following theorem is fundamental for our general purposes.

Theorem 1.1 *Let $(a_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of real numbers with $a_0 = 0$ and such that $\lim_{n\to\infty} a_n = 1$. Let $f : I \longrightarrow I$ be a continuous function strictly increasing in the intervals $[a_{2k}, a_{2k+1}]$, strictly decreasing in the intervals $[a_{2k+1}, a_{2k+2}]$, $k = 0, 1, \ldots$ If in addition $f(1) = 0$ and $|f^{-1}[\{0\}]| < \aleph_0$, then the function $F_l : J \longrightarrow J$ defined as*

$$F_l(x, y) = \begin{cases} (f(x), y) & \text{if } x \in \,]a_{2k}, a_{2k+1}[\\ (f(x), 1-y) & \text{if } x \in \,]a_{2k+1}, a_{2k+2}[\\ (f(x), 0) & \text{if } x = a_{2k+1} \\ (f(x), 1) & \text{if } x = a_{2k} \\ (0, 1) & \text{if } x = 1. \end{cases}$$

is continuous.

Proof We are going to prove the continuity of F_l at each point of J. So, we have to consider several possible cases.

1. First case: $x = a_{2k}$ $(k \in \omega)$.

A canonical neighborhood of $F_l(x, y) = (f(x), 1)$ has the form

$$\begin{aligned} V &=](f(x), 1-\epsilon), (f(x)+\epsilon, 0)[\\ &= \{(f(x), b) : 1-\epsilon < b \le 1\} \cup \{(a, b) : f(x) < a < f(x)+\epsilon, b \in [0, 1]\} \end{aligned}$$

where ϵ is a small positive real number, say $\epsilon < (1 - f(x))/2$.

1.1. First subcase: For $y \in \,]0, 1[$, we take $\delta < \min\{y, 1-y\}/2$. The set $W =](x, y-\delta), (x, y+\delta)[= \{(x, b) : y-\delta < b < y+\delta\}$ is an open interval containing the point (x, y) and such that $F_l((a, b)) = (f(x), 1) \in V$ for all $(a, b) \in W$.

1.2. Second subcase: Assume now that $y = 1$. Since f is continuous and strictly increasing in the interval $[a_{2k}, a_{2k+1}]$, there is a $\delta \in \,]0, (a_{2k+1} - a_{2k})/2[$ such that $f[[x, x+\delta[\,] \subset [f(x), f(x)+\epsilon[$. So, $F_l(a, b) \in V$ if $(a, b) \in \,](x, 1-\epsilon), (x+\delta, 0)[$.

1.3. Third subcase: $y = 0 = x$. Under these circumstances, $W =](0, 0), (0, 1/2)[$ is a neighborhood of $(0, 0)$ and $F_l[W] = \{(f(x), 1)\} \subset V$.

1.4. Fourth subcase: Finally, suppose that $x = a_{2k} > 0$ and $y = 0$. Since f is continuous and decreasing in $[a_{2k-1}, a_{2k}]$, there is a $\delta \in \,]0, (a_{2k} - a_{2k-1})/2[$ such that $f[\,]x-\delta, x]] \subset [f(x), f(x)+\epsilon[$. Now, $W =](a_{2k}-\delta, 1), (a_{2k}, \epsilon)[$ is a neighborhood of (x, y) and $F_l[W] \subset V$. In fact, if $a \in \,]a_{2k} - \delta, a_{2k}[$, $F_l((a, b)) = (f(a), 1-b)$; but, in this case, $f(a) \in \,]f(a_{2k}), f(a_{2k})+\epsilon[$; so, $F_l(a, b) \in V$. If $a = a_{2k}$ and $b < \epsilon$, then $F_l((a_{2k}, b)) = (f(a_{2k}), 1) \in V$.

2. Second case: $x \in \,]a_{2k}, a_{2k+1}[\;\; (k \in \omega)$.
2.1. First subcase: $y \in (0, 1)$. A canonical neighborhood of $F_l((x, y)) = (f(x), y)$ has the form

$$V =](f(x), y - \epsilon), (f(x), y + \epsilon)[$$

where $\epsilon \in [0, \min\{y, 1-y\}/2[$. Let W be equal to $](x, y-\epsilon), (x, y+\epsilon)[$. If $(a, b) \in W$, then $a = x$ and $y - \epsilon < b < y + \epsilon$. So, $F_l((a, b)) = (f(x), b) \in V$.
2.2. Second subcase: $y = 0$. A standard neighborhood of $F_l((x, y)) = (f(x), y) = (f(x), 0)$ has the form $V =](f(x) - \epsilon, 1), (f(x), \epsilon)[$ with $\epsilon < |f(a_{2k}) - f(x)|/2$. Since f is an increasing and continuous function in $]a_{2k}, a_{2k+1}[$, there is δ in the interval $]0, |x - a_{2k}|/2[$ such that $f[\,]x - \delta, x]] \subset]f(x) - \epsilon, f(x)]$. So, for $W =](x - \delta, 1), (x, \epsilon)[$ we have that $F_l[W] \subset V$.
2.3. Third subcase: $y = 1$. In this subcase, we have that a canonical neighborhood of $(f(x), 1)$ has the form $V =](f(x), 1 - \epsilon), (f(x) + \epsilon, 0)[$. Again, since f is an increasing and continuous function in $]a_{2k}, a_{2k+1}[$, there is $\delta \in]0, |x - a_{2k}|/2[$ such that $f[[x, x + \delta[\,] \subset [f(x), f(x) + \epsilon[$. For $W =](x, 1 - \epsilon), (x + \delta, 0)[$, we have $F_l[W] \subset V$.
3. Third case: $x = 1$. The set $V =](0, 1-\epsilon), (\epsilon, 0)[$ is a neighborhood of $F_l((x, y)) = (0, 1)$.

In this case too, we have several subcases:
3.1. First subcase: If $y \in]0, 1[$, we take $\delta \in]0, \min\{y, 1 - y\}/2[$ and we take $W =](1, y - \delta), (1, y + \delta)[$. It happens that $F_l[W] = \{(0, 1)\} \subset V$.
3.2. Second subcase: If $y = 1$, we take $W =](1, 1/2), (1, 1)]$; then we have that $F_l[W] = \{(0, 1)\} \subset V$.
3.3. Third subcase: Finally, assume that $y = 0$. Since f is continuous at 1, $f(1) = 0$ and $|f^{-1}[\{0\}]| < \aleph_0$, there is $\delta > 0$ such that $f[\,]1 - \delta, 1]] \subset [0, \epsilon[$ and 0 does not belong to $f[\,]1 - \delta, 1[\,]$. Let W be the set $](1 - \delta, 1), (1, \delta)[$ and take $(a, b) \in W$. If $a = 1$, then $F_l((a, b)) = (0, 1) \in V$. If $a < 1$, then $F_l((a, b)) = (f(a), z)$ with $f(a) \in]0, \epsilon[$; so, $(f(a), z) \in V$ as well.

The proof of the continuity of F_l at the point (x, y) when $x = a_{2k+1}$ and when $x \in]a_{2k+1}, a_{2k+2}[$ is similar to that given when $x = a_{2k}$ and when $x \in]a_{2k}, a_{2k+1}[$, respectively.

So, we have proved that F_l is a continuous function at each point of J. □

We are frequently going to use the following well known result.

Proposition 1.2 *For a dynamical system (X, φ), the subset $M \subset X$ is minimal if and only if for each $x \in M$, the set $\{\varphi^n(x) : n < \omega\}$ is dense in M.*

2 Non-Cantor minimal sets

2.1 In [4, Example 30, p. 147] it is shown that the classical *middle third* Cantor set C is a minimal set for the function on the interval defined as

$$f(0) = \frac{2}{3}, \quad f(1) = 0, \quad f\left(1 - \frac{2}{3^k}\right) = \frac{1}{3^{k-1}}, \quad f\left(1 - \frac{1}{3^k}\right) = \frac{2}{3^{k+1}} \qquad (k \geq 1),$$

and defined as linear at intermediate points.

Function f satisfies the requirements of Theorem 1.1; thus, we have the following result.

Theorem 2.2 *The set* $C_2 = C \times \{0, 1\}$ *is a non-metrizable minimal set in* (J, F_l).

Proof The space C_2 is separable of uncountable weight, so it is not metrizable. Let f be the function defined in 2.1. Let F_l be the function generated by f as was indicated in Theorem 1.1. In order to prove that C_2 is a minimal set of J with respect to $F_l : J \to J$, we have to verify that for each point $(x, y) \in C_2$ the set $D = \{F_l^n(x, y) : n \in \mathbb{N}\}$ is dense in C_2. However, $\pi_1[D] = \{f^n(x) : n \in \mathbb{N}\}$ where π_1 is the projection to the first coordinate. Since $\{f^n(x) : n \in \mathbb{N}\}$ is dense in C, D is dense in $C_2 = C \times \{0, 1\}$. □

The previous theorem provides the first example of a minimal set in a compact, connected linearly ordered space which is not metrizable. By means of a similar technique to the one used to prove Theorem 2.2, it is possible to construct $2^{\mathfrak{c}}$ non-homeomorphic, non-metrizable infinite minimal sets. In order to carry out this construction, we will first give some preliminary definitions, facts and comments.

2.3 The middle-third Cantor set C is precisely the set of points in $I = [0, 1]$ having a ternary expansion without 1s. That is,

$$C = \left\{ \sum_{i=0}^{\infty} \frac{x_i}{3^{i+1}} \; : \; x_i \in \{0, 2\} \right\}.$$

Let 2^ω be the product of a countable collection of copies of the two-point discrete space $\{0, 1\}$. The function $h : C \to 2^\omega$ defined by

$$h\left(\sum_{i=0}^{\infty} \frac{x_i}{3^{i+1}} \right) = \left(\frac{x_i}{2} \right)_{i<\omega}$$

is a homeomorphism.

For $a \in \{0, 1, 2, 3\}$, let $r(a) = 0$ if $a < 2$, and $r(a) = 1$ if $a \geq 2$. We will denote by $\oplus : 2^\omega \times 2^\omega \to 2^\omega$ *the symbolic addition with carrying* which is defined by

$$(x_i)_{i<\omega} \oplus (y_i)_{i<\omega} = (z_i)_{i<\omega}$$

if and only if $z_0 = x_0 * y_0$, $z_1 = x_1 * y_1 * r(x_0 + y_0)$, $z_2 = x_2 * y_2 * r(x_1 + y_1 + r(x_0 + y_0))$, $z_3 = x_3 * y_3 * r(x_2 + y_2 + r(x_1 + y_1 + r(x_0 + y_0)))$, and so on, where $*$ is the sum in the group of 2-adic integers and $+$ is the usual sum in $\mathbb{Z}$.

Now we define $\sigma : 2^\omega \to 2^\omega$ as $\sigma(\xi) = \xi \oplus \mathbf{1}$ where $\mathbf{1} = (1, 0, 0, 0, \ldots)$. It is easy to verify that the following statement holds.

Claim 2.4 Let $(x_i)_{i<\omega} \in 2^\omega$. Let n_0 be the first natural number $n > 0$ such that $x_n = 0$. If $\sigma((x_i)_{i<\omega}) = (z_i)_{i<\omega}$, then $z_i = x_i$ for every $i > n_0$.

Again consider the function f defined in 2.1. In [4, Example 30, p. 147] it was pointed out that if $x \in C$ has the ternary expansion $\sum_{i=0}^{\infty} \frac{2b_i}{3^{i+1}}$, then $f(x)$ has the ternary expansion $\sum_{i=0}^{\infty} \frac{2c_i}{3^{i+1}}$ determined by the relation $(c_i)_{i<\omega} = \sigma((b_i)_{i<\omega})$. In other words, we have the following:

Claim 2.5 Let $f : I \to I$ be the function defined in 2.1, and consider $h : C \to 2^\omega$ and $\sigma : 2^\omega \to 2^\omega$ as defined in 2.3. Then, $h \circ f \circ h^{-1} = \sigma$.

2.6 Now take a free ultrafilter $\mathcal{U}$ on ω, and let $A_{\mathcal{U}}$ be the subset of 2^ω of all characteristic functions of either a finite subset of ω or of an element U of $\mathcal{U}$; that is:

$$A_{\mathcal{U}} = \{\chi_U \in 2^\omega : U \in Fin \text{ or } U \in \mathcal{U}\}.$$

By 2.4, we obtain:

Claim 2.7 For the function $\sigma : 2^\omega \to 2^\omega$ defined in 2.3, we have that $\sigma[A_{\mathcal{U}}] \subset A_{\mathcal{U}}$ and $\sigma[2^\omega \backslash A_{\mathcal{U}}] \subset 2^\omega \backslash A_{\mathcal{U}}$.

2.8 For $\mathcal{U} \in \omega^*$, we define relation $\sim_{\mathcal{U}}$ on the lexicographic square J by:

$$\begin{aligned}(x, r) \sim_{\mathcal{U}} (y, s) \Leftrightarrow \text{ either } & x = y \text{ and } r = s, \text{ or} \\ & x = y \in C \text{ and } h(x) \in A_{\mathcal{U}}.\end{aligned}$$

The relation $\sim_{\mathcal{U}}$ is an equivalence relation. Let $J_{\mathcal{U}}$ be the set of all the $\sim_{\mathcal{U}}$-equivalence classes, and let $\pi_{\mathcal{U}} : J \to J_{\mathcal{U}}$ be the canonical projection. If we consider $J_{\mathcal{U}}$ with the quotient topology defined by $\pi_{\mathcal{U}}$, then $\pi_{\mathcal{U}} : J \to J_{\mathcal{U}}$ is a continuous function and $J_{\mathcal{U}}$ is a compact connected topological space.

That $J_{\mathcal{U}}$ is a linearly orderable space (in particular, $J_{\mathcal{U}}$ is Hausdorff) is given by the following lemma which can be proved by standard arguments.

Lemma 2.9 *Let $(X, \mathcal{T}, \leq)$ be a linearly ordered topological space. Let $q : X \to Y$ be an onto function. If for every $y \in Y$, $q^{-1}[\{y\}]$ is a $\leq$-interval, then the relation in Y defined by $q(a) \sqsubset q(b)$ if $q(a) \neq q(b)$ and $a < b$ is a well defined linear order relation in Y. Moreover, if for every $y \in Y$, $q^{-1}[\{y\}]$ is a closed $\leq$-interval and if q is an identification, then the quotient topology in Y defined by q coincides with the order topology in Y defined by $\sqsubseteq$.*

Furthermore, the projection $\pi_{\mathcal{U}} : J \to J_{\mathcal{U}}$ is a closed mapping because it is continuous, J is compact and $J_{\mathcal{U}}$ is Hausdorff.

Let $F_{\mathcal{U}} : J_{\mathcal{U}} \to J_{\mathcal{U}}$ be the relation which makes the following diagram commutative

$$\begin{array}{ccc} J & \xrightarrow{\pi_{\mathcal{U}}} & J_{\mathcal{U}} \\ {\scriptstyle F_l}\downarrow & & \downarrow{\scriptstyle F_{\mathcal{U}}} \\ J & \xrightarrow{\pi_{\mathcal{U}}} & J_{\mathcal{U}} \end{array}$$

That is, $F_{\mathcal{U}}(\pi_{\mathcal{U}}(x, r)) = \pi_{\mathcal{U}}(F_l((x, r)))$ for each $(x, r) \in J$.

Claim 2.10 The relation $F_{\mathcal{U}} : J_{\mathcal{U}} \to J_{\mathcal{U}}$, defined above, is a continuous function.

Proof First we will prove that $F_{\mathcal{U}}$ is a function; that is, we must prove that for each two different elements $(x,r),(y,s) \in J$, if $(x,r) \sim_{\mathcal{U}} (y,s)$, then $F_l((x,r)) \sim_{\mathcal{U}} F_l((y,s))$.

We have that $x = y \in C$ and $h(x) \in A_{\mathcal{U}}$. There is $a \in A_{\mathcal{U}}$ such that $x = h^{-1}(a)$. Hence, $h(f(x)) = h(f(h^{-1}(a)) = \sigma(a)$ (2.5). But $a \in A_{\mathcal{U}}$, and thus $\sigma(a) \in A_{\mathcal{U}}$ (2.7). This means that $F_l((x,r)) \sim_{\mathcal{U}} F_l((y,s))$. That is, $F_{\mathcal{U}}$ is a function.

Moreover, $F_{\mathcal{U}}$ is continuous because $F_{\mathcal{U}} \circ \pi_{\mathcal{U}} = \pi_{\mathcal{U}} \circ F_l$, $\pi_{\mathcal{U}}$ is a quotient mapping and F_l is continuous. □

Recall that we denote by C_2 the subspace $C \times \{0,1\}$ of J. Let $C_{\mathcal{U}}$ be the subspace $\pi_{\mathcal{U}}(C_2)$ of $J_{\mathcal{U}}$. Since C_2 is a compact separable space, $C_{\mathcal{U}}$ possesses these properties too. Furthermore, $C_{\mathcal{U}}$ is not metrizable. In order to verify this last assertion, we are going to make some general considerations:

Let $X \subset [0,1]$. Define on J the equivalence relation $(x,r) \sim_X (y,s)$ iff either $(x,r) = (y,s)$ or $x = y \in X$. Let J_X be the quotient space generated by J and π_X, the natural projection defined by $\sim_X$.

Lemma 2.11 (1) *The relation* $\pi_1 : J_X \to [0,1]$ *defined by* $\pi_1(\pi_X((x,r))) = x$ *is an open continuous function.*

(2) *Let* Y *be a subset of* X *and define* $\mathcal{Y} = \{\pi_X((y,s)) : y \in Y, s \in [0,1]\}$. *Then, the function* $\varphi = \pi_1 \restriction \mathcal{Y}$ *is a homeomorphism from the subspace* $\mathcal{Y}$ *of* J_X *to the subspace* Y *of the unit interval* $[0,1]$.

Proof Of course π_1 is a well defined function and it is one-to-one if restricted to $\mathcal{Y}$.

Now, for $(x,r) \in]0,1[\times [0,1]$ (resp., $x = 0$ and $r \in [0,1]$; $x = 1$ and $r \in [0,1]$) and $\epsilon > 0$, the set $V =](x-\epsilon,1),(x+\epsilon,0)[$ (resp., $V = [(0,0),(\epsilon,0)[$; $V =](1-\epsilon,0),(1,1)]$) is open in J and $\pi_X^{-1}\pi_X[V] = V$. So, $\pi_X[V]$ is an open neighborhood of $\pi_X((x,r))$. Moreover, $\pi_1[\pi_X[V]] =]x-\epsilon, x+\epsilon[$ (resp., $\pi_1[\pi_X[V]] = [0,\epsilon,[$; $\pi_1[\pi_X[V]] =]1-\epsilon,1]$). Therefore, π_1 is a continuous function.

On the other hand, for $y,s \in I$,

$$\begin{aligned}
\pi_1[\,]\pi_X((y,s)),\to[\,] &= \;]y,1],\\
\pi_1[\,]\leftarrow,\pi_X((y,s))[\,] &= [0,y[,\\
\varphi[\,]\pi_X((y,s)),\to[\cap\mathcal{Y}] &= \;]y,1]\cap Y \text{ and}\\
\varphi[\,]\leftarrow,\pi_X((y,s))[\cap\mathcal{Y}] &= [0,y[\,\cap\, Y;
\end{aligned}$$

so, π_1 is open and φ is a homeomorphism. □

Observe that the space $\{\pi_X((y,1)) : y \in [0,1]\backslash X\}$ is homeomorphic to the subspace $[0,1]\backslash X$ of the Sorgenfrey line considered with the base constituted by left-closed intervals, and that $\{\pi_X((y,0)) : y \in [0,1]\backslash X\}$ is homeomorphic to the subspace $[0,1]\backslash X$ of the Sorgenfrey line considered with the base of right-closed intervals.

Lemma 2.12 *For every* $Z \subset [0,1]$, *the weight of the subspace* $\mathcal{Z}_0 = \{\pi_X((y,0)) : y \in Z\}$ (*resp.,* $\mathcal{Z}_1 = \{\pi_X((y,1)) : y \in Z\}$) *of* J_X *is equal to* $|Z\backslash X|$.

Proof Let Y be equal to $Z \cap X$. The subspace $\mathcal{Y}_0 = \{\pi_X((y, 0)) : y \in Y\}$ (resp., $\mathcal{Y}_1 = \{\pi_X((y, 1)) : y \in Y\}$) is second countable (Lemma 2.11). Thus, $w(\mathcal{Z}_0) = w(\mathcal{Z}_0 \backslash \mathcal{Y}_0)$ (resp., $w(\mathcal{Z}_1) = w(\mathcal{Z}_1 \backslash \mathcal{Y}_1)$). But, as we have already mentioned, $\mathcal{Z}_0 \backslash \mathcal{Y}_0$ (resp., $\mathcal{Z}_1 \backslash \mathcal{Y}_1$) is homeomorphic to the subspace $Z \backslash Y$ of the Sorgenfrey line. So, the conclusion of this lemma follows. □

Since $I \times \{0, 1\} \subset J$ is hereditarily separable, so $\pi_X[I \times \{0, 1\}]$ is also hereditarily separable. Then, by Lemmas 2.11 and 2.12, we obtain:

Lemma 2.13 *Let $Y_0, Y_1 \subset [0, 1]$. The subspace*

$$Z = \{\pi_X((y, 0)) : y \in Y_0\} \cup \{\pi_X((y, 1)) : y \in Y_1\}$$

of J_X is metrizable if and only if $|(Y_0 \cup Y_1) \backslash X| \leq \aleph_0$.

For each $\mathcal{U} \in \omega^\star$, $|2^\omega \backslash A_{\mathcal{U}}| = 2^\omega$; so, by Lemma 2.13, the following proposition holds.

Corollary 2.14 *For each free ultrafilter $\mathcal{U}$ on ω, $C_{\mathcal{U}}$ is not metrizable.*

Theorem 2.15 *$C_{\mathcal{U}}$ is a non-metrizable minimal set of the dynamical system $(J_{\mathcal{U}}, F_{\mathcal{U}})$.*

Proof We have noted already that $C_{\mathcal{U}}$ is not metrizable Corollary (2.14); so, we only have to show that $C_{\mathcal{U}}$ is minimal.

Because of Proposition 1.2, we must prove that for an arbitrary $c \in C_{\mathcal{U}}$, the set $\{F_{\mathcal{U}}^{n+1}(c) : n < \omega\}$ is dense in $C_{\mathcal{U}}$. Let $a \in C_2$ be such that $\pi_{\mathcal{U}}(a) = c$. So, $\{F_{\mathcal{U}}^{n+1}(c) : n < \omega\} = \{(F_{\mathcal{U}}^{n+1} \circ \pi_{\mathcal{U}})(a) : n < \omega\}$. Observe that, for each $n < \omega$, $(F_{\mathcal{U}}^{n+1} \circ \pi_{\mathcal{U}})(a) = (\pi_{\mathcal{U}} \circ F_l^{n+1})(a)$. Thus, $\{F_{\mathcal{U}}^{n+1}(c) : n < \omega\} = \pi_{\mathcal{U}}[\{F_l^{n+1}(a) : n < \omega\}]$. By Theorem 2.2, $\{F_l^{n+1}(a) : n < \omega\}$ is dense in C_2; therefore, $\{F_{\mathcal{U}}^{n+1}(c) : n < \omega\}$ is dense in $C_{\mathcal{U}}$. □

2.16 For each $\mathcal{U} \in \omega^*$, let us denote by $[\mathcal{U}]$ the set $\{\mathcal{V} \in \omega^* : C_{\mathcal{V}} \cong C_{\mathcal{U}}\}$.

Let $h_{\mathcal{V}}$ be a homeomorphism from $C_{\mathcal{U}}$ onto $C_{\mathcal{V}}$, and consider the functions $\psi_{\mathcal{U}} : A_{\mathcal{U}} \to C_{\mathcal{U}}$ and $\varphi_{\mathcal{V}}$: $C_{\mathcal{V}} \to [0, 1]$ defined by $\psi_{\mathcal{U}}(x) = \pi_{\mathcal{U}}((x, 0))$ and $\varphi_{\mathcal{V}}(\pi_{\mathcal{V}}((x, s)) = x$, respectively. By Lemma 2.11, the function $\psi_{\mathcal{U}}$ is an embedding and $\varphi_{\mathcal{V}}$ is a continuous function; so, $l_{\mathcal{V}} = \varphi_{\mathcal{V}} \circ h_{\mathcal{V}} \circ \psi_{\mathcal{U}} : A_{\mathcal{U}} \to [0, 1]$ is also continuous. Since the space $\pi_{\mathcal{U}}[A_{\mathcal{U}} \times \{0, 1\}]$ is second countable, $h_{\mathcal{V}}[\pi_{\mathcal{U}}[A_{\mathcal{U}} \times \{0, 1\}]]$ has this property too. Because of Lemmas 2.11 and 2.12

$$|(h_{\mathcal{V}} \circ \psi_{\mathcal{U}})[A_{\mathcal{U}}] \backslash \pi_{\mathcal{V}}[A_{\mathcal{V}} \times \{0, 1\}] \cup (\pi_{\mathcal{V}}[A_{\mathcal{V}} \times \{0, 1\}] \backslash (h_{\mathcal{V}} \circ \psi_{\mathcal{U}})[A_{\mathcal{U}}])| \leq \aleph_0.$$

Therefore,

$$|(l_{\mathcal{V}}[A_{\mathcal{U}}] \backslash A_{\mathcal{V}}) \cup (A_{\mathcal{V}} \backslash l_{\mathcal{V}}[A_{\mathcal{U}}])| \leq \aleph_0.$$

On the other hand, if $\mathcal{U}$ and $\mathcal{V}$ are two different free ultrafilters on ω, then $|(A_{\mathcal{U}} \backslash A_{\mathcal{V}}) \cup (A_{\mathcal{V}} \backslash A_{\mathcal{U}})| > \aleph_0$.

Moreover, since $A_{\mathcal{U}}$ is separable, the cardinality of the set of continuous functions from $A_{\mathcal{U}}$ into $[0, 1]$ is less or equal to 2^{ω}.

All these remarks lead us to conclude that $|[\mathcal{U}]| \leq 2^{\omega}$. But $|\omega^*| = 2^{\mathfrak{c}}$, hence, there is a collection $\mathfrak{U} \subset \omega^*$ of cardinality $2^{\mathfrak{c}}$ such that if $\mathcal{U}, \mathcal{V} \in \mathfrak{U}$ and $\mathcal{U} \neq \mathcal{V}$, then $C_{\mathcal{U}}$ is not homeomorphic to $C_{\mathcal{V}}$. So, we have:

Theorem 2.17 *The set $\{C_{\mathcal{U}} : \mathcal{U} \in \mathfrak{U}\}$ is a collection of $2^{\mathfrak{c}}$ pairwise non-homeomorphic, non-metrizable minimal sets.*

3 Final example, remarks and problems

Now we are going to define a compact connected linearly ordered topological space X, and construct a continuous function $F : X \to X$ such that, for each $\mathcal{U} \in \mathfrak{U}$, X contains a copy of $C_{\mathcal{U}}$ which will be minimal in (X, F).

As usual, we say that an ordinal λ is *limit* if it does not have an immediate predecessor. In particular, we will consider 0 as a limit ordinal. An ordinal λ is *even* if $\lambda = \gamma + 2n$ where γ is a limit ordinal and $n < \omega$; and λ is *odd* if $\lambda = \gamma + 2n + 1$ where γ is a limit ordinal and $n < \omega$.

We enumerate the elements in $\mathfrak{U}$ as $\{\mathcal{U}_\lambda : \lambda < 2^{\mathfrak{c}}\}$. For each even ordinal $\lambda \leq 2^{\mathfrak{c}}$ we will denote by J_λ the set $\{((a, b), \lambda) : (a, b) \in J\}$, and for each odd ordinal $\lambda < 2^{\mathfrak{c}}$, $\lambda = \gamma + 2n + 1$, J_λ will be the set $\{(\pi_{\mathcal{U}_{\gamma+n}}(a, b), \lambda) : (a, b) \in J\}$. For each $(a, b) \in J$ and for each $\lambda \leq 2^{\mathfrak{c}}$, we define $\pi_\lambda((a, b)) \in J_\lambda$ as follows:

$$\pi_\lambda((a, b)) = \begin{cases} ((a, b), \lambda) & \text{if } \lambda \text{ is even} \\ (\pi_{\mathcal{U}_{\gamma+n}}(a, b), \lambda) & \text{if } \lambda = \gamma + 2n + 1 \end{cases}$$

Let $Z = \bigcup_{\lambda \leq 2^{\mathfrak{c}}} J_\lambda$. We consider in Z the following relation $\sim_L$: $\pi_\lambda(a, b) \sim_L \pi_\xi(c, d)$ iff one of the following cases happens: (1) $\lambda = \xi$ and $(a, b) = (c, d)$, (2) $\lambda = \xi$, λ is an even ordinal and $a = c = 0$ or $a = c = 1$, (3) $\lambda = \xi + 1$, $a = 0$ and $c = 1$, or (4) $\lambda = \xi = 2^{\mathfrak{c}}$. Let X be the set of $\sim_L$-equivalence classes and let $q : Z \to X$ be the natural projection. We define the following order relation $\leq_L$ in X: For $q\pi_\lambda((a, b)) \neq q\pi_\xi((c, d))$,

$$q(\pi_\lambda(a, b)) <_L q(\pi_\xi(c, d)) \text{ iff either } \lambda < \xi \text{ or } \lambda = \xi < 2^{\mathfrak{c}} \text{ and } (a, b) <_l (c, d),$$

where $<_l$ is the lexicographic order on J.

We provide X with the order topology defined by $\leq_L$. Observe that for each even ordinal $\lambda < 2^{\mathfrak{c}}$, $q[J_\lambda]$ is homeomorphic to the quotient space obtained from J by identifying to one point the set of points in J with first coordinate equal to 0 and to identify to one point the set of points with first coordinate equal to 1. On the other hand, since $0, 1 \in A_{\mathcal{U}_\lambda}$, for each odd ordinal $\lambda = \gamma + 2n + 1 < 2^{\mathfrak{c}}$, $q[J_\lambda]$ is homeomorphic to the space $J_{\mathcal{U}_{\gamma+n}}$. Finally, if $\lambda = 2^{\mathfrak{c}}$, $q[J_\lambda]$ is a one point set, it is the last element in X and it belongs to the closure of its complement in X. Also, observe that the subspace $\{q(\pi_\lambda(0, 0)) : \lambda \leq 2^{\mathfrak{c}}\}$ of X is homeomorphic to the space $[0, 2^{\mathfrak{c}}]$ of all ordinal numbers less or equal to $2^{\mathfrak{c}}$ with their usual order topology. Since each

J_λ is a connected space, the equivalence relation $\sim_L$ makes X a connected space. Furthermore, the compactness of each J_λ and the compactness of $[0, 2^{\mathfrak{c}}]$ makes X compact. So, X is a compact connected linearly ordered topological space.

Now, we construct a convenient continuous function $F : X \to X$. We define:

$$F(q(\pi_\lambda(x, y))) = \begin{cases} q(\pi_\lambda(3x, y)) & \text{if } \lambda \text{ is a limit ordinal } < 2^{\mathfrak{c}} \\ & \text{and } x \in [0, 1/3]; \\ q(\pi_{\lambda+1}(x - \frac{1}{3}, y)) & \text{if } \lambda \text{ is a limit ordinal } < 2^{\mathfrak{c}} \\ & \text{and } x \in [1/3, 1]; \\ q(F_{\mathcal{U}_\lambda}(\pi_\lambda(x, y))) & \text{if } \lambda \text{ is odd;} \\ q(\pi_{\lambda-1}(3x, y)) & \text{if } \lambda \text{ is a non-limit even} \\ & \text{ordinal and } x \in [0, 1/3]; \\ q(\pi_\lambda(3(x - \frac{1}{3}), y)) & \text{if } \lambda \text{ is a non-limit even} \\ & \text{ordinal and } x \in [1/3, 2/3]; \\ q(\pi_{\lambda+1}(2(x - \frac{2}{3}), y)) & \text{if } \lambda \text{ is a non-limit even} \\ & \text{ordinal and } x \in [2/3, 1] \\ q(\pi_\lambda(x, y)) & \text{if } \lambda = 2^{\mathfrak{c}}. \end{cases}$$

It is not difficult to see that F is a well defined function and that $F \restriction q[J_\lambda]$ is continuous for every $\lambda \leq 2^{\mathfrak{c}}$. The proof that F is a continuous function on all of X, can be made following similar arguments to those used in the proof of Theorem 1.1.

As we have already mentioned, for every odd ordinal $\lambda = \gamma + 2n + 1 < 2^{\mathfrak{c}}$, the relation

$$\pi_{\mathcal{U}_{\gamma+n}}(a, b) \to q(\pi_\lambda(a, b))$$

from $J_{\mathcal{U}_{\gamma+n}}$ to $q[J_\lambda]$ is a homeomorphism. The image of $C_{\mathcal{U}_{\gamma+n}}$ under this relation is equal to $q[C_{\mathcal{U}_{\gamma+n}} \times \{\lambda\}]$. So, since $C_{\mathcal{U}_{\gamma+n}}$ is a compact subspace of $J_{\mathcal{U}_{\gamma+n}}$, $C_{\mathcal{U}_{\gamma+n}}$ is homeomorphic to $q[C_{\mathcal{U}_{\gamma+n}} \times \{\lambda\}]$. Moreover, in this case, $F(q\pi_\lambda(a, b)) = qF_{\mathcal{U}_\lambda}(\pi_\lambda(a, b))$; that is, "$F$ has the same behavior as $F_{\mathcal{U}_{\gamma+n}}$" in $q[J_{\gamma+2n+1}]$ for each $\gamma < 2^{\mathfrak{c}}$. Therefore, $q[C_{\mathcal{U}_{\gamma+n}} \times \{\lambda\}]$ becomes a minimal set for (X, F) for all $\gamma < 2^{\mathfrak{c}}$. Denote by C_λ the subspace $q[C_{\mathcal{U}_{\gamma+n}} \times \{\lambda\}]$. So, we have:

Theorem 3.1 *The set $\{C_\lambda : \lambda$ is an odd ordinal $< 2^{\mathfrak{c}}\}$ is a collection of $2^{\mathfrak{c}}$ pairwise non-homeomorphic, non-metrizable minimal sets in the connected compact linearly ordered topological space X, with respect to the dynamical system (X, F).*

Since our space X has cardinality $2^{\mathfrak{c}}$ and two different minimal sets must be disjoint, $2^{\mathfrak{c}}$ is the biggest quantity of pairwise non-homeomorphic minimal sets one can have on X.

To conclude we will mention an open question. It is easy to see that if $\mathcal{U}$ and $\mathcal{V}$ are equivalent ultrafilters, then $C_{\mathcal{U}}$ and $C_{\mathcal{V}}$ are homeomorphic. Also, for each $\mathcal{U} \in \omega^*$, the set $[\mathcal{U}] = \{\mathcal{V} \in \omega^* : C_{\mathcal{V}} \cong C_{\mathcal{U}}\}$ (see 2.16) has size 2^ω. So it is natural to ask:

Problem 3.2 Is $\mathcal{U}$ equivalent to $\mathcal{V}$ whenever $C_{\mathcal{V}} \cong C_{\mathcal{U}}$?

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