GENERALISED MUTUALLY PERMUTABLE PRODUCTS AND SATURATED FORMATIONS, II

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Abstract. A group $G = AB$ is the weakly mutually permutable product of the subgroups A and B, if A permutes with every subgroup of B containing $A \cap B$ and B permutes with every subgroup of A containing $A \cap B$. Weakly mutually permutable products were introduced by the first, second and fourth author and they showed that if G' is nilpotent, A permutes with every Sylow subgroup of B and B permutes with every Sylow subgroup of A, then $G^{\mathfrak{F}} = A^{\mathfrak{F}} B^{\mathfrak{F}}$, where $\mathfrak{F}$ is a saturated formation containing $\mathfrak{U}$, the class of supersoluble groups. In this article we prove results on weakly mutually permutable products concerning $\mathfrak{F}$-residuals, $\mathfrak{F}$-projectors and $\mathfrak{F}$-normalisers which provide new results on mutually permutable products. As an application of some of our arguments, we unify some results on weakly mutually sn-products.

1. Introduction

All groups considered in this article will be finite.

Let a group $G = AB$ be a product of two subgroups A and B. The structural influence of the structure of the subgroups A and B with certain permutability properties on the group G has been of interest to many authors for the past three decades (see [1]). In this article we continue with the investigation on generalised products of finite groups than the ones considered in [1].

We start by recalling some definitions and some notation: a group G is the *mutually permutable* product of the subgroups A and B if $G = AB$ and A permutes with every subgroup of B and B permutes with every subgroup of A; a group G is the *weakly mutually permutable* product of A and B if A permutes with every subgroup V of B such that $A \cap B \leqslant V$, and B permutes with every subgroup U of A such that $A \cap B \leqslant U$; a group G is the *weakly mutually sn-permutable* product of A and B if A permutes with every subnormal subgroup V of B such that $A \cap B \leqslant V$, and B permutes with every subnormal subgroup U of A such that $A \cap B \leqslant U$. The classes of all finite nilpotent and supersoluble groups, are denoted by $\mathfrak{N}$ and $\mathfrak{U}$, respectively.

In [3] some results on mutually permutable products were extended to weakly mutually permutable products. In particular, the following was shown, which is a generalisation of [4, Theorem A]:

Theorem 1.1. [3, Theorem B] *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Let the group $G = AB$ be the weakly mutually permutable product of subgroups A and B. Suppose that A permutes with each Sylow subgroup of B and B permutes with each Sylow subgroup of A. If G' is nilpotent, then $G^{\mathfrak{F}} = A^{\mathfrak{F}} B^{\mathfrak{F}}$.*

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Our objective in this article is to generalise more results on mutually permutable products to weakly mutually permutable ones. In particular, we obtain new results on mutually permutable products as a consequence. In [5] the following was shown:

Theorem 1.2. [5, Theorem 1] *Let $G = AB$ be the mutually permutable product of subgroups A and B. If B is supersoluble and G' is nilpotent, or B is nilpotent, then*

$$G^{\mathfrak{U}} = A^{\mathfrak{U}}.$$

Part of this result was extended in [4], were the authors proved that if $\mathfrak{F}$ is a saturated formation containing the class $\mathfrak{U}$ of supersoluble groups, then the $\mathfrak{F}$-residual respects the operation of forming mutually permutable products with nilpotent commutator subgroup, that is $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}$. However, it turns out that the corresponding result is not true if B is nilpotent, even in the case that $\mathfrak{F}$ is a Fitting class, as the following example shows:

Example 1.3. *Let $\mathfrak{F} = \mathfrak{N}^2$ be the class of metanilpotent groups. Then $\mathfrak{F}$ is a saturated Fitting formation containing $\mathfrak{U}$, which is closed for subgroups. Consider $G = AB$ the symmetric group of degree 4, where B is a Sylow 2-subgroup of G and A is the alternating group of degree 4. Then A and B are mutually permutable. Moreover, A is metanilpotent and B is nilpotent. But $1 = A^{\mathfrak{N}^2} \neq G^{\mathfrak{N}^2} = V$, where V is the Klein four-group.*

However, we have been able to prove the following extension of the result for weakly mutually permutable products.

Theorem A. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation containing $\mathfrak{U}$ such that every group in $\mathfrak{F}$ has a Sylow tower of supersoluble type. Let $G = AB$ be the weakly mutually permutable product of A and B. If B is nilpotent and permutes with each Sylow subgroup of A, then*

$$G^{\mathfrak{F}} = A^{\mathfrak{F}}.$$

As a corollary, we also obtain a result on weakly mutually sn-permutable products.

A widely supersoluble group, or w-*supersoluble* group for short, is defined as a group G such that every Sylow subgroup of G is $\mathbb{P}$-subnormal in G (a subgroup H of a group G is $\mathbb{P}$-subnormal in G whenever either $H = G$ or there exists a chain of subgroups $H = H_0 \leqslant H_1 \leqslant \cdots \leqslant H_{n-1} \leqslant H_n = G$, such that $|H_i{:}H_{i-1}|$ is a prime for every $i = 1, \ldots, n$).

The class of w-supersoluble groups, denoted by $w\mathfrak{U}$, is a subgroup-closed saturated formation containing $\mathfrak{U}$. Moreover w-supersoluble groups have a Sylow tower of supersoluble type (see [8, Corollary]).

We recall some results we proved in [2]:

Theorem 1.4. [2, Theorems A and C, and Corollaries B and D] *Let $\mathfrak{F} = \mathfrak{U}$ or $\mathfrak{F} = w\mathfrak{U}$. Let $G = AB$ be the weakly mutually sn-permutable product of the subgroups A and B, where $A, B \in \mathfrak{F}$. Suppose that B permutes with each Sylow subgroup of A. Then $G \in \mathfrak{F}$, if one of the following holds:*

(a) *B is nilpotent;*
(b) *A permutes with each Sylow subgroup of B and G' is nilpotent.*

We unify these results by proving the following:

Corollary B. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F} \subseteq w\mathfrak{U}$. Let $G = AB$ be the weakly mutually sn-permutable product of the $\mathfrak{F}$-subgroups A and B. Suppose that either B or G' is nilpotent. If B permutes with each Sylow subgroup of A, then the group G belongs to $\mathfrak{F}$.*

The case when G' is nilpotent follows from the fact A and B are metanilpotent and so are supersoluble and hence G is supersoluble by Theorem 1.4.

In [4], the authors showed that unfortunately the $\mathfrak{F}$-projectors and so the $\mathfrak{F}$-normalisers of a mutually permutable product with nilpotent commutator subgroup cannot be obtained from the corresponding projectors of the factor subgroups as the following example shows: Let $G = AB$ be the direct product of a cyclic group $\langle a\rangle$ of order 3 with the alternating group A_4 of degree 4. Let V be the Klein group in A_4. Then G is the mutually permutable product of $A = \mathrm{A}_4$ and $B = \langle a\rangle \times V$. Moreover, B and $G' = V$ are abelian. Note that B is the supersoluble projector of B and a Sylow 3-subgroup A_1 of A_4 is a supersoluble projector of A. But A_1B is not supersoluble.

Some conditions on $\mathfrak{F}$-projectors and $\mathfrak{F}$-normalisers allow us to have the following result:

Theorem C. *Let $\mathfrak{F}$ be a formation. Assume that either $\mathfrak{F} = \mathfrak{U}$ or $\mathfrak{F}$ is a saturated Fitting formation containing $\mathfrak{U}$. Let $G = AB$ be the weakly mutually permutable product of the subgroups A and B. Suppose that G' is nilpotent, A_1 is an $\mathfrak{F}$-normaliser of A such that $A \cap B \leqslant A_1$ and B_1 is an $\mathfrak{F}$-normaliser of B such that $A \cap B \leqslant B_1$, then A_1B_1 is an $\mathfrak{F}$-normaliser of G.*

In the above result, since G' is nilpotent, we have that $G \in \mathfrak{N}\mathfrak{F}$. Applying [6, V, 4.2], the $\mathfrak{F}$-normalisers and the $\mathfrak{F}$-projectors coincide, hence the result is also true for projectors.

2. Preliminary Results

In this section we first recall some properties of weakly mutually permutable products and then prove some results needed in the proof of our main results.

Lemma 2.1. [3, Lemma 2.1] *Let $G = AB$ be the weakly mutually permutable product of subgroups A and B and let N be a normal subgroup of G. Then $G/N = (AN/N)(BN/N)$ is the weakly mutually permutable product of AN/N and BN/N.*

Lemma 2.2. *Let $G = AB$ be the weakly mutually permutable product of subgroups A and B.*

(a) *If H is a subgroup of A such that $A \cap B \leqslant H$ and K is a subgroup of B such that $A \cap B \leqslant K$, then HK is a weakly mutually permutable product of H and K.*
(b) *If $A \cap B = 1$, then G is the totally permutable product of the subgroups A and B, that is, every subgroup of A permutes with every subgroup of B.*
(c) *If B permutes with a Sylow subgroup Q of A, then any subgroup of B containing $A \cap B$ permutes with Q.*

Proof. (a) and (b) are [3, Lemma 2.2]. For (c), if K is such that $A \cap B \leqslant K \leqslant B$, then for a Sylow subgroup Q of A, we have that $QK = Q((A \cap B)K) = (Q(A \cap B))K = K((A \cap B)Q) = KQ$, as required. □

Lemma 2.3. [3, Lemma 2.3] *Let $G = AB$ be the product of the subgroups A and B. If A permutes with every Sylow subgroup of B and B permutes with every Sylow subgroup of A, then $A \cap B$ also permutes with every Sylow subgroup of A and B. In particular, $A \cap B$ is a subnormal subgroup of G.*

Our next lemma studies the behaviour of minimal normal subgroups of weakly mutually permutable products containing the intersection of the factors.

Lemma 2.4. *Let $G = AB$ be the weakly mutually permutable product of subgroups A and B. If N is a minimal normal subgroup of G such that $A \cap B \leqslant N$, then either $A \cap N = B \cap N = 1$ or $N = (N \cap A)(N \cap B)$.*

Proof. Note that $A \cap N$ is a normal subgroup of A such that $A \cap B \leqslant A \cap N$ and so $H = (A \cap N)B$ is a subgroup of G. Observe that $N \cap H = N \cap (A \cap N)B = (A \cap N)(B \cap N)$. Since $N \cap H$ is a normal subgroup of H, we have B normalizes $N \cap H = (A \cap N)(B \cap N)$.

Arguing as above, we have that $K = A(B \cap N)$ is a subgroup of G such that $K \cap N = A(B \cap N) \cap N = (A \cap N)(B \cap N)$. Moreover A normalizes $K \cap N = (A \cap N)(B \cap N)$. Therefore $(A \cap N)(B \cap N)$ is a normal subgroup of G. By the minimality of N, it follows that $A \cap N = B \cap N = 1$ or $N = (N \cap A)(N \cap B)$ as required. □

Lemma 2.5. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation containing $\mathfrak{U}$ such that every group in $\mathfrak{F}$ has a Sylow tower of supersoluble type. Let G be a primitive group and let N be its unique minimal normal subgroup. Assume that G/N belongs $\mathfrak{F}$. If N is a p-group, where p is the largest prime dividing $|G|$, then $N = F(G) = O_p(G)$ is a Sylow p-subgroup of G.*

Proof. It is sufficient to show that N is a Sylow p-subgroup of G. Note that $G = NM$ for some maximal subgroup M of G, $N \cap M = 1$ and $C_G(N) = N$ since G is a primitive soluble group. By [6, A, Theorem 15.6(b)], $O_p(M) = 1$. But $M \cong G/N \in \mathfrak{F}$ which means that $M \in \mathfrak{F}$ and so has a Sylow tower of supersoluble type. Hence a Sylow p-subgroup of M is normal in M and so p does not divide $|M|$, as required. □

Lemma 2.6. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation containing $\mathfrak{U}$ such that every group in $\mathfrak{F}$ has a Sylow tower of supersoluble type. Let $G = AB$ be the weakly mutually permutable product of the subgroups A and B, where B is nilpotent and A is an $\mathfrak{F}$-subgroup. If B permutes with each Sylow subgroup of A, then the group G belongs to $\mathfrak{F}$.*

Proof. Suppose the result is not true and let G be a counterexample with $|G|$ minimal. We shall get to a contradiction by the following steps.

(a) G is a primitive soluble group with a unique minimal normal subgroup N and $N = C_G(N) = F(G) = O_p(G)$ for some prime p.

By [2, Lemma 2.5], G is soluble since A soluble. Let N be a minimal normal subgroup of G. Note that $G/N = (AN/N)(BN/N)$ satisfies the hypotheses of the theorem by Lemma 2.1 and this means that G/N belongs to $\mathfrak{F}$ by the minimality of G. It follows that G is a primitive soluble group since $\mathfrak{F}$ is saturated formation and so G has a unique minimal normal subgroup N with $N = C_G(N) = F(G) = O_p(G)$ for some prime p.

(b) We prove that $N = (N \cap A)(N \cap B)$, BN belongs to $\mathfrak{F}$ and $1 \neq A \cap B \leqslant N$.

If $A \cap B = 1$, then by Lemma 2.2(i), $G = AB$ is the totally permutable product of subgroups A and B . By [1, Theorem 5.2.1], G belongs $\mathfrak{F}$, a contradiction. Hence

$A \cap B \neq 1$. It follows that $A \cap B$ is a nilpotent subnormal subgroup of G using Lemma 2.3. Therefore $A \cap B \leqslant F(G) = N$ and so $N = (N \cap A)(N \cap B)$ by Lemma 2.4. Hence $BN = B(N \cap B)(N \cap A) = B(N \cap A)$ is the weakly mutually permutable product of B and $N \cap A$. Since $N \cap A$ has only one Sylow subgroup, namely itself, B trivially permutes with every Sylow subgroup of $N \cap A$. Notice that BN satisfies the hypotheses of theorem. If $BN < G$, then BN belongs to $\mathfrak{F}$ by the minimality of G. Assume that $G = BN$. Let $1 \neq M \leqslant A \cap B \leqslant N$. Since N is abelian, M is a normal subgroup of N. Hence $N = M^G = M^{NB} = M^B \leqslant B$ and $G = B$, a contradiction. Thus BN belongs to $\mathfrak{F}$, as required.

(c) N is the unique Sylow p-subgroup of G and p is the largest prime dividing $|G|$.

Let q be the largest prime dividing $|G|$ and assume that $q \neq p$. Suppose first that q divides $|BN|$. Note that BN belongs to $\mathfrak{F}$ and so has a Sylow tower of supersoluble type. It follows that BN has a unique Sylow q-subgroup, $(BN)_q$ say. This means that $(BN)_q$ centralises N. Since $C_G(N) = N$, we have that $(BN)_q = 1$ which is a contradiction. Therefore we may assume that q divides $|A|$ but does not divide $|BN|$. Since A also has a Sylow tower of supersoluble type, it follows that A has a unique Sylow q-subgroup, A_q say. This means that A_q is a normal subgroup of $A_q(N \cap A)$. Then $A_q(N \cap B) = A_q(A \cap B)(N \cap B)$ is the weakly mutually permutable product of $A_q(A \cap B)$ and $N \cap B$ by Lemma 2.2. Also, $N \cap B$ permutes with each Sylow subgroup of $A_q(A \cap B)$. Suppose that $A_q(N \cap B) < G$. Then $A_q(N \cap B)$ belongs to $\mathfrak{F}$ by the minimality of G. In particular, $A_q(N \cap B)$ has a unique Sylow q-subgroup since it has a Sylow tower of supersoluble type. Hence A_q is normalised by $N \cap B$. Hence A_q is normalised by $(N \cap A)(N \cap B) = N$. This means that A_q centralises N, a contradiction. We may assume that $A_q(N \cap B) = G$. Then $N \cap B = B$ and so B is an elementary abelian p-group. Moreover, $A = A_q(A \cap B)$. Then $A \cap B = N \cap A$ is a normal Sylow p-subgroup of A. Hence $A \cap B$ is normal in G because B is abelian. By the minimality of N, we have $N = A \cap B$, that is, $G = A_q(N \cap B) = A_q(A \cap B) = A$, a contradiction. Therefore p is the largest prime dividing $|G|$. By Lemma 2.5, N is the unique Sylow p-subgroup of G.

(d) N is a subgroup of A and N is not contained in B.

Suppose that N is contained in B. Then a Hall p'-subgroup $B_{p'}$ of B must centralise $N = C_G(N)$. Hence $B_{p'} = 1$ and B is a p-group. Then $G = AN$. Let $1 \neq M \leqslant A \cap B$. Then $N \leq M^G = M^{AN} = M^A \leqslant A$ and so $G = A$, a contradiction. Therefore N is not contained in B. Hence B has a non-trivial Hall p'-subgroup, $B_{p'}$ say, which is normal in B. Consequently, $AB_{p'} = A(A \cap B)B_{p'}$ is a subgroup of G. Then $1 \neq B_{p'}^G \leqslant AB_{p'}$ and so $N \leqslant AB_{p'}$. Hence $N \leqslant A$, as required.

(e) Final Contradiction

Let $A_{p'}$ be a Hall p'-subgroup of A. If $A_{p'} = 1$, then $G = BN$ belongs to $\mathfrak{F}$ by Step (b), a contradiction. Hence $A_{p'} \neq 1$. Since B permutes with every Sylow subgroup of A, it follows that $A_{p'}B$ is a subgroup of G. By Step (d), N is not contained in B. Hence $A_{p'}B$ is a proper subgroup of G. Since $NA_{p'}B = G$, it follows that $N \cap A_{p'}B = N \cap B$ is normal in G. The minimality of N implies that $N = N \cap B$ or $N \cap B = 1$. By Step (d), $N \neq N \cap B$. Therefore $N \cap B = 1$, and then $A \cap B \leqslant N \cap B = 1$, contradicting Step (b), our final contradiction. $\square$

Theorem 2.7. [7, Theorem 2.2] *Let $\mathfrak{F}$ be a subgroup-closed saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F} \subseteq w\mathfrak{U}$. Let $G = AB$ be a product of $\mathbb{P}$-subnormal subgroups A and B such $A \in \mathfrak{F}$ and B is nilpotent. If B permutes with each Sylow subgroup of A, then G belongs to $\mathfrak{F}$.*

Corollary 2.8. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F} \subseteq w\mathfrak{U}$. Let $G = AB$ be the mutually sn-permutable product of the $\mathfrak{F}$-subgroups A and B, where B is nilpotent. If B permutes with each Sylow subgroup of A, then the group G belongs to $\mathfrak{F}$.*

Proof. By [9, Lemma 4.5], A and B are $\mathbb{P}$-subnormal subgroups of G. Using Theorem 2.7, we have that $G \in \mathfrak{F}$, as required. □

We are in a position to prove Corollary B which we restate below:

Corollary 2.9. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F} \subseteq w\mathfrak{U}$. Let $G = AB$ be the weakly mutually sn-permutable product of the $\mathfrak{F}$-subgroups A and B, where B is nilpotent. If B permutes with each Sylow subgroup of A, then the group G belongs to $\mathfrak{F}$.*

Proof. The argument is the same as in the proof of Lemma 2.6, taking into consideration Corollary 2.8 and some appropriate preliminary results in [2]. □

3. Main Results

In this section we prove our main results, which we shall restate.

Theorem 3.1. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation containing $\mathfrak{U}$ such that every group in $\mathfrak{F}$ has a Sylow tower of supersoluble type. Let $G = AB$ be the weakly mutually permutable product of subgroups A and B. If B is nilpotent and permutes with each Sylow subgroup of A, then*

$$G^{\mathfrak{F}} = A^{\mathfrak{F}}.$$

Proof. Suppose the theorem is not true and let G be a counterexample with $|G|$ as small as possible. We shall get a contradiction by the following steps.

(a) $G^{\mathfrak{F}} = A^{\mathfrak{F}}N$ for each minimal normal subgroup N of G, $F(G) = O_p(G)$ for some prime p and $G^{\mathfrak{F}}$ is an abelian p-group. Moreover, $A \cap B \neq 1$ is a p-group.

Since $AG^{\mathfrak{F}}/G^{\mathfrak{F}} \cong A/(A \cap G^{\mathfrak{F}}) \in \mathfrak{F}$, we have that $A^{\mathfrak{F}} \leqslant G^{\mathfrak{F}}$. Hence $G^{\mathfrak{F}} \neq 1$. Moreover, by Lemma 2.6, we have that $A^{\mathfrak{F}} \neq 1$. Let N be a minimal normal subgroup of G such that $N \leqslant G^{\mathfrak{F}}$. Then G/N is the weakly mutually permutable product of AN/N and BN/N. Moreover, BN/N is nilpotent and permutes with each Sylow subgroup of AN/N. Hence $(G/N)^{\mathfrak{F}} = (AN/N)^{\mathfrak{F}}$ by the minimality of G. This implies that $G^{\mathfrak{F}} = A^{\mathfrak{F}}N$. Let N_1 be a minimal normal subgroup of G such that $N_1 \nleqslant G^{\mathfrak{F}}$. Then $N_1 \cap G^{\mathfrak{F}} = 1$ and $G^{\mathfrak{F}}N_1 = A^{\mathfrak{F}}N_1$. Moreover, $G^{\mathfrak{F}} = A^{\mathfrak{F}}(N_1 \cap G^{\mathfrak{F}}) = A^{\mathfrak{F}}$, a contradiction. This means that every minimal normal subgroup of G is contained in $G^{\mathfrak{F}}$ and so $G^{\mathfrak{F}} = A^{\mathfrak{F}}N$ for each minimal normal subgroup N of G.

We want to show that $G^{\mathfrak{F}}$ is abelian. If $A \cap B = 1$, then $G = AB$ is the totally permutable product of A and B and so $A^{\mathfrak{F}} = G^{\mathfrak{F}}$ by [5, Theorem 1], a contradiction. We may assume that $1 \neq A \cap B \leqslant F(G)$ by Lemma 2.3. Let N be a minimal normal subgroup of G which is contained in $F(G)$. Note that N is abelian. Suppose that N is contained in A. Since $A^{\mathfrak{F}}$ is a normal subgroup of A, N normalizes $A^{\mathfrak{F}}$ and so $A^{\mathfrak{F}}$ is a normal subgroup of $A^{\mathfrak{F}}N = G^{\mathfrak{F}}$. We also have that $G^{\mathfrak{F}}/A^{\mathfrak{F}}$ is abelian, which means that $(G^{\mathfrak{F}})' \leqslant A^{\mathfrak{F}}$. If $(G^{\mathfrak{F}})' \neq 1$, then $A^{\mathfrak{F}}$ contains a minimal normal subgroup N of G and therefore $G^{\mathfrak{F}} = A^{\mathfrak{F}}N = A^{\mathfrak{F}}$, a contradiction. We may assume that $(G^{\mathfrak{F}})' = 1$, that is, $G^{\mathfrak{F}}$ is abelian. Suppose that N is not contained in A. Consider $Y = AN$. Then $Y = A(Y \cap B)$ is the weakly mutually permutable product of A and $Y \cap B$. Moreover, $Y \cap B$ is nilpotent and permutes with each Sylow subgroup of A. If $Y < G$, then $Y^{\mathfrak{F}} = A^{\mathfrak{F}}$ and N normalizes $A^{\mathfrak{F}}$, which implies that $G^{\mathfrak{F}}$ is abelian since $(G^{\mathfrak{F}})' \leqslant A^{\mathfrak{F}}$. We assume that $G = Y = AN$. Since B is nilpotent and $A \cap B$ is a subnormal subgroup of A, $(A \cap B)[A \cap B, A] = (A \cap B)^A \leqslant (A \cap B)^G \leqslant F(G)$ and $[A \cap B, A]$ is contained in A. It follows that $[A \cap B, A]$ is a subnormal nilpotent subgroup of G. By [6, A, 14.3], $[A \cap B, A]$ is normalized by N. Since $[A \cap B, A]$ is a normal subgroup of A, we have that $[A \cap B, A]$ is a normal subgroup of $AN = G$. If $[A \cap B, A] = 1$, then $(A \cap B)^A = A \cap B$ is normalized by A and N, and thus $A \cap B$ is a normal subgroup of G. If $[A \cap B, A] \neq 1$, then it is a normal subgroup of G contained in A and in $F(G)$. In both cases, there exists N a minimal normal subgroup of G contained in $F(G)$ and in A. By the same argument as above, we have that $G^{\mathfrak{F}}$ is abelian.

We now show that $G^{\mathfrak{F}}$ is a p-group. Let N_2 be a minimal normal subgroup of G. Then $N_2 \leqslant G^{\mathfrak{F}}$ is an elementary abelian p-group for some prime p. Since $G^{\mathfrak{F}} = A^{\mathfrak{F}}N_2$ and $G^{\mathfrak{F}}/A^{\mathfrak{F}}$ is a p-group, we have that $O^p(G^{\mathfrak{F}}) \leqslant A^{\mathfrak{F}}$. If $O^p(G^{\mathfrak{F}}) \neq 1$, then $O^p(G^{\mathfrak{F}})$ is a normal subgroup of G, and $A^{\mathfrak{F}}$ contains a minimal normal subgroup of G. This means that $G^{\mathfrak{F}} = A^{\mathfrak{F}}$, a contradiction. Hence $O^p(G^{\mathfrak{F}}) = 1$, that is, $G^{\mathfrak{F}}$ is a p-group for some prime p. Since $Soc(G)$ is contained in $G^{\mathfrak{F}}$, we have that $F(G) = O_p(G)$.

(b) $G^{\mathfrak{F}}$ is contained in B and $A \cap B$ is the unique Sylow p-subgroup of A.

Consider $(A \cap B)A_{p'}$, where $A_{p'}$ is a Hall p'-subgroup of A and let X be a maximal subgroup of A containing $(A \cap B)A_{p'}$. Consider the subgroup $H = XB$ which is the weakly mutually permutable product of X and B. Note that B permutes with all Sylow q-subgroups of X, for $q \neq p$ since they are all Sylow q-subgroups of A and B also permutes with all Sylow p-subgroups of X since they all contain $A \cap B$ (note $A \cap B \leqslant O_p(X)$ which is contained in all Sylow p-subgroups of X). If $G = H$, then $A = X(A \cap B) = X$, a contradiction. Hence H is a proper subgroup of G, hence $H^{\mathfrak{F}} = X^{\mathfrak{F}}$ and so H normalizes $X^{\mathfrak{F}}$. Note that $X^{\mathfrak{F}} \leqslant A^{\mathfrak{F}}$. If $X^{\mathfrak{F}} \neq 1$, then $(X^{\mathfrak{F}})^G \leqslant (X^{\mathfrak{F}})^A \leqslant A^{\mathfrak{F}}$, a contradiction. Hence $H^{\mathfrak{F}} = X^{\mathfrak{F}} = 1$, that is H and X belong to $\mathfrak{F}$. Since X is a maximal subgroup of A, X is an $\mathfrak{F}$-projector of A. Since $A^{\mathfrak{F}}$ is abelian, $A^{\mathfrak{F}}X = A$ and $X \cap A^{\mathfrak{F}} = 1$ by [6, IV, 5.18]. This means that $G = A^{\mathfrak{F}}XB = G^{\mathfrak{F}}H$. By [6, III, 3.2], there exist an $\mathfrak{F}$-projector F of G containing H and so $G = G^{\mathfrak{F}}F$ and $F \cap G^{\mathfrak{F}} = 1$. Hence $G^{\mathfrak{F}} = A^{\mathfrak{F}}(G^{\mathfrak{F}} \cap XB) = A^{\mathfrak{F}}$, a contradiction. Consequently, $A = (A \cap B)A_{p'}$. In particular, $A^{\mathfrak{F}} \leqslant A \cap B \leqslant B$ and $A \cap B$ is the unique Sylow subgroup of A since $A \cap B$ is a subnormal subgroup of A. On the other hand, $1 \neq (A \cap B)^G = (A \cap B)^B \leq B$. Hence there exists N a minimal normal subgroup of G contained in B. Consequently $G^{\mathfrak{F}} = A^{\mathfrak{F}}N \leq B$.

(c) $AF(G)$ is a proper subgroup of G.

Suppose that $G = AF(G)$. Let Z be a maximal subgroup of G such that $A \leqslant Z$. Then $Z = A(Z \cap B)$ is the weakly mutually permutable product of A and $Z \cap B$. Note that $Z \cap B$ permutes with each Sylow subgroup of A by Lemma 2.2. By the minimality of G, we have $Z^{\mathfrak{F}} = A^{\mathfrak{F}}$, that is, $A^{\mathfrak{F}}$ is normal in Z. We also have that $G = ZF(G)$. Suppose that $G^{\mathfrak{F}}$ is not contained in Z. Then $G = G^{\mathfrak{F}}Z$ and so $A^{\mathfrak{F}} = Z^{\mathfrak{F}}$ is normal in G since $A^{\mathfrak{F}}$ is normal in $G^{\mathfrak{F}}$. Therefore $A^{\mathfrak{F}} = G^{\mathfrak{F}}$, a contradiction. We may assume that

$G^{\mathfrak{F}} \leqslant Z$. Let N be a minimal normal subgroup of G. Since $Soc(G) \leqslant G^{\mathfrak{F}}$ by Step (a), we have $N \leqslant G^{\mathfrak{F}}$. Note that $F(G)$ centralizes N. Hence N is also a minimal normal subgroup of Z. This means that either $N \cap Z^{\mathfrak{F}} \in \{1, N\}$. If $N \cap Z^{\mathfrak{F}} = N$, then N is contained in $A^{\mathfrak{F}}$, a contradiction. Suppose that $N \cap Z^{\mathfrak{F}} = 1$. Then $NZ^{\mathfrak{F}}/Z^{\mathfrak{F}}$ is a minimal normal subgroup of $Z/Z^{\mathfrak{F}}$. Since $Z/Z^{\mathfrak{F}} \in \mathfrak{F}$, we have N is $\mathfrak{F}$-central in Z and hence N is also $\mathfrak{F}$-central in G. This means that N is contained in every $\mathfrak{F}$-normalizer of G using [6, V, 3.2]. Let F be such an $\mathfrak{F}$-normalizer of G. Then $G = G^{\mathfrak{F}}F$ and $G^{\mathfrak{F}} \cap F = 1$, a contradiction.

(d) Final contradiction.

We want to show that $G = AP$, where P is the Sylow p-subgroup of G, to obtain our final contradiction. We also want to show that p is the largest prime dividing $|G|$. Suppose that for every q dividing $|B|$ and every Sylow q-subgroup Q of B, we have that AQ is a proper subgroup of G. Then $A(A \cap B)Q$ is the weakly mutually permutable product of A and $(A \cap B)Q$. By Lemma 2.2, $(A \cap B)Q$ permutes with each Sylow subgroup of A. Using the minimality of G, we have $(AQ)^{\mathfrak{F}} = A^{\mathfrak{F}}$. Therefore $A^{\mathfrak{F}}$ is normalized by every Sylow q-subgroup of B, that is, $A^{\mathfrak{F}}$ is normal in G, a contradiction. Hence $G = AQ$ for some Sylow q-subgroup of B, Q say. Suppose that $q \neq p$. Then $A^{\mathfrak{F}}$ is centralized by Q and that means $A^{\mathfrak{F}}$ is a normal subgroup of G, a contradiction. Hence $G = AP$, where P is a Sylow p-subgroup of B. In particular, B is a p-group and since $A = (A \cap B)A_{p'}$, we have that $G = A_{p'}P$ and P is a Sylow p-subgroup of G.

We now show that p is the largest prime dividing $|G|$. Let l be the largest prime dividing $|G|$ and L be a Sylow l-subgroup of G. Suppose $l \neq p$. We may assume that $L \leqslant A$. Note that $LG^{\mathfrak{F}}$ is a normal subgroup of G since $G/G^{\mathfrak{F}} \in \mathfrak{F}$ and hence has a Sylow tower of supersoluble type. Let Z be a maximal subgroup of G containing A. Then $Z = (Z \cap B)A$ is the weakly mutually permutable product of A and $Z \cap B$, and $Z \cap B$ permutes with each Sylow subgroup of A. By the minimality of G, $Z^{\mathfrak{F}} = A^{\mathfrak{F}}$. If $G^{\mathfrak{F}}$ is not contained in Z, then by the same argument in step (c), $A^{\mathfrak{F}}$ is a normal subgroup of G, a contradiction. Hence $G^{\mathfrak{F}}$ is contained in Z. Therefore $LG^{\mathfrak{F}}$ is contained in Z and so $(LG^{\mathfrak{F}})^{\mathfrak{F}} \leqslant Z^{\mathfrak{F}} = A^{\mathfrak{F}}$. If $(LG^{\mathfrak{F}})^{\mathfrak{F}} \neq 1$, then $(LG^{\mathfrak{F}})^{\mathfrak{F}}$ is a normal subgroup of G contained in $A^{\mathfrak{F}}$, a contradiction. This means that $LG^{\mathfrak{F}} \in \mathfrak{F}$. In particular, L is a normal subgroup of G since it is a characteristic subgroup of the normal subgroup $LG^{\mathfrak{F}}$. It follows that $L \leqslant F(G) = O_p(G)$, a contradiction. Therefore p is the largest prime dividing $|G|$. Since $G^{\mathfrak{F}} \leqslant P$ and $P/G^{\mathfrak{F}}$ is a normal subgroup of G, P is a normal subgroup of G. Hence $P = F(G)$ and so $G = AF(G)$, a contradiction to step (c). This contradiction concludes our proof. □

Theorem 3.2. *Let $\mathfrak{F}$ be a formation. Assume that either $\mathfrak{F} = \mathfrak{U}$ or $\mathfrak{F}$ is a saturated Fitting formation containing $\mathfrak{U}$. Let $G = AB$ be the weakly mutually permutable product of the subgroups A and B. Suppose that G' is nilpotent, A_1 is an $\mathfrak{F}$-normaliser of A such that $A \cap B \leqslant A_1$ and B_1 is an $\mathfrak{F}$-normaliser of B such that $A \cap B \leqslant B_1$, then A_1B_1 is an $\mathfrak{F}$-normaliser of G.*

Proof. Suppose the result is not true and let G be a counterexample with $|G|+|A|+|B|$ minimal. If A and B are both $\mathfrak{F}$-groups, then G is an $\mathfrak{F}$-group by [3, Lemma 2.6]. Hence we may assume without loss of generality that $1 \neq A^{\mathfrak{F}}$. Since $\mathfrak{F}$ is a saturated formation, $A^{\mathfrak{F}}$ is not contained in $\Phi(A)$. There is a subgroup T of A such that $F(A^{\mathfrak{F}}/(\Phi(A) \cap A^{\mathfrak{F}})) = T/(\Phi(A) \cap A^{\mathfrak{F}}) \neq 1$. Note that $T \cap \Phi(A) = A^{\mathfrak{F}} \cap \Phi(A)$. Using [6, V, 3.7], it follows that T is a nilpotent subgroup of G. Moreover, since $G^{\mathfrak{F}} \leq G'$, it follows that $G^{\mathfrak{F}}$ is nilpotent. Therefore T is a subnormal subgroup of G. Let M be a maximal subgroup of A such that T is not contained in M. Then $A = TM = A^{\mathfrak{F}}M = F(A)M$. Thus M is an $\mathfrak{F}$-critical maximal subgroup of A. By [6, V, 3.7], every $\mathfrak{F}$-normaliser of M is an $\mathfrak{F}$-normaliser of A. By [6, V, 3.2], $\mathfrak{F}$-normalisers of A are conjugate and so we may assume that $A_1 \leqslant M$. Note that $G = T(MB) = F(G)(MB) = G^{\mathfrak{F}}(MB)$. If $G = MB$, then G is the weakly mutually permutable product of subgroups M and B and also $|G| + |M| + |B| < |G| + |A| + |B|$. By the choice of G, A_1B_1 is an $\mathfrak{F}$-normaliser of G, a contradiction. We may assume that $MB < G$. Note that MB is a maximal $\mathfrak{F}$-critical subgroup of G. We have A_1B_1 is an $\mathfrak{F}$-normaliser of MB. Using [6, V, 3.7], we have that A_1B_1 is an $\mathfrak{F}$-normaliser of G, a contradiction. This concludes our proof. □

As we have said in the introduction, the result is also true under these same hypotheses for projectors.

The above result is not true for saturated Fitting formations containing $\mathfrak{U}$ when B is nilpotent as Example 1.3 shows.

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