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Fluctuations in One-Body Dynamics

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Fluctuations in One-Body Dynamics[*]

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Abstract:

   With a view towards nuclear dynamics at intermediate energies, we develop a transport treatment for systems whose reduced one-body phase-space density exhibits a markovian time evolution. A moment expansion of the corresponding Fokker-Planck equation leads to closed equations for the temporal evolution of the ensemble average of the phase-space occupancy $f(\mathbf{r}, \mathbf{p})$ and its correlation function. The general properties of the associated relaxation processes are illustrated and the utility of the formulation for studies of nuclear collisions is discussed.

1 Introduction

Nuclear collisions provide unique opportunities for creating and studying small many-fermion systems far from equilibrium. In recent years, the feasibility of making detailed experimental studies of nuclear collisions at intermediate energies has spurred the theoretical interest in developing the formal treatment of nuclear dynamics.

Many features of nuclear dynamics at lower energies, where the nucleons remain fairly degenerate, can be understood in terms of one-body models, such as the Time-Dependent Hartree-Fock model or its classical analog, the Vlasov model.[1] In these models, the one-body density matrix is propagated in a one-body (mean) field, which is usually determined self-consistently from the instantaneous one-body information. This treatment is appropriate at relatively low excitation, because the residual direct two-body interactions are suppressed by the Pauli blocking of the final nucleonic orbitals. (This characteristic feature of relatively cold nuclear systems is reflected in the rather long mean free path for the individual nucleons.) Since only the one-body density matrix is retained, these models are best suited for studying the evolution of one-body observables. (Although many-body observables can be calculated by simply assuming that the many-body density matrix is given in terms of the one-body density matrix as for independent particles, the quality of such an approximation would usually be rather poor.)

As the energy of the system is raised, the Pauli blocking becomes less effective (as evidenced by a shortening of the mean free path), and direct two-body collisions between the constituent nucleons grow increasingly important. Over the last several years, much effort has been devoted to incorporating this effect into the dynamical models. One possible avenue is to abandon the mean field approximation altogether and develop a molecular-dynamics model based directly on a specific two-body Hamiltonian, possibly incorporating a momentum-dependent repulsion to simulate the Pauli blocking.[2, 3, 4, 5, 6] Another, on which we shall concentrate in the present study, is to merely augment the Vlasov-type models by the effect of the two-body collisions by means of a Boltzmann-type collision term, incorporating suitable blocking factors. Such an approach was first conceived by Nordheim[7] for the treatment of an electronic gas in solids and soon thereafter also taken by Uehling and Uhlenbeck[8]. More recently, this general method has been adapted to nuclear dynamics under the label Boltzmann-Uehling-Uhlenbeck (*BUU*) or Vlasov-Uehling-Uhlenbeck (*VUU*), to indicate that a Pauli-blocked Boltzmann collision term has been added to the Vlasov equation.[9]

These latter approaches incorporate the average effect of the two-body collisions on the temporal evolution of the reduced one-body distribution, and so they fall in the category of *one-body* treatments. Thus, in this regard they are entirely analogous to pure one-body models, such as Vlasov and *TDHF*, where the system at any time is characterized by its one-body distribution, rather than the full many-body information. Moreover, because the approaches treat only the *average* effect of the two-body collisions, the evolution of the system, as given in terms of its one-body distribution, is entirely deterministic: A specified initial one-body distribution produces a single temporal trajectory in the abstract space of all possible one-body distributions, again

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in full analogy with the Vlasov and *TDHF* models. This basic feature severely limits the practical utility of the models to processes where the actual dynamics exhibits relatively little fluctuation. However, in typical nuclear collisions at intermediate energies, which is the energy regime where the two-body collisions play a significant role, there are generally many different final channels available and so the actual fluctuations can be substantial, particularly when fluctuations in final fragment multiplicity occurs. It is therefore of great practical interest to further develop the treatment so as to allow for dynamical fluctuations to develop.[10] It is on this task the present study is focussed.

Thus, our object of study is the *one-body* density matrix and we seek to develop a formally well-founded, yet practical, treatment of the fluctuations of this quantity induced by the underlying two-body collisions. The main aim is to show how the fluctuations can be incorporated in a one-body treatment within the framework of transport theory. Our treatment is conceptually similar to what is known in classical fluid mechanics as the Boltzmann-Langevin theory of hydrodynamic fluctuations, *i.e.* those fluctuations that appear in collective variables, even in steady states, due to the stochastic behavior of the constituents at the molecular level.[11] Bixon and Zwanzig took a step towards extending the theory of hydrodynamical fluctuations to non-equilibrium situations by deriving the Boltzmann-Langevin equation [12], where the stochastic behavior of the fluid is produced by a fluctuating force incorporated into the standard Boltzmann equation. The characteristics of this force can be explicitly related to the collision kernel. Bixon and Zwanzig furthermore showed that the Boltzmann-Langevin equation describes the fluctuations correctly near equilibrium, where it reproduces the standard hydrodynamical results for the fluctuations in pressure and heat current (for a review, see [13]). Specific studies for non-equilibrium systems, such as a gas subjected to a constant heat flux [14], have demonstrated the validity of the hydrodynamic-fluctuation theory in the presence of strong non-equilibrium constraints, at least when the gas is dilute.

The application of this approach to nuclear dynamics was pioneered by Ayik[15], who derived the statistical properties of the residual interaction by the general projection technique of quantum statistics. Recently, Ayik and Gregoire[16] have applied this general scheme to the particular situation where the residual interaction is approximated by the Uehling-Uhlenbeck collision term. In that work, explicit expressions were derived for the dynamical evolution of the fluctuations in the one-body density matrix. The present work can be seen as a further development along this line, in which we adapt standard techniques from transport theory for the derivation of closed equations for the first and second moments of the one-body phase-space distribution. This approach is motivated by the observation that although the full treatment of the fluctuations of the one-body density are too complex to allow practical calculations, even on the most powerful present computers, most problems of actual interest can be addressed without such degree of detail. While the equation derived for the first moment corresponds to the *BUU* treatment, the equations for the second moments are novel and give access to the variances and covariances of one-body observables within a one-body treatment.

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2 General background

We are interested in the temporal evolution of a many-nucleon system and shall adopt a semi-classical description so that the one-body information is represented in terms of the reduced phase-space density $f(\mathbf{r}, \mathbf{p})$, which can be considered as the classical analogue of the one-body Wigner distribution. In the absense of residual interactions, the evolution of the $f(\mathbf{r}, \mathbf{p})$ is governed by the Vlasov equation. The individual nucleons then move in a common one-body field, described by an effective Hamiltonian which is determined self-consistently from the instantaneous one-body density matrix or prescribed in some other manner. The inclusion of direct two-body collisions subjects the individual nucleons to irregular forces which in turn produces random changes in one-body the phase-space density f. Consequently, in such a scenario, the dynamical evolution exhibits a continual branching and the temporal trajectory of f is no longer a single trajectory but an ever widening bundle of trajectories. We shall seek to treat this diffusive behavior within the framework of transport theory.

2.1 Examples

In order to illustrate the problem addressed, we briefly describe two examples that display similar features, but may be more familiar to the reader.

2.1.1 Extended Hartree-Fock

In ordinary Hartree-Fock theory, the many-body system is represented by a single Slater determinant of the wave functions for the occupied single-particle orbitals k. This standard *HF* treatment can be extended by allowing the occupancies f_k of the individual single-particle orbitals k to take on fractional values between zero and one. A physical motivation for extending the ordinary *HF* to partial occupancy is the desire to encompass the effects of the residual two-particle interaction, such as pairing. The ensuing "generalized Slater determinant" should then be regarded as representing a complicated superposition of true Slater determinants whose occupancy coefficients are either zero or one. It may be helpful to note that the *EHF* system can be represented as the point $(f_1, f_2, \cdots)$ within an N-dimensional unit cube, where N is the number of single-particle orbitals included in the treatment, whereas a pure Slater determinant is situated at one of its corners. (The fact that the physical states are confined within a unit cube follows from the constraint $0 \leq f_k \leq 1$ on the occupancy coefficients.)

The *EHF* treatment can be made time dependent, resulting in time-dependent occupancies $f_k(t)$. The time evolution of the system can then be represented as a trajectory $(f_1(t), f_2(t), \cdots)$ within the unit cube. In the context of our present study, it is essential to note that the evolving *TDEHF* system produces a *single* trajectory in above-mentioned N-dimensional space of generalized Slater determinants, just as *BUU* produces a single trajectory in the space of one-body distributions.

A *diffusive* evolution of the *TDEHF* trajectory can be obtained by alllowing

stochastic changes in the occupancy coefficients $f_k(t)$, in addition the their smooth average change already included in the standard treatment. Within a specific model, the characteristics of the fluctuations in f_k can be derived from the underlying two-body interaction. The implementation of such a program would yield a model producing a stochastic evolution of the system, so that a single initial point will lead to an ever-spreading bundle of trajectories. The methods we are developing in the present work could readily be adapted to such a situation.

It is also important to note that in this specific example, the dynamical variables, the occupancy coefficients f_k, may take on fractional values, in spite of the fact that the system considered consists of fermions. This is so because, as mentioned already, each many-body system, as specified by the occupancy vector $(f_1, f_2, \cdots)$, represents a complicated superposition of pure Slater determinants, each of which has integer occupancy coefficients.

2.1.2   Collective nuclear dynamics

The dynamical situation under consideration is also somewhat similar to what occurs in nuclear fission and reaction processes, where the interest is focussed on a few macroscopic (or collective) variables, such as the shape of the system (or, more generally, the spatial distribution of the matter). In such familiar situations, our specification of the system, for example in terms of its density distribution $\rho(\mathbf{r})$, is incomplete, since there are many microscopically distinct manifestations of the system that have the same spatial distribution, and therefore characterizes an entire ensemble of many-body states. As in the above example the incompleteness in the specification of the system introduces an element of randomness and leads to a stochastic evolution of the macroscopic quantities. Thus, the macroscopic dynamics of a many-body system is inherently diffusive. Much effort has been devoted to developing models and methods for treating this aspect of nuclear dynamics in the context of damped reactions. In the present work we address dynamical problems at somewhat higher energies where the stochastic feature is considerably more pronounced.

2.2   Basic quantities

We consider an ensemble of identical many-body systems, labelled by $n = 1, ..., N$. It is important to note, that although the physical system consists of individual nucleons (and thus has a certain coarseness), the one-particle phase-space density, $f(\mathbf{r}, \mathbf{p})$, appears as a continous (and incompressible) fluid. (Of course, in numerical implementations it is often convenient to represent f in terms of individual test particles, as is commonly done in fluid-dynamical problems.) It is convenient to normalize the one-body phase-space density so that it measures the occupancy of an elementary phase-space cell of volume h^3, where h is Planck's constant. The local spatial density of nucleons is $\rho(\mathbf{r}) = h^{-3} \int d\mathbf{p} f(\mathbf{r}, \mathbf{p})$ and $\int d\mathbf{r}\rho(\mathbf{r}, \mathbf{p}) = A$, where A is total the number of nucleons in the system. In order to facilitate the presentation, we shall ignore the spin-isospin degrees of freedom so that we effectively have a one-component system. It is straightforward to extend the treatment to incorporate spin and isospin into the

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treatment, although the formulas become somewhat more involved.

Assume that the systems have been prepared in the same manner, so that their one-body phase-space densities $f^{(n)}(\mathbf{r}, \mathbf{p}; t = 0)$ are all equal, while the systems generally differ at the many-body level. Due to the stochastic nature of the residual two-body interaction, the individual many-body systems evolve differently and it is convenient to write the phase-space density of an individual system n on the form

$$f^{(n)}(\mathbf{r}, \mathbf{p}; t) = f(\mathbf{r}, \mathbf{p}; t) + \delta f^{(n)}(\mathbf{r}, \mathbf{p}; t) \ . \tag{1}$$

Here $f(\mathbf{r}, \mathbf{p}; t)$ is the average of the one-particle phase-space densities for all the individual systems in the ensemble,

$$f(\mathbf{r}, \mathbf{p}; t) \ \equiv \ \prec f^{(n)}(\mathbf{r}, \mathbf{p}; t) \succ \ = \ \frac{1}{N} \sum_{n=1}^{N} f^{(n)}(\mathbf{r}, \mathbf{p}; t) \ , \tag{2}$$

where we have introduced the notation $\prec \cdot \succ$ to denote the ensemble average. (The symbol $< \cdot >$ is reserved for the expectation value, in accordance with standard notation.) The quantity $\delta f^{(n)}$ in (1) denotes the deviation of the phase-space density for the individual system n from the ensemble average; by definition its ensemble average vanishes, $\prec \delta f^{(n)}(\mathbf{r}, \mathbf{p}; t) \succ = 0$.

In order to characterize the fluctuations of $f^{(n)}$ away from the ensemble average f it is convenient to introduce the following *correlation function* for the one-particle phase-space occupancy,

$$\begin{aligned} \sigma(\mathbf{r}, \mathbf{p}; \mathbf{r}', \mathbf{p}') \ \equiv \ \sigma_{ff'} \ &= \ \prec f^{(n)}(\mathbf{r}, \mathbf{p}) f^{(n)}(\mathbf{r}', \mathbf{p}') \succ \ - f(\mathbf{r}, \mathbf{p}) f(\mathbf{r}', \mathbf{p}') \\ &= \ \prec \delta f^{(n)}(\mathbf{r}, \mathbf{p}) \, \delta f^{(n)}(\mathbf{r}', \mathbf{p}') \succ \ , \end{aligned} \tag{3}$$

where (1) has been used.

Often, the focus is on some specific one-body observable(s), rather than the entire one-body density distribution. Such observables have an expectation value for each of the N systems and these are generally different, because the associated one-body densities $f^{(n)}$ differ. Consider a standard one-body observable of the form $\mathcal{A}(\mathbf{r}, \mathbf{p})$. Its expectation value for the system n can be written

$$\mathcal{A}^{(n)} \ = \ < \mathcal{A} >^{(n)} = \ \mathcal{A}[f^{(n)}] \ = \ \int \frac{d\mathbf{r} d\mathbf{p}}{h^3} \, \mathcal{A}(\mathbf{r}, \mathbf{p}) \, f^{(n)}(\mathbf{r}, \mathbf{p}) \ , \tag{4}$$

and the expectation value of $\mathcal{A}$ calculated with the ensemble-average phase-space occupancy $f(\mathbf{r}, \mathbf{p})$ is denoted by $\mathcal{A} = \mathcal{A}[f]$. It is elementary to show that $\prec \mathcal{A}^{(n)} \succ = \mathcal{A}$,

$$\prec \mathcal{A}^{(n)} \succ = \frac{1}{N} \sum_{n=1}^{N} \int \frac{d\mathbf{r} d\mathbf{p}}{h^3} \, \mathcal{A}(\mathbf{r}, \mathbf{p}) \, f^{(n)}(\mathbf{r}, \mathbf{p}) = \int \frac{d\mathbf{r} d\mathbf{p}}{h^3} \, \mathcal{A}(\mathbf{r}, \mathbf{p}) \, \frac{1}{N} \sum_{n=1}^{N} f^{(n)}(\mathbf{r}, \mathbf{p}) = \mathcal{A} \tag{5}$$

which can also be stated as $\prec \mathcal{A}[f^{(n)}] \succ \ = \ \mathcal{A}[\prec f^{(n)} \succ]$, or, in words, the evaluation of the ensemble average $\prec \cdot \succ$ commutes with the calculation of expectation value $< \cdot >$, for one-body observables.

The covariance between two standard one-body observables $\mathcal{A}$ and $\mathcal{B}$ can be expressed in terms of the correlation function (3),

$$
\begin{aligned}
\sigma_{\mathcal{A}\mathcal{B}} &= \prec \mathcal{A}^{(n)}\mathcal{B}^{(n)} \succ - \prec \mathcal{A}^{(n)} \succ \prec \mathcal{B}^{(n)} \succ \\
&= \frac{1}{N}\sum_{n=1}^{N} \int \frac{d\mathbf{r}d\mathbf{p}}{h^3} \int \frac{d\mathbf{r}'d\mathbf{p}'}{h^3} \, \mathcal{A}(\mathbf{r},\mathbf{p}) \, \mathcal{B}(\mathbf{r}',\mathbf{p}') \, f^{(n)}(\mathbf{r},\mathbf{p})f^{(n)}(\mathbf{r}',\mathbf{p}') - \mathcal{A}\mathcal{B} \\
&= \int \frac{d\mathbf{r}d\mathbf{p}}{h^3} \int \frac{d\mathbf{r}'d\mathbf{p}'}{h^3} \, \mathcal{A}(\mathbf{r},\mathbf{p}) \, \mathcal{B}(\mathbf{r}',\mathbf{p}') \, \sigma(\mathbf{r},\mathbf{p};\mathbf{r}',\mathbf{p}') \, .
\end{aligned}
\tag{6}
$$

2.3 Vlasov propagation

In the absence of two-body collisions, the individual particles move in the (time-dependent) field described by the effective one-body Hamiltonian $H(\mathbf{r},\mathbf{p},t)$, so their trajectories are governed by the equations

$$
\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} \, , \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{r}} \, .
\tag{7}
$$

This scenario corresponds to the Vlasov approximation, the classical analogue of the Time-Dependent Hartree-Fock approximation. The free propagation induces a change in the phase-space occupancy $f(\mathbf{r},\mathbf{p},t)$. With the above equations of motion, the change in the ensemble mean value is given by

$$
\dot{f}(\mathbf{r},\mathbf{p}) = \{H,f\} = \frac{\partial H}{\partial \mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{p}} - \frac{\partial H}{\partial \mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{r}} = -\left[\dot{\mathbf{p}}\cdot\frac{\partial}{\partial \mathbf{p}} + \dot{\mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{r}}\right]f(\mathbf{r},\mathbf{p})
\tag{8}
$$

where $\{\cdot,\cdot\}$ is the standard Poisson bracket. The corresponding evolution of the correlation function (3) is

$$
\begin{aligned}
\dot{\sigma}(\mathbf{r},\mathbf{p};\mathbf{r}',\mathbf{p}') &= \left[\frac{\partial H}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{p}} + \frac{\partial H}{\partial \mathbf{r}'}\cdot\frac{\partial}{\partial \mathbf{p}'} - \frac{\partial H}{\partial \mathbf{p}}\cdot\frac{\partial}{\partial \mathbf{r}} - \frac{\partial H}{\partial \mathbf{p}'}\cdot\frac{\partial}{\partial \mathbf{r}'}\right]\sigma(\mathbf{r},\mathbf{p};\mathbf{r}',\mathbf{p}') \\
&= -\left[\dot{\mathbf{p}}\cdot\frac{\partial}{\partial \mathbf{p}} + \dot{\mathbf{p}}'\cdot\frac{\partial}{\partial \mathbf{p}'} + \dot{\mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{r}} + \dot{\mathbf{r}}'\cdot\frac{\partial}{\partial \mathbf{r}'}\right]\sigma(\mathbf{r},\mathbf{p};\mathbf{r}',\mathbf{p}') \, .
\end{aligned}
\tag{9}
$$

In the subsequent section, we shall consider the stochastic part of the dynamical evolution. In order to simplify the presentation, the above Vlasov terms will be omitted, but they should of course be included in the final equations of motion.

3 Derivation of transport equations

The stochastic contribution to the evolution of the phase-space occupancy is relatively easy to treat when the random force is given in terms of the Uehling-Uhlenbeck collision term.

3.1 Transport formalism

In this subsection we review the necessary elements of transport theory, adapted to our present problem. The dynamical variable under consideration is the reduced one-particle phase-space density distribution, as given in terms of the occupancy function $f(\mathbf{r}, \mathbf{p})$ which takes on values between zero and one. For notational convenience, let us imagine that the one-particle phase space is divided into elementary cells of volume h^3 and enumerated by the index i. The dynamical variables are then the discrete set $f^{(n)} = (f_1^{(n)}, f_2^{(n)}, ...)$ for each system $n = 1, ..., N$ and the expectation value of an observable $\mathcal{A}$ can be written

$$\mathcal{A}^{(n)} = \int \frac{d\mathbf{r}d\mathbf{p}}{h^3} \, \mathcal{A}(\mathbf{r}, \mathbf{p}) \, f^{(n)}(\mathbf{r}, \mathbf{p}) = \sum_i \mathcal{A}_i \, f_i^{(n)} = \mathcal{A}[f^{(n)}] \; . \tag{10}$$

The ensemble $\{f^{(n)}\}$ of one-body phase-space densities is conveniently described by the distribution function

$$\phi[f'] = \phi(f_1', f_2', ...) = \frac{1}{N} \sum_n \delta(f_1^{(n)} - f_1') \, \delta(f_2^{(n)} - f_2') \, \cdots \; , \tag{11}$$

which gives the probability that a randomly selected system n has the specified occupancies $f' = (f_1', f_2', ...)$. (In this subsection, we use primed variables for the arbitrary distribution f' in order to avoid confusion with our convention that f denotes the ensemble average $\prec f^{(n)} \succ$.) It is easy the verify that $\phi[f']$ is normalized to unity, $\int df' \phi[f'] = 1$. The ensemble average of the observable $\mathcal{A}$ can then be written

$$\mathcal{A} = \prec \mathcal{A}^{(n)} \succ = \int df_1' \int df_2' \cdots \mathcal{A}(f_1', f_2', ...) \, \phi(f_1', f_2', ...) = \int df' \, \mathcal{A}[f'] \, \phi[f'] \; , \tag{12}$$

where the last expression is a functional integral over the distribution $f(\mathbf{r}, \mathbf{p})$.

In a transport treatment, the markovian evolution of the ensemble distribution function $\phi[f']$ is given by a master equation, which expresses the rate of change of $\phi(f_1', f_2', ...)$ in terms of specific transition rates between one value set $f' = (f_1', f_2', ...)$ and another $f'' = (f_1'', f_2'', ...)$. For the present review it is convenient to employ the Fokker-Planck approximation, according to which the evolution of $\phi[f']$ is governed by the differential equation

$$\frac{d}{dt}\phi[f'] = -\sum_k \frac{d}{df_k} \, V_k[f'] \, \phi[f'] + \sum_{kl} \frac{d^2}{df_k' df_l'} \, D_{kl}[f'] \, \phi[f'] \; , \tag{13}$$

where $V_i[f']$ are the *drift coefficients* and $D_{ij}[f']$ are the *diffusion coefficents*. The ensemble-average rate of change of the expectation value of a given observable $\mathcal{A}$ is then

$$\dot{\mathcal{A}} = \frac{d}{dt} \prec \mathcal{A}^{(n)} \succ = \sum_i \mathcal{A}_i \, \dot{f}_i = \int df' \, \mathcal{A}[f'] \, \dot{\phi}[f'] \; , \tag{14}$$

where it should be recalled that $f_i \equiv \prec f_i^{(n)} \succ$.

In particular, the ensemble-average rate of change of the occupancy of the phase-space cell i becomes

$$\dot{f}_i \equiv \frac{d}{dt} \prec f_i^{(n)} \succ = \int df' \, f_i' \, \dot{\phi}[f'] = \int df' \, V_i[f'] \, \phi[f'] = \prec V_i[f^{(n)}] \succ \approx \underline{V}_i . \quad (15)$$

Here we have inserted the Fokker-Planck equation of motion (13) and performed a partial integration, exploiting that for finite systems $\phi[f]$ vanishes at the boundary and for periodic systems the boundary contributions cancel. Since the observable f_i is (trivially) linear in f, only the first-order (drift) term contributes. In the last relation, the ensemble average of the drift coefficient has been approximated by its value calculated with the ensemble-average occupancies $f \equiv \prec f^{(n)} \succ$. Here and in the following we underline a quantity to indicate that it should be evaluated using ensemble-average occupancies, so $\underline{V}_i \equiv V_i[\prec f^{(n)} \succ] = V_i[f]$. The above equation (15) then provides a closed equation for the time evolution of the ensemble-average occupancies of the individual phase-space cells i and its solution is often referred to as the *mean trajectory* of the ensemble.

We now turn to the fluctuations around the mean evolution. These are most conveniently described in terms of the covariances (3)

$$\sigma_{ij} = \prec f_i^{(n)} f_j^{(n)} \succ - \prec f_i^{(n)} \succ \prec f_j^{(n)} \succ = \prec \delta f_i^{(n)} \, \delta f_j^{(n)} \succ . \quad (16)$$

By proceeding as above, we find

$$\dot{\sigma}_{ij} = \int df' \, \delta f_i' \, \delta f_j' \, \dot{\phi}[f'] \quad = \quad \prec D_{ij}^{(n)} + D_{ji}^{(n)} \succ + \prec \delta f_i^{(n)} V_j^{(n)} \succ + \prec \delta f_j^{(n)} V_i^{(n)} \succ$$

$$\approx \quad 2 \underline{D}_{ij} + \sum_k (\underline{V}_{jk}' \sigma_{ik} + \underline{V}_{ik}' \sigma_{jk}) . \quad (17)$$

The first term stems from the second-order (diffusion) term in the Fokker-Planck equation and it drives the growth of the fluctuations. We have used the brief notation $D_{ij}^{(n)} \equiv D_{ij}[f^{(n)}]$ for the diffusion coefficients associated with system n, and it has been used that these are symmetric, $D_{ij}^{(n)} = D_{ji}^{(n)}$. The second term arises from the variation of the drift coefficients with their argument f' and it produces a restoring term which resists the growth of σ_{ij} and thus ensures that a stationary value is approached in time (the equilibrium value). To derive the result (17), the drift coefficient has been expanded around the mean occupancy $f = \prec f^{(n)} \succ$,

$$V_i[f'] \approx V_i[f] + \sum_k V_{ik}'[f] \, \delta f_k' = \underline{V}_i + \sum_k \underline{V}_{ik}' \, \delta f_k' , \quad (18)$$

and its linear deviations then lead to the appearance of the covariances in (17). In order to use the equation (17), one must know the derivatives $V_{ij}' \equiv \partial V_i[f']/\partial f_j'$ along the mean trajectory. The same expansion leads to the result (15), since the first-order term gives a vanishing contribution.

Finally, it should be noted that at equilibrium the ensemble averages are stationary in time, $\dot{\phi}[\tilde{f}] = 0$. The equilibrium value $\tilde{f}$ is then characterized by $\underline{V}_i[\tilde{f}] = 0$ for all phase-space points i. This set of conditions yields as many independent equations as

there are phase-space cells, and so it can be solved for the equilibrium occupancies $\bar{f}$. Furthermore, the fluctuations at equilibrium are characterized by $\dot{\sigma}_{ij} = 0$. This latter condition yields a regular ($2N$-dimensional) set of equations for the equilibrium covariances $\tilde{\sigma}_{ij}$.

3.2 Two-body interaction

We assume that the system can be considered locally as a uniform gas of individual quasi-free nucleons. These nucleons may react with each other via a local two-body interaction characterized by the differential cross section $d\sigma/d\Omega$. In the present derivation only elastic scattering is admitted, so that $d\sigma(12 \to 1'2')/d\Omega = d\sigma(12 \leftarrow 1'2')/d\Omega$ and we may write $d\sigma(12 \leftrightarrow 1'2')/d\Omega$ to exhibit this symmetry. Though somewhat tedious, it would be straightforward to extend our treatment to include also inelastic processes, such as Δ formation.

The rate at which nucleons initially situated near the phase-space points $(\mathbf{r}_1, \mathbf{p}_1)$ and $(\mathbf{r}_2, \mathbf{p}_2)$ scatter into phase-space points near $(\mathbf{r}'_1, \mathbf{p}'_1)$ and $(\mathbf{r}'_2, \mathbf{p}'_2)$ is given by

$$
d\nu(12 \to 1'2') = \frac{d\mathbf{r}_1 d\mathbf{p}_1}{h^3} f(\mathbf{r}_1, \mathbf{p}_1) \frac{d\mathbf{r}_2 d\mathbf{p}_2}{h^3} f(\mathbf{r}_2, \mathbf{p}_2) \frac{d\mathbf{r}'_1 d\mathbf{p}'_1}{h^3} \bar{f}(\mathbf{r}'_1, \mathbf{p}'_1) \frac{d\mathbf{r}'_2 d\mathbf{p}'_2}{h^3} \bar{f}(\mathbf{r}'_2, \mathbf{p}'_2)
$$
$$
\times\, w(\mathbf{p}_1 \mathbf{p}_2 \leftrightarrow \mathbf{p}'_1 \mathbf{p}'_2)\, \delta(\mathbf{r}_1 - \mathbf{r}_2)\, \delta(\mathbf{r}_1 - \mathbf{r}'_1)\, \delta(\mathbf{r}_2 - \mathbf{r}'_2) \, . \tag{19}
$$

Here the spatial δ-functions arise from the fact that the interaction is local, and the vacancy factors $\bar{f}'_1$ and $\bar{f}'_2$ take account of the Pauli blocking of the final states. Finally, the basic transition rate is given by

$$
w(\mathbf{p}_1 \mathbf{p}_2 \leftrightarrow \mathbf{p}'_1 \mathbf{p}'_2) = \frac{h^6}{m^2}\, \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1 - \mathbf{p}'_2)\, \delta(\epsilon_1 + \epsilon_2 - \epsilon'_1 - \epsilon'_2)\, \frac{d\sigma(12 \leftrightarrow 1'2')}{d\Omega'_{12}} \, , \tag{20}
$$

where the δ-functions ensure conservation of momentum and energy. Due to these, the integral over the two final momenta can be reduced to an integral over the scattering angle,

$$
\int \frac{d\mathbf{p}'_1}{h^3} \int \frac{d\mathbf{p}'_2}{h^3}\, w(\mathbf{p}_1 \mathbf{p}_2 \leftrightarrow \mathbf{p}'_1 \mathbf{p}'_2) = |\mathbf{v}_1 - \mathbf{v}_2| \int d\Omega'_{12}\, \frac{d\sigma(12 \leftrightarrow 1'2')}{d\Omega'_{12}} \, . \tag{21}
$$

If the integrals over $\mathbf{p}'_1$ and $\mathbf{p}'_2$ are unrestricted, the corresponding integral over Ω' extends over the entire range of 4π. Correspondingly, if a factor of one half is attached on the left-hand side, in order to compensate for the double counting, then the angular integral is limited to one hemisphere.

The expression (19) gives the expected rate for the specific transition process considered. The probability for a collision to occur during a given small time interval Δt is then $P(12 \to 1'2') = d\nu(12 \to 1'2')\Delta t$. Each scattering actually occuring gives rise to local changes of the phase-space distribution $f(\mathbf{r}, \mathbf{p})$. On the basis of the above formulation, it is now a well-defined task to make a numerical implementation of this Pauli-blocked two-body scattering mechanism.

3.3 Occupancy at a single point

Since the two-body interaction is effective only between particles that have different momenta the evolution of the occupancy of a single phase-space cell, including its variance $\sigma_{ff} = \sigma_f^2$ (which is a measure of the fluctuation of the occupancy at that point), can be considered separately from the evolution of the occupancy covariances $\sigma_{ff'}$, which express the correlation between the fluctuations in occupancy at two different points of phase space.

Therefore, we first consider the change of the occupancy at a single phase-space point, as induced by the two-body collisions described above. There are two distinct sources for the change in $f(\mathbf{r}, \mathbf{p})$: 1) scattering *out of* the cell by a particle initially located at $(\mathbf{r}, \mathbf{p})$ and 2) scattering *into* the cell by a particle initially having a different momentum.

The expected rate of change due to the first type of process, the *loss* rate, is obtained by summing (19) over the state of the potential scattering partner, $(\mathbf{r}_2, \mathbf{p}_2)$, as well as over all possible final states, $(\mathbf{r}_1', \mathbf{p}_1')$ and $(\mathbf{r}_2', \mathbf{p}_2')$, yielding

$$
\begin{aligned}
\dot{f}^-(\mathbf{r}_1, \mathbf{p}_1) &= \frac{\Delta f(\mathbf{r}_1, \mathbf{p}_1)}{\Delta t} \\
&= -f(\mathbf{r}_1, \mathbf{p}_1) \int \frac{d\mathbf{r}_2 d\mathbf{p}_2}{h^3} f(\mathbf{r}_2, \mathbf{p}_2) \frac{1}{2} \int \frac{d\mathbf{r}_1' d\mathbf{p}_1'}{h^3} \bar{f}(\mathbf{r}_1', \mathbf{p}_1') \int \frac{d\mathbf{r}_2' d\mathbf{p}_2'}{h^3} \bar{f}(\mathbf{r}_2', \mathbf{p}_2') \\
&\quad \times w(\mathbf{p}_1 \mathbf{p}_2 \leftrightarrow \mathbf{p}_1' \mathbf{p}_2') \, \delta(\mathbf{r}_1 - \mathbf{r}_2) \, \delta(\mathbf{r}_1 - \mathbf{r}_1') \, \delta(\mathbf{r}_2 - \mathbf{r}_2') \qquad (22) \\
&= -f(\mathbf{r}_1, \mathbf{p}_1) \int \frac{d\mathbf{p}_2}{h^3} f(\mathbf{r}_1, \mathbf{p}_2) \frac{1}{2} \int \frac{d\mathbf{p}_1'}{h^3} \bar{f}(\mathbf{r}_1, \mathbf{p}_1') \int \frac{d\mathbf{p}_2'}{h^3} \bar{f}(\mathbf{r}_1, \mathbf{p}_2') \, w(\mathbf{p}_1 \mathbf{p}_2 \leftrightarrow \mathbf{p}_1' \mathbf{p}_2') \\
&= -f(\mathbf{r}_1, \mathbf{p}_1) \, W^-(\mathbf{r}_1, \mathbf{p}_1) \ .
\end{aligned}
$$

The factor of one half occurs in order to compensate for the double counting of the final two-particle states. The expected rate of change due to the second type of process, the *gain* rate, is obtained by summing (19) over the final state of the scattering partner, as well as over the initial states of the two scattering particles, so

$$
\begin{aligned}
\dot{f}^+(\mathbf{r}_1', \mathbf{p}_1') &= \frac{\Delta f(\mathbf{r}_1', \mathbf{p}_1')}{\Delta t} \\
&= \bar{f}(\mathbf{r}_1', \mathbf{p}_1') \frac{1}{2} \int \frac{d\mathbf{r}_1 d\mathbf{p}_1}{h^3} f(\mathbf{r}_1, \mathbf{p}_1) \int \frac{d\mathbf{r}_2 d\mathbf{p}_2}{h^3} f(\mathbf{r}_2, \mathbf{p}_2) \int \frac{d\mathbf{r}_2' d\mathbf{p}_2'}{h^3} \bar{f}(\mathbf{r}_2', \mathbf{p}_2') \\
&\quad \times w(\mathbf{p}_1 \mathbf{p}_2 \leftrightarrow \mathbf{p}_1' \mathbf{p}_2') \, \delta(\mathbf{r}_1 - \mathbf{r}_2) \, \delta(\mathbf{r}_1 - \mathbf{r}_1') \, \delta(\mathbf{r}_2 - \mathbf{r}_2') \qquad (23) \\
&= \bar{f}(\mathbf{r}_1', \mathbf{p}_1') \frac{1}{2} \int \frac{d\mathbf{p}_1}{h^3} f(\mathbf{r}_1', \mathbf{p}_1) \int \frac{d\mathbf{p}_2}{h^3} f(\mathbf{r}_1', \mathbf{p}_2) \int \frac{d\mathbf{p}_2'}{h^3} \bar{f}(\mathbf{r}_1', \mathbf{p}_2') \, w(\mathbf{p}_1 \mathbf{p}_2 \leftrightarrow \mathbf{p}_1' \mathbf{p}_2') \\
&= \bar{f}(\mathbf{r}_1', \mathbf{p}_1') \, W^+(\mathbf{r}_1', \mathbf{p}_1') \ .
\end{aligned}
$$

where the factor of one half compensates for the double counting of initial nucleon pairs. In the above two equations, we have introduced the rate functions $W^\pm$ which can be written in the form

$$
W^-(\mathbf{r}_a, \mathbf{p}_a) =
$$
$$
\frac{1}{2} \int \frac{d\mathbf{p}_1}{h^3} \int \frac{d\mathbf{p}_2}{h^3} \int \frac{d\mathbf{p}_1'}{h^3} \int \frac{d\mathbf{p}_2'}{h^3} \delta(\mathbf{p}_1 - \mathbf{p}_a) f(\mathbf{r}_a, \mathbf{p}_2) \bar{f}(\mathbf{r}_a, \mathbf{p}_1') \bar{f}(\mathbf{r}_a, \mathbf{p}_2') \, w(12 \leftrightarrow 1'2') \ ,
$$

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$$W^+(\mathbf{r}_a, \mathbf{p}_a) =$$
$$\frac{1}{2} \int \frac{d\mathbf{p}_1}{h^3} \int \frac{d\mathbf{p}_2}{h^3} \int \frac{d\mathbf{p}_1'}{h^3} \int \frac{d\mathbf{p}_2'}{h^3} f(\mathbf{r}_a, \mathbf{p}_1) f(\mathbf{r}_a, \mathbf{p}_2) \delta(\mathbf{p}_1' - \mathbf{p}_a) \bar{f}(\mathbf{r}_a, \mathbf{p}_2') \, w(12 \leftrightarrow 1'2') \, .$$

The temporal evolution of $f(\mathbf{r}, \mathbf{p})$ can be regarded as a random walk where the two quantities $\dot{f}^+(\mathbf{r}, \mathbf{p})$ and $\dot{f}^-(\mathbf{r}, \mathbf{p})$ give the respective mean rates of transition into and out of an elementary phase-space cell of volume h^3 situated around $(\mathbf{r}, \mathbf{p})$. The actual number k of such "downward" or "upward" steps during a small time interval Δt is governed by a Poisson distribution, i.e.

$$P_k^- = \frac{(\dot{f}^- \Delta t)^k}{k!} \, e^{-\dot{f}^- \Delta t} \, , \quad P_k^+ = \frac{(\dot{f}^+ \Delta t)^k}{k!} \, e^{-\dot{f}^+ \Delta t} \, . \tag{25}$$

It is now possible to express the transport coefficients for the occupancy $f(\mathbf{r}, \mathbf{p})$. Reading the relation (15) backwards, we see that the drift coefficient for the occupancy at a given point, $V(\mathbf{r}, \mathbf{p})$, is equal to the average rate of change of the occupancy, the average being performed with respect to an ensemble that is prepared so that all its members have the same occupancies at the beginning of the small time interval considered. Furthermore, from (17) we see that the diffusion coefficient is equal to the early growth rate of the corresponding ensemble variance of the value of the occupancy at the particular point considered. Therefore, we find

$$V(\mathbf{r}, \mathbf{p}) = \frac{\Delta f}{\Delta t} = \dot{f}^+(\mathbf{r}, \mathbf{p}) - \dot{f}^-(\mathbf{r}, \mathbf{p}) = \bar{f}(\mathbf{r}, \mathbf{p}) W^+(\mathbf{r}, \mathbf{p}) - f(\mathbf{r}, \mathbf{p}) W^-(\mathbf{r}, \mathbf{p}) \, , \tag{26}$$

$$2D(\mathbf{r}, \mathbf{p}) = \frac{\Delta \sigma_f^2}{\Delta t} = \dot{f}^+(\mathbf{r}, \mathbf{p}) + \dot{f}^-(\mathbf{r}, \mathbf{p}) = \bar{f}(\mathbf{r}, \mathbf{p}) W^+(\mathbf{r}, \mathbf{p}) + f(\mathbf{r}, \mathbf{p}) W^-(\mathbf{r}, \mathbf{p}) \, .$$

We have here used that for a Poisson distribution the variance is equal to the mean value, $\sigma_k^2 \equiv <k^2> - <k>^2 = <k> = \dot{f}^\pm$, together with the fact that the total accumulated variance is the sum of the variances accumulated from the upward and downward steps separately.

It is readily seen from (26) that the derivative of the drift coefficient for the occupancy at a given point with respect to the occupancy at that *same* point is given by $\partial V(\mathbf{r}, \mathbf{p})/\partial f(\mathbf{r}, \mathbf{p}) = -W^+(\mathbf{r}, \mathbf{p}) - W^-(\mathbf{r}, \mathbf{p})$. Using the results (15) and (17) from 3.1, the equations of motion for the mean occupancy the associated variance are then

$$\frac{d}{dt} f = \bar{f} \underline{W}^+ - f \underline{W}^- \, , \tag{27}$$

$$\frac{d}{dt} \sigma_f^2 = 2\underline{D} + 2(\frac{\partial V}{\partial f}) \sigma_f^2 = \bar{f} \underline{W}^+ + f \underline{W}^- - 2(\underline{W}^+ + \underline{W}^-) \sigma_f^2 \, , \tag{28}$$

with an abbreviated but fairly obvious notation.

When the system has reached equilibrium, the ensemble moments of the occupancy remain constant in time, i.e. $\dot{f} = 0$ and $\dot{\sigma}_{ff} = 0$. The first condition yields the relation $\prec f W^- \succ = \prec \bar{f} W^+ \succ$, which amounts to $f \underline{W}^- \approx \bar{f} \underline{W}^+$, as would

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be expected on the basis of detailed balance. The equation for the mean occupancy $f(\mathbf{r}, \mathbf{p})$ can be rewritten on the following form, $\dot{f} = W^+ - (W^- + W^+)f$. From this we immediately see that the instantaneous equilibrium value is $\tilde{f} = W^+/(W^- + W^+)$ and, moreover, that the instantaneous local relaxation time $t_f(\mathbf{r}, \mathbf{p})$ is given by $t_f^{-1} = W^-(\mathbf{r}, \mathbf{p}) + W^+(\mathbf{r}, \mathbf{p})$. These quantities pertain to the system at the particular time at which it is being considered and they change due to the general dynamical evolution of the system. In the special case of statistical equilibrium, the occupancy is given by a Fermi-Dirac distribution, $\tilde{f} = 1/(1 + \exp[(\epsilon - \mu)/\tau])$, so in that situation we have $W^-/W^+ = \exp[(\epsilon - \mu)/\tau]$.

The equilibrium condition for the variance yields $\tilde{\sigma}_f^2 = (\bar{f}W^+ + fW^-)/2(W^+ + W^-)$ for the instantaneous equilibrium variance $\tilde{\sigma}_f^2$. When the mean value has reached equilibrium, this relation amounts to $\tilde{\sigma}_f^2 = f\bar{f}$. This is recognized as the standard result for the variance of the Fermi-Dirac occupancy at equilibrium. In order to examine the approach to equilibrium for the variance of the occupancy, it is instructive to consider its deviation from the instantaneous equilibrium value, $\Delta \equiv \sigma^2 - f\bar{f}$. An elementary calculation readily yields

$$\dot{\Delta} = \dot{\sigma}^2 - \dot{f}\bar{f} - f\dot{\bar{f}} = -2(\underline{W^-} + \underline{W^+})\Delta . \tag{29}$$

This results demonstrates that the variance of the single-particle occupancy relaxes at twice the rate characterizing the relaxation of the mean occupancy, as is ordinarily the case in transport problems.

The relaxation time t_f can be estimated by recalling that $W^-(\mathbf{r}, \mathbf{p})$ is the rate at which a nucleon with the specified momentum $\mathbf{p}$ suffers collisions when moving in the neighborhood of $\mathbf{r}$, *i.e.* $W^- \approx v/\lambda$, where v is the speed of the nucleon and λ is its mean free path (which also depends on the velocities of the surrounding nucleons). For temperatures of a few MeV, and for particles near the Fermi surface, the mean free path is typically 6-8 fm. Using $\bar{f} \approx \frac{1}{2}$, we then find $t_f \approx 0.4 \cdot 10^{-22}$ s for the relaxation of the single-particle occupancy. This time is about an order of magnitude shorter than the duration typical of a nuclear collision at intermediate energy and so we would expect the variances in the single-particle occupancies to track closely their adiabatic values (*i.e.* their instantaneous equilibrium values $\tilde{\sigma}_f^2 = f\bar{f}$).

3.4 Covariance between occupancies at different points

We now turn to the consideration of the correlation between the fluctuations in the single-particle occupancies at two *different* points in phase space, $(\mathbf{r}_a, \mathbf{p}_a)$ and $(\mathbf{r}_b, \mathbf{p}_b)$. The definition of the ensemble covariance is given in (3). The corresponding diffusion coefficient can be calculated by considering the early evolution of the product of the induced changes in occupation at the two phase-space points considered. Since each of these two points can be either an initial state or a final state, there are four distinct types of scattering process contributing and, accordingly, the diffusion coefficient may

be written in the form[1]

$$2D(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b) \equiv \frac{\prec \Delta f(\mathbf{r}_a, \mathbf{p}_a) \Delta f(\mathbf{r}_b, \mathbf{p}_b) \succ}{\Delta t} \tag{30}$$
$$= [f_a f_b W_{ab}^{--} + \bar{f}_a \bar{f}_b W_{ab}^{++} - f_a \bar{f}_b W_{ab}^{-+} - \bar{f}_a f_b W_{ab}^{+-}] \, \delta(\mathbf{r}_{ab}) \ .$$

For convenience, we have here used the condensed notation $f_a \equiv f(\mathbf{r}_a, \mathbf{p}_a)$ and $f_b \equiv f(\mathbf{r}_b, \mathbf{p}_b)$. Furthermore, we have introduced the following four quantities

$$W_{ab}^{--}(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b) \equiv \frac{1}{2} \int \frac{d\mathbf{p}_1'}{h^3} \int \frac{d\mathbf{p}_2'}{h^3} \, \bar{f}(\mathbf{r}_a, \mathbf{p}_1') \bar{f}(\mathbf{r}_b, \mathbf{p}_2') \, w(\mathbf{p}_a \mathbf{p}_b \leftrightarrow \mathbf{p}_1' \mathbf{p}_2'), \tag{31}$$

$$W_{ab}^{++}(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b) \equiv \frac{1}{2} \int \frac{d\mathbf{p}_1}{h^3} \int \frac{d\mathbf{p}_2}{h^3} \, f(\mathbf{r}_a, \mathbf{p}_1) f(\mathbf{r}_b, \mathbf{p}_2) \, w(\mathbf{p}_1 \mathbf{p}_2 \leftrightarrow \mathbf{p}_a \mathbf{p}_b), \tag{32}$$

$$W_{ab}^{-+}(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b) \equiv \int \frac{d\mathbf{p}_2}{h^3} \int \frac{d\mathbf{p}_2'}{h^3} \, f(\mathbf{r}_a, \mathbf{p}_2) \bar{f}(\mathbf{r}_b, \mathbf{p}_2') \, w(\mathbf{p}_a \mathbf{p}_2 \leftrightarrow \mathbf{p}_b \mathbf{p}_2'), \tag{33}$$

$$W_{ab}^{+-}(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b) \equiv \int \frac{d\mathbf{p}_2}{h^3} \int \frac{d\mathbf{p}_2'}{h^3} \, f(\mathbf{r}_a, \mathbf{p}_2) \bar{f}(\mathbf{r}_b, \mathbf{p}_2') \, w(\mathbf{p}_b \mathbf{p}_2 \leftrightarrow \mathbf{p}_a \mathbf{p}_2'). \tag{34}$$

It should be noted that $W_{ab}^{-+} = W_{ba}^{+-}$. Furthermore, these quantities are related to those given in (24) by the following sum rules,

$$W_a^- = W^-(\mathbf{r}, \mathbf{p}_a) = \int \frac{d\mathbf{p}_b}{h^3} \, f(\mathbf{r}, \mathbf{p}_b) \, W_{ab}^{--} = \int \frac{d\mathbf{p}_b}{h^3} \, \bar{f}(\mathbf{r}, \mathbf{p}_b) \, W_{ab}^{-+} \ , \tag{35}$$

$$W_a^+ = W^+(\mathbf{r}, \mathbf{p}_a) = \int \frac{d\mathbf{p}_b}{h^3} \, \bar{f}(\mathbf{r}, \mathbf{p}_b) \, W_{ab}^{++} = \int \frac{d\mathbf{p}_b}{h^3} \, f(\mathbf{r}, \mathbf{p}_b) \, W_{ab}^{+-} \ . \tag{36}$$

The diffusion coefficient drives the growth of the covariance. The restoring terms arise from the dependence of the drift coefficients on the occupancies. From (27) and (24) we find

$$\frac{\partial V(\mathbf{r}_a, \mathbf{p}_a)}{\partial f(\mathbf{r}_b, \mathbf{p}_b)} = -(W_a^+ + W_a^-) \, h^3 \delta(\mathbf{r}_{ab}) \delta(\mathbf{p}_{ab}) \tag{37}$$
$$+ \bar{f}_a(-W_{ab}^{+-} + W_{ab}^{++}) \delta(\mathbf{r}_{ab}) - f_a(-W_{ab}^{-+} + W_{ab}^{--}) \delta(\mathbf{r}_{ab}) \ ,$$

where, for example, $W^+ \equiv W^+(\mathbf{r}, \mathbf{p})$ and $W_{ab}^{+-} \equiv W^{+-}(\mathbf{r}; \mathbf{p}_a, \mathbf{p}_b)$, with $\mathbf{r} \equiv \frac{1}{2}(\mathbf{r}_a + \mathbf{r}_b)$ and $\mathbf{p} \equiv \frac{1}{2}(\mathbf{p}_a + \mathbf{p}_b)$. This derivative differs from the one needed for the variance in section 3.3 because of the conceptual difference between the covariance and the variance (see footnote on page 13).

---

[1] It is important to recognize that the variance and the covariance are conceptually different quantities, because a scattering process is only considerred to have occurrred if the final state differs from the initial one. The fact that scattering can only take place between *different* phase-space points is guaranteed by the vanishing of the relative speed v_{12} when $\mathbf{p}_1 = \mathbf{p}_2$ (so that the scattering probability is zero), and the possibility that particles with different momenta merely exchange their momenta (thus producing no observable change) is suppressed in proportion to the fineness of the phase-space grid employed (or can be explicitly prohibited) and vanishes in the continuum limit. Thus the variance is related to the fluctuations in the scatterings experienced near a single phase-space point, whereas the covariance is a non-local quantity which does *not* reduce to the variance when the two phase-space points on which it depends approach each other - rather, it then tends to zero, because no scattering occurs between particles with nearly identical momenta.

According to (17), the equations of motion for the occupancy covariances are then given by

$$\dot{\sigma}(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b) = 2D(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b)$$

$$+ \int \frac{d\mathbf{r}_c d\mathbf{p}_c}{h^3} \left[\frac{\partial V(\mathbf{r}_b, \mathbf{p}_b)}{\partial f(\mathbf{r}_c, \mathbf{p}_c)} \sigma(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_c, \mathbf{p}_c) + \frac{\partial V(\mathbf{r}_a, \mathbf{p}_a)}{\partial f(\mathbf{r}_c, \mathbf{p}_c)} \sigma(\mathbf{r}_b, \mathbf{p}_b; \mathbf{r}_c, \mathbf{p}_c) \right]$$

$$= (f_a f_b W_{ab}^{--} + \bar{f}_a \bar{f}_b W_{ab}^{++} - f_a \bar{f}_b W_{ab}^{-+} - \bar{f}_a f_b W_{ab}^{+-}) \, \delta(\mathbf{r}_{ab})$$

$$- (W_a^+ + W_a^- + W_b^+ + W_b^-) \, \sigma(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b) \tag{38}$$

$$- \int \frac{d\mathbf{p}_c}{h^3} \left[\left(\bar{f}_b(W_{bc}^{+-} + W_{bc}^{++}) - f_b(W_{bc}^{-+} + W_{bc}^{--}) \right) \sigma(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_c) \right.$$

$$\left. + \left(\bar{f}_a(W_{ac}^{+-} + W_{ac}^{++}) - f_a(W_{ac}^{-+} + W_{ac}^{--}) \right) \sigma(\mathbf{r}_a, \mathbf{p}_c; \mathbf{r}_b, \mathbf{p}_b) \right] .$$

We remind of the fact that the effect of the Vlasov propagation, as given by (9), should be added on the right-hand side of the above equation for the covariance.

3.4.1 Relaxation and equilibrium

The above equation for the covariance $\sigma(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b; t)$ is fairly involved. However, its main features are easy to understand. The first term on the right-hand side acts as the source term for the correlations arising from the scattering of two particles situated near the same point in space (hence the δ-function in the separation $\mathbf{r}_{ab}$). The strength of the source term is simply the total rate at which particles at the two specified phase-space points are engaged with each other, each point acting either as an initial state or as a final state (hence the four different terms). The second line is the diagonal degradation term acting to destroy the correlations in the course of time, as a consequence of the collisions the two particles suffer subsequent to the particular collision that generated the correlation being propagated. The strength of the degradation is recognized as the total rate of scattering into or out of either one of the phase-space cells a and b. The off-diagonal terms represent the effect of the scattering of one of the particles (a or b) with a third particle c with which the other particle (b or a, respectively) is correlated. This contribution is obviously the most tedious one to calculate; the extent to which it may be approximated or ignored requires further study. Finally, the last term (not shown in (38)) represents the Vlasov propagation, *i.e.* the propagation in the absense of two-body scatterings.

If there are no forces associated with the mean field, and if we ignore the off-diagonal degradation terms, the equation of motion (38) takes on a relatively tractable form,

$$\dot{\sigma}(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b; t) = W_{ab} \, \delta(\mathbf{r}_{ab}) - (W_{a+b} + \mathbf{v}_a \cdot \frac{\partial}{\partial \mathbf{r}_a} + \mathbf{v}_b \cdot \frac{\partial}{\partial \mathbf{r}_b}) \, \sigma , \tag{39}$$

where $W_{ab} \equiv f_a f_b W_{ab}^{--} + ...$ and $W_{a+b} \equiv W_a^+ + ...$ for brevity. This equation is amenable to solution by elementary means. By performing a Fourier analysis, it can readily be shown that the stationary solution is given by

$$\tilde{\sigma}(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b) = \theta(\mathbf{r}_{ab} \cdot \hat{\mathbf{v}}_{ab}) \, \delta(\mathbf{r}_{ab}^\perp) \, \frac{W_{ab}}{v_{ab}} \, \exp\left(-\frac{W_{a+b}}{v_{ab}} r_{ab}\right) , \tag{40}$$

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where $\mathbf{r}_{ab}^{\perp} = \mathbf{r}_{ab} - \mathbf{r}_{ab} \cdot \hat{\mathbf{v}}_{ab}$ is the two-dimensional projection of the relative position $\mathbf{r}_{ab}$ onto the plane perpendicular to the direction of relative motion $\hat{\mathbf{v}}_{ab} = \mathbf{v}_{ab}/v_{ab}$. This result can be understood as follows. The truncation function θ together with the constraint function δ act to ensure that correlations between the two particles can only exist provided they have scattered (so that their relative motion is directed away from their center of mass). If they are indeed moving back-to-back, their correlation falls off exponentially as a function of their separation r_{ab}, with a *correlation length* $\lambda_{ab} = v_{ab}/W_{a+b}$, which is the average separation the two particles will have achieved by the time either one of them suffers a next scattering. This quantity is related to the two mean free paths by $\lambda_{ab}^{-1} = \lambda_{a}^{-1} + \lambda_{b}^{-1}$. The maximum correlation thus occurs right after the particles have scattered and is given by W_{ab}/v_{ab}, which is quite reasonable: the correlation grows in proportion to the engagement rate W_{ab} and is inversely proportional to the relative speed v_{ab}.

The mean correlation between two particles with a separation r_{ab} can be obtained by averaging over the direction $\hat{\mathbf{r}}_{ab}$. The corresponding equilibrium value is then

$$< \tilde{\sigma}(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b) >_{\hat{\mathbf{r}}_{ab}} = \frac{1}{4\pi r_{ab}^2} \frac{W_{ab}}{v_{ab}} \exp\left(-\frac{W_{a+b}}{v_{ab}} r_{ab}\right) , \tag{41}$$

as might be intuitively expected. Furthermore, the total amount of correlation of a given particle with any other particle having a specified momentum can be calculated by integrating $\sigma(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b)$ over the relative position $\mathbf{r}_{ab}$. This quantity is identical to the contracted covariance defined below in (43). It then follows that in a constant mean field the equilibrium value of the contracted covariance amounts to

$$
\begin{aligned}
\hat{\sigma}(\mathbf{p}_a, \mathbf{p}_b) &= \int d\mathbf{r}_{ab} \, \tilde{\sigma}(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b) = \frac{W_{ab}}{W_{a+b}} \\
&= \frac{f_a f_b W_{ab}^{--} + \bar{f}_a \bar{f}_b W_{ab}^{++} - f_a \bar{f}_b W_{ab}^{-+} - \bar{f}_a f_b W_{ab}^{+-}}{W_a^+ + W_a^- + W_b^+ + W_b^-} ,
\end{aligned}
\tag{42}
$$

where the full expressions have been exhibited.

3.4.2 Contracted covariances

The above analysis illustrates the most important features of the equation (38) for the evolution of the covariance. The discussion suggests that the covariance typically falls off over a distance of the order of the mean free path, also when the off-diagonal degradation terms are included. Thus, to within this tolerance, the correlation function is local in the positions, and calculational advantage can be gained by eliminating the relative separation $\mathbf{r}_{ab}$ as an argument. In order to achieve this, let us use the symbol $\hat{\ }$ over a two-point quantity to indicate that a contraction has been performed over the spatial separation of its two arguments. For example,

$$\hat{\sigma}(\mathbf{r}; \mathbf{p}_a, \mathbf{p}_b) \equiv \int d\mathbf{r}_{ab} \, \sigma(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b) , \tag{43}$$

$$\widehat{D}(\mathbf{r}; \mathbf{p}_a, \mathbf{p}_b) \equiv \int d\mathbf{r}_{ab} \, D(\mathbf{r}_a, \mathbf{p}_a; \mathbf{r}_b, \mathbf{p}_b) , \tag{44}$$

where $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$ is the separation and $\mathbf{r} = \frac{1}{2}(\mathbf{r}_a + \mathbf{r}_b)$ is the mean position.

The contracted covariances evolve according to the following equation of motion,

$$
\begin{aligned}
\dot{\hat{\sigma}}(\mathbf{r};\mathbf{p}_a,\mathbf{p}_b) = &\quad f_a f_b W_{ab}^{--} + \bar{f}_a \bar{f}_b W_{ab}^{++} - f_a \bar{f}_b W_{ab}^{-+} - \bar{f}_a f_b W_{ab}^{+-} \\
&- (W_a^+ + W_a^- + W_b^+ + W_b^-)\,\hat{\sigma}(\mathbf{r};\mathbf{p}_a,\mathbf{p}_b) \\
&- \int \frac{d\mathbf{p}_c}{h^3} \left[\left(\bar{f}_b(W_{bc}^{+-} + W_{bc}^{++}) - f_b(W_{bc}^{-+} + W_{bc}^{--}) \right) \hat{\sigma}(\mathbf{r};\mathbf{p}_a,\mathbf{p}_c) \right. \\
&\left. + \left(\bar{f}_a(W_{ac}^{+-} + W_{ac}^{++}) - f_a(W_{ac}^{-+} + W_{ac}^{--}) \right) \hat{\sigma}(\mathbf{r};\mathbf{p}_b,\mathbf{p}_c) \right],
\end{aligned}
\tag{45}
$$

where the individual positions $\mathbf{r}_a$ and $\mathbf{r}_b$ have been approximated by their mean $\mathbf{r}$.

4   Illustrative results

In this section, we employ the developed transport treatment to illustrate the relaxation of the phase-space distribution for various idealized scenarios of interest in nuclear dynamics. The computational effort required for solving the transport equations in their full generality is prohibitively large, and for our present purposes we restrict our considerations to systems with translational invariance, corresponding to what was done in ref. [17]. While this simplification reduces the computational complexity considerably, many features of interest in real nuclear processes can still be elucidated.

In order to imitate the momentum-space distribution relevant to a nuclear collision, we shall employ an initial occupancy corresponding to two dislocated Fermi spheres. Let the two spheres be centered at $\mathbf{p} = (0,0,\pm p_0/2)$. The occupancy $f(\mathbf{p};t=0)$ is then unity if the separation between $\mathbf{p}$ and either one of the two centers is less than the Fermi momentum p_F; otherwise it vanishes.

4.1   Local relaxation

Let us first calculate the evolution of the occupancy in a single phase-space point, as given by the contracted version of eqs. (27-28). For the case where $p_0 = p_F$, the *LHS* of fig. 1 displays the occupancy $f(\mathbf{p};t)$ as a function of $\mathbf{p} = (0,0,p_{\parallel})$ and $\mathbf{p} = (p_{\perp},0,0)$, for a number of successive times t. It is seen how the distribution acquires isotropy occurs within a relaxation time of less than 10^{-22} s.

The relaxation of the anisotropy can be brought out by the quadrupole moment

$$
Q(t) = \frac{1}{\rho} \int d\mathbf{p} \; (2p_{\parallel}^2 - p_{\perp}^2) \; f(\mathbf{p};t) .
\tag{46}
$$

This diagnostic quantity is shown in fig. 2 for the above case, which had $p_0 = p_F$, and for $p_0 = 0.5\ p_F$. It is seen that, to a reasonable approximation, Q decreases exponentially and so enables us to extract rather well-determined relaxation times, namely $0.22 \cdot 10^{-22}$ s and $0.5 \cdot 10^{-22}$ s, respectively. These values are consistent with our expectations that the relaxation time is approximately equal to the inverse of the collision rate experienced by a typical particle.

Another illustration of the relaxation is provided by the specific (single-particle) entropy

$$s(t) = -\frac{1}{\rho} \int d\mathbf{p} \; (f \ln f + \bar{f} \ln \bar{f}) \,, \tag{47}$$

which is shown by the dashed curves in fig. 3 for the same two cases. Again an approximately exponential relaxation towards the equilibrium value is obtained. It is instructive to note that a rather similar behavior is exhibited by the average variance

$$< \sigma_f^2 > = \frac{1}{\rho} \int d\mathbf{p} \; f(\mathbf{p}; t) \; \sigma_f^2(\mathbf{p}; t) \,, \tag{48}$$

which is shown in the figure by the solid curves.

The dependence of the variance σ_f^2 on the momentum $\mathbf{p}$ is illustrated on the *RHS* in fig. 1, in the parallel and perpendicular directions. Since the system has been prepared so that the local phase space is either fully occupied or totally empty, there is no fluctuation in the initial occupancy function and so σ_f^2 vanishes at $t = 0$.

A detailed understanding of the evolution of the occupancy illustrated above can be achieved by considering the corresponding transport coefficients. The time dependence of the drift coefficient $V(\mathbf{p}; t)$ and the diffusion coefficient $D(\mathbf{p}; t)$ is shown in fig. 4, in the parallel and perpendicular directions.

4.1.1 Relaxation of a dilute spherical distribution

As discussed in sect. 3.3, the characteristic time for the relaxation of σ_f^2 around its equilibrium value is very short and the variance rarely differs significantly from its instantaneous equilibrium value $f\bar{f}$. To further exhibit this feature, we consider also the evolution of a distribution for which the initial occupancy is one half within a sharp sphere, for all the systems in the ensemble. The initial variance of the occupancy then vanishes, while the instantaneous equilibrium value of the variance is $< f\bar{f} > = 0.25$ at time $t = 0$.

Fig. 5 shows the calculated evolution of the occupancy and the corresponding variance, both plotted as functions of the magnitude of the momentum. An inspection of fig. 5 shows that the occupancy variances relax about twice as fast as the mean occupancies, as is generally expected. The relaxation of the variance can be brought out quantitatively by the corresponding distribution-averaged quantity, as defined in eq. (48). Figure 6 shows the growth of the average variance towards the (time-dependent) adiabatic equilibrium value $< \tilde{\sigma}_f > = < f\bar{f} >$. For this average variance, the relaxation time is approximately $5 \cdot 10^{-23}$, which about half the relaxation time of $< f(\mathbf{p}; t) >$, as expected from eq. (29). Thus, even in this artificially prepared situation, the local fluctuations track closely their instantaneous equilibrium value. Consequently, for situations of practical interest, the variance in occupancy, σ_f^2, may be well approximated by the corresponding adiabatic value $f\bar{f}$.

4.2 Non-local properties

In order to illustrate the non-local properties of the fluctuations, we consider the covariance $\sigma(\mathbf{p}_1, \mathbf{p}_2; t)$ as a function of magnitude of the difference between the two

momentum arguments, $\Delta p = |\mathbf{p}_1 - \mathbf{p}_2|$. For a single time at which equilibrium is nearly reached, the *LHS* of fig. 7 shows the quantity W_{a+b} which acts as a restoring force on the fluctuations. It is shown for specified values of the mean momentum $\mathbf{p} = \mathbf{p}_a - \mathbf{p}_b$, as a function of the magnitude of the difference $\Delta p = |\mathbf{p}_a - \mathbf{p}_b|$, averaged over the direction $\hat{\mathbf{p}}$. It is interesting to note that this quantity does not exhibit much variation, neither as a function of momentum nor in the course of time; its typical size corresponds to a relaxation time of 1-2 10^{-23} s. This feature may allow analytical parametrizations of W_{a+b} to be made, thus simplifying the treatment considerably.

The behavior of the quantity W_{ab}, which provides the driving force for the fluctuations, is shown on the *RHS* of fig. 7. Some structure is evident, namely some anticorrelation for small relative momenta and some positive correlation at large relative momenta. This behavior can be qualitatively understood as follows: In eq. (45) one can split the source term into two parts: The W^{++} and W^{--} terms are responsible for the growth of positive correlations, but are counteracted by the W^{+-} and W^{-+} terms which generate anticorrelations. These two types of term behave differently as functions of $\Delta p = |\mathbf{p}_a - \mathbf{p}_b|$. In the case of W^{--} and W^{++}, Δp is the relative velocity of the colliding pair of particles (see eqs. (31) and (32)); since the collision rates are directly proportional to the modulus of this quantity (see eq. (22)), the source of positive correlations is an increasing function of Δp. For the negative source term (see definitions of W^{-+} and W^{+-} in eqs. (33) and (34)), Δp is the momentum transferred to a particle during the collision, so that small values of transferred momentum can be obtained in collisions between particles with large relative velocity which are the most frequent ones. Of course, these elementary considerations must be augmented by the effect of phase-space availability, if quantitative estimates are to be made.

The origin of the positive peak at $\Delta p \approx 2$ can then be related with the collective depopulation of the outer phase space, for which the positive source terms are large and Pauli blocking less effective. Correspondingly, the negative source term prevails for small Δp.

The resulting covariance $\sigma(\mathbf{p}, \Delta \mathbf{p})$ is shown in fig. 8, for momenta situated within the bulk of the occupied region. It exhibits a structure strikingly similar to that of the driving term W_{ab} shown in fig. 7b. This feature is easy to understand: at the time considered the covariance has relaxed to near the adiabatic value given by W_{ab}/W_{a+b}, and since W_{a+b} has little structure (fig. 7a), the resulting covariance tracks W_{ab}. The center frame in fig. 8 includes the covariance at an early time, $t = 2 \cdot 10^{-23}$ s. It is seen that the build-up of the positive correlation at large values of Δp is already well underway, whereas the negative correlation near zero requires a longer time. This latter feature is a result of the fact that the zone considered must first be depopulated so as to allow scattering into it. The displacement of the positive bump reflects the shrinking of the distribution in the parallel direction associated with the establishment of sperical symmetry. Still this feature closely mirrors the development of W_{ab}, so that the correlations remain close to the adiabatic values throughout the evolution.

Figure 9 again displays the covariance $\sigma(\mathbf{p}, \Delta \mathbf{p})$, but for regions at and above the Fermi level. The features noted above also hold here, but the amplitude of the cor-

relations is significantly larger. A significant increase of the anticorrelation occurs at $\Delta p = 1.5$. This structure arises from collisions between nucleons that originally populate the region of momentum space situated at large values near the beam direction and scatter into the perpendicular direction where the Pauli blocking is less effective, at least early on.

5 Concluding remarks

In the present work, we have developed a formalism for addressing stochastic one-body dynamics within the framework of transport theory. Considering an ensemble of similarly prepared systems, we have derived closed equations of motion for the average reduced one-body phase-space density distribution *and* the associated correlation function. The equation for the mean distribution corresponds to existing models under the labels *BUU* or *VUU*. The formulation of the associated equations of motion for the correlation function is somewhat similar to approaches developed in classical fluid dynamics. Within the context of nuclear dynamics, the incorporation of the second moments is an important advance and extends the utility of nuclear one-body models.

Building on earlier work by Ayik[15] towards the incorporation of fluctuations into *TDHF* theory, Ayik and Gregoire[16] recently made an important advance by considering the semi-classical limit of *TDHF*. Even though the starting point of that work is identical to that of the present paper, the methods employed are significantly different. In both studies, the fluctuations induced by the two-body collisions is considered as a stochastic process. In the work by Ayik and Gregoire, the phenomenon is treated as a Langevin process, with the entire phase-space distribution considered as a Brownian degree of freedom. The correlation function associated with the one-particle distribution may be then deduced from the properties of the fluctuating force. This approach is known as the Boltzmann-Langevin equation in the theory of classical fluids[12], where it has been applied to the treatment of hydrodynamical fluctuations.

In their presentation [16], Ayik and Gregoire derive an explicit expression for the correlation of the fluctuating force. By definition, this is the source term for the fluctuations and it appears in our equations in the form of the diffusion coefficient D (such as eq. (44)). In addition to that term, we have also derived the degradation term, using the stochastic properties of the basic two-body collision term. This restoring term acts to saturate the growth of the fluctuations and is responsible for establishing equilibrium (see eq. (29), for example).

The developed equations of motion for the variances and covariances of the one-body phase-space density may be used to elucidate the character of the relaxation processes occuring in nuclear dynamics. Our present studies indicate that the fluctuations relax rather rapidly, within less than 10^{-22} s, towards the equilibrium distribution associated with the instantaneous mean occupancy. This finding may provide the basis for significant simplifications in the analytical treatment of fluid-dynamical fluctuations.

It should be kept in mind that the presented transport equations have been derived

by expanding around the mean trajectory, so their applicability is limited to processes for which only moderate dynamical branching occurs. Yet, the equations can, in principle, be used to explore the *onset* of instabilities, since these are expected to be signalled by a growth in the magnitude of the calculated fluctuations.

Finally, our formal developments establish a solid basis for implementing numerical simulations of stochastic one-body dynamics. In particular, the formulation gives firm guidance with regard to how the Pauli-blocked two-body scattering should be treated, thus eliminating the need for relying on intuition. Thus, we have established the necessary formal tools for extending nuclear one-body simulation models from their current confines of mean trajectories to the practically more useful realm of dynamical branchings. Not only is such an advance of theoretical interest, but it also allows more direct contact with experimental data on nuclear collision processes.

Acknowledgements

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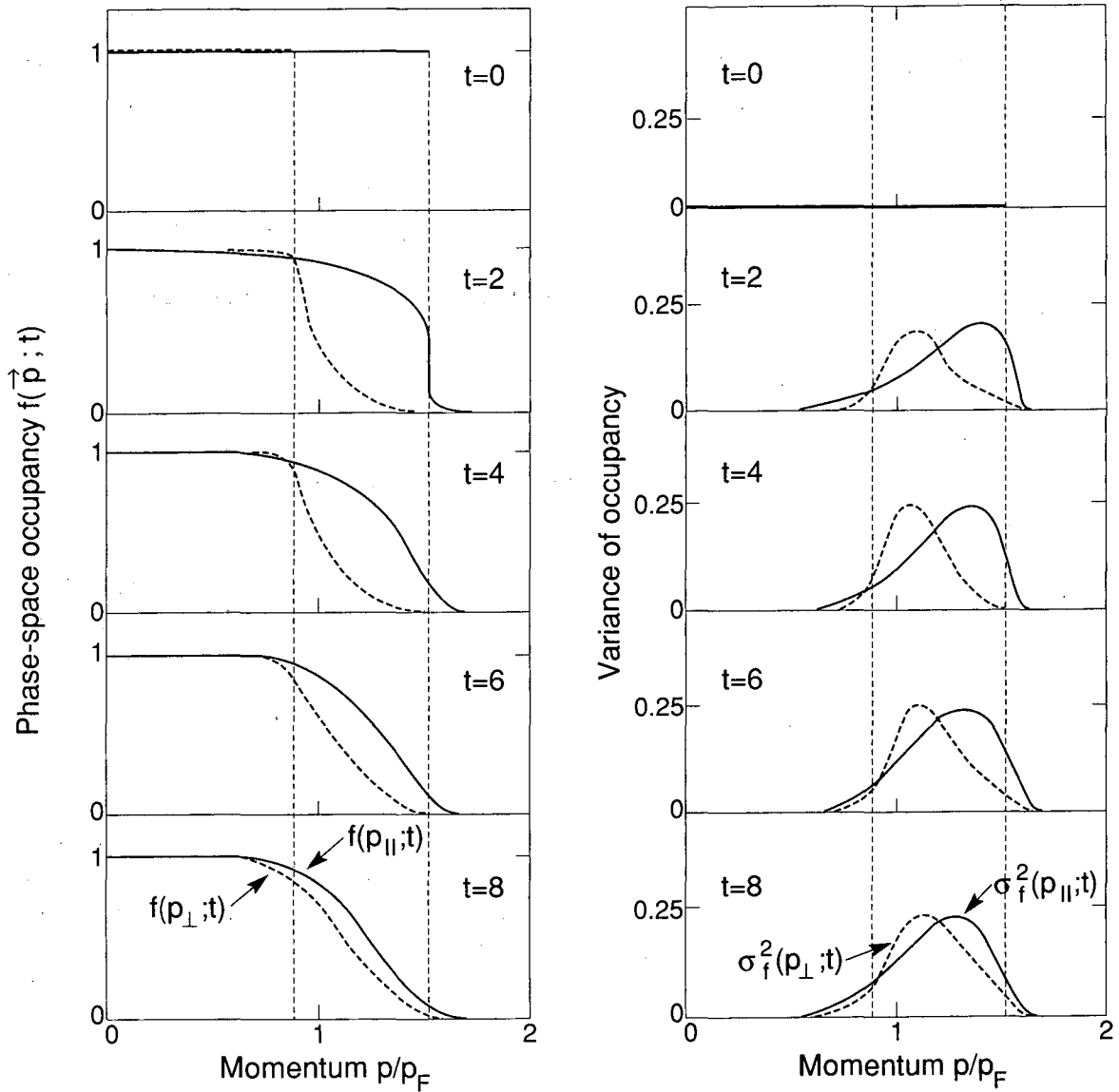
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Figure 1:

For a center separation of $p_0 = p_F$ (corresponding to a bombarding energy of $E_{\text{beam}} \approx$ 37 MeV per nucleon), the *LHS* displays the phase-space occupancy $f(\mathbf{p}; t)$ along the directions parallel and perpendicular to the beam, $\mathbf{p} = (0, 0, p_{\parallel})$ and $\mathbf{p} = (p_\perp, 0, 0)$, respectively, for a number of successive times $t = 0, 2, 4, 6, 8 \cdot 10^{-23}$ s. The time evolution of the variance of the phase-space occupancy, $\sigma_f^2(\mathbf{p}; t)$, is shown on the *RHS*.
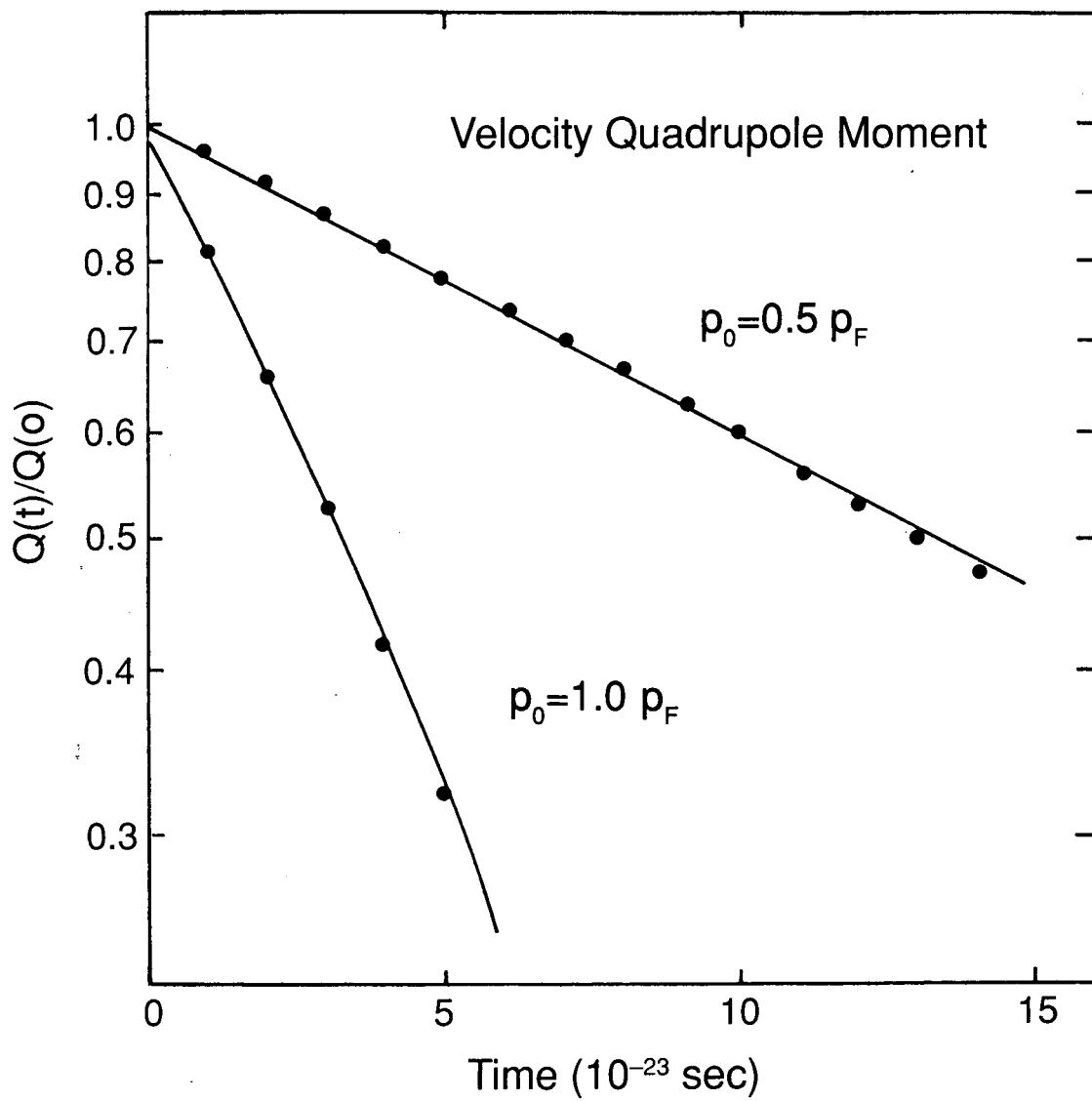
Figure 2:

The time evolution of the quadrupole moment of the momentum distribution, eq. (46), for the case shown in fig. 1 ($p_0 = 1.0\ p_F$) and the corresponding case with $p_0 = 0.5\ p_F$.

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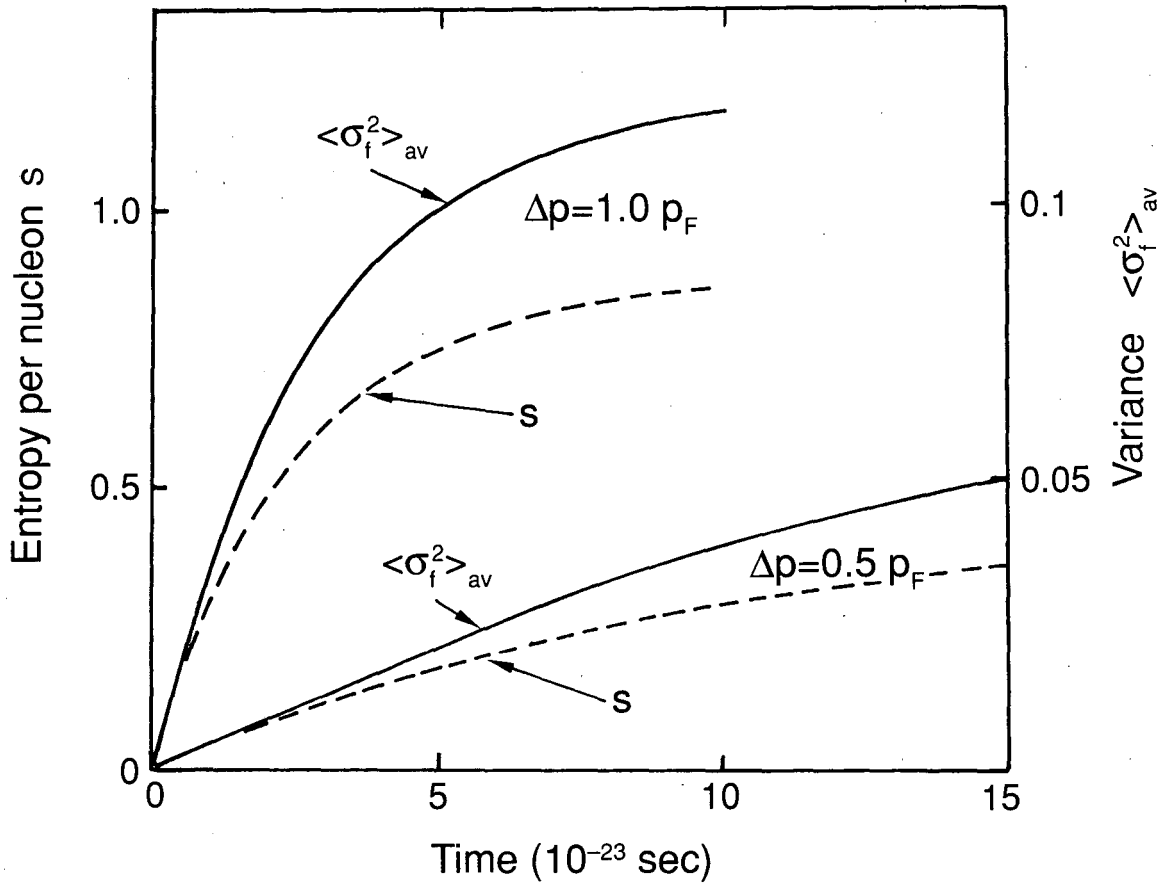
Figure 3:
For the same two cases as in fig. 2, the solid curves show the average value of the variance of the occupancy, eq. (48), while the dashed curves show the specific entropy given by eq. (47).
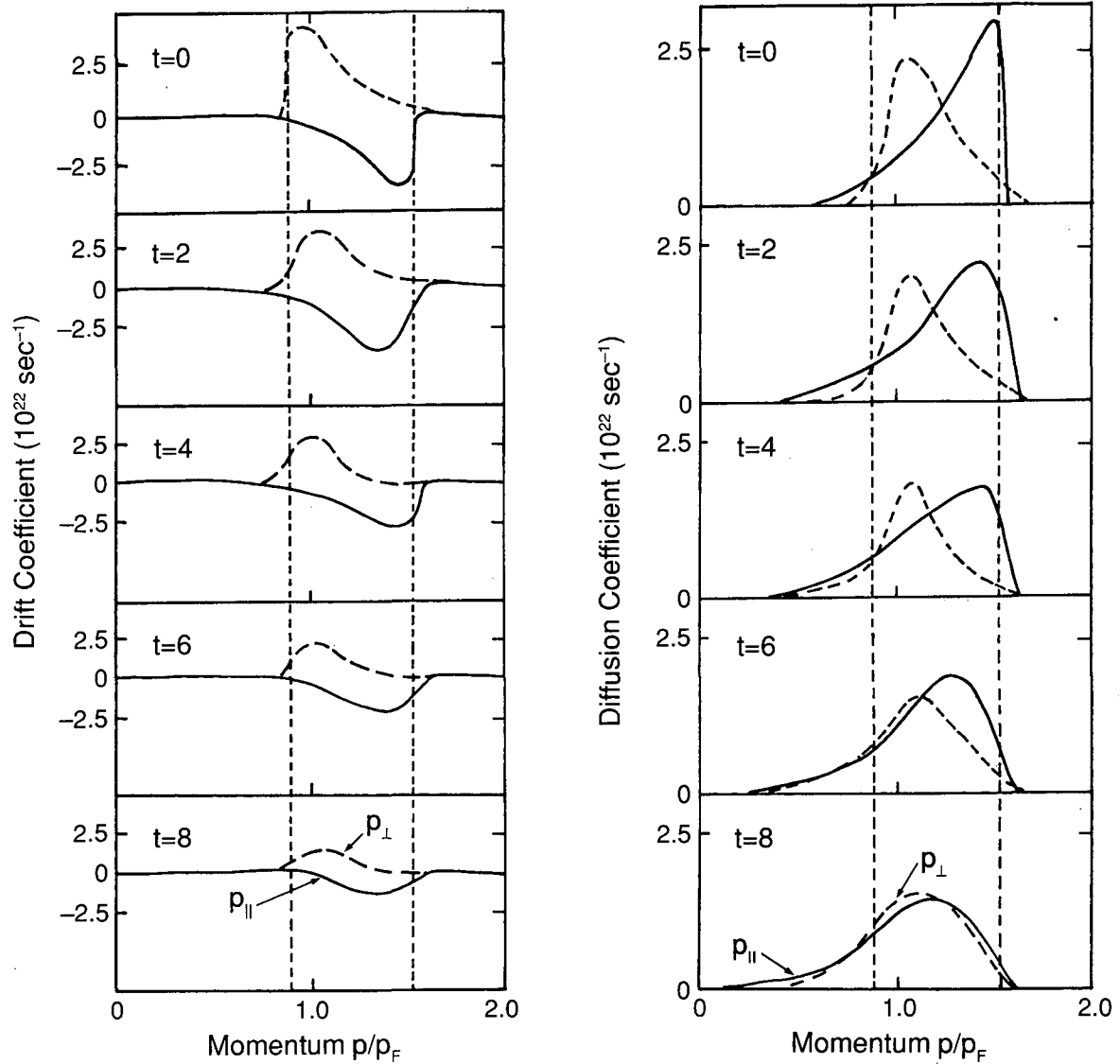
Figure 4:

For the case shown in fig. 1, the figure shows the momentum dependence of the drift coefficient for the phase space occupancy, $V(\mathbf{p};t) = \bar{f}W^+ - fW^-$ *(LHS)* and the diffusion coefficient $D(\mathbf{p};t) = \bar{f}W^+ + fW^-$ *(RHS)*.
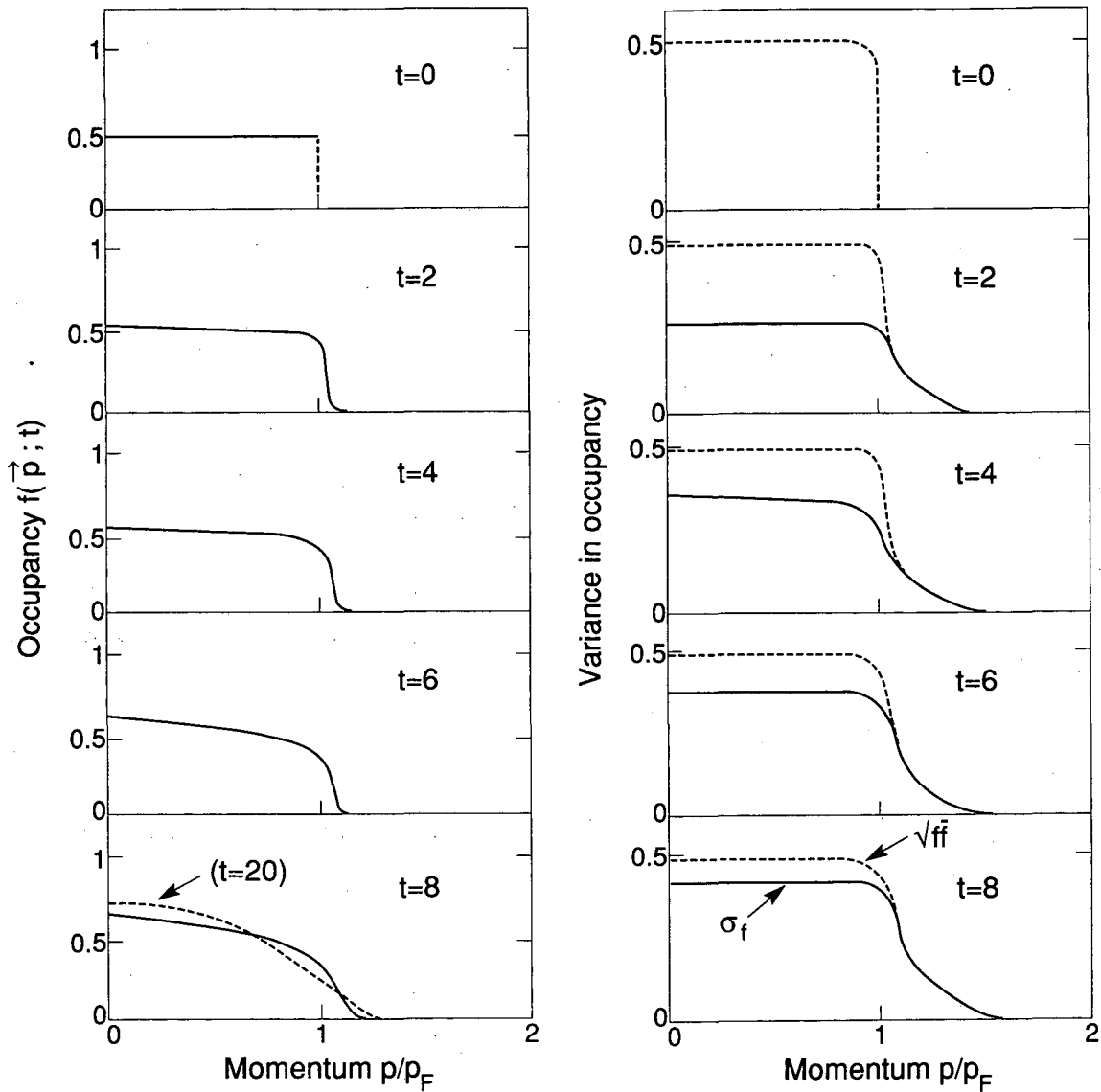
Figure 5:

LHS: The occupancy for an initially sharp spherical distribution with occupancy $f = 0.5$, as a function of the magnitude of the momentum $p = |\mathbf{p}|$, for successive times; the result corresponding to the late time $t = 20 \cdot 10^{-23}$ s is indicated by the dashed curve in the last frame.
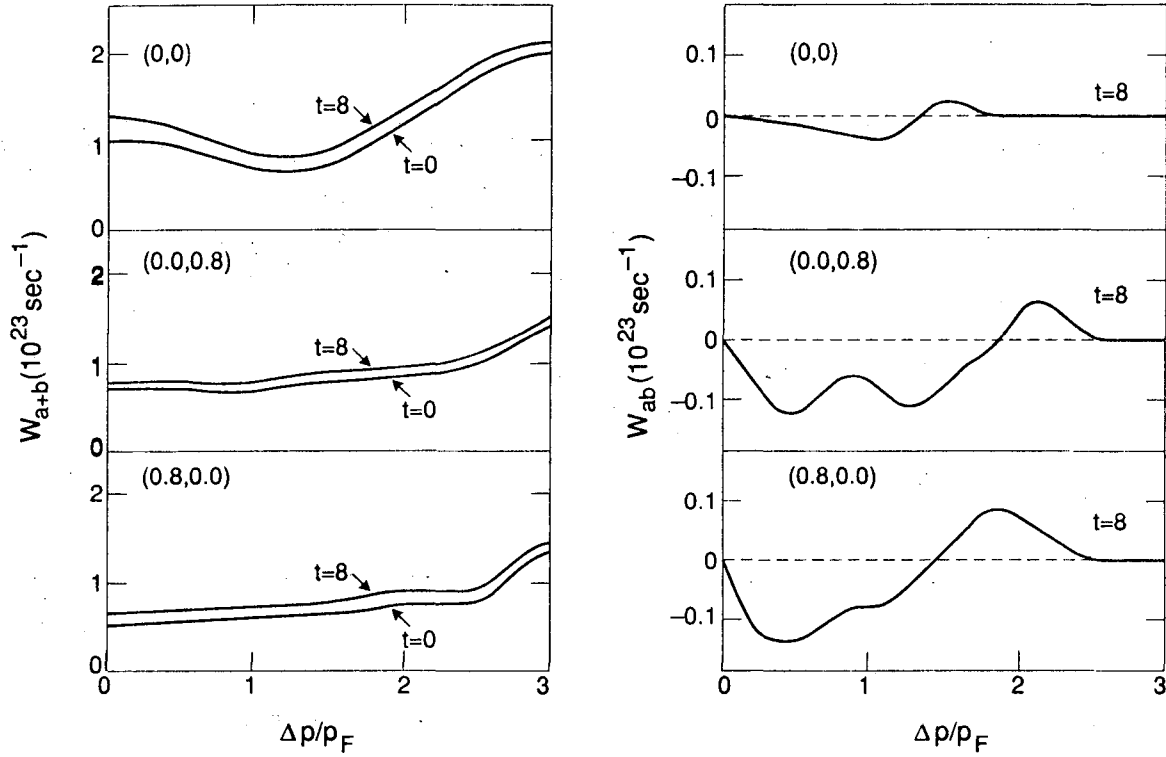
RHS: The dispersion in occupancy (solid curves); the corresponding instantaneous equilibrium value $\tilde{\sigma}_f = \sqrt{f\bar{f}}$ is also shown at each point in time (dashed curves).
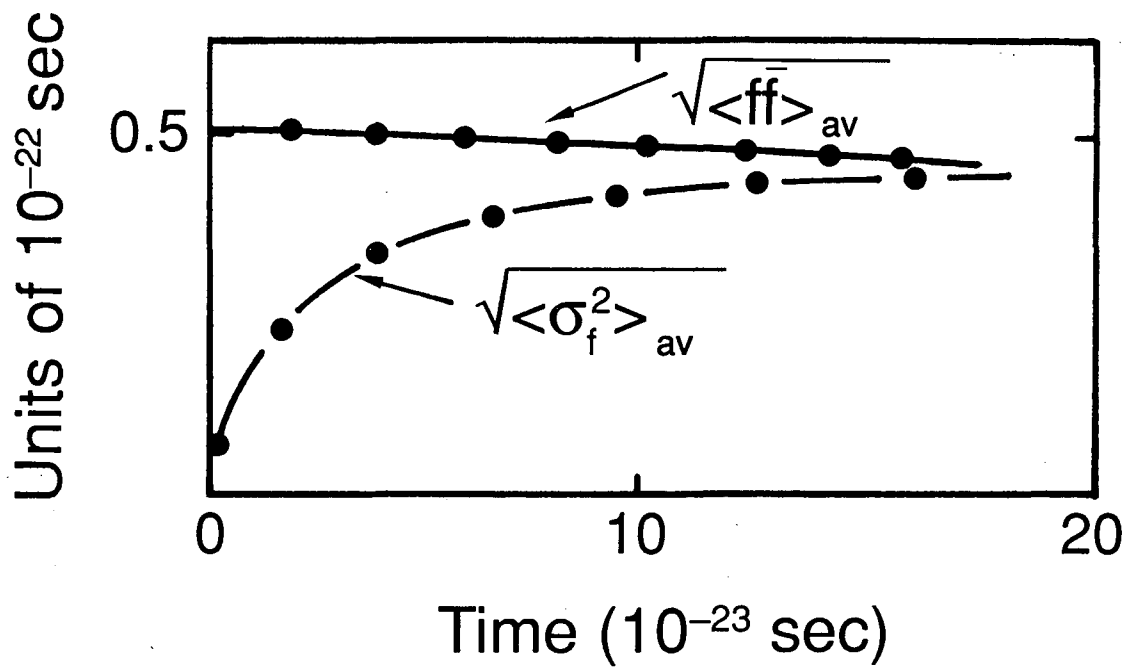
Figure 6:
Comparison between the dynamical evolution of σ_f and its adiabatic limit, for the case illustrated in fig. 5. The values shown represent averages over the entire distribution.
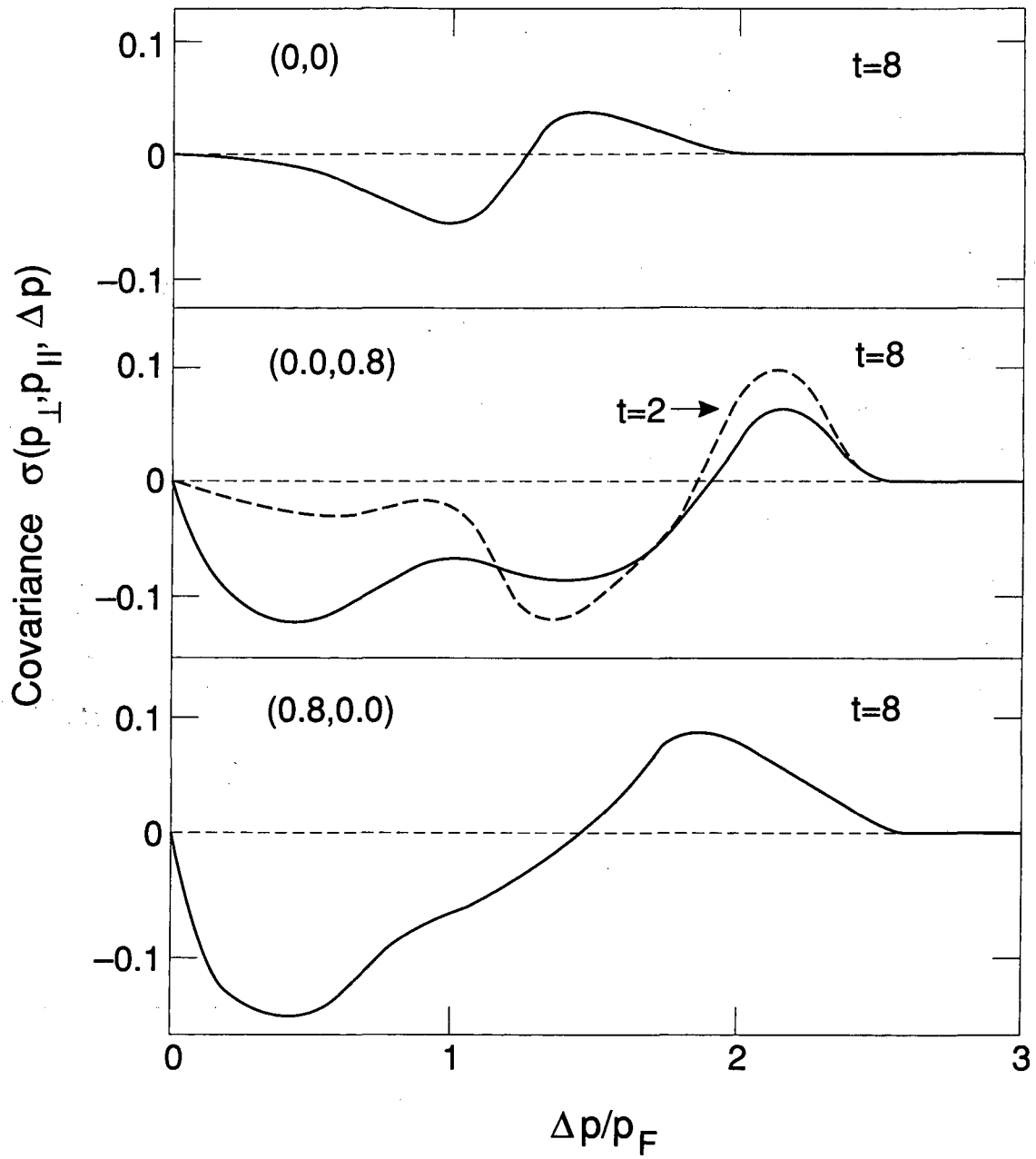
Figure 7:

LHS: The quantity $W_{a+b}(\mathbf{p}, \Delta\mathbf{p})$ degrading the covariance $\sigma(\mathbf{p}, \Delta\mathbf{p})$, shown as a function of the magnitude of the momentum difference $\Delta p = |\mathbf{p}_a - \mathbf{p}_b|$, for specified values of the mean momentum $\mathbf{p} = \mathbf{p}_a + \mathbf{p}_b = (p_\perp, 0, p_{||})$, with the values of $(p_\perp, p_{||})$ chosen as indicated to lie within the bulk of the occupied region. The quantity is shown at the initial time $t = 0$ and at the later time $t = 8 \cdot 10^{-23}$ s when considerable relaxation has occurred.

RHS: The corresponding quantity $W_{ab}(\mathbf{p}, \Delta\mathbf{p})$ driving the covariance $\sigma(\mathbf{p}, \Delta\mathbf{p})$, at $t = 8 \cdot 10^{-23}$ s.

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Figure 8:

The covariance $\sigma(\mathbf{p}, \Delta\mathbf{p})$ as a function of the magnitude of the momentum difference Δp, at specified values of the mean momentum $\mathbf{p} = (p_\perp, 0, p_\parallel)$, with the values $(p_\perp, p_\parallel)$ indicated; they are chosen as in fig. 7. The solid curves are for the time $t = 8 \cdot 10^{-23}$ s, while the dot-dashed curve in the center frame is for $t = 2$.
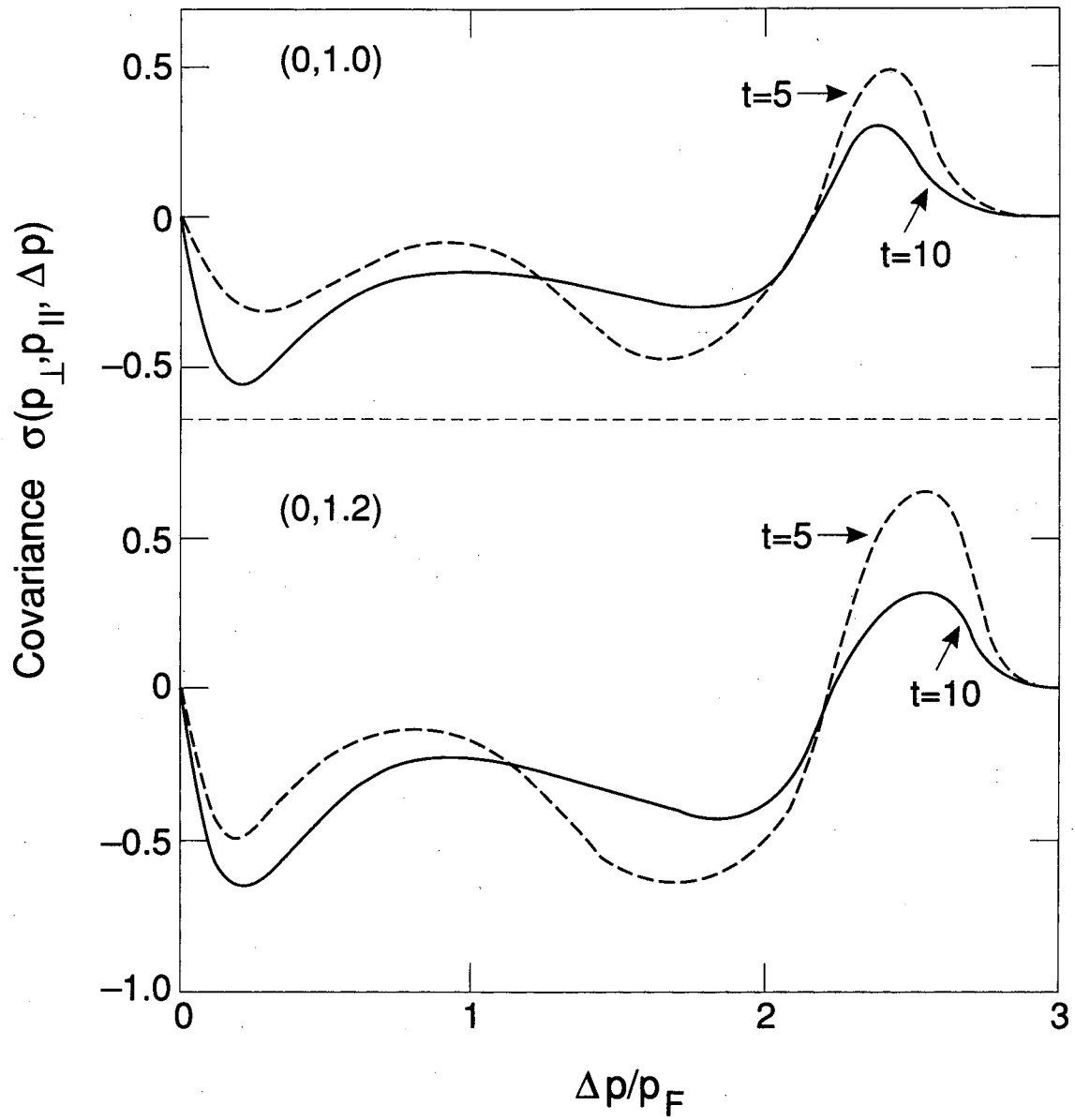
Figure 9:
Similar to fig. 8, but with the covariance considered at the larger mean momenta
$p_{\parallel} = p_F, 1.2\ p_F$, again with $p_{\perp} = 0$.