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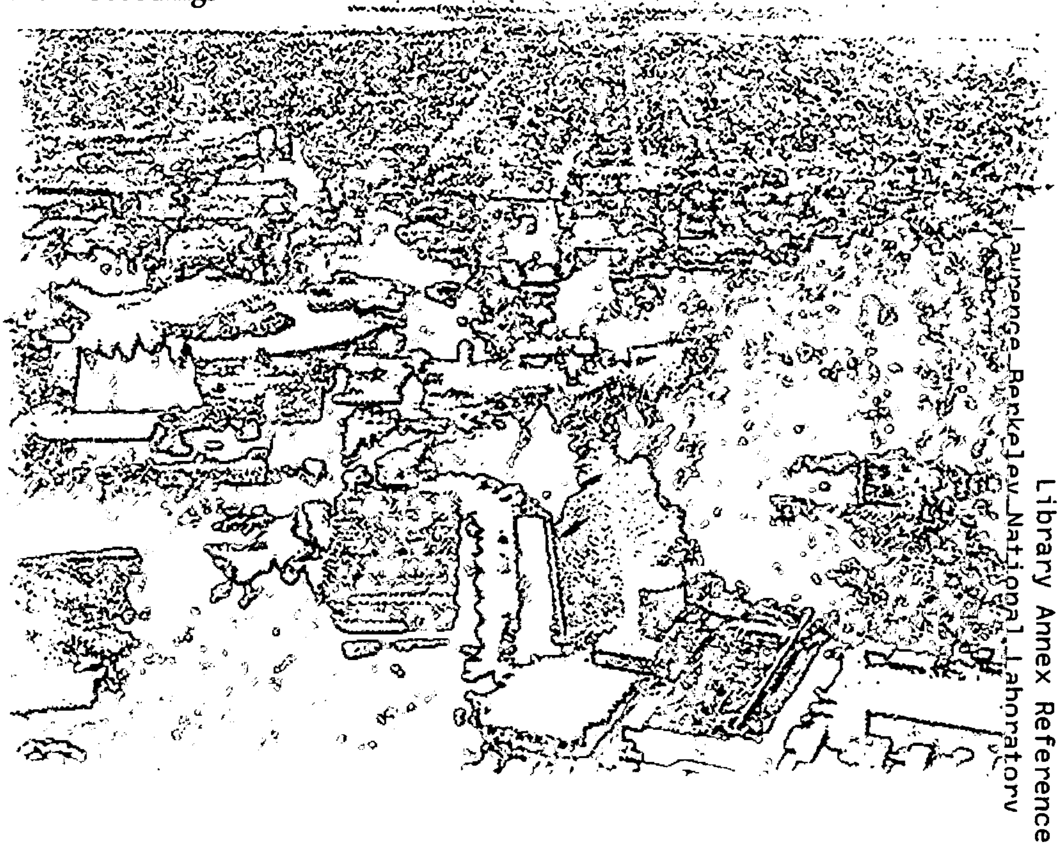

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Some remarks on unilateral matrix equations *

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Abstract

We briefly review the results of our paper [4]: we study certain perturbative solutions of left-unilateral matrix equations. These are algebraic equations where the coefficients and the unknown are square matrices of the same order, or, more abstractly, elements of an associative, but possibly noncommutative algebra, and all coefficients are on the left. Recently such equations have appeared in a discussion of generalized Born-Infeld theories. In particular, two equations, their perturbative solutions and the relation between them are studied, applying a unified approach based on the generalized Bezout theorem for matrix polynomials.

1 Introduction

Left-unilateral matrix equations are algebraic equations of the form

$$P(x) = 0, \text{ with } P(x) \equiv a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n, \tag{1.1}$$

where the coefficients a_r and the unknown x are square matrices of the same order and all coefficients are on the left.

The motivation for studying these kinds of equations is that recently they have appeared in the context of generalized Born-Infeld theories [1, 2]. The construction of a self-dual Lagrangian can be reduced to their solution.

We have proposed in [4] a unified approach to these equations based on the generalized Bezout theorem for matrix polynomials. This enables us to combine the idea of constructing the trace of a perturbative solution in terms of contour

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integrals in the complex plane of the trace of the resolvent of the corresponding matrix, due to A. Schwarz [5], with the idea of applying the basic property of the logarithm, as proposed in [3].

If we define the characteristic polynomial associated to $P(x)$ as

$$P(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \ldots + a_n\lambda^n, \qquad \lambda \in \mathbb{C} \tag{1.2}$$

then the statement of the generalized Bezout theorem [6] is that $\lambda - x$ is a divisor of $P(\lambda) - P(x)$ on the right, i.e. it is possible to find a polynomial $Q(\lambda, x)$, such that

$$P(\lambda) - P(x) = Q(\lambda, x)(\lambda - x). \tag{1.3}$$

In fact, in this particular case it is easy to see that

$$Q(\lambda, x) = \sum_{l=0}^{n-1} \lambda^l \left(\sum_{r=l+1}^{n} a_r x^{r-l-1} \right). \tag{1.4}$$

Notice that all the results described in this paper remain valid in a more general setting, if one considers a_r and x as elements of an associative, but possibly noncommutative, algebra, and an appropriate algebraic definition of the trace as cyclic average (see [2]) is used.

2 Properties of the Trace of Perturbative Solutions

An example of a unilateral equation was studied by A. Schwarz [5]

$$x^n = 1 + \epsilon\left(a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}\right). \tag{2.1}$$

For $\epsilon = 0$ we consider n solutions $e^{\frac{2\pi i k}{n}}1$, $k = 0, \ldots, n-1$. For small ϵ we are interested in finding perturbative solutions around these. An explicit iterative expression for $\operatorname{Tr} x^s$ for the solution $x \overset{\epsilon \to 0}{\to} 1$ is [4]

$$\operatorname{Tr} x^s = \operatorname{Tr} 1 + s \sum_{k=1}^{\infty} \frac{\epsilon^k}{n^k} \sum_{n_0 + \ldots + n_{n-1} = k} \frac{\operatorname{Tr} \mathcal{S}(a_0^{n_0} \ldots a_{n-1}^{n_{n-1}})}{n_0! \ldots n_{n-1}!} \prod_{r=1}^{k-1} \left(s + \sum_{l=1}^{n-1} l n_l - rn \right). \tag{2.2}$$

Some remarks can be made with respect to this formula, before we proceed to its demonstration. Its main feature is that it is symmetrized in the a_r, which enter only through the symmetrized product $\mathcal{S}(a_0^{n_0} \ldots a_n^{n_n})$. This was conjectured in [2] and subsequently proven in [3] and [5]. In (2.2) the normalization of the symmetrized product is chosen in such a way as to give the ordinary product if the factors commute [3].

The formula (2.2) holds for positive as well as for negative values of s. A similar expression could be derived for all the perturbative solutions of the equation.

For the equation of the second order $n = 2$ an explicit expression can be given

$$\mathrm{Tr}\, x = \mathrm{Tr}\left[\epsilon\frac{a_1}{2} + \mathcal{S}\sqrt{1 + \epsilon a_0 + (\epsilon\frac{a_1}{2})^2}\,\right]. \tag{2.3}$$

Let us now sketch a proof of (2.2). We start by applying the Bezout theorem. For (2.1) the characteristic polynomial is

$$P(\lambda) \equiv 1 - \lambda^n + \epsilon\left(a_0 + a_1\lambda + \ldots + a_{n-1}\lambda^{n-1}\right) \tag{2.4}$$

and

$$P(\lambda) = Q(\lambda, x)(\lambda - x) \text{ for } P(x) = 0 \tag{2.5}$$

with

$$Q(\lambda, x) = \sum_{l=0}^{n-1}\left(\sum_{r=l+1}^{n-1} \epsilon a_r x^{r-l-1} - x^{n-l-1}\right)\lambda^l. \tag{2.6}$$

Here, we have chosen one particular solution x of (2.2), namely the one which satisfies $x \overset{\epsilon\to 0}{\to} 1$, but the same technique could be applied to any other of the perturbative solutions.

As a next step, we use the basic property of the trace of the logarithm:

$$\mathrm{Tr}\log P(\lambda) = \mathrm{Tr}\log(Q(\lambda, x)) + \mathrm{Tr}\log(\lambda - x). \tag{2.7}$$

Then, as anticipated in the introduction, we apply Schwarz's idea of making a contour integration in the complex plane and compute the result through the Cauchy theorem.

$$\mathrm{Tr} f(x) = (2\pi i)^{-1}\oint_\Gamma d\lambda\, \mathrm{Tr}\frac{1}{P(\lambda)}P'(\lambda)f(\lambda). \tag{2.8}$$

Here, Γ is a small circle around 1 and $f(\lambda)$ is a function, which is regular for λ near 1. The integration contour is shown in the next figure.

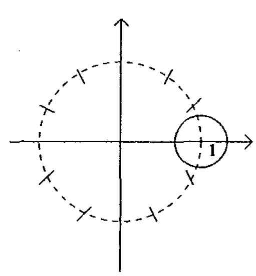

Figure 1: Contour of the integration

We factorize

$$P(\lambda) = (1 - \lambda^n)T(\lambda) \text{ with } T(\lambda) = 1 - \epsilon(\lambda^n - 1)^{-1}\sum_{l=0}^{n-1} a_l\lambda^l \tag{2.9}$$

and perform an integration by parts. In this way (2.8) becomes

$$\mathrm{Tr}\, f(x) = \mathrm{Tr}\, f(1) - \frac{1}{2\pi i} \oint_\Gamma d\lambda\, \mathrm{Tr} \log(T(\lambda)) f'(\lambda)\,. \tag{2.10}$$

We expand the logarithm and make the following change of variable

$$y = \lambda^n\,. \tag{2.11}$$

A small closed curve Γ winding once around 1 still remains a closed curve winding once around 1 after the variable transformation (2.11), so that this is justified. We restrict ourselves to the case $f(\lambda) = \lambda^s$ and get

$$\mathrm{Tr}\, x^s = \mathrm{Tr}\, 1 + \frac{s}{n} \sum_{k=1}^{\infty} \frac{\epsilon^k}{2\pi i k} \oint_\Gamma dy \mathrm{Tr} \left(\frac{\sum_{l=0}^{n-1} a_l y^{\frac{l}{n}}}{y-1} \right)^k y^{\frac{s}{n}-1}. \tag{2.12}$$

In this form we can already see that the result is symmetrized in the coefficients $\{a_i\}$, because they enter only through expressions of the type

$$\left(\sum_{l=0}^{n-1} a_l y^{\frac{l}{n}} \right)^k = \sum_{n_0+\ldots+n_{n-1}=k} \frac{k!}{n_0!\ldots n_{n-1}!} \mathcal{S}(a_0^{n_0} \ldots a_{n-1}^{n_{n-1}}) y^{\frac{1}{n}\sum_{l=1}^{n-1} l n_l}\,, \tag{2.13}$$

which are automatically symmetrized.

Finally, (2.2) is obtained by applying the Cauchy theorem in its more general form

$$(2\pi i)^{-1} \oint_C dy \frac{f(y)}{(y-y_0)^k} = \frac{1}{k-1!} \frac{d^{k-1}}{dy^{k-1}} f(y)|_{y=y_0}\,, \tag{2.14}$$

where C a closed curve winding once around y_0, and $f(y)$ is a function which is regular inside C.

Let us conclude this section by giving some alternative expressions for (2.2). We introduce the notation

$$\left(\begin{array}{c} \alpha \\ k \end{array} \right) = \left\{ \begin{array}{ll} \prod_{r=1}^k \frac{\alpha-r+1}{r} & \text{for } k = 1,2,\ldots \\ 1 & \text{for } k = 0. \end{array} \right. \tag{2.15}$$

Then

$$\mathrm{Tr} x^s = \mathrm{Tr} 1 + \frac{s}{n} \sum_{k=1}^{\infty} \epsilon^k (k-1)! \sum_{n_0+\ldots+n_{n-1}=k} \frac{\mathrm{Tr}\mathcal{S}(a_0^{n_0} \ldots a_{n-1}^{n_{n-1}})}{n_0!\ldots n_{n-1}!} \left(\begin{array}{c} \frac{1}{n}(s-n+\sum_{l=1}^{n-1} l n_l) \\ k-1 \end{array} \right). \tag{2.16}$$

In this form the result can be more easily compared to [3].

Another expression can be given through generalizations of factorials:

$$\mathrm{Tr}\, x = \mathrm{Tr}\, 1 + \sum_{k=1}^{\infty} \frac{\epsilon^k}{n^k} \sum_{n_0+\ldots+n_{n-1}=k} (-1)^{1+[\frac{1}{n}(\sum_{l=0}^{n-1}(l+1)n_{n-1-l}-n-1)]} \tag{2.17}$$

$$N^{(n)} \left(\sum_{l=0}^{n-1} (l+1) n_{n-1-l} - n - 1 \right) \dot{N}^{(n)} \left(\sum_{l=1}^{n-1} l n_l - n + 1 \right) \frac{\mathrm{Tr}\mathcal{S}(a_0^{n_0} \ldots a_{n-1}^{n_{n-1}})}{n_0!\ldots n_{n-1}!},$$

where $[x]$ = Integer Part of x and

$$N^{(n)}(x) = \begin{cases} 1 & \text{for } x < 0 \\ \prod_{r=0}^{[x/n]} x - rn & \text{for } x \geq 0. \end{cases} \tag{2.18}$$

This formula can be more easily compared with the exact expression (2.3) for $n = 2$, as the symbols $N^{(n)}(x)$ are generalizations of the double factorials, which appear in the expansion of the square root.

3 Discussion of another equation

Another unilateral matrix equation is studied in [3]:

$$\Phi = A_0 + A_1\Phi + \ldots A_n\Phi^n . \tag{3.1}$$

Its characteristic polynomial is

$$\lambda - A(\lambda) \text{ where } A(\lambda) = A_0 + A_1\lambda + \ldots A_n\lambda^n. \tag{3.2}$$

We follow a similar procedure as before: we apply the Bezout theorem and perform a contour integration. Then

$$\mathrm{Tr} f(\Phi) = -\frac{1}{2\pi i}\oint_C d\lambda \, \mathrm{Tr}\log(1 - \frac{A(\lambda)}{\lambda}) f'(\lambda), \tag{3.3}$$

where now C is a closed curve winding once around 0. We restrict to the case $f(\lambda) = \lambda^s$, s positive integer, expand the logarithm and apply the Cauchy theorem:

$$\mathrm{Tr}\,\Phi^s = s\mathrm{Tr}\sum_{k=1}^{\infty}\frac{1}{k}(A_0 + \ldots A_n)^k|_{\sum_{l=0}^{n}(l-1)n_l = -s}. \tag{3.4}$$

In this way we recover the result of [3].

To study more closely the relation between equation (2.1) and (3.1) we make the Ansatz

$$x = 1 + \alpha\Phi \text{ with } \alpha^{n-1} = -n . \tag{3.5}$$

Then it is easily seen that

$$A_l = \begin{cases} -\alpha^{l-n}\epsilon\sum_{r=l}^{n-1}\begin{pmatrix} r \\ l \end{pmatrix} a_r & \text{for } l = 0, 1 \\ \alpha^{l-n}\begin{pmatrix} n \\ l \end{pmatrix} - \alpha^{l-n}\epsilon\sum_{r=l}^{n-1}\begin{pmatrix} r \\ l \end{pmatrix} a_r & \text{for } 2 \leq l \leq n . \end{cases} \tag{3.6}$$

Some remarks can be made with respect to (3.4) and (3.6).

For $\epsilon = 0$ the solution of (3.1) is $\Phi = 0$. Therefore it has no sense to invert Φ and s has to be a positive integer. However, negative powers of x can still be computed by expanding (3.5) into a series of positive powers of Φ.

The equation (3.1) depends on $n+1$ coefficients A_r, but there are only n coefficients a_r in (2.1). Therefore in (3.6), $A_n = 1$ is fixed.

Since most of the solutions of the Schwarz's equations are not real, even for real a_n, there is no reason to choose α to be real in (3.5).

The transformation (3.6) is linear and invertible. As a consequence, the result that the series is symmetrized needs to be proven only for one the two equations, and then it follows immediately for the other. However, from (3.6) it is clear that even if ϵ is small, only the first two coefficients A_0 and A_1 need to be small. We do not study the convergence properties of the sums appearing in this note, and we consider them always as formal series, but it is to be expected that the expansions do not hold for the same range of the coefficients.

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