Budget Management in Auctions: Bidding Algorithms and Equilibrium Analysis

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Abstract

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Advertising is the economic engine of the internet. It allows online platforms to fund services that are free at the point of use, while providing businesses the opportunity to target their ads at relevant users. The mechanism of choice for selling these advertising opportunities is real-time auctions: whenever a user visits the platform, an auction is run among interested advertisers, and the winner gets to display their ad to the user. The entire process runs in milliseconds and is implemented via automated algorithms which bid on behalf of the advertisers in every auction. These automated bidders take as input the high-level objectives of the advertiser like value-per-click and budget, and then participate in the auctions with the goal of maximizing the utility of the advertiser subject to budget constraints. Thus motivated, this thesis develops a theory of bidding in auctions under budget constraints, with the goal of informing the design of automated bidding algorithms and analyzing the market-level outcomes that emerge from their simultaneous use.

First, we take the perspective of an individual advertiser and tackle algorithm-design questions. How should one bid in repeated second-price auctions subject to a global budget constraint? What is the optimal way to incorporate data into bidding decisions? Can data be incorporated in a way that is robust to common forms of variability in the market? As we analyze these questions, we go beyond the problem of bidding under budget constraints and develop algorithms for more general online resource allocation problems. In Chapter 2, we study a non-stationary stochastic model of sequential auctions, which despite immense practical importance has received little attention, and propose a natural algorithm for it. With access to just one historical sample per auction/distribution, we show that our algorithm attains (nearly) the same performance as that possible under full knowledge of the distributions, while also being robust to distribution shifts which

typically occur between the sampling and true distributions. Chapter 3 investigates the impact of uncertainty about the total number of auctions on the performance of bidding algorithms. We prove upper bounds on the best-possible performance that can be achieved in the face of such uncertainty, and propose an algorithm that (nearly) achieves this optimal performance guarantee. We also provide a fast method for incorporating predictions about the total number of auctions into our algorithm. All of our proposed algorithms implement some version of FTRL/Mirror-Descent in the dual space, making them ideal for large-scale low-latency markets like online advertising.

Next, we look at the market as a whole and analyze the equilibria which emerge from the simultaneous use of automated bidding algorithms. For example, we address questions like: Does an equilibrium always exist? How does the auction format (first-price vs second-price) impact the structure of the equilibria? Do automated bidding algorithms always efficiently converge to some equilibrium? What are the social welfare properties of these equilibrium outcomes? We systematically examine such questions using a variety of tools, ranging from infinite-dimensional fixed-point arguments for proving existence of structured equilibria, to computational complexity results about finding them. In Chapter 4, we start by establishing the existence of equilibria based on pacing—a practically-popular and theoretically-optimal budget management strategy—for all standard auctions, including first-price and second-price auctions. We then leverage its structure to establish a revenue equivalence result and bound the price of anarchy of liquid welfare. Chapter 5 looks at the market from a computational lens and investigates the complexity of finding pacing-based equilibria. We show that the problem is PPAD complete, which in turn implies the impossibility of polynomial-time convergence of any pacing-based automated bidding algorithms (under standard complexity-theoretic assumptions). Finally, in Chapter 6, we move beyond pacing-based strategies and investigate throttling, which is another popular method for managing budgets in practice. Here, we describe a simple tâtonnement-style algorithm which efficiently converges to an equilibrium in first-price auctions, and show that no such algorithm exists for second-price auctions (under standard complexity-theoretic assumptions). Furthermore, we prove tight bounds on the price of anarchy for liquid welfare, and compare platform revenue under throttling and pacing.

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Chapter 1: Introduction

For millennia, auctions have been used to sell items in a variety of markets, ranging from agricultural produce to government assets. They provide a natural solution to the problem of allocating items among buyers whose valuation (willingness to pay) is unclear/unknown to the seller. And when the internet came along, it was soon discovered that this was exactly the problem internet platforms faced when attempting to monetize their services via advertising. In particular, platforms like Google, Instagram, TikTok etc. have an abundant supply of users who routinely interact with their platforms, and there is demand from advertisers for the attention of those users. A market mechanism is required to match this supply and demand. Crucially, given the heterogeneity of these opportunities (each user and webpage are different), it is very difficult to ascertain the amount that the advertisers would be willing to pay for them. This is exactly the type of setting where auctions shine, and thus have been widely adopted: they are the mechanism by which the vast majority of advertising opportunities are allocated on the internet. Any time a user visits the platform, an auction is run programmatically to determine the advertiser who will get the opportunity to show their ad to the user, and the payment to be charged for it. The entire process runs in the milliseconds that the webpage takes to load after the user query, and occurs billions of times every day across a plethora of platforms.

These online advertising auctions are executed via a complex market ecosystem, composed of publishers/platforms who own the webpages where ads are displayed, advertisers who wish to show their ads to users visiting these webpages, and ad exchanges who are the market makers; not to mention demand-side and supply-side platforms who act as intermediaries. To add to this, advertisers routinely run ad campaigns consisting of tens of thousands of auctions, each one corresponding to a user with some idiosyncratic propensity for favorably interacting with the ad. This ever-increasing complexity, speed and scale of online advertising markets has led to the rise of *automated bidding* (or simply autobidding), which is the practice of using automated data-driven algorithms for bidding in ad auctions. These autobidding algorithms take as input the high-level objectives and constraints of the advertiser, and then bid on their behalf in the auctions with the goal optimizing their objective subject to the specified constraints. The demand for autobidding has led to platforms like Google[1] and Meta[2] themselves offering it as a service to advertisers, which in turn has made autobidding ubiquitous.

Budget management tools are perhaps the most prevalent autobidding tools offered by platforms: the vast majority of advertisers have budgets that limit their total spend across all auctions in the campaign, and they require tools which optimize the use of their budgets. This thesis aims to develop a deeper understanding of budget management, with the dual goal of improving the design of autobidding algorithms and analyzing the market-level outcomes that emerge from their simultaneous use. On the technical front, this translates to developing data-driven algorithms for bidding in repeated auctions under global budget constraints, and understanding the market-level outcomes that result from these algorithms bidding against each other.

1.1 Data-Driven Algorithms for Budget Management

In the first part of the thesis, we focus on the design of data-driven algorithms for bidding in repeated second-price auctions subject to a global budget constraint. It turns out that this problem is closely related to online resource allocation. At its core, both problems involve the same trade-off: spend the budget now or wait for better opportunities later. For greater generality and ease of exposition, we develop algorithms for online resource allocation, and show how the budget management problem can be interpreted as a special case. Departing from previous works, our goal is not to simply optimize performance and sample complexity, but to do it in a way that is robust to the common types of variability which affect online advertising markets. The type of variability we focus on delineates the two chapters in this part of the thesis.

[1] https://support.google.com/google-ads/answer/2979071?hl=en

[2] https://www.facebook.com/business/m/one-sheeters/facebook-bid-strategy-guide/

1.1.1 Chapter 2: Robust Budget Pacing with a Single Sample

Consider an advertiser with a budget of B dollars, which specifies the maximum amount they are willing to spend in some fixed time period (say, a week). Assume that she will participate in T sequential second-price auctions in that period. Her goal is to maximize total utility across all of the auctions in that week, while keeping the cumulative spend below the budget B. How should this advertiser pace her spending? When should she spend aggressively and when should she conserve her budget? How should one design the autobidding algorithm that makes these decisions on her behalf? The answer to these questions depends on the way that the auctions (i.e., advertiser's value and competition) change over time, and the information available to her.

If the environment is adversarial, i.e., nothing is known and the auctions can change arbitrarily with time, then [BG19] showed that no algorithm can achieve sub-linear regret (or even a good asymptotic competitive ratio) against the hindsight-optimal benchmark. In other words, all algorithms perform poorly, and this setting provides very pessimistic guidance for algorithm design. Due to this impossibility result, most of the past works on budget management algorithms—and online resource allocation more generally—have focused on the setting where all auctions are similar on average, i.e., advertiser value and competition are independently and identically distributed. Unsurprisingly, this assumption of stationary/identical distributions leads to algorithms that attempt to evenly spend the budget across all auctions (e.g., [BG19; BLM23]). However, this stationarity assumption is completely untenable in practice because the volume of traffic, demographic of users, rates of conversion etc. change with time [Zho+19], which in turn cause the advertiser's value and competition to change with time. Perhaps even more importantly, evenly spending budget across all auctions can often be a horrible idea, e.g., most businesses do not want to spend the same amount on advertising at 3 am as they do at 3 pm.

Thankfully, even though reality is not stationary, it is not adversarial either: user traffic and the competition for it change with time, but they show some periodicity in their changes, e.g. weeks look similar on average. To deal with this non-stationarity, real-world budget management systems compute a *target expenditure plan* [FG; KMS22], which is a function of time that specifies the recommended amount of spend at each point in time, i.e., such a plan distributes the cumulative daily/weekly budget into smaller chunks of time, appropriately capturing the non-stationarity. A pacing algorithm like a controller is then used to track the plan. This raises an important question:

How many historical samples are required to learn a good target expenditure plan?

To study this question, we study a model where the tuple of an advertiser's value and that of her highest competing bid, in each auction t, are generated from unknown independent time-varying distributions $\{\mathcal{P}_t\}_{t=1}^T$. This is a major departure from the stochastic budget-management/resource-allocation literature, which for the most part assumes stationary distributions (see Section 1.3 for a discussion). We additionally assume that we have access to data from past weeks. This data helps us leverage the periodic nature of internet traffic/auctions. However, the periodicity is almost never exact: weeks look similar on average, but they are not identical on average, i.e., the historical data likely came from distributions $\{\tilde{\mathcal{P}}_t\}_{t=1}^T$ that were different from $\{\mathcal{P}_t\}_{t=1}^T$. This raises another important question:

How should one design an algorithm that is robust to distribution shifts between historical and true distributions?

We measure distribution shift using the total Wasserstein distance between the two sequences of distributions $\{\tilde{\mathcal{P}}_t\}_{t=1}^T$ and $\{\mathcal{P}_t\}_{t=1}^T$. [JLZ20] showed that the regret of all algorithms must degrade super-linearly with the distribution shift. This motivates us to call an algorithm *robust* if its regret degrades with the distribution shift at the optimal linear rate. Surprisingly, we find that both of the aforementioned questions simultaneously admit the most satisfactory solution: we show that *just one* historical sample per distribution is sufficient to learn a good target expenditure plan, and we give an explicit algorithm that can learn and use that plan in a way that is robust to distribution shifts.

Our results. Our contribution is an algorithm that is robust and achieves the near optimal $\tilde{O}(\sqrt{T})$ regret even when it has access to *just one sample* from each distribution $\tilde{\mathcal{P}}_t$, dramatically improving over the state-of-the-art requirement of $T \log T$ samples from each distribution [JLZ20]. Our results go beyond bidding under budget constraints, and apply more broadly to any single-resource online resource allocation problems, like single-leg revenue management and online knapsack.

Key insights. Our algorithm uses the samples to estimate the ideal amount of expenditure to target in each auction and then uses Follow-The-Regularized-Leader in the dual space to follow these targets by shading the advertiser's values appropriately. The key insight driving the reduction in sample complexity from $T \log T$ to 1 per distribution is the following. Prior work by [JLZ20] first learns the sampling distributions $\{\tilde{\mathcal{P}}_t\}_{t=1}^T$ from the samples, then computes the optimal duals on the learned distributions, and finally uses those duals to compute the target expenditures. Our insight is that it is not necessary to learn the entire sampling distribution. Instead, it is far more efficient to directly learn the duals from the samples and construct target expenditure based on those duals (i.e., setting target expenditure at t to be what the dual-based solution consumes at t). Beyond being very efficient with samples, learning the duals from the samples also guarantees robustness to shifts between $\{\tilde{\mathcal{P}}_t\}_{t=1}^T$ and $\{\mathcal{P}_t\}_{t=1}^T$.

Practical implications. Consider T representing a week's worth of auctions. While a total of $T \log T$ samples may be possible to obtain by looking at the past few weeks (given that each week has about T samples to offer), requiring $T \log T$ samples *per distribution* calls for looking at many months into the past. This is because getting a sample for a distribution, where a distribution could correspond to say a particular hour (e.g. Monday 10 AM), entails looking at that same hour from the past week. Asking for $T \log T$ samples for any given hour, when T samples is what we get for the entire week, clearly requires looking at the numerous months into the past. Apart from posing huge storage and operational challenges, given that traffic pattern shifts over time, even gradual shifts would significantly degrade quality as one moves too much into the past. Our ask of one sample per distribution requires looking at just the past week and getting one sample for each hour. Moreover, our algorithm implements pacing, which is a popular method of budget management that multiplicatively shades the value to determine the bid and controls the multiplier to manage budgets. Combined with the fact that it only requires a constant amount of computation after each auction, our algorithm can be readily implemented in practical budget management systems.

Technical contributions. We achieve our result by developing a novel dual-iterate coupling lemma (see Lemma 6) and leveraging it to analyze a leave-one-out thought experiment designed to break challenging correlations which arise from working with one sample per distribution (see Subsection 2.3.3 for details). Additionally, we also prove a novel regret decomposition for Dual FTRL (Theorem 3), which may be of independent interest. Finally, our algorithm does not require solving large linear programs and can be implemented efficiently (see Subsection 2.3.4), which is critical for online advertising since each auction runs in a few milliseconds.

1.1.2 Chapter 3: Online Resource Allocation under Horizon Uncertainty

Almost all of the prior work on bidding in auctions under budget constraints, and more generally online resource allocation, assumes that advertisers know the total number of auctions T (also called the *horizon*) in advance. This assumption is vital for previous algorithms and performance guarantees because it allows them to evenly spend the budget across all auctions. Even in the simple setting where the auctions come from stationary distributions, prior algorithms compute a *per-period resource budget* given by B/T, which crucially requires knowledge of T, and then use it as the target amount to spend in each auction. However, if the horizon T is not known to the decision maker, one can no longer compute this quantity and these previous works fail to offer any meaningful guidance.

Juxtapose this with a world in which viral trends are becoming ever more common, causing online advertising platforms, retailers and service providers to routinely experience traffic spikes. These spikes inject uncertainty into the system and make it difficult to accurately predict the total number of users that will arrive. As the amount of user traffic determines the number of auctions an advertiser participates in, this uncertainty extends to the number of auctions. Moreover, these spikes often present lucrative opportunities for the advertiser, which makes addressing this uncertainty even more pertinent [EKM15]. Crucially, it is usually difficult to predict these spikes, e.g. a news story breaks about COVID-travel bans being lifted, which results in a sudden and large uptick in the number of advertising opportunities for an airline. In fact, search-traffic spikes might be so large that they cause websites to crash[3]. This uncertainty about the horizon T raises the following question:

> *How should one design algorithms that are robust to horizon uncertainty and do not require knowledge of T?*

To address this question, we take the robust-optimization approach and consider a model in which the horizon T is unknown and only assumed to lie in some known uncertainty window $[\tau_1, \tau_2]$. τ_1 and τ_2 parameterize the advertiser's uncertainty about the horizon T. We measure the performance of algorithms using their asymptotic competitive ratio against the hindsight-optimal solution which can be computed with full knowledge of all auctions and the horizon T. In order to tease out the impact of horizon uncertainty on the performance of algorithms, we focus on the setting where the auctions are drawn from stationary distributions.

Our Results. We first show that the performance of any algorithm necessarily degrades with horizon uncertainty. In particular, we prove that no algorithm can achieve a worst-case asymptotic competitive ratio better than $\{e \cdot \ln \ln(\tau_2/\tau_1)\}/\ln(\tau_2/\tau_1)$. We then propose an algorithm that is nearly optimal and always achieves a competitive ratio greater than $1/(1 + \ln(\tau_2/\tau_1))$. Finally, we also provide a method for incorporating predictions about the horizon into our algorithm, and do so in a way that can be tuned to achieve the desired balance between worst-case performance and trust in the prediction. Importantly, all of our results are much more general than bidding under budget constraints, and apply to all online resource allocation problems [BLM23]. Specifically, it includes as special cases various fundamental problems like network revenue management [TVR04], online advertising [Meh13], online linear/convex programming [AD14; AWY14; Dev+11; Kes+14], and assortment optimization under inventory constraints [GNR14].

[3] `https://developers.google.com/search/blog/2012/02/preparing-your-site-for-traffic-spike`

Key Insights. Our impossibility result shows that online resource allocation is qualitatively different than other online optimization problems like online convex optimization. Any algorithm for the latter can be modified using the Doubling Trick to achieve an asymptotic competitive ratio of 1 under horizon uncertainty. In contrast, for online resource allocation, horizon uncertainty imposes a fundamental limit on the performance of algorithms that cannot to bypassed with standard modifications like the Doubling Trick. Algorithms must be designed with the horizon uncertainty in mind, which is what we do. Like Chapter 2, our algorithm first computes a target expenditure plan, and then uses Mirror Descent in the dual space to follow these targets. We characterize the competitive ratio of our algorithm for all target plans, and then simply optimize over all target plans, both in the setting with and without predictions.

Practical Insights. The optimal target expenditure plan is a decreasing sequence. It advocates for front-loading of expenditure: the advertiser should spend more aggressively early on. The intuition being that future opportunities may or may not materialize due to horizon uncertainty, and she should hedge against this uncertainty. Put another way, one should not conserve too much budget in the hopes of a traffic spike, and instead front-load expenditure to lock in sufficient amount of utility in early auctions. In practice, advertisers and platforms use machine learning models to predict the total number of auctions. However, these models are often opaque and uninterpretable (like neural nets). In particular, they can perform horribly in case of traffic spikes and do not provide worst-case guarantees. Thus, one cannot blindly rely on predictions from such models and incorporate them into bidding algorithms. Our algorithm allows the advertiser to strike a balance between completely trusting the prediction and her desire for worst-case guarantees: we provide a method for incorporating these potentially-inaccurate predictions in a way that optimally trades off consistency, which refers to the performance when the predictions are accurate, and worst-case competitive ratio.

Technical Contributions. Our algorithm is a generalization of Dual Mirror Descent [BLM23] that can incorporate target expenditure plans which are time varying. We characterize the competitive ratio of this generalization in semi-closed form for all possible target expenditure plans. This characterization forms the cornerstone of our positive results, and allows us to reduce the problem of finding the optimal algorithm to that of finding the optimal target plan. The latter is much easier than the former and we show that it can be written as an LP. But general-purpose LP solvers can be too computationally expensive in large-scale applications like online advertising. So, we also provide a quadratic-time algorithm for solving the LP.

1.2 Equilibria in Budget Management Systems

Chapter 2 and Chapter 3 focus on budget management from an individual advertiser's perspective, and design algorithms for maximizing utility in repeated auctions subject to budget constraints. What happens when all advertisers simultaneously use these algorithms? This is the question we address in the second half of the thesis, where we zoom out and analyze the entire market. Our focus is on the strategic interactions of the advertisers, each of whom attempts to maximize their own utilities subject to budget constraints. In particular, we analyze the equilibrium outcomes that emerge from these interactions, both from a structural and computational viewpoint. Moreover, we also investigate the impact of the auction format on the equilibrium outcomes, with the aim of informing the rules of advertising auction markets.

1.2.1 Chapter 4: Contextual Standard Auctions with Budgets

Consider the setting in which multiple advertisers are competing against each other for ad slots. Each of them wants to maximize their utility subject to their budget constraint (possibly using some autobidding algorithm). The strategic interactions between advertisers, each of whom is pursuing their own objective, raises many new questions:

> *Does an equilibrium always exist? What is the structure of equilibrium bidding strategies? How do the answers depend on the auction format?*

We would like to analyze the equilibrium points of the market that can arise from the interactions of such buyers. In particular, we are interested in *Bayes-Nash Equilibria (BNE)*, which are market outcomes in which all advertisers are happy with their bidding strategy and none of them wants to deviate unilaterally. These are the points to which individually-optimal autobidding algorithms for budget management stabilize, assuming they do stabilize. Moreover, in this chapter, we go beyond second-price auctions and consider all *standard auctions*, which is the class of all auctions in which the highest bidders wins the item. Importantly, it includes the popular first-price auction format as a special case and allows us to compare it with second-price auction. This raises another important question:

How does the auction format impact the equilibrium outcomes?

Furthermore, till now, our focus has almost entirely been on the advertisers and their utility. But an equally important aspect of these markets is the seller revenue and societal welfare. The former is often the primary objective of the platform and the latter measures the allocative efficiency of the market. In fact, it is often a combination of revenue and welfare, rather than advertiser utility, that determines the mechanism used by the platform. This raises other important questions:

How does the auction format impact the revenue of the platform? How socially-efficient are the equilibria in allocating opportunities to advertisers?

To address these questions, we study a model of standard auctions with contextual values and average budget constraints. The contextual values allow us to incorporate the structured correlation that arises from the same user features being used by all advertisers in determining their values. And average budget constraints capture the impact of long-term budget constraints that are only required to hold across thousands of auctions: if each advertiser statisfies their budget constraints on average in each auction, then concentration arguments imply that they spend close to their global budget across all auctions with high probability.

Our Results. We show that a structured Bayes-Nash equilibrium always exists for all standard auctions. It composes value-pacing, which refers to multiplicatively shading down the value, with the equilibrium strategy for the setting without budget constraints. We then leverage this structure to show a revenue equivalence result: each standard auction yields the same revenue in equilibrium. Finally, we bound the price of anarchy of these equilibria: we show that the liquid welfare (measure of social welfare for settings with budgets) is at least half of what a clairvoyant central planner can attain. In other words, these value-pacing-based equilibria are approximately efficient in allocating opportunities to advertisers.

Key Insights. Our results act as a powerful black-box: they takes as input any Bayes-Nash equilibrium for the commonly-studied i.i.d. setting without budgets, composes it with value-pacing, and outputs a Bayes-Nash equilibrium for our model with contextual values and budgets. Surprisingly, we show that, for a fixed distribution over advertisers and users, the same multiplicative factors can be used by the advertisers to shade their values in the equilibrium strategies for all standard auctions. In other words, these equilibrium strategies are modular: they compose value-pacing, which addresses the budget constraints and is independent of the auction-format, with the i.i.d. BNE strategy that is independent of budgets and captures the strategic misreporting induced by the auction format. This modular structure immediately allows us to extend the well-known revenue-equivalence result for the simple i.i.d. model (without budgets and contexts) to our much more general model with contexts and budgets. The revenue equivalence results and the modular structure also allow us to prove price of anarchy bounds for liquid welfare which hold for all standard auctions.

Practical Implications. The display advertising[4] industry recently switched from second-price auctions to first-price auctions as the method for selling advertising opportunities. Our revenue equivalence result suggests that the transition should not result in a change in revenue for the platform. This is in stark contrast to previous works on standard auctions with strict budget constraints where revenue equivalence does not hold [CG98]. A recent paper of [Gok+22] empirically investigated the revenue impact resulting from this recent switch. [Gok+22] found that, after a brief adjustment period, publishers' revenues under first-price auctions returned to the same levels as

[4]Display advertising refers to graphic advertising through banners, text, images, video, and audio.

they were under second-price auctions before the change. Our theory offers the first principled justification for this empirical finding by establishing revenue equivalence in the presence of contextual values and average budget constraints. Finally, the modular structure of the equilibrium strategies also provides guidance for advertisers in navigating the change in auction formats. The pipeline used for pacing can be composed with the pipeline for bidding in non-truthful auctions, each operating autonomously.

Technical Contributions. The primary technical challenge involved establishing the existence of structured modular bidding strategies for potentially non-truthful auctions. These strategies have a pacing (dual) multiplier for each buyer type, which are uncountably many in cardinality. This leads to an infinite-dimensional equilibrium space even after moving to the simpler dual space. In infinite dimensions, establishing even the simple prerequisites of any fixed-point theorem, namely compactness and continuity, can be an ordeal; one which requires careful topological arguments. While other papers have also analyzed equilibrium strategies in the dual space (see, e.g., [BBW15; GKP12]), these consider settings with finitely-many pacing multipliers in which establishing compactness is a trivial task. The main technical contribution of this chapter is twofold: (i) choosing the right topological space for the pacing multipliers based on their monotonicity properties, (ii) establishing compactness and continuity in this carefully chosen space. As we discuss in Subsection 4.2.3, this choice of topology is far from obvious. In fact, to the best of our knowledge, all of the topologies used in standard fixed-point arguments for infinite-dimensional spaces (see [AB06] for examples) prove insufficient in the setting we consider, which compels us to carefully exploit the structural properties of pacing and work with the topological space of multivariate-functions of bounded variation. We believe that these tools might be useful in other non-atomic games. Finally, our price of anarchy bound is also novel. It does not proceed through the smoothness framework [RST17] and instead leverages the complementary slackness condition of the budget-constrained utility maximization problem of each buyer.

1.2.2 Chapter 5: The Complexity of Pacing for Second-Price Auctions

When instantiated for second-price auctions, the results of Chapter 4 imply the existence of an equilibrium in which all of the buyers pace (i.e., multiplicatively shade) their value to determine their bid. Additionally, our algorithms from Chapter 2 and Chapter 3 also pace values by a multiplicative factor to determine bids, and then update the multiplicative factor to ensure budget consumption in line with the target plan. The use of pacing is not unique to our work: (i) it shows up in all algorithms for bidding in repeated auctions under budget constraints which are optimal for stochastic environments, and (ii) pacing-based equilibria have been shown to exist in other models of advertiser interaction in second-price auctions with budget constraints. This raises a natural question that had long been the subject of conjecture and speculation [Con+18; Bor+07]:

> *Do pacing-based algorithms always converge to pacing-based equilibria? If so, do they converge efficiently (in polynomial time)?*

More generally, most of the attention in prior work had focused on either developing pacing-based algorithms or establishing the existence of pacing-based equilibria, with very little attention being devoted to its computational properties. Understanding the computational complexity of finding pacing-based equilibria is vital, because if one cannot even compute them, then there is very little hope for advertisers, each of whom runs their own algorithm on a computer, to converge to them.

> *What is the computational complexity of finding pacing-based equilibria?*

We initiate the study of these questions by studying the computational complexity of finding pacing equilibria in markets with correlated values. Since a pacing equilibrium always exists, we cannot use P vs NP to characterize the complexity of computing one; the right complexity class is PPAD. Like the well-known complexity class NP, PPAD (*Polynomial Parity Argument in a Directed graph*, introduced by [Pap94]) is a collection of computational problems. As with the definition of NP-hardness and NP-completeness, a problem is said to be PPAD-hard if it is at least as hard as every problem in PPAD; a problem is said to be PPAD-complete if it is contained in PPAD and is PPAD-hard. The analogy to NP extends further: the PPAD-hardness of a problem can be established by providing a polynomial-time reduction *from* a problem already known to be PPAD-hard. One of the quintessential PPAD-complete problems, and the one we will employ in our reductions, is that of computing a Nash equilibrium of a bimatrix game [DGP09; CD06]. The Nash equilibrium problem has been studied extensively for decades and yet, despite much effort, no polynomial-time algorithm is known for it. Moreover, a recent spate of results showed that it is hard to solve, assuming certain strong cryptographic assumptions [BPR15; GPS16; RSS17; HY17; Cho+19]. This has motivated the conjecture that PPAD-hard problems cannot be solved efficiently. If we can show that the problem of finding a pacing equilibrium is PPAD-hard, then it shows that computing a pacing equilibrium is hard, unless all problems in PPAD can be solved efficiently. In the remainder of this subsection, we will assume that PPAD-hard problems are impossible to solve.

Our Results. We prove that the problem of computing pacing equilibria is PPAD complete. In particular, it belongs to the class PPAD—which (informally) means that it is easier than every problem in that class—and it is PPAD-hard—which (informally) means that it is harder than every problem in the class. In fact, we show that the simpler problem of computing even an approximate pacing equilibrium is PPAD-hard. This disproves the conjecture of [Bor+07], who proved polynomial-time convergence of tâtonnement-based dynamics for first-price auctions and predicted similar convergence for second-price auctions. Our hardness results shows that no dynamics always converge for second-price auctions. Moreover, we close the open question of [Con+18], who proved the existence of pacing equilibria but left its computational complexity open.

Key Insights. One of the most important consequences of the PPAD completeness is the impossibility of efficient convergence. We show that no matter which pacing algorithm is employed by the individual advertisers, it will not always converge to an equilibrium in polynomial time. It puts a stop to the quest for pacing algorithms that can be proven to efficiently converge to an equilibrium in all cases.

Practical Implications. Ours is a worst-case impossibility result: it does not say that algorithms will never converge, just that none of them will always converge. In particular, one may still observe convergence to equilibrium in practice. It forces us to refine our expectations and invites further investigation into the features of the market that may lead to convergence, e.g. structured correlation, strong competition etc.

Technical Contributions. We prove the PPAD-hardness of finding approximate pacing equilibria (Theorem 11) by giving a reduction from the problem of finding an ϵ-well-supported Nash equilibrium in win-lose bimatrix games. To prove the PPAD-membership of finding a pacing equilibrium (Theorem 12), we reduce the problem to the algorithmic version of Sperner's Lemma. A direct reduction proves challenging due to the discontinuous way in which the allocation of an item varies with pacing multipliers: In a pacing equilibrium, an item can only be assigned to buyers whose bids are *exactly* equal to the highest bid. Similar issues were encountered in PPAD-membership proofs for market equilibrium computation [VY11]. For this reason, we start by proving the PPAD-membership of finding approximate pacing equilibria, in which items can be allocated smoothly. Then we bootstrap this result to show the PPAD-membership of exact pacing equilibrium.

1.2.3 Chapter 6: Throttling Equilibria in Auction Markets

Hitherto, our focus was on pacing. In Chapter 2 and Chapter 3, we develop algorithms that implemented pacing as a method of budget management, and in Chapter 4 and Chapter 5, we analyze pacing-based equilibria. This focus is well-motivated from (i) a theoretical standpoint: pacing-based algorithms obtain optimal performance guarantees in a variety of environments, and (i) a practical standpoint: it is a popular method of budget management deployed in practice [FG]. However, despite the appealing theoretical and practical properties, budget management is not limited to pacing, and consequently neither is this thesis. In particular, we also analyze throttling, which is another popular method of budget management used in practice.

Unlike pacing, which controls expenditure by multiplicatively shading the value of the advertiser to determine the bid, throttling controls spending by modulating the probability with which an advertiser participates in auctions. In contrast to other budget-management methods like pacing, throttling does not modify the bids of the advertisers to achieve this, which is essential for advertisers aiming to maintain a stable cost-per-opportunity [FG]. Additionally, in practice, many advertisers do not opt into budget-management services that modify their bids, forcing the platform to satisfy their budget constraint by only controlling their participation probability, as in throttling [KMS13]. Importantly, throttling also gives advertisers a more representative sample of users for which they are eligible and their bid is competitive [KMS13]. This is in contrast to budget-management approaches that modify bids, such as pacing, which biases the allocation towards users where the advertiser has a high probability of getting a click, relative to other advertisers. Many advertisers place a premium on the predictability and representative samples offered by unmodified bids, motivating the platforms to offer throttling as a budget-management option.

Consider a market in which all of the advertisers use throttling to manage their budgets, with each advertiser attempting to maximize their utility subject to budget constraints using throttling-based strategies. Like pacing, this strategic interaction of advertisers brings up many natural questions:

> *Do throttling-based equilibria always exist? If so, are they unique? Can they be computed in polynomial time? What is the impact of the auction format? How does throttling compare to pacing?*

We define a *throttling game* with budget-constrained buyers (advertisers) and stochastic good types (user types), in which each buyer chooses the probability with which she participates in the auction, with the goal of maximizing her expected utility while satisfying her budget constraint in expectation. Repeated play of this throttling game captures the repeated online ad auction setting in which each buyer employs throttling to manage their budget. Furthermore, we define the concept of *throttling equilibrium* for this game, show its equivalence to pure strategy Nash equilibrium, and analyze it with an emphasis on its structural and computational properties.

Our Results. For first-price auctions, we show that a unique equilibrium always exists, is well-behaved and can be computed efficiently via tâtonnement-style decentralized dynamics. In contrast, for second-price auctions, we prove that even though an equilibrium always exists, the problem of finding even an approximate equilibrium is PPAD-complete, there can be multiple equilibria, and it is NP-hard to find the revenue maximizing one. We also compare the equilibrium outcomes of throttling to those of pacing, which is the other most popular and well-studied method of budget management. Finally, we characterize the Price of Anarchy of these equilibria for liquid welfare by showing that it is at most 2 for both first-price and second-price auctions, and demonstrating that our bound is tight.

Key Insights. Our results on throttling reinforce what the analysis of pacing suggested: budget management is computationally intractable for second-price auctions. In contrast, throttling equilibria in first-price auctions are extremely well-behaved, both computationally and structurally. Intuitively, this difference stems from the locus of control for expenditure in the two auctions formats: in second-price auctions, competing advertisers determine the payment, whereas the advertiser's own bid determines the payment in first-price auctions. The similarities between throttling and pacing go beyond computational properties; both yield similar revenue in first-price auctions and lead to the same price of anarchy bounds for liquid welfare.

Practical Implications. Given the failure of pacing-based algorithms to always converge in second-price auctions, one might look to other methods of budget management to get such a property. The search for such a method will have to go beyond throttling as well. In contrast, we show that simple tâtonnement-style algorithms for throttling exhibit fast convergence to equilibrium in first-price auctions. Thus, if advertisers use such algorithms, then one can reasonably assume that the market is at equilibrium while performing inference and experiments.

Technical Contributions. We prove that the problem of computing approximate throttling equilibria is PPAD-hard for second-price auctions, even when each good has at most three bids (Theorem 18), by showing a reduction from the PPAD-hard problem of computing an approximate equilibrium of a threshold game [PP21]. Furthermore, we place the problem of computing approximate throttling equilibria in the class PPAD by showing a reduction to the problem of finding a Brouwer fixed point of a Lipschitz mapping from a unit hypercube to itself (Theorem 20); the latter is known to be in PPAD via Sperner's lemma. We provide additional evidence of the computational challenges that afflict throttling for second-price auctions by proving the NP-hardness of finding a revenue-maximizing approximate throttling equilibrium (Theorem 21). We complement these hardness results by describing a polynomial-time algorithm for computing throttling equilibria for the special case in which there are at most two bids on each good (Algorithm 7), thereby precisely delineating the boundary of tractability. On the other hand, for first-price auctions, we show that payment is non-decreasing in the advertiser's own participation probability and non-increasing in competing advertiser's participation probabilities. This allows us show that our algorithm, which increases/decreases the participation probability in response to underspending/overspending, converges to an approximate equilibrium in polynomial time.

1.3 Related Work

Our work lies at the intersection of many major streams of literature. Broadly, the related literature can be categorized into two categories depending on whether their focus is on analyzing online learning algorithms or equilibrium outcomes. Our goal here is not to provide an all-inclusive survey, but to discuss works that are most closely related to ours.

1.3.1 Online Algorithms for Resource Allocation

[BG19] study budget pacing in repeated second-price auctions when the values and competing bids are either i.i.d. according to some unknown distribution or adversarially selected. They propose and analyze Dual Gradient Descent with the constant target sequence, i.e., it always targets B/T for all $t \in [T]$. They show that it attains the optimal regret of $O(\sqrt{T})$ in the i.i.d. stochastic setting, and the optimal parameter-dependent asymptotic competitive ratio (equal to ratio of the per-

period budget to the maximum value) in the adversarial setting. [ZCL08] also study the adversarial setting and provide a pacing-based algorithm that achieves a differently-parameterized competitive ratio which scales as the logarithm of the ratio of the highest-to-lowest return-on-investment, and show that it is optimal. [KMS22] study an episodic setting and provide a density-estimation-based algorithm for learning the target expenditures for each episode. [Gai+22] study the performance of the algorithm of [BG19] for the different objective of value maximization, and against the different benchmark comprised of pacing multipliers which spend the same amount B/T at each time period; they show that that it achieves $O(T^{3/4})$ regret. Moreover, when all of the buyers employ [BG19] to bid, they show that the price of anarchy of liquid welfare is at most 2 even if the algorithms do not converge to equilibrium. [FT23] study first-price auctions and prove price of anarchy bounds that similarly do not require convergence to equilibrium. [Luc+23] extend the work of [Gai+22] to the setting with bandit feedback and an additional ROI constraint. In comparison, our price of anarchy bounds hold only at equilibrium and are independent of the algorithm being used by each of the buyers. Recently, [Wan+23; CCK23] studied bidding in repeated first-price auctions under global budget constraints, and developed primal-dual-style algorithms that achieve good guarantees in stationary stochastic environments.

More generally, budget pacing in second-price auctions is a special case of online linear packing, which in turn is a special case of the online resource allocation problem. Both these problems allow for multiple resources and have been studied extensively; we only provide a broad overview here. For the most part, these problems have also been studied in the i.i.d. stochastic model, or the slightly more general random arrival model (requests are selected by an adversary but arrive in a uniformly random order). [DH09] and [Fel+10] study online linear packing under the random arrival model, and show that learning the dual from the initial requests and then using it to make decisions yields $O(T^{2/3})$ regret. [AWY14] extended these results to show that repeatedly solving for the dual at geometrically increasing intervals yields the optimal $O(\sqrt{T})$ regret. [Dev+11], [GM16] and [Kes+14] also achieve $O(\sqrt{T})$ regret but with a better dependence on the constants and the number of resources. [Dev+11] also achieve $O(\sqrt{T})$ regret when the environment is non-stationary and the optimal expected reward for each distribution is known in advance. However, this quantity cannot be computed with a single sample for non-trivial distributions, and they do not provide guarantees for the sample-access setting. Recently, [JLZ20] initiated the study non-stationary linear packing with access to historical samples, but require $O(T \log T)$ samples per distribution (essentially complete knowledge of the distribution) to achieve $O(\sqrt{T})$ regret.

[AD14] study online resource allocation with concave rewards and convex constraints, and give a dual-descent-based algorithm that achieves $O(\sqrt{T})$ regret. [BLM23] give a Dual Mirror Descent algorithm which attempts to spend B/T at each time step and show that it achieves $O(\sqrt{T})$ regret for the general online allocation problem. Their results also hold for stochastic models that are close to i.i.d. like periodic, ergodic etc.

Another line of work develops algorithms that beat $O(\sqrt{T})$ regret when the problem instance is well-structured (see the recent work of [BKK22] for a discussion). With the exception of [BGV20] and [BF20], all of these works assume complete knowledge of the distributions and/or assume that the distributions are identical. When the number of requests of each type satisfies a concentration property between the trace and the actual requests, [BGV20] and [BF20] show that a constant regret can be achieved for online resource allocation using one sample per distribution. For the budget pacing problem, a type corresponds to a value and competing bid pair. Since complex machine-learning models are typically used to estimate advertiser values to a high precision, this translates to an extremely large number of possible types. Far from concentrating, these large number of types imply that one is unlikely to even observe a type more than once, making their primal-based method ineffective for budget pacing. Moreover, neither [BGV20] nor [BF20] provide any robustness guarantees for possible discrepancies between the sampling and true distributions, and their algorithm requires knowledge of the competing bid. Finally, our results are meaningful when the budget is much larger than the maximum amount one can spend on an auction/request, as is the case for budget pacing. In contrast, the literature on prophet inequalities considers a unit-cost variable-reward online allocation problem where the budget is only large enough to accept one request. See [AKW14; Cor+19; RWW20; Car+22] for a sample-driven treatment of prophet inequalities.

There is also a line of work studying online allocation problems when requests are adversarially chosen. Naturally, the fully-adversarial model subsumes our input model, in which requests are drawn i.i.d. from an unknown distribution and the horizon is uncertain. Therefore, guarantees for adversarial algorithms carry over to our setting. We remark, however, that it is not possible to obtain bounded competitive ratio for the general online allocation problem (see, e.g., [Fel+09]). Notable exceptions are online matching [KVV90], the AdWords problem [Meh+07], or personalized assortment optimization [GNR14], which are linear problems in which rewards are proportional to resource consumption. When rewards are not proportional to resource consumption, there is a stream of literature studying algorithms with parametric competitive ratios. These competitive ratios either depend on the range of rewards (see, e.g., [BQ09; MSL20]) or the ratio of budget to resource consumption (see, e.g., [BLM23]).

Finally, a few very recent papers warrant attention, all of which allow for horizon uncertainty but assume that the distribution of the horizon is known in advance. [Bru+19] study a generalization of online bipartite matching which accounts for ranked preferences over the offline vertices under a variety of input models. They show that a constant competitive ratio cannot be attained under stationary stochastic input when the horizon is completely unknown and use it to justify the known-horizon assumption. Our impossibility result (Theorem 6) establishes a parametrized upper bound on the competitive ratio in terms of the uncertainty τ_2/τ_1 and implies their result as a special case when $\tau_2/\tau_1 \to \infty$. [Ali+20] study the multi-unit prophet-inequality problem in which the resource is perishable, with each unit of the resource exiting the system independently at some time whose distribution is known to the decision maker. When there is one unit of the resource, their model captures horizon uncertainty in the prophet-inequality problem, which is a special case of online resource allocation. Importantly, when there is more than one unit, our models are incomparable. For the single-unit special case, they prove a parameterized upper bound of $\tilde{O}\left(\ln(\tau_2/\tau_1)^{-1}\right)$ on the competitive ratio. In contrast, our upper bound (Theorem 6) holds for the more general regime where the initial resource endowment (number of units of the resource) scales linearly with the horizon and the action space is continuous. This is crucial since the performance guarantees of algorithms for online resource allocation with known-horizon often only hold in this regime [BLM23; Meh+07; TVR04], thereby making the single-unit upper bound inapplicable.

[Bai+22] develop a fluid approximation to the dynamic-programming solution for network revenue management when both the distribution of the request and the horizon are completely known. They show that the asymptotically-tight fluid approximation should attempt to respect the resource constraint for all possible horizon values and not in expectation over the horizon. [AM22] consider a model for network revenue management in which the distribution of the horizon is known and each type of request follows an adversarial or random-order arrival pattern. They also show that the fluid LP relaxation based on the expected value of the horizon can be arbitrarily bad and develop tighter LP relaxations. We do not assume that the type of requests, the distribution of requests or the distribution of the horizon are known ahead of time, and use the hindsight optimal allocation as the benchmark, making our results incomparable even for the special case of network revenue management.

1.3.2 Equilibria in Budget Management Systems

[BBW15] studied budget management in second-price auctions using a *fluid mean-field* model, and showed that in this model existence is guaranteed, and closed-form solutions for equilibria are derived for certain settings. [Bal+21] analyze several different methods for budget management in second-price auctions, including pacing and throttling, and showed existence results for their setting, as well as other analytical and numerical properties. [Con+18] define and study pacing equilibria in second-price auctions. They show that it always exists and study its structural properties. [Con+19] studied the model of [Con+18], but with each auction using a first-price rule. There, pacing equilibrium no longer constitutes best responses, but instead has a market equilibrium interpretation. In the first-price setting, pacing equilibria turn out to be easy to compute, due to a direct relationship to market equilibria. Moreover, for first-price auctions, [Bor+07] describe a simple tâtonnement-style dynamics and prove its efficient convergence to a pacing equilibrium.

They conjectured a similar convergence for second-price auctions, which we disprove. [Bab+20] studied non-quasi-linear agents participating in mechanisms designed for quasi-linear agents. They studied a generalization of budget constraints where agents have a concave disutility in payment, and showed that a Nash equilibria exists which employs multiplicative scaling.

Another direction of research considers buyers with ex-post budget constraints (also called strict budget constraints). There, first price [Kot20], standard auctions [CG98], optimal auctions [PV14], and auctions with combinatorial constraints [GML15] have been studied. In contrast to our revenue equivalence results, [CG98] show that with strict budget constraints first-price auctions yield higher revenue than second-price auctions. These models are different from our setting which only requires budget constraints to hold in expectation at the interim stage. In-expectation budget constraints are more appropriate for modeling repeated ad auctions, and yield simpler and more interpretable equilibrium strategies.

Throttling has also received significant attention in other lines of research. [Aga+14] study throttling in generalized second-price (GSP) auctions from the perspective of a single buyer, provide an algorithm which determines the participation probability based on user traffic forecasts, and analyze its performance empirically on real data from LinkedIn. Similarly, [Xu+15] provide and empirically evaluate practical algorithms for throttling on data from demand-side platforms. [KMS13] use throttling (under the name Vanilla Probabilistic Throttling) as the benchmark in the GSP auction setting to evaluate the budget management algorithm they describe on data from Google, and find that it empirically outperforms throttling on the metrics they study. Importantly, they do not engage in an equilibrium analysis, and their algorithm does not provide a representative sample of the traffic to advertisers. There is also a significant body of work which proposes alternatives to pacing and throttling methods. [Cha+13] study regret-free budget-smoothing policies in which the platform selects the random subset of buyers that participate in the GSP auction for each good. They show that such policies always exist, and, under the small-bids assumption, give an efficient algorithm for the special case of second-price auctions.

Chapter 2: Robust Budget Pacing with a Single Sample

Based on the publication [Bal+23] co-authored with Santiago Balseiro, Vahab Mirrokni, Balasubramanian Sivan and Di Wang.

Motivated by the online advertising industry, this chapter studies the non-stationary stochastic budget management problem: an advertiser repeatedly participates in T second-price auctions, where her value and the highest competing bid are drawn from unknown time-varying distributions, with the goal of maximizing her total utility subject to her budget constraint. In the absence of any information about the distributions, it is known that sub-linear regret cannot be achieved. We assume access to historical samples, with the goal of developing algorithms that are robust to discrepancies between the sampling distributions and the true distributions. In Section 2.2, we first show the sufficiency of a single sample per distribution to obtain $\tilde{O}(\sqrt{T})$ regret. We do so via a simple "Learning the Dual and Earning with It" approach that learns a fixed dual multiplier from historical samples and then uses it to pace (multiplicatively shade) the value for bidding. However, this algorithm turns out to be extremely brittle to distribution shifts between the sampling distributions and the true distributions. In Section 2.3, we propose the Dual Follow-The-Regularized-Leader (FTRL) algorithm and prove our main result: we show that Dual FTRL is robust to distribution shifts and achieves a near-optimal $\tilde{O}(\sqrt{T})$-regret with just one sample per distribution, drastically improving over the best-known sample-complexity of T samples per distribution. For ease of exposition, we prove our results for the more general single-resource online allocation problem with linear rewards/consumptions. It is well-known that bidding in repeated second-price auctions with budgets can be modelled as an instance of this online allocation problem (e.g., see [BLM23]); we provide a formal reduction in Section 2.4 for completeness.

2.1 Model

Notation. We use $\mathbb{R}_+$ and $\mathbb{R}_{++}$ to denote the set of non-negative real numbers and the set of positive real numbers respectively. For $n \in \mathbb{N}$, we use $[n] = \{1, \ldots, n\}$ to denote the set of positive integers less than or equal to n. We use $\mathcal{W}(\cdot, \cdot)$ to denote the Wasserstein distance between two distributions under the metric with which the sample space is endowed.

Online Allocation with a Single Resource and Budget Management. For ease of exposition, we will prove our results for the more general single-resource online allocation problem with linear rewards/consumptions. It is well-known that bidding in repeated second-price auctions with budgets can be modeled as an instance of this online allocation problem (e.g., see [BLM23], or Section 2.4). It also captures the stochastic multi-secretary problem [AG19] as a special case.

Consider a decision maker with an initial budget $B \in \mathbb{R}_{++}$ of a resource, whose goal is to optimally spend it on T sequentially arriving requests. Each request $\gamma = (f, b)$ is comprised of a linear reward function $f : \mathcal{X} \to \mathbb{R}_+$ such that $f(x) = \text{coeff}(f) \cdot x$, and a linear resource consumption function $b : \mathcal{X} \to \mathbb{R}_+$ such that $b(x) = \text{coeff}(b) \cdot x$; where $\mathcal{X} \subseteq \mathbb{R}_+$ is a compact set which denotes the space of possible actions of the decision maker. We will use $\mathcal{S}$ to denote the set of all possible requests and $\Delta(S)$ to denote the set of distributions over $\mathcal{S}$. Moreover, we endow $\mathcal{S}$ with the following metric $d(\cdot, \cdot)$: For any two requests $\gamma = (f, b)$ and $\tilde{\gamma} = (\tilde{f}, \tilde{b})$:

$$d(\gamma, \tilde{\gamma}) = \sup_{x \in \mathcal{X}} \left| f(x) - \tilde{f}(x) \right| + \sup_{x \in \mathcal{X}} \left| b(x) - \tilde{b}(x) \right| .$$

We will assume that $0 \in \mathcal{X}$. This allows the decision maker to avoid spending the resource if she so chooses and ensures feasibility. Moreover, let $\bar{x} = \max_{x \in \mathcal{X}} x$. We will make standard regularity assumptions [JLZ20; BLM23]: there exist $\bar{f}, \bar{b} \in \mathbb{R}_+$ such that $f(x) \leq \bar{f}$ and $b(x) \leq \bar{b}$ for all $x \in \mathcal{X}$. Like [JLZ20], we will also assume that there exists $\kappa \in \mathbb{R}_+$ such that $f(x) \leq \kappa \cdot b(x)$ for all $x \in \mathcal{X}$, i.e., the maximum rate of return from spending the resource is bounded above by κ.

At time $t \in [T]$, the following sequence of events takes place: (i) a request $\gamma_t = (f_t, b_t)$ arrives; (ii) the decision maker observes γ_t and chooses an action $x_t \in \mathcal{X}$ based on the information observed so far; (iii) the request consumes $b_t(x_t)$ amount of the resource and generates a reward of $f_t(x_t)$. The decision maker aims to maximize her rewards subject to her budget constraint. A policy $\{x_t(\cdot)\}_t$ for the decision maker maps requests to actions $x_t : \mathcal{S} \to \mathcal{X}$ based on the available information at each time step, i.e., the action $x_t(\gamma_t)$ at time $t \in [T]$ can depend on the historical requests $\{\gamma_s\}_{s=1}^{t-1}$ and the current request γ_t, but not the future requests $\{\gamma_s\}_{s=t+1}^{T}$. Moreover, a policy is said to be budget-feasible if it respects the budget constraint by ensuring $\sum_{t=1}^{T} b_t(x_t(\gamma_t)) \leq B$ for every sequence $\{\gamma_t\}_t$.

The request γ_t at time t is drawn from a distribution $\mathcal{P}_t \in \Delta(\mathcal{S})$ unknown to the decision maker, independently of the requests at other time steps. We only require the requests $\{\gamma_t\}_t$ to be independent and allow the distributions $\mathcal{P}_t$ to vary arbitrarily across time. We will measure the performance of a policy against the fluid-optimal benchmark, which is defined as:

$$\begin{aligned} \mathrm{FLUID}(\{\mathcal{P}_t\}_t) := \quad \max \quad & \sum_{t=1}^{T} \mathbb{E}[f_t(x_t(\gamma_t))] \\ \text{s.t.} \quad & \sum_{t=1}^{T} \mathbb{E}[b_t(x_t(\gamma_t))] \leq B \\ & x_t : \mathcal{S} \to \mathcal{X} \quad \forall\, t \in [T]\,. \end{aligned}$$

Another benchmark common in the literature on online resource allocation is the expected hindsight optimal solution, which is defined as $\mathbb{E}[\mathrm{OPT}(\{\gamma_t\}_t)]$ for

$$\mathrm{OPT}(\{\gamma_t\}_t) := \max_{x \in \mathcal{X}^T} \sum_{t=1}^{T} f_t(x_t) \text{ s.t. } \sum_{t=1}^{T} b_t(x_t) \leq B\,.$$

It is well-known that $\mathrm{FLUID}(\{\mathcal{P}_t\}) \geq \mathbb{E}[\mathrm{OPT}(\{\gamma_t\}_t)]$, which makes our benchmark the stronger one (we provide a proof in Appendix A.1 for completeness). Hence, our performance guarantees relative to the fluid-optimal benchmark also imply the same guarantees for the expected hindsight-optimal benchmark.

More concretely, we use $R(A|\{\gamma_t\}_t)$ to denote the total reward of a policy A on the request sequence $\{\gamma_t\}_t$, and the performance of an algorithm is measured using its expected regret against the fluid-optimal reward:

$$\text{Regret}(A) := \text{FLUID}(\{\mathcal{P}_t\}_t) - \mathbb{E}\left[R(A|\{\gamma_t\}_t)\right] .$$

Now, if the distributions $\{\mathcal{P}_t\}_t$ are unknown and arbitrary, and no other information about $\{\mathcal{P}_t\}_t$ is available, then the requests $\{\gamma_t\}_t$ can be adversarial. This case has been addressed in [BLM23], where the authors showed no policy can achieve sub-linear regret. In this work, we address the setting in which the decision maker has additional information in the form of historical samples. In particular, we focus on the setting where the decision maker has access to one independent sample $\tilde{\gamma}_t \sim \tilde{\mathcal{P}}_t$ for each $t \in [T]$. We will assume that the $\{\tilde{\gamma}_t\}$ samples are independent of the request sequence $\{\gamma_t\}_t$ and $\{\tilde{\mathcal{P}}_t\}$ are not known to the decision maker. We will show that when the sampling distributions $\{\tilde{\mathcal{P}}_t\}_t$ are not too far from the actual distributions $\{\mathcal{P}_t\}_t$, which is a minimal relaxation over the adversarial setting, it is possible to achieve sub-linear regret. We refer to the collection of samples $\{\tilde{\gamma}_t\}_t$ as a *trace* and allow the actions of the decision-maker to depend on it. Throughout this chapter, we will use $\{\tilde{\gamma}_t\}_t$ to denote the trace and $\{\gamma_t\}_t$ to denote the (random) sequence of requests on which the decision maker wishes to maximize reward.

2.2 Warmup: Learning the Dual and Earning with It

First, let us focus on the simpler case when $\tilde{\mathcal{P}}_t = \mathcal{P}_t$ for all $t \in [T]$, i.e., the sampling distributions are the same as the request distributions. At first glance, it may appear that only having access to one sample from each of request distributions $\mathcal{P}_t$ yields too little information to achieve near-optimal rewards. If one were to attempt to directly learn the optimal solution of FLUID($\{\mathcal{P}_t\}_t$), this initial impression would be accurate because of the high-dimensional nature of the space of all possible solutions $\{x_t(\cdot)\}_t$. Fortunately, we do not need to learn this high-dimensional information and can instead leverage the structure of the problem: the dual space is just one-dimensional and thereby amenable to learning. More precisely, the dual function $D(\mu|\{\mathcal{P}_t\}_t)$ of FLUID($\{\mathcal{P}_t\}_t$) at dual variable $\mu \geq 0$ is given by

$$
\begin{aligned}
&\max_{\{x_t(\cdot)\}_t} \sum_{t=1}^{T} \mathbb{E}[f_t(x_t(\gamma_t))] + \mu \left(B - \sum_{t=1}^{T} \mathbb{E}[b_t(x_t(\gamma_t))] \right) \\
&= \mu \cdot B + \sum_{t=1}^{T} \max_{x_t : \mathcal{S} \to \mathcal{X}} \mathbb{E}\left[f_t(x_t(\gamma_t)) - \mu \cdot b_t(x_t(\gamma_t)) \right] \\
&= \mu \cdot B + \sum_{t=1}^{T} \mathbb{E}\left[\max_{x_t \in \mathcal{X}} \{ f_t(x_t) - \mu \cdot b_t(x_t) \} \right] .
\end{aligned}
$$

Throughout, we assume $\mathrm{argmax}_{x \in \mathcal{X}} \{f(x) - \mu \cdot b(x)\}$ is non-empty for all requests $\gamma \in \mathcal{S}$ and dual solutions $\mu \geq 0$. If we treat the dual variable μ as the per-unit price of the resource, $\max_{x_t \in \mathcal{X}} \{f_t(x_t) - \mu \cdot b_t(x_t)\}$ captures the profit maximization problem. The following terminology would be helpful in working with the dual.

Definition 1. *For a request $\gamma = (f, b)$ and dual variable $\mu \geq 0$, let $x^*(\gamma, \mu)$ be the optimal solution of* $\max_{x \in \mathcal{X}} \{f(x) - \mu \cdot b(x)\}$ *with the largest value of $f(x)$. If there are multiple such solutions, pick one which minimizes $b(x)$. Moreover, let $f^*(\mu) := f(x^*(\gamma, \mu))$ and $b^*(\mu) := b(x^*(\gamma, \mu))$ be the corresponding reward and resource consumption respectively.*

We denote $D(\mu | \{\mathcal{P}_t\}_t) = \mu \cdot B + \sum_{t=1}^{T} \mathbb{E}[f_t^*(\mu) - \mu \cdot b_t^*(\mu)]$. Throughout this chapter, we will repeatedly leverage weak duality, which is a central property of duals. We state the property here and refer the reader to any standard text on convex optimization (e.g., [Ber09]) for a proof.

Proposition 1 (Weak Duality)**.** *For all request distributions $\{\mathcal{P}_t\}_t$ and dual variables $\mu \geq 0$, we have $D(\mu | \{\mathcal{P}_t\}_t) \geq \mathrm{FLUID}(\{\mathcal{P}_t\}_t)$, i.e.,*

$$
\sum_{t=1}^{T} \mathbb{E}[f_t^*(\mu)] \geq \mathrm{FLUID}(\{\mathcal{P}_t\}_t) - \mu \cdot \left(B - \sum_{t=1}^{T} \mathbb{E}[b_t^*(\mu)] \right) .
$$

Observe that $\sum_{t=1}^{T} \mathbb{E}[f_t^*(\mu)]$ is exactly the expected reward the decision maker would receive if she had an infinite budget and she took actions which maximized profit with μ being the per-unit price of the resource. Moreover, $B - \sum_{t=1}^{T} \mathbb{E}[b_t^*(\mu)]$ is the amount by which the decision maker would underspend her budget in expectation if she were to take actions using μ as the price.

Suppose we can find a dual variable $\mu \geq 0$ that satisfies approximate complementary slackness, i.e., μ satisfies at least one of the following statements: (1) $\mu = 0$ and maximizing profit with μ as the per-unit price results in total expenditure less than the budget B ($\sum_{t=1}^{T} b_t^*(\mu) \leq B$) with high probability; (2) $\mu > 0$ and maximizing profit with μ as the per-unit price results in total expenditure close to the budget B ($\sum_{t=1}^{T} b_t^*(\mu) \approx B$) with high probability. Then, if the decision maker were to use μ as the price and make decisions to maximize profit, she will not run out of budget too early and the complementary slackness term $\mu \cdot \left(B - \sum_{t=1}^{T} \mathbb{E}[b_t^*(\mu)]\right)$ would also be small. Therefore, such a μ would yield rewards that are close to FLUID($\{\mathcal{P}_t\}_t$), i.e., yield small regret, as required.

We next describe how such a μ can be learned from the sample trace $\{\tilde{\gamma}_t\}$ when $\tilde{\mathcal{P}}_t = \mathcal{P}_t$ for all $t \in [T]$. We will assume that the distributions satisfy the following mild and standard assumption [DH09; AWY14] to exclude the degenerate case.

Assumption 1 (General Position)**.** *The request sequence $\{\gamma_t\}_t \sim \prod_t \mathcal{P}_t$ is in general position almost surely: For any $\mu \geq 0$, there is at most one request with multiple profit maximizers, i.e.,*

$$\left|\left\{t \in [T] : |\mathrm{argmax}_{x \in \mathcal{X}} \{f_t(x) - \mu \cdot b_t(x)\}| > 1\right\}\right| \leq 1\,.$$

Moreover, the sample trace $\{\tilde{\gamma}_t\}_t \sim \prod_t \tilde{\mathcal{P}}_t$ is also in general position almost surely.

Assumption 1 is made without any loss of generality because, as pointed out in [DH09] and [AWY14], adding an infinitesimally-small perturbation to the reward functions always results in perturbed distributions that satisfy Assumption 1 with only an infinitesimal change in the value of FLUID($\{\mathcal{P}_t\}_t$) (see Appendix A.2 for a formal description). Assumption 1 ensures that there exists a dual solution $\tilde{\mu} \geq 0$ which spends close to the budget B on the trace $\{\tilde{\gamma}_t\}_t$ if it is possible to do so. In fact, as the following lemma shows, the optimal empirical dual solution satisfies this property.

Lemma 1. *Suppose the trace $\{\tilde{\gamma}\}_t \sim \prod_t \tilde{\mathcal{P}}_t$ is in general position, and consider*

$$\tilde{\mu} \in \mathrm{argmin}_{\mu \geq 0} \left\{\mu \cdot B + \sum_{t=1}^{T} \max_{x \in \mathcal{X}} \left\{\tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x)\right\}\right\}\,.$$

Algorithm 1: Learning the Dual and Earning with It

Input: Trace $\{\tilde{\gamma}_t\} \sim \prod_t \tilde{\mathcal{P}}_t$, initial budget $B_1 = B$.

Compute an Optimal Empirical Dual Solution:

$$\tilde{\mu} \in \operatorname{argmin}_{\mu \geq 0} \left\{ \mu \cdot B + \sum_{t=1}^{T} \max_{x \in \mathcal{X}} \left\{ \tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x) \right\} \right\} \tag{2.1}$$

for $t = 1, \ldots, T$ **do**

Receive request $\gamma_t = (f_t, b_t) \sim \mathcal{P}_t$.

Make the primal decision x_t and update the remaining resources B_t:

$$x_t' \in \operatorname{argmax}_{x \in \mathcal{X}} \{f_t(x) - \tilde{\mu} \cdot b_t(x)\} \ ,$$
$$x_t = \begin{cases} x_t' & \text{if } b_t(x_t') \leq B_t \\ 0 & \text{otherwise} \end{cases} ,$$
$$B_{t+1} = B_t - b_t(x_t).$$

end

Then, at least one of the following statements holds:

1. $\tilde{\mu} = 0$ *and* $\sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu}) \leq B + \bar{b}$.
2. $\left| B - \sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu}) \right| \leq \bar{b}$.

Recall that weak duality (Proposition 1) suggests that finding a dual solution which satisfies approximate complementary slackness with high probability would yield reward close to FLUID($\{\mathcal{P}_t\}$). Lemma 1 states that we can compute a dual variable $\tilde{\mu}$ which satisfies approximate complementary slackness on the trace. To finish the argument, we require a uniform convergence bound which shows that expenditure on the trace (or the sequence of requests) is concentrated close to the expected expenditure for all dual variables $\mu \geq 0$.

Theorem 1. *For* $r(T) := 8\bar{b} \cdot \sqrt{T \log(T)}$ *and request distributions* $\{\tilde{\mathcal{P}}_t\}_t$*, the following uniform convergence bound holds*

$$\Pr\left(\sup_{\mu \geq 0} \left| \sum_{t=1}^{T} \tilde{b}_t^*(\mu) - \sum_{t=1}^{T} \mathbb{E}_{\hat{\gamma}_t \sim \tilde{\mathcal{P}}_t} \left[\hat{b}_t^*(\mu) \right] \right| \geq r(T) \right) \leq \frac{1}{T^2} \, .$$

With Theorem 1 in hand, we are now ready to state and prove the regret guarantee for Algorithm 1. It first learns an empirical optimal dual variable $\tilde{\mu}$ from the trace $\{\tilde{\gamma}_t\}_t$, and then uses it as the per-unit price of the resource to take profit-maximizing actions on the request sequence $\{\gamma_t\}_t$.

Theorem 2. *If $\mathcal{P}_t = \tilde{\mathcal{P}}_t$ for all $t \in [T]$, then Algorithm 1 (denoted by A) satisfies* $\mathrm{Regret}(A) \leq 12\kappa\bar{b} + 2\kappa r(T)$.

[AG19] showed that every algorithm must incur a regret of $\Omega(\sqrt{T})$, even when the request distributions are identical (i.e., $\mathcal{P}_t = \mathcal{P}$ for all $t \in [T]$) and known to the decision-maker ahead of time. Thus, Theorem 2 shows the regret of Algorithm 1 achieves a near-optimal dependence on T with just a single sample per distribution, despite the request distributions being unknown and time-varying. However, as the following example demonstrates, this regret bound critically relies on the assumption that $\mathcal{P}_t = \tilde{\mathcal{P}}_t$ for all $t \in [T]$, and is fragile to even slight deviations from it. This fact was also demonstrated in [JLZ20] in a related context which inspired the following example.

Example 1. *Fix a small $\epsilon > 0$, an even horizon T and budget $B = T/2$. Assume actions are accept/reject decisions, i.e., $\mathcal{X} = \{0, 1\}$, and the reward/resource consumption functions are linear with* $\mathrm{coeff}(b) = 1$ *for all $\gamma = (f, b) \in \mathcal{S}$. In this setting, a request is completely determined by the coefficient* $\mathrm{coeff}(f)$ *of its reward function. We will overload notation and use γ to denote this coefficient. Set $\tilde{\mathcal{P}}_t = \mathit{Unif}([1+\epsilon, 1+2\epsilon])$ for all $t \leq T/2+1$ and $\tilde{\mathcal{P}}_t = \mathit{Unif}([1-\epsilon, 1])$ for all $t \geq T/2+2$. Moreover, set $\mathcal{P}_t = \mathit{Unif}([1-\epsilon, 1])$ for all $t \in [T]$. Then, it is easy to see that $\mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t) \leq 3\epsilon$ for all $t \in [T]$. Also, observe that any trace $\{\tilde{\gamma}_t\}_t \sim \prod_t \tilde{\mathcal{P}}_t$ would satisfy $\tilde{\gamma}_t \geq 1+\epsilon$ for all $t \leq T/2+1$ and $\tilde{\gamma}_t \leq 1$ for all $t \geq T/2+2$. Hence, we always have $\tilde{\mu} \geq 1+\epsilon$. On the other hand, we also always have $\gamma_t \leq 1$ for all $t \in [T]$. Therefore, Algorithm 1 sets $x'_t = 0$ for all $t \in [T]$, yielding a reward of 0. Whereas,* $\mathrm{FLUID}(\{\mathcal{P}_t\}_t) \geq (1-\epsilon)\cdot(T/2)$, *thereby making the regret linear in T.*

Since $\epsilon > 0$ was arbitrary in the above example, it shows that even infinitesimally-small differences between the sampling and request distributions can lead to linear regret for Algorithm 1. This is antithetical to our goal of developing robust online algorithms for pacing. Formally, we would like to develop online algorithms that achieve regret which is small and degrades smoothly as $\sum_{t=1}^{T} \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t)$ grows large. Nonetheless, although Algorithm 1 falls short of this goal, it highlights the power of dual-based algorithms. Building on the intuition developed in this section, we next describe and analyze a Dual FTRL algorithm that achieves near-optimal regret while being robust to discrepancies between the sampling distributions $\{\tilde{\mathcal{P}}_t\}_t$ and the request distributions $\{\mathcal{P}_t\}_t$.

2.3 Dual FTRL with Target Rate Estimation

In this section, we will develop an algorithm based on Dual Follow-The-Regularized-Leader (FTRL) that achieves near-optimal regret with a single trace, and is robust to discrepancies between the sampling distributions and request distributions. Now, if one had complete knowledge of the sampling distributions $\{\tilde{\mathcal{P}}_t\}_t$, then one can solve FLUID($\{\tilde{\mathcal{P}}_t\}_t$) to find an optimal solution and run Dual Gradient Descent with the goal of spending the same as the optimal solution at each time step. It is known from [JLZ20] that this approach achieves $O(\max\{\sqrt{T}, \sum_{t=1}^{T} \mathcal{W}(\tilde{\mathcal{P}}_t, \mathcal{P}_t)\})$ regret, thereby making it rate optimal and robust to discrepancies. However, with just a single sample from each of distributions $\tilde{\mathcal{P}}_t$, we are far from having complete knowledge of $\{\tilde{\mathcal{P}}_t\}_t$. Despite this apparent lack of data, a careful analysis of Dual FTRL will allow us to show that it achieves near-optimal regret rate in a robust manner.

2.3.1 Dual Follow-The-Regularized-Leader

The non-stationarity of the request distributions necessitates the need for Dual FTRL that can incorporate target resource consumptions (Algorithm 2). It takes as input a target sequence $\{\lambda_t\}_{t=1}^{T}$ which specifies $\lambda_t \geq 0$ to be the amount of resource Dual FTRL should attempt to consume at time t. Moreover, like FTRL [SS+12; Haz+16], it also takes as input a regularizer $h(\cdot)$, an initial dual variable μ_1 and a step-size η. We will make the standard assumption that the regularizer $h(\cdot)$ is differentiable and is σ-strongly convex in the $\|\cdot\|_1$ norm.

Before stating the performance bound of Algorithm 2, we introduce some preliminaries. Given

Algorithm 2: Dual Follow-The-Regularized-Leader

Input: Initial resource endowment $B_1 = B$, target consumption sequence $\{\lambda_t\}_{t=1}^T$, regularizer $h : \mathbb{R} \to \mathbb{R}$ and step-size η.

Set initial dual solution $\mu_1 = \text{argmin}_{\mu \in [0,\kappa]} h(\mu)$.

for $t = 1, \ldots, T$ **do**

Receive request $\gamma_t = (f_t, b_t) \sim \mathcal{P}_t$.

Make the primal decision x_t and update the remaining resources B_t:

$$x_t' \in \text{argmax}_{x \in \mathcal{X}_t} \{f_t(x) - \mu_t \cdot b_t(x)\} \ , \tag{2.2}$$

$$x_t = \begin{cases} x_t' & \text{if } b_t(x_t') \le B_t \\ 0 & \text{otherwise} \end{cases} ,$$

$$B_{t+1} = B_t - b_t(x_t).$$

Obtain a sample sub-gradient of the dual function $D(\mu|\mathcal{P}_t, \lambda_t)$: $g_t = \lambda_t - b_t(x_t')$.

Update the dual iterate with FTRL:

$$\mu_{t+1} = \text{argmin}_{\mu \in [0,\kappa]} \left\{ \eta \sum_{r=1}^{t} g_r \cdot \mu + h(\mu) \right\} , \tag{2.3}$$

end

a budget of β_t for period $t \in [T]$, the optimal expected reward which can be collected in period t is captured by the following fluid optimization problem:

$$\begin{aligned} \text{FLUID}(\mathcal{P}_t, \beta_t) \coloneqq \max \quad & \mathbb{E}[f_t(x_t(\gamma_t))] \\ \text{s.t.} \quad & \mathbb{E}[b_t(x_t(\gamma_t))] \le \beta_t \\ & x_t : \mathcal{S} \to \mathcal{X} \, . \end{aligned}$$

The dual function of $\text{FLUID}(\mathcal{P}_t, \beta_t)$ is given by

$$D(\mu|\mathcal{P}_t, \beta_t) \coloneqq \mu \cdot \beta_t + \mathbb{E}\left[\max_{x \in \mathcal{X}} \{f_t(x) - \mu \cdot b_t(x)\} \right] ,$$

for any $\mu \ge 0$. Then, by weak duality, we have $\text{FLUID}(\mathcal{P}_t, \beta_t) \le D(\mu|\mathcal{P}_t, \beta_t)$ for all $\mu \ge 0$. Moreover, since dual functions are always convex (they are the suprema of linear functions), the dual function $D(\cdot|\mathcal{P}_t, \beta_t)$ is convex.

Theorem 3 states a general regret bound for Algorithm 2 with an arbitrary target sequence $\{\lambda_t\}_t$ and against a general benchmark $\sum_{t=1}^T D(\mu_t|\mathcal{P}_t, \beta_t)$. Since $\text{FLUID}(\mathcal{P}_t, \beta_t) \leq D(\mu|\mathcal{P}_t, \beta_t)$ by weak duality, Theorem 3 also characterizes the performance against the weaker benchmark $\sum_{t=1}^T \text{FLUID}(\mathcal{P}_t, \beta_t)$, which is simply the optimal expected reward the decision maker would collect if she spent β_t at time t.

Theorem 3. *Consider Algorithm 2 with target consumption sequence $\{\lambda_t\}_t$, regularizer $h(\cdot)$ and step-size η. Then, for a benchmark sequence $\{\beta_t\}_t$, we have*

$$\mathbb{E}\left[\left\{\sum_{t=1}^T D(\mu_t|\mathcal{P}_t, \beta_t)\right\} - R(A|\{\gamma_t\}_t)\right] \leq R_1 + R_2 + R_3 ,$$

where

- $R_1 = \kappa\bar{b} + \frac{2(\bar{b}+\bar{\lambda})^2}{\sigma} \cdot \eta T + \frac{d_R}{\eta}$, *for* $\bar{\lambda} = \max_t \lambda_t$ *and* $d_R = \max\{h(0) - h(\mu_1), h(\kappa) - h(\mu_1)\}$.
- $R_2 = \kappa \cdot \left(\left\{\sum_{t=1}^T \lambda_t\right\} - B\right)^+$,
- $R_3 = \mathbb{E}\left[\sum_{t=1}^T \mu_t \cdot (\beta_t - \lambda_t)\right]$.

Theorem 3 decomposes the regret of Algorithm 2 into three terms, where (i) R_1 is simply the regret associated with the FTRL algorithm in the OCO setting [Haz+16]; (ii) R_2 captures the overspending error, which is large whenever the total target consumption $\sum_{t=1}^T \lambda_t$ is in excess of the budget B; (iii) R_3 captures the underestimation error, which is a weighted sum over the amounts by which the target sequence $\{\lambda_t\}_t$ underestimates the benchmark sequence $\{\beta_t\}_t$, with weights equal to the dual iterates μ_t. Observe that there is an inherent tension between the overspending error R_2 and the underestimation error R_3—R_2 can be made smaller by making the target consumptions $\{\lambda_t\}_t$ smaller, but this in turn makes R_3 bigger, and vice versa. To obtain the desired performance guarantees for Algorithm 2 (see Theorem 4), we need to carefully choose the benchmark sequence $\{\beta_t\}_t$ and the target sequence $\{\lambda_t\}_t$, which is what we do next (see (2.4)). We go on to show that

- Our choice of target sequence does not overspend too much. In particular, it satisfies $R_2 \leq \kappa \cdot \bar{b}$ (see (2.5)).

- Our choice of benchmark sequence $\{\beta_t\}_t$ ensures

$$\text{FLUID}(\{\mathcal{P}_t\}_t) - \sum_{t=1}^{T} D(\mu_t|\mathcal{P}_t, \beta_t) \leq \tilde{O}\left(\max\left\{\sqrt{T}, \sum_{t=1}^{T} \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t)\right\}\right)$$

  i.e., the benchmark in Theorem 3 is at most $\tilde{O}\left(\max\left\{\sqrt{T}, \sum_{t=1}^{T} \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t)\right\}\right)$ larger than our desired benchmark $\text{FLUID}(\{\mathcal{P}_t\}_t)$ (see Lemma 2 and the discussion that follows).

- Moreover, our choice of the sequences in combination with an intricate argument, consisting of a coupling lemma and a leave-one-out thought experiment, allows use to prove $R_3 = O(\sqrt{T})$ (see Subsection 2.3.3).

Finally in Subsection 2.3.4, we combine everything to prove the desired regret bound for Algorithm 2.

2.3.2 Choosing the Target and Benchmark Sequences

We define the target and benchmark sequences using the empirical optimal dual solution computed from the trace $\{\tilde{\gamma}_t\}_t$:

$$\tilde{\mu} \in \text{argmin}_{\mu \geq 0} \left\{\mu \cdot B + \sum_{t=1}^{T} \max_{x \in \mathcal{X}} \left\{\tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x)\right\}\right\}.$$

If there are multiple minimizers, set $\tilde{\mu}$ to be the smallest one. Given the empirical optimal dual solution $\tilde{\mu}$, the target and benchmark sequences are defined as

$$\beta_t = \mathbb{E}_{\hat{\gamma}_t \sim \tilde{\mathcal{P}}_t}\left[\hat{b}_t^*(\tilde{\mu})\right] \quad \text{and} \quad \lambda_t = \tilde{b}_t^*(\tilde{\mu}), \tag{2.4}$$

where $\hat{\gamma}_t = (\hat{f}_t, \hat{b}_t)$. In other words, the benchmark sequence is the expected consumption and the target sequence is the empirical consumption on the trace if we were to make profit-maximizing decisions using the empirical optimal dual solution $\tilde{\mu}$ as the price of the resource. Instead of learning the empirical optimal dual $\tilde{\mu}$ and directly making decisions with it like we did in Algorithm 1, we use $\tilde{\mu}$ to learn the empirical consumptions $\{\lambda_t\}_t$ and use Algorithm 2 to track this target. Unlike the former, we will show that the latter approach is robust to discrepancies between the sampling and request distributions, while maintaining the same $\tilde{O}(\sqrt{T})$-regret guarantee. Importantly, note that the benchmark sequence $\{\beta_t\}_t$ cannot be computed in practice because it requires full knowledge of the request distributions. Algorithm 2 respects this limitation and does not require knowledge of the benchmark sequence $\{\beta_t\}_t$; we only use it for our analysis.

Our choice of $\{\lambda_t\}_t$ and Lemma 1 immediately imply

$$R_2 = \kappa \cdot \left(\left\{ \sum_{t=1}^{T} \lambda_t \right\} - B \right)^+ \leq \kappa \cdot \bar{b}\,. \tag{2.5}$$

Next, we show that, for our choice of $\{\beta_t\}_t$, the benchmark $\sum_{t=1}^{T} D(\mu_t|\mathcal{P}_t, \beta_t)$ of Theorem 3 is not too far from the desired benchmark FLUID($\{\mathcal{P}_t\}$).

Lemma 2. *For any dual variable $\tilde{\mu} \geq 0$, dual iterates $\{\mu_t\}_t \in [0, \kappa]^T$ and benchmark sequence $\{\beta\}_t$ with $\beta_t = \mathbb{E}_{\hat{\gamma}\sim\tilde{\mathcal{P}}_t}[\hat{b}^*(\tilde{\mu})]$, we have*

$$\sum_{t=1}^{T} D(\mu_t|\mathcal{P}_t, \beta_t) \geq \text{FLUID}(\{\mathcal{P}_t\}_t) - \tilde{\mu} \cdot \left(B - \sum_{t=1}^{T} \beta_t \right) - 2(1+\kappa) \cdot \sum_{t=1}^{T} \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t)\,.$$

Observe that Theorem 1 implies that, with probability at least $1 - 1/T^2$, we have

$$\left| \sum_{t=1}^{T} \beta_t - \sum_{t=1}^{T} \lambda_t \right| \leq r(T).$$

Combining this with Lemma 1 yields

$$\begin{aligned} \tilde{\mu} \cdot \left(B - \sum_{t=1}^{T} \beta_t \right) &\leq \tilde{\mu} \cdot \left(r(T) + B - \sum_{t=1}^{T} \lambda_t \right) \\ &= \tilde{\mu} \cdot r(T) + \tilde{\mu} \cdot \left(B - \sum_{t=1}^{T} \lambda_t \right) \\ &\leq \tilde{\mu} \cdot \left(r(T) + \bar{b} \right) \end{aligned}$$

$$\leq \kappa \cdot r(T) + \kappa \bar{b}\,, \tag{2.6}$$

thereby showing that the benchmark $\sum_{t=1}^{T} D(\mu_t|\mathcal{P}_t, \beta_t)$ of Theorem 3 is not too far from the desired benchmark FLUID($\{\mathcal{P}_t\}$). In order to establish the desired regret and robustness guarantees for Algorithm 2, all that remains to show is that $R_3 \leq \tilde{O}(\sqrt{T})$. However, as we demonstrate in the next subsection, this step is rife with challenges.

2.3.3 Bounding R_3

We begin with a brief discussion of the challenges involved in bounding R_3. It is illuminating to consider the slightly more permissive setting in which the decision maker has access to two sample traces: suppose in addition to trace $\{\tilde{\gamma}_t\}_t \sim \prod_t \tilde{\mathcal{P}}_t$, we had access to an additional trace $\{\tilde{\tilde{\gamma}}_t\}_t \sim \prod_t \tilde{\mathcal{P}}_t$. Then, we could compute $\tilde{\mu}$ using $\{\tilde{\tilde{\gamma}}_t\}_t$ as follows

$$\tilde{\mu} \in \operatorname{argmin}_{\mu \geq 0} \left\{ \mu \cdot B + \sum_{t=1}^{T} \max_{x \in \mathcal{X}} \left\{ \tilde{\tilde{f}}_t(x) - \mu \cdot \tilde{\tilde{b}}_t(x) \right\} \right\} ,$$

making it completely independent of $\{\tilde{\gamma}_t\}_t$. With this modified $\tilde{\mu}$, we continue to define $\{\beta_t\}_t, \{\gamma_t\}_t$ as before (see (2.4)). As a consequence, we get that μ_s is completely determined by $\{\gamma_t\}_{t=1}^{s-1}$ and $\{\lambda_t\}_{t=1}^{s-1}$, with the latter being completely determined by $\tilde{\mu}$ and $\{\tilde{\gamma}_t\}_{t=1}^{s-1}$. This makes μ_s independent of λ_s conditional on $\tilde{\mu}$, and consequently yields

$$\mathbb{E}\left[\mu_s \cdot (\beta_s - \lambda_s) |\, \tilde{\mu}, \{\tilde{\gamma}_t, \gamma_t\}_{t=1}^{s-1}\right] = \mu_s \cdot \left(\beta_s - \mathbb{E}\left[\tilde{b}_s^*(\tilde{\mu}) |\, \tilde{\mu}\right]\right) = 0\,.$$

Thus, we can apply the Tower Rule of conditional expectations to get

$$\begin{aligned} R_3 &= \sum_{s=1}^{T} \mathbb{E}\left[\mu_s \cdot (\beta_s - \lambda_s)\right] \\ &= \sum_{s=1}^{T} \mathbb{E}\left[\mathbb{E}\left[\mu_s \cdot (\beta_s - \lambda_s) |\, \tilde{\mu}, \{\tilde{\gamma}_t, \gamma_t\}_{t=1}^{s-1}\right]\right] \\ &= 0\,. \end{aligned}$$

It is straightforward to see that the bounds on R_1 and R_2 established in the previous subsection continue to hold in this two-trace setting. Therefore, two traces allow us to achieve the near-optimal $\tilde{O}(\sqrt{T})$-regret while being robust to discrepancies between $\tilde{\mathcal{P}}_t$ and $\mathcal{P}_t$.

Although moving from two traces to one trace might appear to be a minor change, it introduces correlations that make the proof much more difficult. Observe that Algorithm 2 determines μ_s using $\{\lambda_t\}_{t=1}^{s-1}$, all of which depend on $\tilde{\mu}$, which in turn is computed using the request $\tilde{\gamma}_s$. Furthermore, λ_s directly depends on $\tilde{\gamma}_s$. Thus, μ_s and λ_s are intricately correlated with each other, which breaks the aforementioned argument for the two-trace setting. Nonetheless, R_3 can still be shown to be small, as we note in the following lemma and prove in the remainder of this subsection.

Lemma 3. *For all $s \in [T]$, we have*

$$R_3 = \sum_{s=1}^{T} \mathbb{E}\left[\mu_s \cdot (\beta_s - \lambda_s)\right] \leq \frac{4\eta \bar{b}^2}{\sigma} \cdot T \quad .$$

We prove Lemma 3 in the remainder of this subsection. The following lemma will find repeated use in the proof. In keeping with economic intuition, it shows that increasing the price (dual variable) leads to smaller consumption under the profit-maximzing decision.

Lemma 4 (Monotonicity)**.** *For $\mu > \mu'$, request $\gamma = (f, b) \in \mathcal{S}$, $x \in \operatorname{argmax}_{z\in\mathcal{X}}\{f(z) - \mu \cdot b(z)\}$ and $x' \in \operatorname{argmax}_{z\in\mathcal{X}}\{f(z) - \mu' \cdot b(z)\}$, we have $b(x) \leq b(x')$.*

Fix an $s \in [T]$. We will get around the correlation between μ_s and $\tilde{\gamma}_s$ by conducting the following leave-one-out thought experiment: suppose we remove the s-th sample $\tilde{\gamma}_s$, compute $\tilde{\mu}$ on the remaining trace $\{\tilde{\gamma}_t\}_{t\neq s}$, and run Algorithm 2 with the resulting target sequence. More precisely, in this thought experiment, we set $\tilde{\mathcal{P}}_s$ to be the distribution which always serves the request $\gamma = (f, b)$ with $f(x) = b(x) = 0$ for all $x \in \mathcal{X}$. Thus, $\tilde{f}_s(x) = \tilde{b}_s(x) = \tilde{f}^*(\mu) = \tilde{b}^*(\mu) = 0$ for all $x \in \mathcal{X}$ and $\mu \geq 0$. We will use the superscript $(-s)$ to denote the various variables in this thought experiment:

- $\tilde{\mu}^{(-s)} \in \operatorname{argmin}_{\mu\geq 0} \mu \cdot B + \sum_{t\neq s} \max_{x\in\mathcal{X}}\{\tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x)\}$. If there are multiple minimizers, set $\tilde{\mu}^{(-s)}$ to be the smallest one amongst them.

- $\lambda_t^{(-s)} = \tilde{b}_t^* \left(\tilde{\mu}^{(-s)}\right)$ for all $t \in [T]$.
- $\mu_t^{(-s)}$ is the t-th iterate of Algorithm 2 with the target consumption sequence $\left\{\lambda_t^{(-s)}\right\}_t$.

We begin by characterizing the impact of this change on the target consumption sequence.

Lemma 5. *For every sample trace $\{\tilde{\gamma}_t\}_t$, we have $\tilde{\mu} \geq \tilde{\mu}^{(-s)}$ and $\lambda_t \leq \lambda_t^{(-s)}$ for all $t \neq s$. Moreover,* $\sum_{t=1}^{s-1} \left|\lambda_t^{(-s)} - \lambda_t\right| \leq 3\bar{b}$.

Lemma 5 shows that the target sequences $\{\lambda_t\}_t$ and $\{\lambda_t^{(-s)}\}_t$ are close to each other. Next, we couple the dual iterates μ_t and $\mu_t^{(-s)}$ generated by Algorithm 2 to show that they never stray too far from each other whenever the target sequences are close.

Lemma 6 (Dual Iterate Coupling)**.** *Let $\{\mu_t\}_t$ and $\{\mu_t'\}_t$ denote the iterates generated by Algorithm 2 on the request sequence $\{\gamma_t\}_t$ for the target sequences $\{\lambda_t\}_t$ and $\{\lambda_t'\}_t$ respectively. Assume that the initial iterates are the same, i.e., $\mu_1 = \mu_1'$. Then, for all $s \in [T]$, we have*

$$\left|\mu_s - \mu_s'\right| \leq \frac{\eta}{\sigma} \cdot \left\{\sum_{t=1}^{s-1} |\lambda_t - \lambda_t'|\right\} + \frac{\eta}{\sigma} \cdot \bar{b} \,.$$

Applying Lemma 6 with $\lambda_t' = \lambda_t^{(-s)}$ and using Lemma 5 yields $\left|\mu_s - \mu_s'\right| \leq \frac{\eta}{\sigma} \cdot \left\{3\bar{b}\right\} + \frac{\eta}{\sigma} \cdot \bar{b} = \frac{4\eta\bar{b}}{\sigma}$. Combining this with the fact that $|\beta_s - \lambda_s| \leq \bar{b}$, we get

$$\begin{aligned}\mathbb{E}\left[\mu_s \cdot (\beta_s - \lambda_s)\right] &= \mathbb{E}\left[\left(\mu_s - \mu_s^{(-s)}\right) \cdot (\beta_s - \lambda_s)\right] + \mathbb{E}\left[\mu_s^{(-s)} \cdot (\beta_s - \lambda_s)\right] \\ &\leq \frac{4\eta\bar{b}}{\sigma} \cdot \bar{b} + \mathbb{E}\left[\mu_s^{(-s)} \cdot (\beta_s - \lambda_s)\right] \,. \end{aligned} \tag{2.7}$$

The next lemma shows that the second term is non-positive. Its proof critically leverages the fact that the iterate $\mu_s^{(-s)}$ is independent of the s-th sample in the trace $\tilde{\gamma}_s$ (which is used to determine λ_s). This is in stark contrast to μ_s which depends on $\tilde{\gamma}_s$, and demonstrates the merit of our leave-one-out thought experiment.

Lemma 7. $\mathbb{E}\left[\mu_s^{(-s)} \cdot (\beta_s - \lambda_s)\right] \leq 0$ *for all $s \in [T]$.*

Lemma 7 in combination with (2.7) yields $\mathbb{E}\left[\mu_s \cdot (\beta_s - \lambda_s)\right] \leq 4\eta\bar{b}^2/\sigma$. Summing over all $s \in [T]$ finishes the proof of Lemma 3.

2.3.4 Putting It All Together

We have bounded R_1, R_2 and R_3, and related the benchmark $\sum_{t=1}^{T} D(\mu_t|\mathcal{P}_t, \beta_t)$ from Theorem 3 to our desired benchmark FLUID($\{\mathcal{P}_t\}_t$). Combining everything yields the following performance guarantee for Algorithm 2.

Theorem 4. *Let A be Algorithm 2 with target sequence $\{\lambda_t\}_t$, where $\lambda_t = \tilde{b}_t^*(\tilde{\mu})$ (as defined in* (2.4)*), regularizer $h(\cdot)$ and step-size $\eta = \sqrt{d_R/T}$, where $d_R = \max\{h(0) - h(\mu_1), h(\kappa) - h(\mu_1)\}$. Then,*

$$\text{Regret}(A) \leq C_1\sqrt{T\log(T)} + C_2\sum_{t=1}^{T}\mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t)\,.$$

where $C_1 = \frac{12\bar{b}^2\sqrt{d_R}}{\sigma} + \sqrt{d_R} + 12\kappa\bar{b}$ and $C_2 = 2(1+\kappa)$.

Observe that the regret of Dual FTRL satisfies $\text{Regret}(A) = \tilde{O}(\sqrt{T})$ whenever $\sum_{t=1}^{T}\mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t) = \tilde{O}(\sqrt{T})$. In other words, Dual FTRL achieves near-optimal regret with a single trace as long as the total discrepancy $\sum_{t=1}^{T}\mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t)$ is not too large. Finally, we would also like to note that our algorithm is extremely efficient computationally. In particular, due to the equivalence of FTRL and “Lazy" Online Mirror Descent (OMD) (see [Haz+16]), each dual update in (2.3) can be computed in constant time by running Lazy OMD. Moreover, given a trace $\{\tilde{\gamma}_t\}$ which is sorted in increasing order of bang-per-buck $\text{coeff}(\tilde{f}_t)/\text{coeff}(\tilde{b}_t)$, the target sequence $\{\lambda_t\}_t$ can be computed in $O(T)$ steps (see Appendix A.3 for details).

2.4 Application to Budget Pacing

Here, we discuss how the budget pacing problem fits as a special case of the online resource allocation problem that we study in this chapter. Consider the setting in which a budget-constrained advertiser repeatedly participates in T second-price auctions. For simplicity, assume that all ties are broken in favor of this advertiser. Let v_t and d_t denote her value and the highest competing bid in the t-th auction respectively. Moreover, let B denote her budget, which represents the maximum amount she is willing to spend over all T auctions.

We will assume that the tuple (v_t, d_t) is drawn from some distribution $\mathcal{P}_t$, independently of all other auctions. Now, observe that every bid of the advertiser results in one of two possible outcomes: (i) she bids greater than or equal to d_t, wins the auction, gains utility $v_t - d_t$ and pays d_t; (ii) she bids strictly less than d_t, loses the auction, gains zero value and pays zero. Thus, corresponding to the tuple (v_t, d_t), we can define a corresponding request γ_t with linear reward function $f_t(x) = (v_t - d_t) \cdot x$ and linear consumption function $b_t(x) = d_t \cdot x$, for the action space $x \in \{0, 1\} = \{\text{lose}, \text{win}\}$. Similarly, corresponding to the sample trace of tuples $\{(\tilde{v}_t, \tilde{d}_t)\}_t$, we can define a trace $\{\tilde{\gamma}_t\}_t$ for the online allocation problem. This defines a corresponding instance of the online allocation problem. Since every bid either results in either a win or loss, the maximum expected utility (value - payment) that the advertiser can earn subject to her budget constraint is bounded above by FLUID($\{\mathcal{P}_t\}_t$) for this instance. Finally, consider step t of Algorithm 3 on this instance. The decision x_t is calculated as $x_t \in \operatorname{argmax}_{x \in \mathcal{X}_t} \{f_t(x) - \mu_t \cdot b_t(x)\}$. Therefore, $x_t = 1$ if $v_t - d_t \geq \mu d_t$, or equivalently $v_t/(1 + \mu_t) \geq d_t$, and $x_t = 0$ otherwise. Observe that, in a second price auction, if the advertiser bids $v_t/(1 + \mu_t)$, she will win ($x_t = 1$) if $v_t/(1 + \mu_t) \geq d_t$ and lose ($x_t = 0$) otherwise. Thus, by bidding $v_t/(1 + \mu_t)$, she can simulate the actions of Algorithm 3 for the online allocation instance. Moreover, she does not require knowledge of the competing bid d_t to compute her bid, which is crucial because d_t is not known in practice. Once the auction is over, the expenditure $b_t(x_t) = d_t \cdot x_t$ is revealed to the advertiser. She can then use it to update the dual iterate according to (2.3).

Chapter 3: Online Resource Allocation under Horizon Uncertainty

Based on the publication [BKK23b] co-authored with Santiago Balseiro and Christian Kroer.

In this chapter, we relax the assumption that the total number of auctions T (also called the horizon) is known to the decision maker in advance, and develop algorithms for general stochastic online resource allocation which are robust to this uncertainty. In Section 3.3.1, we show that no online algorithm can achieve a greater than $\tilde{O}\left(\ln(\tau_2/\tau_1)^{-1}\right)$ fraction of the hindsight optimal reward (Theorem 6). This upper bound holds even when (i) there is only 1 type of resource, (ii) the decision maker receives the same request at each time step, (iii) this request is known to the decision maker ahead of time, (iv) the request has a smooth concave reward function and linear resource consumption, (v) τ_1 is arbitrarily large, and (vi) the initial resource endowment $B = \Theta(\tau_1)$ scales with the horizon. In particular, unlike the known-horizon setting, vanishing regret is impossible to achieve under horizon uncertainty, leading us to focus on developing algorithms with a good asymptotic competitive ratio (fraction of the hindsight optimal reward).

Dual mirror descent is a natural algorithm for the known-horizon case introduced by [BLM23], who build on a long line of primal-dual algorithms for online allocation problems [AD14; Dev+11; GM16]. It maintains a price (i.e., dual variable) for each resource and then dynamically updates them with the goal of consuming the per-period resource budget at each step—if the resource is being over-consumed, increase its price; and vice-versa. As stated earlier, this approach fails if the horizon is not known because the per-period budget cannot be computed ahead of time. A natural approach to handle horizon uncertainty is to use dual mirror descent with some proxy horizon $T^* \in [\tau_1, \tau_2]$ in the hopes of getting good performance for all $T \in [\tau_1, \tau_2]$. Unfortunately, as we show in Section 3.1.1, this approach can be extremely sub-optimal, not just for dual mirror descent but for any algorithm which is optimal for the known-horizon setting. Thus, the unknown-horizon setting calls for new algorithms. Our main insight is that, even though one cannot compute the

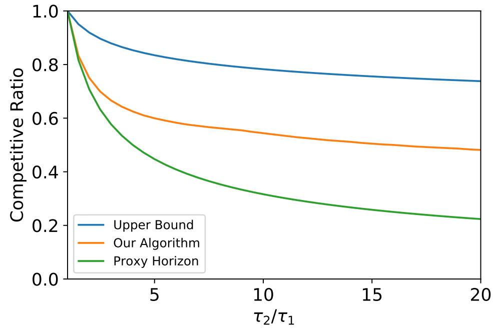

Figure 3.1: Plot with (i) our upper bound on the best-possible competitive ratio (Theorem 6) which scales as $\tilde{O}\left(\ln(\tau_2/\tau_1)^{-1}\right)$, (ii) the (asymptotic) competitive ratio of our algorithm which scales as $\Omega\left(\ln(\tau_2/\tau_1)^{-1}\right)$ (Algorithm 3 with target sequence from Algorithm 4), and (iii) an upper bound on the competitive ratio of algorithms that are optimal for the known-horizon setting when used with some proxy horizon $T^* \in [\tau_1, \tau_2]$ (Section 3.1.1), which scales as $\left(\sqrt{\tau_2/\tau_1}\right)^{-1}$. Even for small values of τ_2/τ_1, our algorithm significantly outperforms previous ones.

per-period resource budget and target its consumption, it is possible to compute a time-varying sequence of target consumptions which, if consumed at those rates, perform well no matter what the horizon turns out to be. To achieve this, we propose and analyze *Variable Target Dual Mirror Descent* in Section 3.2, which takes a sequence of target consumptions as input and dynamically updates the prices to hit those targets. One of our primary technical contributions is generalizing the analysis of dual mirror descent to develop a fundamental bound that allows for general target consumption sequences. We leverage this bound to show that there exists a simple time-varying target consumption sequence which can be described in closed form and achieves a near-optimal $\Omega\left(\ln(\tau_2/\tau_1)^{-1}\right)$ asymptotic competitive ratio when deployed with Algorithm 3, matching the upper bound up to logarithmic factors.

Variable Target Dual Mirror Descent reduces the complex problem of finding an algorithm which maximizes the competitive ratio to the much simpler problem of finding the optimal target consumption sequence. We develop an algorithm to solve the latter efficiently (Algorithm 4), leading to substantial gains over previous algorithms even for small values of τ_2/τ_1 (see Figure 3.1). Importantly, Algorithm 4 does not require one to solve computationally-expensive linear programs (LPs), which can be desirable in time-sensitive applications. Finally, in Section 3.4, we use the Algorithms-with-Predictions framework [MV20] to study incorporating (potentially inaccurate) predictions about the horizon with the goal of performing well if the prediction comes true, while also ensuring a good competitive ratio no matter what the horizon turns out to be. We show that the problem of computing the optimal target consumption sequence for the goal of optimally incorporating predictions can also be solved efficiently using Algorithm 4. Our algorithm allows the decision maker to account for the level of confidence she has in the predictions, and smoothly interpolate between the known-horizon and uncertainty-window settings.

3.1 Model

Notation We use $\mathbb{R}_+$ to denote the set of non-negative real numbers and $\mathbb{R}_{++}$ to denote the set of positive real numbers. We use a^+ to denote $\max\{a, 0\}$. For a vector $v \in \mathbb{R}^m$ and a scalar $a \in \mathbb{R} \setminus \{0\}$, v/a denotes the scalar multiplication of v by $1/a$. For vectors $u, v \in \mathbb{R}^m$ and a relation $R \in \{\geq, >, \leq, <\}$, we write $u \; R \; v$ whenever the relation holds component wise: $u_i \; R \; v_i$ for all $i \in [m]$.

We consider a general online resource allocation problem with m resources, in which requests arrive sequentially. At time t, a request $\gamma_t = (f_t, b_t, \mathcal{X}_t)$ arrives, which is composed of a reward function $f_t : \mathcal{X}_t \to \mathbb{R}_+$, a resource consumption function $b_t : \mathcal{X}_t \to \mathbb{R}_+^m$ and a compact action set $\mathcal{X}_t \subset \mathbb{R}_+^d$. We assume that $0 \in \mathcal{X}_t$ and $b_t(0) = 0$ for all t. This ensures that the decision maker has the option to not spend any resources at each time step. We make no assumptions about the convexity/concavity of either f_t, b_t or $\mathcal{X}_t$.

Let $\mathcal{S}$ represent the set of all possible requests. We make standard regularity assumptions: there exist constants $\bar{f}, \bar{b} \in \mathbb{R}_+$ such that, for every request $\gamma = (f, b, \mathcal{X}) \in \mathcal{S}$, we have $|f(x)| \leq \bar{f}$ and $\|b(x)\|_\infty \leq \bar{b}$ for all $x \in \mathcal{X}$. Furthermore, we assume that the requests are drawn i.i.d. from some distribution $\mathcal{P}$ over $\mathcal{S}$, both of which are not assumed to be known to the decision maker. The decision maker has a known initial resource endowment (or budget) of $B = (B_1, \ldots, B_m) \in \mathbb{R}_{++}^m$, where B_i denotes the initial amount of resource i available to the decision maker. We will assume that $B_i \geq 2\bar{b}$ for all $i \in [m]$.

Let T denote the total number of requests that will arrive (also called the horizon). We will use

$\rho_T = B/T$ to denote the per-period resource endowment that is available to the decision maker. In contrast to previous work, we do not assume that T (or its distribution) is known to the decision maker. Looking ahead, this uncertainty is what makes our problem much harder than vanilla online resource allocation where the horizon is known, as evidenced by the fact that no algorithm can attain even a positive competitive ratio when nothing is known about the horizon T (see Theorem 6), which is a far-cry from the no-regret property exhibited by algorithms for the known-horizon setting.

At each time step $t \le T$, the following sequence of events take place: (i) A request $\gamma_t = (f_t, b_t, \mathcal{X}_t)$ arrives and is observed by the decision maker; (ii) The decision maker selects an action $x_t \in \mathcal{X}_t$ from the action set based on the information seen so far; (iii) The decision maker receives a reward of $f_t(x_t)$ and the request consumes $b_t(x_t)$ resources. The goal of the decision maker is to take actions that maximize her total reward while keeping the total consumption of resources below the initial resource endowment. More concretely, an online algorithm (for the decision maker) chooses an action $x_t \in \mathcal{X}_t$ at each time step $t \le T$ based on the current request $\gamma_t = (f_t, b_t, \mathcal{X}_t)$ and the history observed so far $\{\gamma_s, x_s\}_{s=1}^t$ such that the resource constraints $\sum_{t=1}^T b_t(x_t) \le B$ are satisfied almost surely w.r.t. $\vec{\gamma} \sim \mathcal{P}^T$. Our results continue to hold even if the actions $\{x_t\}_t$ are randomized, but we work with deterministic actions for ease of exposition. Since we assume that T is not known to the decision maker, the actions of the online algorithm cannot depend on T. The total reward of algorithm A on a sequence of requests $\vec{\gamma} = (\gamma_1, \dots, \gamma_T)$ is denoted by $R(A|T, \vec{\gamma}) = \sum_{t=1}^T f_t(x_t)$.

We measure the performance of an algorithm for the decision maker by comparing it to the hindsight optimal solution computed with access to all the requests and the value of T. More concretely, for a horizon T and a sequence of requests $\vec{\gamma} = (\gamma_1, \dots, \gamma_T)$, the hindsight-optimal reward $\mathrm{OPT}(T, \vec{\gamma})$ is defined as the optimal value of the following hindsight optimization problem:

$$\mathrm{OPT}(T, \vec{\gamma}) := \max_{x \in \prod_t \mathcal{X}_t} \sum_{t=1}^{T} f_t(x_t) \quad \text{subject to} \quad \sum_{t=1}^{T} b_t(x_t) \le B\,. \tag{3.1}$$

We define the performance ratio of an algorithm A for horizon T and request distribution $\mathcal{P}$ as

$$c(A|T,\mathcal{P}) := \frac{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[R(A|T,\vec{\gamma})\right]}{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[\mathrm{OPT}(T,\vec{\gamma})\right]}\,.$$

Throughout this paper, we will assume that the horizon T belongs to an uncertainty window $[\tau_1,\tau_2]$ which is known to the decision maker. This assumption is necessary because it is impossible to attain non-trivial performance guarantees in the absence of an upper bound on the horizon (see Theorem 6). Moreover, we will assume that there exists a constant $\kappa > 0$ such that $\mathbb{E}[\mathrm{OPT}(T,\vec{\gamma})] \geq \kappa \cdot T$ for all $T \in [\tau_1,\tau_2]$. The assumption that $\mathbb{E}[\mathrm{OPT}(T,\vec{\gamma})] = \Omega(T)$ is common in the literature on online resource allocation with bandit feedback (see [Sli+19] for a survey). A mild sufficient condition for this assumption to hold is the existence of some mapping from request to actions which achieves positive expected reward: $\exists\, x : \mathcal{S} \to \mathcal{X}$ such that $\mathbb{E}_{\gamma\sim\mathcal{P}}[f(x(\gamma))] > 0$.[1]

Next, we describe the models of horizon uncertainty we consider in this chapter.

Uncertainty Window. Here, we assume that the decision maker is not aware of the exact value of T and it can lie anywhere in the known uncertainty window $[\tau_1,\tau_2]$. This approach is motivated by the literature on robust optimization, where it is often assumed that the exact value of the parameter is unknown but it is constrained to belong to some known uncertainty set [BTN02]. Our goal here is to capture settings with large horizon uncertainty where it is difficult to predict the total number of requests with high accuracy. In such settings, it is often easier to generate confidence intervals than precise estimates. For this model of horizon uncertainty, we measure the performance of an algorithm A by its competitive ratio $c(A)$, which we define as

$$c(A) := \inf_{\mathcal{P}} \min_{T\in[\tau_1,\tau_2]} c(A|T,\mathcal{P})\,.$$

[1] To see how this, define $\psi := \mathbb{E}_{\gamma\sim\mathcal{P}}[f(x(\gamma))] > 0$ and set $x_t = x(\gamma_t)$ starting from $t = 1$ till some resource runs out. Since $\|b(x)\|_\infty \leq \bar{b}$, resource j will last at least $\lfloor B_j/\bar{b}\rfloor$ time steps, which in combination with $B \geq 2\bar{b}$ implies

$$\mathbb{E}_{\vec{\gamma}}[\mathrm{OPT}(T,\vec{\gamma})] \geq \min_{j\in[m]}\lfloor B_j/\bar{b}\rfloor \cdot \mathbb{E}_{\gamma\sim\mathcal{P}}[f(x(\gamma))] \geq \min_{j\in[m]}\left(\frac{B_j}{\bar{b}} - 1\right)\psi \geq \min_{j\in[m]}\frac{B_j\psi}{2\bar{b}} = \min_{j\in[m]}\frac{\rho_{T,j}\psi}{2\bar{b}}\cdot T\,.$$

Since $\rho_T \geq \rho_{\tau_2}$ for all $T \leq \tau_2$, setting $\kappa = \min_j \rho_{\tau_2,j}\psi/2\bar{b}$ yields $\mathbb{E}[\mathrm{OPT}(T,\vec{\gamma})] \geq \kappa\cdot T$ for all $T \in [\tau_1,\tau_2]$.

We also say that an algorithm A is $c(A)$-competitive if it has a competitive ratio of $c(A)$. The competitive ratio of our algorithm degrades at a near-optimal logarithmic rate as τ_2/τ_1 grows large, and consequently yields good performance even for conservative estimates of the uncertainty window.

Algorithms with Predictions. We also consider a model of horizon uncertainty, inspired by the Algorithms-with-Predictions framework, which interpolates between the previously studied known-horizon model and the uncertainty-window model described above. This framework assumes that the decision maker has access to a prediction $T_P \in [\tau_1, \tau_2]$ about the horizon but no assumptions are made about the accuracy of this prediction. In particular, the goal is to develop algorithms that perform well when the prediction is accurate (consistency) while maintaining worst-case guarantees (competitiveness). For this setting, our algorithm allows the decision maker to smoothly trade-off consistency and competitiveness depending on her preferences.

3.1.1 Why do we need a new algorithm?

As discussed in Section 1.3, online resource allocation and its special cases have been extensively studied in the literature. Perhaps one of the algorithms from the literature continues to perform well under horizon uncertainty? We show below that previously-studied algorithms can be exponentially worse than our algorithm. Consider an uncertainty window $[\tau_1, \tau_2]$, where $\tau_1, \tau_2 \in \mathbb{Z}_+$. Consider an online algorithm A which takes as input the horizon and is optimal (defined precisely later) for stochastic online resource allocation when the horizon is known. Suppose we pick some horizon $T^* \in [\tau_1, \tau_2]$ before the first request arrives and run algorithm A with T^* in the hope of getting good performance for all horizons $T \in [\tau_1, \tau_2]$. As we show next, this approach performs much worse than our algorithm even when there is only one resource ($m = 1$), the same request arrives at all time steps, and the decision-maker knows this to be the case.

Let B be the initial resource endowment, $\mathcal{X}_t = [0, \max\{1, B/\tau_1\}]$ be the action set for all $t \in [\tau_2]$, and $\mathcal{P}_r$ be the deterministic distribution that always serves the request (f, b) where $f(x) = x^r$ for a fixed $r \in (0, 1)$ and $b(x) = x$. Observe that all the requests are the same, the decision-maker knows this fact, and she takes her first action after observing the request. In particular, the decision maker completely knows the deterministic request after the first request arrives and before she takes her first action. Moreover, if she employs an algorithm for the known-horizon setting with horizon T^* as the input and we have $T = T^*$, then the algorithm has as much information (about the request and the horizon) available before making its first decision as it would in hindsight. This motivates us to call an algorithm **optimal for the known-horizon setting** if it takes the same actions as the hindsight optimal $\text{OPT}(T^*, \vec{\gamma})$ on this instance when $T = T^*$ and it is given horizon T^* as the input. The dual-descent algorithm of [BLM23] (when appropriately initialized), and all of the primal methods based on solving the fluid approximation (e.g., [JK12; AWY14] and [BBP21]) satisfy this definition of optimality. Let A be such an optimal algorithm.

Consequently, if $\{x_t\}_t$ are the actions of algorithm A, then $\{x_t\}_t$ is an optimal solution to the hindsight-optimization problem $\text{OPT}(T^*, \vec{\gamma})$ (see equation (3.1)). In Lemma 9, we will show that the concavity of f implies that $x_t^* = B/T^*$ for all $t \leq T^*$ and $x_t^* = 0$ for $t > T^*$ is the unique hindsight optimal solution of $\text{OPT}(T^*, \vec{\gamma})$, which implies that $x_t = x_t^*$ for all $t \geq 1$. Now, recall that algorithm A does not know the horizon T and is non-anticipating. Consequently, it will take the actions $\{x_t\}_t$ no matter what T turns out to be. This is because, if $T \leq T^*$, then it does not know that T is different from T^*, and if $T > T^*$, then it has run out of budget by time step T^*.

The performance ratio of algorithm A for $T = \tau_1$ is given by

$$c(A|\tau_1, \mathcal{P}_r) = \frac{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}_r^{\tau_1}}\left[R(A|\tau_1, \vec{\gamma})\right]}{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}_r^{\tau_1}}\left[\text{OPT}(\tau_1, \vec{\gamma})\right]} = \frac{(B/T^*)^r \cdot \tau_1}{(B/\tau_1)^r \cdot \tau_1} = \left(\frac{\tau_1}{T^*}\right)^r$$

and for $T = \tau_2$ is given by

$$c(A|\tau_2, \mathcal{P}_r) = \frac{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}_r^{\tau_2}}\left[R(A|\tau_2, \vec{\gamma})\right]}{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}_r^{\tau_2}}\left[\text{OPT}(\tau_2, \vec{\gamma})\right]} = \frac{(B/T^*)^r \cdot T^*}{(B/\tau_2)^r \cdot \tau_2} = \left(\frac{T^*}{\tau_2}\right)^{1-r}.$$

Finally, observe that:

- If $T^* > \sqrt{\tau_1\tau_2}$, then $\inf_{r\in(0,1)} c(A|\tau_1, \mathcal{P}_r) = \lim_{r\to 1} c(A|\tau_1, \mathcal{P}_r) = \tau_1/T^* \leq \left(\sqrt{\tau_2/\tau_1}\right)^{-1}$.
- If $T^* \leq \sqrt{\tau_1\tau_2}$, then $\inf_{r\in(0,1)} c(A|\tau_2, \mathcal{P}_r) = \lim_{r\to 0} c(A|\tau_2, \mathcal{P}_r) = T^*/\tau_2 \leq \left(\sqrt{\tau_2/\tau_1}\right)^{-1}$.

Therefore, we get that the competitive ratio of algorithm A is bounded above by $(\sqrt{\tau_2/\tau_1})^{-1}$ for all values of $\{\tau_1, \tau_2\}$. In stark contrast, if τ_1 is large and $B = \Theta(\tau_1)$, we will show that our online algorithm achieves a competitive ratio greater than $(1 + \ln(\tau_2/\tau_1))^{-1}$, which is exponentially better than algorithm A. Even for small values of τ_2/τ_1, our algorithm significantly outperforms previous algorithms (see Figure 3.1).

We have shown that a proxy horizon does not allow us to use algorithms which are optimal for the known-horizon setting to obtain good performance in the face of horizon uncertainty. Perhaps one can use the Doubling Trick instead? The Doubling Trick involves running an algorithm designed for the known-horizon setting repeatedly on time-intervals of increasing lengths. More precisely, given an optimal (or low-regret) algorithm A for the known horizon setting, run it separately on the intervals $[1, T^*], [T^*, 2T^*], \ldots, [2^k T^*, 2^{k+1} T^*]$ for some $T^* \geq 1$. Unfortunately, as we alluded to in the Introduction, the Doubling Trick does not work for online resource allocation. This is because, unlike online convex optimization [Haz+16; SS+12] where the problem decouples and the regret from the different intervals is simply added together to get total regret, the online resource allocation problem has global resource constraints and does not decouple.

In particular, if we were to run an algorithm A with low-regret in the known-horizon setting on the interval $[1, T^*]$, it will attempt to deplete all of the available resources by time T^* (because unused resources have no value to A after T^*), which in turn implies that we will not have sufficient resource capacity to even run algorithm A on latter intervals $[T^*, 2T^*], \ldots, [2^k T^*, 2^{k+1} T^*]$. The crux of the problem is that the Doubling Trick does not take the resources capacities into account: since we only have a finite amount of resources, one cannot repeatedly run algorithm A because it will consume the entire resource capacity on every run (if possible). Additionally, note that the benchmark in online resource allocation is the optimal solution in hindsight considering all requests till time T, which is very different from the sum of the benchmark optimal solutions in the intervals $[1, T^*], [T^*, 2T^*], \ldots, [2^k T^*, 2^{k+1} T^*]$. One can potentially come up with sophisticated versions of the Doubling Trick that allocate the resource endowment between the intervals in non-trivial ways. But the aforementioned lack of decomposability of the benchmark across intervals means that analyzing such heuristics would be far from straightforward. In fact, one of our primary contributions is a general performance guarantee for dual mirror descent with arbitrary allocation of the resource endowment across time steps (Theorem 5), which allows one to analyze such heuristics. Finally, in online convex optimization, the Doubling Trick allows one to convert an algorithm for the known-horizon setting into one for the unknown-horizon setting while maintaining the same asymptotic competitive ratio of 1. However, as we will show in Theorem 6, it is impossible to achieve the same competitive ratio in the known-horizon and unknown-horizon settings for the online resource allocation problem. Thus, the simple Doubling Trick cannot be applied to online resource allocation, necessitating the need for novel techniques beyond the ones developed for online convex optimization.

3.2 The Algorithm

In this section, we describe our dual-mirror-descent-based master algorithm. As the name suggests, this algorithm maintains and updates dual variables, using them to compute the action x_t at time t. Moreover, the algorithm is parameterized by a *target consumption sequence*.

Definition 2. *We call a sequence* $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_t, \ldots, \lambda_{\tau_2})$ *a target consumption sequence if* $\lambda_t \in \mathbb{R}_+^m$ *for all* $t \leq \tau_2$, $\lambda_1 > 0$ *and* $\sum_{t=1}^{\tau_2} \lambda_t \leq B$.

Here, $\lambda_t \in \mathbb{R}_+^m$ denotes the target amount of resources that one wants to consume in the t-th time period. $\sum_{t=1}^{\tau_2} \lambda_t \leq B$ ensures that the budget never runs out if one is able to hit these target consumptions. Given a target consumption sequence $\vec{\lambda}$, we use $\bar{\lambda} = \max_{t,j} \lambda_{t,j}$ to denote the largest target consumption of any resource at any time step.

We will be showing performance guarantees for our algorithms in terms of the target consumption sequence, and then provide algorithms for computing the optimal target sequence in later sections.

3.2.1 The Dual Problem

The Lagrangian dual problem to the hindsight optimization problem (3.1) is obtained by moving the resource constraints to the objective using multipliers $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}_+^m$. Intuitively, the dual variable μ_j acts as the price of resource j and accounts for the opportunity cost of consuming resource j. This allows us to define the objective function of the dual optimization problem:

$$\begin{aligned} D(\mu|T, \vec{\gamma}) &:= \sup_{x \in \prod_t \mathcal{X}_t} \left\{ \sum_{t=1}^{T} f_t(x_t) + \mu^\top \left(B - \sum_{t=1}^{T} b_t(x_t) \right) \right\} \\ &= \sum_{t=1}^{T} \sup_{x_t \in \mathcal{X}_t} \left\{ f_t(x_t) + \mu^\top (\rho_T - b_t(x_t)) \right\} \\ &= \sum_{t=1}^{T} \left(f_t^*(\mu) + \mu^\top \rho_T \right) , \end{aligned}$$

where the second equation follows because the objective is separable and $\rho_T = B/T$, and the last by defining the opportunity-cost-adjusted reward to be $f_t^*(\mu) := \sup_{x \in \mathcal{X}_t} \{f_t(x) - \mu^\top b_t(x)\}$. The dual problem is simply $\min_{\mu \in \mathbb{R}_+} D(\mu|T, \vec{\gamma})$. Importantly, we get weak duality: $\mathrm{OPT}(T, \vec{\gamma}) \leq D(\mu|T, \vec{\gamma})$ for all dual solutions μ (we provide a proof in Proposition 11 of Appendix B.1).

Recall that, in our definition of competitive ratio (3.4), we are interested in the expectation of $\mathrm{OPT}(T, \vec{\gamma})$ when $\vec{\gamma} \sim \mathcal{P}^T$. Weak duality allows us to bound this quantity from above as

$$\mathbb{E}_{\vec{\gamma} \sim \mathcal{P}^T} [\mathrm{OPT}(T, \vec{\gamma})] \leq \mathbb{E}_{\vec{\gamma} \sim \mathcal{P}^T} [D(\mu|T, \vec{\gamma})] = \sum_{t=1}^{T} \left(\mathbb{E}_{\gamma \sim \mathcal{P}} \left[f_t^*(\mu) \right] + \rho_T^\top \mu \right) . \tag{3.2}$$

This motivates us to define the following single-period dual function with target consumption $\lambda \in \mathbb{R}_+^m$ as $\mathcal{D}(\mu|\lambda, \mathcal{P}) := \mathbb{E}_{\gamma \sim \mathcal{P}} \left[f^*(\mu) \right] + \lambda^\top \mu$. The following lemma notes some important properties of the single-period dual objective.

Lemma 8. *$\mathcal{D}(\mu|\lambda, \mathcal{P})$ is convex in $\mu \in \mathbb{R}_+^m$ for every $\lambda \in \mathbb{R}_+^m$. Moreover, for every $\mu \in \mathbb{R}_+$ and $T \geq 1$, the following properties hold:*

(a) Separability: $\mathbb{E}_{\vec{\gamma} \sim \mathcal{P}^T} [D(\mu|T, \vec{\gamma})] = T \cdot \mathcal{D}(\mu|\rho_T, \mathcal{P})$

Algorithm 3: Variable Target Dual Mirror Descent Algorithm

Input: Initial dual solution μ_1, initial resource endowment $B_1 = B$, target consumption sequence $\vec{\lambda}$, reference function $h : \mathbb{R}^m \to \mathbb{R}$, and step-size η.

for $t = 1, \ldots, T$ **do**

Receive request $(f_t, b_t, \mathcal{X}_t) \sim \mathcal{P}$.

Make the primal decision x_t and update the remaining resources B_t:

$$\tilde{x}_t \in \operatorname{argmax}_{x \in \mathcal{X}_t} \left\{ f_t(x) - \mu_t^\top b_t(x) \right\} , \tag{3.3}$$

$$x_t = \begin{cases} \tilde{x}_t & \text{if } b_t(\tilde{x}_t) \le B_t \\ 0 & \text{otherwise} \end{cases} ,$$

$$B_{t+1} = B_t - b_t(x_t).$$

Obtain a sub-gradient of the dual function: $g_t = \lambda_t - b_t(\tilde{x}_t)$.

Update the dual variable by mirror descent: $\mu_{t+1} = \arg\min_{\mu \in \mathbb{R}_+^m} g_t^\top \mu + \frac{1}{\eta} V_h(\mu, \mu_t)$, where $V_h(x, y) = h(x) - h(y) - \nabla h(y)^\top (x - y)$ is the Bregman divergence.

end

(b) Sub-homogeneity: For $a \in [0, 1]$, $a \cdot \mathcal{D}(\mu|\lambda, \mathcal{P}) \le \mathcal{D}(\mu|a \cdot \lambda, \mathcal{P})$.

(c) Monotonicity: If $\lambda \le \kappa$, *then* $\mathcal{D}(\mu|\lambda, \mathcal{P}) \le \mathcal{D}(\mu|\kappa, \mathcal{P})$.

3.2.2 Variable Target Dual Mirror Descent

Algorithm 3 is a highly-flexible stochastic dual descent algorithm that allows the decision maker to specify the target consumption sequence $\vec{\lambda}$, in addition to the initial dual variable μ_1, the reference function $h(\cdot)$ and the step-size η needed to specify the mirror-descent procedure. This flexibility allows us to seamlessly analyze a variety of different algorithms using the same framework. As is standard in the literature on mirror descent [SS+12; Haz+16], we require the reference function $h(\cdot)$ to be either differentiable or essentially smooth [BBC01], and be σ-strongly convex in the $\|\cdot\|_1$ norm. Moreover, Algorithm 3 is efficient when an optimal solution for the per-period optimization problem in equation (3.3) can be computed efficiently. This is possible for many applications, see [BLM23] for details.

The algorithm maintains a dual variable μ_t at each time step, which acts as the price of the resources and accounts for the opportunity cost of spending them at time t. Then, for a request

$\gamma_t = (f_t, b_t, \mathcal{X}_t)$ at time t, it chooses the action x_t that maximizes the opportunity-cost-adjusted reward $x_t \in \arg\max_{x \in \mathcal{X}_t} \left\{ f_t(x) - \mu_t^\top b_t(x) \right\}$. As our goal here is to build intuition, we ignore the minor difference between $\tilde{x}_t$ and x_t which ensures that we never overspend resources. The dual variable is updated using mirror descent with reference function $h(\cdot)$, step-size η, and using $g_t = \lambda_t - b_t(x_t)$ as a subgradient. Intuitively, mirror descent seeks to make the subgradients as small as possible, which in our settings translates to making the expected resource consumption in period t as close as possible to the target consumption λ_t. As a result, the target consumption sequence can be interpreted as the ideal expected consumption per period, and the algorithm seeks to track these rates of consumption.

We conclude by discussing some common choices for the reference function. If the reference function is the squared-Euclidean norm, i.e., $h(\mu) = \|\mu\|_2^2$, then the update rule is $\mu_{t+1} = \max\{\mu_t - \eta g_t, 0\}$ and the algorithm implements subgradient descent. If the reference function is the negative entropy, i.e., $h(\mu) = \sum_{j=1}^m \mu_j \log(\mu_j)$, then the update rule is $\mu_{t+1,j} = \mu_{t,j} \exp\left(-\eta g_{t,j}\right)$ and the algorithm implements multiplicative weights.

3.2.3 Performance Guarantees

In this section, we provide worst-case performance guarantees of our algorithm for arbitrary target consumption sequences. Before stating our result, we provide further intuition about our algorithm. Consider the single-period dual function with target consumption λ, given by

$$\mathcal{D}(\mu|\lambda, \mathcal{P}) = \mathbb{E}_{\gamma \sim \mathcal{P}} \left[f^*(\mu) \right] + \lambda^\top \mu = \mathbb{E}_{\gamma \sim \mathcal{P}} \left[\sup_{s \in \mathcal{X}} \left\{ f(x) - \mu^\top b(x) \right\} \right] + \lambda^\top \mu \,.$$

Then, by Danskin's Theorem, its subgradient is given by $\mathbb{E}_{\gamma \sim \mathcal{P}}[\lambda - b(x_\gamma(\mu))] \in \partial_\mu \mathcal{D}(\mu|\lambda, \mathcal{P})$ where $x_\gamma(\mu) \in \arg\max_{x \in \mathcal{X}} \{f(x) - \mu^\top b(x)\}$ is an optimal decision when the request is $\gamma = (f, b, \mathcal{X})$ and the dual variable is μ. Therefore, $g_t = \lambda_t - b_t(x_t)$ is a random unbiased sample of the subgradient of $\mathcal{D}(\mu|\lambda, \mathcal{P})$. Now, if we wanted to minimize the dual objective $\mathbb{E}[D(\mu|T, \vec{\gamma})] = \sum_{t=1}^T \mathcal{D}(\mu|\rho_T, \mathcal{P})$ for some known T, we can run mirror descent on the function $\mathcal{D}(\mu|\rho_T, \mathcal{P})$ by setting $\lambda_t = \rho_T$ for all $t \leq T$. This is exactly the approach taken by [BLM23]. Unfortunately, this method does not extend to our setting because the horizon T is unknown.

Observe that, since mirror descent guarantees vanishing regret even against adversarially generated losses, it continues to give vanishing regret in the dual space even when the single-period duals $\mathcal{D}(\mu|\lambda_t, \mathcal{P})$ vary with time due to the changing target consumptions. However, when $\lambda_t \neq \rho_T$ for some $t \leq T$, it is no longer the case that $\sum_{t=1}^{T} \mathcal{D}(\mu|\lambda_t, \mathcal{P})$ provides an upper bound on the hindsight optimization problem. The crux of the following result involves overcoming this difficulty and comparing the time-varying single-period duals with the hindsight optimal solution for all T simultaneously, leading to a performance guarantee for Algorithm 3.

Theorem 5. *Consider Algorithm 3 with initial dual solution μ_1, initial resource endowment $B_1 = B$, a target consumption sequence $\vec{\lambda}$, reference function $h(\cdot) : \mathbb{R}^m \to \mathbb{R}$, and step-size η. For any $T \geq 1$, if we set*

$$c(\vec{\lambda}, T) := \frac{1}{T} \sum_{t=1}^{T} \min \left\{ \min_{1 \leq j \leq m} \frac{\lambda_{t,j}}{\rho_{T,j}}, 1 \right\}, \tag{3.4}$$

then it holds that

$$\mathbb{E}_{\vec{\gamma} \sim \mathcal{P}^T} \left[c(\vec{\lambda}, T) \cdot \mathrm{OPT}(T, \vec{\gamma}) - R(A|T, \vec{\gamma}) \right] \leq C_1^{(T)} + C_2 \eta T + \frac{C_3^{(T)}}{\eta}. \tag{3.5}$$

where $C_1^{(T)} = \frac{\bar{f}\bar{b}}{\underline{\rho}_T}$, $C_2 = \frac{(\bar{b}+\bar{\lambda})^2}{2\sigma}$, $C_3^{(T)} = \max \left\{ V_h(\mu, \mu_1) : \mu \in \{0, (\bar{f}/\underline{\rho}_T)e_1, \ldots, (\bar{f}/\underline{\rho}_T)e_m\} \right\}$. Here, $e_j \in \mathbb{R}^m$ is the j-th unit vector and $\underline{\rho}_T = \min_j \rho_{T,j}$.

The proof proceeds in multiple steps. First, we write the rewards collected by Algorithm 3 as a sum of per-period duals and complementary-slackness terms. Next, we use weak duality to upper bound the expected value of the hindsight optimal reward $\mathbb{E}[\mathrm{OPT}(T, \vec{\gamma})]$ in terms of the expected hindsight dual. These two steps are common to all primal-dual analyses, but past techniques offer no guidance beyond this point. The core difficulty is that the expected hindsight dual is equal to the sum of per-period duals with target consumption ρ_T, whereas the lower bound on the performance of our algorithm is in terms of per-period duals with target consumptions λ_t. Importantly, this difficulty does not arise in past works because the target consumptions $\lambda_t = \rho_T$, which makes the two terms directly comparable. Our main technical insight lies in using Lemma 8 to manipulate the per-period duals and then carefully choosing the right dual solution in order to compare the two terms. Moreover, one also needs to take into account the fact that the resources may run out before the horizon T is reached, and Algorithm 3 does not accumulate rewards after this point. Since the point at which the budget runs out depends on the target consumption sequence, we also establish a bound on the loss from depleting the resources too early which applies to variable target sequences. We believe that Algorithm 3 and our proof techniques distill the core tradeoffs of the problem and can be used more broadly. The full proof is in Appendix B.1.

Theorem 5 is the bedrock of our positive results. It allows us to drastically simplify the design of algorithms: instead of searching for the optimal algorithm, we can focus on the much simpler problem of selecting the optimal target consumption sequence. The following result provides a key step in this direction by showing that $c(\vec{\lambda}, T)$ is the asymptotic performance ratio of Algorithm 3 with target sequence $\vec{\lambda}$.

Proposition 2. *Let A be Algorithm 3 with initial dual solution μ_1, initial resource endowment $B_1 = B$, a target consumption sequence $\vec{\lambda}$, reference function $h(\cdot) : \mathbb{R}^m \to \mathbb{R}$, and step-size η. Set $C_1' = \max_{T \in [\tau_1, \tau_2]} C_1^{(T)}$ and $C_3' = \max_{T \in [\tau_1, \tau_2]} C_3^{(T)}$. Then, with step size $\eta = \sqrt{C_3' / \{C_2 \tau_2\}}$, the following statement holds for all $T \in [\tau_1, \tau_2]$:*

$$\inf_{\mathcal{P}} c(A|T, \mathcal{P}) = \inf_{\mathcal{P}} \frac{\mathbb{E}_{\vec{\gamma} \sim \mathcal{P}^T} \left[R(A|T, \vec{\gamma})\right]}{\mathbb{E}_{\vec{\gamma} \sim \mathcal{P}^T} \left[\mathrm{OPT}(T, \vec{\gamma})\right]} \geq c(\vec{\lambda}, T) - \epsilon\,,$$

where

$$\epsilon = \frac{C_1'}{\kappa \tau_1} + 2 \cdot \frac{\sqrt{(\tau_2/\tau_1) C_2 C_3'}}{\kappa \sqrt{\tau_1}}\,.$$

Remark 1. *To convert the guarantee in Proposition 2 to an asymptotic guarantee, one needs to consider the regime where the initial resources scale with the horizon as $B = \Omega(\tau_2)$, which ensures* *that $\rho_{\tau_2} = \Omega(1)$ and the constants C_1' and C_2' remain bounded. In which case, if we let τ_1 grow large while ensuring $\tau_2 = O(\tau_1)$, we can make ϵ arbitrarily small. In particular, $\epsilon = O(\tau_1^{-1/2})$. The assumption that the initial resources scales linearly with the number of requests is pervasive in the literature and well-motivated in applications such as internet advertising [Meh13] and revenue management [TVR04]. Moreover, an error of $\epsilon = \Omega(\tau_1^{-1/2})$ is unavoidable even for the case when the horizon is known, i.e., $\tau_1 = \tau_2$ (see [AG19]).*

Remark 2. *In applications where it might be difficult to estimate the constants C_2 and C_3', one can use the step size $\eta = 1/\sqrt{\tau_2}$ to get*

$$\epsilon = \frac{C_1'}{\kappa\tau_1} + \frac{\sqrt{(\tau_2/\tau_1)} \cdot (C_2 + C_3')}{\kappa\sqrt{\tau_1}},$$

which yields similar asymptotic rates.

Having characterized the performance of Algorithm 3 in terms of the target sequence, we next optimize it for the models of horizon uncertainty discussed in Section 3.1. Although we will only discuss two models of uncertainty, we would like to note that Theorem 5 and Proposition 2 are very general tools that can be applied more broadly. In particular, observe that

$$c(\vec{\lambda}, T) = \frac{1}{T}\sum_{t=1}^{T} \min\left\{\min_{1\le j\le m} \frac{\lambda_{t,j}}{\rho_{T,j}}, 1\right\}$$

is a concave function of $\vec{\lambda}$ for all $T \geq 1$. This is because each term in the sum is a minimum of a finite collection of linear functions of $\vec{\lambda}$. Consequently, any performance measure of Algorithm 3 which is a concave non-decreasing function of the performance ratios $\{c(A|T, \mathcal{P})\}_T$ is a concave function of the target sequence $\vec{\lambda}$ in light of Proposition 2. We pause to emphasize this important transition we have made in this section: we reduced the extremely complex problem of designing an algorithm for online resource allocation under horizon uncertainty to a concave optimization problem with the power to handle a variety of constraints and objectives. In the next section, we show that this reduction is without much loss in the uncertainty-window setting—picking the optimal target consumption sequence leads to a near-optimal competitive ratio in the uncertainty-window model.

3.3 Uncertainty Window

Motivated by robust optimization, in this section, we take the uncertainty-set approach to modeling horizon uncertainty. In particular, we assume that the decision maker is not aware of the exact value of T but knows it can lie anywhere in the known uncertainty window $[\tau_1, \tau_2]$. Recall that we measure the performance of an algorithm A in this model by its competitive ratio $c(A)$, which is defined as

$$c(A) \coloneqq \min_{\mathcal{P}} \min_{T \in [\tau_1, \tau_2]} c(A|T, \mathcal{P}) = \inf_{T \in [\tau_1, \tau_2]} \frac{\mathbb{E}_{\vec{\gamma} \sim \mathcal{P}^T} [R(A|T, \vec{\gamma})]}{\mathbb{E}_{\vec{\gamma} \sim \mathcal{P}^T} [\mathrm{OPT}(T, \vec{\gamma})]} .$$

3.3.1 Upper Bound on Competitive Ratio

We begin by showing that no online algorithm can attain a competitive ratio of $c(A) = 1$ whenever $\tau_2/\tau_1 > 1$ and, moreover, when τ_2/τ_1 is large the competitive ratio degrades at a rate of $e \cdot \ln\ln(\tau_2/\tau_1)/\ln(\tau_2/\tau_1)$. In other words, the competitive ratio of every algorithm degrades to 0 as τ_2/τ_1 grows large. In fact, we prove that this upper bound on the best-possible competitive ratio holds even when (i) there is only 1 resource, (ii) the decision maker receives the same request at each time step, (iii) this request is known to the decision maker ahead of time, and (iv) the request has a smooth concave reward function and linear resource consumption.

For the purposes of this subsection, set the number of resources to $m = 1$. Consider an arbitrary initial resource endowment $B \in \mathbb{R}^m_{++}$. For every $r \in (0, 1)$, define the singleton request space $\mathcal{S}_r = \{(f_r, I, \mathcal{X})\}$, where $\mathcal{X} = [0, \max\{1, B/\tau_1\}]$, and $f_r(x) = x^r$, $I(x) = x$ for all $x \in \mathcal{X}$. Note that f_r is concave for all $r \in (0, 1)$. Let $\mathcal{P}_r$ be the canonical distribution on $\mathcal{S}_r$ that serves the request $(f_r, I, \mathcal{X})$ with probability one. Since all requests are identical, we abuse notation and use $\mathrm{OPT}(T, r)$ (similarly $R(A|T, r)$) to denote the hindsight-optimal reward $\mathrm{OPT}(T, \vec{\gamma})$ (total reward $R(A|T, \vec{\gamma})$ of algorithm A) when $\vec{\gamma} \sim \mathcal{P}_r^T$, i.e., $\gamma_t = (f_r, I, \mathcal{X})$ for all $t \leq T$.

Before stating the upper bound, we would like to note that randomization only makes the performance of any online algorithm worse. To see this consider any non-deterministic online algorithm A and let $x(A)_t$ denote the random variable which captures the action taken by A at time t. Define A' to be the online algorithm which takes the action $x(A')_t = \mathbb{E}[x(A)_t]$ at time t. Then, due to the strict concavity of f_r, we have $f_r(x(A')_t) > \mathbb{E}[f_r(x(A)_t)]$, and from the linearity of expectation, we have $\sum_{t=1}^{\tau_2} x(A')_t = \mathbb{E}[\sum_{t=1}^{\tau_2} x(A)_t] \leq B$. Therefore, the deterministic algorithm A' attains strictly greater reward. Consequently, we will focus only on deterministic online algorithms for the remainder of this subsection. We are now ready to state the main result of this section.

Theorem 6. *For all $r \in (0,1)$ and $1 \leq \tau_1 \leq \tau_2$, every online algorithm A satisfies*

$$\min_{T \in [\tau_1, \tau_2]} \frac{R(A|T,r)}{\mathrm{OPT}(T,r)} \leq \frac{1}{\left(1 + (1-r)^{1/r} \cdot \ln(\tau_2/\tau_1) + \ln\left(\frac{\tau_1}{\tau_1+1}\right)\right)^r} .$$

In particular, when $r = 1 - \{1/\ln\ln(\tau_2/\tau_1)\}$ and $\tau_2/\tau_1 > e^e$, every online algorithm A satisfies

$$\min_{T \in [\tau_1, \tau_2]} \frac{R(A|T,r)}{\mathrm{OPT}(T,r)} \leq \frac{e \cdot \ln\ln(\tau_2/\tau_1)}{\ln(\tau_2/\tau_1)} .$$

The above bounds hold even for online algorithms that have prior knowledge of $\mathcal{P}_r$ before time $t = 1$.

Remark 3. *Note that the upper bound on the competitive ratio established in Theorem 6 degrades to zero as τ_2/τ_1 grows large. In particular, a positive competitive ratio cannot be obtained if no upper-bound on the horizon T is known, thereby necessitating the need for a known uncertainty window.*

Figure 3.1 plots the value of the upper bound on the competitive ratio as a function of τ_2/τ_1 for $\tau_2/\tau_1 \in [1, 20]$.

We now discuss the main ideas behind the proof of Theorem 6. It suffices to prove the stronger statement in the theorem that holds for online algorithms with prior knowledge of $(r, \mathcal{P}_r)$ before time $t = 1$. Consequently, we assume that online algorithms have this prior knowledge in the remainder of this section. Any algorithm without this knowledge can only do worse. We begin by utilizing the concavity of f_r to evaluate the optimal reward, which we note in the following lemma.

Lemma 9. *For* $r \in (0,1)$ *and* $T \in [\tau_1, \tau_2]$, *we have* $\mathrm{OPT}(T,r) = T \cdot (B/T)^r = B^r \cdot T^{1-r}$. *Moreover,* $x_t = B/T$ *is the unique hindsight optimal solution.*

Because the reward function f_r is concave, it is optimal to spread resources uniformly over time and the optimal action with the benefit of hindsight is $x_t = B/T$. Next, we provide an alternative characterization of the competitive ratio that is more tractable.

Lemma 10. *For* $r \in (0,1)$*and* $1 \le \tau_1 \le \tau_2$, *we have*

$$\sup_A \min_{T \in [\tau_1, \tau_2]} \frac{R(A|T,r)}{\mathrm{OPT}(T,r)} = \max \left\{ c \in [0,1] \,\middle|\, \tau_1 \cdot f_r^{-1}\left(c \cdot \frac{\mathrm{OPT}(\tau_1, r)}{\tau_1}\right) + \sum_{t=\tau_1+1}^{\tau_2} f_r^{-1}\left(c \cdot \Delta\,\mathrm{OPT}(t,r)\right) \le B \right\},$$

where $\Delta\,\mathrm{OPT}(t,r) = \mathrm{OPT}(t,r) - \mathrm{OPT}(t-1,r)$ *and the* sup *is taken over all online algorithms.*

We present a proof sketch of Lemma 10 here. The main step in the proof involves showing that, for a given competitive ratio c, the minimum amount of resources that any online algorithm A needs to be spend in order to satisfy $\min_{T \in [\tau_1, \tau_2]} R(A|T,r)/\mathrm{OPT}(T,r) \ge c$ is given by

$$\tau_1 \cdot f_r^{-1}\left(c \cdot \frac{\mathrm{OPT}(\tau_1, r)}{\tau_1}\right) + \sum_{t=\tau_1+1}^{\tau_2} f_r^{-1}\left(c \cdot \Delta\,\mathrm{OPT}(t,r)\right).$$

This is because f_r is concave for all $r \in (0,1)$ and the resource consumption function I is linear, which together imply that the marginal bang-per-buck $f_r'(x)$ (amount of reward per marginal unit of resource spent) decreases with x. As a consequence, an online algorithm that does not have any knowledge of T (other than $T \in [\tau_1, \tau_2]$) and needs to satisfy $R(A|T,r) \ge c \cdot \mathrm{OPT}(T,r)$ for all $T \in [\tau_1, \tau_2]$ would spend the minimum amount of resources in doing so if (i) it attains a reward of $c \cdot \mathrm{OPT}(\tau_1, r)$ by evenly spending resources in the first τ_1 steps, and (ii) it spends just enough resources to attain a reward of $c \cdot \Delta\,\mathrm{OPT}(t,r)$ at each time step $t \ge \tau_1 + 1$. Proving (ii) requires showing that $\Delta\,\mathrm{OPT}(t,r)$ decreases with an increase in t, which follows from Lemma 9. In particular, this ensures that obtaining all of $\Delta\,\mathrm{OPT}(t,r)$ at time t is cheaper than obtaining some of that reward at an earlier time $t' < t$. However, the proof requires a sophisticated water-filling argument to show that the aforementioned greedy strategy of minimizing the amount of resources at each time step leads to globally-minimal spending. Finally, combining Lemma 10 and Lemma 9 yields

$$\tau_1 \cdot \left(c^* \cdot \frac{B^r \tau_1^{1-r}}{\tau_1}\right)^{1/r} + \sum_{t=\tau_1+1}^{\tau_2} \left(c^* \cdot [B^r t^{1-r} - B^r (t-1)^{1-r}]\right)^{1/r} \leq B$$

for $c^* = \sup_A \min_{T \in [\tau_1, \tau_2]} R(A|T, r)/\mathrm{OPT}(T, r)$. The above equation specifies an upper bound on c^*, which upon simplification leads to Theorem 6.

We conclude by noting that the upper bound of Theorem 6 can be extended to the popular setting of online resource allocation with random linear rewards and consumptions (see Appendix B.2.4 for details). Moreover, the upper bound of Theorem 6 holds even when the horizon T is drawn from a distribution $\mathscr{T}$ supported on $[\tau_1, \tau_2]$ and this distribution is known to the decision maker. A proof based on strong duality can be found in Appendix B.2.5.

3.3.2 Optimizing the Target Sequence

Having shown that no algorithm can attain a competitive ratio better than $\tilde{O}(1/\ln(\tau_2/\tau_1))$, we now show that Algorithm 3 with an appropriately chosen target consumption sequence $\vec{\lambda}$ can achieve a competitive ratio of $\Omega(1/\ln(\tau_2/\tau_1))$ for sufficiently large τ_1 and B. In light of Proposition 2, we can optimize the competitive ratio of Algorithm 3 by finding the target consumption sequence which maximizes $\min_{T \in [\tau_1, \tau_2]} c(\vec{\lambda}, T)$, i.e., we need to solve the following maximin problem:

$$\max_{\vec{\lambda}} \min_{T \in [\tau_1, \tau_2]} c(\vec{\lambda}, T) = \max_{\vec{\lambda}} \min_{T \in [\tau_1, \tau_2]} \frac{1}{T} \sum_{t=1}^{T} \min \left\{ \min_{1 \leq j \leq m} \frac{\lambda_{t,j}}{\rho_{T,j}}, 1 \right\} .$$

The following proposition restates the above maximin problem as an LP.

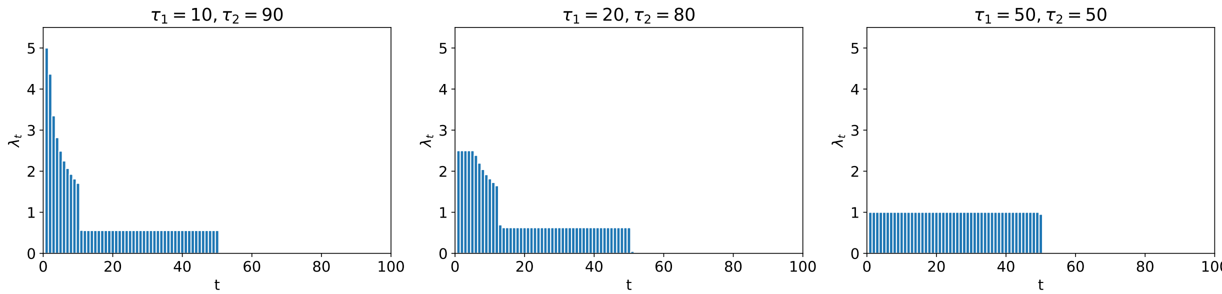

Figure 3.2: The optimal target consumption sequence for various possible uncertainty windows centered on $T = 50$. Here, number of resources $m = 1$ and initial resource endowment $B = 50$.

Proposition 3. *For budget B and uncertainty window $[\tau_1, \tau_2]$, we have*

$$
\begin{aligned}
\max_{\vec{\lambda}} \min_{T \in [\tau_1, \tau_2]} c(\vec{\lambda}, T) \quad = \quad \max_{z,y,\lambda} \quad & z \\
s.t. \quad & z \leq \frac{1}{T} \sum_{t=1}^{T} y_{T,t} && \forall T \in [\tau_1, \tau_2] \\
& y_{T,t} \leq \frac{\lambda_{t,j}}{\rho_{T,j}} && \forall j \in [m], T \in [\tau_1, \tau_2], t \in [T] \\
& y_{T,t} \leq 1 && \forall T \in [\tau_1, \tau_2], t \in [T] \\
& \sum_{t=1}^{\tau_2} \lambda_t \leq B \\
& \lambda \geq 0
\end{aligned}
$$

Proposition 3 states that we can efficiently compute the optimal target consumption sequence by solving an LP. Figure 3.2 plots the optimal target sequences from Proposition 3 for different uncertainty windows. The optimal target consumption sequences are decreasing as the algorithm consumes resources more aggressively early on to prevent having too many leftover resources if the horizon ends being short. Moreover, as the uncertainty window becomes more narrow, the consumption sequence becomes less variable.

To see that Algorithm 3 with the optimal target consumption sequence from the above LP has an asymptotic competitive ratio of $\Omega(1/\ln(\tau_2/\tau_1))$, consider the following target consumption

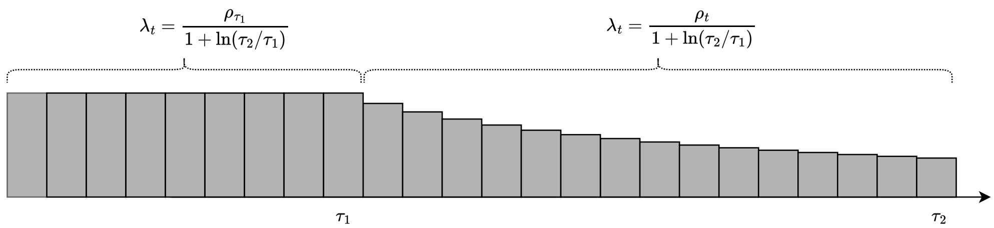

Figure 3.3: A simple target consumption sequence that achieves a competitive ratio of $1/(1 + \ln(\tau_2/\tau_1))$. The height of the bars represents λ_t.

sequence (depicted in Figure 3.3):

$$\lambda_t := \begin{cases} \frac{1}{1+\ln(\tau_2/\tau_1)} \cdot \frac{B}{\tau_1} = \frac{1}{1+\ln(\tau_2/\tau_1)} \cdot \rho_{\tau_1} & \text{if } t \leq \tau_1 \,, \\ \frac{1}{1+\ln(\tau_2/\tau_1)} \cdot \frac{B}{t} = \frac{1}{1+\ln(\tau_2/\tau_1)} \cdot \rho_t & \text{if } \tau_1 + 1 \leq t \leq \tau_2 \,. \end{cases} \tag{3.6}$$

It is easy to see that it satisfies the budget constraint:

$$\sum_{t=1}^{\tau_2} \lambda_t = \frac{B}{1 + \ln(\tau_2/\tau_1)} \cdot \left(\tau_1 \cdot \frac{1}{\tau_1} + \sum_{t=\tau_1+1}^{\tau_2} \frac{1}{t}\right) \leq \frac{B}{1 + \ln(\tau_2/\tau_1)} \cdot \left(1 + \ln\left(\frac{\tau_2}{\tau_1}\right)\right) = B \,.$$

Moreover, since $\rho_t \geq \rho_T$ for all $t \leq T$ and $T \in [\tau_1, \tau_2]$, we get

$$\frac{1}{T} \sum_{t=1}^{T} \min\left\{\min_{1\leq j\leq m} \frac{\lambda_{t,j}}{\rho_{T,j}}, 1\right\} \geq \frac{1}{T} \sum_{t=1}^{T} \frac{1}{1 + \ln(\tau_2/\tau_1)} = \frac{1}{1 + \ln(\tau_2/\tau_1)} \,,$$

where the inequality follows from the fact that $\rho_T \leq \rho_t$ for all $t \in [\tau_1, T]$ and the definition of λ as given in (3.6).

Since $\vec{\lambda}$ from (3.6) is just one possible choice of the target consumption sequence, we have

$$\max_{\vec{\lambda}} \min_{T\in[\tau_1,\tau_2]} c(\vec{\lambda}, T) \geq \frac{1}{1 + \ln(\tau_2/\tau_1)} \,.$$

Therefore, we get that Algorithm 3 in combination with the target consumption sequence returned by the LP in Proposition 3 achieves a degradation of $1/(1 + \ln(\tau_2/\tau_1))$ in the competitive ratio as a function of the multiplicative uncertainty τ_2/τ_1, which is optimal up to constants and

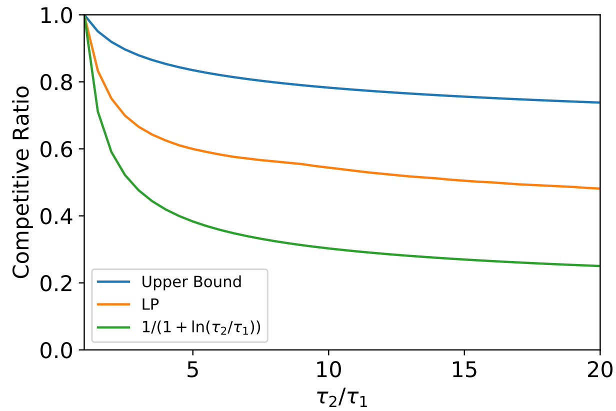

Figure 3.4: The competitive ratios achieved using the target consumption sequence from the LP in Proposition 3, and the simple one defined in (3.6) that yields a competitive ratio of $1/(1+\ln(\tau_2/\tau_1))$.

a $\ln \ln(\tau_2/\tau_1)$ factor. In fact, as Figure 3.4 shows, the target consumption sequence from the LP performs much better than $1/(1 + \ln \tau_2/\tau_1)$, even for small values of τ_2/τ_1. In Section 3.5, we will give a faster algorithm which leverages the structure of the problem to optimize the target sequence and does not require solving an LP.

We conclude with a discussion on the structural similarity of the results of this subsection with those of [BS14], who studied the dynamic pricing problem (special case of online resource allocation) under demand shifts. They worked under the assumption that the request distribution is perfectly known, and showed that the optimal dynamic programming solution has a non-increasing resource consumption sequence when the horizon is uncertain. The target consumption sequences described in this section are also non-increasing, leading to a similar structural insight for the more general online resource allocation problem with unknown request distribution.

3.4 Incorporating Predictions about the Horizon

In the previous section, we did not assume that we had any information about the horizon T other than the fact that it belonged to the uncertainty window $[\tau_1, \tau_2]$. This may be too pessimistic in settings where the environment is well behaved and machine learning algorithms can be deployed to make predictions about the horizon. In this section, we show that our Variable Target Dual Descent algorithm allows us to easily incorporate predictions by optimizing the target sequences. We formulate an LP to optimize the target sequence which allows the decision-maker to smoothly interpolate between the uncertainty-window setting and the known-horizon setting, thereby catering to different levels of confidence in the prediction.

First, we define the performance metrics we will use to measure the performance of an online algorithm capable of incorporating predictions. These metrics are pervasive in the Algorithms-with-Predictions literature (see [MV20] for an excellent survey) and are aimed at capturing the performance of the algorithm both when the prediction is accurate and in the worst case when the instance bears no resemblance to the prediction. To this end, in addition to the competitive-ratio metric defined in Section 3.1, which captures the worst-case performance, we introduce the notion of consistency to capture the performance of the algorithm when the prediction is accurate. Let $T_P \in [\tau_1, \tau_2]$ denote the predicted value of the horizon and let $A(T_P)$ denote algorithm A when provided with the prediction T_P. We say that an algorithm A is β-consistent on prediction T_P and γ-competitive if it satisfies

$$c(A(T_P)|T_P, \mathcal{P}) = \frac{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[R(A(T_P)|T_P, \vec{\gamma})\right]}{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[\mathrm{OPT}(T_P, \vec{\gamma})\right]} \geq \beta\,,$$

and

$$\inf_{T\in[\tau_1,\tau_2]} c(A(T_P)|T, \mathcal{P}) = \inf_{T\in[\tau_1,\tau_2]} \frac{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[R(A(T_P)|T, \vec{\gamma})\right]}{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[\mathrm{OPT}(T, \vec{\gamma})\right]} \geq \gamma\,,$$

for all request distributions $\mathcal{P}$. In other words, an algorithm which is β-consistent on prediction T_P and γ-competitive is guaranteed to get a β fraction of the hindsight optimal reward in expectation if the prediction comes true and it is guaranteed to attain a γ fraction of the hindsight optimal reward for every horizon $T \in [\tau_1, \tau_2]$ (whether or not it conforms to the prediction). Consistency and competitiveness are conflicting objectives and different decision makers might have different preferences over them. In particular, increasing consistency usually leads to lower competitiveness. Consequently, our goal is to find an algorithm which can trade off the two quantities, allowing us to interpolate between the known-horizon and the uncertainty-window settings.

Once again, the versatility of Algorithm 3 and its ability to reduce the problem of finding the optimal algorithm to that of finding the optimal target consumption sequence comes to the fore. In particular, Proposition 2 implies that Algorithm 3 with target consumption sequence $\vec{\lambda}(T_P)$ for prediction $T_P \in [\tau_1, \tau_2]$ is β'-consistent for prediction T_P and γ'-competitive with

$$\beta' = c(\vec{\lambda}(T_P), T_P) - \epsilon \quad \text{and} \quad \gamma' = \inf_{T \in [\tau_1, \tau_2]} c(\vec{\lambda}(T_P), T) - \epsilon \,.$$

Therefore, given a prediction T_P and a required level of competitiveness $\gamma' = \gamma - \epsilon$, we need to solve the following optimization problem in order to maximize consistency while achieving γ'-competitiveness:

$$\max_{\vec{\lambda}} c(\vec{\lambda}, T_P) \quad \text{s.t.} \quad \inf_{T \in [\tau_1, \tau_2]} c(\vec{\lambda}, T) \geq \gamma \,.$$

As in the uncertainty-window setting, we can rewrite this as an LP.

Proposition 4. *For budget B, uncertainty window $[\tau_1, \tau_2]$, predicted horizon $T_P \in [\tau_1, \tau_2]$ and required level of competitiveness $\gamma' = \gamma - \epsilon$, we have*

$$\begin{aligned}
\max_{\vec{\lambda}} \quad & c(\vec{\lambda}, T_P) & = \quad \max_{\lambda, y} \quad & \frac{1}{T_P} \sum_{t=1}^{T_P} y_{T_P, t} & \\
s.t. \quad & \min_{T \in [\tau_1, \tau_2]} c(\vec{\lambda}, T) \geq \gamma & s.t. \quad & \gamma \leq \frac{1}{T} \sum_{t=1}^{T} y_{T,t} & \forall T \in [\tau_1, \tau_2] \\
& & & y_{T,t} \leq \frac{\lambda_{t,j}}{\rho_{T,j}} & \forall j \in [m], T \in [\tau_1, \tau_2], t \in [T] \\
& & & y_{T,t} \leq 1 & \forall T \in [\tau_1, \tau_2], t \in [T] \\
& & & \sum_{t=1}^{\tau_2} \lambda_t \leq B & \\
& & & \lambda \geq 0 &
\end{aligned}$$

Remark 4. *Our framework can also accommodate distributional predictions about the horizon, leading to a similar LP with the only difference being an additional expectation over the predicted*

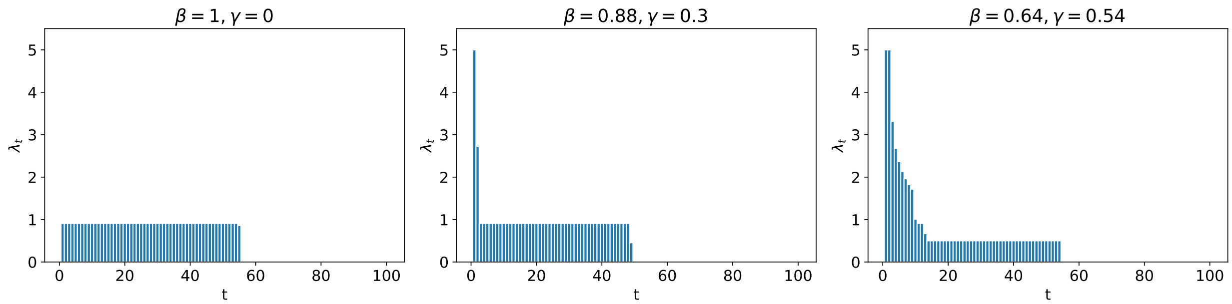

Figure 3.5: The optimal target consumption sequence for various values of required levels of competitiveness γ. Here, $m = 1$, $\tau_1 = 10$, $\tau_2 = 100$, $B = 50$ and $T_P = 55$. Moreover, 0.54 is the competitive ratio of the optimal target sequence, i.e., 0.54 is the optimal value of the LP in Proposition 3. The sequences lose consistency and gain competitiveness from left to right.

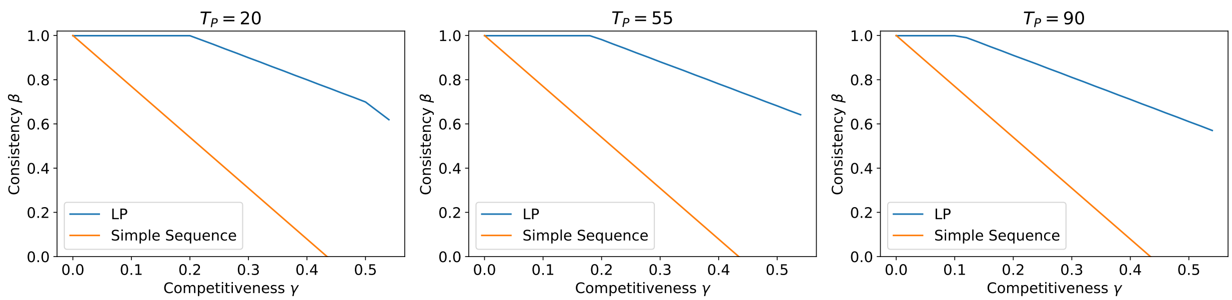

Figure 3.6: The consistency-competitiveness curves for the LP from Proposition 4 and the simple target sequence from (3.7), with predicted horizon $T_P \in \{20, 55, 90\}$. Here, $m = 1$, $\tau_1 = 10$, $\tau_2 = 100$ and $B = 50$. Consistency $\beta = 1$ corresponds to the known-horizon setting and competitiveness $\gamma = 0.54$ corresponds to the largest possible competitiveness which can be obtained by optimizing the target sequence (Proposition 3).

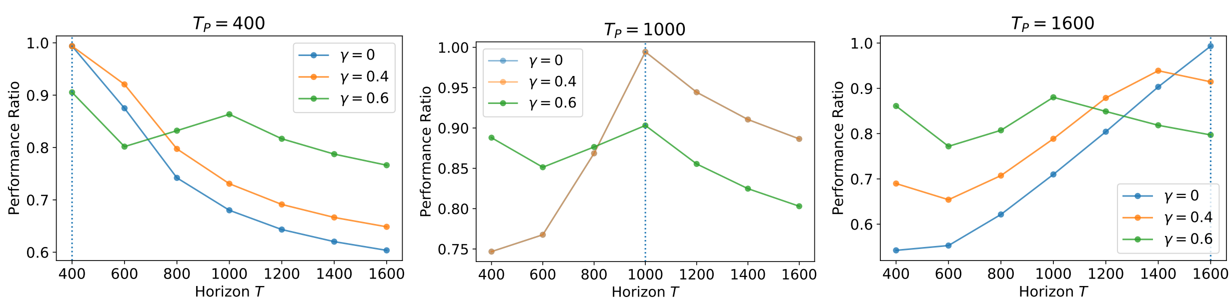

Figure 3.7: Average performance ratio (over 100 runs) of Algorithm 3, with target sequence from the LP in Proposition 4 for different values of γ, on the uniform multi-secretary problem.

horizon T_P in the objective.

Observe that, when $\gamma = 0$ and the decision maker does not desire robustness, the LP in Proposition 4 would output $\vec{\lambda}$ with $\lambda_t = \rho_{T_P}$ for $t \leq T_P$ and $\lambda_t = 0$ otherwise. Algorithm 3 with this target consumption sequence is exactly the algorithm of [BLM23], which yields a consistency of $\beta = 1$. On the other extreme is γ being equal to the output of the LP in Proposition 3, in which case the LP in Proposition 4 would output a target sequence $\vec{\lambda}$ which maximizes the competitive ratio $\min_{T\in[\tau_1,\tau_2]} c(\vec{\lambda}, T)$. For values of γ in between the two extremes, the LP in Proposition 4 outputs a target consumption sequence which attempts to balance the two objectives, as can be seen in Figure 3.5. This allows the decision maker to interpolate between the known-horizon and the uncertainty-window settings (see Figure 3.6).

Now, suppose the required level of competitiveness $\gamma' = \gamma - \epsilon$ is such that $\gamma = \alpha\cdot(1+\ln(\tau_2/\tau_1))^{-1}$ for some $\alpha \in [0, 1]$. Then, for predicted horizon $T_P \in [\tau_1, \tau_2]$, consider the following simple target consumption sequence

$$\lambda_t := \begin{cases} \frac{\alpha}{1+\ln(\tau_2/\tau_1)} \cdot \frac{B}{\tau_1} + (1-\alpha)\cdot\frac{B}{T_P} = \frac{\alpha}{1+\ln(\tau_2/\tau_1)}\cdot\rho_{\tau_1} + (1-\alpha)\cdot\rho_{T_P} & \text{if } t \le \tau_1\,, \\ \frac{\alpha}{1+\ln(\tau_2/\tau_1)} \cdot \frac{B}{t} + (1-\alpha)\cdot\frac{B}{T_P} = \frac{\alpha}{1+\ln(\tau_2/\tau_1)}\cdot\rho_{t} + (1-\alpha)\cdot\rho_{T_P} & \text{if } \tau_1 + 1 \le t \le T_P\,, \\ \frac{\alpha}{1+\ln(\tau_2/\tau_1)} \cdot \frac{B}{t} = \frac{\alpha}{1+\ln(\tau_2/\tau_1)}\cdot\rho_{t} & \text{if } T_P + 1 \le t \le \tau_2\,. \end{cases} \tag{3.7}$$

The target sequence $\vec{\lambda}$ is simply a sum of two target sequences: (i) The first part is an α-scaled-down version of the simple target sequence from (3.6), which ensures $\alpha\cdot(1+\ln(\tau_2/\tau_1)^{-1})$ competitiveness; (ii) The second is a $(1-\alpha)$-scaled-down version of the target sequence which spends $\rho_{T_P} = B/T_P$ evenly and is optimal when the prediction were true. $\vec{\lambda}$, as defined in (3.7), is a feasible solution to the optimization of Proposition 4, which allows us to establish the following closed-form guarantee.

Proposition 5. *Let ϵ be as in Proposition 2. Consider a target level of competitiveness $\gamma - \epsilon$, where $\gamma = \alpha\cdot(1+\ln(\tau_2/\tau_1))^{-1}$ for some $\alpha \in [0, 1]$. Let $\vec{\lambda}(T_P)$ be an optimal solution of the LP in Proposition 4 and let $A(T_P)$ denote Algorithm 3 with the target sequence $\vec{\lambda}(T_P)$. Then, for every request distribution $\mathcal{P}$ and predicted horizon $T_P \in [\tau_1, \tau_2]$, we have*

$$c(A(T_P)|T_P, \mathcal{P}) \ge \left(1 - \alpha + \frac{\alpha}{1+\ln(\tau_2/\tau_1)}\right) - \epsilon \quad \textit{and} \quad \inf_{T,T_P} c(A(T_P)|T, \mathcal{P}) \ge \frac{\alpha}{1+\ln(\tau_2/\tau_1)} - \epsilon\,.$$

Note that the target sequence in (3.7) is just one particular target sequence and the LP in Proposition 4 computes the optimal target sequence, and consequently the latter always performs better. This domination in performance is depicted in Figure 3.6, where the consistency-competitiveness curve the simple sequence (in orange) lies entirely below the curve from Proposition 4 (blue curve).

Numerical Experiment. We evaluated our algorithm (Algorithm 3 with target sequence from Proposition 4) on the multi-secretary problem with uniform rewards and the results are summarized in Figure 3.7. In this experiment, the request distribution captures the uniform multi-secretary problem: each request $\gamma = (f, b, \mathcal{X})$ has reward $f(x) = r \cdot x$ for $r \sim \text{Unif}([0, 1])$, consumption $b(x) = x$ and an accept/reject action space $\mathcal{X} = \{0, 1\}$. Moreover, $\tau_1 = 400$, $\tau_2 = 1600$, $B = 500$, $\eta = 0.03$, $\mu_1 = 0.5$ and $h(\cdot) = \|\cdot\|_2$. As expected, smaller values of γ lead to better performance when the true horizon T is close to the prediction T_P, but this comes at the expense of lower worst-case reward (minimum competitive ratio over all possible values of the horizon $T \in [\tau_1, \tau_2]$). Recall that $\gamma = 0$ represents the algorithm of [BLM23] with horizon T_P. Our experiment demonstrates its fragility to traffic spikes: if the number of requests turns out to be 3 times the predicted traffic of $T_P = 400$, the algorithm of [BLM23] achieves a drastically lower performance ratio than our algorithm with $\gamma = 0.6$.

3.5 Bypassing the LP: A Faster Algorithm

Even though the LPs of Proposition 3 and Proposition 4 compute the optimal target consumption sequence in polynomial time, they do not exploit the structure of the problem and are not desirable in large-scale domains like internet advertising where speed is of the essence. To remedy this, we next develop a faster algorithm to compute the optimal target consumption sequence; this algorithm more directly exploits the structure of the problem. The algorithm (Algorithm 4) will rely on the following observation about $c(\vec{\lambda}, T)$:

$$c(\vec{\lambda}, T) = \frac{1}{T} \sum_{t=1}^{T} \min\left\{ \min_{1 \le j \le m} \frac{\lambda_{t,j}}{\rho_{T,j}}, 1 \right\} \le \min_{1 \le j \le m} \frac{1}{T} \sum_{t=1}^{T} \min\left\{ \frac{\lambda_{t,j}}{\rho_{T,j}}, 1 \right\} = \min_{1 \le j \le m} \frac{1}{B_j} \sum_{t=1}^{T} \min\left\{ \lambda_{t,j}, \rho_{T,j} \right\} \tag{3.8}$$

Algorithm 4: Optimal Target Consumption Sequence

Input: Budget $B \in \mathbb{R}^m_{++}$, uncertainty window $[\tau_1, \tau_2]$, prediction T_P, required level of consistency $\beta \in [0,1]$ and required level of competitiveness $\gamma \in [0, \beta]$.

Initialize: $\lambda_{t,j} \leftarrow 0 \quad \forall t \in [\tau_2], j \in [m]$

for $T = \tau_2$ *to* τ_1 **do**

  for $t = 1$ *to* T **do**

$$\lambda_{t,j} \leftarrow \begin{cases} \lambda_{t,j} + \min\left\{\rho_{T,j} - \lambda_{t,j}, \beta \cdot B_j - \sum_{s=1}^{T} \lambda_{s,j}\right\}^+ & \text{if } T = T_P, \\ \lambda_{t,j} + \min\left\{\rho_{T,j} - \lambda_{t,j}, \gamma \cdot B_j - \sum_{s=1}^{T} \lambda_{s,j}\right\}^+ & \text{if } T \neq T_P \end{cases} \tag{3.9}$$

  end

end

Return: TRUE if $\sum_{t=1}^{\tau_2} \lambda_t \leq B_j$; else FALSE.

where the last equality follows from multiplying and diving by $\rho_{T,j} = B_j/T$. Moreover, note that the above inequality is tight when $\frac{\lambda_{t,j}}{\rho_{T,j}} = \frac{\lambda_{t,k}}{\rho_{T,k}}$ for all $j, k \in [m], t \in [T]$.

Therefore, any target sequence $\vec{\lambda}$ which is β-consistent for prediction T_P, i.e., $c(\vec{\lambda}, T_P) \geq \beta$, and γ-competitive, i.e. $\min_{T \in [\tau_1, \tau_2]} c(\vec{\lambda}, T) \geq \gamma$, satisfies the following inequalities for all $j \in [m]$:

$$\sum_{t=1}^{T_P} \min\{\lambda_{t,j}/\rho_{T_P,j}\} \geq \beta \cdot B_j \quad \text{and} \quad \sum_{t=1}^{T} \min\{\lambda_{t,j}/\rho_{T,j}\} \geq \gamma \cdot B_j \quad \forall\, T \in [\tau_1, \tau_2]\,.$$

Algorithm 4 minimizes $\sum_{t=1}^{\tau_2} \lambda_{t,j}$ while maintaining the above property. And as a consequence, we can show that β consistency on T_P and γ competitiveness are attainable if and only if Algorithm 4 returns TRUE. Given this property, it is a straightforward exercise to use binary search in conjunction with Algorithm 4 to compute the optimal solution to the LPs in Proposition 3 and Proposition 4 up to arbitrary precision (For completeness, we provide details in Appendix B.5).

Theorem 7. *Given budget $B \in \mathbb{R}^m_{++}$, uncertainty window $[\tau_1, \tau_2]$, prediction T_P, required level of consistency $\beta \in [0,1]$ and required level of competitiveness $\gamma \in [0, \beta]$ as input, let $\vec{\lambda}^*$ be the sequence computed by Algorithm 4. Then,*

1. $c(\vec{\lambda}^*, T_P) \geq \beta$ *and* $\min_{T \in [\tau_1, \tau_2]} c(\vec{\lambda}^*, T) \geq \gamma$
2. $\sum_{t=1}^{\tau_2} \lambda_t^* \leq B$ *if and only if there exists a target consumption sequence* $\vec{\lambda}'$ *(with* $\sum_{t=1}^{\tau_2} \lambda'_t \leq B$*) which satisfies* $c(\vec{\lambda}', T_P) \geq \beta$ *and* $\min_{T \in [\tau_1, \tau_2]} c(\vec{\lambda}', T) \geq \gamma$.

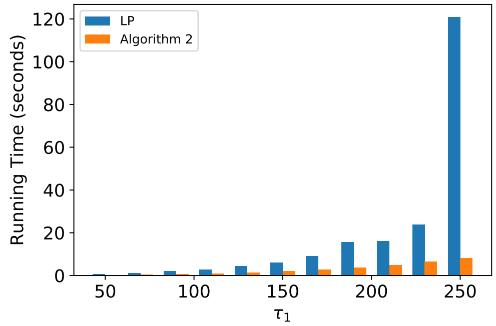

Figure 3.8: A comparison of the running times of the LP from Proposition 3 solved using Gurobi Optimizer version 9.1.2 build v9.1.2rc0 (mac64) and Algorithm 4 run on Python 3.7.6 without the use of any advanced libraries. The minimum runtime from 10 runs was selected for GUROBI and the maximum runtime from 10 runs was selected for Algortihm 4. Both algorithms were limited to a single thread to ensure parity of computational resources. Here, $\tau_2 = 3 \cdot \tau_1$ and $B = 1.5 \cdot \tau_1$ for all values of τ_1. For $\tau_1 \geq 300$, GUROBI did not terminate with a solution even after 10 min, while Algorithm 4 consistently did so under 10 seconds.

Observe that there can be at most τ_2^2 updates of the target sequence $\vec{\lambda}$ (as given in (3.9)) during the run of Algorithm 4. One can maintain and iteratively update $\sum_{s=1}^{T} \lambda_{s,j}$ after the completion of each iteration of the inner and outer **For** loops to perform the update in constant time. Therefore, the runtime complexity of Algorithm 4 is $O(m \cdot \tau_2^2)$, which is faster than any known general-purpose LP solver applied to the LP in Proposition 3 or Proposition 4. We also empirically observed this difference in running times between the LP of Proposition 3 and Algorithm 4 (see Figure 3.8).

3.6 Conclusion

We develop and analyze a generalized version of dual descent which can incorporate variable target consumption sequences (Algorithm 3), thereby reducing the complicated problem of finding an algorithm for online resource allocation under horizon uncertainty to the much simpler (and convex) problem of optimizing the target sequence. We then demonstrate the power of this reduction by showing that, with the optimal target sequence, Algorithm 3 achieves a near-optimal competitive ratio when only upper and lower bounds on the horizon are known. We also provide a way to smoothly interpolate between the previously-studied known-horizon setting and the uncertainty-window setting through the Algorithms-with-Predictions framework, thereby providing a robust approach to online allocation which allows the decision-maker to tailor the degree to robustness to their requirements. Our algorithms have the added advantage of simplicity and speed because they do not require the decision-maker to solve any large linear programs.

We leave open the problem of closing the gap between our lower and upper bounds on the competitive ratio by accounting for the $e \cdot \ln\ln(\tau_2/\tau_1)$ discrepancy. Although this gap is not large asymptotically, closing it will likely result in a deeper understanding of the problem. It would also be interesting to explore whether algorithms which operate in the primal space can similarly benefit from employing a variable target consumption sequence. Finally, when both the distribution of requests and the distribution of the horizon are known in advance, it is worth studying if it is possible to achieve a constant/logarithmic regret against an appropriately defined benchmark (see for example [BW20; VB19] for similar results when the horizon is known).

Chapter 4: Contextual Standard Auctions with Budgets

Based on the publication [BKK23a] co-authored with Santiago Balseiro and Christian Kroer.

This chapter marks a change in perspective. Till now, we have focused on the budget management problem faced by an individual advertiser. Now, we will take a broader look at the market as a whole, and analyze the market-level outcomes that emerge from the strategic interactions of advertisers who are all simultaneously attempting to maximize their utility subject to budget constraints. The current chapter focuses on investiagting the equilibria that emerge in such markets.

In Section 4.2, we establish the existence of a well-structured Bayes-Nash equilibrium for all standard auction. The equilibrium bidding strategy paces (i.e., multiplcatively shades) the value and composes it with the equilibrium strategy for the setting without budget constraints. Then, in Section 4.3, we leverage this modular structure to prove a revenue equivalence results, which states that the average revenue to the platform is the same under all standard auction formats, even in the presence of budgets. Finally, we prove a price of anarchy bound for liquid welfare in Section 4.4 and conclude with some structural properties (Section 4.5) and numerical experiments (Section 4.6).

4.1 Model

We consider the setting in which a seller (i.e., the advertising platform) plans to sell an indivisible item to one of n buyers (i.e., the advertisers) using an auction. We adopt a feature-based valuation model for the buyer. More precisely, the item type is represented using a vector α belonging to the set $A \subset \mathbb{R}^d$, where each component of α can be interpreted as a feature. We also refer to α as the *context*. Each buyer type is represented using a vector (w, B) belonging to the set $\Theta \subset \mathbb{R}^{d+1}$ of possible buyer types, where the last component B denotes her budget and the first d components w capture the weights she assigns to each of the d features. The value (maximum willingness to pay) that buyer type (w, B) has for item α is given by the inner product $w^T\alpha$. For simplicity of notation and ease of exposition, we will state our results under this linear relationship between values and the features, but our model and results can be extended to accommodate non-linear response functions (such as the logistic function) that are commonly used in practice (see Appendix C.7 for a more detailed discussion). We will use $\omega = \max_{(w,B)\in\Theta,\alpha\in A} w^T\alpha$ to denote the maximum value that a buyer can have for an item.

We assume that the context of the item to be auctioned is drawn from some distribution F over the set of possible items types A. Furthermore, the type for every buyer is drawn according to some distribution G over the set of possible buyer types Θ, independently of the other buyers and the choice of the item. Note that, by virtue of our context-based valuation model, the values of the n buyers for the item need not be independent. In line with standard models used for Bayesian analysis of auctions, we will assume that both G and F are common knowledge, while maintaining that the realized type vector (w, B) associated with a buyer is her private information. Our model allows budgets to be random and correlated with the buyers' weight vector. In addition, we will assume that buyers are unaware of the type of their competing buyers—this implies budgets are private.

To fix ideas, we first consider the case of a first-price auction with reserve prices and then discuss how our results extend to standard auctions in Section 4.3. In a first-price auction, the seller allocates the item to the highest bidder whenever her bid is above the reserve price and the winning bidder pays her bid. We assume the seller discloses the item type α to the n buyers before bids are solicited from them. As a result, the bid of a buyer on item α can depend on α. We use $r : A \to \mathbb{R}$ to specify the publicly known context-dependent reserve prices, where $r(\alpha)$ denotes the reserve price on item type α.

The budget of a buyer represents an upper bound on the amount she would like to pay in the auction. We only require that each buyer satisfy her budget constraint in expectation over the item type and competing buyer types. Similar assumptions have been made in the literature (see, e.g., [GKP12; AH13; BBW15; Bal+21; Con+18]). The motivation behind this modeling choice is that budget constraints are often enforced on average by advertising platforms. For example, Google Ads allows daily budgets to be exceeded by a factor of two in any given day, but, over the course of month, the total expenditure never exceeds the daily budget times the days in the month.[1] In-expectation budget constraints are also motivated by the fact that, in practice, buyers typically participate in a large number of auctions and many buyers use stationary bidding strategies. Thus, by the law of large numbers, our model can be interpreted as collapsing multiple, repeated auctions in which item types are drawn i.i.d. from F into a single one-shot auction with in-expectation constraints.

Notation. We will use $\mathbb{R}_+$ and $\mathbb{R}_{\geq 0}$ to denote the set of strictly positive and non-negative real numbers, respectively. We will use G_w to denote the marginal distribution of w when $(w, B) \sim G$, i.e., $G_w(K) := G(\{(w, B) \in \Theta \mid w \in K\})$ for all Borel sets $K \subset S$. In a similar vein, we will use Θ_w to denote the set of $w \in \mathbb{R}^d$ such that $(w, B) \in \Theta$ for some $B \in \mathbb{R}$. (Here we abuse notation by using w both as a weight vector variable and as a subscript to denote the projection of a buyer type onto the first d dimensions). Unless specified otherwise, $\|\cdot\|$ denotes the Euclidean norm.

Assumptions. We will assume that there exist $U, B_{\min} > 0$ such that the set of possible buyer types Θ is given by $\Theta = (0, U)^d \times (B_{\min}, U)$. In a similar vein, we also assume that the set of possible item types A is a subset of the positive orthant $\mathbb{R}^d_+$. We will restrict our attention to $d \geq 2$, which is the regime in which our feature-vector based valuation model yields interesting insights. To completely specify the aforementioned probability spaces, we endow A, Θ and Θ_w with the Lebesgue $\sigma-$algebra. Moreover, we will assume that the distributions F and G have density functions. Note that the distribution G can be any distribution on Θ, including one with probability zero on some regions of Θ. Thus we can address any buyer distribution, so long as it has a density and is supported on a bounded subset of the strictly-positive orthant with a positive lower bound on the possible budgets. Similarly, F can capture a wide variety of item distributions.

[1]Google Ads Help page defines "Average Daily Budget": `https://support.google.com/google-ads/answer/6312?hl=en`

It is worth noting that any distribution that lacks a density can be approximated arbitrarily well by a distribution with a density, thereby extending the reach of our results to arbitrary distributions.

4.1.1 Equilibrium Concept

Consider the decision problem faced by a buyer type $(w, B) \in \Theta$ if we fix the bidding strategies of all competing buyers on all possible item types: She wishes to bid on the items in a way that maximizes her expected utility while satisfying her budget constraint in expectation (where the expectation is taken over items and competing buyers' types). As is true in the well-studied standard budget-free i.i.d. setting ([Kri09]), her optimal strategy depends on the strategies used by the other buyer types. In the standard setting, the symmetric Bayes-Nash equilibrium is an appealing solution concept for the game formed by these interdependent decision problems faced by the buyers. We adopt a similar approach and define the symmetric Bayes-Nash equilibrium for our model. A strategy $\beta^* : \Theta \times A \to \mathbb{R}_{\geq 0}$ (a mapping that specifies what each buyer type should bid on every item) is a Symmetric First-Price Equilibrium if, almost surely over all buyer types, using β^* is an optimal solution to a buyer type's decision problem when all other buyer types also use it.

Definition 3. *A strategy $\beta^* : \Theta \times A \to \mathbb{R}_{\geq 0}$ is called a Symmetric First-Price Equilibrium (SFPE) if $\beta^*(w, B, \alpha)$ (as a function of α) is an optimal solution to the following optimization problem almost surely w.r.t. $(w, B) \sim G$:*

$$\begin{aligned} \max_{b:A\to\mathbb{R}_{\geq 0}} \quad & \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}} \left[(w^T\alpha - b(\alpha))\, \mathbb{1}\left\{ b(\alpha) \geq \max\left(r(\alpha), \{\beta^*(\theta_i, \alpha)\}_i \right) \right\} \right] \\ \text{s.t.} \quad & \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}} \left[b(\alpha)\, \mathbb{1}\left\{ b(\alpha) \geq \max\left(r(\alpha), \{\beta^*(\theta_i, \alpha)\}_i \right) \right\} \right] \leq B. \end{aligned}$$

In the buyer's optimization problem the buyer wins whenever her bid $b(\alpha)$ is higher than the reserve price $r(\alpha)$ and all competiting bids $\beta^*(\theta_i, \alpha)$ for $i = 1, \ldots, n-1$. Because of the first-price auction payment rule, each bidder pays her bid whenever she wins. For convenience, in the above definition, we are using an infeasible tie-breaking rule which allocates the entire good to every highest bidder. This is inconsequential, and can be replaced by any arbitrary tie-breaking rule, because we will later show (see part (d) of Lemma 34) that ties are a zero-probability event under our value-pacing-based equilibria.

In our solution concept, it is sufficient that advertisers have Bayesian priors over the maximum competing bid $\max_i\{\beta^*(\theta_i, \alpha)\}$ to determine a best response. This is aligned with practice as many advertising platforms provide bidders with historical bidding landscapes, which advertisers can use to optimize their bidding strategies [Goo].[2] Additionally, we require that budgets are satisfied in expectation over the contexts and buyer types. Connecting back to our repeated auctions interpretation, one can assume competitors' types to be fixed throughout the horizon while contexts are drawn i.i.d. in each auction. In this case, our solution concept would be appropriate if buyers cannot observe the types of competitors and, in turn, employ stationary strategies that do not react to the market dynamics. Such stationary strategies are appealing because they deplete budgets smoothly over time and are simple to implement. Moreover, it has been previously established that stationary policies approximate well the performance of dynamic policies in non-strategic settings when the number of auctions is large and the maximum value of each auction is small relative to the budget (see, e.g., [TVR06]).

When the types of bidders is fixed throughout the horizon, a bidder who employs a dynamic strategy could, in principle, profitably deviate by inferring the competitors' types and using this information to optimally shade her bids. Implementing such strategies in practice is challenging because many platforms do not disclose the identity of the winner nor the bids of competitors in real-time (as we discussed above, they mostly provide historical information that is aggregated over many auctions). Moreover, when the number of bidders is large and each bidder competes with a random subset of bidders, such deviations can be shown to not be profitable using mean-field techniques (see, e.g., [IJS14; BBW15]) in our contextual value model as long as values are independent across time. Therefore, our model can be alternatively interpreted as one in which there is a large population of active buyers and each buyer competes with a random subset of

[2] See, for example, https://www.blog.google/products/admanager/rolling-out-first-price-auctions-google-ad-manager-partners/.

buyers. This assumption is well motivated in the context of internet advertising markets because the number of advertisers actively bidding is typically large and, because of sophisticated targeting technologies, advertisers often participate only in a fraction of all auctions.

4.1.2 Ties and the Role of Contexts

Before moving onto the proof of existence of SFPE, we would like to shed some light on the role played by contexts in our model and results. The assumption that the feature vectors α are drawn from a distribution F which has a density is necessary for our results to hold. In fact, if there was only one deterministic context, an SFPE may fail to exist: we provide an example in Appendix C.1. The root cause behind the absence of a well-behaved equilibrium in this example is the tension between the proclivity of budgets to cause ties with positive probability (as we demonstrate in Section 4.5) and the potential lack of equilibria for first-price auctions under value distributions that cause ties with a positive probability. Our example in Appendix C.1 does admit a symmetric equilibrium for second-price auction, thereby demonstrating the added complexity of dealing with first-price auctions.

Issues of tie-breaking have previously come up in a line of related work on pacing-based equilibria in second-price auctions [Bor+07; BBW15; Con+18; Bab+20], where they were addressed by methods that are some version of randomly perturbing the value of each buyer and enforcing the budget constraint on average over these perturbations. This causes ties to become zero-probability events. It is possible to prove our existence and revenue equivalence results for the case of one deterministic context with value perturbations. However, unlike second-price auctions where bidding truthfully is a dominant strategy, value perturbation is not well-suited for first-price auctions because, even in the absence of budgets, the first-price auction equilibrium strategy would depend on the perturbations. Moreover, our structural results (Proposition 8 and Proposition 9) may not hold for arbitrary perturbations and would require an unjustifiably-strong assumption that carefully coordinates the perturbations across buyer types. That being said, if one is willing to ignore ties, our results continue to hold for a single deterministic context and the reader can safely continue with that setting in mind.

4.2 Existence of Symmetric First-Price Equilibrium

In this section, we study the existence of SFPE, and show that this existence is achieved by a compelling solution which is interpretable. We do so in several steps. First, we define a natural parameterized class of value-pacing-based strategies. Then, assuming that the buyer types are using a strategy from this class, we establish strong duality for the optimization problem faced by each buyer type and characterize the primal optimum in terms of the dual optimum. This leads to a substantial simplification of the analysis because it allows us to work in the much simpler dual space. Finally, we establish the existence of a value-pacing-based SFPE by a fixed-point argument over the space of dual-multipliers.

4.2.1 Value-Pacing-Based Strategies

In this dissertation, pacing refers to multiplicatively scaling down a quantity.[3] Consider a function $\mu : \Theta \to \mathbb{R}_{\geq 0}$, which we will refer to as the *pacing function*. We define the *paced weight vector* of a buyer with type (w, B) to be $w/(1 + \mu(w, B))$, which is simply the true weight vector w scaled down by the factor $1/(1 + \mu(w, B))$. Similarly, we define the *paced value* of a buyer type (w, B) for item α as $w^T \alpha/(1 + \mu(w, B))$. We will use pacing to ensure that the budget constraints of all buyer types are satisfied, and at the same time, maintain the best response property at equilibrium. The motivation for using pacing as a budget management strategy will become clear in the next section, where we show that the best response of a buyer to other buyers using a value-pacing-based strategy is to also use a value-pacing-based strategy. Before defining the strategy, we set up some preliminaries.

Consider a pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$ and an item $\alpha \in A$. Let λ_α^μ denote the distribution of paced values $w^T \alpha/(1 + \mu(w, B))$ for item α when $(w, B) \sim G$. Let H_α^μ denote the distribution of the highest value $Y := \max\{X_1, \ldots, X_{n-1}\}$ among $n - 1$ buyers, when each $X_i \sim \lambda_\alpha^\mu$ is drawn

[3] We use the term value-pacing-based strategies to differentiate it from bid-pacing/bid-shading, which has previously been studied in the context of truthful auctions [Bor+07; BBW15; Bal+21; Con+18; Con+19].

independently for $i \in \{1, \ldots, n-1\}$. Observe that $H^\mu_\alpha((-\infty, x]) = \lambda^\mu_\alpha((-\infty, x])^{n-1}$ for all $\alpha \in A$ because the random variables are i.i.d.

For a given item $\alpha \in A$, when $x \geq r(\alpha)$, define the following bidding function,

$$\sigma^\mu_\alpha(x) := x - \int_{r(\alpha)}^{x} \frac{H^\mu_\alpha(s)}{H^\mu_\alpha(x)} ds,$$

where we interpret $\sigma^\mu_\alpha(x)$ to be 0 if $H^\mu_\alpha(x) = 0$. Moreover, when $x < r(\alpha)$, define $\sigma^\mu_\alpha(x) := x$ (we make this choice to ensure that no value below the reserve price gets mapped to a bid above the reserve price, while maintaining continuity). Note that $\sigma^\mu_\alpha(x) = \mathbb{E}[\max(Y, r) \mid Y < x]$. If λ^μ_α has a density, then σ^μ_α is the same as the single-auction equilibrium strategy for a standard first-price auction without budgets, when the buyer values are drawn i.i.d. from λ^μ_α and the item has a reserve price of $r(\alpha)$ (see, e.g., section 2.5 of [Kri09]). Our value-pacing-based strategy uses $\sigma^\mu_\alpha(x)$ as a building block, by composing it with value-pacing:

Definition 4. *The value-pacing-based strategy $\beta^\mu : \Theta \times A \to \mathbb{R}_{\geq 0}$ for pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$ is given by*

$$\beta^\mu(w, B, \alpha) := \sigma^\mu_\alpha\left(\frac{w^T \alpha}{1 + \mu(w, B)}\right) \qquad \forall\, (w, B) \in \Theta, \alpha \in A$$

The bid $\beta^\mu(w, B, \alpha)$ is the amount that a non-budget-constrained buyer with type (w, B) would bid on item α if she acted as if her paced value was her true value (this is captured by the use of the paced value as the argument for σ^μ_α), and believed that the rest of the buyers were also acting in this way (this is captured by the use of σ^μ_α). Therefore, our strategy has a simple interpretation: bidders pace their values and then bid as in a first-price auction in which competitors' values are also paced. Consequently, under our strategy bidders are shading their values twice: first when determining their paced values $w^T\alpha/(1 + \mu(w, B))$ to account for budget constraints and then again when adopting the bidding function σ^μ_α for the first-price auction. The bidding strategy σ^μ_α optimally trades off two effects: on the one hand, bidding too close to their paced values leaves no utility to buyers because they pay their bid in case of winning and, on the other hand, bidding too low decreases payments at the expense of also decreasing the chance of winning.

Observe that value-pacing-based strategies greatly reduce the degrees of freedom in the system. Instead of specifying a bidding strategy, which is a function, for each buyer type, we only need to specify a scalar, $\mu(w, B)$ for each buyer type. In addition, our dual characterization allow us to optimize over the space of all bidding strategies without imposing any restriction on the class of admissible functions. Having defined value-pacing-based strategies, we are now ready to state our main existence result.

Theorem 8. *There exists a pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$ such that the value-pacing-based strategy $\beta^{\mu} : \Theta \times A \to \mathbb{R}_{\geq 0}$ is a Symmetric First-Price Equilibrium (SFPE).*

Before proceeding with the proof of Theorem 8, we note some of its practical prescriptions: (i) It recommends that buyers should pace their value to manage their budgets. As we will later show, the equilibrium pacing functions for first-price auctions are identical to the ones for second-price auctions. This suggests that pacing-based-budget-management techniques developed for second-price auctions (like [BG19]) can be used for first-price auctions to compute the paced valued. (ii) Advertising platforms typically provide bidding landscapes to the buyers which allow them to compute the optimal bid for a given value. Given a context α, if $\mathbb{P}^{\mu}_{\alpha}$ represents the equilibrium bidding landscape (distribution of highest competing bids), then we have

$$\sigma^{\mu}_{\alpha}(x) \in \operatorname{argmax}_b \, (x - b) \, \mathbb{P}^{\mu}_{\alpha}(b)$$

Thus, the paced value can be combined with the landscape to compute the optimal bid $\beta^{\mu}(w, B, \alpha)$.

We provide the proof of Theorem 8 in the remaining subsections. First, in Subsection 4.2.2, we show that, if all of the competing buyers are assumed to employ a value-pacing-based strategy, then strong duality holds for the budget-constrained utility maximization problem faced by each buyer type. This allows us to drastically simplify the equilibrium strategy space of each buyer type from a function (mapping contexts to bids) to a single scalar (the dual variable $\mu(w, B)$). Next, in Subsection 4.2.3, we prove the existence of a value-pacing-based equilibrium strategy by proving a fixed-point theorem in the dual space of pacing functions. Despite our simplifying move to the dual space, establishing a fixed point is by no means a straightforward task because we are still left with a dual variable for each buyer type and there are (uncountable) infinitely many of those. This leads to an infinite-dimensional fixed-point problem which requires careful topological analysis. We find that the commonly-employed general-purpose topologies fail for our problem, and this motivates us to carefully exploit the structure of pacing to select the right topology.

4.2.2 Strong Duality and Best Response Characterization

We start by considering the optimization problem faced by an individual buyer with type (w, B) when all competing buyers use the value-pacing-based strategy with pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$. Denoting by $Q^\mu(w, B)$ the optimal expected utility of such a buyer, we have

$$\begin{aligned} Q^\mu(w, B) = \max_{b:A\to\mathbb{R}_{\geq 0}} \quad & \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}} \left[(w^T\alpha - b(\alpha))\, \mathbb{1}\left\{ b(\alpha) \geq \max\left(r(\alpha), \{\beta^\mu(\theta_i, \alpha)\}_i \right) \right\} \right] \\ \text{s.t.} \quad & \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}} \left[b(\alpha)\, \mathbb{1}\left\{ b(\alpha) \geq \max\left(r(\alpha), \{\beta^\mu(\theta_i, \alpha)\}_i \right) \right\} \right] \leq B. \end{aligned}$$

Our goal in this section is to show that the value-pacing-based strategy put forward in Definition 4 is a best response when competitors are pacing their bids according to a pacing function μ.

Remark 5. *Compare $Q^\mu(w, B)$ to the definition of a SFPE (Definition 3), and observe that, if we were able to show that there exists $\mu : \Theta \to \mathbb{R}_{\geq 0}$ such that $\beta^\mu(w, B, \cdot)$ is an optimal solution to $Q^\mu(w, B)$ almost surely w.r.t. $(w, B) \sim G$, then β^μ would be an SFPE.*

For $\mu : \Theta \to \mathbb{R}_{\geq 0}$ and $(w, B) \in \Theta$, consider the Lagrangian optimization problem of $Q^\mu(w, B)$ in which we move the budget constraint to the objective using the Lagrange multiplier $t \geq 0$. We use t to denote the multiplier of one buyer in isolation to distinguish from μ, which is a function giving a multiplier for every buyer type. Denoting by $q^\mu(w, B, t)$ the dual function, we have that

$$q^\mu(w, B, t) = \max_{b(\cdot)} \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}} \left[(w^T\alpha - (1+t)b(\alpha))\, \mathbb{1}\left\{ b(\alpha) \geq \max\left(r(\alpha), \{\beta^\mu(\theta_i, \alpha)\}_i \right) \right\} \right] + tB$$

$$= (1+t) \max_{b(\cdot)} \mathbb{E}_{\alpha, \{\theta_i\}_{i=1}^{n-1}} \left[\left(\frac{w^T \alpha}{1+t} - b(\alpha) \right) \mathbb{1} \left\{ b(\alpha) \geq \max \left(r(\alpha), \{\beta^\mu(\theta_i, \alpha)\}_i \right) \right\} \right] + tB.$$

The dual problem of $Q^\mu(w, B)$ is given by $\min_{t \geq 0} q^\mu(w, B, t)$.

The next lemma states that the optimal solution to the Lagrangian optimization problem is a value-pacing-based strategy. More specifically, for every pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$, buyer type (w, B) and dual multiplier t, the value pacing based strategy $\sigma_\alpha^\mu \left(w^T \alpha / (1+t) \right)$ is an optimal solution to the Langrangian relaxation of $Q^\mu(w, B)$ corresponding to multiplier t. Note that, in general, t need not be equal to $\mu(w, B)$.

Lemma 11. *For pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$, buyer type $(w, B) \in \Theta$ and dual multiplier $t \geq 0$,*

$$\sigma_\alpha^\mu \left(\frac{w^T \alpha}{1+t} \right) \in \text{argmax}_{b(\cdot)} \ \mathbb{E}_{\alpha, \{\theta_i\}_{i=1}^{n-1}} \left[\left(\frac{w^T \alpha}{1+t} - b(\alpha) \right) \mathbb{1} \left\{ b(\alpha) \geq \max \left(r(\alpha), \{\beta^\mu(\theta_i, \alpha)\}_i \right) \right\} \right].$$

In the proof of Lemma 11, we actually show something stronger than the statement of Lemma 11: the value-pacing-based strategy is optimal point-wise for each α and not just in expectation over α. This follows from the observation that once we fix an item α, we are solving the best response optimization problem faced by a buyer with value $w^T \alpha / (1+t)$ in the standard i.i.d. setting [Kri09] with competing buyer values being drawn from λ_α^μ and under the assumption that the competing buyers use the strategy σ_α^μ. If λ_α^μ had a strictly positive density, then the optimality of $\sigma_\alpha^\mu \left(w^T \alpha / (1+t) \right)$ would be a direct consequence of the definition of a symmetric BNE in the standard i.i.d. setting. Even though the standard results cannot be used directly because of the potential absence of a density in the situation outlined above, we show that it is possible to adapt the techniques used in the proof of Proposition 2.2 of [Kri09] to show Lemma 11.

Using Lemma 11, we can simplify the expression for the dual function $q^\mu(w, B, t)$. First, note that because σ_α^μ is non-decreasing the highest competing bid can be written as

$$\max_{i=1,\ldots,n-1} \{\beta^\mu(\theta_i, \alpha)\} = \max_{i=1,\ldots,n-1} \left\{ \sigma_\alpha^\mu \left(\frac{w_i^T \alpha}{1 + \mu(\theta_i)} \right) \right\} = \sigma_\alpha^\mu \left(Y \right),$$

where $Y \sim H_\alpha^\mu$ is the maximum of $n-1$ paced values. Therefore, using that $\sigma_\alpha^\mu\left(w^T\alpha/(1+t)\right)$ is an optimal bidding strategy we get that

$$\begin{aligned}
q^\mu(w,B,t) &= (1+t)\,\mathbb{E}_\alpha \mathbb{E}_{Y\sim H_\alpha^\mu}\left[\left(\frac{w^T\alpha}{1+t} - \sigma_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right)\right)\,\mathbb{1}\left\{\sigma_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right) \geq \max\left(r(\alpha), \sigma_\alpha^\mu(Y)\right)\right\}\right] + tB \\
&= (1+t)\,\mathbb{E}_\alpha \mathbb{E}_{Y\sim H_\alpha^\mu}\left[\left(\frac{w^T\alpha}{1+t} - \sigma_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right)\right)\,\mathbb{1}\left\{\frac{w^T\alpha}{1+t} \geq \max\left(r(\alpha), Y\right)\right\}\right] + tB \\
&= (1+t)\,\mathbb{E}_\alpha\left[\left(\frac{w^T\alpha}{1+t} - \sigma_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right)\right)\,H_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right)\mathbb{1}\left\{\frac{w^T\alpha}{1+t} \geq r(\alpha)\right\}\right] + tB \\
&= (1+t)\,\mathbb{E}_\alpha\left[\mathbb{1}\left\{\frac{w^T\alpha}{1+t} \geq r(\alpha)\right\}\int_{r(\alpha)}^{\frac{w^T\alpha}{1+t}} H_\alpha^\mu(s)ds\right] + tB\,,
\end{aligned}$$

where the second equation follows from part (c) of Lemma 34, the third from taking expectations with respect to Y, and the last from our formula for σ_α^μ.

We now present the main result of this subsection, which characterizes the optimal solution of $Q^\mu(w,B)$ in terms of the optimal solution of the dual problem. The idea of using value-pacing-based strategies as candidates for the equilibrium strategy owes its motivation to Proposition 6. It establishes that if all the other buyers are using a value-pacing-based strategy, with some pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$, then a value-pacing-based strategy is a best response for a given buyer (w,B).

Proposition 6. *There exists $\Theta' \subset \Theta$ such that $G(\Theta') = 1$ and for all pacing functions $\mu : \Theta \to \mathbb{R}_{\geq 0}$ and buyer types $(w,B) \in \Theta'$, if t^* is an optimal solution to the dual problem, i.e., if $t^* \in \operatorname{argmin}_{t^*\geq 0} q^\mu(w,B,t)$, then $\sigma_\alpha^\mu\left(w^T\alpha/(1+t^*)\right)$ is an optimal solution for the optimization problem $Q^\mu(w,B)$.*

In Proposition 6, the pacing parameter t^* used for pacing in the best response can, in general, be different from $\mu(w,B)$. This caveat requires a fixed-point argument to resolve, which will be the subject matter of the next subsection.

Remark 6. *Restricting to the measure-one set Θ' is without loss. Recall that according to Definition 3, a strategy constitutes a SFPE if, almost surely over $(w,B) \sim G$, using β^* is an optimal solution to their optimization problem when all other buyer types also use it. As a consequence* *of this definition, we will show that it suffices to show strong duality for a subset of buyer types $\Theta' \subset \Theta$ such that $G(\Theta') = 1$. In the absence of reserve prices $r(\alpha)$ for the items, Proposition 6 holds for all $(w, B) \in \Theta$. Reserve prices introduce some discontinuities in the utility and payment term. The subset $\Theta' \subset \Theta$ captures a collection of buyer types for which these discontinuities are inconsequential, while maintaining $G(\Theta') = 1$.*

Observe that $Q^{\mu}(w, B)$ is not a convex optimization problem, so in order to prove the above theorem, we cannot appeal to the well-known strong duality results established for convex optimization. Instead, we will use Theorem 5.1.5 of [BHM98], which states that, to prove optimality of $\sigma_{\alpha}^{\mu}\left(w^T\alpha/(1+t^*)\right)$ for $Q^{\mu}(w, B)$, it suffices to show primal feasibility of $\sigma_{\alpha}^{\mu}\left(w^T\alpha/(1+t^*)\right)$, dual feasibility of t^*, Lagrange optimality of $\sigma_{\alpha}^{\mu}\left(w^T\alpha/(1+t^*)\right)$ for multiplier t^*, and complementary slackness. Our approach will be to show these required properties by combining the differentiability of the dual function with first order optimality conditions for one dimensional optimization problems. The key observation here is that the derivative of the dual function is equal to the difference between the budget of the buyer and her expected expenditure. Therefore, at optimality, the first-order conditions of the dual problem imply feasibility of the value-based pacing strategy. To prove differentiability we leverage that in our game the distribution of competing bids is absolutely continuous, which is critical for our results to hold.

For $t^* \in \operatorname{argmin}_{t\geq 0} q^{\mu}(w, B, t)$, if we apply the first-order optimality conditions for an optimization problem with a differentiable objective function over the domain $[0, \infty)$, we get

$$\frac{\partial q^{\mu}(w, B, t^*)}{\partial t} \geq 0, \qquad t^* \geq 0, \qquad t^* \cdot \frac{\partial q^{\mu}(w, B, t^*)}{\partial t} = 0\,.$$

The first condition can be shown to imply primal feasibility, the second implies dual feasibility, and the third implies complementary slackness. Combining this with Lemma 11, which establishes Lagrange optimality, and applying Theorem 5.1.5 of [BHM98] yields Proposition 6. The complete proof of Proposition 6 can be found in Appendix C.2.

4.2.3 Fixed Point Argument

In light of Proposition 6, the proof of Theorem 8 (the existence of a value-pacing-based SFPE) boils down to showing that there exists a pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$ such that, almost surely w.r.t. $(w, B) \sim G$, $\mu(w, B)$ is an optimal solution to the dual optimization problem $\min_{t \geq 0} q^{\mu}(w, B, t)$. In other words, given that everybody else acts according to μ, a buyer (w, B) that wishes to minimize the dual function is best off acting according to μ. More specifically, in Proposition 6 we showed that, starting from a pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$, if $\mu^*(w, B)$ constitutes an optimal solution to the dual problem $\min_{t \geq 0} q^{\mu}(w, B, t)$ almost surely w.r.t. $(w, B) \sim G$, then $\sigma_{\alpha}^{\mu}\left(w^T\alpha/(1 + \mu^*(w, B))\right)$ is an optimal solution for the optimization problem $Q^{\mu}(w, B)$ almost surely w.r.t. $(w, B) \sim G$. In other words, bidding according to σ_{α}^{μ} while pacing according to $\mu^* : \Theta \to \mathbb{R}_{\geq 0}$ is a utility-maximizing strategy for buyer $(w, B) \sim G$ almost surely, given that other buyers bid according to σ_{α}^{μ} with paced values obtained from μ. The following theorem establishes the existence of a pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$ for which μ itself fills the role of μ^* in the previous statement, thereby implying the optimality of $\sigma_{\alpha}^{\mu}\left(w^T\alpha/(1 + \mu(w, B))\right)$ almost surely w.r.t. $(w, B) \sim G$.

Proposition 7. *There exists $\mu : \Theta \to \mathbb{R}_{\geq 0}$ such that $\mu(w, B) \in \operatorname{argmin}_{t \geq 0} q^{\mu}(w, B, t)$ almost surely w.r.t. $(w, B) \sim G$.*

We prove the above statement using an infinite-dimensional fixed-point argument on the space of pacing functions with a carefully chosen topology. Informally, we need to show that the correspondence that maps a pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$ to the set of dual-optimal pacing functions, $\mu^* : \Theta \to \mathbb{R}_{\geq 0}$ which satisfy $\mu^*(w, B) \in \operatorname{argmin}_{t \geq 0} q^{\mu}(w, B, t)$, has a fixed point. However, unlike finite-dimensional fixed-point arguments, establishing the sufficient conditions of convexity and compactness needed to apply infinite-dimensional fixed point theorems requires a careful topological argument.

Lemma 36 in the appendix shows that all dual optimal functions $\mu^* : \Theta \to \mathbb{R}_{\geq 0}$ map to a range that is a subset of $[0, \omega/B_{\min}]$. Therefore, any pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$ that is a fixed point, i.e., satisfies $\mu(w, B) \in \operatorname{argmin}_{t \geq 0} q^{\mu}(w, B, t)$ almost surely w.r.t. $(w, B) \sim G$, must also satisfy range(μ) $\subset [0, \omega/B_{\min}]$. Hence, it suffices to restrict our attention to pacing functions of the form $\mu : \Theta \to [0, \omega/B_{\min}]$.

Consider the set of all potential pacing functions

$$\mathcal{X} = \{\mu \in L_1(\Theta) \mid \mu(w, B) \in [0, \omega/B_{\min}] \; \forall \; (w, B) \in \Theta\},$$

where $L_1(\Theta)$ is the space of functions $f : \Theta \to \mathbb{R}$ with finite L_1 norm w.r.t. the Lebesgue measure. Here, by L_1 norm of f w.r.t. the Lebesgue measure, we mean $\|f\|_{L_1} = \int_{\Theta} |f(\theta)| d\theta$. Our goal is to find a $\mu \in \mathcal{X}$ such that almost surely w.r.t $(w, B) \sim G$ we have

$$\mu(w, B) \in \operatorname{argmin}_{t \in [0, \omega/B_{\min}]} q^{\mu}(w, B, t).$$

Dealing with infinitely many individual optimization problems $\min_{t \in [0, \omega/B_{\min}]} q^{\mu}(w, B, t)$, one for each (w, B), makes the analysis hard. To remedy this issue, we combine these optimization problems by defining the objective $f : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, for all $\mu, \hat{\mu} \in \mathcal{X}$, as follows

$$f(\mu, \hat{\mu}) := \mathbb{E}_{(w,B)}[q^{\mu}(w, B, \hat{\mu}(w, B))].$$

For a fixed $\mu \in \mathcal{X}$, we then get a single optimization problem $\min_{\hat{\mu} \in \mathcal{X}} f(\mu, \hat{\mu})$ over functions in $\mathcal{X}$, instead of one optimization problem for each of the infinitely-many buyer types $(w, B) \in \Theta$. Later, in Lemma 15, we will show that any optimal solution to the combined optimization problem is also an optimal solution to the individual optimization problems almost surely w.r.t $(w, B) \sim G$. Thus, shifting our attention to the combined optimization problem is without any loss (because sub-optimality on zero-measure sets is tolerable).

With f as above, we proceed to define the correspondence that is used in our fixed-point argument. The *optimal solution correspondence* $C^* : \mathcal{X} \rightrightarrows \mathcal{X}$ is given by $C^*(\mu) := \arg\min_{\hat{\mu} \in \mathcal{X}} f(\mu, \hat{\mu})$ (which could be empty) for all $\mu \in \mathcal{X}$. In Lemma 15, we will show that the proof of Proposition 7 boils down to showing that C^* has a fixed point, which will be our next step.

Our proof will culminate with an application of the Kakutani-Glicksberg-Fan theorem, on a suitable version of C^*, to show the existence of a fixed point. An application of this result (or any other infinite dimensional fixed point theorem) requires intricate topological considerations. In particular, we need to endow $\mathcal{X}$ with a topology that satisfies the following conditions:

I. $\mathcal{X}$ is compact, convex and $C^*(\mu)$ is a non-empty subset of $\mathcal{X}$ for all $\mu \in \mathcal{X}$.

II. C^* is a Kakutani map, i.e., it is upper hemicontinuous, and $C^*(\mu)$ is compact and convex for all $\mu \in \mathcal{X}$.

In the case of infinite dimensions, bounded sets in many spaces, such as the $L_p(\Omega)$ spaces, are not compact. In particular, $\mathcal{X}$ is not compact as a subset of $L_p(\Omega)$ for any $1 \leq p \leq \infty$. One possible way around it would be to consider the weak* topology on $\mathcal{X} \subset L_\infty(\Omega)$, in which bounded sets are compact. This choice runs into trouble because it is difficult to show the upper hemicontinuity of C^* (property II) under the weak convergence notion of the weak* topology. Alternatively, one could impose structural properties and restrict to a subset of $\mathcal{X}$, such as the space of Lipschitz functions, in which both compactness and continuity can be established. The issue with this approach is that the correspondence operator may, in general, not preserve these properties, i.e., property I might not hold. For example, even if μ is Lipschitz, $C^*(\mu)$ might not contain any Lipschitz functions.

We would like to strike a delicate balance between properties I and II by picking a space in which we can establish compactness of $\mathcal{X}$ and upper hemicontinuity of C^*, while, at the same time, ensuring that $C^*(\mu)$ contains at least one element from this space. It turns out that the right space that works for our proof is the space of bounded variation. To motivate this topology on the space of pacing functions, we state some properties of the "smallest" dual optimal pacing function. For $\mu : \Theta \to [0, \omega/B_{\min}]$, we define $\ell^\mu : \Theta \to [0, \omega/B_{\min}]$ as

$$\ell^\mu(w, B) := \min \left\{ s \in \operatorname{argmin}_{t \in [0, \omega/B_{\min}]} q^\mu(w, B, t) \right\}$$

for all $(w, B) \in \Theta$. The minimum always exists because $q^\mu(w, B, t)$ is continuous as a function of t (see Corollary 3 in the appendix for a proof) and the feasible set of the dual problem is compact.

We first show that ℓ^μ varies nicely with w and B along individual components:

Lemma 12. *For $\mu : \Theta \to [0, \omega/B_{\min}]$, the following statements hold:*

1. *$\ell^\mu : \Theta \to [0, \omega/B_{\min}]$ is non-decreasing in each component of w.*
2. *$\ell^\mu : \Theta \to [0, \omega/B_{\min}]$ is non-increasing as a function of B.*

The proof applies results from comparative statics, which characterize the way the optimal solutions change with the parameters, to the family of optimization problems $\min_{t\in[0,\omega/B_{\min}]} q^\mu(w, B, t)$ parameterized by $(w, B) \in \Theta$.

Now we wish to show bounded variation of ℓ^μ. It is a well-known fact that monotonic functions of one variable have finite total variation. Moreover, functions of bounded total variation also form the dual space of the space of continuous functions with the L_∞ norm, which allows us to invoke the Banach-Alaoglu Theorem to establish compactness in the weak* topology. These results for single variable functions, although not directly applicable to the multivariable setting, act as a guide in choosing the appropriate topology for our setting.

Since pacing functions take as input several variables, we need to look at multivariable generalizations of total variation. To this end, we state one of the standard definitions (there are multiple equivalent ones) of total variation for functions of several variables (see section 5.1 of [EG15]) and then follow it up by a lemma which gives a bound on the total variation of the component-wise monotonic function ℓ^μ.

Definition 5. *For an open subset $\Omega \subset \mathbb{R}^n$, the total variation of a function $u \in L_1(\Omega)$ is given by*

$$V(u, \Omega) := \sup\left\{\int_\Omega u(\omega)\,\mathrm{div}\,\phi(\omega)d\omega \;\middle|\; \phi \in C_c^1(\Omega, \mathbb{R}^n), \|\phi\|_\infty \leq 1\right\}$$

where $C_c^1(\Omega, \mathbb{R}^n)$ is the space of continuously differentiable vector functions ϕ of compact support contained in Ω and $\mathrm{div}\,\phi = \sum_{i=1}^n \frac{\partial \phi_i}{\partial x_i}$ is the divergence of ϕ.

Lemma 13. *For any pacing function $\mu : \Theta \to [0, \omega/B_{\min}]$, the following statements hold:*

1. $\ell^\mu \in L_1(\Theta)$.

2. $V(\ell^\mu, \Theta) \le V_0$ *where* $V_0 := (d+1)U^{d+1}\omega/B_{\min}$ *is a fixed constant.*

Motivated by the above lemma, we define the set of pacing functions that will allow us to use our fixed-point argument. Define $\mathcal{X}_0 = \{\mu \in \mathcal{X} \mid V(\mu, \Theta) \le V_0\}$ to be the subset of pacing functions with variation at most V_0. Note that $\ell^\mu \in \mathcal{X}_0$. Define $C_0^* : \mathcal{X}_0 \rightrightarrows \mathcal{X}_0$ as $C_0^*(\mu) := \operatorname{argmin}_{\hat{\mu} \in \mathcal{X}_0} f(\mu, \hat{\mu})$ for all $\mu \in \mathcal{X}_0$. We now state the properties satisfied by $\mathcal{X}_0$ that make it compatible with the Kakutani-Fan-Glicksberg fixed-point theorem.

Lemma 14. *The following statements hold:*

1. $\mathcal{X}_0$ *is non-empty, compact and convex as a subset of* $L_1(\Theta)$.

2. $f : \mathcal{X}_0 \times \mathcal{X}_0 \to \mathbb{R}$ *is continuous when* $\mathcal{X}_0 \times \mathcal{X}_0$ *is endowed with the product topology.*

3. $C_0^* : \mathcal{X}_0 \rightrightarrows \mathcal{X}_0$ *is upper hemi-continuous with non-empty, convex and compact values.*

Finally, with the above lemma in place, we can apply the Kakutani-Fan-Glicksberg theorem to establish the existence of a $\mu \in \mathcal{X}_0$ such that $\mu \in C_0^*(\mathcal{X}_0)$. The following lemma completes the proof of Proposition 7 by showing that the fixed point is also almost surely optimal for each type. It follows from the fact that for $\mu \in \mathcal{X}_0$ that satisfy $\mu \in C_0^*(\mu)$, we have $\ell^\mu \in C_0^*(\mu)$.

Lemma 15. *If* $\mu \in C_0^*(\mu) = \operatorname{argmin}_{\hat{\mu} \in \mathcal{X}_0} f(\mu, \hat{\mu})$*, then* $\mu(w, B)$ *is almost surely optimal for each type, i.e.,* $\mu(w, B) \in \operatorname{argmin}_{t \in [0, \omega/B_{\min}]} q^\mu(w, B, t)$ *a.s. w.r.t.* $(w, B) \sim G$.

As mentioned earlier, Proposition 7, combined with Proposition 6, implies Theorem 8.

4.3 Standard Auctions and Revenue Equivalence

In this section, we move beyond first-price auctions and generalize our results to anonymous standard auctions with reserve prices. An auction $\mathcal{A} = (Q, M)$, with allocation rule $Q : \mathbb{R}^n_{\ge 0} \to [0, 1]^n$, payment rule $M : \mathbb{R}^n_{\ge 0} \to \mathbb{R}^n_{\ge 0}$ and reserve price r, is called an *anonymous standard auction* if the following conditions are satisfied:

- *Highest bidder wins.* When the buyers bid $(b_1, \ldots, b_n)$, the allocation received by buyer i is given by $Q_i(b_1, \ldots, b_n) = \mathbb{1}(b_i \geq r, b_i \geq b_j \; \forall j \in [n])$, for all $i \in [n]$.

- *Anonymity.* The payments made by a buyer do not depend on the identity of the buyer. More formally, if the buyers bid $(b_1, \ldots, b_n)$, then for any permutation π of $[n]$ and buyer $i \in [n]$, we have $M_i(b_1, \ldots, b_n) = M_{\pi(i)}(b_{\pi(1)}, \ldots, b_{\pi(n)})$, i.e., the payment made by the ith buyer before the bids are permuted equals the payment made by the bidder $\pi(i)$ after the bids have been permuted.

As in our definition of SFPE, we are using an infeasible tie-breaking rule which allocates the entire good to every highest bidder. As with SFPE, ties are a zero-probability event under our value-pacing-based equilibria, and our results hold for arbitrary tie-breaking rules.

For consistency of notation, we will modify the above notation slightly to better match the one used in previous sections. Exploiting the anonymity of auction $\mathcal{A}$, we will denote the payment made by a buyer who bids b, when the other $n-1$ buyers bid $\{b_i\}_{i=1}^{n-1}$, by $M\left(b, \{b_i\}_{i=1}^{n-1}\right)$, i.e., we use the first argument for the bid of the buyer under consideration and the other arguments for the competitors' bids. Also, as the reserve price completely determines the allocation rule of a standard auction, in the rest of the section, we will omit the allocation rule while discussing anonymous standard auctions and represent them as a tuple $\mathcal{A} = (r, M)$ of reserve price and payment rule.

To avoid delving into the inner workings of the auction, we assume the existence of an *oracle* that takes as an input an atomless distribution $\mathcal{H}$ over $[0, \omega]$ and outputs a bidding strategy $\psi^{\mathcal{H}} : [0, \omega] \to \mathbb{R}$ satisfying the following properties:

1. The strategy $\psi^{\mathcal{H}}$ is a single-auction equilibrium for the auction $\mathcal{A}$ when the values are drawn i.i.d. from $\mathcal{H}$, i.e.,

$$\psi^{\mathcal{H}}(x) \in \operatorname{argmax}_{b \geq 0} \mathbb{E}_{X_i \sim \mathcal{H}} \left[x \, \mathbb{1}\{b \geq \max(r, \{\psi^{\mathcal{H}}(X_i)\}_i)\} - M\left(b, \{\psi^{\mathcal{H}}(X_i)\}_i\right) \right] .$$

2. The strategy $\psi^{\mathcal{H}}(x)$ is non-decreasing in x, and $\psi^{\mathcal{H}}(x) \geq r$ if and only if $x \geq r$.

3. The payoff for a bidder who has zero value for the object is zero at the single-auction equilibrium.

4. The distribution of $\psi^{\mathcal{H}}(x)$, when $x \sim \mathcal{H}$, is atomless.

Our results will produce a pacing-based equilibrium bidding strategy for budget-constrained buyers by invoking $\psi^{\mathcal{H}}$ as a black box. To make the discussion more concrete, let $\mathcal{A}$ to be a second-price auction with reserve price r. For a given atomless distribution $\mathcal{H}$, define $\psi^{\mathcal{H}}(v) = v$ to be the truthful bidding strategy. Then, $\psi^{\mathcal{H}}$ is a single-auction equilibrium because bidding truthfully is a dominant strategy in second-price auctions. Moreover, $\psi^{\mathcal{H}}$ is non-decreasing, $\psi^{\mathcal{H}}(x) \geq r$ if and only if $x \geq r$, a bidder with zero value bids zero to attain a payoff of zero, and finally the distribution of $\psi^{\mathcal{H}}(x)$ when $x \sim \mathcal{H}$ is simply $\mathcal{H}$, which is atomless. Thus, second-price auctions with reserve prices satisfy the above assumptions.

In our analysis, we allow the seller to condition on the feature vector and choose a different mechanism for each context $\alpha \in A$. Let $\{\mathcal{A}_\alpha = (r(\alpha), M_\alpha)\}_{\alpha \in A}$ be a family of anonymous standard auctions such that $\alpha \mapsto r(\alpha)$ is measurable. Moreover, suppose that for any measurable bidding function $\alpha \mapsto b(\alpha)$ and any collection of measurable competing bidding functions $\alpha \mapsto b_i(\alpha)$ for $i \in [n-1]$, the payment function $\alpha \mapsto M_\alpha\left(b(\alpha), \{b_i(\alpha)\}_{i=1}^{n-1}\right)$ is also measurable. Below, we define the equilibrium notion for the family $\{\mathcal{A}_\alpha\}_{\alpha \in A}$ of anonymous standard auctions.

Definition 6. *A strategy $\beta^* : \Theta \times A \to \mathbb{R}$ is called a* Symmetric Equilibrium *for the family of standard auctions $\{\mathcal{A}_\alpha\}_{\alpha \in A}$, if $\beta^*(w, B, \alpha)$ (as a function of α) is an optimal solution to the following optimization problem almost surely w.r.t. $(w, B) \sim G$.*

$$\begin{aligned}
\max_{b:A\to\mathbb{R}_{\geq 0}} \quad & \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}}\left[w^T\alpha\,\mathbb{1}\{b(\alpha) \geq \max(r(\alpha), \{\beta^*(\theta_i, \alpha)\}_i)\} - M_\alpha\left(b(\alpha), \{\beta^*(w_i, B_i, \alpha)\}_i\right)\right] \\
s.t. \quad & \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}}\left[M_\alpha\left(b(\alpha), \{\beta^*(w_i, B_i, \alpha)\}_i\right)\right] \leq B\,.
\end{aligned}$$

Observe that the above definition reduces to Definition 3 if we take $\{\mathcal{A}_\alpha\}_{\alpha \in A}$ to be the set of first-price auctions with reserve price $r(\alpha)$. Next, we show that the equilibrium existence and characterization results of the previous sections apply to all standard auctions that satisfy the required assumptions. To do this, we first need to define value-pacing strategies for anonymous standard auctions. These are a natural generalization of the value-pacing-based strategies used for first-price auctions.

Recall that, for a pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$ and $\alpha \in A$, λ_α^μ denotes the distribution of paced values for item α, and H_α^μ denotes the distribution of the highest value for α, among $n-1$ buyers. For ease of notation, we will use ψ_α^μ to denote the single-auction equilibrium strategy for auction $\mathcal{A}_\alpha$ when values are drawn from $\mathcal{H} = \lambda_\alpha^\mu$ or more formally $\psi_\alpha^\mu := \psi_\alpha^{\lambda_\alpha^\mu}$. For a pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$, $(w, B) \in \Theta$ and $\alpha \in A$, define

$$\Psi^\mu(w, B, \alpha) := \psi_\alpha^\mu \left(\frac{w^T \alpha}{1 + \mu(w, B)} \right), \tag{4.1}$$

to be our candidate equilibrium strategy. This strategy is well-defined because, by Lemma 34, λ_α^μ is atom-less almost surely w.r.t. α. As before, the bid $\Psi^\mu(w, B, \alpha)$ is the amount a non-budget-constrained buyer with type (w, B) would bid on item α if her paced value was her true value, when competitors are pacing their values accordingly. In other words, bidders in the proposed equilibrium first pace their values, and then bid according to the single-auction equilibrium of auction $\mathcal{A}_\alpha$ in which competitors' values are also paced.

With the definition of value-pacing-based strategies in place, we can now state the main result of this section. Recall that, $C_0^* : \mathcal{X}_0 \rightrightarrows \mathcal{X}_0$ is given by $C_0^*(\mu) := \arg\min_{\hat{\mu} \in \mathcal{X}_0} f(\mu, \hat{\mu})$ for all $\mu \in \mathcal{X}_0$, where f is the expected dual function in the case of a first-price auction, as defined in Section 4.2.3.

Theorem 9 (Revenue and Pacing Equivalence)**.** *For any pacing function $\mu \in \mathcal{X}_0$ such that $\mu \in C_0^*(\mu)$ is an equilibrium pacing function for first-price auctions, the value-pacing-based strategy $\Psi^\mu : \Theta \times A \to \mathbb{R}_{\geq 0}$ is a Symmetric Equilibrium for the family of auctions $\{\mathcal{A}_\alpha\}_{\alpha \in A}$. Moreover, the expected payment made by buyer θ under this equilibrium strategy is equal to the expected payment made by buyer θ in first-price auctions under the equilibrium strategy $\beta^\mu : \Theta \times A \to \mathbb{R}_{\geq 0}$,* *i.e.,*

$$\begin{aligned}&\mathbb{E}_{\alpha,\{(\theta_i)\}_{i=1}^{n-1}}\left[M_\alpha\left(\Psi^\mu(\theta,\alpha),\{\Psi^\mu(\theta_i,\alpha)\}_i\right)\right]\\=&\mathbb{E}_{\alpha,\{(\theta_i)\}_{i=1}^{n-1}}\left[\beta^\mu(\theta,\alpha)\ \mathbb{1}\{\beta^\mu(\theta,\alpha)\geq\max(r(\alpha),\{\beta^\mu(\theta_i,\alpha)\}_i)\}\right]\end{aligned}$$

The key step in the proof involves showing that the dual of the budget-constrained utility-optimization problem faced by a buyer is identical for all standard auctions, when the other buyers use the equilibrium strategy Ψ^μ of the standard auction under consideration. To establish this key step, we exploit the separable structure of the Lagrangian optimization problem and apply the known utility equivalence result for standard auctions in the single-auction i.i.d. setting, once for each item $\alpha \in A$. Then, we establish the analogue of Proposition 6 for standard auctions. Combining this with $\mu \in C_0^*(\mu)$ yields Theorem 9.

Our revenue equivalence relies on three critical assumptions: risk-neutrality, independence of weight vectors, and symmetry. As in the classical setting, revenue equivalence would fail if buyers are risk averse (see, e.g., [Kri09]). We emphasize that, in contrast to the classical revenue equivalence result, buyers' values $w^T\alpha$ are not independent. Our result does require that weight vectors are independent across buyers. Buyers in our model are ex-ante homogeneous since buyer types are drawn from the same population. We remark, however, that buyers are heterogenous in the interim sense: the buyers competing in an auction can have different budgets and weight vectors. Revenue equivalence would fail is buyers are ex-ante heterogenous, i.e., if competitors are drawn from different populations.

Before ending this section, we state some important implications of Theorem 9. If the pacing function μ allows the buyers to satisfy their budget constraints in some standard auction, then the same pacing function μ allows the buyers to satisfy their budgets in every other standard auction. In other words, the equilibrium pacing functions are the same for all standard auctions. This means that in order to calculate an equilibrium pacing function μ that satisfies $\mu \in C_0^*(\mu)$, it suffices to compute it for any standard auction (in particular, one could consider a second-price auction for which bidding truthfully is a dominant-strategy equilibrium in the absence of budget constraints). This fact is especially pertinent in view of the recent shift in auction format used for selling display ads from second-price auctions to first-price auctions, because it states that, in equilibrium, the buyers can use the same pacing function even after the change. Moreover, the same pacing function continues to work even if the family $\{\mathcal{A}_\alpha\}_{\alpha \in A}$ is an arbitrary collection of first-price and second-price auctions (or any other combination of standard auctions), i.e., Theorem 9 states that, not only can one pacing function be used to manage budgets in first-price and second-price auctions, the same pacing function also works in the intermediate transitions stages, in which buyers may potentially participate in some mixture of these auctions.

Another important takeaway is that all standard auctions with the same allocation rule yield the same revenue to the seller. We remark, however, that the revenue of the seller does depend on the allocation, and the seller could thus maximize her revenue by optimizing over the reserve prices. We leave the question of optimizing the auction design as a future research direction.

The revenue-equivalence in the presence of in-expectation budget constraints is driven by the invariance of the pacing function over all standard auctions and the classical revenue equivalence result for the unconstrained i.i.d. setting, which shows that—on average—payments are the same across standard auctions. While revenue equivalence is known to hold for standard auctions without budget constraints, [CG98] showed that, when budget constraints are hard, first-price auctions lead to higher revenue than second-price auctions. The intuition for their result is that because bids are higher in second-price auctions than first-price auctions, hard budget constraints are more likely to bind in the former, which reduces the seller's revenue. Surprisingly, Theorem 9 shows that when budgets constraints are in expectation (and values are feature-based), we recover revenue equivalence. To better understand the difference between the two types of constraints, consider the following example:

Example 2. *Consider two buyers with values drawn uniformly from the unit interval* $[0, 1]$. *Moreover, let the budget of the buyer with value* v *be given by* $0.5 + \epsilon v$ *for some small* $\epsilon > 0$. *First, observe that, in the absence of budget constraints, bidding truthfully is a dominant strategy in a* *second-price auction and bidding half of one's value is a Bayes-Nash equilibrium in a first-price auction. Moreover, from the standard revenue-equivalence result, a buyer with value x spends* $x^2/2$ *in expectation over the other buyer's type in both auctions. Now, since this expected expenditure is less than* $1/2$ *for all types, the in-expectation budget constraints are non-binding and the equilibria remain unchanged even when in-expectation budget constraints are imposed. On the other hand, consider the case when the budget constraints are hard. The first-price auction equilibrium remains unchanged because every buyer type bids less than* 0.5*, so the constraint is always satisfied. But, for second-price auction, this is not the case: With hard budget constraints, the equilibrium strategy for the buyers is to bid the minimum of their value and budget, thereby leading to lower revenue compared to the truthful-bidding equilibrium.*

We conclude this section with a discussion of extensions and alternative models. Firstly, even though we only consider anonymous standard auctions in this work, our equilibrium existence and revenue equivalence results can be extended to other anonymous allocation rules Q which (i) admit an oracle that outputs an equilibrium bidding strategy for traditional i.i.d. setting and satisfies properties (1)-(4) listed at the beginning of this section, (ii) lead to continuous non-decreasing interim-allocation rules for every buyer-item pair when other buyers follow a value-pacing-based strategy analogous to the one defined in equation (4.1). Secondly, the argument developed in the section also implies the existence of value-pacing-based equilibria and revenue equivalence for standard auctions in the symmetric special case of the models studied in [BBW15] and [Bal+21], which consider buyers with ex-ante budget constraints that hold in expectation over a buyer's own value and the values of others (see Appendix C.3.1 for a detailed description).

4.4 Worst-Case Efficiency Guarantees

In this section, we use our framework to characterize the Price of Anarchy, i.e., the worst-case ratio of the efficiency of a pacing equilibrium relative to the efficiency of the best possible allocation. We measure efficiency of an allocation using the notion of *liquid welfare* introduced by [DL14], which captures the maximum revenue that can be extracted by a seller who knows the values in advance. We use liquid welfare as a measure of efficiency instead of social welfare because the latter can have arbitrarily small Price of Anarchy (see Appendix C.4 for an example). Throughout this section, we assume that the reserve price is zero for each item, i.e., $r(\alpha) = 0$ for all $\alpha \in A$.

We begin by defining the appropriate notion of liquid welfare of an allocation for our model motivated by the original definition of [DL14]. Here, an allocation simply refers to a measurable function $x : A \times \Theta^n \to \Delta^n$, where $\Delta^n = \{y \in \mathbb{R}^n_+ \mid \sum_{k=1}^n y_k = 1\}$ is the n-simplex, and $x_i(\alpha, \vec{\theta})$ denotes the fraction of the item α allocated to buyer i when the buyer types are given by the profile $\vec{\theta} = (\theta_1, \dots, \theta_n)$. In our setting, the liquid welfare of a buyer is equal to the minimum of the value obtained by the buyer from the allocation and her budget.

Definition 7. *For an allocation $x : A \times \Theta^n \to \Delta^n$, we define its liquid welfare as*

$$\mathrm{LW}(x) = \sum_{i=1}^n \mathbb{E}_{\theta_i} \left[\min \left\{ \mathbb{E}_{\alpha, \theta_{-i}} [w_i^T \alpha \cdot x_i(\alpha, \theta_i, \theta_{-i})], B_i \right\} \right] .$$

Next, we define Price of Anarchy with respect to liquid welfare for pacing-based equilibria. Our definition is an instantiation of the general definition of Price of Anarchy introduced in [KP99]. Before proceeding with the definition, it is worth noting an important consequence of our revenue equivalence result (Theorem 9): Given an equilibrium pacing function μ, i.e., a fixed point of C_0^*, the allocation under the equilibrium parameterized by μ is the same for all standard auctions. Thus, the equilibrium allocation is determined by the pacing function and is independent of the pricing rule of the standard auction, which is reflected in the following definition. For an equilibrium pacing function μ, we use x^μ to denote the allocation under the equilibrium parameterized by μ; again, this allocation is the same for all standard auctions without reserve prices.

Definition 8. *The Price of Anarchy (PoA) of pacing-based equilibria (for all standard auctions) is defined as the ratio of the worst-case liquid welfare across all pacing equilibria, and the optimal* *liquid welfare*

$$\text{PoA} = \frac{\inf_{\mu:\mu\in C_0^*(\mu)} \text{LW}(x^{\mu})}{\sup_x \text{LW}(x)}$$

where the supremum in the denominator is taken over all measurable allocations x.

Since the PoA of pacing-based equilibria does not depend on the payment rule, we can work with the most convenient standard auction to prove a lower bound on the PoA, which in this case happens to be the second-price auction. [Aza+17] study the PoA of pure-strategy Nash equilibria of second-price auctions in a non-Bayesian multi-item setting with budgets, and provide a lower bound of 1/2 for it. Unfortunately, their result hinges on the "no over-budgeting" assumption that requires the sum of equilibrium bids to be bounded above by the budget, which need not hold for pacing-based equilibria, thereby necessitating new proof ideas. Moreover, their bound may be vacuous for some parameter values because a pure-strategy Nash equilibrium is not guaranteed to exist in their setting. To get around this, they study mixed-strategy and Bayes-Nash equilibria, and bound their PoA, but the lower bound they obtain for these equilibria is much worse (less than 0.02). Our model does not suffer from the problem of existence: a pure-strategy pacing-based equilibrium is always guaranteed to exist (Theorem 8). This makes the following lower bound on the PoA, which provides a worst-case guarantee of $1/2$, more appealing.

Theorem 10. *The PoA of pacing-based equilibria of any standard auction is greater or equal to* $1/2$*.*

The proof, which is in Appendix C.4, leverages the complementary slackness condition of pacing-based equilibria to bound the PoA. Interestingly, our proof does not use a hypothetical deviation to another bidding strategy, a technique commonly found in PoA bounds (see [RST17] for a survey); and thus may be of independent interest.

4.5 Structural Properties

In this section, we will show that pacing-based equilibria satisfy certain monotonicity and geometric properties related to the space of value vectors. It is worth noting that, in light of the revenue equivalence result of the preceding section, the properties established in this section hold for pacing equilibria of *all standard auctions*. As in Section 4.4, we will assume that the reserve price for each item is zero, i.e., $r(\alpha) = 0$ for all $\alpha \in A$. Without this assumption, similar results hold, but they become less intuitively appealing and harder to state. Moreover, we will also assume that the support of G, denoted by $\delta(G)$, is a convex compact subset of $\mathbb{R}_+^{d+1}$. This assumption is made to avoid having to specify conditions on the pacing multipliers of types with probability zero of occurring. Moreover, we consider a pacing function $\mu : \Theta \to [0, \omega/B_{\min}]$ such that $\mu(w, B)$ is the unique optimal solution for the dual minimization problem for each (w, B) in the support of G, i.e., $\mu(w, B) = \operatorname{argmin}_{t \in [0,\omega/B_{\min}]} q^{\mu}(w, B, t)$ for all $(w, B) \in \delta(G)$. We remark that we are assuming that the best response is unique rather than the equilibrium being unique. The former can be shown to hold under fairly general conditions.

First, in Lemma 12 we showed that the pacing function associated with an SFPE is monotone in the buyer type. In particular, when the best response is unique, this result implies that $\mu(w, B)$ is non-decreasing in each component of the weight vector w and non-increasing in the budget B. Intuitively, if the budget decreases, a buyer needs to shade bids more aggressively to meet her constraints. Alternatively, when the weight vector increases, the advertiser's paced values increase, which would result in more auctions won and higher payments. Therefore, to meet her constraints the advertiser would need to respond by shading bids more aggressively. Furthermore, when the best response is unique, it can also be shown that μ is continuous (see Lemma 42 in the appendix).

The next theorem further elucidates the structure imposed on μ by virtue of it corresponding to the optima of the family of dual optimization problems parameterized by (w, B). In what follows, we will refer to a buyer (w, B) with $\mu(w, B) = 0$ as an *unpaced buyer*, and call her a *paced buyer* otherwise.

Proposition 8. *Consider a unit vector $\hat{w} \in \mathbb{R}^d_+$ and budget $B > 0$ such that $w/\|w\| = \hat{w}$, for some $(w, B) \in \delta(G)$. Then, the following statements hold,*

1. *Paced buyers with budget B and weight vectors lying along the same unit vector $\hat{w}$ have identical paced feature vectors in equilibrium. Specifically, if $(w_1, B), (w_2, B) \in \delta(G)$, with $w_1/\|w_1\| = w_2/\|w_2\| = \hat{w}$ and $\mu(w_1, B), \mu(w_2, B) > 0$, then $w_1/(1 + \mu(w_1, B)) = w_2/(1 + \mu(w_2, B))$.*

2. *Suppose there exists an unpaced buyer $(w, B) \in \delta(G)$ with $w/\|w\| = \hat{w}$ and $\mu(w, B) = 0$. Let $w_0 = \operatorname{argmax}\{\|w\| \mid w \in \mathbb{R}^d;\ \mu(w, B) = 0$ and $w/\|w\| = \hat{w}\}$ be the largest unpaced weight vector along the direction $\hat{w}$. Then, all paced weight vectors get paced down to w_0, i.e., $w/(1 + \mu(w, B)) = w_0$ for all $w \in \delta(G)$ with $w/\|w\| = \hat{w}$ and $\mu(w, B) > 0$.*

In combination with complementary slackness, the first part states that, in equilibrium, buyers who have the same budget, have positive pacing multipliers, and have feature vectors which are scalar multiples of each other, get paced down to the same type at which they exactly spend their budget. In other words, scaling up the feature vector of a budget-constrained buyer, while keeping her budget the same, does not affect the equilibrium outcome. The second case of Proposition 8 addresses the directions of buyers that have a mixture of paced and unpaced buyers. In this case, there is a critical buyer type who exactly spends her budget when unpaced, and all buyer types that have weight vectors with larger norm (but the same budget) get paced down to this critical buyer type, i.e., their paced weight vector equals the critical buyer type's weight vector in equilibrium. The buyer types which have a smaller norm are unpaced.

Our non-atomic model also allows us to answer the following question: Keeping the competition fixed, how should an advertiser modify her targeting criteria or ad (as captured by the weight vector) in order to maximize her utility? This result is especially important for online display ad auctions, where the weight vector is estimated with the goal of predicting the click-through-rate (CTR) and advertisers routinely modify their ads to attract more clicks. The following theorem states that the gradient w.r.t. the weight vector of the equilibrium utility of a buyer with type (w, B) is given by the expected feature vector that she wins in equilibrium. This is because strong duality (Proposition 6) implies that the utility of every buyer type is given by the optimal dual value $q^{\mu}(w, B, \mu(w, B))$. From a practical perspective, an advertiser should focus on improving the weights of those features which have the largest average among the contexts won. It is worth noting that these quantities can be easily computed using data available to an advertiser.

Proposition 9. *Assume that A is compact. Let $\mu : \Theta \to \mathbb{R}_{\geq 0}$ be an equilibrium pacing function, i.e., $\mu : \Theta \to \mathbb{R}_{\geq 0}$ such that $\mu(w, B) \in \operatorname{argmin}_{t \geq 0} q^{\mu}(w, B, t)$ almost surely w.r.t. $(w, B) \sim G$. Then, for all $(w, B) \in \Theta$, we have $\nabla_w q^{\mu}(w, B, \mu(w, B)) = \mathbb{E}_{\alpha, \{\theta_i\}_{i=1}^{n-1}} \left[\alpha \, \mathbb{1} \left\{\beta^{\mu}((w, B), \alpha) \geq \beta^{\mu}(\theta_i, \alpha)_i \ \forall i\right\}\right]$.*

4.6 Analytical Example and Numerical Experiments

In this section, we illustrate our theory by providing a stylized example in which we can determine the equilibrium bidding strategies in closed form, and then conduct some numerical experiments to verify our theoretical results. The purpose of the analytical example is to confirm our structural results and also help validate that our numerical procedures converge to an approximate version of the equilibrium strategies proposed in this chapter.

4.6.1 Analytical Example

We provide an instructive (albeit stylized) example with two-dimensional feature vectors to illustrate the structural property described in Section 4.5. For $1 \leq a < b$, define the set of buyer types as (see the blue region in Figure 4.1 for a visualization of this set)

$$\Theta := \left\{ (w, B) \in \mathbb{R}^2_{\geq 0} \times \mathbb{R}_+ \middle| a \leq \|w\| \leq b, \ B = \frac{2\|w\| - w_1 - w_2}{\pi \|w\|} \right\}.$$

In this example weight vectors lie in the intersection of a disk with the non-negative quadrant. Observe that all buyer types whose weight vectors are co-linear (i.e., they lie along the same unit vector) have identical budgets. Let the number of buyers in the auction be $n = 2$. Moreover, define the set of item types as the two standard basis vectors $A := \{e_1, e_2\}$. Finally, let G (distribution over buyer types) and F (distribution over item types) be the uniform distribution on Θ and A

respectively. Since A is discrete and F does not have a density, this example does not satisfy the assumptions we made in our model. Nonetheless, in the next claim, we show that not only does a pacing equilibrium exists, but we can also state it in closed form. The proof of the claim can be found in Appendix C.6.

Claim 1. *The pacing functions $\mu : \Theta \to \mathbb{R}$ defined as $\mu(w, B) = \|w\|-1$, for all $(w, B) \in \Theta$, is an equilibrium, i.e., β^{μ}, as given in Definition 3, is a SFPE.*

Since $H^{\mu}_{\alpha}(\cdot)$ is a strictly increasing function for all $\alpha \in A$, it is easy to check that $\mu(w, B)$ is the unique optimal to the dual optimization problem $\min_{t\in[0,\omega/B_{\min}]} q(\mu, w, B, t)$ for all $(w, B) \in \delta(G)$. Therefore, this example falls under the purview of part 1 of Proposition 8. As expected, conforming to Proposition 8, the buyers whose weight vectors are co-linear get paced down to the same point on the unit arc, as shown in Figure 4.1.

4.6.2 Numerical Experiments

We now describe the simulation-based experiments we conducted to verify our theoretical results. As is necessitated by computer simulations, we studied a discretized version of our problem in these experiments. More precisely, in our experiments, we used discrete approximations to the buyer type distribution G and the item type distribution F. Moreover, for all item types α, we set the reserve price $r(\alpha) = 0$. One of the primary objectives of our simulations is to demonstrate that, despite the discretization, a buyer type can obtain her optimal bidding strategy by finding the optimal solution to the dual problem, as our theory suggests. In other words, to compute an equilibrium it suffices to best-respond in the dual space which has the advantage of being much simpler than the primal space. To do so, for each discretized instance, we run best-response dynamics in the dual space by iterating over buyer types; computing each buyer type's optimal dual solution while keeping everyone else's pacing-based strategy fixed and then using this optimal dual solution to determine her pacing-based bidding strategy. This approach is not guaranteed to converge. In fact, due to the discretization, strong duality may fail to hold and a pure strategy equilibrium may not even exist. Nevertheless, despite the lack of theoretical guarantees, our experiments demonstrate that our analytical results and the dual best-response algorithm they inspire continue to work well in discrete settings.

As a first step, and to validate our best-response dynamics, we ran the algorithm on the discrete approximation of the example discussed in Subsection 4.6.1, for which we had already analytically determined a pacing equilibrium in Claim 1. The problem was discretized by picking 320 points lying in the set of buyer types Θ defined in Subsection 4.6.1. In Figure 4.1, we provide plots for the case when $a = 2, b = 3$. We see that the theoretical predictions from Claim 1 are replicated almost exactly by the solution computed by best-response iteration on the discretized problem. Moreover, co-linear buyer types converge to the same paced type vector, thereby validating Proposition 8.

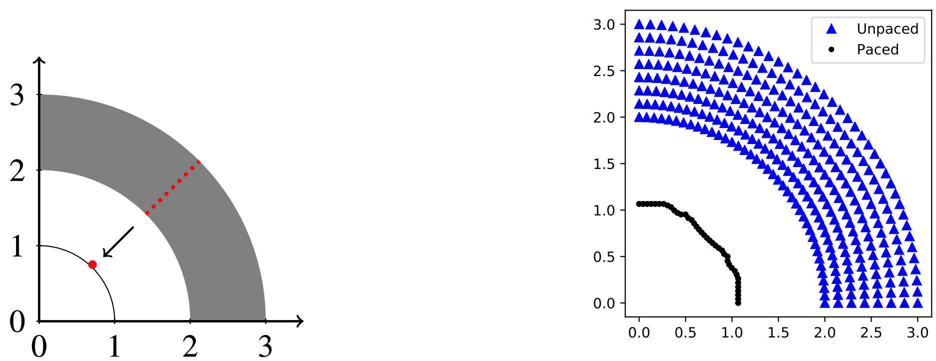

Figure 4.1: The example from Section 4.6.1 with $a = 2, b = 3$. The unpaced and paced buyer weight vectors are uniformly distributed in the gray (triangle) and black (circle) region, respectively. Each plot shows the distribution of two-dimensional buyer weight vectors. The weight vectors before pacing are depicted in gray (triangles) and the paced weight vectors are depicted in black (circles). The left plot shows the theoretical results of Subsection 4.6.1. In the left plot, the buyer weight vectors lying on the dotted line get paced down to the point. The right plot shows the results of best-response iteration on the corresponding discretized problem.

We conducted experiments to verify the structural properties described in Proposition 8. Here we consider instances with $n = 3$ buyers per auction, $d = 2$ features, the buyer type distribution G given by the uniform distribution on $(1, 2) \times (1, 2) \times \{0.6\}$ and the item type distribution F given by the uniform distribution on the one-dimensional simplex $\{(x, y) \mid x, y \geq 0;\ x + y = 1\}$. These were discretized taking a uniform grid with 10 points along each dimension. The results are portrayed in Figure 4.2.

The structural properties discussed in Proposition 8 are clearly evident in Figure 4.2. In this

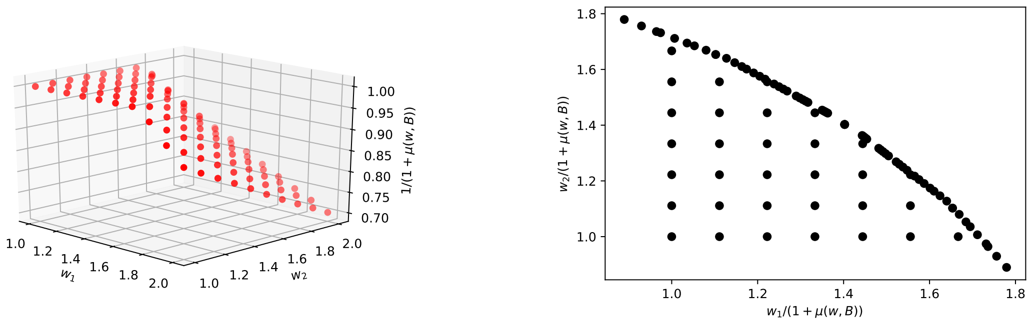

Figure 4.2: The left plot depicts how the multiplicative shading factor $1/(1 + \mu(w, B))$ varies with buyer weight vector w (budget $B = 0.6$ is the same for every buyer type). On the right, we plot the paced weight vectors of the buyer types.

scenario, the buyer types are uniformly distributed on $(1, 2) \times (1, 2) \times \{0.6\}$ and, as a consequence, all buyers have identical budgets equal to 0.6. At equilibrium, it can be seen that the co-linear buyer types (i.e., buyers whose weight vectors w are co-linear) who have a positive multiplier get paced down to the critical buyer type who exactly spends her budget. Moreover, at equilibrium, the boundary that separates the paced buyer types from the unpaced buyer types—the curve in which the critical buyer types lie—can be clearly observed in the left-hand plot in Figure 4.2. Finally, we constructed random discrete instances by uniformly sampling 50 buyer weight vectors and 20 item feature vectors from the square $(1, 2) \times (1, 2)$, and setting the number of buyers to be $N = 3$ and the budget of all buyer types to be $B = 2$. We found that our dual-based dynamics always converged within 250 iterations to pacing-based bidding strategies which on average were within 2.5% of the utility-maximizing budget feasible bidding strategy.

4.7 Conclusion

This chapter introduces a natural contextual valuation model and characterizes the equilibrium bidding behavior of budget-constrained buyers in first-price auctions in this model. We extend this result to other standard auctions and establish revenue equivalence among them. Due to the extensive focus on second-price auctions, previous works endorse bid-pacing as the framework of choice for budget management in the presence of strategic buyers. Our results suggest that value-pacing, which coincides with bid-pacing in second-price auctions, is an appropriate framework to manage budgets across all standard auctions.

An important open question we leave unanswered is that of optimizing the reserve prices to maximize seller revenue under equilibrium bidding. In general, optimizing under equilibrium constraints is usually challenging, so it is interesting to explore whether our model possesses additional structure that allows for tractability. Another related question is that of characterizing the revenue-optimal mechanism for our model. Our contextual-value model can capture multi-item auctions with additive valuations as a special case (by interpreting each context as a different item), which is a notoriously hard setting for revenue maximization, even in the absence of budget constraints. Investigating dynamics in first-price auctions with strategic budget-constrained buyers is another interesting open direction worth exploring. We also leave open the question of efficient computation of the pacing-based equilibria discussed in this chapter. Addressing this question will likely require choosing a suitable method of discretization and tie-breaking, without which equilibrium existence may not be guaranteed (see, e.g., [Con+18; Bab+20]). Finally, another interesting research direction is to develop conditions that guarantee uniqueness of an equilibrium. In light of recent results by [Con+18], we conjecture that, without further assumptions, the equilibrium would generally not be unique.

Chapter 5: Complexity of Pacing for Second-Price Auctions

Based on the publication [CKK23] co-authored with Xi Chen and Christian Kroer.

In this chapter, we analyze equilibria of pacing-based budget management systems from a computational perspective. In Section 5.2, we first show that the problem of finding an *approximate* pacing equilibrium is PPAD-hard. Our notion of approximation relaxes the definition of (exact) pacing equilibria in two ways: (i) buyers who bid *close* to (but not necessarily exactly equal to) the highest bid may also win fractions of an item; (ii) each buyer either spends *most* of her budget, or her pacing multiplier is *close* to one. We use two parameters δ and γ to capture these two relaxations quantitatively and such a solution is called a (δ, γ)-*approximate pacing equilibrium* (see Definition 10).

Theorem 11. *For any constant $c > 0$, finding a (δ, γ)-approximate pacing equilibrium in a second-price pacing game with n players is PPAD-hard when $\delta = \gamma = 1/n^c$.*

Next, in Section 5.3, we first prove that finding a pacing equilibrium is in PPAD. In particular, this implies that, when values and budgets of buyers are rational in the game, there always exists a pacing equilibrium in which every entry is rational and can be written using polynomially many bits. (In contrast, the existence proof of [Con+18] uses a convergence argument, from which it is not clear whether an equilibrium with rational entries always exists.)

Theorem 12. *Finding a pacing equilibrium in a second-price pacing game is in PPAD.*

Note that, by virtue of being a relaxation, finding an approximate pacing equilibrium is in PPAD as a direct consequence of Theorem 12. Similarly, the PPAD-hardness of finding an exact pacing equilibrium follows from Theorem 11.

5.1 Model

We start with the definition of Second-price Pacing Games. In a *Second-price Pacing Game* (SPP game as a shorthand) $G = (n, m, (v_{ij}), (B_i))$, there are n buyers and m (indivisible) goods. Each good is sold through independent (single slot) second-price auctions. We use $v_{ij} \geq 0$, $i \in [n]$ and $j \in [m]$, to denote the value of good j to buyer i, and $B_i > 0$ to denote the budget of buyer i. We will require (1) for each $j \in [m]$, $v_{ij} > 0$ for some $i \in [n]$, and (2) for each $i \in [n]$, $v_{ij} > 0$ for some $j \in [m]$. Each buyer i plays the game by picking a *pacing multiplier* $\alpha_i \in [0, 1]$ and then bidding $\alpha_i v_{ij}$ on good j for each $j \in [m]$.

To finish describing the game, one approach is to specify a tie-breaking rule: a rule that determines the probabilities with which a good is allocated among the highest bidders. However, [Con+18] showed that the choice of tie-breaking rule affects equilibrium existence. This motivated them to introduce an equilibrium notion called the pacing equilibrium, which is not concerned with any specific tie-breaking rule, but instead includes the probability distribution used to allocate each good as part of the equilibrium (see Definition 9). We will take a similar approach and work with pacing equilibrium, focusing on its computational aspects. It is worth pointing out that this only makes our hardness results stronger because they apply to any tie-breaking rule (such as the one used by [Bor+07], which works via random perturbations; see Section 5.2.3 for a detailed discussion of the implications of our hardness results).

With slight abuse of notation, we will write $x_{ij} \geq 0$ to denote the *fraction* of good j allocated to buyer i, which, in our indivisible goods regime, should be interpreted to mean the probability of allocating good j to buyer i. Therefore, the allocation should always satisfy $\sum_{i \in [n]} x_{ij} \leq 1$ for all $j \in [m]$. In addition, only buyers i with the *highest* bid for good j can have $x_{ij} > 0$ and they pay for good j under the *second-price* rule.

Formally, when buyers use pacing multipliers $\alpha = (\alpha_1, \ldots, \alpha_n)$, we let $h_j(\alpha) = \max_{i \in [n]} \alpha_i v_{ij}$ denote the *highest* bid on good j and $p_j(\alpha)$ denote the *second highest bid* on good j, i.e., $p_j(\alpha)$ is the second largest element among $\alpha_1 v_{1j}, \ldots, \alpha_n v_{nj}$ (in particular, $p_j(\alpha) = h_j(\alpha)$ when there is a tie for the highest bid). Only buyers who bid $h_j(\alpha)$ can purchase (fractions of) good j under the price $p_j(\alpha)$. Thus, under an allocation $x = (x_{ij})$, the total payment of buyer i is given by $\sum_{j\in[m]} x_{ij} p_j(\alpha)$, which should not exceed the budget B_i of buyer i.

Next, we define the notion of *pacing equilibria* [Con+18] of SPP games. A pacing equilibrium consists of a tuple of pacing multipliers $\alpha = (\alpha_i)$ and an allocation $x = (x_{ij})$ of goods that satisfy the two conditions described above (i.e., only buyers with the highest bid can be allocated a good and their budgets are satisfied, as captured in (a) and (c) below). In addition, we require (b) the full allocation of any good with a positive bid and (d) that there is no unnecessary pacing: if a buyer i does not spend her whole budget, then her pacing multiplier should be one. Intuitively, this makes sense because if her budget is not binding, then she should participate as if each auction is a regular second-price auction.

Definition 9 (Pacing Equilibria)**.** *Given an SPP game* $G = (n, m, (v_{ij}), (B_i))$*, we say* (α, x) *with* $\alpha = (\alpha_i) \in [0,1]^n$*,* $x = (x_{ij}) \in [0,1]^{nm}$ *and* $\sum_{i\in[n]} x_{ij} \le 1$ *for all* $j \in [m]$ *is a* pacing equilibrium *if*

(a) Only buyers with the highest bid win the good: $x_{ij} > 0$ *implies* $\alpha_i v_{ij} = h_j(\alpha)$*.*

(b) Full allocation of each good with a positive bid: $h_j(\alpha) > 0$ *implies* $\sum_{i\in[n]} x_{ij} = 1$*.*

(c) Budgets are satisfied: $\sum_{j\in[m]} x_{ij} p_j(\alpha) \le B_i$*.*

(d) No unnecessary pacing: $\sum_{j\in[m]} x_{ij} p_j(\alpha) < B_i$ *implies* $\alpha_i = 1$*.*

We will work with an *approximate* version of pacing equilibria in both of our PPAD-hardness and PPAD-membership results. In an approximate pacing equilibrium, we make two relaxations on (b) and (d); the two parameters used to capture these two relaxations are δ and γ, respectively.

Definition 10 (Approximate Pacing Equilibria)**.** *Given an SPP game* $G = (n, m, (v_{ij}), (B_i))$ *and parameters* $\delta, \gamma \in [0,1)$*, we say* (α, x)*, with* $\alpha = (\alpha_i) \in [0,1]^n$*,* $x = (x_{ij}) \in [0,1]^{nm}$ *and* $\sum_{i\in[n]} x_{ij} \le 1$ *for all* $j \in [m]$*, is a* (δ, γ)-approximate pacing equilibrium *of* G *if*

(a) *Only buyers* close *to the highest bid win the good:* $x_{ij} > 0$ *implies* $\alpha_i v_{ij} \geq (1-\delta)h_j(\alpha)$.

(b) *Full allocation of each good with a positive bid:* $h_j(\alpha) > 0$ *implies* $\sum_{i\in[n]} x_{ij} = 1$.

(c) *Budgets are satisfied:* $\sum_{j\in[m]} x_{ij} p_j(\alpha) \leq B_i$.

(d) Not too much *unnecessary pacing:* $\sum_{j\in[m]} x_{ij} p_j(\alpha) < (1-\gamma)B_i$ *implies* $\alpha_i \geq 1-\gamma$.

For convenience we will write (δ,γ)-*approximate PE* to denote (δ,γ)-approximate pacing equilibrium, and write γ-approximate PE to denote $(0,\gamma)$-approximate PE. It is clear from the definition that when $\delta = \gamma = 0$, (δ,γ)-approximate PE captures the exact pacing equilibria of a SPP game.

Remark 7. *We can incorporate reserve prices in our model. Definition 9 can be extended in a natural way to model the presence of reserve prices (see Definition 20). All our results continue to hold with this extension. We refer the reader to Appendix D.3 for a full discussion.*

5.1.1 Connections to Dynamics, Best Response and Nash Equilibrium

Before moving on to our results, we motivate the definition of pacing equilibrium by connecting it more concretely to practice and previous work. Consider a collection of n buyers that participate repeatedly in T second-price auctions. For each auction $t \in [T]$, the good to be sold is drawn from a collection of m possible goods, with good j being selected with probability $d_j > 0$. Moreover, suppose the value v'_{ij} that buyer i has for good j is given by $\epsilon_{ij} v_{ij}/d_j$ for some $v_{ij} \geq 0$, where ϵ_{ij} is drawn independently for each buyer-good pair from some continuous distribution supported over $[1-\delta, 1]$. The ϵ_{ij} component of the value can also be thought of as a perturbation that arises from errors in estimating the click-through-rate (probability of a click) which is a crucial factor in determining the value of an advertiser in internet advertising. Finally, let B'_i denote the budget of buyer i, which is the maximum amount she is willing to spend over all T auctions.

[BG19] prove that, if we fix the bidding strategy of the other buyers, then it is optimal for a buyer to use pacing-based strategy to bid. The optimal pacing-based algorithm of [BG19] iteratively updates the pacing multiplier and satisfies the following properties: (i) If the buyer spends less than her per-period budget $B_i = B'_i/T$ in an iteration, her pacing multiplier is increased, and if the payment is greater than her per-period budget, then the multiplier is decreased; (ii) The pacing multiplier is constrained to belong to $[0, 1]$ because bidding more than the value leads to negative utility. These properties are also satisfied by the algorithm proposed by [Bor+07] and forms the basis of pacing algorithms used in practice which aim to smooth the expenditure of a buyer by evenly spending the budget over all auctions, i.e., aim to spend the per-period budget in each period if possible. If all of the buyers use an algorithm that satisfies these properties, the system can only stabilize when all of the buyers satisfy the no-unnecessary-pacing condition.

The no-unnecessary-pacing condition and the optimality of pacing stem from strong duality, as argued in [BBW15] and [BG19]. We provide a brief overview of their argument here. When T is large and $B'_i = \Theta(T)$, as is the case in online advertising, concentration arguments kick in and the problem of repeatedly bidding in T auctions can be interpreted as repeatedly bidding in the following single-shot game: Each buyer wishes to maximize her expected utility (value − payment) while keeping her expenditure below $B_i = B'_i/T$ in expectation over the randomness in the values (see [BBW15; BG19] for more details). This single-shot game captures the crux of the problem and its variants have been extensively studied in the literature [BBW15; Bal+21; Bab+20]. In fact, [BG19] show that, under some fairly stringent assumptions, their algorithm efficiently converges to an approximate pacing equilibrium of this single-shot game when all of the buyers employ it. But, these assumptions require independence of values across buyers and strong monotonicity of payments as a function of the pacing multipliers, both of which are unlikely to hold in practice. As we show in this chapter, if PPAD $\neq$ P, then the convergence can no longer be efficient in the absence of these assumptions. In the rest of this subsection, we will restrict our focus to this single-shot game and connect it to SPP games and pacing equilibria.

Fix buyer i and let f_j denote the highest bid from buyers other than i on good j. Then, the optimization problem faced by buyer i in the single-shot game is given by

$$\max_b \quad \sum_{j=1}^{m} d_j \cdot \mathbb{E}_{v'_{ij}, f_j} \left[(v'_{ij} - f_j) \mathbf{1}(b(j, v'_{ij}) \geq f_j) \right]$$

$$\text{s.t.} \quad \sum_{j=1}^{m} d_j \cdot \mathbb{E}_{v'_{ij}, f_j} \left[f_j \cdot \mathbf{1}(b(j, v'_{ij}) \geq f_j) \right] \leq B_i$$

where $b(j, \cdot)$ denotes the bidding strategy of buyer i for good j. Assume that the distribution of f_j conditioned on v'_{ij} (value of buyer i for good j) is continuous. Then, using the strong-duality argument of Chapter 4, it can be shown that strong duality holds, where the dual problem is given by

$$\begin{aligned}
&\min_{\mu_i \geq 0} \mu_i \cdot B + \max_b \sum_{j=1}^{m} d_j \cdot \mathbb{E}_{v'_{ij}, f_j} \left[(v'_{ij} - (1 + \mu_i) f_j) \mathbf{1}(b(j, v'_{ij}) \geq f_j) \right] \\
=& \min_{\mu_i \geq 0} \mu_i \cdot B + (1 + \mu_i) \max_b \sum_{j=1}^{m} d_j \cdot \mathbb{E}_{v'_{ij}, f_j} \left[\left(\frac{v'_{ij}}{1 + \mu_i} - f_j \right) \mathbf{1}(b(j, v'_{ij}) \geq f_j) \right]
\end{aligned}$$

Therefore, if $\mu_i^* \geq 0$ is the optimal dual solution, then an optimal bidding strategy for buyer i is $b(j, v'_{ij}) = v'_{ij}/(1 + \mu_i^*)$ (i.e., to pace her value with the multiplier $\alpha_i = 1/(1 + \mu_i^*)$) since it is optimal for the inner Lagrangian optimization problem over b. Note that this argument does not require other buyers to use a pacing-based strategy. Thus, it establishes that a pacing-based best response always exists.

Strong duality also implies that any optimal primal-dual solution pair satisfies complementary slackness: $\mu_i^* = 0$ if

$$\sum_{j=1}^{m} d_j \cdot \mathbb{E}_{v'_{ij}, f_j} \left[f_j \cdot \mathbf{1}(v'_{ij}/(1 + \mu_i^*) \geq f_j) \right] < B_i \,.$$

The fixed-point argument of [BBW15] further shows that a pacing-based Nash equilibrium exists for the single-shot game where all of the buyers use pacing with multipliers $\alpha_i = 1/(1 + \mu_i)$. Moreover, if a collection of feasible dual multipliers satisfy complementary slackness and the corresponding pacing-based strategies satisfy the budget constraints, then they form a Nash equilibrium of the single-shot game described above. Now, let $\alpha_i = 1/(1 + \mu_i^*)$ be a collection of equilibrium pacing multipliers. Then, the complementary slackness condition for buyer i can equivalently be written as a no-unnecessary-pacing condition: $\alpha_i = 0$ if

$$\sum_{j=1}^{m} d_j \cdot \mathbb{E}_{v'_{ij}, f_j} \left[f_j \cdot \mathbf{1}(\alpha_i v'_{ij} \geq f_j) \right] < B_i$$

As a consequence, every pacing equilibrium of this single-shot game is also a Nash equilibrium, where we define a pacing equilibrium to be any collection of pacing multipliers that satisfy the no-unnecessary-pacing condition and satisfy the budget constraint. Even if one has no interest in duality, the no-unnecessary-pacing condition is also extremely desirable in practice when the platform manages the budget of the buyer on her behalf — it ensures that the platform bids the value of the buyer on each good unless doing so would violate her budget. Thus, as outlined above, pacing equilibrium is an important refinement of Nash equilibrium for the single-shot game in both theory and practice.

Next, we connect pacing equilibria in single-shot games to approximate pacing equilibria in SPP games. Observe that, when all of the buyers use pacing to bid, $f_j = \max_{k \neq i} \alpha_k \epsilon_{kj} v_{kj} / d_j$. Hence, the expected payment of buyer i in this single-shot game can be rewritten as

$$\mathbb{E}_{\{\epsilon_{ij}\}_{i,j}} \left[\sum_{j=1}^{m} \left\{ \max_{k \neq i} \alpha_k \epsilon_{kj} v_{kj} \right\} \mathbf{1} \left(\epsilon_{ij} \alpha_i v_{ij} \geq \max_{k \neq i} \epsilon_{kj} \alpha_k v_{kj} \right) \right]$$

If we ignore the perturbations ϵ_{ij}, this is exactly the payment of buyer i in the SPP game with values v_{ij} and pacing multipliers α_i. To account for the perturbations and connect the single-shot game to the SPP game, we can define a perturbed SPP game (like [Bor+07]) as one in which (i) the value of buyer i for good j is given by $\epsilon_{ij} v_{ij}$; (ii) each item is sold through second-price auction; (iii) the strategy of each buyer is her pacing multiplier $\alpha_i \in [0, 1]$; (iv) ϵ_{ij} are drawn i.i.d. from some distribution with a positive density over $[1 - \delta, 1]$; (v) each buyer wishes to maximize her expected utility while satisfying her budget constraint in expectation over the perturbations ($-\infty$ utility if the budget constraint is violated). We define an approximate pacing equilibrium of this perturbed SPP game as simply a collection of budget-feasible pacing multipliers that satisfy the not-too-much-unnecessary-condition (see Appendix D.4). Recall that approximate pacing equilibrium of SPP games allows for arbitrary allocation between all buyers close to the highest bid, and therefore includes the allocation induced by perturbations as a special case. In Appendix D.4, we use this fact to show that computing a pacing equilibrium of perturbed SPP games is harder than computing an approximate pacing equilibrium in (unperturbed) SPP games, and therefore PPAD-hard due to Theorem 13.

Finally, as we make δ smaller, this perturbed SPP game gets closer to a true SPP game. Unfortunately, the duality-based arguments of existence (like those given in [BBW15] and Chapter 4) break down when $\delta = 0$ because ties are no longer a zero-probability event. The following example shows that a pacing equilibrium may not exist in this case under the uniform tie-breaking rule.

Example 3. *Consider a setting with two buyers and one good.* $v_{11} = 1$, $v_{21} = v \gg 1$ *and* $B_1 = \infty$, $B_2 = 1/4$. *Then, in any pacing equilibrium we have* $\alpha_1 = 1$ *because of the no-unnecessary-pacing condition. Now, if* $\alpha_2 \geq 1/v$, *then buyer 2 spends at least* $1/2$ *due to the uniform tie-breaking rule, which violates her budget. Hence,* $\alpha_2 < 1/v_2$ *and buyer two wins nothing and spends 0, thereby violating the no-unnecessary pacing condition.*

[Con+18] show that a pacing equilibrium does exist if the ties are broken carefully, which was their motivation behind making the tie-breaking rule a part of the equilibrium concept. This equilibrium tie-breaking rule can be thought of as the limiting expected allocation in the perturbed equilibrium as δ approaches zero. They also show that, in an unperturbed SPP game, if we fix the bids of other buyers and allow a buyer to pick her bids along with the fraction of each good she wants, it is a best-response for her to use pacing to bid because it allows her to win goods that yield the highest value per unit cost—using the multiplier α_i ensures that a buyer wins a good if and only if α_i times her value is greater than the second-highest bid, i.e., if the value per unit cost is above $1/\alpha_i$. In the one-shot game studied by [Con+18], pacing may not be a best response if the m good types arrive one-by-one and a buyer can change her bid to cause other buyers to drop out of later auctions due to budget exhaustion. In the repeated auction setting that motivates our single-stage pacing game, the budget constraint is over all T auctions. Therefore, deviating in a single-stage game would not cause other advertisers to drop out of that game[1]. [Con+18] also provide a discussion on the undesirable properties of Nash equilibria in SPP games enroute to motivating pacing equilibria as a more desirable solution concept. Nevertheless, we would like to note that our hardness result can be extended to Nash equilibria: In Appendix D.4, we prove that computing a Nash equilibrium of the perturbed SPP game is also PPAD-hard. We do so by showing that a minor modification of the game constructed in our hardness reduction for Theorem 13 only admits Nash equilibria that are also pacing equilibria.

5.2 Hardness Results

In this section we investigate the hardness of computing approximate pacing equilibria and show that the problem is PPAD-hard for second-price pacing games. Our most general result (Theorem 11) shows that the problem of finding a (δ, γ)-approximate PE in a SPP game is PPAD-hard, even when δ and γ are polynomially small in the number of players.

Our result is shown by reducing the problem of computing a Nash equilibriun in a $\{0, 1\}$-cost bimatrix game to that of finding a (δ, γ)-approximate PE in a corresponding SPP game. Because we wish to show the result for (δ, γ)-approximate PE, we must start our reduction from such approximate PE. In order to manage the resulting approximation factors, we are forced to introduce a number of additional bookkeeping gadgets, and correspondingly work with the problem of computing ϵ-well-supported Nash equilibria of $\{0, 1\}$-cost bimatrix games, as opposed to standard Nash equilibria. Taken together, all these facts lead to a longer proof that may obfuscate the main ideas underlying our reduction. To better highlight the key ideas in our reduction and motivate our techniques, we are going to start by proving that finding an *exact* pacing equilibrium in a SPP game is PPAD-hard, by showing a reduction from the problem of finding an exact Nash equilibrium in a

[1] One can imagine a strategy where a buyer consistently bids higher in an attempt to run other buyers out of budget early with the goal of winning goods for cheap later on. Although interesting, an analysis of these strategies would require studying a complicated incomplete-information extensive-form game, which is not the focus of this work. Moreover, in practice, these pacing algorithms are predominantly implemented by platforms who have no incentive to take advantage of some advertisers on behalf of other ones. Finally, while such strategies may seem appealing in toy examples, in a large-scale market, where the budget of an individual advertiser is small relative to the whole market, such an approach is unlikely to be possible.

$\{0,1\}$-cost bimatrix game.

5.2.1 Hardness of Finding Exact Pacing Equilibria

Our reduction will be from the problem of computing a Nash equilibrium in a $\{0,1\}$-cost bimatrix game. Let Δ_n denote the set of probability distributions over $[n]$. The input of the bimatrix problem is a pair of cost matrices $A, B \in \{0,1\}^{n\times n}$ and the goal is to find a Nash equilibrium $(x, y) \in \Delta_n \times \Delta_n$, meaning that x minimizes cost given y, i.e. $x^T Ay \le \hat{x}^T Ay$ for all $\hat{x} \in \Delta_n$, and similarly y minimizes cost given x, i.e. $x^T By \le x^T B\hat{y}$ for all $\hat{y} \in \Delta_n$. Equivalently, (x, y) is a Nash equilibrium if $x_i > 0$ for any $i \in [n]$ implies that $\sum_j A_{ij} y_j \le \sum_j A_{kj} y_j$ for all $k \in [n]$, and $y_j > 0$ for any $j \in [n]$ implies that $\sum_i x_i B_{ij} \le \sum_i x_i B_{ik}$ for all $k \in [n]$. This problem is known to be PPAD-complete [CTV07].

Given a $\{0,1\}$-cost bimatrix game (A, B) with $A, B \in \{0,1\}^{n\times n}$, we would like to construct an SPP game G in time polynomial in n, such that every exact PE of G can be mapped back to a Nash equilibrium of the bimatrix game (A, B) in polynomial time.

Before proceeding further, we informally describe some important aspects of the construction to provide some intuition. First, in the SPP game G, we will encode the pair (x, y) of mixed strategies in Δ_n using pacing multipliers. For each player $p \in \{1,2\}$ in the bimatrix game (A, B) and each (pure) strategy $s \in [n]$, there will be a corresponding buyer $\mathbb{C}(p, s)$ in the SPP game G, whose pacing multiplier $\alpha(\mathbb{C}(p, s))$ will be used to encode the probability with which player p plays strategy s in the bimatrix game (A, B). For now, take x to be the distribution obtained by normalizing $\alpha(\mathbb{C}(1, s))$, i.e., $x_t = \alpha(\mathbb{C}(1, t))/\sum_s \alpha(\mathbb{C}(1, s))$, and define y similarly using $\alpha(\mathbb{C}(2, s))$; we will discuss the issues with this proposal and ways to fix them momentarily.

Second, in order to capture the best response condition of Nash equilibria, we need to encode the cost borne by player $p \in \{1,2\}$ when playing a given strategy $s \in [n]$ against the mixed strategy of the other player. For simplicity, let us focus on $p = 1$. We will create a set of n *expenditure goods* $E(1, s)_1, \dots, E(1, s)_n$ for each pure strategy s of player 1. We will set buyer $\mathbb{C}(1, s)$'s value at 1 for each of the expenditure goods $E(1, s)_1, \dots, E(1, s)_n$. Additionally, each buyer $\mathbb{C}(2, t)$ will value $E(1,s)_t$ at νA_{st}, where $\nu = 1/(16n)$ is set to be so small that $\mathbb{C}(1,s)$ always wins all the goods $E(1,s)_1, \ldots, E(1,s)_n$ under any PE of G. This means that, in any PE with multipliers $\alpha(\mathbb{C}(p,s))$, buyer $\mathbb{C}(1,s)$ pays a total of $\nu \sum_t \alpha(\mathbb{C}(2,t))A_{st}$ for the expenditure goods $E(1,s)_1, \ldots, E(1,s)_n$, which captures player 1's cost for playing strategy s in (A,B), when player 2 uses the mixed strategy that plays each t with probability defined by $\alpha(\mathbb{C}(2,t))$ after normalization.

Finally we need to make sure that the best response condition of Nash equilibria holds for a strategy pair (x,y) obtained from multipliers $\alpha(\mathbb{C}(p,s))$ in any PE of G, i.e., only best-response strategies are played with positive probability. This poses a challenge because pacing multipliers are never zero in a pacing equilibrium, so we can't use them directly to encode probabilities in x and y (which need to be zero for strategies which are not best responses). To get around this issue, we will use thresholds to encode entries of (x,y) using $\alpha(\mathbb{C}(p,s))$. More formally, we add a *threshold buyer* and a set of *threshold goods* to G to make sure that $\alpha(\mathbb{C}(p,s)) \geq 1/2$ in any PE of G. This allows us to encode x by normalizing $\alpha(\mathbb{C}(1,s)) - 1/2$ and y by normalizing $\alpha(\mathbb{C}(2,s)) - 1/2$. The most challenging part of the construction is to have buyers/goods work together to ensure that both $\alpha(\mathbb{C}(1,s)) - 1/2$ and $\alpha(\mathbb{C}(2,s)) - 1/2$, $s \in [n]$, are not identically zero. We accomplish this by creating a set of *normalization goods* for each buyer $\mathbb{C}(p,s)$, with the property that each buyer $\mathbb{C}(p,s)$ spends approximately $\sum_{t=1}^{n} \alpha(\mathbb{C}(p,t))$ on her normalization goods. This, in combination with a carefully chosen budget and the 'No unnecessary pacing' condition, ensures that $\{\alpha(\mathbb{C}(p,s)) - 1/2\}_s$ are not identically zero. Then, we can follow the plan described in the last paragraph to encode the cost of player p playing s using the expenditure of buyer $\mathbb{C}(p,s)$ on $E(p,s)_1, \ldots, E(p,s)_n$, with careful calibration via the use of thresholds. This finally helps us enforce the best response condition of Nash equilibria on (x,y) in (A,B) by comparing total expenditures of buyers $\mathbb{C}(p,s)$ and using implications from such comparisons.

We now formally define the SPP game G in the next section, and then the following sections show the hardness result based on G.

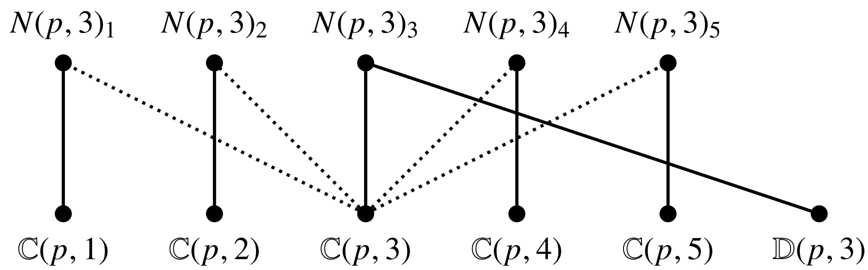

Figure 5.1: Normalization goods for $p \in \{1,2\}$ and $s = 3$, when $n = 5$. A buyer having a non-zero value for a good is represented by a line connecting the two. Solid line denotes a value of 1 and dotted line denotes a value of 2.

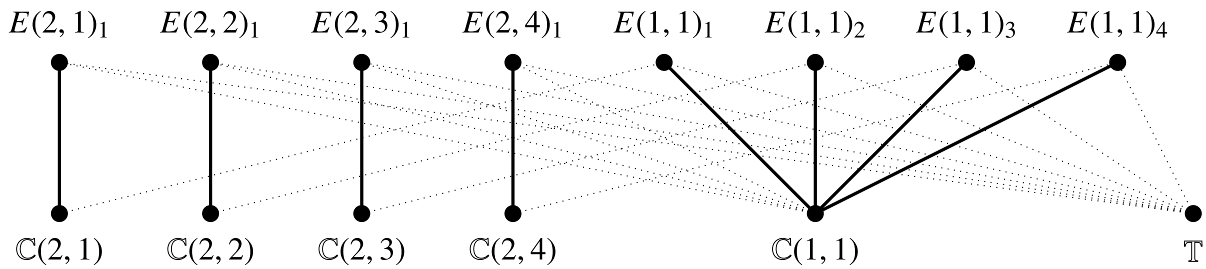

Figure 5.2: All the expenditure goods for which buyer $\mathbb{C}(1,1)$ has a non-zero value, when $n = 4$. A buyer having a non-zero value for a good is represented by a line connecting the two. Solid lines denote a value of 1 and dotted lines denote values which are smaller than $v = 1/(16n)$.

The SPP Game

The game G has the following set of goods:

- Normalization goods: n goods $\{N(p,s)_1, \ldots, N(p,s)_n\}$ for each $p \in \{1,2\}$ and $s \in [n]$.
- Expenditure goods: n goods $\{E(p,s)_1, \ldots, E(p,s)_n\}$ for each $p \in \{1,2\}$ and $s \in [n]$.
- Threshold goods: 1 good $T(p,s)$ for each $p \in \{1,2\}$ and $s \in [n]$.

Set $v = 1/(16n)$. The set of buyers in G is defined as follows, where we write $V(\cdot,\cdot)$ to denote the value of a good (the second component) to a buyer (the first component):

- Buyer $\mathbb{C}(p,s)$, $p \in \{1,2\}$ and $s \in [n]$: $\mathbb{C}(p,s)$ has positive values for the following goods:

– Normalization goods: $V(\mathbb{C}(p,s), N(p,s)_i) = 2$ for all $i \in [n] \setminus \{s\}$;
$V(\mathbb{C}(p,s), N(p,s)_s) = 1$; and $V(\mathbb{C}(p,s), N(p,t)_s) = 1$ for all $t \in [n] \setminus \{s\}$.

– Threshold good $T(p,s)$: $V(\mathbb{C}(p,s), T(p,s)) = 2n^4$.

– Expenditure goods: $V(\mathbb{C}(p,s), E(p,s)_i) = 1$ for all $i \in [n]$.
For $p = 1$: $V(\mathbb{C}(1,s), E(2,t)_s) = \nu B_{st}$ for all $t \in [n]$.
For $p = 2$: $V(\mathbb{C}(2,s), E(1,t)_s) = \nu A_{ts}$ for all $t \in [n]$.

For $p = 1$, the budget of $\mathbb{C}(1,s)$ is $n/2 + n^4 + 1/4 - \nu + \sum_{t\in[n]} \nu A_{st}/2$;
For $p = 2$, the budget of $\mathbb{C}(2,s)$ is $n/2 + n^4 + 1/4 - \nu + \sum_{t\in[n]} \nu B_{ts}/2$.

- Threshold Buyer $\mathbb{T}$: $\mathbb{T}$ has positive values only for the following goods:

  – Threshold goods: $V(\mathbb{T}, T(p,s)) = n^4$ for each $p \in \{1,2\}$ and $s \in [n]$.

  – Expenditure goods: $V(\mathbb{T}, E(1,s)_t) = \nu A_{st}/2$ and $V(\mathbb{T}, E(2,s)_t) = \nu B_{ts}/2$ for all $s,t \in [n]$.

  $\mathbb{T}$ has budget n^7 (high enough so that $\alpha(\mathbb{T}) = 1$ in any PE).

- Dummy buyers $\mathbb{D}(p,s)$, $p \in \{1,2\}$ and $s \in [n]$: The budget of $\mathbb{D}(p,s)$ is ν and she only values the normalization good $N(p,s)_s$ at $V(\mathbb{D}(p,s), N(p,s)_s) = 1$.

It is clear from the definition of G that it can be constructed from (A,B) in polynomial time.

Structure of Pacing Equilibria of G

With the definition of G in place, we start by showing some auxiliary structural results on the PE of G; these will be used to construct strategies for the bimatrix game. Let $\mathcal{E}$ be a PE of the SPP game G. We will use $\alpha(b)$ to denote the pacing multiplier of buyer b in $\mathcal{E}$. Observe that, from the definition of pacing equilibria, we can conclude that $\alpha(\mathbb{T}) = 1$ in $\mathcal{E}$; otherwise $\mathbb{T}$ needs to spend all her budget of n^7, which is impossible given that no buyer has value more than $2n^4$ for any good. The following lemma establishes bounds on $\alpha(\mathbb{C}(p, s))$ in $\mathcal{E}$.

Lemma 16. *For each* $p \in \{1, 2\}$ *and* $s \in [n]$, $1/2 \leq \alpha(\mathbb{C}(p, s)) < 1$ *and* $\alpha(\mathbb{D}(p, s)) = \alpha(\mathbb{C}(p, s))$.

Proof. Suppose for some $p \in \{1, 2\}$ and $s \in [n]$, we have $\alpha(\mathbb{C}(p, s)) < 1/2$. Then $\mathbb{C}(p, s)$ doesn't win any part of threshold good $T(p, s)$. Observe that she has value at most 2 for every other good. Given that there are only $O(n^2)$ goods in G, she cannot possibly spend all her budget (which is $\Omega(n^4)$). Here, we have used the fact that the payment is smaller than her bid on every item that she wins because of the second-price auction format, which in turn is always smaller than her value. This contradicts the assumption that $\mathcal{E}$ is a PE of G. Therefore, $\alpha(\mathbb{C}(p, s)) \geq 1/2$.

Next we prove $\alpha(\mathbb{D}(p, s)) = \alpha(\mathbb{C}(p, s))$. Suppose $\alpha(\mathbb{D}(p, s)) > \alpha(\mathbb{C}(p, s))$ for some $p \in \{1, 2\}$ and $s \in [n]$. Then, buyer $\mathbb{D}(p, s)$ wins all of good $N(p, s)_s$ at price $\alpha(\mathbb{C}(p, s)) \geq 1/2$ because $\mathbb{D}(p, s)$ and $\mathbb{C}(p, s)$ both value $N(p, s)_s$ at 1, and the rest of the buyers have zero value for it. This violates her budget constraint and leads to a contradiction. Therefore, $\alpha(\mathbb{D}(p, s)) \leq \alpha(\mathbb{C}(p, s))$. Moreover, if $\alpha(\mathbb{D}(p, s)) < \alpha(\mathbb{C}(p, s))$ (which implies $\alpha(\mathbb{D}(p, s)) < 1$) then her expenditure is zero. This violates the no unnecessary pacing condition. Hence, $\alpha(\mathbb{D}(p, s)) = \alpha(\mathbb{C}(p, s))$ must hold. Observe that, in particular, this means that the price of $N(p, s)_s$ is $\alpha(\mathbb{C}(p, s))$.

Finally suppose $\alpha(\mathbb{C}(p, s)) = 1$ for some $p \in \{1, 2\}$, $s \in [n]$. Then she wins the following goods:

- All of normalization goods $N(p, s)_t$ for each $t \neq s$ because $\mathbb{C}(p, s)$ has the higher value for them, and she spends at least $1/2$ on each of them because $\alpha(\mathbb{C}(p, t)) \geq 1/2$ by the first part of the proof.

- Part of normalization good $N(p, s)_s$ by spending at least $1 - \nu$. This is because $N(p, s)_s$ has price 1, she shares it with $\mathbb{D}(p, s)$, and buyer $\mathbb{D}(p, s)$ only has budget ν.

- All of threshold good $T(p, s)$ by spending n^4 because she has the higher value.

- All of expenditure good $E(p,s)_t$, for each $t \in [n]$, by spending at least $\nu A_{st}/2$ if $p = 1$ and $\nu B_{ts}/2$ if $p = 2$ because she has the higher value.

Hence, the total expenditure of $\mathbb{C}(p,s)$ is at least $(n-1)/2 + 1 - \nu + n^4 + \sum_t \nu A_{st}/2$ if $p = 1$ and at least $(n-1)/2 + 1 - \nu + n^4 + \sum_t \nu B_{ts}/2$ if $p = 2$. In both cases, the budget constraint is violated, leading to a contradiction. Therefore, the lemma holds. $\square$

The above lemma implies that every $\mathbb{C}(p,s)$ is paced in $\mathcal{E}$ (i.e. $\alpha(\mathbb{C}(p,s)) < 1$), thereby implying that their total expenditures must exactly equal their budgets. Additionally, we have the following corollary which will be used in the proof of Lemma 18.

Corollary 1. *For each $p \in \{1,2\}$ and $s \in [n]$, $\mathbb{C}(p,s)$ spends exactly $\alpha(\mathbb{C}(p,s)) - \nu$ on $N(p,s)_s$.*

Next let $x'_s = \alpha(\mathbb{C}(1,s)) - 1/2$ and $y'_s = \alpha(\mathbb{C}(2,s)) - 1/2$ for each $s \in [n]$. The following lemma will allow us to normalize x' and y' to obtain probability distributions x and y.

Lemma 17. *The following inequalities hold: $\sum_{s\in[n]} x'_s > 0$ and $\sum_{s\in[n]} y'_s > 0$.*

Proof. We show $\sum_s x'_s > 0$. The proof of $\sum_s y'_s > 0$ is completely analogous. Suppose $\sum_s x'_s = 0$. Then, $\alpha(\mathbb{C}(1,s)) = 1/2$ for all $s \in [n]$ because $\alpha(\mathbb{C}(1,s)) = 1/2$ by Lemma 16. We argue below that $\mathbb{C}(1,1)$ violates the no-unnecessary-pacing condition.

To see this, observe $\mathbb{C}(1,1)$ only wins a non-zero fraction of the following goods, and spends:

- At most $1/2$ on each normalization good $N(1,1)_t$, $t \in [n]$, because the highest competing bid is 1/2 on these goods.

- At most n^4 on the threshold good $T(1,1)$ because that is the highest competing bid.

- At most νA_{1t} on each expenditure good $E(1,1)_t$, $t \in [n]$, because that is the highest possible competing bid.

Hence, the total expenditure of $\mathbb{C}(1,1)$ is at most $n/2 + n^4 + \sum_t \nu A_{1t}$, which is strictly less than her budget of $n/2 + n^4 + 1/4 - \nu + \sum_t \nu A_{1t}/2$, a contradiction. $\square$

Extracting Bimatrix Game Equilibria from G

Now, we are ready to define the mixed strategies (x, y) for the bimatrix game (A, B). Set player 1's mixed strategy x to be $x_s = x'_s / \sum_i x'_i$ and player 2's mixed strategy y to be $y_s = y'_s / \sum_i y'_i$. These are valid mixed strategies because of Lemma 16 and Lemma 17. The next lemma shows that (x, y) is indeed a Nash equilibrium of (A, B).

Lemma 18. (x, y) *is a Nash equilibrium for the bimatrix game* (A, B)*.*

Proof. Suppose there are $s, s^* \in [n]$ such that $x_s > 0$ but $\sum_t A_{st} y_t > \sum_t A_{s^*t} y_t$ (the proof for y is analogous). Using $x_s > 0$, buyer $\mathbb{C}(1, s)$ spends non-zero amounts on the following goods:

- $\alpha(\mathbb{C}(1, t))$ on the normalization good $N(1, s)_t$ for each $t \neq s$ because $\mathbb{C}(1, s)$ has a bid strictly greater than 1, which is the value and an upper bound on the bid of $\mathbb{C}(1, t)$.
- $\alpha(\mathbb{C}(1, s)) - \nu$ on the normalization good $N(1, s)_s$ because she shares the good with $\mathbb{D}(1, s)$ who has a budget of ν.
- n^4 on the threshold good $T(1, s)$ because her bid is strictly greater than n^4.
- $\alpha(\mathbb{C}(2, t)) \cdot \nu A_{st}$ on the expenditure good $E(1, s)_t$ for each $t \in [n]$.

Therefore, the total expenditure of buyer $\mathbb{C}(1, s)$ is given by

$$\sum_{t \in [n]} \alpha(\mathbb{C}(1, t)) + n^4 - \nu + \sum_{t \in [n]} \alpha(\mathbb{C}(2, t)) \cdot \nu A_{st}$$
$$= \sum_{t \in [n]} \alpha(\mathbb{C}(1, t)) + n^4 - \nu + \sum_{t \in [n]} \nu A_{st}/2 + \sum_{t \in [n]} y_t \nu A_{st}$$

Note that the RHS above after replacing s with s^*:

$$\sum_{t \in [n]} \alpha(\mathbb{C}(1, t)) + n^4 - \nu + \sum_{t \in [n]} \nu A_{s^*t}/2 + \sum_{t \in [n]} y_t \nu A_{s^*t}$$

is an upper bound for the total expenditure of buyer $\mathbb{C}(1, s^*)$ (no matter whether $x_{s^*} > 0$ or not).

As a result, the total expenditure of $\mathbb{C}(1, s)$ minus that of $\mathbb{C}(1, s^*)$ is at least

$$\left(\sum_{t\in[n]} \nu A_{st}/2 + \sum_{t\in[n]} y_t \nu A_{st}\right) - \left(\sum_{t\in[n]} \nu A_{s^*t}/2 + \sum_{t\in[n]} y_t \nu A_{s^*t}\right) > \sum_{t\in[n]} \nu A_{st}/2 - \sum_{t\in[n]} \nu A_{s^*t}/2$$

using the assumption that $\sum_t A_{st} y_t > \sum_t A_{s^*t} y_t$. On the other hand, the budget of $\mathbb{C}(1, s)$ minus that of $\mathbb{C}(1, s^*)$ is equal to the RHS above. This is a contradiction because both buyers should have their total expenditures equal to their budgets. This finishes the proof of the lemma. □

Thus, given a $\{0, 1\}$-cost bimatrix game (A, B), we have defined an SPP game G which satisfies the following properties: (i) G can be constructed in polynomial time; (ii) any PE $\mathcal{E}$ of G can be used to construct a Nash equilibrium (x, y) of (A, B) in polynomial time. As a result, the problem of finding an exact pacing equilibrium in a second-price pacing game is PPAD-hard.

5.2.2 Hardness of Finding Approximate Pacing Equilibria

We next state our main hardness result, which extends the PPAD-hardness of finding pacing equilibria to the approximate case of finding (δ, γ)-approximate pacing equilibria.

Theorem 13. *The problem of computing a (δ, γ)-approximate PE of an SPP game $G = (n, m, (v_{ij}), (B_i))$ with $\delta = \gamma = 1/n^7$ is PPAD-hard.*

The proof is relegated to Appendix D.1. It uses similar ideas but entails more involved bookkeeping to incorporate approximations introduced in (δ, γ)-approximate PE. Theorem 11 follows from Theorem 13 by standard padding arguments (i.e., adding dummy buyers to the game).

5.2.3 Implications of The Hardness Result

Before concluding this section, we discuss some implications of our hardness results. In [Bor+07], the authors introduced a natural bidding heuristic for optimizing the utility of budget-constrained agents who repeatedly participate in day-long auction campaigns for m items, where the set of agents and items remains the same every day. The heuristic maintains a pacing multiplier for each agent, which is increased by a small amount if the buyer ran out of her daily budget before the end of the previous day, and decreased otherwise. They use random perturbation to avoid instabilities, which gives an agent who bids close to the highest bid a fraction of the item in expectation. If we ignore the intra-day temporal aspects of their model, their setting can be thought of as repeatedly playing the perturbed SPP game from Section 5.1.1 every day. In Theorem 1 of [Bor+07], they prove that their heuristic efficiently converges for first-price auctions. Furthermore, they conjecture the convergence of the heuristic for second-price auctions to pacing multipliers which satisfy the following conditions: (i) Every agent runs out of her daily-budget close to the end of the day; (ii) Every agent either spends most of her daily budget or has a pacing multiplier close to one. In Theorem 27 of Appendix D.4, we show that Theorem 13 implies that computing an approximate pacing equilibrium of the perturbed SPP game is also PPAD hard. As a consequence, if PPAD ≠ P, then ALGORITHM 1 of [Bor+07] does not always converge efficiently for second-price auctions, i.e., the number of days/time-steps required for convergence cannot scale as a polynomial function of the input size and $(1/\delta, 1/\gamma)$ in the worst-case. In other words, we have shown that Theorem 1 of [Bor+07] cannot be extended to second-price auctions in any way that maintains efficient convergence unless PPAD = P, thereby making progress towards their open conjecture.

Moreover, recall from Section 5.1.1 that if all of the buyers employ pacing algorithms, like the one proposed by [BG19], and the resulting dynamics converge, then they will converge to an approximate pacing equilibrium. Our hardness result (Theorem 13 and Theorem 27) implies that there exists a (correlated) value distribution such that the algorithm of [BG19], which is optimal for a single buyer against an adversarial/stochastic competition, does not converge efficiently to an equilibrium when employed by all the buyers, unless PPAD=P.

Our hardness results are also pertinent to the relationship between pacing equilibria and market equilibria. In Proposition 5 of [Con+18], the authors show that every pacing equilibrium in a second-price pacing game has an equivalent supply-aware market equilibrium with linear utilities, where supply-aware means that the buyers are aware of the supplies of each item and choose their demand set accordingly. Thus, the relationship between pacing equilibria and market equilibria, in combination with Theorem 13, implies that there exists a refinement of the set of supply-aware market equilibria with linear utilities which is PPAD-hard to compute.

5.3 Existence of Pacing Equilibria and Membership in PPAD

We prove Theorem 12 in this section, i.e., the problem of finding a pacing equilibrium of an SPP game is in PPAD. One consequence of this result is that every SPP game with rational values v_{ij} and budgets B_i has a pacing equilibrium (α, x) with rational entries.

Our plan is as follows. We first introduce a restricted version of approximate pacing equilibria called *smooth* (δ, γ)*-approximate PE* (see Definition 11), which will only be used in Section 5.3.1. We prove in Section 5.3.1 that the problem of finding a smooth (δ, γ)-approximate PE (when δ and γ are input parameters encoded in binary) is in PPAD. Given that the smooth version (Definition 11) is a restriction of (δ, γ)-approximate PE (Definition 10), this implies that the problem of computing a (δ, γ)-approximate PE is in PPAD.

Next we give in Section 5.3.2 an efficient algorithm that can round any $(\delta, \gamma/2)$-approximate PE into a γ-approximate PE when δ is sufficiently small. This, combined with the PPAD-membership of (δ, γ)-approximate PE, shows that the problem of computing γ-approximate PE is also in PPAD.

Finally we show in Section 5.3.3 that, when γ is sufficiently small, any γ-approximate PE of G can be used to build a linear program which can then be solved to obtain an *exact* pacing equilibrium of G. It follows that the problem of computing an exact pacing equilibrium is in PPAD.

5.3.1 PPAD Membership of Computing (δ, γ)- Approximate Equilibria

We start with the definition of *smooth* (δ, γ)*-approximate PE.* It is a refinement of (δ, γ)-approximate PE in which the pacing multipliers (α_i) *fully determine* the allocations (x_{ij}). Note that this is not the case for (δ, γ)-approximate PE in general: potentially there can be (δ, γ)-approximate PE with identical multipliers but different allocations. The smooth version we consider below, on the other hand, specifies the allocations as continuous functions of multipliers.

Definition 11 (Smooth Approximate Pacing Equilibria)**.** *Given an SPP game* $G = (n, m, (v_{ij}), (B_i))$ *and two parameters* $\delta \in (0,1), \gamma \in [0,1)$*, we say that* (α, x) *with* $\alpha = (\alpha_i) \in [0,1]^n$*,* $x = (x_{ij}) \in [0,1]^{nm}$ *and* $\sum_{i\in[n]} x_{ij} \le 1$ *for all* $j \in [m]$ *is a* smooth (δ,γ)-approximate PE *of* G *if*

(a) Only buyers close to the highest bid win the good and the allocation x is completely specified by α*: For each* $i \in [n]$ *and* $j \in [m]$*,* x_{ij} *(as a function of* α*) is given by*

$$x_{ij}(\alpha) := \frac{[\alpha_i v_{ij} - (1-\delta)h_j(\alpha)]^+}{\sum_{r\in[n]}[\alpha_r v_{rj} - (1-\delta)h_j(\alpha)]^+}$$

where $[y]^+$ *is* y *if* $y \ge 0$ *and* 0 *otherwise. (We assume by default that* $0/0 = 0$*.)*

(b) Budgets are satisfied: $\sum_{j\in[m]} x_{ij}(\alpha)p_j(\alpha) \le B_i$*.*

(c) Not too much unnecessary pacing: $\sum_{j\in[m]} x_{ij}(\alpha)p_j(\alpha) < (1-\gamma)B_i$ *implies* $\alpha_i \ge 1-\gamma$*.*

Observe from the definition that, if (α, x) is a smooth (δ,γ)-approximate PE of an SPP game G, then it must be a (δ,γ)-approximate PE of G as well. Therefore, the PPAD membership of computing a smooth (δ,γ)-approximate PE in an SPP game implies directly the PPAD membership for (δ,γ)-approximate PE. A similar statement holds for establishing their existence.

The main tools we will use are Sperner's Lemma and the search problem it defines.

High-dimensional Sperner's Lemma. We review Sperner's lemma. Consider a $(n-1)$-dimensional simplex $S = \{\sum_{i=1}^n \alpha_i v_i \,|\, \alpha_i \ge 0, \sum_{i=1}^n \alpha_i = 1\}$, where $v_1, \ldots, v_n$ are n vertices of S. A triangulation of S is a partition of S into smaller subsimplices such that any two subsimplices either are disjoint or share a full face of a certain dimension. A *Sperner coloring* T of a triangulation of S is then an assignment of n colors $\{1, \ldots, n\}$ to vertices of the triangulation (union of the vertices of subsimplices that make up the triangulation) such that

- Vertices of the original simplex S each receive a different color: $T(v_i) = i$ for each $i \in [n]$.
- Vertices on each face of S are colored using only the colors of the vertices defining that face: For any vertex $u = \sum_i \beta_i v_i$ in the triangulation, we have $T(u) \ne j$ if $\beta_j = 0$.

A *panchromatic* subsimplex of T is one in the triangulation whose vertices have all the n colors.

Sperner's Lemma: Every Sperner coloring T of any triangulation of S has a panchromatic subsimplex.

Before proceeding with the formal proof of PPAD membership (with its added burden of rigorously attending to complexity-theoretic details), we provide an informal argument for the existence of smooth (δ, γ)-approximate PE which forms the basis of its PPAD membership proof. Let G be an SPP game and S be the standard simplex $S = \{\beta = (\beta_1, \dots, \beta_n) \,|\, \beta_i \geq 0, \sum_i \beta_i = 1\}$ from now on. We will assign a color to each point $\beta \in S$ (informally) as follows: Construct a pacing multiplier $\alpha_i(t) = t\beta_i$ for each $i \in [n]$, where t is a scalar. Increase t, starting at 0, and instruct each buyer $i \in [n]$ to say "Stop" when either $\alpha_i(t) = 1$ or $\sum_j x_{ij}(\alpha(t))p_j(\alpha(t)) = B_i$ happens. Color β with k if buyer k is the first to say "Stop" (with tie breaking done arbitrarily, e.g., taking the smallest such k).

Let $t^*(\beta)$ be the value of t at which some buyer says "Stop" for the first time. Then the buyer that says "Stop" first is either spending her budget or is not paced, i.e. she satisfies both the budget constraint (b) and the 'No unnecessary pacing' condition (c) (see Definition 9). Now, by taking a triangulation of S, it is easy to verify that the coloring described above induces a Sperner coloring and thus, Sperner's lemma implies the existence of a panchromatic subsimplex Q. It follows from our coloring that every buyer says "Stop" at one of the vertices of Q and hence, every buyer satisfies (b) and (c) of Definition 9 at one of its vertices. By proving the Lipschitzness of $t^*(\beta)$ and the total expenditures of buyers, both as functions of β, we show that when the triangulation is *fine* enough, any point β in a panchromatic subsimplex yields a (δ, γ)-approximate PE of G.

With the blueprint of the proof in place, we now proceed with the formal proof that places the problem of computing smooth (δ, γ)-approximate PE in PPAD. Let S be the standard simplex as above, and we consider *Kuhn's triangulation* of S [Kuh60; DQS12]. Given any $\omega > 0$ with $1/\omega$ being an integer, Kuhn's triangulation uses S_ω as its vertices, where S_ω consists of all points $\beta \in S$ whose coordinates β_i are integer multiples of ω. Kuhn's triangulation also has the property that any two vertices of a subsimplex of the triangulation has ℓ_∞-distance at most 2ω.

A proof of the following PPAD membership result can be found in [EY10] (see the proof of item 2 of Proposition 2.2; note that on page 2548 they reduce the problem they are interested in to the problem of finding a panchromatic subsimplex in a Sperner coloring over Kuhn's triangulation and then show the latter is in PPAD):

Theorem 14. *Given a Boolean circuit*[2] *that encodes a Sperner coloring $T : S_\omega \to [n]$ of Kuhn's triangulation for some ω and n, the problem of finding a panchromatic subsimplex is in PPAD.*

We prove the PPAD membership of the problem of finding a smooth (δ, γ)-approximate PE by giving a polynomial-time reduction to the problem described in Theorem 14. Given an SPP game $G = (n, m, (v_{ij}), (B_i))$ and parameters δ and γ (which we assume without loss of generality that $\delta, \gamma < 1/4$), we set the parameter ω to be

$$\omega = \frac{\min(B_{\min}, 1)}{\left(2^{|G|}/\delta\right)^{10,000}} \cdot \frac{\gamma}{2}$$

where $B_{\min} := \min_{i\in[n]} B_i$ and $|G|$ denotes the number of bits needed to represent G. We define a coloring $T : S_\omega \to [n]$, following ideas described in the sketch of existence above, and prove that T satisfies the following properties:

Lemma 19.
1. *T is a Sperner coloring;*
2. *Every panchromatic subsimplex of T in the triangulation can be used to compute a smooth (δ, γ)-approximate PE of the SPP game G in polynomial time.*
3. *There is a polynomial-time algorithm that outputs $T(\beta)$ on inputs G, ω, δ and $\beta \in S_\omega$.*

The PPAD membership of computing a smooth (δ, γ)-approximate PE in an SPP game follows directly by combining Theorem 14 and Lemma 19.

We now give the definition of the coloring $T : S_\omega \to [n]$. Let $\beta = (\beta_1, \ldots, \beta_n)$ be a vertex of S_ω. Set $\alpha_i(t) = t\beta_i$, where t is a positive scalar. As discussed earlier, we set the color $T(\beta)$ of β by

[2] The circuit has $O(n \log(1/\omega))$ input variables to encode a point of S_ω and has $\lceil \log n \rceil$ output gates to encode the output of the Sperner coloring T.

increasing t, starting at 0, and instructing each buyer i to say "Stop" when either $\alpha_i(t) = 1$ or

$$\sum_{j\in[m]} x_{ij}(\alpha(t)) \cdot p_j(\alpha(t)) = B_i.$$

The color $T(\beta)$ of β is set to be $k \in [n]$ if buyer k is the first buyer to say "Stop" (with arbitrary tie breaking, e.g., by taking the smallest such k).

More formally, recall that for $t > 0$,

$$x_{ij}(\alpha(t)) = \frac{[t\beta_i v_{ij} - (1-\delta)\max_k t\beta_k v_{kj}]^+}{\sum_r [t\beta_r v_{rj} - (1-\delta)\max_k t\beta_k v_{kj}]^+} = \frac{[\beta_i v_{ij} - (1-\delta)\max_k \beta_k v_{kj}]^+}{\sum_r [\beta_r v_{rj} - (1-\delta)\max_k \beta_k v_{kj}]^+} = x_{ij}(\beta),$$

which does not depend on t. Also, for $t \geq 0$, $p_j(\alpha(t)) = tp_j(\beta)$, where we write $p_j(\beta)$ to denote the second largest element among $\beta_1 v_{1j}, \ldots, \beta_n v_{nj}$. For each buyer $i \in [n]$, define

$$t_i(\beta) = \min\left\{\frac{1}{\beta_i}, \frac{B_i}{\sum_j x_{ij}(\beta)p_j(\beta)}\right\},$$

where the first term is $+\infty$ if $\beta_i = 0$ and the second term is $+\infty$ if $\sum_j x_{ij}(\beta)p_j(\beta) = 0$. Note that $t_i(\beta)$ is exactly the value of t at which buyer i would say "Stop" in the informal coloring procedure described earlier. Given our assumption of $B_i > 0$, we have $t_i(\beta) > 0$ for all $i \in [n]$. Additionally, define $t^*(\beta) = \min_{i\in[n]} t_i(\beta)$. Given that β_i's sum to 1, we have that $t^*(\beta) \leq n$ because $\beta_i \geq 1/n$ for some $i \in [n]$. We record the discussion as the following lemma:

Lemma 20. *For every $\beta \in S_\omega$ we have $0 < t^*(\beta) \leq n$.*

Finally, the color $T(\beta)$ of $\beta \in S_\omega$ is set to be the smallest $i \in [n]$ such that $t_i(\beta) = t^*(\beta)$. We are now ready to prove Lemma 19.

Proof of Lemma 19. Part (3) of Lemma 19 follows from the description of T. To prove part (1) (T is a Sperner coloring), consider a vertex $\beta \in S_\omega$ on the facet of S opposite to the vertex e_i, i.e., $\beta_i = 0$. Hence, $t_i(\beta) = \infty$, which by Lemma 20 implies that $T(\beta) \neq i$ given that $t^*(\beta) \leq n$.

To prove part (2), we show that if q is a vertex of any panchromatic subsimplex of T, then (α, x) must be a smooth (δ, γ)-approximate PE of G where $\alpha = t^*(q) \cdot q$ and $x = (x_{ij})$ has $x_{ij} = x_{ij}(q)$.

First it follows from the definition of $t^*(\beta)$ and $x_{ij}(\beta)$ that $\alpha_i \in [0,1]$ and $x_{ij} \in [0,1]$. Conditions (a) and (b) of Definition 11 also trivially hold for all vertices of the triangulation. It suffices to prove (c) for all $i \in [n]$, which means the complementarity condition that either $\alpha_i \geq 1-\gamma$ or the expenditure of buyer i is at least $(1-\gamma)B_i$. Fix an arbitrary $i \in [n]$.

For this purpose we note that given the subsimplex is panchromatic, it has a vertex q' such that $T(q') = i$, which implies that if we used q' to define α' and x' (i.e. $\alpha' = t^*(q') \cdot q'$ and $x'_{ij} = x_{ij}(q')$), then they would satisfy the above complementarity condition for buyer i with $\gamma = 0$. The following claim shows that both the multiplier $t^*(\beta) \cdot \beta_i$ and the total expenditure of buyer i:

$$\sum_{j\in[m]} x_{ij}(\beta) \cdot p_j(t^*(\beta) \cdot \beta) = t^*(\beta) \sum_{j\in[m]} x_{ij}(\beta) \cdot p_j(\beta)$$

are smooth as functions of β. Intuitively this allows us to use the complementarity condition for buyer i at q' to show that the same condition holds at q *approximately* given that $\|q - q'\|_\infty \leq 2\omega$ (as a property of subsimplices in Kuhn's triangulation).

Claim 2. *Let $L = (2^{|G|}/\delta)^{10{,}000}$. Then for any panchromatic subsimplex S_0 of T and buyer $i \in [n]$, the following Lipschitz conditions hold for all $\beta, \beta' \in S_0$:*

$$\left| t^*(\beta) \cdot \beta_i - t^*(\beta') \cdot \beta'_i \right| \leq L \cdot \|\beta - \beta'\|_\infty \quad \textit{and}$$

$$\left| t^*(\beta) \sum_{j\in[m]} x_{ij}(\beta) \cdot p_j(\beta) - t^*(\beta') \sum_{j\in[m]} x_{ij}(\beta') \cdot p_j(\beta') \right| \leq L \cdot \|\beta - \beta'\|_\infty$$

We use Claim 2 to finish the proof of the lemma and consign the claim's proof to Appendix D.2. Given $T(q') = i$, one of the following two cases holds:

- $t^*(q') \cdot q'_i = 1$, which by Claim 2 and our chocie of ω implies

$$\alpha_i = t^*(q) \cdot q_i \geq 1 - 2L\omega \geq 1 - \gamma$$

- $t^*(q') \sum_j x_{ij}(q')p_j(q') = B_i$, which in combination with Claim 2 and our choice of ω implies that the expenditure of buyer i exceeds $(1-\gamma)B_i$:

$$t^*(q) \sum_{j \in [m]} x_{ij}(q) \cdot p_j(q) \geq B_i - 2L\omega \geq B_i - B_{\min}\gamma \geq (1-\gamma)B_i.$$

Since $i \in [n]$ was arbitrary, this finishes the proof that (α, x) is a smooth (δ, γ)-approximate approximate PE. □

5.3.2 PPAD Membership of Computing γ- approximate PE

Consider an SPP game $G = (n, m, \{v_{ij}\}_{i,j}, \{B_i\}_i)$. As before, we will use $|G|$ to denote the number of bits required to represent G. The main result of this subsection shows that (informally) when δ is small enough, any $(\delta, \gamma/2)$-approximate PE of G can be efficiently rounded to a γ-approximate PE. It follows from the PPAD membership of (δ, γ)-approximate PE established in the previous subsection that the problem of computing a γ-approximate PE is in PPAD as well.

Before presenting the rounding algorithm, we motivate the main idea behind it. Observe that the major difference between (δ, γ)-approximate PE and γ-approximate PE is the ability of buyers that don't have the highest bid to win the good in the former. In order to round a (δ, γ')-approximate PE (α^*, x^*) to obtain a γ-approximate PE (α', x') of G (where $\gamma' = \gamma/2$ in the rest of this subsection), we set $x' = x^*$ and need to round α^* to α' to ensure that all the winners are tied for the highest bid and at the same time, the multiplier and total expenditure of each buyer changes only slightly.

We now present an informal argument that demonstrates how this is achieved in our rounding algorithm when there are only two buyers ($n = 2$). Define the set of all valuation ratios

$$\tilde{\mathcal{V}} = \left\{ \frac{v_{ar}}{v_{br}} : a, b \in [n], r \in [m] \text{ such that } v_{ar}, v_{br} > 0 \right\}.$$

Set δ to be small enough: for all $y, z \in \mathcal{V}$ with $yz > 1$, we have $(1-\delta)^2 yz > 1$. Consider a (δ, γ')-approximate PE (α^*, x^*). Assume without loss of generality that there is a good j such that $\alpha_1^* v_{1j} = c_j \alpha_2^* v_{2j}$ and $1 - \delta \leq c_j \leq 1/(1-\delta)$. If no such good j exists then every good is fully allocated to the buyer with the highest bid because only bidders with bids greater than $(1-\delta)$ times the highest bid can win the item in a (δ, γ')-approximate PE, and thus, (α^*, x^*) is already a γ'-approximate PE. We show that after scaling the pacing multiplier of buyer 2 from α_2^* to $c_j\alpha_2^*$ (and letting $\alpha' = (\alpha_1^*, c_j\alpha_2^*)$ be the new multipliers), (α', x^*) satisfies the property that $x_{i\ell}^* > 0$ for any i and ℓ implies buyer i has the highest bid for good ℓ. This is trivially true for good $\ell = j$ given that the two buyers are now tied on good j. The remaining goods can be divided into two categories and we argue about each one separately:

- Consider good ℓ such that $\alpha_1^* v_{1\ell} = c_\ell \alpha_2^* v_{2\ell}$ and c_ℓ satisfies either $c_\ell < 1-\delta$ or $c_\ell > 1/(1-\delta)$. Given that we only changed the multiplier of buyer 2 by a factor of $1-\delta \le c_j \le 1/(1-\delta)$, the highest bidder does not change. Moreover, the highest bidder won the entire good in the (δ, γ')-approximate PE because $1-\delta \le c_j \le 1/(1-\delta)$ and continues to do so in the (δ, γ')-approximate PE because the allocation does not change.

- Consider a good ℓ such that $\alpha_1^* v_{1\ell} = c_\ell \alpha_2^* v_{2\ell}$ and c_ℓ satisfies $(1-\delta) \le c_\ell \le 1/(1-\delta)$. Then, we can write $\alpha_1^*/\alpha_2^* = c_j(v_{2j}/v_{1j}) = c_\ell(v_{2\ell}/v_{1\ell})$, which implies $(c_j/c_\ell)(v_{2j}/v_{1j})(v_{1\ell}/v_{2\ell}) = 1$. Observe that $c_j/c_\ell \in [(1-\delta)^2, 1/(1-\delta)^2]$. Thus, by our choice of δ, we get $(v_{2j}/v_{1j})(v_{1\ell}/v_{2\ell}) = 1$, which implies $c_j = c_\ell$. Hence, both buyers are tied in good ℓ.

To finish the proof that (α', x^*) is a γ-approximate PE, it suffices to show that the budget constraint and the not too much unnecessary pacing condition still hold approximately after the small scaling of α_2^*. In the rest of this subsection, we extend the aforementioned line of reasoning to design a rounding algorithm for the general setting, and prove its correctness.

Building on $\tilde{\mathcal{V}}$ defined above, we can define the set of valuation ratio products

$$\mathcal{V} = \Big\{ y_1 y_2 \dots y_k : k \in [2n] \text{ and } y_i \in \tilde{\mathcal{V}} \text{ for each } i \in [k] \Big\},$$

i.e., $\mathcal{V}$ consists of all products of no more than $2n$ numbers from $\tilde{\mathcal{V}}$. Given G and $\gamma \in [0,1)$ (with $\gamma' = \gamma/2$), we choose $\delta \in [0,1)$ to be small enough to satisfy the following two conditions:

$$(1-\delta)^{2^n} > (1-\gamma') \text{ and } (1-\delta)^{2^n} z > 1 \text{ for all } z \in \mathcal{V} \text{ such that } z > 1.$$

It suffices to set δ to be $1/2^N$ where N is polynomial in $|G|$ and $\log(1/\gamma)$.

Let (α^*, x^*) be a (δ, γ')-approximate PE of $G = (n, m, (v_{ij}), (B_i))$, where $\gamma' = \gamma/2$ and δ satisfies the two conditions above. We will use W_j to denote the winners of the good j under x^*: W_j consists of buyers i with $x^*_{ij} > 0$. Moreover, recall that $h_j(\alpha)$ denotes the highest bid on good j when the pacing multipliers are given by α. Our rounding algorithm is presented in Algorithm 5. The polynomial reduction then follows from the following performance guarantee of the rounding algorithm, which we prove in the rest of the subsection:

Lemma 21 (Correctness)**.** *The rounding algorithm takes (α^*, x^*), δ and G as input and runs in polynomial time. Let α' be the tuple of multipliers returned by the rounding algorithm. Then (α', x^*) is a γ-approximate PE of G.*

The rounding algorithm maintains an undirected graph $\mathcal{G}$ over vertices $[n]$ as buyers. $\mathcal{G}$ starting out with an empty edge set and edges are added according to Algorithm 5 to keep track of the rounding-updates performed on α. We use $C_{\mathcal{G}}(i)$ to denote the connected component of i in the graph $\mathcal{G}$. The algorithm also maintains an edge labeling $I(\cdot)$ that maps each edge of the graph $\mathcal{G}$ to a good $j \in [m]$ (which intuitively is the good that caused the creation of this edge). We remark that the labeling $I(\cdot)$ is only relevant for the analysis of the algorithm below. Now, we proceed to prove Lemma 21.

Lemma 22. *Suppose in the t_0 iteration of the while loop, $\{i, k\}$ is the edge that was just added to $\mathcal{G}$ with $I(\{i, k\}) = j$, then at the end of this iteration we have $C_{\mathcal{G}}(i) = C_{\mathcal{G}}(k)$ and*

$$\frac{\alpha_i}{\alpha_k} = \frac{v_{kj}}{v_{ij}}. \tag{\#}$$

Moreover, (#) holds for all iterations $t \geq t_0$.

Proof. We prove the lemma using induction on the iterations on the while loop. For the base case $t = t_0$, note that (#) holds at the end of the iteration due to Step 2 of Algorithm 5. Moreover,

Algorithm 5: Rounding Algorithm

Initialize: Graph $\mathcal{G} = (V, E)$ with $V = [n]$ and $E = \emptyset$; $\alpha = \alpha^*$

While there exists a good $j \in [m]$ and a buyer $i \in W_j$ such that $\alpha_i v_{ij} < h_j(\alpha)$, i.e., i does not have the highest bid on j but wins a positive fraction of it:

1. Pick $k, i \in [n]$ and $j \in [m]$ such that $i \in W_j$ and $\alpha_i v_{ij} < \alpha_k v_{kj} = h_j(\alpha)$
2. Set $\alpha_a \leftarrow (h_j(\alpha)/\alpha_i v_{ij}) \cdot \alpha_a$ for every buyer $a \in C_{\mathcal{G}}(i)$
3. Set $E \leftarrow E \cup \{\{i, k\}\}$ and $I(\{i, k\}) = j$

Return: $\alpha' := (1 - \delta)^{2^n} \alpha$

since edge $\{i, k\}$ is added to $\mathcal{G}$ in Step 3, we also have $C_{\mathcal{G}}(i) = C_{\mathcal{G}}(k)$ at the end of iteration t_0. Moreover, since no edges are removed during the run of Algorithm 5, $\{i, k\} \in E$ for iterations after t_0, and hence $C_{\mathcal{G}}(i) = C_{\mathcal{G}}(k)$ at the end of all iterations $t \geq t_0$. Suppose (#) holds at the end of iteration $t - 1$ for some $t - 1 \geq t_0$. Then, either both α_i and α_k will both be updated identically or neither of them will be updated because $C_{\mathcal{G}}(i) = C_{\mathcal{G}}(k)$, thereby maintaining (#). This completes the induction and establishes the lemma. □

Next we prove that at the end of each iteration, bids for the same good from buyers in the same component of $\mathcal{G}$ are either tied or not very close.

Lemma 23. *After each iteration of the while loop, and for each good $j \in [m]$, all buyers from the same connected component of $\mathcal{G}$ are either tied for j, or their bids for j are multiplicatively separated by a factor larger than $(1 - \delta)^{2^n}$.*

Proof. Let $\mathcal{G}$ be the current graph and $a, b \in [n]$ be two buyers in the same connected component of $\mathcal{G}$. Assuming $\alpha_a v_{aj} > \alpha_b v_{bj}$ for some $j \in [m]$, we show below that $(1-\delta)^{2^n} \alpha_a v_{aj} > \alpha_b v_{bj}$ from which the lemma follows. Given that a and b are connected in $\mathcal{G}$, we write $\{a, i_1\}, \{i_1, i_2\}, \ldots,$ $\{i_L, b\}$ to denote a path from a to b in $\mathcal{G}$ with $L < n$. Then, using Lemma 22, we can write

$$1 > \frac{\alpha_b v_{bj}}{\alpha_a v_{aj}} = \frac{v_{bj}}{v_{aj}} \cdot \frac{\alpha_{i_1}}{\alpha_a} \frac{\alpha_{i_2}}{\alpha_{i_1}} \frac{\alpha_{i_3}}{\alpha_{i_2}} \cdots \frac{\alpha_b}{\alpha_{i_L}} = \frac{v_{bj}}{v_{aj}} \cdot \frac{v_{aI(\{a,i_1\})}}{v_{i_1 I(\{a,i_1\})}} \frac{v_{i_1 I(\{i_1,i_2\})}}{v_{i_2 I(\{i_1,i_2\})}} \cdots \frac{v_{i_L I(\{i_L,b\})}}{v_{bI(\{i_L,b\})}}$$

Hence, $\alpha_a v_{aj}/\alpha_a v_{bj} \in \mathcal{V}$ and $\alpha_a v_{aj}/\alpha_b v_{bj} > 1$. Therefore, our choice of δ implies that

$$(1-\delta)^{2^n} \cdot \frac{\alpha_a v_{aj}}{\alpha_b v_{bj}} > 1\,,$$

as required. □

Initially (in α^*) we have every $i \in W_j$ has $\alpha_i^* v_{ij} \geq (1-\delta)h_j(\alpha^*)$ (given that (α^*, x^*) is a (δ, γ')-approximate PE). The next lemma shows that, at the end of each iteration, $\alpha_i v_{ij}$ of every $i \in W_j$ (note that W_j is always defined using the original allocation x^*) remains not far from $h_j(\alpha)$.

Lemma 24. *After t iterations of the while loop, every $j \in [m]$ and $i \in W_j$ satisfy*

$$\alpha_i v_{ij} \geq (1-\delta)^{2^t} \cdot h_j(\alpha).$$

Proof. The proof follows from induction. The base case of $t = 0$ follows from definition.

Suppose the statement holds after $(t-1)$ iterations, and let's focus on some $j \in [m]$ and $i \in W_j$ during the t-th iteration. By our inductive hypothesis, we have

$$\alpha_i v_{ij} \geq (1-\delta)^{2^{t-1}} \cdot h_j(\alpha)$$

before the start of the t-th iteration. On the other hand, note that all changes to α occur in step 2 of the while loop, and moreover, all such changes result in an increase of some entries of α. It also follows from the inductive hypothesis and the choices of k, i, j in step 1 of the while loop that entries of α can only go up by a multiplicative factor of at most $1/(1-\delta)^{2^{t-1}}$. Therefore, after the t-th iteration, we have

$$\alpha_i v_{ij} \geq (1-\delta)^{2^{t-1}} \cdot (1-\delta)^{2^{t-1}} \cdot h_j(\alpha) = (1-\delta)^{2^t} \cdot h_j(\alpha).$$

This completes the induction step. □

Lemmas 23 and 24 imply that, in each of the first n iterations of the while loop, buyers i and k

picked in step 1 must belong to different connected components of $\mathcal{G}$. As a result, there are at most $n-1$ iterations of the while loop given that we merge two connected components in each loop. On the one hand, this implies that the rounding algorithm terminates in polynomial time. On the other hand, at the termination of the while loop, for every good $j \in [m]$, we have $\alpha_i v_{ij} = h_j(\alpha)$ for all $i \in W_j$, i.e., every winner of j under x^* has the highest bid for j.

The next lemma shows that the α' returned by the rounding algorithm is close to α^*.

Lemma 25. *Let α' be the tuple of multipliers returned by the rounding algorithm. Then*

$$(1-\delta)^{2^n}\alpha^* \le \alpha' \le \alpha^*.$$

Proof. By Lemma 24, in iteration t of the while loop, each entry of α either stays the same or increases multiplicatively by a factor of at most $1/(1-\delta)^{2^{t-1}}$. As there are at most $n-1$ iterations of the while loop, we have for every $i \in [n]$:

$$(1-\delta)^{2^n} \cdot \alpha_i^* \le \alpha_i' := (1-\delta)^{2^n} \cdot \alpha_i \le (1-\delta)^{2^n} \prod_{t=1}^{n-1} \frac{1}{(1-\delta)^{2^{t-1}}} \cdot \alpha_i^* \le \alpha_i^*.$$

This finishes the proof of the lemma. ◻

We are now ready to prove Lemma 21.

Proof of Lemma 21. We have already shown that the algorithm runs in polynomial time. Assuming that (α^*, x^*) is a (δ, γ')-approximate PE of $\mathcal{G}$, we show that (α', x^*) is a γ-approximate PE of $\mathcal{G}$ by establishing conditions (a)-(d) of Definition 10. Using Lemma 25, we have $\alpha' \in [0,1]^n$. Condition (a) has already been established earlier using Lemmas 23 and 24. Condition (b) holds because we kept the same allocation x^* and given how we obtain α' from α^*, the set of goods j with $h_j(\alpha^*) > 0$ is the same as that in α'. Condition (c) follows easily from Lemma 25. So it suffices to verify that (d) holds with γ.

To see this we have for each buyer $i \in [n]$ that either $\alpha_i^* \ge 1-\gamma'$ or $\sum_j x_{ij}^* p_j(\alpha^*) \ge (1-\gamma')B_i$.

For the former case, we have from Lemma 25 that

$$\alpha'_i \geq (1-\delta)^{2^n} \cdot (1-\gamma') > (1-\gamma')^2 \geq 1-\gamma$$

using $(1-\delta)^{2^n} > 1-\gamma'$ from the choice of δ and that $\gamma = 2\gamma'$. For the latter case, it follows from Lemma 25 and our choice of δ that

$$p_j(\alpha') \geq (1-\delta)^{2^n} \cdot p_j(\alpha^*) > (1-\gamma') \cdot p_j(\alpha^*)$$

for all $j \in [m]$. Here we have used the fact that $p_j((1-\delta)^{2^n}\alpha^*) = (1-\delta)^{2^n} p_j(\alpha^*)$ and $p_j(\alpha) \geq p_j(\tilde{\alpha})$ whenever $\alpha \geq \tilde{\alpha}$. As a result, the total expenditure of buyer i in (α', x^*) is

$$\sum_{j\in[m]} x^*_{ij} \cdot p_j(\alpha') > (1-\gamma') \sum_{j\in[m]} x^*_{ij} \cdot p_j(\alpha^*) \geq (1-\gamma')^2 B_i \geq (1-\gamma)B_i.$$

Therefore, we have shown that (α', x^*) is a γ-approximate PE of G. □

5.3.3 PPAD Membership of Computing Exact Pacing Equilibria

In the last subsection we showed that the problem of finding a γ-approximate PE of a second-price pacing game G is in PPAD. Finally we show in this subsection that the problem of finding an exact equilibrium of a pacing game is also in PPAD. To this end, we show that when γ is small enough (though with bit length polynomial in $|G|$), any γ-approximate PE(α', x') of G can be "rounded" into an exact equilibrium by solving a linear program defined using support information extracted from (α', x'). This technique is similar to the one used in [EY10; VY11] and [FR+20]. For this purpose we recall the following fact about linear programs:

Fact 1. *There is a polynomial $r(\cdot)$ with the following property. Let* LP *be a linear program that minimizes a non-negative variable γ. Then an optimal solution of* LP *has either $\gamma = 0$ or $\gamma \geq 1/2^{r(|\mathsf{LP}|)}$, where $|\mathsf{LP}|$ denotes the number of bits needed to represent* LP.

Given a γ-approximate PE (α', x') of $G = (n, m, (v_{ij}), (B_i))$ (for some sufficiently small γ to be specified later), we extract from (α', x') the following support information:

1. $I' \subseteq [n]$ consists of buyers $i \in [n]$ who are almost unpaced, i.e., $\alpha'_i \geq 1 - \gamma$. Given that (α', x') is a γ-approximate PE, condition (d) of Definition 10 implies that

$$\sum_{j \in [m]} x'_{ij} p_j(\alpha') \geq (1 - \gamma) B_i, \quad \text{for all } i \notin I.$$

2. For each $j \in [m]$, W'_j is the set of buyers $i \in [n]$ with $x'_{ij} p_j(\alpha') > 0$ (which implies $\alpha'_i v_{ij} = h_j(\alpha')$). These are buyers who win good j and pay a positive amount for it.

3. For each $j \in [m]$, let $s_j \in [n]$ be the smallest index i such that $\alpha'_i v_{ij} = h_j(\alpha')$, i.e., s_j is the smallest index among the buyers who have the highest bid in good j.

4. For each $j \in [m]$, let $t_j \in [n]$ be the smallest index $i \neq s_j$ such that $\alpha'_i v_{ij} = \max_{k \neq s_j} \alpha'_k v_{kj}$ (so we have that $\alpha'_{t_j} v_{t_j j} = p_j(\alpha')$).

On the other hand, given any $I \subseteq [n]$, $W = (W_j \subseteq [n] : j \in [n])$, $s = (s_j \in [n] : j \in [m])$, and $t = (t_j \in [n] : j \in [m])$, we use $\mathsf{LP}(I, W, s, t)$ to denote the following linear program on $n + nm + 1$ variables $\alpha = (\alpha_i : i \in [n])$, $q = (q_{ij} : i \in [n], j \in [m])$ and τ (where each variable q_{ij} captures the amount buyer i pays for good j):

minimize τ

$\tau \geq 0, \alpha_i \in [0, 1], q_{ij} \geq 0$ for all $i \in [n]$ and $j \in [m]$

$q_{ij} = 0$ for all $j \in [m]$ and $i \notin W_j$

$\alpha_{s_j} v_{s_j j} \geq \alpha_k v_{kj}$ for all $j \in [m]$ and $k \in [n]$

$\alpha_{t_j} v_{t_j j} \geq \alpha_k v_{kj}$ for all $j \in [m]$ and $k \neq s_j \in [n]$

(a) $\alpha_i v_{ij} \geq \alpha_{s_j} v_{s_j j}$ for all $j \in [m]$ and $i \in W_j$

(b) $\sum_{k \in [n]} q_{kj} = \alpha_{t_j} v_{t_j j}$ for all $j \in [m]$

(c) $\sum_{j\in[m]} q_{ij} \leq B_i$ for all $i \in [n]$

(d) $\alpha_i \geq 1-\tau$ for all $i \in I$ and $\sum_{j\in[m]} q_{ij} \geq (1-\tau)B_i$ for all $i \notin I$

Here, (a) ensures that the buyers in W_j have the highest bid on good j; (b) ensures that the total payment of all buyers for good j is equal to the second highest bid; (c) ensures that the budgets are satisfied; and (d) ensures that the not-too-much-unnecessary-pacing condition is satisfied. The lemma below follows directly from the definition of γ-approximate PE and the way I', W', s' and t' are extracted from (α', x').

Lemma 26. *Suppose (α', x') is a γ-approximate PE of G. Then (α', q', γ) is a feasible solution to the linear program $\mathsf{LP}(I', W', s', t')$, where $q' = (q'_{ij})$ with $q'_{ij} = x'_{ij} p_j(\alpha')$.*

On the other hand, the next lemma shows that if $\mathsf{LP}(I, W, s, t)$ has a feasible solution $(\alpha, q, 0)$ for some I, W, s and t, then (α, x) is an exact pacing equilibrium, where $x = (x_{ij})$ and $x_{ij} = q_{ij}/p_j(\alpha)$ if $p_j(\alpha) > 0$; when $p_j(\alpha) = 0$ we set $x_{s_j j} = 1$ and $x_{ij} = 0$ for all other i.

Lemma 27. *If $(\alpha, q, 0)$ is a feasible solution to $\mathsf{LP}(I, W, s, t)$, then (α, x) is an exact equilibrium.*

Proof. Let $(\alpha, q, 0)$ be a feasible solution to $\mathsf{LP}(I, W, s, t)$. Set α to be the pacing multipliers of buyers in G and define the allocation $x = (x_{ij})$ as above. Then, the LP constraints imply that the highest bid on good j is $h_j(\alpha) = \alpha_{s_j} v_{s_j j}$ and the second highest bid is $p_j(\alpha) = \alpha_{t_j} v_{t_j j}$. Next we note that, in the latter case, the constraints of the LP force the set of winners $\{i \mid x_{ij} > 0\}$ of good $j \in [m]$ to be a subset of W_j. This is because $x_{ij} > 0$ implies $q_{ij} > 0$ and $q_{ij} = 0$ for all $i \notin W_j$. Now, it is straightforward to see that constraints (a)-(d), in combination with $\tau = 0$, imply that (α, x) satisfies the corresponding conditions (a)-(d) of Definition 9. □

Given the definition of $\mathsf{LP}(I, W, s, t)$, there is a polynomial $r'(\cdot)$ such that

$$\max_{I,W,s,t} |\mathsf{LP}(I, W, s, t)| \leq r'(|G|).$$

Now we can set γ to be smaller than $1/2^{r(r'(|G|))}$ (with bit length still polynomial in $|G|$). To finish the proof of Theorem 12, we let (α', x') be a γ-approximate PE of G. It follows from Lemma 26 that (α', q', γ) is a feasible solution to $\mathsf{LP}(I', W', s', t')$. Next it follows from Fact 1 that this linear program has a feasible solution $(\alpha, q, 0)$ and the latter can be computed in polynomial time. Lemma 27 shows that (α, x), which can be computed in polynomial time, is a pacing equilibrium of G.

5.4 Conclusion

We studied the computational complexity of pacing equilibria in second-price pacing games with multiplicative pacing. Our results show that finding a pacing equilibrium, whether exact or approximate, is a PPAD-complete problem. As discussed previously, these results close the open problem from [Con+18] on the complexity of pacing equilibria, and make progress towards resolving the conjecture of [Bor+07] by showing that their dynamics is unlikely to converge efficiently in second-price auctions. More generally, our results show that algorithms for budget-smoothing in auctions, an important problem for Internet advertising, cannot be expected to efficiently find even approximate pacing equilibria in the worst case.

There are several interesting future questions and implications to investigate based on our work. Perhaps most importantly, we would like to understand exactly when budget-smoothing becomes hard. As discussed in the literature review, [BG19] developed regret minimization algorithms for the case of i.i.d. and continuous stochastic valuations. Yet our results imply that for general correlated valuations convergence cannot occur efficiently. The question is now which types of correlated stochastic valuations admit efficient algorithms, and which types are hard. It would also be interesting to understand whether other methods of budget smoothing (such as those discussed by [Bal+21]) lead to PPAD-complete equilibrium problems as well.

In the direction of positive results, our PPAD membership proof suggests that complementary pivoting may be a fruitful research direction for computing pacing equilibria. This is especially pertinent because approaches based on mixed-integer programming seem to scale poorly [Con+18].

Chapter 6: Throttling Equilibria in Auction Markets

Based on the publication [CKK21] co-authored with Xi Chen and Christian Kroer.

This chapter goes beyond pacing and investigates the other most popular method of budget management: throttling. In Section 6.2, we analyze first-price auctions. We show that a throttling equilibrium always exists, and characterize it as the maximal element in the set of participation probabilities that result in all buyers satisfying their budgets (Theorem 15). Furthermore, we use this characterization to establish its uniqueness. On the computational front, we describe decentralized dynamics in which buyers repeatedly play the throttling game and make simple tâtonnement-style adjustments to their participation probabilities based on their expected expenditure (Algorithm 6). We show that these tâtonnement-style dynamics converge to an approximate throttling equilibrium in polynomial time (Theorem 16).

In Section 6.3, we study second-price auctions, We begin by establishing that a throttling equilibrium always exists for second-price auctions (Theorem 17), but find that it may not be unique, and for some games all throttling equilibria can be irrational. Next, we prove results about the computational complexity of finding throttling equilibria, which requires the use of terminology from computational complexity theory. Before summarizing those results, we make a note for readers who may not be familiar with complexity theory: In order to make our results more accessible, we provide an informal description of them at the head of every subsection, in an attempt to avoid letting complexity-theoretic terminology obscure the conclusions derived from the result. Continuing on with the summary of our results, we prove that the problem of computing approximate throttling equilibria is PPAD-hard even when each good has at most three bids (Theorem 18), by showing a reduction from the PPAD-hard problem of computing an approximate equilibrium of a threshold game [PP21]. As a consequence, we show that, unlike first-price auctions, no dynamics can converge in polynomial time to a second-price throttling equilibrium (assuming PPAD-complete problems cannot be solved in polynomial time). Furthermore, we place the problem of computing approximate throttling equilibria in the class PPAD by showing a reduction to the problem of finding a Brouwer fixed point of a Lipschitz mapping from a unit hypercube to itself (Theorem 20); the latter is known to be in PPAD via Sperner's lemma (e.g. see [CD06]). We provide additional evidence of the computational challenges that afflict throttling for second-price auctions by proving the NP-hardness of finding a revenue-maximizing approximate throttling equilibrium (Theorem 21). We complement these hardness results by describing a polynomial-time algorithm for computing throttling equilibria for the special case in which there are at most two bids on each good (Algorithm 7), thereby precisely delineating the boundary of tractability.

In Section 6.4, we compare the equilibrium outcomes of throttling with those of pacing. We show that, for first-price auctions, the revenue of the unique throttling equilibrium and the unique pacing equilibrium, although incomparable directly, are always within a factor of 2 of each other (Theorem 23). Moreover, we find that pacing and throttling equilibria share a remarkably similar computational and structural landscape, as summarized in Table 6.1 and Table 6.2. In view of this comparison, our work can be seen as providing the analogous set of results for throttling equilibria that [Bor+07; Con+18; Con+19] proved for pacing equilibria. Our results reaffirm what the analysis of pacing suggested: budget management for first-price auctions is more well-behaved as compared to second-price auctions.

Existence	**Rationality**	**Multiplicity**	**Computational Complexity**	**Efficient Dynamics**
Always	Not always	Always unique	Poly.-time for approx. eq.	For approx. eq.
Always [Con+19]	Always [Con+19]	Always unique [Con+19]	Poly.-time for exact eq. [Con+19]	For approx. eq. [Bor+07]

Table 6.1: Comparison of throttling (top row) and pacing equilibria (bottom row) for **first-price** auctions.

Existence	Rationality	Multiplicity	Computational Complexity	Revenue Max.
Always	Not always	Possibly infinite	PPAD-complete for approx. eq	NP-hard
Always [Con+18]	Always [Chapter 5]	Possible [Con+18]	PPAD-complete for both exact and approx. eq. [Chapter 5]	NP-hard [Con+18]

Table 6.2: Comparison of throttling equilibria (top row) and pacing equilibria (bottom row) for **second-price** auctions.

Finally, in Section 6.5, we analyze the price of anarchy of liquid welfare [DL14; Aza+17]. We show that the liquid welfare under any throttling equilibrium is at most a factor of 2 away from the liquid welfare that can be obtained by a central planner with complete information of the buyers bids/values, i.e., the Price of Anarchy is at most 2. We do so for both first-price and second-price auctions. Moreover, we provide examples to show that this bound is tight for both auction formats.

6.1 Model

Consider a seller who has m types of goods to sell, and n budget constrained buyers who are interested in buying these goods. The seller runs an auction amongst the buyers in order to make the sale. We assume that the type of good to be sold is drawn from some known distribution $d = (d_1, \dots, d_m)$, i.e., the good to be sold is of type j with probability d_j. Buyer i bids $\tilde{b}_{ij}$ on good type j, for all $i \in [n], j \in [m]$, and has a per-auction budget of $B_i > 0$. To control their budget expenditure, each buyer i is associated with a *throttling parameter* $\theta_i \in [0, 1]$, which represents the probability with which she participates in the auction: each buyer i independently flips a biased coin which comes up heads with probability θ_i, and submits their bid $\tilde{b}_{ij}$ if the coin comes up heads, while submitting no bid if the coin comes up tails.

We focus on the setting where each buyer wishes to satisfy her budget constraint in expectation, i.e., buyer i would like to spend less than B_i in expectation over the good types and participation coin flips of all buyers. Requiring the budget constraints to be satisfied in expectation draws its motivation from the large number of auctions that are run by online-advertising platforms, in conjunction with concentration arguments, and has been employed by previous works on budget management in online auctions (see, e.g., [GKP12; AH13; BBW15; Bal+21; BG19; Con+18]). Additionally, in this chapter, we restrict our attention to the two most commonly-used auction formats in online advertising: first-price auctions and second-price auctions. In a first-price auction, the participating buyer with the highest bid wins the good and pays her bid, whereas in a second-price auction, the participating buyer with the highest bid wins the good and pays the second-highest bid among the participating buyers. Our model can be interpreted as a discrete version of the one defined in [Bal+21].

Before proceeding further, we introduce some additional notation that allows us to capture the stochastic nature of the good types via a rescaling of the bids, thereby allowing us to analyze the setting as a deterministic multi-good auction problem: Set $b_{ij} := d_j \tilde{b}_{ij}$ for all $i \in [n], j \in [m]$. Since the participation of buyers is independent of the good type and we are only concerned with expected payments, the good type distribution $d = (d_j)_j$ and the bids $\{\tilde{b}_{ij}\}_{i,j}$ are consequential only insofar as they determine $\{b_{ij}\}_{i,j}$. Therefore, with some abuse of terminology, going forward, we refer to b_{ij} as the bid of buyer i on good j (instead of $\tilde{b}_{ij}$, which will no longer be used).[1] Furthermore, to simplify our analysis, we will assume that ties are broken lexicographically, i.e., the smaller buyer number wins in case of a tie. Our results continue to hold for all other tie-breaking priority orders over the buyers (even when they are different for each good). The lexicographic tie-breaking rule allows for simplified notation, albeit with some abuse: We will write $b_{ij} > b_{kj}$ to mean that either b_{ij} is strictly greater than b_{kj}, or $b_{ij} = b_{kj}$ and $i < k$. Finally, we refer to any tuple $\big(n, m, (b_{ij}), (B_i)\big)$ as a *throttling game*.

In online-advertising auctions, the buyers (or, more typically in practice, the platform on behalf of the buyers) attempt to satisfy their budget constraints by adjusting their throttling parameters. This naturally leads to a game where each buyer's strategy is her throttling parameter. We use $p(\theta)_{ij}$ to denote the expected payment of buyer i on good j when buyers use $\theta = (\theta_1, \ldots, \theta_n)$ to decide their participation probabilities. Let X_i be a random variable such that $X_i = 1$ if buyer i

[1]This deterministic view is equivalent to the model of [Con+18], except that we focus on probabilistic throttling for managing budgets, whereas they focus on multiplicative pacing. Their model can similarly be viewed as a stochastic setting.

participates and $X_i = 0$ if buyer i does not participate. Then, by our modeling assumptions, X_i is a Bernoulli(θ_i) random variable. More concretely, $p(\theta)_{ij}$ can be defined as follows:

- First-price auction: $p(\theta)_{ij} = \mathbb{E}\left[X_i b_{ij} \prod_{k:b_{kj}>b_{ij}}(1 - X_k)\right] = \theta_i b_{ij} \prod_{k:b_{kj}>b_{ij}}(1 - \theta_k)$
- Second-price auction:

$$p(\theta)_{ij} = \mathbb{E}\left[\sum_{\ell:b_{\ell j}<b_{ij}} b_{\ell j} X_i X_\ell \prod_{k\neq i:b_{kj}>b_{\ell j}} (1 - X_k)\right] = \sum_{\ell:b_{\ell j}<b_{ij}} b_{\ell j}\theta_i\theta_\ell \prod_{k\neq i:b_{kj}>b_{\ell j}} (1 - \theta_k)$$

We overload $p(\theta)_{ij}$ to represent the expected payment in both auction formats; the auction format will be clear from the context. We assume here that the participation probability of a buyer across goods is perfectly correlated for simplicity (X_i is the same for all j). Any other correlation structure, e.g. independent across goods, would also lead to the same results due to linearity of expectation. Next, we define the equilibrium concept which will be the main object of study in this work.

Definition 12 (Throttling Equilibrium)**.** *Given a throttling game $\left(n, m, (b_{ij}), (B_i)\right)$, a vector of throttling parameters $\theta = (\theta_1, \ldots, \theta_n) \in [0, 1]^n$ is called an δ-approximate throttling equilibrium if:*

1. *Budget constraints are satisfied: $\sum_j p(\theta)_{ij} \leq B_i$ for all $i \in [n]$*
2. *No unnecessary throttling occurs: If $\sum_j p(\theta)_{ij} < B_i$, then $\theta_i = 1$*

The above definition applies to both first-price and second-price auctions using the corresponding payment rule $p(\theta)_{ij}$. Definition 12 draws its motivation from the fact that, in a natural utility model, throttling equilibria are essentially equivalent to pure Nash Equilibria, which we describe next. Consider a throttling game $\left(n, m, (b_{ij}), (B_i)\right)$. Fix an auction format: either first-price or second-price. Suppose buyer i has value v_{ij} for good type j for all $i \in [n], j \in [m]$. We make the natural assumption that the buyers bid less than their value, i.e., $d_i v_{ij} \geq b_{ij}$ for all $i \in [n], j \in [m]$ in second-price auctions and strictly less than their value, i.e., $d_i v_{ij} > b_{ij}$ for all $i \in [n], j \in [m]$

in first-price auctions. Define a new game G in which each buyer i's strategy is her throttling parameter θ_i and her utility function is given by

$$u_i(\theta) = \begin{cases} \sum_j \left(v_{ij} d_i \theta_i \prod_{k: b_{kj} > b_{ij}} (1 - \theta_k) - p(\theta)_{ij} \right) & \text{if } \sum_j p(\theta)_{ij} \leq B_i \\ -\infty & \text{otherwise} \end{cases}$$

This utility is simply the expected quasi-linear utility modified to ascribe a utility value of negative infinity for budget violations. Since all of the buyers get a non-negative utility from winning any good, increasing the throttling parameter improves utility so long as the budget constraint is satisfied. This makes it easy to see that every throttling equilibrium of the throttling game $\left(n, m, (b_{ij}), (B_i)\right)$ is a Nash equilibrium of the corresponding game G. Furthermore, it is also straightforward to check that a pure Nash equilibrium of G is not a throttling equilibrium only in the following scenario: There is a buyer who spends 0 in the Nash equilibrium and has a throttling parameter strictly less than 1. In this scenario, setting the throttling parameter of all such buyers to 1 yields a throttling equilibrium with exactly the same expected allocation and payment for all the buyers as under the Nash equilibrium. Hence, given a throttling game $\left(n, m, (b_{ij}), (B_i)\right)$, for every pure Nash equilibrium of the corresponding game G, there is a throttling equilibrium with the same expected allocation and revenue.

We conclude this section by defining an approximate version of throttling equilibrium, which allows us to side-step issues of irrationality that can plague exact equilibria (see Example 10 and Example 11).

Definition 13 (Approximate Throttling Equilibrium)**.** *Given a throttling game $\left(n, m, (b_{ij}), (B_i)\right)$, a vector of throttling parameters $\theta = (\theta_1, \ldots, \theta_n)$ is called an δ-approximate throttling equilibrium if:*

1. *Budget constraints are satisfied: $\sum_j p(\theta)_{ij} \leq B_i$ for all $i \in [n]$*

2. *Not too much unnecessary throttling occurs: If $\sum_j p(\theta)_{ij} < (1 - \delta)B_i$, then $\theta_i \geq 1 - \delta$*

6.2 Throttling in First-price Auctions

We begin by studying throttling equilibria in the first-price auction setting. We start by showing that, for first-price auctions, there always exists a unique throttling equilibrium. We then describe a simple and efficient tâtonnement-style algorithm for approximate throttling equilibria.

6.2.1 Existence of First-Price Throttling Equilibria

To show existence, we will characterize the throttling equilibrium as a component-wise maximum of the set of all budget-feasible throttling parameters. This argument is inspired from the technique used in [Con+19] for pacing equilibria in first-price auctions. We use the following definition to make the argument precise.

Definition 14 (Budget-feasible Throttling Parameters)**.** *Given a throttling game $\big(n, m, (b_{ij}), (B_i)\big)$, a vector of throttling parameters $\theta \in [0,1]^n$ is called budget-feasible if every buyer satisfies her budget constraints, i.e., $\sum_j p(\theta)_{ij} \le B_i$ for all buyers $i \in [n]$.*

The following lemma captures the crucial fact that the component-wise maximum of two budget-feasible throttling parameters is also budget-feasible.

Lemma 28. *Given a throttling game $\big(n, m, (b_{ij}), (B_i)\big)$, if $\theta, \tilde{\theta} \in [0,1]^n$ are budget-feasible, then $\max(\theta, \tilde{\theta}) := (\max(\theta_i, \tilde{\theta}_i))_i$ is also budget-feasible.*

Proof. Fix $i \in [m]$ and $j \in [m]$. Without loss of generality, we assume that $\theta_i \ge \tilde{\theta}_i$. Observe that

$$p(\max(\theta, \tilde{\theta}))_{ij} = \prod_{k: b_{kj} > b_{ij}} (1 - \max(\theta_k, \tilde{\theta}_k))\theta_i b_{ij} \le \prod_{k: b_{kj} > b_{ij}} (1 - \theta_k)\theta_i b_{ij} = p(\theta)_{ij}$$

Summing over all goods $j \in [m]$ completes the proof. □

The maximality property shown in Lemma 28 is analogous to the maximality property of multiplicative pacing: there it is also the case that component-wise maxima over pacing vectors preserves budget feasibility for first-price auctions, and this was used by [Con+19] to show several structural properties of pacing equilibria. Next we show that the maximality property allows us to establish the existence and uniqueness of throttling equilibria for first-price auction.

Theorem 15. *For every throttling game* $\big(n, m, (b_{ij}), (B_i)\big)$*, there exists a unique throttling equilibrium* $\theta^* \in [0,1]^n$ *which is given by* $\theta^*_i = \sup\{\theta_i \in [0,1] \mid \exists\, \theta_{-i} \text{ s.t. } \theta = (\theta_i, \theta_{-i}) \text{ is budget-feasible}\}$.

Proof. Set $\theta^*_i = \sup\{\theta_i \in [0,1] \mid \exists\, \theta_{-i} \text{ s.t. } \theta = (\theta_i, \theta_{-i}) \text{ is budget-feasible}\}$ for all $i \in [n]$. First, we show that θ^*_i is budget-feasible. Observe that the function $\theta \mapsto \big(\sum_j p(\theta)_{1j}, \dots, \sum_j p(\theta)_{nj}\big)$ is continuous. Therefore, the pre-image of the set $\bigtimes_{i=1}^n [0, B_i]$ under this function is closed. In other words, the set of all budget-feasible throttling parameters is closed. Fix $\epsilon > 0$. For all $i \in [n]$, by the definition of θ^*_i, there exists $\theta^{(i)} \in [0,1]^n$ which is budget feasible and $\theta^{(i)}_i > \theta^*_i - \epsilon$. Iterative application of Lemma 28 yields the budget-feasibility of the vector θ defined by $\theta_i = \max_{k \in [n]} \theta^{(k)}_i$. Moreover, $\theta_i > \theta^*_i - \epsilon$ for all $i \in [n]$ because $\theta^{(i)}_i > \theta^*_i - \epsilon$ for all $i \in [n]$. Since $\epsilon > 0$ was arbitrary, we have shown that there exists a sequence of budget-feasible throttling parameters which converges to θ^*, which implies that θ^* is budget-feasible because the set of budget-feasible throttling parameters is closed.

Next, we show that θ^* also satisfies the 'No unnecessary pacing' property. Suppose there exists $i \in [n]$ such that $\sum_j p(\theta^*)_{ij} < B_i$ and $\theta^*_i < 1$. Then, by the continuity of $\theta \mapsto \sum_j p(\theta)_{ij}$, there exists θ_i such that $\theta^*_i < \theta_i < 1$ and $\sum_j p(\theta_i, \theta^*_{-i})_{ij} < B_i$, which contradicts the definition of θ^*. Therefore, for all $i \in [n]$, we have $\sum_j p(\theta^*)_{ij} < B_i$ implies $\theta^*_i = 1$. Thus, θ^* is a throttling equilibrium.

Finally, we prove uniqueness of θ^*. Suppose there is a throttling equilibrium $\theta \in [0,1]^n$ such that $\theta_i \neq \theta^*_i$ for some $i \in [n]$. Then, the set of buyers $C \subset [n]$ for whom $\theta^*_i > \theta_i$ is non-empty. Note that $\theta_i < 1$ for all $i \in C$. Hence, every buyer in C spends her entire budget under θ. On the other hand, since $\theta^*_i > \theta_i$ for all $i \in C$ and $\theta^*_i = \theta_i$ for $i \notin C$, the total payment made by buyers in C under θ^* is strictly greater than the payment under θ, which contradicts the budget-feasibility of θ^*. Therefore, θ^* is the unique throttling equilibrium. ◻

We conclude this subsection by noting that in Appendix E.1 we describe a throttling game for which the unique throttling equilibrium has irrational throttling parameters for some buyers. In

Algorithm 6: Dynamics for First-price Auction

Input: Throttling game $\big(n, m, (b_{ij}), (B_i)\big)$ and approximation parameter $\delta \in (0, 1/2)$
Initialize: $\theta_i = \min\{B_i/(2\sum_j b_{ij}), 1\}$ for all $i \in [n]$
While there exists a buyer $i \in [n]$ such that $\theta_i < 1 - \delta$ and $\sum_j p(\theta)_{ij} < (1 - \delta)B_i$:

- For all $i \in [n]$ such that $\theta_i < 1 - \delta$ and $\sum_j p(\theta)_{ij} < (1 - \delta)B_i$, set $\theta_i \leftarrow \theta_i/(1 - \delta)$

Return: θ

other words, rational throttling equilibrium need not always exist. Since irrational numbers cannot be represented exactly with a finite number of bits in the standard floating point representation, it leads us to consider algorithms for finding *approximate* throttling equilibrium instead.

6.2.2 An Algorithm for Computing Approximate First-Price Throttling Equilibria

In this subsection, we define a simple tâtonnement-style algorithm and prove that it yields an approximate throttling equilibrium in polynomial time.

Before proceeding to prove the correctness and efficiency of Algorithm 6, we note some of its properties. Typically, in online advertising auctions, buyers participate in a large number of auctions throughout their campaign, and the platform periodically updates their throttling parameters to ensure that they don't finish their budgets prematurely and loose out on valuable advertising opportunity. The above algorithm is especially suited for this setting due to its decentralized and easy-to-implement updates to the throttling parameter. Moreover, it also lends credence to the notion of throttling equilibrium as a solution concept because the update step in Algorithm 6 can be implemented independently by the buyers, resulting in decentralized dynamics which converge to a throttling equilibrium in polynomially-many steps. We refer the reader to [Bor+07] for a detailed model under which Algorithm 6 can be naturally interpreted as decentralized dynamics for online advertising auctions.

In the next lemma, we show that, throughout the run of Algorithm 6, all the buyers always satisfy their budget constraints.

Lemma 29. *Consider the run of Algorithm 6 on the throttling game $\big(n, m, (b_{ij}), (B_i)\big)$ and ap-*

proximation parameter $\delta \in (0, 1/2)$. Then, after every iteration of the while loop, we have $\sum_j p(\theta)_{ij} \leq B_i$ for all $i \in [n]$.

Proof. We prove the lemma using induction on the number of iterations of the while loop. Note that $\sum_j p(\theta)_{ij} \leq B_i$ for all $i \in [n]$ before the first iteration of the while loop by virtue of our initialization. Let θ and θ' represent the throttling parameters after the t-th iteration and the $(t+1)$-th iteration of the while loop. Suppose $\sum_j p(\theta)_{ij} \leq B_i$ for all $i \in [n]$. To complete the proof by induction, we need to show that $\sum_j p(\theta')_{ij} \leq B_i$ for all $i \in [n]$. Consider a buyer i. Suppose $\sum_j p(\theta)_{ij} \geq (1-\delta)B_i$. By the update step of the algorithm, $\theta'_i = \theta_i$ and $\theta'_j \geq \theta_j$ for $j \neq i$. Therefore,

$$\sum_j p(\theta')_{ij} = \sum_j \prod_{k:b_{kj}>b_{ij}} (1-\theta'_k)\theta'_i b_{ij} \leq \sum_j \prod_{k:b_{kj}>b_{ij}} (1-\theta_k)\theta_i b_{ij} = \sum_j p(\theta)_{ij} \leq B_i$$

On the other hand, suppose $\sum_j p(\theta)_{ij} < (1-\delta)B_i$. Then, by the update step of the algorithm, $\theta'_i \leq \theta_i/(1-\delta)$ and $\theta'_j \geq \theta_j$ for $j \neq i$. Therefore,

$$\sum_j p(\theta')_{ij} = \sum_j \prod_{k:b_{kj}>b_{ij}} (1-\theta'_k)\theta'_i b_{ij} \leq \frac{1}{1-\delta} \cdot \sum_j \prod_{k:b_{kj}>b_{ij}} (1-\theta_k)\theta_i b_{ij} = \frac{1}{1-\delta} \cdot \sum_j p(\theta)_{ij} < B_i$$

This completes the proof of $\sum_j p(\theta')_{ij} \leq B_i$ for all buyers $i \in [n]$. □

We conclude this subsection by proving the correctness and efficiency of Algorithm 6.

Theorem 16. *Given a throttling game $\big(n, m, (b_{ij}), (B_i)\big)$ and an approximation parameter $\delta \in (0, 1/2)$ as input, Algorithm 6 returns a δ-approximate throttling equilibrium in polynomial time.*

Proof. Since each iteration of the while loop only performs basic arithmetic operations, to establish a polynomial run-time complexity, it suffices to show that the while loop terminates in polynomially-many steps. Note that $c = \min_i \min\{B_i/(2\sum_j b_{ij}), 1\}$ is a lower bound on the initial value of the throttling parameter of every buyer. Due to the termination condition of the while loop and the update step, at each iteration of the while loop, there exists a buyer $i \in [n]$ whose throttling parameter is updated, i.e., there exists $i \in [n]$ such that $\theta_i < 1-\delta$ and $\theta_i \leftarrow \theta_i/(1-\delta)$. Therefore, the number of iteration of the while loop T satisfies the following relationships:

$$\frac{c}{(1-\delta)^{T/n}} \leq 1 \iff T \leq \frac{n \log(1/c)}{\log(1/(1-\delta))} \leq \frac{n \log(1/c)}{\delta}$$

The second sequence of inequalities upper bounds the number of iterations of the while loop by a polynomial function of the problem size.

To complete the proof, it suffices to show that at the termination of the while loop, θ is a δ-approximate throttling equilibrium. Budget-feasibility follows from Lemma 29, and the 'Not too much unnecessary throttling' condition is satisfied by virtue of the termination condition. □

6.3 Throttling in Second-price Auctions

In this section, we study throttling equilibria in second-price auctions. We begin by establishing the existence of exact throttling equilibria. Next, we show that, in contrast to first-price auctions, it is PPAD-hard to compute an approximate throttling equilibrium. To complete the characterization of its complexity, we then place the problem in PPAD. Moreover, we also show that, unlike first-price auctions, multiple throttling-equilibria can exist for second-price auctions, and finding the revenue-maximizing one is NP-hard. Finally, we complement these negative results with an efficient algorithm for the case when each good has at most two positive bids.

6.3.1 Existence of Second-Price Throttling Equilibria

The following theorem establishes the existence of an exact throttling equilibrium for every bid profile by invoking Brouwer's fixed-point theorem on an appropriately defined function.

Theorem 17. *For every throttling game* $\left(n, m, (b_{ij}), (B_i)\right)$*, there exists a throttling equilibrium* $\theta^* \in [0,1]^n$.

Proof. First, observe that we can write the expected payment of buyer i on good j under θ as

$$p(\theta)_{ij} = \sum_{\ell : b_{\ell j} < b_{ij}} b_{\ell j} \theta_i \theta_\ell \prod_{k \neq i : b_{kj} > b_{\ell j}} (1 - \theta_k) = \theta_i \cdot \sum_{\ell : b_{\ell j} < b_{ij}} b_{\ell j} \theta_\ell \prod_{k \neq i : b_{kj} > b_{\ell j}} (1 - \theta_k) = \theta_i \cdot p(1, \theta_{-i})_{ij} \tag{6.1}$$

Define $f : [0,1]^n \to [0,1]^n$ as

$$f_i(\theta) = \min \left\{ \frac{B_i}{\sum_j p(1, \theta_{-i})_{ij}}, 1 \right\} \quad \forall \theta \in [0,1]^n$$

where, we assume that $f_i(\theta) = 1$ if $\sum_j p(1, \theta_{-i})_{ij} = 0$. Note that $p(1, \theta_{-i})_{ij}$ is a continuous function of θ because it is a polynomial. Therefore, f is continuous as a function of θ and hence, by Brouwer's fixed-point theorem, there exists a θ^* such that $f(\theta^*) = \theta^*$. As a consequence, for all buyers $i \in [n]$, we get the following equivalent statements

$$f_i(\theta^*) = \theta_i^* \iff \theta_i^* \cdot \sum_j p(1, \theta_{-i}^*)_{ij} < B_i \text{ implies } \theta_i^* = 1 \iff \sum_j p(\theta^*)_{ij} < B_i \text{ implies } \theta_i^* = 1$$

where the last equivalence follows from equation 6.1. Moreover, by definition of f, we get

$$\theta_i^* = f_i(\theta^*) \leq \frac{B_i}{\sum_j p(1, \theta_{-i})_{ij}}$$

which in conjunction with equation 6.1 implies $\sum_j p(\theta^*)_{ij} \leq B_i$. Therefore, θ^* is a throttling equilibrium. □

Even though the above theorem establishes the existence of a throttling equilibrium for every throttling game, in Appendix E.1 we give an example of a throttling game for which all equilibria have a buyer with an irrational throttling parameter. This prompts us to study the problem of computing *approximate* throttling equilibria, which we do in the following subsections.

6.3.2 Complexity of Finding Approximate Second-Price Throttling Equilibria

In the previous subsection, by way of our existence proof, we reduced the problem of finding an approximate throttling equilibrium to that of finding a Brouwer fixed point of the function f; but this is of little use if we want to actually compute an approximate throttling equilibrium: no known algorithm can compute a Brouwer fixed point in polynomial time and it is believed to be a hard problem. This is because the problem of computing an approximate Brouwer fixed point is a complete problem for the class PPAD [Pap94]; informally, this means that it is as hard as any other problem in the class, such as computing Nash equilibria of bimatrix games [DGP09; CD06] or computing a market equilibrium under piece-wise linear concave utilities [CT09; VY11]. These problems have eluded a polynomial-time algorithm for decades despite intensive effort.

However, through our reduction we have only shown that the problem of computing an approximate throttling equilibrium is easier than the problem of computing a Brouwer fixed point by showing that any algorithm for the latter can be employed to solve the former. Perhaps, computing an approximate throttling equilibrium is strictly easier? Unfortunately, this is not the case and the goal of this subsection is to prove it. More precisely, we show that the problem of finding an approximate throttling equilibrium is PPAD-hard, which in informal terms means that it is as hard as any other problem in the class PPAD, in particular that of computing a Brouwer fixed point. Before stating the hardness result itself, we note a consequence of particular importance: Under the assumption that PPAD-hard problems cannot be solved in polynomial time, no dynamics can efficiently converge to an approximate throttling equilibrium in polynomial time, which is in stark contrast to throttling in first-price auctions. Now, we state the main result of the section.

Theorem 18. *There is a positive constant $\delta < 1$ such that the problem of finding a δ-approximate throttling equilibrium in a throttling game is PPAD-hard. This holds even when the number of buyers with non-zero bids for each good is at most three.*

The proof of Theorem 18 uses *threshold games*, introduced recently by [PP21]. They showed that the problem of finding an approximate equilibrium in a threshold game is PPAD-complete.

Definition 15 (Threshold game of [PP21])**.** *A threshold game is defined over a directed graph $\mathcal{G} = ([n], E)$. Each node $i \in [n]$ represents a player with strategy space $x_i \in [0, 1]$. Let N_i be the set of nodes $j \in [n]$ with $(j, i) \in E$. Then $(x_i : i \in [n]) \in [0, 1]^n$ is an ϵ-*approximate equilibrium *if every x_i satisfies*

$$x_i \in \begin{cases} [0, \epsilon] & \sum_{j \in N_i} x_j > 0.5 + \epsilon \\ [1 - \epsilon, 1] & \sum_{j \in N_i} x_j < 0.5 - \epsilon \\ [0, 1] & \sum_{j \in N_i} x_j \in [0.5 - \epsilon, 0.5 + \epsilon] \end{cases}$$

Theorem 19 (Theorem 4.7 of [PP21])**.** *There is a positive constant $\epsilon < 1$ such that the problem of finding an ϵ-approximate equilibrium in a threshold game is PPAD-hard. This holds even when the in-degree and out-degree of each node is at most three in the threshold game.*

Given a threshold game $\mathcal{G} = ([n], E)$, we let O_i denote the set of nodes $j \in V$ with $(i, j) \in E$. So $|N_i|, |O_i| \leq 3$ for every $i \in [n]$. To prove Theorem 18, we need to construct a throttling game $\mathcal{I}_{\mathcal{G}}$ from $\mathcal{G}$ such that any approximate throttling equilibrium of $\mathcal{I}_{\mathcal{G}}$ corresponds to an approximate equilibrium of the threshold game. Before rigorously diving into the construction, we give an informal description to build intuition.

With each node of $\mathcal{G}$, we will associate a collection of buyers and goods, with the goal of capturing the corresponding strategy and equilibrium conditions of the threshold game. Fix a node $i \in [n]$. We will define a strategy buyer $S(i)$ and set the strategy x_i for node i to be proportional to $1 - \theta_{S(i)}$. Next, in order to implement the equilibrium condition of the threshold game, our goal will be to define buyers and goods such that the linear form $\sum_{j \in N_i} x_j$ ends up being proportional to the total payment of a buyer who we will refer to as the threshold buyer $T(i)$. For each in-neighbour $j \in N_i$, we will define a neighbour good $G(i, j)$, for which the strategy buyer of the neighbour $S(j)$ places the highest bid of 6 and the threshold buyer $T(i)$ places a bid of 5. Furthermore, for a reason that will become clear shortly, the strategy buyer $S(i)$ places a bid of 4 on $G(i, j)$. With these bids, the payment made by the threshold buyer $T(i)$ on all the neighbour goods $\{G(i, j)\}_j$ is proportional

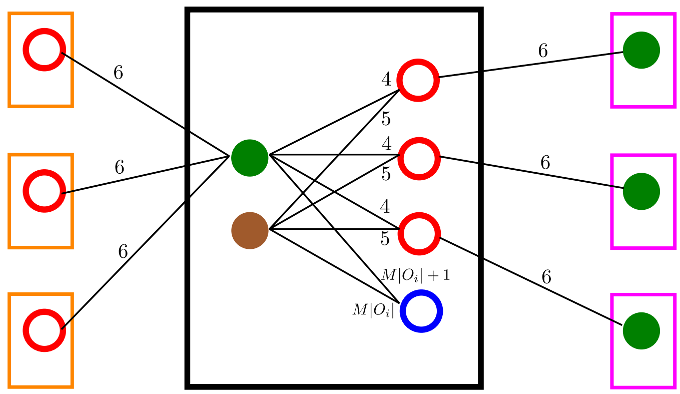

Figure 6.1: A diagrammatic representation of the non-zero bids made and received by the buyers and goods corresponding to a particular node of the threshold game. Consider a particular node of the threshold game, represented here by the black rectangle. It has three out neighbours, depicted as orange rectangles, and three in neighbours, depicted as pink rectangles. The hollow circles represent goods and the solid circles represent buyers. Corresponding to each node of the threshold game, there is one strategic buyer, shown here in green, and one threshold buyer, shown in brown. Furthermore, for each node of the threshold game, there are three neighbour goods, shown in red, and one reciprocal good, shown in blue. The lines represent non-zero bids.

to $\theta_{T(i)}\theta_{S(i)} \sum_{j \in N_i}(1 - \theta_S(j)) = \theta_{T(i)}\theta_{S(i)} \sum_{j \in N_i} x_j$. Now, if we are somehow able to ensure that the throttling parameter of the strategy buyer $\theta_{S(i)}$ is inversely proportional to the throttling parameter of the threshold buyer $\theta_{T(i)}$, then the payment made by the threshold buyer on all the neighbour goods will be proportional to $\sum_{j \in N_i} x_j$, as desired. To achieve this, we define a reciprocal good $R(i)$. Finally, we set the budget of the threshold buyer $T(i)$ in such a way that comparing it to her payment, which is proportional to $\sum_{j \in N_i} x_j$, is tantamount to making a comparison between $\sum_{j \in N_i} x_j$ and 0.5. The challenging part of the reduction lies in setting up the bids and budgets in a way that ensures that this comparison leads to an enforcement of the equilibrium condition of the threshold game.

With this high level overview of the reduction in place, we move on to the rigorous construction of the throttling game $\mathcal{I}_\mathcal{G}$ from $\mathcal{G}$. First, $\mathcal{I}_\mathcal{G}$ contains the following set of goods:

- For each $i \in [n]$ and $j \in N_i$, there is a *neighbor good* $G(i, j)$.
- For each $i \in [n]$, there is a *reciprocal good* $R(i)$.

Next, setting two constants M and δ as $M = 160/\delta$ and $\delta = \min\{\epsilon/(3+\epsilon), \epsilon/2, 1/4\}$, the throttling game $\mathcal{I}_{\mathcal{G}}$ has the following set of buyers:

- For each $i \in [n]$, there is a *threshold buyer* $T(i)$ who has budget $1/2$ and has non-zero bids only on the following goods: $b(T(i), G(i, j)) = 5$ for all $j \in N_i$; and $b(T(i), R(i)) = M|O_i|$.
- For each $i \in [n]$, there is a *strategy buyer* $S(i)$ who has budget $M|O_i|/2$ and has non-zero bids only on the following goods: $b(S(i), R(i)) = M|O_i|+1$; $b(S(i), G(i, j)) = 4$ for all $j \in N_i$; and $b(S(i), G(j, i)) = 6$ for all $j \in O_i$.

It is clear that $\mathcal{I}_{\mathcal{G}}$ can be constructed from $\mathcal{G}$ in polynomial time, and the number of buyers with non-zero bids for each good is at most three. Let θ be any δ-approximate throttling equilibrium of $\mathcal{I}_{\mathcal{G}}$ and use it to define $(x_i : i \in [n]) \in [0, 1]^n$ as follows:

$$x_i = \min\{2(1 - \theta_{S(i)}), 1\}, \quad \text{for all } i \in [n]. \tag{6.2}$$

To complete the reduction, we will show that $(x_i : i \in [n])$ is an ϵ-approximate equilibrium of the threshold game $\mathcal{G}$. Since we are considering a particular θ, we will suppress the dependence on θ of the payment made by buyer B on good G and simply denote it by $p(B, G)$. The next lemma notes the payment terms of buyers in $\mathcal{I}_{\mathcal{G}}$.

Lemma 30. *For all $i \in [n]$, we have*

1. $p(T(i), G(i, j)) = \left(1 - \theta_{S(j)}\right) \theta_{T(i)}\theta_{S(i)}4$, *for all $j \in N_i$; and* $p(T(i), R(i)) = 0$.
2. $p(S(i), R(i)) = \theta_{S(i)}\theta_{T(i)}M|O_i|$; $p(S(i), G(i, j)) = 0$, *for all $j \in N_i$; and*

$$p(S(i), G(j, i)) = \theta_{S(i)} \left[\theta_{T(j)}5 \ + \ (1 - \theta_{T(j)})\theta_{S(j)}4\right], \quad \textit{for all } j \in O(i).$$

In the next lemma, we bound the total payment made by strategy buyer $S(i)$ on the neighbor goods and provide lower bounds on the throttling parameters:

Lemma 31. *For all $i \in [n]$, we have $\theta_{S(i)} \geq (1-2\delta)/2$, $\theta_{T(i)} \geq 1/32$, and the total payment of $S(i)$ on the neighbor goods satisfies:*

$$|O_i|\theta_{S(i)} \leq \sum_{j \in N_i} p(S(i), G(i,j)) + \sum_{j \in O_i} p(S(i), G(j,i)) \leq 5|O_i|\theta_{S(i)}$$

Proof. For $i \in [n]$, using Lemma 30, we get

$$\sum_{j \in N_i} p(S(i), G(i,j)) + \sum_{j \in O_i} p(S(i), G(j,i)) = \sum_{j \in O_i} \theta_{S(i)} \left[\theta_{T(j)} 5 \ + \ (1-\theta_{T(j)})\theta_{S(j)} 4\right] \leq 5|O_i|\theta_{S(i)}.$$

Suppose $\theta_{S(i)} < (1-2\delta)/2$ for some $i \in [n]$. Then, the total payment made by $S(i)$ is at most

$$\theta_{S(i)}\theta_{T(i)} M|O_i| + 5|O_i|\theta_{S(i)} < \frac{(1-2\delta)M|O_i|}{2} + 5|O_i| < (1-\delta) \cdot \frac{M|O_i|}{2}$$

using $M > 10/\delta$ by our choice of M. This contradicts the definition of δ-approximate throttling equilibrium because $S(i)$ has a budget of $M|O_i|/2$. Therefore, we have $\theta_{S(i)} \geq (1-2\delta)/2$ and in particular, $\theta_{S(i)} \geq 1/4$ for all $i \in [n]$ using $\delta \leq 1/4$. Hence,

$$\begin{aligned}
\sum_{j \in N_i} p(S(i), G(i,j)) + \sum_{j \in O_i} p(S(i), G(j,i)) &= \sum_{j \in O_i} \theta_{S(i)} \left[\theta_{T(j)} 5 \ + \ (1-\theta_{T(j)})\theta_{S(j)} 4\right] \\
&> \sum_{j \in O_i} \theta_{S(i)} \left[\theta_{T(j)}\theta_{S(j)} 4 \ + \ (1-\theta_{T(j)})\theta_{S(j)} 4\right] \\
&= 4\theta_{S(i)} \cdot \sum_{j \in O_i} \theta_{S(j)} \geq |O_i|\theta_{S(i)}.
\end{aligned}$$

Suppose, $\theta_{T(i)} < 1/32$. By Lemma 30 the total payment of threshold buyer $T(i)$ is at most

$$\sum_{j \in N_i} 4\theta_{T(i)} \leq 12\theta_{T(i)} < \frac{3}{8} \leq \frac{1-\delta}{2}$$

using $|N_i| \leq 3$ and $\delta \geq 1/4$. Hence, we have obtained a contradiction with the definition of δ-approximate throttling equilibria. Therefore, $\theta_{T(i)} \geq 1/32$. □

We are now ready to complete the reduction.

Lemma 32. $(x_i : i \in [n])$, *as defined in (6.2), is an* ϵ*-approximate equilibrium of the threshold game* $\mathcal{G}$.

Proof. Fix an $i \in [n]$. First consider the case when $\sum_{j \in N_i} x_j > 0.5 + \epsilon$. Assume for a contradiction that $x_i > \epsilon$. Then, $\theta(S(i)) < 1 - (\epsilon/2) \leq 1 - \delta$ using $\delta \leq \epsilon/2$. By the definition of δ-approximate throttling equilibrium, the total payment of threshold buyer $S(i)$ is at least $(1-\delta)M|O_i|/2$. Combining this observation with Lemma 30 and Lemma 31, we get

$$\theta_{S(i)} \left[\theta_{T(i)} M|O_i| + 5|O_i|\right] \geq (1-\delta) \cdot \frac{M|O_i|}{2}$$

and thus (using $\theta_{T(i)} \geq 1/32$ from Lemma 31 and our choice of $M = 160/\delta$),

$$\theta_{S(i)} \geq \frac{1-\delta}{2\theta_{T(i)} + (10/M)} \geq \frac{1}{2\theta_{T(i)}} \cdot \frac{1-\delta}{1+\delta} \quad \Longrightarrow \quad \theta_{T(i)}\theta_{S(i)} \geq \frac{1}{2(1+2\epsilon)}$$

using $\delta \leq \epsilon/2$ and $\epsilon < 1$. Moreover, note that $\sum_{j \in N_i} x_j > 0.5 + \epsilon$ implies

$$\sum_{j \in N_i} 2\left(1 - \theta_{S(j)}\right) > (1+2\epsilon)/2$$

Combining the above statements allows us to bound the total payment of buyer $T(i)$:

$$\sum_{j \in N_i} \left(1 - \theta_{S(j)}\right) \theta_{T(i)}\theta_{S(i)} 4 \geq \frac{4}{2(1+2\epsilon)} \cdot \sum_{j \in N_i} \left(1 - \theta_{S(j)}\right) > \frac{1}{2}.$$

This yields a contradiction because $T(i)$ has budget $1/2$. Hence $x_i \leq \epsilon$ when $\sum_{j \in N_i} x_j > 0.5 + \epsilon$. Next consider the case of $\sum_{j \in N_i} x_j < 0.5 - \epsilon$. The budget constraint of $S(i)$ and Lemma 31 yield

$$\theta_{S(i)} \left[\theta_{T(i)} M|O_i| + |O_i|\right] \leq \frac{M|O_i|}{2}$$

which implies that

$$\theta_{S(i)} \left[\theta_{T(i)} M|O_i| + \theta_{T(i)}|O_i|\right] \leq \frac{M|O_i|}{2} \quad \Longrightarrow \quad \theta_{T(i)}\theta_{S(i)} \leq \frac{1}{2(1+(1/M))} < \frac{1}{2}. \tag{6.3}$$

By Lemma 31, we have $\theta_{S(j)} \geq (1-2\delta)/2$ and thus, $2(1-\theta_{S(j)}) \leq 1+2\delta$. Multiplying both sides by $(1-2\delta)$ yields $2(1-\theta_{S(j)})(1-2\delta) \leq 1-4\delta^2 < 1$. In other words, we have

$$2(1-\theta_{S(j)})(1-2\delta) < \min\{2(1-\theta_{S(j)}), 1\} = x_j \tag{6.4}$$

for every $j \in [n]$. This together with $\sum_{j\in N_i} x_j < 0.5-\epsilon$ implies

$$(1-2\delta)\sum_{j\in N_i} 2\left(1-\theta_{S(j)}\right) < (1-2\epsilon)/2.$$

Therefore, we get that the total payment of $T(i)$ satisfies the following bound

$$\sum_{j\in N_i}\left(1-\theta_{S(j)}\right)\theta_{T(i)}\theta_{S(i)}4 < \sum_{j\in N_i} 2\left(1-\theta_{S(j)}\right) < \frac{1-2\epsilon}{2(1-2\delta)} \leq (1-\delta)\cdot\frac{1}{2}$$

using $\delta \leq \epsilon/2$. As a consequence of the definition of δ-approximate throttling equilibria, we have that $\theta_{T(i)} \geq 1-\delta$. Finally, using (6.3) and (6.4), we have

$$x_i > 2(1-\theta_{S(i)})(1-2\delta) \geq 2\left(1-\frac{1}{2\theta_{T(i)}}\right)(1-2\delta) \geq \frac{(1-2\delta)^2}{1-\delta} > \frac{1-4\delta}{1-\delta} \geq 1-\epsilon,$$

where the last inequality follows from $\delta \leq \epsilon/(3+\epsilon)$. □

This completes the reduction, and thereby the proof of Theorem 18, because we have shown that for any δ-approximate throttling equilibrium of the throttling game $\mathcal{I}_{\mathcal{G}}$, the strategy $(x_i)_i$ is an ϵ-approximate equilibrium of the threshold game $\mathcal{G}$.

PPAD Membership of Approximate Second-Price Throttling Equilibria Next, we show that the problem of computing a δ-approximate throttling equilibrium belongs to PPAD by showing a reduction to BROUWER: the problem of computing an approximate fixed point of a Lipschitz continuous function from a n-dimensional unit cube to itself, which known to be in PPAD [CD06]. Its proof is motivated by the argument for existence of exact throttling equilibria given in Theorem 17 and can be found in Appendix E.2.1

Theorem 20. *The problem of computing an approximate throttling equilibrium is in PPAD.*

6.3.3 NP-hardness of Revenue Maximization under Throttling

To further strengthen our hardness result, next we establish the NP-hardness of computing the revenue-maximizing approximate throttling equilibrium. With revenue being one of the primary concerns of advertising platforms, this result provides further evidence of the computational difficulties which plague throttling equilibria in second-price auctions. We begin by defining the decision version of the revenue-maximization problem.

Definition 16 (REV)**.** *Given a throttling game G and target revenue R as input, decide if there exists a δ-approximate throttling equilibrium of G, for any $\delta \in [0, 1)$, with revenue greater than or equal to R.*

Note that we allow for arbitrarily bad approximations to the throttling equilibrium by allowing δ to be any number in $[0, 1)$. Theorem 21 states the problem of finding the revenue maximizing approximate throttling equilibrium is NP-hard. Its based on a reduction from 3-SAT to REV and has been relegated to Appendix E.2.2.

Theorem 21. *REV is NP-hard.*

6.3.4 An Algorithm for Second-Price Throttling Equilibria with Two Buyers Per Good

Next, we contrast the hardness results of the previous subsection with an algorithm for the case when each good receives at most two non-zero bids. Since goods with only one positive bid never result in a payment, without loss of generality, we can assume that every good has exactly two buyers with positive bids. More precisely, in this subsection, we will assume that $|\{i \in [n] \mid b_{ij} > 0\}| = 2$ for all $j \in [m]$. This special case demarcates the boundary of tractability for computing throttling equilibria in second-price auctions: Our PPAD-hardness result (Theorem 18) holds for the slightly more general case of each good receiving at most three positive bids. We begin by describing the algorithm (Algorithm 7).

Algorithm 7: Algorithm for the Two Buyer Case

Input: Throttling game $\big(n, m, (b_{ij}), (B_i)\big)$ and parameter $\gamma > 0$
Initialize: $\theta_i = \min\{B_i/(2\sum_j b_{ij}), 1\}$ for all $i \in [n]$
While there exists a buyer $i \in [n]$ such that $\theta_i < 1-\gamma$ and $\sum_j p(\theta)_{ij} < (1-\gamma)^3 B_i$:

1. For all $i \in [n]$ such that $\theta_i < 1-\gamma$ and $\sum_j p(\theta)_{ij} < (1-\gamma)^2 B_i$, set $\theta_i \leftarrow \theta_i/(1-\gamma)$
2. For all $i \in [n]$ such that $\sum_j p(\theta)_{ij} > B_i$, set $\theta_i \leftarrow (1-\gamma)\theta_i$

Return: θ

The following theorem, whose proof can be found in Appendix E.2.3, establishes the correctness and polynomial runtime of Algorithm 7.

Theorem 22. *Algorithm 7 returns a* $(1-3\gamma)$*-approximate throttling equilibrium in time which is polynomial in the size of the instance and* $1/\gamma$.

6.4 Comparing Pacing and Throttling

In this section, we compare two of the most popular budget management strategies: multiplicative pacing and throttling. First, we restate the definition of pacing equilibrium, as it appears in [Con+18; Con+19]. Under pacing, each buyer i has a pacing parameter α_i and, she bids $\alpha_i b_{ij}$ on good j. Let $p_j(\alpha)$ denote the price on good j when all of the buyers use pacing, i.e., $p_j(\alpha)$ is the highest (second-highest) element among $\{\alpha_i v_{ij}\}_i$ for first-price (second-price) auctions. Then, a tuple $((\alpha_i), (x_{ij}))$ of pacing parameters and allocations x_{ij} is called a pacing equilibrium if the following hold:

(a) Only buyers with the highest bid win the good: $x_{ij} > 0$ implies $\alpha_i v_{ij} = \max_i \alpha_i b_{ij}$.

(b) Full allocation of each good with a positive bid: $\max_i \alpha_i b_{ij} > 0$ implies $\sum_{i\in[n]} x_{ij} = 1$.

(c) Budgets are satisfied: $\sum_{j\in[m]} x_{ij} p_j(\alpha) \leq B_i$.

(d) No unnecessary pacing: $\sum_{j\in[m]} x_{ij} p_j(\alpha) < B_i$ implies $\alpha_i = 1$.

Comparing Pacing and Throttling in First-Price Auctions We begin with a comparison of pacing equilibria and throttling equilibria for first-price auctions. In [Con+19], the authors show that a unique pacing equilibrium always exists in first-price auctions and characterize it as the largest element in the collection of all budget-feasible vectors of pacing parameters. In Theorem 15, we show the analogous result for throttling using similar techniques. However, unlike pacing equilibrium, which is known to be rational [Con+19], there exist throttling games where the throttling equilibrium is irrational as we demonstrate through Example 10. Furthermore, in [Bor+07], the authors develop tâtonnement-style dynamics similar to those described in Algorithm 6, which converge to an approximate pacing equilibrium in polynomial time. In combination with Theorem 16, this provides evidence supporting the tractability of budget management for first-price auctions.

The uniqueness of pacing equilibrium and throttling equilibrium in first-price auctions is conducive to comparison, which we carry out for revenue. More specifically, in Theorem 23, we show that the revenue under the pacing equilibrium and the throttling equilibrium are always within a multiplicative factor of 2 of each other. Let REV(PE) and REV(TE) denote the revenue under the unique pacing equilibrium and the unique throttling equilibrium respectively.

Theorem 23. *For any throttling game $\big(n, m, (b_{ij}), (B_i)\big)$, the revenue from the pacing equilibrium and the revenue from the throttling equilibrium are always within a factor of 2 of each other, i.e., REV(PE) $\leq 2 \times$ REV(TE) and REV(TE) $\leq 2 \times$ REV(PE).*

Proof. Consider a throttling game $\big(n, m, (b_{ij}), (B_i)\big)$. Let $\theta = (\theta_i)_i$ be the unique throttling equilibrium (TE) and $\alpha = (\alpha_i)_i$ be the unique pacing equilibrium (PE) for this game. We will use $p_j(\theta)$ and $p_j(\alpha)$ to denote the (expected) payment made to the seller on good j under the TE and PE respectively. Then, REV(TE) $= \sum_j p_j(\theta)$ and REV(PE) $= \sum_j p_j(\alpha)$.

First, we show that REV(PE) $\leq 2 \times$ REV(TE). Let $N := \{i \in [n] \mid \theta_i = 1\}$ be the set of buyers who are not budget constrained under the TE. Moreover, define

$$M := \{j \in [m] \mid \exists\, i \in N \text{ such that } i \text{ wins a non-zero fraction of } j \text{ under the PE } \alpha\}$$

Note that, since $\theta_i = 1$ and $\alpha_i \le 1$ for all $i \in N$, we get that the TE yields a higher revenue for the seller on all goods in the set M, i.e., $p_j(\theta) \ge p_j(\theta)$ for all $j \in N$. Therefore, REV(TE) $\ge \sum_{j \in M} p_j(\theta) \ge \sum_{j \in M} p_j(\alpha)$. Furthermore, the definition of throttling equilibrium implies that every buyer $i \notin N$ spends her entire budget B_i under the TE. Hence, by our choice of M, we get $\sum_{j \notin M} p_j(\alpha) \le \sum_{i \notin N} B_i \le$ REV(TE). Combining the two statements yields REV(PE) $\le 2 \times$ REV(TE), as desired.

Next, we complete the proof by showing that REV(TE) $\le 2 \times$REV(PE). Let $S = \{i \in [n] \mid \alpha_i = 1\}$ be the set of buyers who are not budget constrained under the PE. Note that, for all $i \in S$ and $j \in [m]$, buyer i bids b_{ij} under the PE, which implies $p_j(\alpha) \ge \max_{i \in S} b_{ij}$ for all goods $j \in [m]$. Therefore, for any good $j \in [m]$, the total payment made by buyers in the set S under the TE is at most $p_j(\alpha)$. As a consequence, the total payment made by buyers in S under the TE is at most REV(PE). Furthermore, the buyers not in S completely spend their budget under the TE so the total payment made by buyers not in S under the TE is also at most REV(PE). Hence, we have the desired inequality REV(TE) $\le 2 \times$ REV(PE). □

In Appendix E.3, we give examples to demonstrate that REV(TE) can be arbitrarily close to $2 \times$ REV(PE), and REV(PE) can be arbitrarily close to $(4/3) \times$ REV(TE). In other words, for Theorem 23, the inequality REV(TE) $\le 2 \times$ REV(PE) is tight and the inequality REV(PE) $\le 2 \times$ REV(TE) is not too loose.

Comparing Pacing and Throttling for Second-Price Auctions This subsection is devoted to the comparison of pacing equilibria and throttling equilibria in second-price auctions. We begin by noting that, in stark contrast to first-price auctions, there could be infinitely many throttling equilibria for second-price auctions as the following example demonstrates.

Example 4. *There are 2 goods and 2 buyers. The bids are given by $b_{11} = b_{22} = 2$, $b_{12} = b_{21} = 1$, and the budgets are given by $B_1 = B_2 = 1/2$. Then, it is straightforward to check that any pair of throttling parameters $\theta_1, \theta_2 \in [1/2, 1]$ such that $\theta_1 \theta_2 = 1/2$ forms a throttling equilibrium.*

[Con+18] demonstrate that multiplicity (although only finitely-many different equilibria) also shows up for pacing equilibria in second-price auctions, which in combination with the multiplicity of throttling equilibria bodes unfavorably for potential comparisons of revenue. The similarities do not end with multiplicity: the problems of computing an approximate pacing equilibrium and computing an approximate throttling equilibrium are both PPAD-complete (Chapter 5). As a consequence, we get that, unlike first-price auctions, no dynamics can converge to an approximate equilibrium in polynomial time for second-price auctions under either budget-management approach (assuming PPAD-hard problems cannot be solved in polynomial time). Furthermore, finding the revenue maximizing throttling equilibrium and finding the revenue maximizing pacing equilibrium are both NP-hard problems [Con+18]. However, unlike throttling equilibria, a rational pacing equilibrium always exists (Chapter 5).

6.5 Price of Anarchy

In this section, we study the efficiency of throttling equilibria in first-price and second-price auctions. We will use liquid welfare [DL14] to measure efficiency. It is an alternative to social welfare which is more suitable for settings with budget constraints, and it reduces to social welfare when budgets are infinite.

Definition 17. *For an allocation $y = \{y_{ij}\}$, where $y_{ij} \in [0, 1]$ denotes the probability of allocating good j to buyer i, its liquid welfare $\mathrm{LW}(x)$ is defined as*

$$\mathrm{LW}(y) = \sum_{i=1}^{n} \min \left\{ \sum_{j=1}^{m} b_{ij} y_{ij}, B_i \right\}$$

Remark 8. *Liquid welfare is traditionally defined as the amount of revenue that can be extracted from budget-constrained buyers with full knowledge of their values. If buyer i was assumed to have value v_{ij} for good j, this is given by*

$$\sum_{i=1}^{n} \min \left\{ \sum_{j=1}^{m} v_{ij} y_{ij}, B_i \right\} .$$

However, since our model does not assume a valuation structure, we define LW(y) *to capture the amount of revenue that can be extracted from budget-constrained buyers with full knowledge of their bids if no buyer could be charged more than her bid for any good. It reverts to the traditional definition when* $b_{ij} = v_{ij}$.

Let $y(\theta)$ denote the allocation when the buyers use the throttling parameters $\theta = (\theta_1, \dots, \theta_n)$, and let Θ^* be the set of all throttling equilibria. Price of Anarchy [KP99], which we define next, is the ratio of the worst-case liquid welfare of throttling equilibria to the best-possible liquid welfare that can be attained by any allocation. It measures the worst-case loss in efficiency incurred due the strategic behavior of agents when compared to the optimal outcome that could be achieved by a central planner.

Definition 18. *The Price of Anarchy (PoA) of throttling equilibria for liquid welfare is given by*

$$PoA = \frac{\sup_{y \in (\Delta^n)^m} LW(y)}{\inf_{\theta \in \Theta^*} \mathrm{LW}(y(\theta))}$$

We begin by establishing an upper bound on the Price of Anarchy for both first-price and second-price auctions. Its proof critically leverages the no-unnecessary-throttling condition of throttling equilibria, and is inspired by the Price of Anarchy result of Chapter 4 for pacing equilibria.

Theorem 24. *For both first-price and second-price auctions, we have* $PoA \leq 2$.

Next, we show that the upper bound on the PoA established in Theorem 24 is tight for both first-price and second-price auctions. We do so by demonstrating particular instances for which the bound is tight, starting with the second-price auction format.

Example 5. *Consider a second-price auction with* $m + 1$ *buyers and* m *goods for some* $m \in \mathbb{Z}_+$. *Each of the first* m *buyers bid* 1 *for the* m *goods respectively and have a budget of* ∞, *i.e., for*

$i \in [m]$, we have

$$b_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} ,$$

and $B_i = \infty$ (any $B > 2m$ would suffice). The last buyer has bid $b_{m+1,j} = m$ for each of the goods $j \in [m]$ and has a budget of $B_{m+1} = m + \epsilon$ for some small $\epsilon > 0$. In any throttling equilibrium $\theta \in \Theta$, we have $\theta_i = 1$ for all $i \in [m]$ because of the no-unnecessary-throttling condition. Since the sum of all the second-highest bids is m and buyer $m + 1$ has the highest bid for every good, she cannot possibly spend her entire budget of $B_{m+1} = m + \epsilon$ and we must also have $\theta_{m+1} = 1$ by the no-unnecessary throttling condition. Therefore, there is a unique throttling equilibrium θ such that $\theta_i = 1$ for all $i \in [m + 1]$ and it has liquid welfare given by

$$\mathrm{LW}(y(\theta)) = \left(\sum_{i=1}^{m} \min \{y_{ii}(\theta), B_i\} \right) + \min \left\{ \sum_{j=1}^{m} m \cdot y_{m+1,j}(\theta), m + \epsilon \right\} = m + \epsilon$$

because $y_{m+1,j}(\theta) = 1$ for all $j \in [m]$. On the other hand, consider the allocation y such that $y_{ii} = 1$ for all $i \in [m - 1]$ and $y_{m+1,m} = 1$. It has liquid welfare given by

$$\mathrm{LW}(y) = \left(\sum_{i=1}^{m-1} \min \{y_{ii}, B_i\} \right) + \min \left\{ m \cdot y_{m+1,m}, m + \epsilon \right\} = m - 1 + m = 2m - 1 .$$

Hence, the PoA is at least $(2m - 1)/(m + \epsilon)$. As m and ϵ were arbitrary, we can consider the limit when $m \to \infty$ and $\epsilon \to 0$, which yields the required lower bound of PoA ≥ 2.

Observe that in the previous example none of the buyers were throttled ($\theta_i = 1$), which indicates that the lower bound is driven more by the second-price auction format than the specific budget management method, and applies to other methods like pacing. Next, we show that our bound is tight for first-price auctions.

Example 6. *Consider a first-price auction with $m + 1$ buyers and $m + 1$ goods, for some $m \in \mathbb{Z}_+$. Each of the first m buyers bid 1 for the first m goods respectively and bid m on good $m + 1$, and* *have a budget of* 1*, i.e., for each* $i \in [m]$*, we have*

$$b_{ij} = \begin{cases} 1 & \text{if } j = i \\ m & \text{if } j = m+1 \\ 0 & \text{otherwise} \end{cases},$$

and $B_i = 1$*. Moreover, buyer* $m+1$ *has value* $b_{m+1,m+1} = m$ *for the* $(m+1)$*-th good and* $b_{m+1,j} = 0$ *for all* $j \in [m]$*, with* $B_{m+1} = \infty$*.*

Consider a throttling equilibrium $\theta \in \Theta$*. We begin by showing that* $\theta_i < 1$ *for all* $i \in [m]$*. For contradiction, suppose not. Let i be the smallest index such that* $\theta_i = 1$*. Then, buyer i spends 1 on good i and spends* $m \cdot \prod_{k=1}^{i-1}(1-\theta_k) > 0$ *on good* $m+1$ *(we use the lexicographic tie-breaking rule), which makes her total expenditure strictly greater than her budget of* $B_i = 1$*, thereby yielding the required contradiction. Hence,* $\theta_i < 1$ *for all buyers* $i \in [m]$*, and consequently, the no-unnecessary-throttling condition implies that their total expected expenditure is exactly 1, i.e., the following equivalent statements hold*

$$\theta_i \cdot 1 + \left(\prod_{k=1}^{i-1}(1-\theta_k)\right) \cdot \theta_i \cdot m = 1 \iff \theta_i = \frac{1}{1 + \left(\prod_{k=1}^{i-1}(1-\theta_k)\right) \cdot m}. \tag{6.5}$$

Moreover, since their expenditure is $B_i = 1$*, that is also their contribution towards the liquid welfare. Let* $g(i) := \prod_{k=1}^{i}(1-\theta_k)$ *denote the probability that the first i buyers do not participate. Next, observe that* $\theta_{m+1} = 1$ *because of the no-unnecessary-throttling condition and* $B_{m+1} = \infty$*. Therefore, due to the lexicographic tie-breaking rule, buyer m wins good* $m+1$ *with probability* $g(m) = \prod_{k=1}^{m}(1-\theta_k)$*. Hence, the liquid welfare of* θ *is given by*

$$\mathrm{LW}(y(\theta)) = m \cdot 1 + g(m) \cdot m$$

On the other hand, the allocation y which awards good i to buyer i for all $i \in [m+1]$ *has* $\mathrm{LW}(y) =$

$m + m = 2m$. Consequently, we have

$$PoA \geq \frac{2m}{m + g(m) \cdot m} = \frac{2}{1 + g(m)} .$$

To show $PoA \geq 2$, it suffices to show $\lim_{m\to\infty} g(m) = 0$, which is what we do next. First, observe that (6.5) implies the following recursion for $g(\cdot)$:

$$g(i) = (1 - \theta_i)g(i-1) = \frac{\left(\prod_{k=1}^{i-1}(1-\theta_k)\right) \cdot m}{1 + \left(\prod_{k=1}^{i-1}(1-\theta_k)\right) \cdot m} \cdot g(i-1) = \frac{g(i-1)^2 \cdot m}{1 + g(i-1) \cdot m} .$$

We will inductively show that $g(i) \leq 1 - i/(m + \sqrt{m})$. Set $b = 1/(m + \sqrt{m})$ The base case $i = 1$ follows because $\theta_1 = 1/(1 + m)$ (see (6.5)). Suppose $g(i-1) \leq 1 - (i-1)/(m + \sqrt{m})$ for some $i \in [m]$. Then, we have

$$g(i) = \frac{g(i-1)^2 \cdot m}{1 + g(i-1) \cdot m} = \frac{m}{\frac{1}{g(i-1)^2} + \frac{m}{g(i-1)}} \leq \frac{m}{\frac{1}{(1-b(i-1))^2} + \frac{m}{1-b(i-1)}} = \frac{m \cdot (1 - bi + b)^2}{1 + m(1 - bi + b)} .$$

To complete the induction, it suffices to show:

$$\begin{aligned}
&\frac{m \cdot (1 - bi + b)^2}{1 + m(1 - bi + b)} \leq 1 - bi \\
\iff\ & m(1 + b^2 i^2 + b^2 - 2bi + 2b - 2b^2 i) \leq 1 - bi + m(1 + b^2 i^2 - 2bi + b - b^2 i) \\
\iff\ & 1 - bi + m(b^2 i - b - b^2) \geq 0 \\
\iff\ & 1 - m\left(\frac{1}{m + \sqrt{m}} + \frac{1}{(m + \sqrt{m})^2}\right) - i\left(\frac{1}{m + \sqrt{m}} - \frac{m}{(m + \sqrt{m})^2}\right) \geq 0 \\
\iff\ & 1 - m\left(\frac{1}{m + \sqrt{m}} + \frac{1}{(m + \sqrt{m})^2}\right) - i\left(\frac{\sqrt{m}}{(m + \sqrt{m})^2}\right) \geq 0
\end{aligned}$$

To see why the last inequality in the above equivalence chain holds, observe that:

$$\begin{aligned}
&1 - m\left(\frac{1}{m+\sqrt{m}} + \frac{1}{(m+\sqrt{m})^2}\right) - i\left(\frac{\sqrt{m}}{(m+\sqrt{m})^2}\right)\\
\geq &1 - m\left(\frac{1}{m+\sqrt{m}} + \frac{1}{(m+\sqrt{m})^2}\right) - m\left(\frac{\sqrt{m}}{(m+\sqrt{m})^2}\right)\\
= &\frac{(m+\sqrt{m})^2 - m(m+\sqrt{m}) - m - m\sqrt{m}}{(m+\sqrt{m})^2}\\
= &\frac{m^2 + m + 2m\sqrt{m} - m^2 - m\sqrt{m} - m - m\sqrt{m}}{(m+\sqrt{m})^2}\\
= &0
\end{aligned}$$

which completes the induction. Hence, $g(m) \leq 1 - m/(m+\sqrt{m})$ *and* $\lim_{m\to\infty} g(m) = 0$*, as required.*

6.6 Conclusion

We defined the notion of a throttling equilibrium and studied its properties for both first-price and second-price auctions. Through our analysis of computational and structural properties, we found that throttling equilibria in first-price auctions satisfy the desirable properties of uniqueness and polynomial-time computability. In contrast, we showed that for second-price auctions, equilibrium multiplicity may occur, and computing a throttling equilibrium is PPAD hard. This disparity between the two auction formats is reinforced when we compare throttling and pacing: our results show that the properties of throttling equilibrium across the two formats have a striking similarity to the properties of first-price versus second-price pacing equilibrium. Finally, we also showed that the Price of Anarchy of throttling equilibria for liquid welfare is bounded above by 2 for both first-price and second-price auctions, and that this bound is tight for both auction formats. Altogether, this provides a comprehensive analysis of the equilibria which arise from the use of throttling as a method of budget management.

There are many interesting directions for future work, such as what happens when a combination of pacing and throttling-based buyers exist in the market, whether the combination of throttling and pacing behaves well for second-price auctions, whether second-price throttling equilibria can be computed efficiently under some natural assumptions on the bids, and whether the tractability of budget management in first-price auctions holds more generally beyond throttling and pacing.

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Appendix A: Appendix to Chapter 2

A.1 Fluid Benchmark is Stronger

Proposition 10. *For any collection of request distributions* $\{\mathcal{P}_t\}_t$*, we have* $\mathbb{E}_{\{\gamma_t\}_t}[\mathrm{OPT}(\{\gamma_t\}_t)] \leq \mathrm{FLUID}(\{\mathcal{P}_t\}_t)$.

Proof. Fix any request sequence $\{\gamma_t\}_t$ and let $\{x_t^*\}_t$ be an optimal solution to the corresponding hindsight optimization problem $\mathrm{OPT}(\{\gamma_t\}_t)$. Then, $x_t(\gamma) = x_t^*$ for all $\gamma \in \mathbb{S}$ is a feasible solution of $\mathrm{FLUID}(\{\mathcal{P}_t\})$ and consequently, we have $\mathrm{OPT}(\{\gamma_t\}_t) \leq \mathrm{FLUID}(\{\mathcal{P}_t\}_t)$. Since the request sequence $\{\gamma_t\}_t$ was arbitrary, we have $\mathbb{E}_{\{\gamma_t\}_t}[\mathrm{OPT}(\{\gamma_t\}_t)] \leq \mathrm{FLUID}(\{\mathcal{P}_t\}_t)$, as required. □

A.2 General Position

Given any collection of request distributions $\{\mathcal{P}_t\}_t$ (which may or may not satisfy Assumption 1), we can define perturbed distributions $\{\hat{\mathcal{P}}_t\}_t$ to capture the distribution of perturbed requests $\hat{\gamma}_t = (\hat{f}_t, \hat{b}_t)$ generated using the following two step procedure: (i) Draw a request $\gamma_t = (f_t, b_t)$ according to the unperturbed distributions $\mathcal{P}_t$; (ii) Add a perturbation by setting $\hat{f}_t(x) = f_t(x) + \epsilon_t \cdot x$ for all $x \in \mathcal{X}$, where $\epsilon_t \sim \mathrm{Unif}([0, a])$, and leave the consumption function unchanged $\hat{b}_t(\cdot) = b_t(\cdot)$. Then, $\{\hat{\mathcal{P}}_t\}_t$ satisfy Assumption 1 and $\left|\mathrm{FLUID}(\{\mathcal{P}_t\}_t) - \mathrm{FLUID}(\{\hat{\mathcal{P}}_t\}_t)\right| \leq a \cdot T$, where $a > 0$ can be made arbitrarily small.

A.3 Efficiently Computing the Target Sequence

In this section, we describe an efficient procedure for computing the empirical optimal dual solution $\tilde{\mu}$ and the target sequence $\{\lambda_t\}_t$. Consider a trace $\{\tilde{\gamma}_t\}_t$ and set $\tilde{\gamma}_0 = (f_0, b_0)$ such that $f_0(x) = b_0(x) = 0$ for all $x \in \mathcal{X}$. Without loss of generality, we will assume that $\{\tilde{\gamma}_t\}_t$ is sorted in

Algorithm 8: Learning the Dual from the Trace

Input: Trace $\{\tilde{\gamma}_t\}_t$ in general position and sorted in increasing order of $\text{coeff}(\tilde{f}_t)/\text{coeff}(\tilde{b}_t)$.
Initialize: Total payment $P = 0$ and target sequence $\lambda_t = 0$ for all $t \in [T]$.
for $t = T, \ldots, 0$ **do**
  if $P + \tilde{b}_t(\bar{x}) > B$ **then**
    Set $\lambda_t \leftarrow \tilde{b}_t(\bar{x})$, and set $\tilde{\mu} = \text{coeff}(\tilde{f}_t)/\text{coeff}(\tilde{b}_t)$. **Break**.
  end
  else
    Update total payment $P \leftarrow P + \tilde{b}_t(\bar{x})$ and set $\lambda_t \leftarrow \tilde{b}_t(\bar{x})$.
  end
end
return *Dual variable* $\tilde{\mu}$, *target sequence* $\{\lambda_t\}_t$

increasing order of $\text{coeff}(\tilde{f}_t)/\text{coeff}(\tilde{b}_t)$ (assume $0/0 = 0$), i.e.,

$$\frac{\text{coeff}(\tilde{f}_s)}{\text{coeff}(\tilde{b}_s)} \leq \frac{\text{coeff}(\tilde{f}_t)}{\text{coeff}(\tilde{b}_t)} \quad \forall\, s \leq t\,.$$

This can be easily achieved by maintaining a sorted array with $O(\log(T))$ insertion time or sorting the array with $O(T\log(T))$ processing time. Moreover, since the trace $\{\tilde{\gamma}_t\}_t$ is in general position by Assumption 1, all the $\text{coeff}(\tilde{f}_t)/\text{coeff}(\tilde{b}_t)$ are distinct for $t \in [T]$.

Theorem 25. *$\tilde{\mu}$ returned by Algorithm 8 is the smallest element in*

$$\text{argmin}_{\mu\geq 0}\, \mu \cdot B + \sum_{t=1}^{T} \max_{x\in\mathcal{X}} \left\{\tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x)\right\}.$$

Moreover, $\lambda_t = b_t^*(\tilde{\mu})$ for all $t \in [T]$.

Proof. Set $q(\mu) = \mu \cdot B + \sum_{t=1}^{T} \max_{x\in\mathcal{X}} \left\{\tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x)\right\}$. First, we show that the dual variable $\tilde{\mu}$ is smallest element in $\text{argmin}_{\mu\geq 0}\, q(\mu)$. To do so, we consider the following two cases:

- $\tilde{\mu} = 0$. In this case, the 'If' condition implies that there exists $s \in [T]$ such that $\sum_{t=s}^{T} \tilde{b}_t(\bar{x}) \leq B$ and $\text{coeff}(\tilde{f}_t) = 0$ for all $t < s$. Moreover, we have $0 \in \text{argmax}_{x\in\mathcal{X}} \left\{\tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x)\right\}$ for all $t < s$ and $\bar{x} \in \text{argmax}_{x\in\mathcal{X}} \left\{\tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x)\right\}$ for all $t \geq s$. Now, note that the set of sub-gradients of the maximum of a collection of linear functions is equal to convex hull of gradients of all the linear functions which are binding (for example, see Chapter 5 of [Ber09]). Therefore, $B - \sum_{t=s}^{T} \tilde{b}(\bar{x}) \in \partial q(0)$. Since $B - \sum_{t=s}^{T} \tilde{b}(\bar{x}) \geq 0$, the definition of a subgradient implies that $q(0) \leq q(\mu)$ for all $\mu \geq 0$. Hence, we have shown that $\tilde{\mu}$ is the smallest minimizer of $q(\cdot)$, as required.

- $\tilde{\mu} > 0$. In this case, the 'If' condition implies that there exists $s \in [T]$ such that $\sum_{t=s+1}^{T} \tilde{b}_t(\bar{x}) < B$, $\sum_{t=s}^{T} \tilde{b}_t(\bar{x}) > B$ and $\tilde{f}_s(x) - \tilde{\mu} \cdot \tilde{b}_s(x) = 0$ for all $x \in \mathcal{X}$. Moreover, we have $0 \in \text{argmax}_{x \in \mathcal{X}} \left\{ \tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x) \right\}$ for all $t < s$, $\bar{x} \in \text{argmax}_{x \in \mathcal{X}} \left\{ \tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x) \right\}$ for all $t > s$ and $\{0, \bar{x}\} \subseteq \text{argmax}_{x \in \mathcal{X}} \left\{ \tilde{f}_s(x) - \mu \cdot \tilde{b}_s(x) \right\}$. Now, select a $\lambda \in [0, 1]$ such that

$$B - \lambda \cdot \tilde{b}_s(0) + (1 - \lambda) \cdot \tilde{b}_s(\bar{x}) + \sum_{t=s+1}^{T} \tilde{b}_t(\bar{x}) = 0 \,.$$

  Now, note that the set of sub-gradients of the maximum of a collection of linear functions is equal to convex hull of gradients of all the linear functions which are binding (for example, see Chapter 5 of [Ber09]). Therefore, $0 = B - \lambda \cdot \tilde{b}_s(0) + (1 - \lambda) \cdot \tilde{b}_s(\bar{x}) + \sum_{t=s+1}^{T} \tilde{b}_t(\bar{x}) \in \partial q(\tilde{\mu})$. Consequently, the definition of a subgradient implies that $q(0) \leq q(\mu)$ for all $\mu \geq 0$. Finally, consider any $\mu < \tilde{\mu}$. Then, $\tilde{f}_s(x) - \mu \cdot \tilde{b}_s(x) > 0$, which further implies $\{\bar{x}\} = \text{argmax}_{x \in \mathcal{X}} \left\{ \tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x) \right\}$ for all $t \geq s$. Therefore, for any $\{x_t\}_t$ such that $x_t \in \text{argmax}_{x \in \mathcal{X}} \left\{ \tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x) \right\}$, we have $B - \sum_{t=1}^{T} \tilde{b}_t(x_t) > 0$. Hence, $v > 0$ for all $v \in \partial q(\mu)$ and consequently μ is a minimizer of $q(\cdot)$. Hence, we have shown that $\tilde{\mu}$ is the smallest minimizer of $q(\cdot)$, as required.

Finally, we show that $\lambda_t = \tilde{b}_t^*(\tilde{\mu})$ for all $t \in [T]$. Let s be the value of t at which the 'For' loop terminates. From the definition of $\tilde{\mu}$, we have $\{\bar{x}\} = \text{argmax}_{x \in \mathcal{X}} \left\{ \tilde{f}_t(x) - \tilde{\mu} \cdot \tilde{b}_t(x) \right\}$ for all $t > s$, $\mathcal{X} = \text{argmax}_{x \in \mathcal{X}} \left\{ \tilde{f}_s(x) - \tilde{\mu} \cdot \tilde{b}_s(x) \right\}$ and $\{0\} = \text{argmax}_{x \in \mathcal{X}} \left\{ \tilde{f}_t(x) - \tilde{\mu} \cdot \tilde{b}_t(x) \right\}$ for all $t < s$. Therefore, $\tilde{b}_t^*(\tilde{\mu}) = \tilde{b}_t(\bar{x}) = \lambda_t$ for all $t \geq s$ and $\tilde{b}_t^*(\tilde{\mu}) = \tilde{b}_t(0) = \lambda_t$ for all $t < s$. □

A.4 Missing Proofs from Section 2.2

A.4.1 Proof of Lemma 1

Proof of Lemma 1. Define $q(\mu) = \mu \cdot B + \sum_{t=1}^{T} \max_{x\in\mathcal{X}} \left\{\tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x)\right\}$. Then, $q(\cdot)$ is a convex function of μ because the maximum of a collection of linear function is convex and the sum of convex function is also convex [Ber09]. Since $\tilde{\mu} \in \operatorname{argmin}_{\mu\geq 0} q(\mu)$, first-order condition of optimality (Proposition 5.4.7 of [Ber09]) implies that one of the following statements holds:

(i) $\tilde{\mu} = 0$ and there exists $v \in \partial q(0)$ such that $v \geq 0$.

(ii) Zero is a sub-differential of q at $\tilde{\mu}$, i.e., $0 \in \partial q(\tilde{\mu})$.

Recall that the trace $\{\tilde{\gamma}_t\}_t$ is assumed to be in general position with probability one. Therefore, there is at most one time step for which $\operatorname{argmax}_{x\in\mathcal{X}} f_t(x) - \tilde{\mu} \cdot b_t(x)$ is not unique. Let s be that time step. Now, note that the set of sub-gradients of the maximum of a collection of linear functions is equal to convex hull of gradients of all the linear functions which are binding (for example, see Chapter 5 of [Ber09]). Hence, $v \in \partial q(\tilde{\mu})$ implies the existence of $\mathbb{D}_s \in \Delta(\mathcal{X})$ such that

$$\text{Support}(\mathbb{D}_s) \subseteq \operatorname{argmax}_{x\in\mathcal{X}} \tilde{f}_s(x) - \tilde{\mu} \cdot \tilde{b}_s(x) \quad \text{and} \quad v = B - \mathbb{E}_{x\sim\mathbb{D}_s}[\tilde{b}_s(x)] - \sum_{t\neq s}^{T} \tilde{b}_t^*(\tilde{\mu})\,.$$

where $x_t^*(\gamma_t, \tilde{\mu})$ is the optimal solution to $\max_{x\in\mathcal{X}} \tilde{f}_t(x) - \tilde{\mu} \cdot \tilde{b}_t(x)$ as described in Definition 1. Since $0 \leq b_s(x) \leq \bar{b}$ for all $x \in \mathcal{X}$, we get

$$\left|B - v - \sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu})\right| \leq \bar{b}\,,$$

where we have used $0 \leq b_t(x) \leq \bar{b}$ for all $x \in \mathcal{X}$ and $t \in [T]$. Therefore, statements (i) and (ii) imply that either $\tilde{\mu} = 0$ and $\bar{b} + B - \sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu}) \geq v \geq 0$, or $\left|B - \sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu})\right| \leq \bar{b}$, as required. □

A.4.2 Proof of Theorem 1

Proof of Theorem 1. Define the hypothesis class

$$\mathcal{F} := \{(f, b) \mapsto b^*(\mu) \mid \mu \geq 0\}.$$

Let Rad(·) denote Radmacher complexity. Then, we know from PAC learning theory (for example see Chapter 26 of [SSBD14]) that

$$\Pr_{\{\tilde{\gamma}\}_t \sim \prod_t \tilde{\mathcal{P}}_t} \left(\sup_{\mu \geq 0} \left| \sum_{t=1}^{T} \tilde{b}_t^*(\mu) - \sum_{t=1}^{T} \mathbb{E}_{\hat{\gamma}_t \sim \tilde{\mathcal{P}}_t} \left[\hat{b}_t^*(\mu) \right] \right| \geq r(T) \right) \leq \frac{1}{T^2}. \tag{A.1}$$

for

$$r(T) \geq 2T \cdot \mathbb{E}_{\{\hat{\gamma}_t\}_t \sim \prod_t \tilde{\mathcal{P}}_t} \left[\mathrm{Rad}(\mathcal{F} \circ \{\hat{\gamma}_t\}_t) \right] + 2\bar{b} \cdot \sqrt{T \log(2T)}.$$

Let $H(\{\hat{\gamma}_t\}_t) = \left\{ \{\hat{b}_t^*(\mu)\}_t \mid \mu \geq 0 \right\}$ denote the set of all possible resource expenditure vectors that can be generated from a trace $\{\hat{\gamma}_t\}_t$, then

$$\begin{aligned} \mathbb{E}_{\{\hat{\gamma}_t\}_t \sim \prod_t \tilde{\mathcal{P}}_t} \left[\mathrm{Rad}(\mathcal{F} \circ \{\hat{\gamma}_t\}_t) \right] &= \mathbb{E}_{\{\hat{\gamma}_t\}_t \sim \prod_t \tilde{\mathcal{P}}_t} \left[\mathrm{Rad}(H(\{\hat{\gamma}_t\}_t)) \right] \\ &= \frac{1}{T} \cdot \mathbb{E}_{\{\hat{\gamma}_t\}_t \sim \prod_t \tilde{\mathcal{P}}_t} \mathbb{E}_{\vec{\sigma}} \left[\sup_{\mu \geq 0} \sum_{t=1}^{T} \sigma_t \cdot \hat{b}_t^*(\mu) \right], \end{aligned} \tag{A.2}$$

where $\{\sigma_t\}_t$ are independent Radmacher random variables.

For a linear function $f : \mathbb{R} \to \mathbb{R}$, let coeff$(f)$ denote its coefficient. Moreover, let $\bar{x} = \max_{x \in \mathcal{X}} x$. Then, observe that for a request $\gamma = (f, b)$ and dual variable $\mu \geq 0$, we have

$$x^*(\gamma, \mu) = \begin{cases} \bar{x} & \text{if } \mathrm{coeff}(f) - \mu \cdot \mathrm{coeff}(b) \geq 0 \text{ and } \mathrm{coeff}(f) \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, for any request $\gamma = (f, b)$ with coeff$(f) \neq 0$ and coeff$(b) \neq 0$, there exists a critical

$\mu^* = \text{coeff}(f)/\text{coeff}(b)$ such that

$$b^*(\mu) = \begin{cases} b(\bar{x}) & \text{if } \mu \le \mu^* \\ 0 & \text{if } \mu > \mu^* \end{cases}.$$

For $\gamma = (f, b)$ with $\text{coeff}(f) = 0$ or $\text{coeff}(b) \neq 0$, we have $b^*(\mu) = 0$ for all $\mu \geq 0$. Consider the trace $\{\hat{\gamma}_t\}_t$, and let μ_t^* be the critical dual solution for request $\hat{\gamma}_t$ as defined above. Then, the assumption that the trace is in general position (Assumption 1) implies that the critical points $\{\mu_t^*\}_t$ are distinct. Consequently, we get that $\{\hat{b}_t^*(\mu)\}_t$ remains constant whenever μ lies between any two critical points. Since the total number of critical points is T, we get that $|H(\{\hat{\gamma}_t\}_t)| \leq T$. Therefore, Massart Lemma applies and we get

$$\text{Rad}(H(\{\hat{\gamma}_t\}_t)) \leq \bar{b} \cdot \sqrt{\frac{2\log(T)}{T}}.$$

Combining this with (A.1) and (A.2) yields the theorem. □

A.4.3 Proof of Theorem 2

Proof of Theorem 2. By Assumption 1, the request sequence $\{\gamma_t\}$ is in general position almost surely. Therefore, there is at most 1 time step such that $f_t(x_t') \neq f_t^*(\tilde{\mu})$, call it s. Let ζ_A be the first time step t in which $B_{t+1} \leq \bar{b}$, i.e., $\sum_{t=1}^{\zeta_A} b_t^*(\tilde{\mu}) \geq B - \bar{b}$. Then,

$$\begin{aligned}
\mathbb{E}\left[R(A|\{\gamma_t\}_t)\right] &= \mathbb{E}\left[\sum_{t=1}^{\zeta_A} f_t(x_t')\right] \\
&\geq \mathbb{E}\left[\sum_{t=1}^{\zeta_A} f_t^*(\tilde{\mu})\right] - |f_s^*(\tilde{\mu}) - f_s(x_s')| \\
&\geq \mathbb{E}\left[\sum_{t=1}^{T} f_t^*(\tilde{\mu})\right] - \mathbb{E}\left[\sum_{t=\zeta_A+1}^{T} f_t^*(\tilde{\mu})\right] - \bar{f} \\
&\geq \mathbb{E}\left[\sum_{t=1}^{T} f_t^*(\tilde{\mu})\right] - \mathbb{E}\left[\kappa \cdot \sum_{t=\zeta_A+1}^{T} b_t^*(\tilde{\mu})\right] - \bar{f}
\end{aligned}$$

$$\geq D(\tilde{\mu}|\{\mathcal{P}_t\}_t) - \mathbb{E}\left[\tilde{\mu} \cdot \left(B - \sum_{t=1}^{T} b_t^*(\tilde{\mu})\right)\right] - \mathbb{E}\left[\kappa \cdot \sum_{t=\zeta_A+1}^{T} b_t^*(\tilde{\mu})\right] - \bar{f}$$
$$\geq \text{FLUID}(\{\mathcal{P}_t\}_t) - \mathbb{E}\left[\tilde{\mu} \cdot \left(B - \sum_{t=1}^{T} b_t^*(\tilde{\mu})\right)\right] - \mathbb{E}\left[\kappa \cdot \sum_{t=\zeta_A+1}^{T} b_t^*(\tilde{\mu})\right] - \bar{f}\,.$$

Therefore,

$$\text{Regret}(A) \leq \mathbb{E}\left[\kappa \cdot \sum_{t=\zeta_A+1}^{T} b_t^*(\tilde{\mu})\right] + \mathbb{E}\left[\tilde{\mu} \cdot \left(B - \sum_{t=1}^{T} b_t^*(\tilde{\mu})\right)\right] + \bar{f}\,.$$

In the remainder of the proof, we bound the first two terms on the RHS.

For the first term, observe that

$$\begin{aligned} \sum_{t=\zeta_A+1}^{T} b_t^*(\tilde{\mu}) &\leq \left(\sum_{t=1}^{T} b_t^*(\tilde{\mu})\right) - (B - \bar{b}) \\ &= \left(\sum_{t=1}^{T} b_t^*(\tilde{\mu}) - \sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu})\right) - \left(B - \sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu})\right) + \bar{b} \\ &\leq \left|\sum_{t=1}^{T} b_t^*(\tilde{\mu}) - \sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu})\right| + 2 \cdot \bar{b}\,, \end{aligned} \tag{A.3}$$

where the first inequality follows from the definition of ζ_A and the last inequality follows from Lemma 1.

For the second term, observe that Lemma 1 implies

$$\begin{aligned} \tilde{\mu} \cdot \left(B - \sum_{t=1}^{T} b_t^*(\tilde{\mu})\right) &= \tilde{\mu} \cdot \left(B - \sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu})\right) + \tilde{\mu} \cdot \left(\sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu}) - \sum_{t=1}^{T} b_t^*(\tilde{\mu})\right) \\ &\leq \tilde{\mu} \cdot \bar{b} + \tilde{\mu} \cdot \left|\sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu}) - \sum_{t=1}^{T} b_t^*(\tilde{\mu})\right|\,. \end{aligned}$$

Note that $\tilde{\mu} \leq \kappa$. This is because the definition of κ implies that $\max_{x \in \mathcal{X}} f(x) - \mu \cdot b(x) = 0$ for all $\gamma = (f, b) \in \mathbb{S}$ and $\mu \geq \kappa$. Hence,

$$\mu \cdot B + \sum_{t=1}^{T} \max_{x \in \mathcal{X}} \left\{\tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x)\right\} = \mu \cdot B < \kappa \cdot B = \kappa \cdot B + \sum_{t=1}^{T} \max_{x \in \mathcal{X}} \left\{\tilde{f}_t(x) - \kappa \cdot \tilde{b}_t(x)\right\}$$

for all $\mu > \kappa$. Therefore, we get

$$\tilde{\mu} \cdot \left(B - \sum_{t=1}^{T} b_t^*(\tilde{\mu}) \right) \leq \kappa \cdot \bar{b} + \kappa \cdot \left| \sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu}) - \sum_{t=1}^{T} b_t^*(\tilde{\mu}) \right| . \tag{A.4}$$

Define G to be the good event to be one in which the total expenditures under the trace and the requests sequence are close:

$$\sup_{\mu \geq 0} \left| \sum_{t=1}^{T} \tilde{b}_t^*(\mu) - \sum_{t=1}^{T} b_t^*(\mu) \right| \leq r(T) .$$

Then, Theorem 1 and Union Bound imply that $\Pr(G^c) \leq 2/T^2$ and $\Pr(G) \geq 1/2/T^2$. Finally, we can put it all together to get the required bound:

$$\begin{aligned}
\text{Regret}(A) &\leq \mathbb{E}\left[\kappa \cdot \sum_{t=\zeta_A+1}^{T} b_t^*(\tilde{\mu}) \right] + \mathbb{E}\left[\tilde{\mu} \cdot \left(B - \sum_{t=1}^{T} b_t^*(\tilde{\mu}) \right) \right] + \bar{f} \\
&\leq \mathbb{E}\left[\kappa \cdot \left| \sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu}) - \sum_{t=1}^{T} b_t^*(\tilde{\mu}) \right| \right] + 2\kappa\bar{b} + \mathbb{E}\left[\kappa \cdot \left| \sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu}) - \sum_{t=1}^{T} b_t^*(\tilde{\mu}) \right| \right] + \kappa\bar{b} + \kappa\bar{b} \\
&= 2\kappa \cdot \mathbb{E}\left[\left| \sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu}) - \sum_{t=1}^{T} b_t^*(\tilde{\mu}) \right| \,\middle|\, G \right] \Pr(G) + 2\kappa \cdot \mathbb{E}\left[\left| \sum_{t=1}^{T} \tilde{b}_t^*(\tilde{\mu}) - \sum_{t=1}^{T} b_t^*(\tilde{\mu}) \right| \,\middle|\, G^c \right] \Pr(G^c) + 4\kappa\bar{b} \\
&\leq 2\kappa \cdot r(T) + 2\kappa \cdot 2T\bar{b} \cdot \frac{2}{T^2} + 4\kappa\bar{b} \\
&\leq 12\kappa\bar{b} + 2\kappa r(T) .
\end{aligned}$$

□

A.5 Missing Proofs from Section 2.3

A.5.1 Proof of Theorem 3

Proof of Theorem 3. Let ζ_A be the first time less than T for which $\sum_{t=1}^{\zeta_A} b_t(x_t) + \bar{b} \geq B$. Set $\zeta_A = T$ if this inequality is never satisfied. Then, $x_t = x_t'$ for all $t \leq \zeta_A$ and $\sum_{t=1}^{\zeta_A} b_t(x_t') \geq B - \bar{b}$.

First, observe that

$$R(A|\{\gamma_t\}_t) \geq \sum_{t=1}^{\zeta_A} f_t(x_t') = \sum_{t=1}^{T} f_t(x_t') - \sum_{t=\zeta_A+1}^{T} f_t(x_t') \geq \sum_{t=1}^{T} f_t(x_t') - \kappa \cdot \sum_{t=\zeta_A+1}^{T} b_t(x_t') . \tag{A.5}$$

Next observe that, for all $t \in [T]$, μ_t is independent of γ_t because μ_t is completely determined by $\{\gamma_1, \ldots, \gamma_{t-1}\}$. Hence,

$$
\begin{aligned}
&\mathbb{E}_{\gamma_t}\left[f_t(x'_t) \mid \mu_t\right] \\
&= \mathbb{E}_{\gamma_t}\left[f_t(x'_t) + \mu_t \cdot (\beta_t - b_t(x'_t)) \mid \mu_t\right] - \mathbb{E}_{\gamma_t}\left[\mu_t \cdot (\lambda_t - b_t(x'_t)) \mid \mu_t\right] - \mathbb{E}_{\gamma_t}\left[\mu_t \cdot (\beta_t - \lambda_t) \mid \mu_t\right] \\
&= \mathbb{E}_{\gamma_t}\left[D(\mu_t|\mathcal{P}_t, \beta_t)|\mu_t\right] - \mathbb{E}_{\gamma_t}\left[\mu_t \cdot (\lambda_t - b_t(x'_t)) \mid \mu_t\right] - \mathbb{E}_{\gamma_t}\left[\mu_t \cdot (\beta_t - \lambda_t) \mid \mu_t\right] \,.
\end{aligned}
$$

Taking unconditional expectations on both sides and applying the tower rule yields

$$
\mathbb{E}\left[f_t(x'_t)\right] = \mathbb{E}\left[D(\mu_t|\mathcal{P}_t, \beta_t)\right] - \mathbb{E}\left[\mu_t \cdot (\lambda_t - b_t(x'_t))\right] - \mathbb{E}\left[\mu_t \cdot (\beta_t - \lambda_t)\right] \,.
$$

Summing over $t \in [T]$, we get

$$
\sum_{t=1}^{T} \mathbb{E}[f_t(x'_t)] = \sum_{t=1}^{T} \mathbb{E}\left[D(\mu_t|\mathcal{P}_t, \beta_t)\right] - \sum_{t=1}^{T} \mathbb{E}\left[\mu_t \cdot (\lambda_t - b_t(x'_t))\right] - \sum_{t=1}^{T} \mathbb{E}\left[\mu_t \cdot (\beta_t - \lambda_t)\right] \,. \quad \text{(A.6)}
$$

Therefore, (A.5) and (A.6) together imply

$$
\begin{aligned}
\mathbb{E}\left[\left\{\sum_{t=1}^{T} D(\mu_t|\mathcal{P}_t, \beta_t)\right\} - R(A|\{\gamma_t\}_t)\right] &\leq \mathbb{E}\left[\sum_{t=1}^{T} \mu_t \cdot (\lambda_t - b_t(x'_t)) + \kappa \cdot \sum_{t=\zeta_A+1}^{T} b_t(x'_t)\right] \\
&\quad + \sum_{t=1}^{T} \mathbb{E}[\mu_t \cdot (\beta_t - \lambda_t)] \,. \qquad \text{(A.7)}
\end{aligned}
$$

FTRL Regret Bound. Define $w_t(\mu) := \mu \cdot (\lambda_t - b_t(x'_t))$. Then, Algorithm 2 can be seen as running FTRL with linear losses $\{w_t(\cdot)\}_t$. The gradients of these loss functions are given by $\nabla w_t(\mu) = \lambda_t - b_t(x'_t)$, which satisfy $\|\nabla w_t(\mu)\|_\infty \leq \|b_t(x'_t)\|_\infty + \|\lambda_t\|_\infty \leq \bar{b} + \bar{\lambda}$. Therefore, the regret bound for FTRL implies that for all $\mu \geq 0$:

$$
\sum_{t=1}^{T} w_t(\mu_t) - w_t(\mu) \leq E(T, \mu) \,, \qquad \text{(A.8)}
$$

where $E(T, \mu) = \frac{2(\bar{b}+\bar{\lambda})^2}{\sigma}\eta \cdot T + \frac{h(\mu)-h(\mu_1)}{\eta}$ is the regret bound of FTRL after T iterations [Haz+16].

Equivalently, we can write

$$\sum_{t=1}^{T} \mu_t \cdot (\lambda_t - b_t(x'_t)) \leq E(T, \mu) + \sum_{t=1}^{T} \mu \cdot (\lambda_t - b_t(x'_t)) \qquad \forall\, \mu \geq 0.$$

Now, consider the following two cases:

- Case 1: $\zeta_A = T$. Here, setting $\mu = 0$ yields

$$\sum_{t=1}^{T} \mu_t \cdot (\lambda_t - b_t(x'_t)) + \kappa \cdot \sum_{t=\zeta_A+1}^{T} b_t(x'_t) \leq E(T, 0)\,.$$

- Case 2: $\zeta_A < T$. Then, $\sum_{t=1}^{\zeta_A} b_t(x'_t) \geq B - \bar{b}$. Hence, setting $\mu = \kappa$ yields

$$\begin{aligned}
\sum_{t=1}^{T} \mu_t \cdot (\lambda_t - b_t(x'_t)) + \kappa \cdot \sum_{t=\zeta_A+1}^{T} b_t(x'_t) &\leq E(T, \kappa) + \sum_{t=1}^{T} \kappa \cdot (\lambda_t - b_t(x'_t)) + \kappa \cdot \sum_{t=\zeta_A+1}^{T} b_t(x'_t) \\
&= E(T, \kappa) + \kappa \cdot \left(\sum_{t=1}^{T} \lambda_t - \sum_{t=1}^{\zeta_A} b_t(x'_t) \right) \\
&\leq E(T, \kappa) + \kappa \cdot \left(\left\{ \sum_{t=1}^{T} \lambda_t \right\} - (B - \bar{b}) \right) \\
&= E(T, \kappa) + \kappa \bar{b} + \kappa \cdot \left(\left\{ \sum_{t=1}^{T} \lambda_t \right\} - B \right)\,.
\end{aligned}$$

Combining the two cases implies that for all values of ζ_A we have

$$\sum_{t=1}^{T} \mu_t \cdot (\lambda_t - b_t(x'_t)) + \kappa \cdot \sum_{t=\zeta_A+1}^{T} b_t(x'_t) \leq \max\{E(T, 0), E(T, \kappa)\} + \kappa \bar{b} + \kappa \cdot \left(\left\{ \sum_{t=1}^{T} \lambda_t \right\} - B \right)^{+}\,. \tag{A.9}$$

Finally, combining (A.7) and (A.9) yields

$$\begin{aligned}
\mathbb{E}\left[\left\{ \sum_{t=1}^{T} D(\mu_t | \mathcal{P}_t, \beta_t) \right\} - R(A | \{\gamma_t\}_t) \right] \leq{}& \max\{E(T, 0), E(T, \kappa)\} + \kappa \bar{b} + \kappa \cdot \left(\left\{ \sum_{t=1}^{T} \lambda_t \right\} - B \right)^{+} \\
&+ \sum_{t=1}^{T} \mathbb{E}[\mu_t \cdot (\beta_t - \lambda_t)]\,.
\end{aligned}$$

Plugging in the definition of $E(T, \mu)$ finishes the proof. $\square$

A.5.2 Proof of Lemma 2

Proof of Lemma 2. Consider any two request distributions $\mathcal{P}, \tilde{\mathcal{P}} \in \Delta(\mathbb{S})$. Then, by the definition of the Wasserstein metric, there exists a joint probability distribution Q, with marginals $\mathcal{P}$ and $\tilde{\mathcal{P}}$ on the first and second factors respectively, such that

$$\mathcal{W}(\mathcal{P}, \tilde{\mathcal{P}}) = \mathbb{E}_{(\gamma,\tilde{\gamma})\sim Q}\left[\sup_x |f(x) - \tilde{f}(x)| + \sup_x |b(x) - \tilde{b}(x)|\right].$$

Let $x^*(\gamma, \mu)$ be the optimal solution of $\max_{x\in X} f(x) - \mu \cdot b(x)$ for request $\gamma = (f, b)$ as described in Definition 1. Then, for any $\mu \in [0, \kappa]$ and $x : \mathbb{S} \to \mathcal{X}$, we have

$$\begin{aligned}
&\mathbb{E}_{(\gamma,\tilde{\gamma})\sim Q}\left[\left|f(x(\gamma)) - \mu \cdot b(x(\gamma)) - \left\{\tilde{f}(x(\gamma)) - \mu \cdot \tilde{b}(x(\gamma))\right\}\right|\right] \\
\leq&\mathbb{E}_{(\gamma,\tilde{\gamma})\sim Q}\left[\left|f(x(\gamma)) - \tilde{f}(x(\gamma))\right| + \mu \cdot \left|b(x(\gamma)) - \tilde{b}(x(\gamma))\right|\right] \\
\leq&\mathcal{W}(\mathcal{P}, \tilde{\mathcal{P}}) + \kappa \cdot \mathcal{W}(\mathcal{P}, \tilde{\mathcal{P}}) \\
=&(1 + \kappa) \cdot \mathcal{W}(\mathcal{P}, \tilde{\mathcal{P}})\,.
\end{aligned} \tag{A.10}$$

Now, for $t \in [T]$, we can use (A.10) to write

$$\begin{aligned}
&D(\mu_t|\tilde{\mathcal{P}}, \beta_t) - D(\mu_t|\mathcal{P}, \beta_t) \\
=&\mathbb{E}_{\tilde{\gamma}\sim\tilde{\mathcal{P}}}[\tilde{f}(x^*(\tilde{\gamma}, \mu_t)) - \mu_t \cdot \tilde{b}(x^*(\tilde{\gamma}, \mu_t)) + \mu_t \cdot \beta_t] - \mathbb{E}_{\gamma\sim\mathcal{P}}[f(x^*(\gamma, \mu_t)) - \mu_t \cdot b(x^*(\gamma, \mu_t)) + \mu_t \cdot \beta_t] \\
\leq&\mathbb{E}_{\tilde{\gamma}\sim\tilde{\mathcal{P}},\gamma\sim\mathcal{P}}[\tilde{f}(x^*(\tilde{\gamma}, \mu_t)) - \mu_t \cdot \tilde{b}(x^*(\tilde{\gamma}, \mu_t)) + \mu_t \cdot \beta_t - \{f(x^*(\tilde{\gamma}, \mu_t)) - \mu_t \cdot b(x^*(\tilde{\gamma}, \mu_t)) + \mu_t \cdot \beta_t\}] \\
\leq&\mathbb{E}_{(\gamma,\tilde{\gamma})\sim Q}\left[\left|f(x^*(\tilde{\gamma}, \mu_t)) - \mu_t \cdot b(x^*(\tilde{\gamma}, \mu_t)) - \left\{\tilde{f}(x^*(\tilde{\gamma}, \mu_t)) - \mu_t \cdot \tilde{b}(x^*(\tilde{\gamma}, \mu_t))\right\}\right|\right] \\
\leq&(1 + \kappa) \cdot \mathcal{W}(\mathcal{P}, \tilde{\mathcal{P}})\,,
\end{aligned}$$

where the first inequality follows from the definition of $x^*(\gamma, \mu_t)$ and the second inequality follows from the fact that $(\mathcal{P}, \tilde{\mathcal{P}})$ are the marginals of Q. As a consequence, we get

$$
\begin{aligned}
\sum_{t=1}^{T} D(\mu_t|\mathcal{P}_t, \beta_t) &= \sum_{t=1}^{T} D(\mu_t|\tilde{\mathcal{P}}_t, \beta_t) - \sum_{t=1}^{T} \left\{ D(\mu_t|\tilde{\mathcal{P}}_t, \beta_t) - D(\mu_t|\mathcal{P}_t, \beta_t) \right\} \\
&\geq \sum_{t=1}^{T} D(\mu_t|\tilde{\mathcal{P}}_t, \beta_t) - (1+\kappa) \cdot \sum_{t=1}^{T} \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t) \\
&\geq \sum_{t=1}^{T} \mathrm{FLUID}(\tilde{\mathcal{P}}_t, \beta_t) - (1+\kappa) \cdot \sum_{t=1}^{T} \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t)\,, \qquad \text{(A.11)}
\end{aligned}
$$

where the second inequality follows from weak duality.

Next, observe that the definition of $\beta_t = \mathbb{E}_{\hat{\gamma} \sim \tilde{\mathcal{P}}_t}[\hat{b}^*(\tilde{\mu})]$ implies that $x^*(\tilde{\gamma}_t, \tilde{\mu})$ is a feasible to solution to the optimization problem which defines $\mathrm{FLUID}(\tilde{\mathcal{P}}_t, \beta_t)$. Hence,

$$
\begin{aligned}
\sum_{t=1}^{T} \mathrm{FLUID}(\tilde{\mathcal{P}}_t, \beta_t) &\geq \sum_{t=1}^{T} \mathbb{E}_{\tilde{\gamma}_t \sim \tilde{\mathcal{P}}_t} \left[\tilde{f}_t(x^*(\tilde{\gamma}_t, \tilde{\mu})) \right] \\
&= \sum_{t=1}^{T} \mathbb{E}_{\tilde{\gamma}_t \sim \tilde{\mathcal{P}}_t} \left[\tilde{f}_t(x^*(\tilde{\gamma}_t, \tilde{\mu})) - \tilde{\mu} \cdot \tilde{b}_t(x^*(\tilde{\gamma}_t, \tilde{\mu})) \right] + \tilde{\mu} \cdot \sum_{t=1}^{T} \mathbb{E}_{\tilde{\gamma}_t \sim \tilde{\mathcal{P}}_t} \left[\tilde{b}_t(x^*(\tilde{\gamma}_t, \tilde{\mu})) \right] .
\end{aligned}
$$

Let $\{x_t(\cdot)\}_t$ be an optimal solution for $\mathrm{FLUID}(\{\mathcal{P}_t\})$. Then, for all $t \in [T]$, we have

$$
\begin{aligned}
\mathbb{E}_{\tilde{\gamma}_t \sim \tilde{\mathcal{P}}_t} \left[\tilde{f}_t(x^*(\tilde{\gamma}_t, \tilde{\mu})) - \tilde{\mu} \cdot \tilde{b}_t(x^*(\tilde{\gamma}_t, \tilde{\mu})) \right] &= \mathbb{E}_{(\gamma_t, \tilde{\gamma}_t) \sim Q} \left[\tilde{f}_t(x^*(\tilde{\gamma}_t, \tilde{\mu})) - \tilde{\mu} \cdot \tilde{b}_t(x^*(\tilde{\gamma}_t, \tilde{\mu})) \right] \\
&\geq \mathbb{E}_{(\gamma_t, \tilde{\gamma}_t) \sim Q} \left[\tilde{f}_t(x_t(\gamma_t)) - \tilde{\mu} \cdot \tilde{b}_t(x(\gamma_t)) \right] \\
&\geq \mathbb{E}_{(\gamma_t, \tilde{\gamma}_t) \sim Q} \left[f_t(x_t(\gamma_t)) - \tilde{\mu} \cdot b_t(x(\gamma_t)) \right] - (1+\kappa) \cdot \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t) \\
&= \mathbb{E}_{\gamma_t \sim \mathcal{P}_t} \left[f_t(x_t(\gamma_t)) - \tilde{\mu} \cdot b_t(x(\gamma_t)) \right] - (1+\kappa) \cdot \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t)\,,
\end{aligned}
$$

where the first inequality follows from the definition of $x^*(\tilde{\gamma}_t, \tilde{\mu})$ and the second inequality follows from (A.10). Therefore,

$$
\begin{aligned}
&\sum_{t=1}^{T} \mathrm{FLUID}(\tilde{\mathcal{P}}_t, \beta_t) \\
&\geq \sum_{t=1}^{T} \mathbb{E}_{\gamma_t} \left[f_t(x_t(\gamma_t)) - \tilde{\mu} \cdot b_t(x(\gamma_t)) \right] - (1+\kappa) \cdot \sum_{t=1}^{T} \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t) + \tilde{\mu} \cdot \sum_{t=1}^{T} \mathbb{E}_{\tilde{\gamma}_t} \left[\tilde{b}_t(x^*(\tilde{\gamma}_t, \tilde{\mu})) \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{t=1}^{t} \mathbb{E}_{\gamma_t \sim \mathcal{P}_t}[f_t(x(\gamma_t))] - \tilde{\mu} \cdot \left(\sum_{t=1}^{T} \mathbb{E}_{\gamma_t}[b_t(x(\gamma_t))] - \sum_{t=1}^{T} \mathbb{E}_{\tilde{\gamma}_t}[\tilde{b}_t^*(\tilde{\mu})] \right) - (1+\kappa) \cdot \sum_{t=1}^{T} \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t) \\
&\geq \text{FLUID}(\{\mathcal{P}_t\}_t) - \tilde{\mu} \cdot \left(B - \sum_{t=1}^{T} \beta_t \right) - (1+\kappa) \cdot \sum_{t=1}^{T} \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t) \,,
\end{aligned}
$$

where the second inequality follows from the feasibility of the optimal solution $\{x_t(\cdot)\}_t$, i.e., $\sum_{t=1}^{T} \mathbb{E}_{\gamma_t}[b_t(x(\gamma_t))] \leq B$. Combining this with (A.11) yields

$$
\sum_{t=1}^{T} D(\mu_t | \mathcal{P}_t, \beta_t) \geq \text{FLUID}(\{\mathcal{P}_t\}_t) - \tilde{\mu} \cdot \left(B - \sum_{t=1}^{T} \beta_t \right) - 2(1+\kappa) \cdot \sum_{t=1}^{T} \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t) \,,
$$

as required. □

A.5.3 Proof of Lemma 4

Proof of Lemma 4. The definition of x and x' implies

$$
f(x) - \mu \cdot b(x) \geq f(x') - \mu \cdot b(x') \quad \text{and} \quad f(x') - \mu' \cdot b(x') \geq f(x) - \mu' \cdot b(x) \,.
$$

Combining the two inequalities, we get

$$
\begin{aligned}
& f(x) - \mu \cdot b(x) - \{f(x) - \mu' \cdot b(x)\} \geq f(x') - \mu \cdot b(x') - \{f(x') - \mu' \cdot b(x')\} \\
\Longrightarrow \; & (\mu - \mu') \cdot (b(x') - b(x)) \geq 0 \,.
\end{aligned}
$$

The lemma follows from the last inequality because $\mu - \mu' > 0$. □

A.5.4 Proof of Lemma 5

Proof of Lemma 5. Define

$$
q(\mu) := \mu \cdot B + \sum_{t=1}^{T} \max_{x \in \mathcal{X}} \left\{ \tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x) \right\} \quad \text{and} \quad q^{(-s)}(\mu) := \mu \cdot B + \sum_{t \neq s} \max_{x \in \mathcal{X}} \left\{ \tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x) \right\} \,.
$$

We start by proving $\tilde{\mu} \geq \tilde{\mu}^{(-s)}$. For contradiction, suppose $\tilde{\mu} < \tilde{\mu}^{(-s)}$. Consider the following two cases:

- Case I: $0 \in \text{argmax}_{x\in\mathcal{X}} \tilde{f}_s(x) - \tilde{\mu}\cdot\tilde{b}_s(x)$. Then, we must have $0 \in \text{argmax}_{x\in\mathcal{X}} \tilde{f}_s(x) - \mu\cdot\tilde{b}_s(x)$ for all $\mu \geq \tilde{\mu}$. This is because, for $\mu \geq \tilde{\mu}$, Lemma 4 implies that $\tilde{b}_s(x') \leq \tilde{b}_s(0) = 0$ for all $x' \in \text{argmax}_{x\in\mathcal{X}} \tilde{f}_s(x) - \mu\cdot\tilde{b}_s(x)$, and $\tilde{f}_s(x) \leq \kappa\cdot\tilde{b}_s(x)$ for all $x \in \mathcal{X}$. Therefore, $q(\mu) = q^{(-s)}(\mu)$ for all $\mu \geq \tilde{\mu}$. Since $\tilde{\mu}$ is a minimizer of $q(\cdot)$ and $\tilde{\mu}^{(-s)} > \tilde{\mu}$, we get that

$$q^{(-s)}(\tilde{\mu}) = q(\tilde{\mu}) \leq q(\tilde{\mu}^{(-s)}) = q^{(-s)}\left(\tilde{\mu}^{(-s)}\right).$$

  On the other hand, $\tilde{\mu}^{(-s)}$ is a minimizer of $q^{(-s)}(\cdot)$, which implies $q^{(-s)}\left(\tilde{\mu}^{(-s)}\right) \leq q^{(-s)}(\tilde{\mu})$. Therefore, $q^{(-s)}\left(\tilde{\mu}^{(-s)}\right) \leq q^{(-s)}(\tilde{\mu})$, which contradicts the fact that $\tilde{\mu}^{(-s)}$ is the smallest minimizer of $q^{(-s)}(\cdot)$.

- Case II: $0 \notin \text{argmax}_{x\in\mathcal{X}} \tilde{f}_s(x) - \tilde{\mu}\cdot\tilde{b}_s(x)$. Since $f_s(x) \leq \kappa\cdot b_s(x)$ for all $x \in \mathcal{X}$, we get that $\tilde{b}_s(x') > 0$ for all $x' \in \text{argmax}_{x\in\mathcal{X}} \tilde{f}_s(x) - \tilde{\mu}\cdot\tilde{b}_s(x)$. Consider any sequences of optimal action sequences $\{x_t\}_t$ and $\{x_t^{(-s)}\}_t$ such that for all $t \in [T]$, we have

$$x_t \in \text{argmax}_{x\in\mathcal{X}} \tilde{f}_t(x) - \tilde{\mu}\cdot\tilde{b}_t(x) \quad \text{and} \quad x_t^{(-s)} \in \text{argmax}_{x\in\mathcal{X}} \tilde{f}_t(x) - \tilde{\mu}^{(-s)}\cdot\tilde{b}_t(x).$$

  Then, Lemma 4 implies that $\tilde{b}_t(x_t) \geq \tilde{b}_t(x_t^{(-s)})$ for all $t \neq s$. Therefore,

$$B - \sum_{t=1}^{T}\tilde{b}_t(x_t) = \left\{B - \sum_{t\neq s}\tilde{b}_t(x_t)\right\} - b_t(x_t) < B - \sum_{t\neq s}\tilde{b}_t(x_t) \leq B - \sum_{t\neq s}\tilde{b}_t(x_t^{(-s)}). \quad \text{(A.12)}$$

  Now observe that, since $q(\cdot)$ (and $q^{(-s)}(\cdot)$) are the maxima of a collection of linear functions, its sub-gradient is given by the convex hull of gradients of all the linear functions which are binding (for example, see Chapter 5 of [Ber09]). Therefore, $\partial q(\tilde{\mu})$ (and $\partial q^{(-s)}\left(\tilde{\mu}^{(-s)}\right)$) is a convex hull of terms of the form $B - \sum_{t=1}^{T}\tilde{b}_t(x_t)$ for some optimal action sequence $\{x_t\}_t$ (and $\{x_t^{(-s)}\}_t$). Since $\tilde{\mu}^{(-s)} > \tilde{\mu} \geq 0$, first-order optimality conditions imply that $0 \in \partial q^{(-s)}\left(\tilde{\mu}^{(-s)}\right)$.

Therefore, (A.12) implies that $v < 0$ for all $v \in \partial q(\tilde{\mu})$. This contradicts the optimality of $\tilde{\mu}$ for $q(\cdot)$.

As we have obtained a contradiction in both cases, we get that $\tilde{\mu} \geq \tilde{\mu}^{(-s)}$, as required. Moreover, $\lambda_t \leq \lambda_t^{(-s)}$ for all $t \neq s$ follows immediately from Lemma 4. Hence, to finish the proof, it suffices to show the final inequality in the following chain:

$$\sum_{t=1}^{s-1} \left|\lambda_t^{(-s)} - \lambda_t\right| \leq \sum_{t \neq s} \left|\lambda_t^{(-s)} - \lambda_t\right| = \sum_{t \neq s} \lambda_t^{(-s)} - \sum_{t \neq s} \lambda_t \leq 3\bar{b}\,. \tag{A.13}$$

Note that, Lemma 1 implies that at least one of the following conditions hold

1. $\tilde{\mu} = 0$ and $\sum_{t=1}^{T} \lambda_t \leq B + \bar{b}$.
2. $\left|B - \sum_{t=1}^{T} \tilde{\lambda}_t\right| \leq \bar{b}$.

If $\tilde{\mu} = 0$, then $\tilde{\mu}^{(-s)} = 0$ because $\tilde{m}u^{(-s)} \leq \tilde{\mu}$. Therefore, in that case $\lambda_t^{(-s)} = \lambda_t = \tilde{b}_t^*(0)$ for all $t \neq s$ and (A.13) follows.

Suppose $\left|B - \sum_{t=1}^{T} \tilde{\lambda}_t\right| \leq \bar{b}$. Observe that Lemma 1 applied to the trace $\{\hat{\gamma}_t\}_t$, where $\hat{\gamma}_t = \tilde{\gamma}_t$ for all $t \neq s$ and $\hat{\gamma}_s = (0, 0)$, implies that at least one of the following conditions hold:

1. $\tilde{\mu}^{(-s)} = 0$ and $\sum_{t \neq s} \lambda_t^{(-s)} \leq B + \bar{b}$.
2. $\left|B - \sum_{t \neq s} \tilde{\lambda}_t^{(-s)}\right| \leq \bar{b}$.

Therefore, $\sum_{t \neq s} \lambda_t^{(-s)} - \sum_{t \neq s} \lambda_t \leq B + \bar{b} - \sum_{t \neq s} \lambda_t \leq \bar{b} + \bar{b} + \lambda_s \leq 3\bar{b}$, as required to establish (A.13). □

A.5.5 Proof of Lemma 6

Proof of Lemma 6. It is known that FTRL is equivalent to Lazy Online Mirror Descent (for example, see [Haz+16]). In particular, if we let $V_h(x, y) = h(x) - h(y) - \nabla h(y)^\top (x - y)$ denote the Bregman divergence w.r.t. $h(\cdot)$, then the FTRL update (2.3) of Algorithm 2 can be equivalently written as:

$$\theta_s = \nabla h(\mu_s)$$
$$\theta_{s+1} = \theta_t - \eta \cdot g_s = \theta_t - \eta \cdot (\lambda_s - b_s(x'_t))$$
$$\mu_{t+1} = \text{argmin}_{\mu \in [0,\kappa]} V_h(\mu, (\nabla h)^{-1}(\theta_{s+1}))\,.$$

where $x'_t \in \text{argmax}_{x \in \mathcal{X}} f_s(x) - \mu_s \cdot b_s(x)$. We will use $\{\mu'_t\}_t$ and $\{\theta'_t\}_t$ to represent the dual and mirror iterates of Algorithm 2 with target sequence $\{\lambda'_t\}_t$:

$$\theta'_s = \nabla h(\mu'_s)$$
$$\theta'_{s+1} = \theta'_t - \eta \cdot g_s = \theta'_t - \eta \cdot (\lambda'_s - b_s(y'_t))$$
$$\mu'_{t+1} = \text{argmin}_{\mu \in [0,\kappa]} V_h(\mu, (\nabla h)^{-1}(\theta'_{s+1}))\,.$$

where $y'_t \in \text{argmax}_{x \in \mathcal{X}} f_s(x) - \mu'_s \cdot b_s(x)$.

We will first use induction on s to prove the following statement,

$$\left|\theta_s - \theta'_s\right| \leq \eta \cdot \left\{\sum_{t=1}^{s-1} |\lambda_t - \lambda'_t|\right\} + \eta \cdot \bar{b}\,. \tag{A.14}$$

The base case $s = 1$ follows directly from our assumption that the initial iterates $\theta_1 = \nabla h(\mu_1) = \nabla h(\mu'_1) = \theta_2$ are the same.

Suppose (A.14) holds for $s \in [T-1]$ (Induction Hypothesis). Define $\theta_{s+1/2} = \theta_s + \eta \cdot b_s(x'_t)$ and $\theta'_{s+1/2} = \theta'_s + \eta \cdot b_s(y'_t)$. W.l.o.g. assume that $\theta_s \geq \theta'_s$. Due to the invertibility of ∇h, we get that $\mu_s \geq \mu'_s$, and consequently Lemma 4 implies $b(x'_t) \leq b(y'_t)$. Consider the following cases:

- Case I: $\theta'_{s+1/2} \leq \theta_{s+1/2}$. Then, $\theta_{s+1/2} - \theta'_{s+1/2} = \theta_s - \theta'_s + \eta \cdot (b(x'_t) - b(y'_t)) \leq \theta_s - \theta'_s$ because $b(x'_t) \leq b(y'_t)$.

- Case II: $\theta'_{s+1/2} \geq \theta_{s+1/2}$. Then, $\theta'_{s+1/2} - \theta_{s+1/2} = \theta'_s - \theta_s + \eta \cdot (b(x'_t) - b(y'_t)) \leq \eta \cdot \bar{b}$ because $\theta'_s \leq \theta_s$ and $b(y'_t) - b(x'_t) \leq \bar{b}$.

Therefore, in both cases we have

$$|\theta_{s+1/2} - \theta'_{s+1/2}| \leq \max\{\theta_s - \theta'_s, \eta \cdot \bar{b}\} \leq \eta \cdot \left\{\sum_{t=1}^{s-1} |\lambda_t - \lambda'_t|\right\} + \eta \cdot \bar{b}\,.$$

where we used the induction hypothesis in the second inequality. Consequently, we can write

$$\begin{aligned}
|\theta_{s+1} - \theta'_{s+1}| &= \left|\theta_{s+1/2} - \eta \cdot \lambda_s + (\theta'_{s+1/2} - \eta \cdot \lambda'_s)\right| \\
&\leq |\theta_{s+1/2} - \theta'_{s+1/2}| + \eta \cdot |\lambda_s - \lambda'_s| \\
&\leq \eta \cdot \left\{\sum_{t=1}^{s} |\lambda_t - \lambda'_t|\right\} + \eta \cdot \bar{b}\,.
\end{aligned}$$

Hence, we have established (A.14) for all $s \in [T]$. Now, since h is σ-strongly convex and differentiable, we have

$$\nabla h(x) - \nabla h(y) \geq \sigma \cdot (x - y) \qquad \forall\, x \geq y\,.$$

Therefore, we have

$$\left|(\nabla h)^{-1}(\theta_s) - (\nabla h)^{-1}(\theta'_s)\right| \leq \frac{1}{\sigma} \cdot |\theta_s - \theta'_s|\,. \tag{A.15}$$

To finish the proof, we will use the fact that Bregman projections are contractions in one dimensions, which we prove next. Consider any $x < 0$, then for any $\mu \in [0, \kappa]$, we have

$$V_h(\mu, x) - V_h(0, x) = h(\mu) - h(0) - \nabla h(x)^\top (\mu - 0) \geq h(\mu) - h(0) - \nabla h(0)^\top (\mu - 0) \geq 0\,,$$

where the inequality follows from $\nabla h(0) \geq \nabla h(x)$ (convexity of $h(\cdot)$). Therefore,

$$\operatorname{argmin}_{\mu \in [0,\kappa]} V_h(\mu, x) = 0 = \operatorname{argmin}_{\mu \in [0,\kappa]} |\mu - x|\,.$$

Similarly, for $x > \kappa$ and $\mu \in [0, \kappa]$, we have

$$V_h(\mu, x) - V_h(\kappa, x) = h(\mu) - h(\kappa) - \nabla h(x)^\top(\mu - \kappa) \geq h(\mu) - h(\kappa) - \nabla h(\kappa)^\top(\mu - \kappa) \geq 0\,,$$

where the inequality follows from $\nabla h(x) \geq \nabla h(\kappa)$ (convexity of $h(\cdot)$). Therefore,

$$\operatorname{argmin}_{\mu\in[0,\kappa]} V_h(\mu, x) = \kappa = \operatorname{argmin}_{\mu\in[0,\kappa]}|\mu - x|\,.$$

Consequently, we have shown that $\operatorname{argmin}_{\mu\in[0,\kappa]} V_h(\mu, x) = \operatorname{argmin}_{\mu\in[0,\kappa]}|\mu - x|$, i.e., the Bregman project is identical to the Euclidean projection in one dimension. Since Euclidean projection is a contraction, we get

$$\begin{aligned}\left|\mu_s - \mu'_s\right| &= \left|\operatorname{argmin}_{\mu\in[0,\kappa]} V_h(\mu, (\nabla h)^{-1}(\theta_{s+1})) - \operatorname{argmin}_{\mu\in[0,\kappa]} V_h(\mu, (\nabla h)^{-1}(\theta'_{s+1}))\right| \\ &= \left|\operatorname{argmin}_{\mu\in[0,\kappa]}|\mu - (\nabla h)^{-1}(\theta_{s+1}))| - \operatorname{argmin}_{\mu\in[0,\kappa]}|\mu - (\nabla h)^{-1}(\theta'_{s+1}))|\right| \\ &\leq \left|(\nabla h)^{-1}(\theta_s) - (\nabla h)^{-1}(\theta'_s)\right|\,.\end{aligned}$$

Finally, combining this with (A.14) and (A.15), we get

$$|\mu_s - \mu'_s| \leq \frac{\eta}{\sigma}\cdot\left\{\sum_{t=1}^{s-1}|\lambda_t - \lambda'_t|\right\} + \frac{\eta}{\sigma}\cdot\bar{b}\,,$$

as required. □

A.5.6 Proof of Lemma 7

Proof of Lemma 7. Using the definitions of λ_s and β_s, we can write

$$\mathbb{E}\left[\lambda_s|\tilde{\mu}^{(-s)}\right] = \mathbb{E}\left[\tilde{b}^*_s(\tilde{\mu})\middle|\,\tilde{\mu}^{(-s)}\right] \quad \text{and} \quad \mathbb{E}\left[\beta_s|\,\tilde{\mu}^{(-s)}\right] = \mathbb{E}\left[\mathbb{E}_{\hat{\gamma}\sim\tilde{\mathcal{P}}_t}\left[\hat{b}^*_s(\tilde{\mu})\right]\middle|\,\tilde{\mu}^{(-s)}\right]\,.$$

Fix a trace $\{\tilde{\gamma}_t\}_t$. Observe that for any request

$$x^*(\tilde{\gamma}_s, \tilde{\mu}) = \begin{cases} \bar{x} & \text{if } \operatorname{coeff}(\tilde{f}_s) - \tilde{\mu} \cdot \operatorname{coeff}(\tilde{b}_s) \geq 0 \text{ and } \operatorname{coeff}(\tilde{f}_s) \neq 0 \\ 0 & \text{otherwise} \end{cases},$$

and

$$x^*(\tilde{\gamma}_s, \tilde{\mu}^{(-s)}) = \begin{cases} \bar{x} & \text{if } \operatorname{coeff}(\tilde{f}_s) - \tilde{\mu}^{(-s)} \cdot \operatorname{coeff}(\tilde{b}_s) \geq 0 \text{ and } \operatorname{coeff}(\tilde{f}_s) \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

From Lemma 5, we know that $\tilde{\mu}^{(-s)} \leq \tilde{\mu}$. Now, if $\operatorname{coeff}(\tilde{f}_s) = 0$, then $x^*(\tilde{\gamma}_s, \tilde{\mu}) = x^*(\tilde{\gamma}_s, \tilde{\mu}^{(-s)}) = 0$. Assume that $\operatorname{coeff}(\tilde{f}_s) > 0$ (and thus $\operatorname{coeff}(\tilde{b}_s) > 0$ because $f_s(x) \leq \kappa \cdot b(x)$), let $A := \{\mu \geq 0 \mid \operatorname{coeff}(\tilde{f}_s) - \mu \cdot \operatorname{coeff}(\tilde{b}_s) < 0\}$ be the set of all dual variables that lead to $x^*(\tilde{\gamma}_s, \mu) = 0$.

Define the dual functions:

$$q(\mu) := \mu \cdot B + \sum_{t=1}^{T} \max_{x \in \mathcal{X}} \left\{ \tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x) \right\} \text{ and } q^{(-s)}(\mu) := \mu \cdot B + \sum_{t \neq s} \max_{x \in \mathcal{X}} \left\{ \tilde{f}_t(x) - \mu \cdot \tilde{b}_t(x) \right\}.$$

For contradiction, suppose $\tilde{\mu} \in A$ and $\tilde{\mu}^{(-s)} \notin A$. Since A is open, there exists a point $\mu \in A$ such that $\mu = \alpha \cdot \tilde{\mu} + (1 - \alpha) \cdot \tilde{\mu}^{(-s)}$ for some $\alpha \in (0, 1)$. Moreover, observe that the minimality of $\tilde{\mu}^{(-s)}$ implies $q^{(-s)}\left(\tilde{\mu}^{(-s)}\right) \leq q^{(-s)}(\mu)$ and $q^{(-s)}\left(\tilde{\mu}^{(-s)}\right) \leq q^{(-s)}(\tilde{\mu})$. Therefore, as $q^{(-s)}$ is convex, we get

$$q^{(-s)}(\mu) \leq \alpha \cdot q^{(-s)}(\tilde{\mu}) + (1 - \alpha) \cdot q^{(-s)}\left(\tilde{\mu}^{(-s)}\right) \leq \alpha \cdot q^{(-s)}(\tilde{\mu}) + (1 - \alpha) \cdot q^{(-s)}(\tilde{\mu}) = q^{(-s)}(\tilde{\mu}).$$

Now, observe that $q(\mu) = q^{(-s)}(\mu)$ for all $\mu \in A$. Therefore, $q(\mu) \leq q(\tilde{\mu})$, which contradicts the fact that $\tilde{\mu}$ is the smallest minimizer of $q(\cdot)$. Hence, either $\tilde{\mu}, \tilde{\mu}^{(-s)} \in A$ or $\tilde{\mu}, \tilde{\mu}^{(-s)} \in A^c$, and as a consequence, we get $\tilde{b}_s^*(\tilde{\mu}) = \tilde{b}_s^*\left(\tilde{\mu}^{(-s)}\right)$. Furthermore, combining $\tilde{\mu} \geq \tilde{\mu}^{(-s)}$ (from Lemma 5) and

Lemma 4, we also get $\hat{b}_s^*\left(\tilde{\mu}^{(-s)}\right) \geq \hat{b}_s^*(\tilde{\mu})$ for every $\hat{\gamma}_s \in \mathbb{S}$. Therefore,

$$
\begin{aligned}
\mathbb{E}\left[\lambda_s|\tilde{\mu}^{(-s)}\right] &= \mathbb{E}\left[\tilde{b}_s^*(\tilde{\mu})\middle|\, \tilde{\mu}^{(-s)}\right] \\
&= \mathbb{E}\left[\tilde{b}_s^*\left(\tilde{\mu}^{(-s)}\right)\middle|\, \tilde{\mu}^{(-s)}\right] \\
&= \mathbb{E}_{\hat{\gamma}_s\sim\tilde{\mathcal{P}}_s}\left[\hat{b}_s^*\left(\tilde{\mu}^{(-s)}\right)\middle|\, \tilde{\mu}^{(-s)}\right] \\
&\geq \mathbb{E}\left[\mathbb{E}_{\hat{\gamma}_s\sim\tilde{\mathcal{P}}_s}\left[\hat{b}_s^*(\tilde{\mu})\middle|\, \tilde{\mu}^{(-s)}\right]\right] \\
&= \mathbb{E}\left[\beta_s|\, \tilde{\mu}^{(-s)}\right] ,
\end{aligned}
$$

where the third equality follows from the fact that $\tilde{\gamma}_s$ and $\tilde{\mu}^{(-s)}$ are independent of each other, which allows us to rename the variable from $\tilde{\gamma}_s \sim \tilde{\mathcal{P}}_s$ to $\hat{\gamma}_s \sim \tilde{\mathcal{P}}_s$. We combine this with the Tower Property of conditional expectations to finish the proof:

$$
\mathbb{E}\left[\mu_s^{(-s)}\cdot(\beta_s-\lambda_s)\right] = \mathbb{E}\left[\mu_s^{(-s)}\cdot\mathbb{E}\left[(\beta_s-\lambda_s)\middle|\,\tilde{\mu}^{(-s)}\right]\right] = \mathbb{E}\left[\mu_s^{(-s)}\cdot\left(\mathbb{E}\left[\beta_s\middle|\,\tilde{\mu}^{(-s)}\right]-\mathbb{E}\left[\lambda_s\middle|\,\tilde{\mu}^{(-s)}\right]\right)\right] \leq 0\,.
$$

□

A.5.7 Proof of Theorem 4

Proof of Theorem 4. Theorem 3 and (2.6) together imply that, with probability at least $1 - 1/T^2$, we have

$$
\begin{aligned}
\text{Regret}(A) &= \text{FLUID}(\{\mathcal{P}_t\}_t) - \mathbb{E}_{\{\gamma_t\}_t\sim\prod_t\mathcal{P}_t}[R(A|\{\gamma_t\}_t)] \\
&\leq R_1 + R_2 + R_3 + \kappa\cdot r(T) + \kappa\bar{b} + 2(1+\kappa)\cdot\sum_{t=1}^{T}\mathcal{W}(\mathcal{P}_t,\tilde{\mathcal{P}}_t)\,.
\end{aligned}
$$

From our choice of step size $\eta = \sqrt{d_R/T}$ and the observation that $\bar{\lambda} = \max_t \lambda_t \leq \bar{b}$, we get that

$$
R_1 = \kappa\bar{b} + \frac{2(\bar{b}+\bar{\lambda})^2}{\sigma}\cdot\eta T + \frac{d_R}{\eta} \leq \kappa\bar{b} + \left(\frac{8\bar{b}^2}{\sigma}+1\right)\cdot\sqrt{d_r T}\,.
$$

From (A.3), we know that

$$R_2 = \kappa \cdot \left(\left\{ \sum_{t=1}^{T} \lambda_t \right\} - B \right)^+ \leq \kappa \bar{b} \, .$$

Moreover, from Lemma 7 and $\eta = \sqrt{d_R/T}$, we know that

$$R_3 = \sum_{s=1}^{T} \mathbb{E}\left[\mu_s \cdot (\beta_s - \lambda_s)\right] \leq \frac{4\eta \bar{b}^2}{\sigma} \cdot T = \frac{4\bar{b}^2}{\sigma} \cdot \sqrt{d_r T} \quad .$$

Combining the above inequalities and plugging in $r(T) = 8\bar{b}\sqrt{T \log(T)}$, we get that

$$\text{Regret}(A) \leq 3\kappa\bar{b} + \left(\frac{12\bar{b}^2}{\sigma} + 1\right) \cdot \sqrt{d_r T} + 8\kappa\bar{b}\sqrt{T\log(T)} + 2(1+\kappa) \cdot \sum_{t=1}^{T} \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t)$$

with probability at least $1 - 1/T^2$. On the other than, we always have $\text{Regret}(T) \leq \bar{f}T \leq \kappa\bar{b}T$. Hence, we get

$$\begin{aligned}
\text{Regret}(A) &\leq \left(1 - \frac{1}{T^2}\right) \cdot \left[3\kappa\bar{b} + \left(\frac{12\bar{b}^2}{\sigma} + 1\right) \cdot \sqrt{d_r T} + 8\kappa\bar{b}\sqrt{T\log(T)} + 2(1+\kappa) \cdot \sum_{t=1}^{T} \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t)\right] + \frac{\kappa\bar{b}T}{T^2} \\
&\leq 4\kappa\bar{b}\sqrt{T\log(T)} + \left(\frac{12\bar{b}^2}{\sigma} + 1\right) \cdot \sqrt{d_r} \cdot \sqrt{T\log(T)} + 8\kappa\bar{b}\sqrt{T\log(T)} + 2(1+\kappa) \cdot \sum_{t=1}^{T} \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t) \\
&= C_1 \cdot \sqrt{T\log(T)} + C_2 \cdot \sum_{t=1}^{T} \mathcal{W}(\mathcal{P}_t, \tilde{\mathcal{P}}_t)
\end{aligned}$$

where $C_1 = \frac{12\bar{b}^2\sqrt{d_R}}{\sigma} + \sqrt{d_R} + 12\kappa\bar{b}$ and $C_2 = 2(1+\kappa)$. □

Appendix B: Appendix to Chapter 3

B.1 Proofs for Section 3.2

We begin by formally stating and proving weak duality

Proposition 11. *For every* $\mu \in \mathbb{R}_+$*,* $T \geq 1$ *and* $\vec{\gamma} \in \mathbb{S}^T$*, we have* $\mathrm{OPT}(T, \vec{\gamma}) \leq D(\mu|T, \vec{\gamma})$*.*

Proof. Consider any $x \in \prod_t \mathcal{X}_t$ such that $\sum_{t=1}^T b_t(x_t) \leq B$. Then, for $\mu \geq 0$, we have

$$D(\mu|T, \vec{\gamma}) \geq \left\{ \sum_{t=1}^{T} f_t(x_t) \right\} + \mu^\top \left(B - \sum_{t=1}^{T} b_t(x_t) \right) \geq \sum_{t=1}^{T} f_t(x_t)$$

Since $x \in \prod_t \mathcal{X}_t$ satisfied $\sum_{t=1}^T b_t(x_t) \leq B$ was otherwise arbitrary, we have shown $\mathrm{OPT}(T\vec{\gamma}) \leq D(\mu|T, \vec{\gamma})$. □

B.1.1 Proof of Lemma 8

Proof of Lemma 8. The convexity of $\mathbb{D}(\cdot|\lambda, \mathcal{P})$ follows from part (a) and the fact that the dual objective $D(\mu|T, \vec{\gamma})$ is always convex since it is a supremum of a collection of linear functions.

(a) Shown in equation 3.2.

(b) For $a \in [0, 1]$, we have

$$\mathbb{D}(\mu|a \cdot \lambda, \mathcal{P}) - a \cdot \mathbb{D}(\mu|\lambda, \mathcal{P}) = (1 - a)\mathbb{E}_{\gamma\sim\mathcal{P}}\left[f^*(\mu)\right] \geq 0$$

where we have used the fact that $f^*(\mu) \geq 0$ for all $\gamma = (f, b, \mathcal{X}) \in \mathbb{S}$), which holds because $0 \in \mathcal{X}$.

(c) For $\lambda \leq \kappa$, we have

$$\mathbb{D}(\mu|\kappa, \mathcal{P}) - \mathbb{D}(\mu|\lambda, \mathcal{P}) = (\kappa - \lambda)^\top \mu \geq 0$$

where the inequality follows from the fact that $\kappa - \lambda,\ \mu \geq 0$. □

B.1.2 Proof of Theorem 5

Proof of Theorem 5. Fix an arbitrary $T \geq 1$. To simplify notation, define

$$\frac{\lambda_t}{\rho_T} := \min_{1 \leq j \leq m} \frac{\lambda_{t,j}}{\rho_{T,j}}$$

We split the proof into four steps: (1) The first step involves lower bounding the performance of our algorithm in terms of single-period duals and the complementary slackness term; (2) The second step involves bounding the complementary slackness term using standard regret analysis of mirror descent; (3) The third step involves bounding the optimal from above using the single-period dual; (4) The final step puts it all it all together. Our proof significantly generalizes the proof of [BLM23], who established this result for the special case of a constant target consumption rate $\lambda_t = \rho_T$ for all $t \leq T$. The main technical contribution of the current proof is establishing a general performance guarantee for dual-mirror descent for *all* target consumption sequences, which will prove critical in getting an asymptotically-near-optimal competitive ratio for our model. This involves establishing a novel target-rate-dependent lower bound on the algorithm's reward (Step 1), a novel target-rate-dependent upper bound on the optimal reward (Step 3), and a new way to put these bounds together (Step 4).

Step 1: Lower bound on algorithm's reward. Consider the filtration $\mathcal{F} = \{\sigma(\xi_t)\}_t$, where $\xi_t = \{\gamma_1, \ldots, \gamma_t\}$ is the set of all requests seen till time t and $\sigma(\xi_t)$ is the sigma algebra generated by it. Note that Algorithm 3 only collects rewards when there are enough resources left. Let ζ_A be first time less than T for which there exists a resource j such that $\sum_{t=1}^{\zeta_A} b_{t,j}(x_t) + \bar{b} \geq B_j$. Here,

$\zeta_A = T$ if this inequality is never satisfied. Observe that ζ_A is a stopping time w.r.t. $\mathcal{F}$ and it is defined so that we cannot violate the resource constraints before ζ_A. In particular, $x_t = \tilde{x}_t$ for all $t \leq \zeta_A$. Therefore, we get

$$f_t(x_t) = f^*(\mu_t) + \mu_t^\top b_t(x_t)$$

Observe that μ_t is measurable w.r.t. $\sigma(\xi_{t-1})$ and γ_t is independent of $\sigma(\xi_{t-1})$, which allows us to take conditional expectation w.r.t. $\sigma(\xi_{t-1})$ to write

$$\begin{aligned}\mathbb{E}\left[f_t(x_t)|\sigma(\xi_{t-1})\right] &= \mathbb{E}_{\gamma_t\sim\mathcal{P}}\left[f^*(\mu_t)\right] + \mu_t^\top\lambda_t + \mu_t^\top\left(\mathbb{E}\left[b_t(x_t)|\sigma(\xi_{t-1})\right] - \lambda_t\right)\\ &= \tilde{D}(\mu_t|\lambda_t,\mathcal{P}) - \mathbb{E}\left[\mu_t^\top\left(\lambda_t - b_t(x_t)\right)|\sigma(\xi_{t-1})\right] \end{aligned} \tag{B.1}$$

where the second equality follows the definition of the single-period dual function.

Define $Z_t = \sum_{s=1}^t \mu_s^\top\left(\lambda_s - b_s(x_s)\right) - \mathbb{E}\left[\mu_s^\top\left(\lambda_s - b_s(x_s)\right)|\sigma(\xi_{s-1})\right]$. Then, $\{Z_t\}_t$ is a martingale w.r.t. the filtration $\mathcal{F}$ because $Z_t \in \sigma(\xi_t)$ and $\mathbb{E}[Z_{t+1}|\sigma(\xi_t)] = Z_t$. As ζ_A is a bounded stopping time w.r.t. $\mathcal{F}$, the Optional Stopping Theorem yields $\mathbb{E}\left[Z_{\zeta_A}\right] = 0$. Therefore,

$$\mathbb{E}\left[\sum_{t=1}^{\zeta_A}\mu_t^\top\left(\lambda_t - b_t(x_t)\right)\right] = \mathbb{E}\left[\sum_{t=1}^{\zeta_A}\mathbb{E}\left[\mu_t^\top\left(\lambda_t - b_t(x_t)\right)|\sigma(\xi_{t-1})\right]\right].$$

A similar argument yields

$$\mathbb{E}\left[\sum_{t=1}^{\zeta_A} f_t(x_t)\right] = \mathbb{E}\left[\sum_{t=1}^{\zeta_A}\mathbb{E}\left[f_t(x_t)|\sigma(\xi_{t-1})\right]\right].$$

Hence, summing over (B.1) and taking expectations, we get

$$\begin{aligned}&\mathbb{E}\left[\sum_{t=1}^{\zeta_A} f_t(x_t)\right]\\ &= \mathbb{E}\left[\sum_{t=1}^{\zeta_A}\tilde{D}(\mu_t|\lambda_t,\mathcal{P})\right] - \mathbb{E}\left[\sum_{t=1}^{\zeta_A}\mu_t^\top\left(\lambda_t - b_t(x_t)\right)\right]\end{aligned} \tag{B.2}$$

$$
\begin{aligned}
&\geq \mathbb{E}\left[\sum_{t=1}^{\zeta_A} \tilde{D}\left(\mu_t | \min\{\lambda_t/\rho_T, 1\} \cdot \rho_T, \mathcal{P}\right)\right] - \mathbb{E}\left[\sum_{t=1}^{\zeta_A} \mu_t^\top \left(\lambda_t - b_t(x_t)\right)\right] \\
&\geq \mathbb{E}\left[\sum_{t=1}^{\zeta_A} \min\{\lambda_t/\rho_T, 1\} \cdot \tilde{D}\left(\mu_t | \rho_T, \mathcal{P}\right)\right] - \mathbb{E}\left[\sum_{t=1}^{\zeta_A} \mu_t^\top \left(\lambda_t - b_t(x_t)\right)\right] \\
&= \mathbb{E}\left[\sum_{t=1}^{\zeta_A} \min\{\lambda_t/\rho_T, 1\} \cdot \sum_{t=1}^{\zeta_A} \frac{\min\{\lambda_t/\rho_T, 1\}}{\sum_{s=1}^{\zeta_A} \min\{\lambda_s/\rho_T, 1\}} \cdot \tilde{D}\left(\mu_t | \rho_T, \mathcal{P}\right)\right] - \mathbb{E}\left[\sum_{t=1}^{\zeta_A} \mu_t^\top \left(\lambda_t - b_t(x_t)\right)\right] \\
&\geq \mathbb{E}\left[\sum_{t=1}^{\zeta_A} \min\{\lambda_t/\rho_T, 1\} \cdot \tilde{D}\left(\bar{\mu}_{\zeta_A} | \rho_T, \mathcal{P}\right)\right] - \mathbb{E}\left[\sum_{t=1}^{\zeta_A} \mu_t^\top \left(\lambda_t - b_t(x_t)\right)\right] \qquad \text{(B.3)}
\end{aligned}
$$

where

$$
\bar{\mu}_{\zeta_A} = \sum_{t=1}^{\zeta_A} \frac{\min\{\lambda_t/\rho_T, 1\} \cdot \mu_t}{\sum_{s=1}^{\zeta_A} \min\{\lambda_s/\rho_T, 1\}}.
$$

The first inequality follows from part (c) of Lemma 8, the second inequality follows from part (b) of Lemma 8 and the third inequality follows from the convexity of the single-period dual function (Lemma 8).

Step 2: Complementary slackness. Define $w_t(\mu) := \mu^\top(\lambda_t - b_t(x_t))$. Then, Algorithm 3 can be seen as running online mirror descent on the choice of the dual variables with linear losses $\{w_t(\cdot)\}_t$. The gradients of these loss functions are given by $\nabla w_t(\mu) = \lambda_t - b_t(x_t)$, which satisfy $\|\nabla w_t(\mu)\|_\infty \leq \|b_t(x_t)\|_\infty + \|\lambda(t)\|_\infty \leq \bar{b} + \bar{\lambda}$. Therefore, Proposition 5 of [BLM23] implies that for all $\mu \in \mathbb{R}_+^m$:

$$
\sum_{t=1}^{\zeta_A} w_t(\mu_t) - w_t(\mu) \leq E(\zeta_A, \mu) \leq E(T, \mu)\,, \qquad \text{(B.4)}
$$

where $E(t, \mu) = \frac{1}{2\sigma}(\bar{b} + \bar{\lambda})^2 \eta \cdot t + \frac{1}{\eta} V_h(\mu, \mu_1)$ is the regret bound of the online mirror descent algorithm after t iterations, and the second inequality follows because $\zeta_A \leq T$ and the error term $E(t, \mu)$ is increasing in t.

Step 3: Upper bound on the optimal reward. For every $\zeta_A \in [1, T]$, we have

$$\begin{aligned}
&\mathbb{E}_{\vec{\gamma}\sim\mathbb{D}^T}\left[c(\vec{\lambda},T)\cdot \mathrm{OPT}(T,\vec{\gamma})\right] \\
=&\frac{\sum_{s=1}^{T}\min\{\lambda_s/\rho_T,1\}}{T}\cdot \mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[\mathrm{OPT}(T,\vec{\gamma})\right] && \left(\text{Defn. of } c(\vec{\lambda},T)\right)\\
=&\frac{\sum_{s=1}^{\zeta_A}\min\{\lambda_s/\rho_T,1\}}{T}\cdot \mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[\mathrm{OPT}(T,\vec{\gamma})\right] + \frac{\sum_{s=\zeta_A+1}^{T}\min\{\lambda_s/\rho_T,1\}}{T}\cdot \mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[\mathrm{OPT}(T,\vec{\gamma})\right]\\
\le&\sum_{s=1}^{\zeta_A}\min\{\lambda_s/\rho_T,1\}\cdot\frac{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}[\mathrm{OPT}(T,\vec{\gamma})]}{T} + \sum_{s=\zeta_A+1}^{T}\min\{\lambda_s/\rho_T,1\}\cdot\bar{f} && \left(\mathrm{OPT}(T)\le T\cdot\bar{f}\right)\\
\le&\sum_{s=1}^{\zeta_A}\min\{\lambda_s/\rho_T,1\}\cdot\frac{\mathbb{E}_{\vec{\gamma}\sim\mathbb{D}^T}\left[D(\bar{\mu}_{\zeta_A}|T,\vec{\gamma})\right]}{T} + \sum_{s=\zeta_A+1}^{T}\min\{\lambda_s/\rho_T,1\}\cdot\bar{f} && (\text{weak duality})\\
=&\sum_{t=1}^{\zeta_A}\min\{\lambda_t/\rho_T,1\}\cdot\tilde{D}\left(\bar{\mu}_{\zeta_A}|\rho_T,\mathcal{P}\right) + \sum_{s=\zeta_A+1}^{T}\min\{\lambda_s/\rho_T,1\}\cdot\bar{f} && \text{(B.5)}
\end{aligned}$$

where the last equality follows from part (a) of Lemma 8.

Step 4: Putting it all together. Combining the results from steps 1-3 yields:

$$\begin{aligned}
&\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[c(\vec{\lambda},T)\cdot\mathrm{OPT}(T,\vec{\gamma}) - R(A|T,\vec{\gamma})\right]\\
\le&\,\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[c(\vec{\lambda},T)\cdot\mathrm{OPT}(T,\vec{\gamma}) - \sum_{t=1}^{\zeta_A} f_t(x_t)\right]\\
\le&\,\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[c(\vec{\lambda},T)\cdot\mathrm{OPT}(T,\vec{\gamma}) - \sum_{t=1}^{\zeta_A}\min\{\lambda_t/\rho_T,1\}\cdot\tilde{D}\left(\bar{\mu}_{\zeta_A}|\rho_T,\mathcal{P}\right) + \sum_{t=1}^{\zeta_A}\mu_t^\top\left(\lambda_t - b_t(x_t)\right)\right] && \text{(Equation B.2)}\\
\le&\,\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[\sum_{s=\zeta_A+1}^{T}\min\{\lambda_s/\rho_T,1\}\cdot\bar{f} + \sum_{t=1}^{\zeta_A} w_t(\mu_t)\right] && \text{(Equation B.5)}\\
\le&\,\underbrace{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[\sum_{s=\zeta_A+1}^{T}\min\{\lambda_s/\rho_T,1\}\cdot\bar{f} + \sum_{t=1}^{\zeta_A} w_t(\mu) + E(T,\mu)\right]}_{\clubsuit} && \text{(Equation B.4)}
\end{aligned}$$

for all $\mu \in \mathbb{R}_+^m$. All that remains to complete the proof is choosing the right μ. If $\zeta_A = T$ (no resource was completely depleted), set $\mu = 0$. If $\zeta_A < T$, then there exists a resource $j \in [m]$ that nearly got depleted, i.e., $\sum_{t=1}^{\zeta_A} b_{t,j}(x_t) + \bar{b} \ge B_j$. Moreover, recall that the definition of a target consumption sequence implies $\sum_{t=1}^{T}\lambda_{t,j} \le B_j$. Thus, $\sum_{t=1}^{\zeta_A} b_{t,j}(x_t)) \ge -\bar{b} + \sum_{t=1}^{T}\lambda_{t,j}$. Therefore, setting $\mu = (\bar{f}/\underline{\rho}_T)e_j$, where $e_j \in \mathbb{R}_m$ is the j-th unit vector, yields:

$$
\begin{aligned}
\sum_{t=1}^{\zeta_A} w_t(\mu) &= \sum_{t=1}^{\zeta_A} \frac{\bar{f}}{\underline{\rho}_T} e_j^\top(\lambda_t - b_t(x_t)) \\
&= \frac{\bar{f}}{\underline{\rho}_T} \cdot \left(\sum_{t=1}^{\zeta_A} \lambda_{t,j} - \sum_{t=1}^{\zeta_A} b_{t,j}(x_t) \right) \\
&\leq \frac{\bar{f}}{\underline{\rho}_T} \cdot \left(\bar{b} - \left\{ \sum_{t=1}^{T} \lambda_{t,j} - \sum_{t=1}^{\zeta_A} \lambda_{t,j} \right\} \right) \\
&= \frac{\bar{f}\bar{b}}{\underline{\rho}_T} - \bar{f} \cdot \sum_{t=\zeta_A+1}^{T} \frac{\lambda_{t,j}}{\underline{\rho}_T} \\
&\leq \frac{\bar{f}\bar{b}}{\underline{\rho}_T} - \bar{f} \cdot \sum_{t=\zeta_A+1}^{T} \min\{\lambda_t/\rho_T, 1\}
\end{aligned}
$$

Here we use that $\frac{\lambda_{t,j}}{\underline{\rho}_T} \geq \min\{\lambda_t/\rho_T, 1\}$. This follows because $\min_j a_j / \min_j b_j \geq \min_j a_j/b_j$. Finally, if we put everything together, we get

$$
\clubsuit \leq \frac{\bar{f}\bar{b}}{\underline{\rho}_T} + E(T,\mu) \leq \frac{\bar{f}\bar{b}}{\underline{\rho}_T} + \frac{1}{2\sigma}(\bar{b}+\bar{\lambda})^2 \eta \cdot T + \frac{1}{\eta} V_h(\mu, \mu_1)\,,
$$

where we have used the definition of $E(T,\mu)$. The theorem follows from observing that all of our choices of μ in the above discussion lie in the set $\{0, (\bar{f}/\underline{\rho}_T)e_1, \ldots, (\bar{f}/\underline{\rho}_T)e_m\}$. □

B.1.3 Proof of Proposition 2

Proof of Proposition 2. Setting $\eta = \sqrt{C_3'/\{C_2\tau_2\}}$ in Theorem 5 yields

$$
\begin{aligned}
\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[c(\vec{\lambda},T)\cdot \mathrm{OPT}(T,\vec{\gamma}) - R(A|T,\vec{\gamma})\right] &\leq C_1^{(T)} + C_2 T\sqrt{\frac{C_3'}{C_2\tau_2}} + C_3^{(T)}\sqrt{\frac{C_2\tau_2}{C_3'}} \\
&\leq C_1' + \sqrt{C_2 C_3'\tau_2} + \sqrt{C_2 C_3'\tau_2}\,.
\end{aligned}
$$

Dividing both sides by $\mathbb{E}[\text{OPT}(T, \vec{\gamma})]$ and using $\mathbb{E}[\text{OPT}(T, \vec{\gamma})] \geq \kappa \cdot T$, we get

$$\frac{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[c(\vec{\lambda}, T) \cdot \text{OPT}(T, \vec{\gamma}) - R(A|T, \vec{\gamma})\right]}{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}[\text{OPT}(T, \vec{\gamma})]} \leq \frac{C_1' + 2 \cdot \sqrt{C_2 C_3' \tau_2}}{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}[\text{OPT}(T, \vec{\gamma})]}$$
$$\iff c(\vec{\lambda}, T) - \frac{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[R(A|T, \vec{\gamma})\right]}{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}[\text{OPT}(T, \vec{\gamma})]} \leq \frac{C_1'}{\kappa T} + \frac{2 \cdot \sqrt{C_2 C_3' \tau_2}}{\kappa T}.$$

Therefore, rearranging terms and using $T \geq \tau_1$ we get

$$\begin{aligned}
\frac{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}\left[R(A|T, \vec{\gamma})\right]}{\mathbb{E}_{\vec{\gamma}\sim\mathcal{P}^T}[\text{OPT}(T, \vec{\gamma})]} &\geq c(\vec{\lambda}, T) - \left(\frac{C_1'}{\kappa T} + 2 \cdot \frac{\sqrt{C_2 C_3' \tau_2}}{\kappa T}\right) \\
&\geq c(\vec{\lambda}, T) - \left(\frac{C_1'}{\kappa T} + 2 \cdot \frac{\sqrt{(\tau_2/\tau_1) C_2 C_3' \tau_1}}{\kappa T}\right) \\
&\geq c(\vec{\lambda}, T) - \left(\frac{C_1'}{\kappa \tau_1} + 2 \cdot \frac{\sqrt{(\tau_2/\tau_1) C_2 C_3' \tau_1}}{\kappa \tau_1}\right) \\
&\geq c(\vec{\lambda}, T) - \left(\frac{C_1'}{\kappa \tau_1} + 2 \cdot \frac{\sqrt{(\tau_2/\tau_1) C_2 C_3'}}{\kappa \sqrt{\tau_1}}\right)
\end{aligned}$$

as required. □

B.2 Proofs and Extensions for Section 3.3.1

To prove Theorem 6, it suffices to prove the stronger statement which holds for online algorithms with prior knowledge of $(r, \mathcal{P}_r)$ before time $t = 1$. Consequently, we assume that online algorithms have this prior knowledge in the remainder of this section. Any algorithm without this knowledge can only do worse.

B.2.1 Proof of Lemma 9

Proof of Lemma 9. Fix $T \in [\tau_1, \tau_2]$. As the resource consumption function is given by $I(\cdot)$, we get that

$$\text{OPT}(T, r) = \max_x \sum_{t=1}^{T} f_r(x_t) \quad \text{subject to} \quad \sum_{t=1}^{T} x_t \leq B$$

Let x be a feasible solution of the above optimization problem. Then,

$$\sum_{t=1}^{T} f_r(x_t) = T \cdot \frac{\sum_{t=1}^{T} f_r(x_t)}{T} \leq T \cdot f_r\left(\frac{\sum_{t=1}^{T} x_t}{T}\right) \leq T \cdot f_r\left(\frac{B}{T}\right) \tag{B.6}$$

where the first inequality follows from the concavity of f_r and the second inequality follows from the resource constraint $\sum_{t=1}^{T} x_t \leq B$. Hence, we get that $x_t^* = B/T$ for all $t \leq T$ is an optimal solution to the above optimization problem and as a consequence, $\mathrm{OPT}(T, r) = T \cdot f_r(B/T) = B^r \cdot T^{1-r}$. Moreover, we have that $x_t^* = B/T$ is the unique optimal solution because f_r is strictly concave and increasing for $r \in (0, 1)$, and therefore (i) The first inequality in (B.6) is strict whenever $x_t \neq x_s$ for some $s, t \in [T]$; (ii) $f_r(\sum_{t=1}^{T} x_t/T) < f(B/T)$ whenever $\sum_{t=1}^{T} x_t < B$. □

B.2.2 Proof of Lemma 10

Proof of Lemma 10. We begin by noting that our use of max instead of sup is justified in the right-hand side of the equality in Lemma 10 because f_r^{-1} is continuous for all $r \in (0, 1)$. Now, fix $r \in (0, 1)$ and $1 \leq \tau_1 \leq \tau_2$. Let

$$c \in \operatorname{argmax}\left\{c' \;\middle|\; \tau_1 \cdot f_r^{-1}\left(c' \cdot \frac{\mathrm{OPT}(\tau_1, r)}{\tau_1}\right) + \sum_{t=\tau_1+1}^{\tau_2} f_r^{-1}\left(c' \cdot \Delta\,\mathrm{OPT}(t, r)\right) \leq B\right\},$$

and define

$$x_t := \begin{cases} f_r^{-1}\left(c \cdot \frac{\mathrm{OPT}(\tau_1, r)}{\tau_1}\right) & \text{if } t \leq \tau_1 \\ f_r^{-1}\left(c \cdot \Delta\,\mathrm{OPT}(t, r)\right) & \text{if } \tau_1 < t \leq \tau_2 \end{cases} \tag{B.7}$$

Then, by definition of c, we have $\sum_{t=1}^{\tau_2} I(x_t) = \sum_{t=1}^{\tau_2} x_t \leq B$. Moreover, observe that $\mathrm{OPT}(\tau_1, r)/\tau_1 = (B/\tau_1)^r$ and

$$\Delta\,\mathrm{OPT}(t, r) = \mathrm{OPT}(t, r) - \mathrm{OPT}(t-1, r) = B^r \cdot \left(t^{1-r} - (t-1)^{1-r}\right) \leq B^r \cdot \frac{1-r}{(t-1)^r}$$

for all $t \geq \tau_1 + 1$. To see why the second-last inequality holds, note that the Intermediate Value Theorem applied to the function $t \mapsto t^{1-r}$ between the points t and $t-1$ yields the existence of an $s \in [t-1, t]$ such that $t^{1-r} - (t-1)^{1-r} = (1-r)/s^r$, which implies $t^{1-r} - (t-1)^{1-r} \leq (1-r)/(t-1)^r$.

As a consequence, we get

$$x_t \leq \begin{cases} c^{1/r}\{B/\tau_1\} & \text{if } t \leq \tau_1 \\ c^{1/r}\{B/(t-1)\} & \text{if } \tau_1 < t \leq \tau_2 \end{cases} \tag{B.8}$$

Combining the above inequalities with the definition of c yields $c \leq 1$. Hence, we have $x_t \in \mathcal{X}$ for all $t \in [\tau_2]$.

Consider the algorithm that selects action x_t at time t. Then, in $T \in [\tau_1, \tau_2]$ time steps, it receives a reward of

$$\begin{aligned} \sum_{t=1}^{T} f_r(x_t) &= \sum_{t=1}^{\tau_1} f_r\left(f_r^{-1}\left(c \cdot \frac{\mathrm{OPT}(\tau_1, r)}{\tau_1}\right)\right) + \sum_{t=\tau_1+1}^{T} f_r\left(f_r^{-1}\left(c \cdot \Delta\,\mathrm{OPT}(t, r)\right)\right) \\ &= c \cdot \mathrm{OPT}(\tau_1, r) + \sum_{t=\tau_1+1}^{T} c \cdot \Delta\,\mathrm{OPT}(t, r) \\ &= c \cdot \mathrm{OPT}(T, r)\,. \end{aligned}$$

Therefore, we have shown that

$$\sup_A \min_{T \in [\tau_1, \tau_2]} \frac{R(A|T, r)}{\mathrm{OPT}(T, r)} \geq \max\left\{c \in [0, 1] \;\middle|\; \tau_1 \cdot f_r^{-1}\left(c \cdot \frac{\mathrm{OPT}(\tau_1, r)}{\tau_1}\right) + \sum_{t=\tau_1+1}^{\tau_2} f_r^{-1}\left(c \cdot \Delta\,\mathrm{OPT}(t, r)\right) \leq B\right\}$$

Next, we prove the above inequality in the opposite direction. Consider an online algorithm A such that

$$\frac{R(A|T, r)}{\mathrm{OPT}(T, r)} \geq c \quad \forall\, T \in [\tau_1, \tau_2],$$

for some constant $c > 0$. Let $x(A)_t$ represent the action taken by A at time t. Since the action of an online algorithm cannot depend on future information, $x(A)_t$ represents the action taken by algorithm A for all horizons $T \geq t$. Let $\{x(\tilde{A})_t\}_t$ represent the sequence obtained by sorting $\{x(A)_t\}_t$ in decreasing order, and let $\tilde{A}$ represent the algorithm that takes action $x(\tilde{A})_t$ at time t. Then, we have

$$\frac{\sum_{t=1}^{T} f_r(x(\tilde{A})_t)}{\mathrm{OPT}(T, r)} \geq \frac{\sum_{t=1}^{T} f_r(x(A)_t)}{\mathrm{OPT}(T, r)} \geq c \quad \forall\, T \in [\tau_1, \tau_2]\,,$$

which allows us to assume that $\{x(A)_t\}_t$ is sorted in decreasing order without loss of generality.

Since $\sum_{t=1}^{\tau_2} x(A)_t \leq B$, to complete the proof it suffices to show that

$$\sum_{t=1}^{\tau_2} x_t = \tau_1 \cdot f_r^{-1}\left(c \cdot \frac{\mathrm{OPT}(\tau_1, r)}{\tau_1}\right) + \sum_{t=\tau_1+1}^{\tau_2} f_r^{-1}\left(c \cdot \Delta\,\mathrm{OPT}(t, r)\right) \leq \sum_{t=1}^{\tau_2} x(A)_t$$

where the equality follows from the definition of x_t (B.7). We will prove this via induction by inductively proving the following statement for all $T \in [\tau_1, \tau_2]$:

$$\sum_{t=1}^{T} x_t = \tau_1 \cdot f_r^{-1}\left(c \cdot \frac{\mathrm{OPT}(\tau_1, r)}{\tau_1}\right) + \sum_{t=\tau_1+1}^{T} f_r^{-1}\left(c \cdot \Delta\,\mathrm{OPT}(t, r)\right) \leq \sum_{t=1}^{T} x(A)_t$$

To do so, we will maintain variables $\{w(T)_t\}_{t \leq T}$ that we initialize to be 0 and update inductively. At a high level, they capture a water-filling procedure. Suppose there is a container corresponding to each time step t with a capacity of $x(A)_t$. We assume that these containers can be connected to each other so that water always goes to the lowest level, which corresponds to the highest marginal reward since f_r is concave. Moreover, we will assume that container T becomes available at time T and is connected to containers $t < T$ at that point. Finally, we also freeze the newly-added water at the end of each time step to inductively use the properties of the water level from the previous time step. We would like to caution the reader that this water-filling interpretation is just a tool that guided our intuition, and the mathematical quantities defined below may not match it exactly.

Base Case $T = \tau_1$: Let $\{w(\tau_1)_t\}_{t=1}^{\tau_1}$ be a decreasing sequence that satisfies the following properties:

I. $\sum_{t=1}^{\tau_1} f_r(w(\tau_1)_t) = c \cdot \mathrm{OPT}(\tau_1, r)$.

II. $w(\tau_1)_t \leq x(A)_t$ for all $t \leq \tau_1$.

III'. $w(\tau_1)_t < w(\tau_1)_1 \implies w(\tau_1)_t = x(A)_t$.

Such a sequence is guaranteed to exist because $\{x(A)_t\}_{t=1}^{\tau_1}$ satisfies properties (II - III') trivially, and (I) can be satisfied as $\sum_{t=1}^{\tau_1} f_r(x(A)_t) \geq c \cdot \mathrm{OPT}(\tau_1, r)$ and f_r is a continuous increasing function. If $\{w(\tau_1)_t\}_{t=1}^{\tau_1}$ is a constant sequence, then property (I) implies

$$w(\tau_1)_t = f_r^{-1}\left(c \cdot \frac{\mathrm{OPT}(\tau_1, r)}{\tau_1}\right) = x_{\tau_1} \quad \forall\, t \leq \tau_1$$

Suppose $\{w(\tau_1)_t\}_{t=1}^{\tau_1}$ is not a constant sequence. Then, the strict concavity of f_r implies that

$$f_r\left(\frac{\sum_{t=1}^{\tau_1} w(\tau_1)_t}{\tau_1}\right) > \frac{\sum_{t=1}^{\tau_1} f_r(w(\tau_1)_t)}{\tau_1} = \frac{c \cdot \mathrm{OPT}(\tau_1, r))}{\tau_1},$$

which implies

$$w(\tau_1)_1 \geq \frac{\sum_{t=1}^{\tau_1} w(\tau_1)_t}{\tau_1} > f_r^{-1}\left(c \cdot \frac{\mathrm{OPT}(\tau_1, r)}{\tau_1}\right) = x_{\tau_1}. \tag{B.9}$$

Therefore, we have established:

III. $w(\tau_1)_t < x_t \implies w(\tau_1)_t = x(A)_t$.

IV. $\sum_{t=1}^{\tau_1} w(\tau_1)_t \geq \sum_{t=1}^{\tau_1} x_t = \tau_1 \cdot x_{\tau_1}$.

where (III) follows follows trivially when $\{w(\tau_1)_t\}_{t=1}^{\tau_1}$ is a constant sequence and follows from (III') and $w(\tau_1)_1 > x_t = x_{\tau_1}$ otherwise, and (IV) also follows trivially when $\{w(\tau_1)_t\}_{t=1}^{\tau_1}$ is a constant sequence and follows from (B.9) otherwise.

Induction Hypothesis $\tau_1 \leq T < \tau_2$: Suppose there exists a decreasing sequence $\{w(T)_t\}_{t=1}^{T}$ that satisfies the following properties:

I. $\sum_{t=1}^{T} f_r\left(w(T)_t\right) = c \cdot \mathrm{OPT}(T, r)$.

II. $w(T)_t \leq x(A)_t$ for all $t \leq T$.

III. $w(T)_t < x_t \implies w(T)_t = x(A)_t$.

IV. $\sum_{t=1}^{T} w(T)_t \geq \sum_{t=1}^{T} x_t$.

Induction Step $T+1$**:** If $x(A)_{T+1} \geq x_{T+1}$, then set $w(T+1)_{T+1} = x_{T+1}$ and $w(T+1)_t = w(T)_t$ for all $t \leq T$. In this case, it is easy to see that conditions (I-IV) hold for $\{w(T+1)_t\}_t$. Next, assume $x(A)_{T+1} < x_{T+1}$. In this case, set $w(T+1)_{T+1} = x(A)_{T+1}$. Moreover, let $\{w(T+1)_t\}_{t=1}^{T}$ be a decreasing sequence that satisfies the following properties:

I. $\sum_{t=1}^{T} f_r(w(T+1)_t) = c \cdot \mathrm{OPT}(T, r) + f_r(x_{T+1}) - f_r(x(A)_{T+1})$.

II. $w(T+1)_t \leq x(A)_t$ for all $t \leq T$.

III'. $w(T)_t \leq w(T+1)_t$ for all $t \leq T$.

Such a sequence is guaranteed to exist because $\{x(A)_t\}_{t=1}^{T}$ satisfies property (II) trivially, (III') as a consequence of the inductive hypothesis, and (I) can be satisfied because f_r is a continuous increasing function and

$$\begin{aligned}\sum_{t=1}^{T+1} f_r(x(A)_t) \geq c \cdot \mathrm{OPT}(T+1, r) &\iff \sum_{t=1}^{T+1} f_r(x(A)_t) \geq c \cdot \mathrm{OPT}(T, r) + c \cdot \Delta\,\mathrm{OPT}(T+1, r) \\ &\iff \sum_{t=1}^{T} f_r(x(A)_t) \geq c \cdot \mathrm{OPT}(T, r) + f_r(x_{T+1}) - f_r(x(A)_{T+1}).\end{aligned}$$

Observe that (III') and $w(T+1)_{T+1} = x(A)_{T+1}$ implies

III. $w(T+1)_t < x_t \implies w(T+1)_t = x(A)_t$

Now, only (IV) remains. First, note that the Intermediate Value Theorem applied to $t \mapsto t^{1-r}$ implies

$$\Delta\,\mathrm{OPT}(T+1, r) = B^r \cdot [(T+1)^{1-r} - T^{1-r}] \leq B^r \cdot \frac{1-r}{T^{1-r}} \leq B^r \cdot [T^{1-r} - (T-1)^{1-r}] = \Delta\,\mathrm{OPT}(T, r),$$

and as a consequence, we get $x_{T+1} \leq x_T$. This further implies

$$\begin{aligned}\left[\sum_{t=1}^{T} w(T+1)_t - \sum_{t=1}^{T} w(T)_t\right] f_r'(x_{T+1}) &\geq \left[\sum_{t=1}^{T} w(T+1)_t - \sum_{t=1}^{T} w(T)_t\right] f_r'(x_T) \\ &\geq \sum_{t=1}^{T} \left[w(T+1)_t - w(T)_t\right] f_r'(w(T)_t) \\ &\geq \sum_{t=1}^{T} \left[f_r(w(T+1)_t) - f_r(w(T)_t)\right] \\ &= f(x_{T+1}) - f(x(A)_{T+1}) \\ &\geq \left[x_{T+1} - x(A)_{T+1}\right] f_r'(x_{T+1})\end{aligned}$$

where the first inequality follows from the concavity of f_r and the fact that $x_{T+1} \leq x_T$; the second inequality follows from concavity of f_r and the observation that the induction hypothesis and (III') imply $w(T)_t = w(T+1)_t = x(A)_t$ whenever $w(T)_t < x_T \leq x_t$, i.e., $w(T+1)_t - w(T)_t > 0$ implies $w(T)_t \geq x_T$; the third and the fourth inequalities follow from the Intermediate Value Theorem applied to f_r; and the equality follows from (I). Therefore, we get (IV) by using the inductive hypothesis and $w(T+1)_{T+1} = x(A)_{T+1}$:

$$\sum_{t=1}^{T+1} w(T+1) \geq \sum_{t=1}^{T} w(T)_t + x_{T+1} - x(A)_{T+1} + w(T+1)_{T+1} \geq \sum_{t=1}^{T+1} x_t$$

This concludes the induction step and thereby the proof, since (II) and (IV) together imply $\sum_{t=1}^{T} x_t \leq \sum_{t=1}^{T} x(A)_t$. □

B.2.3 Proof of Theorem 6

Proof of Theorem 6. Combining Lemma 10 and Lemma 9 yields

$$\tau_1 \cdot \left(c^* \cdot \frac{B^r \tau_1^{1-r}}{\tau_1}\right)^{1/r} + \sum_{t=\tau_1+1}^{\tau_2} \left(c^* \cdot \left[B^r t^{1-r} - B^r (t-1)^{1-r}\right]\right)^{1/r} \leq B\,, \tag{B.10}$$

where $c^* = \sup_A \min_{T \in [\tau_1, \tau_2]} R(A|T, r)/\mathrm{OPT}(T, r)$. First, note that the Intermediate Value Theorem applied to $t \mapsto t^{1-r}$ implies

$$t^{1-r} - (t-1)^{1-r} \geq \frac{1-r}{t^r} \quad \forall\, T \geq \tau_1 + 1,$$

which allows us to derive the following inequality from (B.10):

$$\begin{aligned} (c^*)^{1/r} + \sum_{t=\tau_1+1}^{\tau_2} \left(c^* \cdot \frac{1-r}{t^r}\right)^{1/r} &\leq 1 \\ \iff (c^*)^{1/r} &\leq \frac{1}{1 + (1-r)^{1/r} \sum_{t=\tau_1+1}^{\tau_2} 1/t} \end{aligned}$$

Using $\sum_{t=\tau_1+1}^{\tau_2} 1/t \geq \ln(\tau_2/(\tau_1+1)) = \ln(\tau_2/\tau_1) + \ln(\tau_1/(\tau_1+1))$ and $(1-r)^{1/r} < 1$, we get the first half of Theorem 6:

$$c^* \leq \frac{1}{\left(1 + (1-r)^{1/r} \cdot \ln(\tau_2/\tau_1) + (1-r)^{1/r} \ln\left(\frac{\tau_1}{\tau_1+1}\right)\right)^r} \leq \frac{1}{\left(1 + (1-r)^{1/r} \cdot \ln(\tau_2/\tau_1) + \ln\left(\frac{\tau_1}{\tau_1+1}\right)\right)^r}.$$

To get the second half, note that $1 + \ln(\tau_1/(\tau_1+1)) \geq 0$, which allows us to write:

$$c^* \leq \frac{1}{(1-r)\ln(\tau_2/\tau_1)^r}$$

Finally, define $g : (0,1) \to \mathbb{R}$ as $g(r) = (1-r)\ln(\tau_2/\tau_1)^r$. Then,

$$\begin{aligned} &\ln(g(r)) = \ln(1-r) + r \cdot \ln\ln(\tau_2/\tau_1) \\ \implies &\frac{g'(r)}{g(r)} = -\frac{1}{1-r} + \ln\ln(\tau_2/\tau_1) \\ \implies &g'(r) = -\ln(\tau_2/\tau_1)^r + \ln\ln(\tau_2/\tau_1) g(r) \\ \implies &g''(r) = -\ln\ln(\tau_2/\tau_1)\ln(\tau_2/\tau_1)^r + \ln\ln(\tau_2/\tau_1) f(r) = -\ln\ln(\tau_2/\tau_1) \cdot r \cdot \ln(\tau_2/\tau_1)^r \end{aligned}$$

Hence, for $\tau_2/\tau_1 > e^e$, g is concave and is maximized at $r = 1 - (1/\ln\ln(\tau_2/\tau_1))$. Plugging in

$r = 1 - (1/\ln\ln(\tau_2/\tau_1))$ yields

$$c^* \leq \frac{\ln\ln(\tau_2/\tau_1)}{\ln(\tau_2/\tau_1)^{1-\frac{1}{\ln\ln(\tau_2/\tau_1)}}} = \frac{e \cdot \ln\ln(\tau_2/\tau_1)}{\ln(\tau_2/\tau_1)}$$

which completes the proof. □

B.2.4 Randomized Upper Bound with Linear Rewards and Consumptions

The upper bound of Theorem 6 can be extended to the popular special case of linear rewards and linear consumption. Fix $r \in (0, 1)$ and $B = \tau_1$. We show that there exists a request distribution, with linear rewards and random linear resource consumption functions, that behaves like $(f_r, I, \mathcal{X})$ in expectation. Define a consumption rate CDF H as

$$H(s) = \begin{cases} 0 & \text{if } s \leq 0 \\ s^{\frac{r}{1-r}} & \text{if } 0 \leq s \leq 1 \\ 1 & \text{if } s \geq 1 \end{cases}.$$

Consider the following request distribution: the reward of every request is r^r, the linear resource-consumption rate s is drawn randomly from H, and the action set is $[0, 1]$ (to represent the fraction of the request accepted), i.e., for request $(r^r, s, [0, 1])$ and action $y \in [0, 1]$, the reward is $r^r \cdot y$ and the amount of resource consumed is $s \cdot y$. It is relatively straightforward to see that the optimal action at each time-step is to set a threshold x and accept a request $(r^r, s, [0, 1])$ (set $y = 1$) if and only if the consumption rate s is less than or equal to the threshold x. For such a threshold x, the expected cost is given by

$$\mathbb{E}_{s\sim H}[s \cdot \mathbb{1}(s \leq x)] = xH(x) - \int_0^x H(s)ds = x^{\frac{1}{1-r}} - (1 - r) \cdot x^{\frac{1}{1-r}} = r \cdot x^{\frac{1}{1-r}}$$

and the expected reward is given by $r^r \cdot H(x) = r^r \cdot x^{\frac{r}{1-r}}$. Therefore, the expected reward is equal to the expected cost raised to the power r, and consequently this request distribution behaves like

$(f_r, I, \mathcal{X})$ in expectation.

B.2.5 Upper Bound with Distributional Knowledge of Horizon

In this appendix, we show that the upper bound of Theorem 6 holds even when the horizon T drawn from a distribution $\mathbb{T}$ supported on $[\tau_1, \tau_2]$ and this distribution is known to the decision maker. This is because the reward functions $f_r(\cdot)$ are concave and $I(\cdot)$ is linear, which makes the problem of maximizing the competitive ratio a convex problem that satisfies strong duality. As we note in the following theorem, the dual variables give rise to a distribution $\mathbb{T}_r$ of the horizon which leads to the same expected performance ratio as the competitive ratio.

Proposition 12. *For every $r \in (0, 1)$, there exists a distribution $\mathbb{T}_r$ of the horizon T such that*

$$\sup_A \mathbb{E}_{T\sim\mathbb{T}_r}\left[\frac{R(A|T, r)}{\mathrm{OPT}(T, r)}\right] = \sup_A \min_{T\in[\tau_1,\tau_2]} \frac{R(A|T, r)}{\mathrm{OPT}(T, r)},$$

where $\sup_A$ denotes the supremum over all online algorithms A with the knowledge of the horizon distribution $\mathbb{T}_r$ and the request distribution $\mathcal{P}_r$.

Proof of Proposition 12. Fix $r \in (0, 1)$. We begin by showing that for all $r \in (0, 1)$, we have

$$\begin{aligned}
\sup_A \min_{T\in[\tau_1,\tau_2]} \frac{R(A|T, r)}{\mathrm{OPT}(T, r)} \quad = \quad \max_{z, x_t} \quad & z \\
\text{s.t.} \quad & z \le \frac{\sum_{t=1}^{T} f_r(x_t)}{\mathrm{OPT}(T, r)} && \forall T \in [\tau_1, \tau_2] \\
& \sum_{t=1}^{\tau_2} x_t \le B \\
& x_t \in \mathcal{X} && \forall t \in [\tau_2]
\end{aligned} \tag{B.11}$$

The RHS is a convex program because f_r is concave. Let A be any online algorithm. Let $x(A)_t$ denote the action taken by A at time t. Then, $x_t = x(A)_t$ and

$$z = \min_{T\in[\tau_1,\tau_2]} \frac{\sum_{t=1}^{T} f_r(x(A)_t)}{\mathrm{OPT}(T, r)} = \min_{T\in[\tau_1,\tau_2]} \frac{R(A|T, r)}{\mathrm{OPT}(T, r)}$$

is a feasible solution of the convex program, thereby establishing the '≤' direction. The other direction is equally straightforward: Any feasible solution (x_t, z) of the convex program naturally gives rise to an algorithm which takes action x_t at time t and achieves the desired competitive ratio.

Note that $x_t = 1/2$ and $z = 0$ is a feasible solution of the convex program that satisfies

$$z < \frac{\sum_{t=1}^{T} f_r(x_t)}{\text{OPT}(T, r)} \quad \forall T \in [\tau_1, \tau_2]$$

Therefore, by Slater's condition (see [BHM98]), we get that strong duality holds and there exists an optimal dual multiplier $\{p_T\}_{T=\tau_1}^{\tau_2}$ associated with the constraints in (B.11) such that $p_T \geq 0$ for all $T \in [\tau_1, \tau_2]$ and

$$\begin{aligned}
\sup_A \min_{T\in[\tau_1,\tau_2]} \frac{R(A|T,r)}{\text{OPT}(T,r)} \quad = \quad \max \quad & z + \sum_{T=\tau_1}^{\tau_2} p_T \left(\frac{\sum_{t=1}^{T} f_r(x_t)}{\text{OPT}(T,r)} - z \right) \\
\text{s.t.} \quad & \sum_{t=1}^{\tau_2} x_t \leq B \\
& x_t \in \mathcal{X} \qquad\qquad \forall t \in [\tau_2] \\
= \quad \max \quad & z \left(1 - \sum_{T=\tau_1}^{\tau_2} p_T \right) + \sum_{T=\tau_1}^{\tau_2} p_T \cdot \frac{\sum_{t=1}^{T} f_r(x_t)}{\text{OPT}(T,r)} \\
\text{s.t.} \quad & \sum_{t=1}^{\tau_2} x_t \leq B \\
& x_t \in \mathcal{X} \qquad\qquad \forall t \in [\tau_2]
\end{aligned}$$

Observe that, since z is an unrestricted variable, we need $\sum_{T=\tau_1}^{\tau_2} p_T = 1$ to ensure that the restated (Lagrangian) optimization problem is bounded, which is necessary because the LHS is bounded. Hence, we get

$$\begin{aligned}
\sup_A \min_{T\in[\tau_1,\tau_2]} \frac{R(A|T,r)}{\text{OPT}(T,r)} \quad = \quad \max \quad & \sum_{T=\tau_1}^{\tau_2} p_T \cdot \frac{\sum_{t=1}^{T} f_r(x_t)}{\text{OPT}(T,r)} \\
\text{s.t.} \quad & \sum_{t=1}^{\tau_2} x_t \leq B \\
& x_t \in \mathcal{X} \qquad\qquad \forall t \in [\tau_2]
\end{aligned}$$

Let $\mathbb{T}_r$ be the distribution which picks horizon T with probability p_T for all $T \in [\tau_1, \tau_2]$. Then, once again, we can use the equivalence between feasible solutions of the convex program and the actions of an online algorithm ($x_t = x(A)_t$) to get

$$\begin{aligned}
\sup_A \min_{T\in[\tau_1,\tau_2]} \frac{R(A|T,r)}{\mathrm{OPT}(T,r)} \quad = \quad \max \quad & \sum_{T=\tau_1}^{\tau_2} p_T \cdot \frac{\sum_{t=1}^T f_r(x_t)}{\mathrm{OPT}(T,r)} \quad = \quad \sup_A \mathbb{E}_{T\sim\mathbb{T}_r}\left[\frac{R(A|T,r)}{\mathrm{OPT}(T,r)}\right] \\
\text{s.t.} \quad & \sum_{t=1}^{\tau_2} x_t \leq B \\
& x_t \in \mathcal{X} \quad \forall t \in [\tau_2]
\end{aligned}$$

as required. □

Combining Proposition 12 and Theorem 6 immediately yields the following corollary.

Corollary 2. *For all $r \in (0,1)$ and $1 \leq \tau_1 \leq \tau_2$, there exists a horizon distribution $\mathbb{T}_r$ such that every online algorithm A with knowledge of $\mathbb{T}_r$ satisfies*

$$\mathbb{E}_{T\sim\mathbb{T}_r}\left[\frac{R(A|T,r)}{\mathrm{OPT}(T,r)}\right] \leq \frac{1}{\left(1 + (1-r)^{1/r} \cdot \ln(\tau_2/\tau_1) + \ln\left(\frac{\tau_1}{\tau_1+1}\right)\right)^r} .$$

In particular, for $r = 1 - \{1/\ln\ln(\tau_2/\tau_1)\}$, $\tau_2/\tau_1 > e^e$ and $\tau_1 \geq 1$, we have

$$\mathbb{E}_{T\sim\mathbb{T}_r}\left[\frac{R(A|T,r)}{\mathrm{OPT}(T,r)}\right] \leq \frac{e \cdot \ln\ln(\tau_2/\tau_1)}{\ln(\tau_2/\tau_1)} .$$

The above bounds hold even for online algorithms that have prior knowledge of $\mathcal{P}_r$ before time $t = 1$.

B.3 Proof of Proposition 3

Proof. Consider a target consumption sequence λ. Set

$$y_{T,t} = \min\left\{\min_{1\leq j\leq m} \frac{\lambda_{t,j}}{\rho_{T,j}}, 1\right\} \quad \text{and} \quad z = \min_{T\in[\tau_1,\tau_2]} \frac{1}{T} \cdot \sum_{t=1}^{T} \min\left\{\min_{1\leq j\leq m} \frac{\lambda_{t,j}}{\rho_{T,j}}, 1\right\} .$$

Then, (λ, z, y) is a feasible solution of the LP with objective value $\min_{T\in[\tau_1,\tau_2]} c(\vec{\lambda}, T)$. Hence, we get LHS ≤ RHS. To prove LHS ≥ RHS, consider any feasible solution (λ, z, y). Then, λ is a target consumption sequence and

$$z \le \min_{T\in[\tau_1,\tau_2]} \frac{1}{T} \cdot \sum_{t=1}^{T} y_{T,t} \le \min_{T\in[\tau_1,\tau_2]} \frac{1}{T} \cdot \sum_{t=1}^{T} \min\left\{\min_{1\le j\le m} \frac{\lambda_{t,j}}{\rho_{T,j}}, 1\right\} = \min_{T\in[\tau_1,\tau_2]} c(\vec{\lambda}, T)$$

where the first inequality follows constraints

$$z \le \frac{1}{T}\sum_{t=1}^{T} y_{T,t} \qquad \forall T \in [\tau_1, \tau_2]\,,$$

and the second inequality follows from constraints

$$y_{T,t} \le \frac{\lambda_{t,j}}{\rho_{T,j}} \qquad \forall j \in [m], T \in [\tau_1, \tau_2], t \in [T]$$

$$y_{T,t} \le 1 \qquad \forall T \in [\tau_1, \tau_2], t \in [T]\,.$$

Therefore, we have LHS ≥ RHS, as required. □

B.4 Proofs of Section 3.4

B.4.1 Proof of Proposition 4

Proof. Consider a target consumption sequence λ such that $\min_{T\in[\tau_1,\tau_2]} c(\vec{\lambda}, T) \ge \gamma$. Set

$$y_{T,t} = \min\left\{\min_{1\le j\le m} \frac{\lambda_{t,j}}{\rho_{T,j}}, 1\right\}\,.$$

Then, (λ, y) is a feasible solution of the LP with objective value

$$\frac{1}{T_P}\sum_{t=1}^{T_P} y_{T_P,t} = \frac{1}{T_P}\sum_{t=1}^{T_P} \min\left\{\min_{1\le j\le m} \frac{\lambda_{t,j}}{\rho_{T_P,j}}, 1\right\} = c(\vec{\lambda}, T_P)\,.$$

Hence, we get LHS ≤ RHS.

To prove LHS ≥ RHS, consider any feasible solution (λ, y). Then, λ is a target consumption sequence and we have

$$y_{T,t} \leq \min\left\{\min_{1\leq j\leq m} \frac{\lambda_{t,j}}{\rho_{T,j}}, 1\right\} ,$$

for all $T \in [\tau_1, \tau_2], t \in [T]$, where the inequality follows from constraints

$$y_{T,t} \leq \frac{\lambda_{t,j}}{\rho_{T,j}} \qquad \forall j \in [m], T \in [\tau_1, \tau_2], t \in [T]$$

$$y_{T,t} \leq 1 \qquad \forall T \in [\tau_1, \tau_2], t \in [T] .$$

Therefore, we get

$$c(\vec{\lambda}, T_P) = \frac{1}{T_P} \sum_{t=1}^{T_P} \min\left\{\min_{1\leq j\leq m} \frac{\lambda_{t,j}}{\rho_{T_P,j}}, 1\right\} = c(\vec{\lambda}, T_P) = \frac{1}{T_P} \sum_{t=1}^{T_P} y_{T_P,t}$$

and

$$c(\vec{\lambda}, T) = \frac{1}{T} \sum_{t=1}^{T} \min\left\{\min_{1\leq j\leq m} \frac{\lambda_{t,j}}{\rho_{T,j}}, 1\right\} = c(\vec{\lambda}, T_P) \geq \frac{1}{T} \sum_{t=1}^{T} y_{T,t} \geq \gamma .$$

Consequently, we have $\min_{T\in[\tau_1,\tau_2]} c(\vec{\lambda}, T) \geq \gamma$ and the objective of the LP is at most $c(\vec{\lambda}, T_P)$. Hence, LHS ≥ RHS as required. □

B.4.2 Proof of Proposition 5

Proof of Proposition 5. It is straightforward to see that $\vec{\lambda}$ satisfies the budget constraint:

$$\begin{aligned}
\sum_{t=1}^{T} \lambda_t &= \sum_{t=1}^{\tau_1} \frac{\alpha}{1 + \ln(\tau_2/\tau_1)} \cdot \frac{B}{\tau_1} + \sum_{t=\tau_1+1}^{\tau_2} \frac{\alpha}{1 + \ln(\tau_2/\tau_1)} \cdot \frac{B}{t} + \sum_{t=1}^{T_P} (1 - \alpha) \cdot \frac{B}{T_P} \\
&= \frac{\alpha B}{1 + \ln(\tau_2/\tau_1)} \cdot \left(\frac{\tau_1}{\tau_1} + \sum_{t=\tau_1+1}^{\tau_2} \frac{1}{t}\right) + (1 - \alpha) \cdot B \\
&\leq \frac{\alpha B}{1 + \ln(\tau_2/\tau_1)} \cdot (1 + \ln(\tau_2/\tau_1)) + (1 - \alpha) \cdot B
\end{aligned}$$

$$= B\,.$$

Moreover, note that for any $T \in [\tau_1, \tau_2]$, we have

$$c(\vec{\lambda}, T) = \frac{1}{T}\sum_{t=1}^{T} \min\left\{\min_{1\le j\le m} \frac{\lambda_{t,j}}{\rho_{T,j}}, 1\right\} \ge \frac{1}{T}\sum_{t=1}^{T} \frac{\alpha}{1+\ln(\tau_2/\tau_1)} = \frac{\alpha}{1+\ln(\tau_2/\tau_1)}\,,$$

and

$$c(\vec{\lambda}, T_P) = \frac{1}{T_P}\sum_{t=1}^{T_P} \min\left\{\min_{1\le j\le m} \frac{\lambda_{t,j}}{\rho_{T_P,j}}, 1\right\} \ge \frac{1}{T_P}\sum_{t=1}^{T_P} 1-\alpha+\frac{\alpha}{1+\ln(\tau_2/\tau_1)} = 1-\alpha+\frac{\alpha}{1+\ln(\tau_2/\tau_1)}\,,$$

where we have used the fact that $\rho_t \ge \rho_T$ for all $t \le T$. Hence, we have shown that Algorithm 3 with target sequence $\vec{\lambda}$ is $(\gamma-\epsilon)$-competitive, where $\gamma = \alpha\cdot(1+\ln(\tau_2/\tau_1))^{-1}$, and $(1-\alpha)$-consistent on prediction T_P. Since $\vec{\lambda}$ is one possible choice of the target consumption sequence, the proposition holds. □

B.5 Proofs for Section 3.5

B.5.1 Proof of Theorem 7

Proof of Theorem 7. To simplify exposition, we define

$$a_T = \begin{cases} \beta & \text{if } T = T_P \\ \gamma & \text{if } T \ne T_P \end{cases}$$

Hence,

$$c(\vec{\lambda}', T_P) \ge \beta \quad \text{and} \min_{T\in[\tau_1,\tau_2]} c(\vec{\lambda}', T) \ge \gamma \quad \iff c(\vec{\lambda}', T) \ge a_T \quad \forall\, T \in [\tau_1, \tau_2]\,.$$

We start by proving some important properties of Algorithm 4. First, we show that

$$\frac{\lambda_{t,j}}{\rho_{T,j}} = \frac{\lambda_{t,k}}{\rho_{T,k}} \qquad \forall \quad j, k \in [m], t \in [T]$$

throughout the run of the algorithm. We do so via induction on each update of $\vec{\lambda}$ (see (3.9)). Initially, $\vec{\lambda} = 0$ so the statement holds trivially. Suppose it holds before the update. Observe that

$$\min \left\{ \rho_{T,j} - \lambda_{t,j}, a_T \cdot B_j - \sum_{s=1}^{T} \lambda_{s,j} \right\} = \rho_{T,j} \min \left\{ 1 - \frac{\lambda_{t,j}}{\rho_{T,j}}, a_T \cdot T - \sum_{s=1}^{T} \frac{\lambda_{s,j}}{\rho_{T,j}} \right\} .$$

Let $\vec{\lambda}'$ be the sequence after the update in (3.9). Then, we have

$$\frac{\lambda'_{t,j}}{\rho_{T,j}} = \frac{\lambda_{t,j}}{\rho_{T,j}} + \min \left\{ 1 - \frac{\lambda_{t,j}}{\rho_{T,j}}, a_T \cdot T - \sum_{s=1}^{T} \frac{\lambda_{s,j}}{\rho_{T,j}} \right\} .$$

Since the RHS is the same for all $j \in [m]$ by the induction hypothesis, the statement holds after the update, thereby completing the induction step. Therefore, we have

$$c(\vec{\lambda}^*, T) = \frac{1}{T} \sum_{t=1}^{T} \min \left\{ \min_{1 \le j \le m} \frac{\lambda^*_{t,j}}{\rho_{T,j}}, 1 \right\} = \min_{1 \le j \le m} \frac{1}{T} \sum_{t=1}^{T} \min \left\{ \frac{\lambda^*_{t,j}}{\rho_{T,j}}, 1 \right\} = \min_{1 \le j \le m} \frac{1}{B_j} \sum_{t=1}^{T} \min \left\{ \lambda^*_{t,j}, \rho_{T,j} \right\} . \tag{B.12}$$

We prove an intermediate lemma that will prove useful later

Lemma 33. *For each outer* **For** *loop counter T, the inner* **For** *loop always maintains $\lambda_{t,j} \le \rho_{T,j}$ and one of the following holds at its termination:*

- $\sum_{s=1}^{T} \lambda_{s,j} = a_T \cdot B_j$.
- $\sum_{s=1}^{T} \lambda_{s,j} \ge a_T \cdot B_j$ *held before the first iteration and $\lambda_{t,j}$ was not modified during any of the iterations of the inner* **For** *loop.*

Proof. This is because, for each resource $j \in [m]$, exactly one of the following cases holds after each iteration of the inner **For** loop in which $\lambda_{t,j}$ was modified:

- $\lambda_{t,j} = \rho_{T,j}$ and $\sum_{s=1}^{T} \lambda_{s,j} < a_T \cdot B_j$

- $\sum_{s=1}^{T} \lambda_{s,j} = a_T \cdot B_j$

Now, suppose $\sum_{s=1}^{T} \lambda_{s,j} < a_T \cdot B_j$ at termination of the inner **For** loop. Then, $\lambda_{t,j} = \rho_{T,j}$ for all $t \in [T]$ and $a_T \cdot B_j - \sum_{s=1}^{T} \lambda_{s,j} \leq B_j - T \cdot \rho_{T,j} \leq 0$, which contradicts $\sum_{s=1}^{T} \lambda_{s,j} < a_T \cdot B_j$. Hence, the lemma holds. □

In both cases, at termination we have

$$\sum_{s=1}^{T} \lambda_{s,j} \geq a_T \cdot B_j \quad \forall j \in [m] \,. \tag{B.13}$$

As we only ever increase $\vec{\lambda}$ in (3.9), we get

$$\sum_{t=1}^{T} \min\{\lambda^*_{t,j}, \rho_{T,j}\} \geq a_T \cdot B_j \quad \forall\, T \in [\tau_1, \tau_2]$$

for all $j \in [m]$. Therefore, $c(\vec{\lambda}^*, T) \geq a_T$ for all $T \in [\tau_1, \tau_2]$ by (B.12). Part (1) of the theorem follows as a direct consequence and we focus on part (2) in the remainder

We are now ready to prove the theorem. We begin with the “only if" direction. Suppose $\sum_{t=1}^{\tau_2} \lambda^*_t \leq B$. Since we have shown that $c(\vec{\lambda}^*, T) \geq a_T$ for all $T \in [\tau_1, \tau_2]$, this makes $\vec{\lambda}' = \vec{\lambda}^*$ the required target consumption sequence.

For the other direction, assume that there exists a target consumption sequence $\vec{\lambda}^o$ (with $\sum_{t=1}^{\tau_2} \lambda^o_t \leq B$) which satisfies $c(\vec{\lambda}^o, T) \geq a_T$ for all $T \in [\tau_1, \tau_2]$. Let $\vec{\lambda}'$ be the target consumption sequence which minimizes $\sum_{k=1}^{m} \sum_{t=1}^{T} t \cdot \lambda^o_{t,j}$ among all such sequences, i.e.,

$$\begin{aligned}
\vec{\lambda}' \in \operatorname{argmin}_{\vec{\lambda}^o} \quad & \sum_{k=1}^{m} \sum_{t=1}^{T} t \cdot \lambda^o_{t,j} & \\
\text{s.t.} \quad & c(\vec{\lambda}^o, T) \geq a_T & \forall\, T \in [\tau_1, \tau_2] \\
& \sum_{t=1}^{\tau_2} \lambda^o_t \leq B &
\end{aligned}$$

By (3.8), we get

$$\sum_{t=1}^{T} \min\{\lambda'_{t,j}, \rho_{T,j}\} \geq a_T \cdot B_j \quad \forall\, T \in [\tau_1, \tau_2], j \in [m]\,.$$

To prove $\sum_{t=1}^{\tau_2} \lambda_t^* \leq B$, it suffices to show that $\lambda^*_{t,j} \leq \lambda'_{t,j}$ for all $t \in [t]$, $j \in [m]$. For contradiction, suppose the latter does not hold. In what follows, we will use $\vec{\lambda}^{(T^*)}$ to denote the value of $\vec{\lambda}$ at the end of the T-th iteration of the outer **For** loop.

Let $T = T^*$ and $t = t^*$ be the outer and inner **For** loop counters respectively for the update (3.9) at the end of which $\lambda_{t^*,k} > \lambda'_{t^*,k}$ for some resource $k \in [m]$ for the first time during the run of Algorithm 4. Since $c(\vec{\lambda}', T^*) \geq a_T$, (3.8) implies that $\sum_{t=1}^{T^*} \min\{\lambda'_{t,k}, \rho_{T,k}\} \geq a_T \cdot B_k$. Since $\lambda_{t^*,k}$ had to be modified to get $\lambda_{t^*,k} > \lambda'_{t^*,k}$ for the first time, Lemma 33 implies that the inner **For** loop will terminate with $\sum_{t=1}^{T^*} \min\left\{\lambda^{(T^*)}_{t,k}, \rho_{T,k}\right\} = a_T \cdot B_k$. Therefore, there must exist a $t^* < s \leq T^*$ such that $\lambda'_{s,k} > \lambda^{(T^*)}_{s,k}$ after the T^*-th iteration of the outer **For** loop.

Now, pick $\nu < \min\left\{\lambda'_{s,k} - \lambda^{(T^*)}_{s,k}, \lambda^{(T^*)}_{t^*,k} - \lambda'_{t^*,k}\right\}$ and define a new target consumption sequence $\vec{\lambda}''$ which is exactly the same as $\vec{\lambda}'$ except $\lambda''_{t^*,k} = \lambda'_{t^*,k} + \nu$ and $\lambda''_{s,k} = \lambda'_{s,k} - \nu$. Since $t^* < s$, we get

$$\sum_{j=1}^{m} \sum_{t=1}^{T} t \cdot \lambda''_{t,j} < \sum_{j=1}^{m} \sum_{t=1}^{T} t \cdot \lambda'_{t,j}\,.$$

If we can show that $c(\vec{\lambda}'', T) \geq a_T$ for all $T \in [\tau_1, \tau_2]$, we will contradict the minimality of $\vec{\lambda}'$. To see this, consider the following cases

- $T > T^*$: $\lambda'' \geq \lambda^{(T)}$ by definition of T^*, t^* and ν. Hence, (B.13) implies

  $$\sum_{t=1}^{T} \min\{\lambda''_{t,j}, \rho_{T,j}\} \geq \sum_{t=1}^{T} \min\{\lambda^{(T)}_{t,j}, \rho_{T,j}\} \geq a_T \cdot B_j \quad \forall\, j \in [m]\,,$$

  and consequently $c(\vec{\lambda}'', T) \geq a_T$.

- $T \leq T^*$: Observe that $\lambda''_{t^*,k} \leq \lambda^{(T)}_{t^*,k} \leq \rho_{T^*,k} \leq \rho_{T,k}$. Recall that $\lambda''_{t^*,k} = \lambda'_{t^*,k} + \nu$ and

$\lambda''_{s,k} = \lambda'_{s,k} - \nu$ where $t^* < s$, and $\lambda''_{t,j} = \lambda'_{t,j}$ otherwise. Therefore,

$$\sum_{t=1}^{T} \min\{\lambda''_{t,j}, \rho_{T,j}\} \geq \sum_{t=1}^{T} \min\{\lambda'_{t,j}, \rho_{T,j}\} \geq a_T \cdot B_j \quad \forall\, j \in [m]\,,$$

and consequently $c(\vec{\lambda}'', T) \geq a_T$.

Thus we have established the required contradiction, thereby completing the proof. □

B.5.2 Binary Search Procedure

We explain how to use Algorithm 4 to find an ε-approximate solution to the LP in Proposition 4. A similar procedure can be used to compute an ε-approximate solution to the LP in Proposition 3.

Consider a required level of competitiveness $\gamma \geq 0$. Then, we can run binary search to find the highest consistency that can be achieved by any target consumption sequence which is γ competitive as follows:

- Initialize $\ell = 0$ and $u = 1$
- Set $\beta = (u + \ell)/2$. Run Algorithm 4. If it returns TRUE, set $\ell = \beta$, otherwise set $u = \beta$. Repeat this step till $u - \ell \leq \varepsilon$.

Let $\vec{\lambda}'$ be the optimal solution of the LP in Proposition 4, i.e.,

$$\vec{\lambda}' \in \operatorname{argmax}_{\vec{\lambda} \geq 0} c(\vec{\lambda}, T_P) \quad \text{s.t.} \min_{T \in [\tau_1, \tau_2]} c(\vec{\lambda}, T) \geq \gamma \text{ and } \sum_{t=1}^{\tau_2} \lambda_t \leq B\,.$$

Then, part (2) of Theorem 7 implies that Algorithm 4 returns TRUE if and only if $\beta \leq c(\vec{\lambda}', T_P)$. Consequently, $\ell \leq c(\vec{\lambda}', T_P) \leq u$ at all times during the run of the binary search procedure, which further implies that $\ell \geq c(\vec{\lambda}', T_P) - \varepsilon$ at termination. Let $\vec{\lambda}^*$ be the sequence computed by Algorithm 4 when given $\beta = \ell$. Then, Theorem 7 implies that

$$c(\vec{\lambda}^*, T_P) \geq c(\vec{\lambda}^*, T_P) - \varepsilon, \quad \min_{T \in [\tau_1, \tau_2]} c(\vec{\lambda}^*, T) \geq \gamma \quad \text{and } \sum_{t=1}^{\tau_2} \lambda^*_t \leq B\,,$$

as required.

Appendix C: Appendix to Chapter 4

C.1 Counter Example for Deterministic Context

Example 7. *Consider an auction with $n = 2$ budget-constrained buyers per auction. Buyers draw their value v uniformly from the interval $[0, 1]$ and each with a budget of $1/8$, i.e., $T = \{(v, 1/8) \in \mathbb{R}^2 \mid 0 \leq v \leq 1\}$ is the type space where the first component denotes the value and the second one denotes the budget. (A uniform distribution of values can be achieved by a number of fixed contexts and weight vector distributions, for example suppose the item context is $\alpha = (1)$ and the weight vectors w are distributed uniformly in $[0, 1]$. This would yield values $v = w^T \alpha$ that are uniformly distributed) As in our model, the buyers would like to satisfy their budget constraint in expectation at the interim stage: A buyer with value v would like to spend less than $1/8$ in expectation over the value of the other buyer. Moreover, assume that the ties are broken uniformly. We will show that there does not exist a symmetric continuous non-decreasing Bayes-Nash equilibrium strategy $\beta : T \to \mathbb{R}_{\geq 0}$ for this example.*

Let F denote the distribution of bids under β. We first show that F must contain an atom. For contradiction, suppose not, i.e., F is atomless. Since F is atomless β should strictly increasing. Then, the probability that a buyer with value v wins the item in equilibrium is given by v. This follows because the bidder with the highest value wins when strategies are symmetric and strictly increasing together with the fact that values are uniformly distributed. Therefore, if the buyer with value 1 bids b, her expected expenditure is given by b, which must be less than or equal to $1/8$ due to the budget constraint. Hence, $\beta(v) \leq 1/8$ for all $v \in [0, 1]$. It is easy to see that the optimal bid for any buyer with value $v \in [1/2, 1]$, in response to the other buyer using β, is $1/8$. This contradicts the assumption that F is atomless.

Hence, F has an atom b^*. As β is non-decreasing, there exists an interval $[x, x+\epsilon]$, where $\epsilon > 0$, *such that $\beta(v) = b^*$ for all $v \in [x, x+\epsilon]$ and $\beta(v) < b^*$ for all $v < x$. If $b^* = x = 0$, then bidding infinitesimally more than b^* is strictly better for a buyer with value $x + \epsilon$ because her probability of winning increases by at least $\epsilon/2$ without violating her budget constraint, thereby contradicting the fact that β is a BNE. Hence, we have $0 < b^* = \beta(x) < x$, because if $b^* = x$, then bidding slightly less than b^* would give the buyer with value x a higher utility. Finally, the continuity of β implies that, for a buyer with a value that is infinitesimally smaller than x, it is optimal to bid b^* since it increases her probability of winning by at least $\epsilon/2$ with only an infinitesimal increase in bid. This contradicts the definition of a BNE, thereby implying that no symmetric continuous non-decreasing BNE strategy exists for this example.* □

It is worth noting that a BNE does exist for the above example if the seller employs the second-price auction. In particular, we claim that the following strategy forms a BNE for the second-price auction:

$$\beta(v) = \begin{cases} 1/4 & \text{if } v \geq 1/4 \\ v & \text{if } v < 1/4 \end{cases}$$

First, observe that if a buyer bids $b > 1/4$, her total expected expenditure (expectation over the other buyer's value) is given by

$$\frac{3}{4} \cdot \frac{1}{4} + \int_0^{1/4} v dv = \frac{7}{32}$$

which is strictly greater than $1/8$. Therefore no buyer can bid strictly more than $1/4$ without violating her budget constraint. Moreover, a buyer who bids exactly $1/4$ spends

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4} + \int_0^{1/4} v dv = \frac{1}{8}$$

and satisfies her budget constraint.

Consider a buyer with value $v > 1/4$ and suppose the competing buyer bids using β. As argued above, her budget constraints her to select a bid $b \leq 1/4$. Her utility from bidding $b < 1/4$ is given by $v \cdot b - b^2/2$, which is at most $v \cdot (1/4) - (1/4)^2/2 = v/4 - 1/32$. On the other hand, the utility she receives from bidding $b = 1/4$ is given by

$$v \left[\frac{1}{2} \cdot \frac{3}{4} + \frac{1}{4} \right] - \frac{1}{8}$$

which is strictly greater than $v/4 - 1/32$ because $v > 1/4$. Next, consider a buyer with value $v \leq 1/4$. If she ignores her budget constraint, it is a weakly dominant strategy to bid her value. As we have shown above, bidding her value also respects her budget constraint and is therefore a best response. Hence, we have shown that, if the other buyer bids using β, it is a best response for any buyer with value $v > 1/4$ to bid $1/4$ and for any buyer with value $v \leq 1/4$ to bid v, as desired.

C.2 Existence of Symmetric First-Price Equilibrium

C.2.1 Preliminaries on Continuity

The following lemma establishes the almost sure continuity of the CDF of the distribution of the maximum of paced values H^μ_α, which is used extensively in our analysis.

Lemma 34. *For every $\mu : \Theta \to \mathbb{R}_{\geq 0}$, the following properties hold:*

a. λ^μ_α and H^μ_α have continuous CDFs almost surely w.r.t. $\alpha \sim F$

b. σ^μ_α is continuous almost surely w.r.t. $\alpha \sim F$

c. σ^μ_α is non-decreasing. Furthermore, for $x \in [0, \omega]$ and $\alpha \in A$ such that H^μ_α is continuous, the following statement holds almost surely w.r.t. $Y \sim H^\mu_\alpha$,

$$\mathbb{1}\{x \geq r(\alpha), x \geq Y\} = \mathbb{1}\left\{\sigma^\mu_\alpha(x) \geq r(\alpha),\ \sigma^\mu_\alpha(x) \geq \sigma^\mu_\alpha(Y)\right\}$$

d. Almost surely w.r.t. $\alpha \sim F$, when $x_1, x_2 \sim \lambda^\mu_\alpha$ i.i.d., the probability of $\sigma^\mu_\alpha(x_1) = \sigma^\mu_\alpha(x_2)$ is zero.

Part (a) states that the distributions of paced values are atomless almost surely w.r.t. the items $\alpha \sim F$. This property is crucial because it allows us to leverage the known result establishing the existence of a symmetric equilibrium in the i.i.d. setting under arbitrary tie-breaking rules, which holds only if the distribution of values is atom-less. Part (b) is a direct consequence of the definition of σ_α^μ. Part (c) follows from part (a). Part (c) says that when everyone uses the strategy σ_α^μ, with probability 1, a buyer who has paced value x for item α has the highest bid if and only if she has the highest paced value, which plays a key role in our analysis. Finally, part (d), says that ties are a zero probability event when players use the value-pacing-based strategy. We will need the following lemma to prove Lemma 34

Lemma 35. *Consider a set $Y = \{y_\alpha\}_{\alpha \in I}$ with $y_\alpha > 0$, where I is an index set. If I is uncountable, then there exists a countable sequence $\{\alpha_n\}_{n \in \mathbb{N}} \subset I$ such that $\sum_{n \in \mathbb{N}} y_{\alpha_n} = \infty$.*

Proof. Rewrite I as $I = \cup_{n \in \mathbb{Z}_+} \{\alpha \in I \mid y_\alpha \geq 1/n\}$. It is a well-known fact that a countable union of countable sets is countable (see Theorem 2.12 of [Rud+64]). Therefore, in order for I to be uncountable, there must exist n_0 such that $\{\alpha \in I \mid y_\alpha \geq 1/n_0\}$ is uncountable. It follows that we can find a countable sequence $\{y_{\alpha_n}\}_{n \in \mathbb{N}}$ such that $y_{\alpha_n} \geq 1/n_0$ for all $n \in \mathbb{N}$. For this sequence, $\sum_{n \in \mathbb{N}} y_{\alpha_n} = \infty$. ◻

We now state the proof of Lemma 34.

Proof of Lemma 34.

a. Consider a pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$. Let $\alpha_1, \alpha_2 \in A$ be linearly independent feature vectors and $x_1, x_2 \in [0, \omega]$ be two possible item values. We consider the set of buyer types which have paced value x_1 for α_1 and paced value x_2 for α_2, i.e., define

$$S := \left\{ (w, B) \in \Theta \middle| \frac{w^T \alpha_1}{1 + \mu(w, B)} = x_1;\ \frac{w^T \alpha_2}{1 + \mu(w, B)} = x_2 \right\}$$

Observe that, for $(w, B) \in S$ and $c := x_1/x_2$, we have $w^T \alpha_1 = c \cdot w^T \alpha_2$. Therefore, the set $T = \{w \in \Theta_w \mid w^T(\alpha_1 - c\alpha_2) = 0\}$ is a superset of the set S_w. Hence, $S \subset T \times (B_{\min}, U)$, which in combination with the assumption that G has a density implies $G(S) = 0$.

Define $J = \left\{\alpha/\|\alpha\| \mid \exists\, x_\alpha > 0 \text{ s.t. } G(\{(w, B) \mid w^T\alpha/(1 + \mu(w, B) = x_\alpha\}) > 0)\right\}$. Suppose J is uncountable. Then, by Lemma 35, there exists a countable sequence $\{\alpha_m\}_{m\in\mathbb{N}}$ and $\{x_{\alpha_m}\}_{m\in\mathbb{N}}$ such that $\alpha_i/\|\alpha_i\| \neq \alpha_j/\|\alpha_j\|$ for all $i \neq j$ and

$$\sum_m G(\{(w, B) \mid w^T\alpha_m/(1 + \mu(w, B) = x_{\alpha_m}\}) > 0) = \infty.$$

Set $S_m := \{(w, B) \mid w^T\alpha_m/(1 + \mu(w, B) = x_{\alpha_m}\}$. We have shown above that $G(S_i \cap S_j) = 0$ for all $i \neq j$. Therefore, for all $m \geq 1$, we have $G(S_m \cap\ (\cup_{j<m} S_j)) = 0$, which implies $G(S_m \cap\ (\cup_{j<m} S_j)^C) = G(S_m)$. This contradicts $G(\cup_m S_m) \leq 1$ as $G(\cup_m S_m) = \sum_m G(S_m \cap (\cup_{j<m} S_j)^C) = \sum_m G(\alpha_m^T s = x_{\alpha_m}) = \infty$. Hence, J is countable. Observe that

$$\left\{\frac{\alpha}{\|\alpha\|} \,\middle|\, \lambda_\alpha^\mu \text{ has an atom}\right\} \subset J$$

As F has a density, we get $F(\text{cone}(J)) = 0$. Therefore, λ_α^μ has no atoms almost surely w.r.t. $\alpha \in A$, i.e., λ_α^μ has a continuous CDF almost surely w.r.t. $\alpha \sim F$. Moreover, this implies that H_α^μ has a continuous CDF almost surely w.r.t. $\alpha \sim F$.

b. Follows from the fact that the integral of every bounded function is continuous.

c. Using Lemma 2.2.8 from [Dur19], we can write

$$\sigma_\alpha^\mu(x) = x - \int_{r(\alpha)}^x \frac{H_\alpha^\mu(s)}{H_\alpha^\mu(x)} ds = r(\alpha) + \int_{r(\alpha)}^x \frac{H_\alpha^\mu(x) - H_\alpha^\mu(s)}{H_\alpha^\mu(x)} ds = \mathbb{E}_{Y\sim H_\alpha^\mu}\left[\max\{Y, r(\alpha)\} \mid Y < x\right]$$

From the last term, it can be easily seen that σ_α^μ is non-decreasing.

Observe that $\mathbb{1}(x \geq r(\alpha), x \geq Y) \leq \mathbb{1}\left(\sigma_\alpha^\mu(x) \geq r(\alpha), \sigma_\alpha^\mu(x) \geq \sigma_\alpha^\mu(Y)\right)$ always holds as σ_α^μ is non-decreasing and $\sigma_\alpha^\mu(r(\alpha)) = r(\alpha)$. Moreover,

$$\begin{aligned}\mathbb{1}(x \geq r(\alpha), x \geq Y) < \mathbb{1}\left(\sigma_\alpha^\mu(x) \geq r(\alpha), \sigma_\alpha^\mu(x) \geq \sigma_\alpha^\mu(Y)\right) &\implies x \geq r(\alpha), x < Y,\ \sigma_\alpha^\mu(x) \geq \sigma_\alpha^\mu(Y)\\ &\implies x \geq r(\alpha), x < Y,\ \sigma_\alpha^\mu(x) = \sigma_\alpha^\mu(Y)\end{aligned}$$

because $\sigma_\alpha^\mu(x) \geq r(\alpha)$ if and only if $x \geq r(\alpha)$, and σ_α^μ is non-decreasing.

Therefore, it is enough to show for $\alpha \in A$ such that H_α^μ is continuous and $x \geq r(\alpha)$, we have

$$H_\alpha^\mu\left(\{y \in [0,\omega] \mid x < y, \sigma_\alpha^\mu(x) = \sigma_\alpha^\mu(y)\}\right) = 0$$

Suppose the above statement doesn't hold for some $\alpha \in A$ such that H_α^μ is continuous and $x \geq r(\alpha)$. Then, for $y = \sup\{t > x \mid \sigma_\alpha^\mu(t) = \sigma_\alpha^\mu(x)\}$, we have $\sigma_\alpha^\mu(y) = \sigma_\alpha^\mu(x)$ (as σ_α^μ is continuous) and $H_\alpha^\mu((x,y]) > 0$. First, consider the case when $H_\alpha^\mu(x) > 0$. Observe that

$$\begin{aligned}\sigma_\alpha^\mu(y) - \sigma_\alpha^\mu(x) &= y - x - \int_{r(\alpha)}^{y} \frac{H_\alpha^\mu(s)}{H_\alpha^\mu(y)} ds + \int_{r(\alpha)}^{x} \frac{H_\alpha^\mu(s)}{H_\alpha^\mu(x)} ds \\ &= y - x - \int_{x}^{y} \frac{H_\alpha^\mu(s)}{H_\alpha^\mu(y)} ds + \left(\frac{1}{H_\alpha^\mu(x)} - \frac{1}{H_\alpha^\mu(y)}\right) \int_{r(\alpha)}^{x} H_\alpha^\mu(s) ds \\ &> \left(\frac{1}{H_\alpha^\mu(x)} - \frac{1}{H_\alpha^\mu(y)}\right) \int_{r(\alpha)}^{x} H_\alpha^\mu(s) ds\end{aligned}$$

where the last inequality follows from $H_\alpha^\mu(y) - H_\alpha^\mu(x) = H_\alpha^\mu((x,y]) > 0$. Therefore, $\sigma_\alpha^\mu(y) > \sigma_\alpha^\mu(x)$ because $H_\alpha^\mu(y) > H_\alpha^\mu(x)$, which contradicts $\sigma_\alpha^\mu(y) = \sigma_\alpha^\mu(x)$.

Next, consider the case when $H_\alpha^\mu(x) = 0$. Then, $H_\alpha^\mu(x) = 0$ and $H_\alpha^\mu(y) = H_\alpha^\mu(x) + H_\alpha^\mu((x,y]) > 0$. Note that

$$\sigma_\alpha^\mu(y) H_\alpha^\mu(y) = y H_\alpha^\mu(y) - \int_{r(\alpha)}^{y} H_\alpha^\mu(s) ds = \int_{r(\alpha)}^{y} [H_\alpha^\mu(y) - H_\alpha^\mu(s)] ds + r(\alpha) H_\alpha^\mu(y)$$

Hence, $\sigma_\alpha^\mu(y) = 0$ if and only if $H_\alpha^\mu(s) = H_\alpha^\mu(y)$ for all $s \in [r(\alpha), y]$ and $r(\alpha) = 0$. As $H_\alpha^\mu(0) = 0$ and $H_\alpha^\mu(y) > 0$, we get $\sigma_\alpha^\mu(y) > 0$, which contradicts $\sigma_\alpha^\mu(y) = \sigma_\alpha^\mu(x)$.

d. Consider $\alpha \in A$ such that λ_α^μ has a continuous CDF and $P_{x \sim \lambda_\alpha^\mu}(\sigma_\alpha^\mu(x) = c) > 0$ for some $c \geq 0$. Then, by the definition of σ_α^μ, it must be that $c \geq r(\alpha)$. Moreover, if we let $x_0 = \inf\{x \mid \sigma_\alpha^\mu(x) = c\}$, then $P_{x \sim \lambda_\alpha^\mu}(\sigma_\alpha^\mu(x) = c) > 0$ implies

$$H_\alpha^\mu\left(\{y \in [0,\omega] \mid x_0 < y, \sigma_\alpha^\mu(x_0) = \sigma_\alpha^\mu(y)\}\right) > 0\,.$$

This contradicts the fact we proved as part of the proof of part (c): for $\alpha \in A$ such that H^μ_α is continuous and $x \geq r(\alpha)$, we have

$$H^\mu_\alpha \left(\{y \in [0, \omega] \mid x < y, \sigma^\mu_\alpha(x) = \sigma^\mu_\alpha(y)\}\right) = 0$$

Therefore, when $x \sim \lambda^\mu_\alpha$, the CDF of $\sigma^\mu_\alpha(x)$ is continuous, and hence, if $x_1, x_2 \sim \lambda^\mu_\alpha$ i.i.d., then the probability of $\sigma^\mu_\alpha(x_1) = \sigma^\mu_\alpha(x_2)$ is zero. Part (d) follows from combining this fact with part (a). □

C.2.2 Strong Duality and Characterizing an Optimal Pacing Strategy

We begin with the proof of Lemma 11.

Proof of Lemma 11. Note that bidding more than the highest competing bid with a positive probability is not optimal, i.e., if $\mathbb{P}_\alpha(b(\alpha) > \sigma^\mu_\alpha(\omega)) > 0$, then b is not optimal. Therefore, we can restrict our attention to b such that $0 \leq b(\alpha) \leq \sigma^\mu_\alpha(\omega)$ a.s. w.r.t. $\alpha \sim F$. Now, consider such a b. As $\sigma^\mu_\alpha(0) = 0$ and σ_α is continuous a.s. w.r.t. $\alpha \sim F$, by the Intermediate Value Theorem, there exists $z(\alpha) \in [0, \omega]$ such that $\sigma^\mu_\alpha(z(\alpha)) = b(\alpha)$.

Therefore, with $x(\alpha) := w^T\alpha/(1+t)$, we have

$$\begin{aligned}
&\max_{b(.)} \mathbb{E}_{\alpha, \{\theta_i\}_{i=1}^{n-1}} \left[\left(\frac{w^T\alpha}{1+t} - b(\alpha)\right) \mathbb{1}\{b(\alpha) \geq \max(r(\alpha), \{\beta(\theta_i, \alpha)\}_i)\}\right] \\
&= \max_{b(.)} \mathbb{E}_\alpha \mathbb{E}_{Y \sim H^\mu_\alpha} \left[(x(\alpha) - b(\alpha))\, \mathbb{1}\left\{b(\alpha) \geq \max(r(\alpha), \sigma^\mu_\alpha(Y))\right\}\right] \\
&= \max_{z(.)} \mathbb{E}_\alpha \mathbb{E}_{Y \sim H^\mu_\alpha} \left[\left(x(\alpha) - \sigma^\mu_\alpha(z(\alpha))\right) \mathbb{1}\left\{\sigma^\mu_\alpha(z(\alpha)) \geq \max(r(\alpha), \sigma^\mu_\alpha(Y))\right\}\right] \\
&= \max_{z(.)} \mathbb{E}_\alpha \mathbb{E}_{Y \sim H^\mu_\alpha} \left[\left(x(\alpha) - \sigma^\mu_\alpha(z(\alpha))\right) \mathbb{1}\left\{z(\alpha) \geq \max(r(\alpha), Y)\right\}\right] \\
&= \max_{z(.)} \mathbb{E}_\alpha \left[\left(x(\alpha) - \sigma^\mu_\alpha(z(\alpha))\right) H^\mu_\alpha\left(z(\alpha)\right)\right) \mathbb{1}\left\{z(\alpha) \geq r(\alpha)\right\}\right]
\end{aligned}$$

where the third equality follows from part (c) of Lemma 34. Hence, to prove the claim, it is enough to show that for all $\alpha \in A$, we have

$$x(\alpha) \in arg \max_{z(.)} \left(x(\alpha) - \sigma_\alpha^\mu(z(\alpha))\right) H_\alpha^\mu \left(z(\alpha)\right)) \mathbb{1} \left\{z(\alpha) \geq r(\alpha)\right\}$$

The above statement holds trivially for α such that $x(\alpha) < r(\alpha)$, because $\sigma_\alpha^\mu(t) \geq r(\alpha)$ when $t \geq r(\alpha)$. Consider $\alpha \in A$ for which $x(\alpha) \geq r(\alpha)$. Then, for $z(\alpha) \geq r(\alpha)$,

$$\begin{aligned}\left(x(\alpha) - \sigma_\alpha^\mu(z(\alpha))\right) H_\alpha^\mu \left(z(\alpha)\right)) &= x(\alpha) H_\alpha^\mu(z(\alpha)) - z(\alpha) H_\alpha^\mu(z(\alpha)) + \int_{r(\alpha)}^{z(\alpha)} H_\alpha^\mu(s) ds \\ &= (x(\alpha) - z(\alpha)) H_\alpha^\mu(z(\alpha)) + \int_{r(\alpha)}^{z(\alpha)} H_\alpha^\mu(s) ds\end{aligned}$$

Therefore, for $z(\alpha) \geq r(\alpha)$, we have

$$\left(x(\alpha) - \sigma_\alpha^\mu(x(\alpha))\right) H_\alpha^\mu(x(\alpha)) - \left(x(\alpha) - \sigma_\alpha^\mu(z(\alpha))\right) H_\alpha^\mu(z(\alpha)) = (z(\alpha) - x(\alpha)) H_\alpha^\mu(z(\alpha)) - \int_{x(\alpha)}^{z(\alpha)} H_\alpha^\mu(s) ds \geq 0$$

where the inequality holds regardless of $z(\alpha) \geq x(\alpha)$ or $x(\alpha) \geq z(\alpha)$. Furthermore, for $z(\alpha) < r(\alpha)$, we have

$$\left(x(\alpha) - \sigma_\alpha^\mu(x(\alpha))\right) H_\alpha^\mu \left(x(\alpha)\right)) \mathbb{1} \left\{x(\alpha) \geq r(\alpha)\right\} \geq \left(x(\alpha) - \sigma_\alpha^\mu(z(\alpha))\right) H_\alpha^\mu \left(z(\alpha)\right)) \mathbb{1} \left\{z(\alpha) \geq r(\alpha)\right\} = 0$$

Hence, $z(\alpha) = x(\alpha)$ is optimal, which completes the proof. □

In the rest of the sub-section, we build towards the proof of Proposition 6. Recall that the dual objective function is given by

$$q^\mu(w, B, t) = (1+t)\mathbb{E}_\alpha \left[\mathbb{1}\left\{\frac{w^T\alpha}{1+t} \geq r(\alpha)\right\} \int_{r(\alpha)}^{\frac{w^T\alpha}{1+t}} H_\alpha^\mu(s) ds\right] + tB$$

We will prove Proposition 6 by first establishing the differentiability of the dual objective function, and then invoking the first-order optimality conditions for the dual-optimal solution. Lemma 38 will establish the differentiability of the dual objective function. To prove it, we will need the convexity of the dual objective function (Part 1 of Lemma 36), the existence of a bounded dual-optimal solution (Part 2 of Lemma 36), and the differentiability of the indicator function

$$\mathbb{1}\left\{\frac{w^T\alpha}{1+t} \geq r(\alpha)\right\}$$

as a function of t almost surely w.r.t. $(w, B) \sim G$, which is implied by the continuity of the CDF of $w^T\alpha/r(\alpha)$, when $\alpha \sim F$ (Lemma 37).

Lemma 36. *For* $\mu : \Theta \to \mathbb{R}_{\geq 0}$ *and* $(w, B) \in \Theta$*:*

1. $q^\mu(w, B, t)$ *is convex as a function of* t*.*

2. $\min_{t\geq 0} q^\mu(w, B, t) = \min_{t\in[0,\omega/B]} q^\mu(w, B, t)$

Proof.

1. The objective function of the dual problem of a maximization problem is convex.

2. As $H^\mu_\alpha(s) \leq 1$ for all $\alpha \in A$ and $s \in \mathbb{R}$, the following inequalities hold

$$0 \leq \mathbb{1}\left\{\frac{w^T\alpha}{1+t} \geq r(\alpha)\right\} \int_{r(\alpha)}^{\frac{w^T\alpha}{1+t}} H^\mu_\alpha(s)ds \leq \omega \qquad \forall\, t \geq 0, \alpha \in A$$

If $t > \omega/B$, then, $q^\mu(w, B, t) \geq tB > \omega \geq q^\mu(w, B, 0)$. Hence, $q^\mu(w, B, t)$, as a function of t, has its minimum in the interval $[0, \omega/B]$.

□

Let K be the distribution of $\alpha/r(\alpha)$ when $\alpha \sim F$, assuming $1/r(\alpha) = 1$ when $r(\alpha) = 0$. For $w \in \Theta_w$, let K_w be the distribution of $w^T\gamma$ when $\gamma \sim K$, i.e., $K_w(\mathcal{B}) := K(\{\gamma \mid w^T\gamma \in \mathcal{B}\})$ for all Borel sets $\mathcal{B} \subset \mathbb{R}$.

Lemma 37. K_w *has a continuous CDF almost surely w.r.t.* $w \sim G_w$*.*

Proof. Let $w_1, w_2 \in \Theta_w$ be linearly independent weight vectors and $x_1, x_2 \in \mathbb{R}_{\geq 0}$. We consider the set of items α which satisfy $w_1^T\alpha/r(\alpha) = x_1$ and $w_2^T\alpha/r(\alpha) = x_2$. Define

$$S := \left\{\alpha \in A \,\middle|\, \frac{w_1^T\alpha}{r(\alpha)} = x_1;\ \frac{w_2^T\alpha}{r(\alpha)} = x_2\right\}$$

Observe that, for $\alpha \in S$ and $c := x_1/x_2$, we have $w_1^T\alpha = c \cdot w_2^T\alpha$. Therefore, the set $T = \{\alpha \in A \mid (w_1 - cw_2)^T\alpha = 0\}$ is a superset of the set S. Hence, since F has a density, we get $F(S) = 0$.

Define $J = \left\{w/\|w\| \,\middle|\, \exists\, x_w > 0 \text{ s.t. } F(w^T\alpha/r(\alpha) = x_w) > 0\right\}$. Suppose J is uncountable. Then, by Lemma 35, there exists a countable sequence $\{w_m\}_{m\in\mathbb{N}}$ and $\{x_{w_m}\}_{m\in\mathbb{N}}$ such that $w_i/\|w_i\| \neq w_j/\|w_j\|$ for all $i \neq j$ and

$$\sum_m F(w_m^T\alpha/r(\alpha) = x_{w_m}) = \infty.$$

Set $S_m := \{\alpha \mid w_m^T\alpha/r(\alpha) = x_{w_m}\}$. We have shown above that $F(S_i \cap S_j) = 0$ for all $i \neq j$. Therefore, for all $m \geq 1$, we have $F(S_m \cap (\cup_{j<m}S_j)) = 0$, which implies $F(S_m \cap (\cup_{j<m}S_j)^C) = F(S_m)$. This contradicts $F(\cup_m S_m) \leq 1$ as $F(\cup_m S_m) = \sum_m F(S_m \cap (\cup_{j<m}S_j)^C) = \sum_m F(\alpha_m^T s = x_{\alpha_m}) = \infty$. Hence, J is countable. Observe that

$$\left\{\frac{w}{\|w\|} \,\middle|\, K_w \text{ has an atom}\right\} \subset J$$

As G_w has a density, we get $G_w(\text{cone}(J)) = 0$. Therefore, K_w has no atoms almost surely w.r.t. $w \sim G_w$, i.e., K_w has a continuous CDF almost surely w.r.t. $w \sim G_w$. □

Definition 19. *Define $\Theta' \subset \Theta$ to be the set of $(w, B) \in \Theta$ for which K_w has a continuous CDF.*

The following lemma establishes differentiability of the dual objective function.

Lemma 38. *For all pacing functions $\mu : \Theta \to \mathbb{R}_{\geq 0}$ and buyer types $(w, B) \in \Theta'$, the dual objective $q^\mu(w, B, t)$ is differentiable as a function of t for $t > -1/2$. Moreover,*

$$\frac{\partial q^\mu(w, B, t)}{\partial t} = B - \mathbb{E}_\alpha\left[\sigma_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right) H_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right) \mathbb{1}\left\{\frac{w^T\alpha}{1+t} \geq r(\alpha)\right\}\right]$$

Proof. Fix a pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$ and a buyer $(w, B) \in \Theta'$. Define

$$g(t, \alpha) := \mathbb{1}\left\{\frac{w^T\alpha}{1+t} \geq r(\alpha)\right\} \int_{r(\alpha)}^{\frac{w^T\alpha}{1+t}} H_\alpha^\mu(s)ds \qquad \forall\, t > -1/2, \alpha \in A$$

Note that $x \mapsto \mathbb{1}(x \geq r(\alpha)) \int_{r(\alpha)}^{x} H_\alpha^\mu(s)ds$ is a non-decreasing convex function because H_α^μ is non-decreasing. Moreover, it is easy to verify using the second order sufficient condition that $t \mapsto \frac{w^T\alpha}{1+t}$ is convex. As $t \mapsto g(t,\alpha)$ is a composition of these aforementioned functions, it is convex for each α.

Fix $t_0 > -1/2$. Using Lemma 37 and the definition of Θ', we can write

$$
\begin{aligned}
F\left(\left\{\alpha \in A \mid \frac{w^T\alpha}{1+t_0} = r(\alpha)\right\}\right) &= F\left(\left\{\alpha \in A \mid r(\alpha) > 0; \frac{w^T\alpha}{r(\alpha)} = 1+t_0\right\}\right) + F\left(\left\{\alpha \in A \mid r(\alpha) = 0; w^T\alpha = 0\right\}\right) \\
&\leq K\left(\left\{\gamma \mid w^T\gamma = 1+t_0\right\}\right) + F\left(\left\{\alpha \in A \mid w^T\alpha = 0\right\}\right) \\
&= K_w(1+t_0) + 0 \\
&= 0
\end{aligned}
$$

Using Theorem 7.46 of [SDR09], we get that $\mathbb{E}_\alpha[g(t,\alpha)]$ is differentiable w.r.t t at t_0 and

$$
\frac{\partial}{\partial t}\mathbb{E}_\alpha[g(t_0,\alpha)] = \mathbb{E}_\alpha\left[\frac{\partial g(t_0,\alpha)}{\partial t}\right].
$$

Therefore, the dual objective $q^\mu(w,B,t)$ is differentiable as a function of t for $t > -1/2$, and

$$
\begin{aligned}
\frac{\partial q^\mu(w,B,t_0)}{\partial t} &= \mathbb{E}_\alpha\left[g(t_0,\alpha)\right] + (1+t)\frac{\partial}{\partial t}\mathbb{E}_\alpha[g(t_0,\alpha)] + B \\
&= \mathbb{E}_\alpha\left[g(t_0,\alpha)\right] + (1+t)\mathbb{E}_\alpha\left[\frac{\partial g(t_0,\alpha)}{\partial t}\right] + B \\
&= \mathbb{E}_\alpha\left[\mathbb{1}\left\{\frac{w^T\alpha}{1+t_0} \geq r(\alpha)\right\}\int_{r(\alpha)}^{\frac{w^T\alpha}{1+t_0}} H_\alpha^\mu(s)ds\right] \\
&\quad + (1+t)\mathbb{E}_\alpha\left[\frac{-w^T\alpha}{(1+t_0)^2}H_\alpha^\mu\left(\frac{w^T\alpha}{1+t_0}\right)\mathbb{1}\left\{\frac{w^T\alpha}{1+t_0} \geq r(\alpha)\right\}\right] + B \\
&= B - \mathbb{E}_\alpha\left[\left\{\frac{w^T\alpha}{1+t_0}H_\alpha^\mu\left(\frac{w^T\alpha}{1+t_0}\right) - \int_{r(\alpha)}^{\frac{w^T\alpha}{1+t_0}} H_\alpha^\mu(s)ds\right\}\mathbb{1}\left\{\frac{w^T\alpha}{1+t_0} \geq r(\alpha)\right\}\right]
\end{aligned}
$$

□

Corollary 3. *For all pacing functions $\mu : \Theta \to \mathbb{R}_{\geq 0}$ and buyer types $(w,B) \in \Theta'$, $q^\mu(w,B,t)$ is continuous as a function of t for t for $t > -1/2$.*

Corollary 4. *For all pacing functions $\mu : \Theta \to \mathbb{R}_{\geq 0}$ and buyer types $(w, B) \in \Theta'$, $\text{argmin}_{t\in[0,\omega/B]}\, q^{\mu}(w, B, t)$ is non-empty and compact.*

Corollary 3 is a direct consequence of Lemma 38 and Corollary 4 follows from Weierstrass Theorem. Finally, having established the required lemmas, we are ready to prove Proposition 6.

Proof of Proposition 6. Let $t^* \in \text{argmin}_{t\in[0,\omega/B]}\, q^{\mu}(w, B, t)$. According to Theorem 5.1.5 from [BHM98], in order to prove Proposition 6, it suffices to show the following conditions:

(i) Primal feasibility:

$$\mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}}\left[\sigma_{\alpha}^{\mu}\left(\frac{w^T\alpha}{1+t^*}\right)\,\mathbb{1}\left\{\sigma_{\alpha}^{\mu}\left(\frac{w^T\alpha}{1+t^*}\right) \geq \max\left(r(\alpha), \{\beta^{\mu}(\theta_i, \alpha)\}_i\right)\right\}\right] \leq B$$

(ii) Dual feasibility: $t^* \geq 0$

(iii) Lagrangian Optimality:

$$\sigma_{\alpha}^{\mu}\left(\frac{w^T\alpha}{1+t^*}\right) \in \text{argmax}_{b(.)}\ \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}}\left[(w^T\alpha - (1+t)b(\alpha))\,\mathbb{1}\{b(\alpha) \geq \max(r(\alpha), \{\beta^{\mu}(\theta_i, \alpha)\}_i)\}\right] + tB$$

(iv) Complementary slackness:

$$t^*.\left\{B - \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}}\left[\sigma_{\alpha}^{\mu}\left(\frac{w^T\alpha}{1+t^*}\right)\,\mathbb{1}\left\{\sigma_{\alpha}^{\mu}\left(\frac{w^T\alpha}{1+t^*}\right) \geq \max\left(r(\alpha), \{\beta^{\mu}(\theta_i, \alpha)\}_i\right)\right\}\right]\right\} = 0$$

First, we simplify the expression for the expected expenditure used in the sufficient conditions (i)-(iv) stated above:

$$\begin{aligned}
&\mathbb{E}_{\alpha,\{(\theta_i)\}_{i=1}^{n-1}}\left[\sigma_{\alpha}^{\mu}\left(\frac{w^T\alpha}{1+t^*}\right)\,\mathbb{1}\left\{\sigma_{\alpha}^{\mu}\left(\frac{w^T\alpha}{1+t^*}\right) \geq \max\left(r(\alpha), \{\beta^{\mu}(\theta_i, \alpha)\}_i\right)\right\}\right]\\
=&\mathbb{E}_{\alpha,\{(w_i,B_i)\}_{i=1}^{n-1}}\left[\sigma_{\alpha}^{\mu}\left(\frac{w^T\alpha}{1+t^*}\right)\,\mathbb{1}\left\{\sigma_{\alpha}^{\mu}\left(\frac{w^T\alpha}{1+t^*}\right) \geq \max\left(r(\alpha), \left\{\sigma_{\alpha}^{\mu}\left(\frac{w_i^T\alpha}{1+\mu(w_i, B_i)}\right)\right\}_i\right)\right\}\right]\\
=&\mathbb{E}_{\alpha,\{(w_i,B_i)\}_{i=1}^{n-1}}\left[\sigma_{\alpha}^{\mu}\left(\frac{w^T\alpha}{1+t^*}\right)\,\mathbb{1}\left\{\frac{w^T\alpha}{1+t^*} \geq \max\left(r(\alpha), \left\{\frac{w_i^T\alpha}{1+\mu(w_i, B_i)}\right\}_i\right)\right\}\right]
\end{aligned}$$

$$=\mathbb{E}_\alpha\left[\sigma_\alpha^\mu\left(\frac{w^T\alpha}{1+t^*}\right)H_\alpha^\mu\left(\frac{w^T\alpha}{1+t^*}\right)\mathbb{1}\left\{\frac{w^T\alpha}{1+t^*}\geq r(\alpha)\right\}\right]$$

In the rest of the proof, we establish the aforementioned sufficient conditions (i)-(iv). Note that t^* satisfies the following first order conditions of optimality

$$\frac{\partial q^\mu(w,B,t^*)}{\partial t}\geq 0 \qquad t^*\geq 0 \qquad t^*\cdot\frac{\partial q^\mu(w,B,t^*)}{\partial t}=0 \tag{C.1}$$

Using Lemma 38, we can write

$$\frac{\partial q^\mu(w,B,t^*)}{\partial t}=B-\mathbb{E}_\alpha\left[\sigma_\alpha^\mu\left(\frac{w^T\alpha}{1+t^*}\right)H_\alpha^\mu\left(\frac{w^T\alpha}{1+t^*}\right)\mathbb{1}\left\{\frac{w^T\alpha}{1+t^*}\geq r(\alpha)\right\}\right]$$

To establish the sufficient conditions (i)-(iv), observe that (after simplification) conditions (i), (ii) and (iv) are the same as (C.1), and condition (iii) is a direct consequence of Lemma 11, thereby completing the proof of Proposition 6. □

C.2.3 Fixed Point Argument

Proof of Lemma 12.

1. First, observe that

$$\begin{aligned}q^\mu(w,B,t)&=\mathbb{E}_\alpha\left[\mathbb{1}\left\{\frac{w^T\alpha}{1+t}\geq r(\alpha)\right\}\int_{r(\alpha)}^{\frac{w^T\alpha}{1+t}}(1+t)H_\alpha^\mu(s)ds+tB\right]\\&=\mathbb{E}_\alpha\left[\mathbb{1}\left\{\frac{w^T\alpha}{1+t}\geq r(\alpha)\right\}\int_{(1+t)r(\alpha)}^{w^T\alpha}H_\alpha^\mu\left(\frac{y}{1+t}\right)dy+tB\right]\end{aligned}$$

Consider $(w^L,B),(w^H,B)\in\Theta'$ such that $w_i^L<w_i^H$ and $w_{-i}^L=w_{-i}^H$, for some $i\in[d]$. Moreover, consider $t^L,t^H\in[0,\omega/B_{\min}]$ such that $t^L<t^H$. As H_α^μ is a non-decreasing function, it is straightforward to check that $-q^\mu(w,B,t)$ has increasing differences w.r.t. w_i

and t:

$$q^\mu(w^H, B, t^L) - q^\mu(w^L, B, t^L) \geq q^\mu(w^H, B, t^H) - q^\mu(w^L, B, t^H)$$

Theorem 10.7 of [Sun+96] in combination with the definition of ℓ^μ imply $\ell^\mu(w^H, B) \geq \ell^\mu(w^L, B)$.

2. Consider $(w, B^L), (w, B^H) \in \Theta'$ such that $B^L < B^H$ and and $t^L, t^H \in [0, \omega/B_{\min}]$ such that $t^L < t^H$. Then, $-q^\mu(w, B, t)$ has increasing differences w.r.t. $-B$ and t:

$$q^\mu(w, B^H, t^H) - q^\mu(w, B^L, t^H) = (B^H - B^L)t^H \geq (B^H - B^L)t^L = q^\mu(w, B^H, t^L) - q^\mu(w, B^L, t^L)$$

Theorem 10.7 of [Sun+96] and the definition of ℓ^μ imply $\ell^\mu(w, B^H) \leq \ell^\mu(w, B^L)$.

□

Proof of Lemma 13.

1. Theorem 1 of [Idc94] implies measurability of ℓ^μ. Moreover, ℓ^μ is bounded by definition.

2. Consider $\phi \in C_c^1(\Theta, \mathbb{R}^n)$ such that $\|\phi\|_\infty \leq 1$. Then,

$$\begin{aligned}
V(\ell^\mu, \Theta) &= \int_\Theta \ell^\mu(\theta)\mathbf{div}\phi(\theta)d\theta \\
&= \sum_{i=1}^{d+1} \int_\Theta \ell^\mu(\theta)\frac{\partial \phi(\theta)}{\partial \theta_i}d\theta \\
&= \sum_{i=1}^{d+1} \int_{\theta_{-i}} \int_{\theta_i} \ell^\mu(\theta)\frac{\partial \phi(\theta)}{\partial \theta_i}d\theta_i d\theta_{-i} \\
&= \sum_{i=1}^{d+1} \int_{\theta_{-i}} \int_{\theta_i} -\phi(\theta_i, \theta_{-i})d\ell^\mu(\theta_i)d\theta_{-i} \\
&\leq \sum_{i=1}^{d+1} \int_{\theta_{-i}} \int_{\theta_i} d\ell^\mu(\theta_i)d\theta_{-i} \\
&\leq \sum_{i=1}^{d+1} \int_{\theta_{-i}} \frac{\omega}{B_{\min}}d\theta_{-i}
\end{aligned}$$

$$\leq (d+1)U^{d+1}\frac{\omega}{B_{\min}}$$

where the third equality follows from Fubini's Theorem. The sufficient conditions for Fubini's Theorem to hold are satisfied because $|\ell^{\mu}\mathbf{div}\phi|$ is bounded. Moreover, the fourth equality follows from the integration by parts for Lebesgue-Stieltjes integral and the fact that ϕ evaluates to 0 at the boundaries of Θ because ϕ is compactly supported. □

Proof of Lemma 14. We start by noting that, as G has a density, if a sequence converges almost surely (or in L_1) under the Lebesgue measure on Θ, then it converges almost surely (or in L_1) under G.

1. If $\mu(\theta) = 0$ for all $\theta \in \Theta$, then $\mu \in \mathcal{X}_0$. Hence, $\mathcal{X}_0$ is non-empty. Consider $\mu_1, \mu_2 \in \mathcal{X}_0$ and $a \in [0,1]$. Then, $a\mu_1 + (1-a)\mu_2 \in [0, \omega/B_{\min}]$ and for $\phi \in C_c^1(\Omega, \mathbb{R}^n)$ s.t. $\|\phi\|_\infty \leq 1$, we have

$$\int_\Omega \{a\mu_1 + (1-a)\mu_2\}(\theta)\mathbf{div}\phi(\theta)d\theta = a\int_\Omega \mu_1(\theta)\mathbf{div}\phi(\theta)d\theta + (1-a)\int_\Omega \mu_2(\theta)\mathbf{div}\phi(\theta)d\theta$$
$$\leq \frac{(d+1)U^{d+1}\omega}{B_{\min}}$$

Hence, $\mathcal{X}_0$ is convex.

Consider a sequence $\{\mu_n\} \subset \mathcal{X}_0$ and $\mu \in L_1(\Theta)$ such that $\mu_n \xrightarrow{L_1} \mu$. Then, there exists a subsequence $\{n_k\}$ such that $\mu_{n_k} \xrightarrow{\text{a.s.}} \mu$ as $k \to \infty$. Hence, range$(\mu) \subset [0, \omega/B_{\min}]$. Moreover, by the semi-continuity of total variation (Remark 3.5 of [AFP00]), we have

$$V(\mu, \Theta) \leq \liminf_{n\to\infty} V(\mu_n, \Theta) \leq (d+1)U^{d+1}\omega/B_{\min}$$

Therefore, $\mathcal{X}_0$ is closed. To see why $\mathcal{X}_0$ is compact, consider a sequence $\{\mu_n\} \subset \mathcal{X}_0$. Then, by Theorem 3.23 of [AFP00], there exists a subsequence $\{n_k\}$ and $\mu \in BV(\Theta)$ such that μ_{n_k} converges to μ in the weak* topology, which implies convergence in $L_1(\Theta)$ (Proposition 3.13 of [AFP00]). Combining this with the fact that $\mathcal{X}_0$ is closed, completes the proof of compactness of $\mathcal{X}_0$.

2. For contradiction, suppose f is not continuous. Then, there exists $\epsilon > 0$, a sequence $\{(\mu_n, \hat{\mu}_n)\}_n \subset \mathcal{X}_0 \times \mathcal{X}_0$ and $(\mu, \hat{\mu}) \in \mathcal{X}_0 \times \mathcal{X}_0$ such that $\lim_{n\to\infty}(\mu_n, \hat{\mu}_n) = (\mu, \hat{\mu})$ and $|f(\mu_n, \hat{\mu}_n) - f(\mu, \hat{\mu})| \geq \epsilon$ for all $n \in \mathbb{N}$. As $\mu_n \xrightarrow{L_1} \mu$, there exists a subsequence $\{n_k\}_k$ such that $\mu_{n_k} \xrightarrow{a.s.} \mu$ when $k \to \infty$. Moreover, $\hat{\mu}_n \xrightarrow{L_1} \hat{\mu}$ implies $\hat{\mu}_{n_k} \xrightarrow{L_1} \hat{\mu}$. Therefore, there exists a subsequence $\{n_{k_l}\}_l$ such that $\hat{\mu}_{n_{k_l}} \xrightarrow{a.s.} \hat{\mu}$ and $\mu_{n_{k_l}} \xrightarrow{a.s.} \mu$ as $l \to \infty$. Here, we have repeatedly used the fact that L_1 convergence implies the existence of a subsequence that converges a.s. Hence, after relabelling for ease of notation, we can write that there exists $\epsilon > 0$, a sequence $\{(\mu_n, \hat{\mu}_n)\}_n \subset \mathcal{X}_0 \times \mathcal{X}_0$ and $(\mu, \hat{\mu}) \in \mathcal{X}_0 \times \mathcal{X}_0$ such that $\mu_n \xrightarrow{a.s.} \mu$, $\hat{\mu}_n \xrightarrow{a.s.} \hat{\mu}$ and $|f(\mu_n, \hat{\mu}_n) - f(\mu, \hat{\mu})| \geq \epsilon$ for all $n \in \mathbb{N}$.

   First, observe that $\mu_n \xrightarrow{a.s.} \mu$ implies $w^T\alpha/(1 + \mu_n(w, B)) \xrightarrow{a.s.} w^T\alpha/(1 + \mu(w, B))$ and hence, $\lambda_\alpha^{\mu_n} \xrightarrow{d} \lambda_\alpha^{\mu}$ for all $\alpha \in A$. As λ_α^{μ} is continuous almost surely w.r.t. α, by the definition of convergence in distribution, we get that $\lim_{n\to\infty} \lambda_\alpha^{\mu_n}(s) = \lambda_\alpha^{\mu}(s)$ for all $s \in \mathbb{R}$ a.s. w.r.t. $\alpha \sim F$. Therefore, $\lim_{n\to\infty} H_\alpha^{\mu_n}(s) = H_\alpha^{\mu}(s)$ for all $s \in \mathbb{R}$, a.s. w.r.t. $\alpha \sim F$.

   Also, note that $\lambda_\alpha^{\hat{\mu}_n}$ and $\lambda_\alpha^{\hat{\mu}}$ are atom-less almost surely w.r.t. α. Let $\bar{A} \subset A$ be the set of α such that $\lim_{n\to\infty} H_\alpha^{\mu_n}(s) = H_\alpha^{\mu}(s)$ for all $s \in \mathbb{R}$ and $\{\lambda_\alpha^{\hat{\mu}_n}, \lambda_\alpha^{\hat{\mu}}\}$ are atom-less. Therefore, $F(\bar{A}) = 1$. For $s \in \mathbb{R}$ and $\alpha \in \bar{A}$, we get

$$\lim_{n\to\infty} \mathbb{1}\left\{\frac{w^T\alpha}{1 + \hat{\mu}_n(w, B)} \geq s \geq r(\alpha)\right\} = \mathbb{1}\left\{\frac{w^T\alpha}{1 + \hat{\mu}(w, B)} \geq s \geq r(\alpha)\right\}$$

   a.s. w.r.t. $(w, B) \sim G$. Note that the set of measure zero on which the above equality doesn't hold may depend on α, s.

   Fix $s \in [0, \omega]$ and $\alpha \in \bar{A}$. Combining these a.s. convergence statements, we get

$$\begin{aligned}&\lim_{n\to\infty}(1 + \hat{\mu}_n(w, B))H_\alpha^{\mu_n}(s)\mathbb{1}\left\{\frac{w^T\alpha}{1 + \hat{\mu}_n(w, B)} \geq s \geq r(\alpha)\right\}\\ =&(1 + \hat{\mu}(w, B))H_\alpha^{\mu}(s)\mathbb{1}\left\{\frac{w^T\alpha}{1 + \hat{\mu}(w, B)} \geq s \geq r(\alpha)\right\}\end{aligned}$$

a.s. w.r.t. $(w, B) \sim G$.

Furthermore, we can use the Dominated Convergence Theorem (as the sequence is bounded) to show

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}_{(w,B)}\left[(1+\hat{\mu}_n(w,B))H_\alpha^{\mu_n}(s)\mathbb{1}\left\{\frac{w^T\alpha}{1+\hat{\mu}_n(w,B)} \geq s \geq r(\alpha)\right\}\right] \\
=&\mathbb{E}_{(w,B)}\left[(1+\hat{\mu}(w,B))H_\alpha^{\mu}(s)\mathbb{1}\left\{\frac{w^T\alpha}{1+\hat{\mu}(w,B)} \geq s \geq r(\alpha)\right\}\right]
\end{aligned}$$

Keep $s \in [0, \omega]$ fixed and apply the Dominated Convergence Theorem for a second time to obtain,

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}_\alpha\left[\mathbb{E}_{(w,B)}\left[(1+\hat{\mu}_n(w,B))H_\alpha^{\mu_n}(s)\mathbb{1}\left\{\frac{w^T\alpha}{1+\hat{\mu}_n(w,B)} \geq s \geq r(\alpha)\right\}\right]\right] \\
=&\mathbb{E}_\alpha\left[\mathbb{E}_{(w,B)}\left[(1+\hat{\mu}(w,B))H_\alpha^{\mu}(s)\mathbb{1}\left\{\frac{w^T\alpha}{1+\hat{\mu}(w,B)} \geq s \geq r(\alpha)\right\}\right]\right]
\end{aligned}$$

Finally, apply the Dominated Convergence Theorem for the third time to obtain,

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^\omega \mathbb{E}_\alpha\left[\mathbb{E}_{(w,B)}\left[(1+\hat{\mu}_n(w,B))H_\alpha^{\mu_n}(s)\mathbb{1}\left\{\frac{w^T\alpha}{1+\hat{\mu}_n(w,B)} \geq s \geq r(\alpha)\right\}\right]\right] ds \\
=&\int_0^\omega \mathbb{E}_\alpha\left[\mathbb{E}_{(w,B)}\left[(1+\hat{\mu}(w,B))H_\alpha^{\mu}(s)\mathbb{1}\left\{\frac{w^T\alpha}{1+\hat{\mu}(w,B)} \geq s \geq r(\alpha)\right\}\right]\right] ds
\end{aligned}$$

As we are dealing with non-negative random variables, we can apply Fubini's Theorem to rewrite the above statement as

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}_{(w,B)}\mathbb{E}_\alpha\left[(1+\hat{\mu}_n(w,B))\mathbb{1}\left(\frac{w^T\alpha}{1+\hat{\mu}_n(w,B)} \geq r(\alpha)\right)\int_{r(\alpha)}^{\frac{w^T\alpha}{1+\hat{\mu}_n(w,B)}} H_\alpha^{\mu_n}(s)ds\right] \\
=&\lim_{n\to\infty} \mathbb{E}_{(w,B)}\mathbb{E}_\alpha\left[\int_0^\omega (1+\hat{\mu}_n(w,B))H_\alpha^{\mu_n}(s)\mathbb{1}\left\{\frac{w^T\alpha}{1+\hat{\mu}_n(w,B)} \geq s \geq r(\alpha)\right\} ds\right] \\
=&\mathbb{E}_{(w,B)}\mathbb{E}_\alpha\left[\int_0^\omega (1+\hat{\mu}(w,B))H_\alpha^{\mu}(s)\mathbb{1}\left\{\frac{w^T\alpha}{1+\hat{\mu}(w,B)} \geq s \geq r(\alpha)\right\} ds\right] \\
=&\mathbb{E}_{(w,B)}\mathbb{E}_\alpha\left[(1+\hat{\mu}(w,B))\mathbb{1}\left\{\frac{w^T\alpha}{1+\hat{\mu}(w,B)} \geq r(\alpha)\right\}\int_{r(\alpha)}^{\frac{w^T\alpha}{1+\hat{\mu}(w,B)}} H_\alpha^{\mu}(s)ds\right]
\end{aligned}$$

Moreover, applying Dominated Convergence Theorem to $\hat{\mu}_n \xrightarrow{a.s.} \hat{\mu}$ yields $\lim_{n\to\infty} E_{(w,B)}[\hat{\mu}_n(w,B)B] = E_{(w,B)}[\hat{\mu}(w,B)B]$. Together, the above statements imply $\lim_{n\to\infty} f(\mu_n,\hat{\mu}_n) = f(\mu,\hat{\mu})$, which is a contradiction.

3. Part (2) allows us to invoke the Berge Maximum Theorem (Theorem 17.31 of [AB06]), which implies that C_0^* is upper hemi-continuous with non-empty and compact values. Next, we show that $C_0^*(\mu)$ is also convex. Fix $\mu \in \mathcal{X}$. Consider $\hat{\mu}_1, \hat{\mu}_2 \in C^*(\mu)$ and $\lambda \in [0,1]$. Then, by part (1) of Lemma 36, we have

$$\begin{aligned} f\,(\mu, \lambda\hat{\mu}_1 + (1-\lambda)\hat{\mu}_2) &= \mathbb{E}_{(w,B)}[q^{\mu}(w,B,\lambda\hat{\mu}_1(w,B) + (1-\lambda)\hat{\mu}_2(w,B))] \\ &\leq \lambda\mathbb{E}_{(w,B)}[q^{\mu}(w,B,\hat{\mu}_1(w,B))] + (1-\lambda)\mathbb{E}_{(w,B)}[q^{\mu}(w,B,\hat{\mu}_2(w,B))] \\ &= \lambda f(\mu,\hat{\mu}_1) + (1-\lambda) f(\mu,\hat{\mu}_2) \end{aligned}$$

Hence, $\lambda\hat{\mu}_1 + (1-\lambda)\hat{\mu}_2 \in C^*(\mu)$. □

Proof of Lemma 15. Recall that in, in Lemma 13, we showed that $\ell^{\mu} \in \mathcal{X}_0$. Therefore, as $\mu \in C^*(\mu)$,

$$\mathbb{E}_{(w,B)}[q^{\mu}(w,B,\ell^{\mu}(w,B))] \geq \mathbb{E}_{(w,B)}[q^{\mu}(w,B,\mu(w,B))]$$

On the other hand, by the definition of ℓ^{μ}, we get that

$$q^{\mu}(w,B,\ell^{\mu}(w,B)) \leq q^{\mu}(w,B,\mu(w,B)) \qquad \forall\,(w,B) \in \Theta$$

Hence, combining the two statements yields $q^{\mu}(w,B,\ell^{\mu}(w,B)) = q^{\mu}(w,B,\mu(w,B))$ a.s. w.r.t. $(w,B) \sim G$, which completes the proof. □

C.3 Standard Auctions and Revenue Equivalence

In this section, we extend our results for first-price auctions to all anonymous standard auctions and establish revenue equivalence among them by proving Theorem 9. To do this, we will show that the dual of the optimization problem faced by each buyer type is identical for all anonymous standard auctions, by exploiting the structure of the Lagrangian problem and the known revenue equivalence results from the standard i.i.d. setting [Kri09]. This concurrence of the dual problems for all anonymous standard auctions allows us to directly apply Theorem 8 to reduce the proof of Theorem 9 to showing strong duality for the optimization problem faced by the buyer types.

For buyer type $(w, B) \in \Theta$, we will use $R(w, B)$ to denote the following optimization problem:

$$\begin{aligned} R^\mu(w,B) := \max_{b:A\to\mathbb{R}_{\geq 0}} \quad & \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}} \left[w^T\alpha \cdot \mathbb{1}\{b(\alpha) \geq \max(r(\alpha), \{\Psi^\mu(\theta_i,\alpha)\}_i)\} - M_\alpha(b(\alpha), \{\Psi^\mu(\theta_i,\alpha)\}_i) \right] \\ \text{s.t.} \quad & \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}} \left[M_\alpha \left(b(\alpha), \{\Psi^\mu(\theta_i,\alpha)\}_i \right) \right] \leq B \end{aligned}$$

Then the dual optimization problem (or simply the dual problem) of $R^\mu(w, B)$ is given by

$$\min_{t\geq 0} \max_{b:A\to\mathbb{R}_{\geq 0}} \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}} \left[w^T\alpha \cdot \mathbb{1}\{b(\alpha) \geq \max(r(\alpha), \{\Psi^\mu(\theta_i,\alpha)\}_i) - (1+t) M_\alpha(b(\alpha), \{\Psi^\mu(\theta_i,\alpha)\}_i) \} \right] + tB$$

The following lemma characterizes the optimal solution to the Lagrangian problem.

Lemma 39. *For all $t \geq 0$,*

$$\psi_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right) \in arg\max_{b(.)} \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}} \left[\frac{w^T\alpha}{1+t} \cdot \mathbb{1}(b(\alpha) \geq \max\left(r(\alpha), \{\Psi^\mu(\theta_i,\alpha)\}_i\right)) - M_\alpha(b(\alpha), \{\Psi^\mu(\theta_i,\alpha)\}_i) \right]$$

Proof. Consider an $\alpha \in A$ such that λ_α^μ is atom-less. Then, using the assumptions on auction $\mathcal{A}$, we can write

$$\psi_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right) \in arg\max_{t\in\mathbb{R}} \mathbb{E}_{\{X_i\}_{i=1}^{n-1}\sim\lambda_\alpha^\mu} \left[\left(\frac{w^T\alpha}{1+t} \cdot \mathbb{1}(t \geq \max(r(\alpha), \{\psi_\alpha^\mu(X_i)\}_i)) - M_\alpha\left(t, \{\psi_\alpha^\mu(X_i)\}_i\right) \right) \right]$$

Combining this with the definition of Ψ^μ, we get

$$\psi_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right) \in arg\max_{t\in\mathbb{R}} \mathbb{E}_{\{\theta_i\}_{i=1}^{n-1}}\left[\frac{w^T\alpha}{1+t}\cdot \mathbb{1}(t \geq \max(r(\alpha), \{\Psi^\mu(\theta_i,\alpha)\}_i)) - M_\alpha(t, \{\Psi^\mu(\theta_i,\alpha)\}_i)\right]$$

To complete the proof, note that λ_α^μ is atom-less a.s. w.r.t. α by part (a) of Lemma 34. □

We take a short interlude to state and prove a lemma which will help us simplify the expression for the dual optimization problem of $R^\mu(w, B)$.

Lemma 40. *For $\alpha \in A$ such that λ_α^μ is continuous,*

$$\mathbb{1}\left\{\psi_\alpha^\mu(x) \geq \max(r(\alpha), \psi_\alpha^\mu(Y))\right\} = \mathbb{1}\left\{x \geq \max(r(\alpha), Y)\right\} \qquad a.s.\ Y \sim H_\alpha^\mu,\ \forall\, x \in [0,\omega]$$

Proof. As ψ_α^μ is non-decreasing, $\mathbb{1}\left\{\psi_\alpha^\mu(x) \geq \max(r(\alpha), \psi_\alpha^\mu(Y))\right\} \geq \mathbb{1}\left\{x \geq \max(r(\alpha), Y)\right\}$ always holds. Suppose there exists $\alpha \in A$ such that λ_α^μ is continuous and $x \in [0,\omega]$ for which $\mathbb{1}\left\{\psi_\alpha^\mu(x) \geq \max(r(\alpha), \psi_\alpha^\mu(Y))\right\} > \mathbb{1}\left\{x \geq \max(r(\alpha), Y)\right\}$ with positive probability w.r.t. $Y \sim H_\alpha^\mu$. Observe that $\psi_\alpha^\mu(x) \geq r$ implies $x \geq r$, by the assumptions made on ψ_α^μ. Therefore,

$$\mathbb{1}\left\{\psi_\alpha^\mu(x) \geq \max(r(\alpha), \psi_\alpha^\mu(Y))\right\} > \mathbb{1}\left\{x \geq \max(r(\alpha), Y)\right\} \implies Y > x,\ x \geq r(\alpha),\ \psi_\alpha^\mu(x) \geq \psi_\alpha^\mu(Y)$$

Hence, there exists $\alpha \in A$ such that λ_α is continuous and $x \in [r(\alpha), \omega]$ for which

$$H_\alpha^\mu\left(\{y \in [0,\omega] \mid y > x, \psi_\alpha^\mu(y) \leq \psi_\alpha^\mu(x)\}\right) > 0$$

As $y > x$ implies $\psi_\alpha^\mu(y) \geq \psi_\alpha^\mu(x)$, we get

$$H_\alpha^\mu\left(\{y \in [0,\omega] \mid \psi_\alpha^\mu(y) = \psi_\alpha^\mu(x)\}\right) > 0$$

which contradicts the assumption that ψ_α^μ has a atom-less distribution. Hence, the lemma holds. □

Next, we proceed to prove that the dual of $R^\mu(w, B)$ is the same as the dual of the optimization problem $Q^\mu(w, B)$ associated to first-price auctions. Consider an α for which λ_α^μ is continuous. Then, the expected utility $U_\alpha^\mu(x)$ of a bidder with value x in auction $\mathcal{A}$, when the values of the other agents are drawn i.i.d. from λ_α^μ and every bidder employs strategy ψ_α^μ, is given by

$$\begin{aligned}
U_\alpha^\mu(x) &:= \mathbb{E}_{\{X_i\}_{i=1}^{n-1}\sim\lambda_\alpha^\mu}\left[x \cdot \mathbb{1}\left\{\psi_\alpha^\mu(x) \geq \max(r(\alpha), \{\psi_\alpha^\mu(X_i)\}_i)\right\} - M_\alpha(\psi_\alpha^\mu(x), \{\psi_\alpha^\mu(X_i)\}_i)\right] \\
&= \mathbb{E}_{\{X_i\}_{i=1}^{n-1}\sim\lambda_\alpha^\mu}\left[x\, \mathbb{1}\left\{x \geq \max(r(\alpha), \{X_i\}_i)\right\}\right] - m_\alpha(x) \\
&= xH_\alpha^\mu(x)\mathbb{1}\{x \geq r(\alpha)\} - m_\alpha^\mu(x)
\end{aligned}$$

where the second line follows from Lemma 40 and $m_\alpha^\mu(x) = \mathbb{E}_{\{X_i\}_{i=1}^{n-1}\sim\lambda_\alpha^\mu}\left[M_\alpha(\psi_\alpha^\mu(x), \{\psi_\alpha^\mu(X_i)\}_i)\right]$.

Then, from the arguments given in section 5.1.2 of Krishna, we get

$$U_\alpha^\mu(x) = \int_0^x H_\alpha^\mu(s)\mathbb{1}\{s \geq r(\alpha)\}ds = \mathbb{1}\{x \geq r(\alpha)\}\int_{r(\alpha)}^x H_\alpha^\mu(s)ds$$

which further implies

$$m_\alpha^\mu(x) = xH_\alpha^\mu(x)\mathbb{1}\{x \geq r(\alpha)\} - U_\alpha^\mu(x) = \mathbb{1}\{x \geq r(\alpha)\}\left(xH_\alpha^\mu(x) - \int_{r(\alpha)}^x H_\alpha^\mu(s)ds\right)$$

Then, using Lemma 39 and Lemma 40, the value that the objective function of the dual problem of $R^\mu(w, B)$ takes at $t \geq 0$ is given by:

$$\begin{aligned}
&\max_{b:A\to\mathbb{R}_{\geq 0}} \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}}\left[w^T\alpha \cdot \mathbb{1}\{b(\alpha) \geq \max(r(\alpha), \{\Psi^\mu(\theta_i,\alpha)\}_i)\} - (1+t)M_\alpha(b(\alpha), \{\Psi^\mu(\theta_i,\alpha)\}_i)\right] + tB \\
=&(1+t)\max_{b:A\to\mathbb{R}_{\geq 0}} \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}}\left[\frac{w^T\alpha}{1+t}\cdot \mathbb{1}(b(\alpha) \geq \max\left(r(\alpha), \{\Psi^\mu(\theta_i,\alpha)\}_i)\right) - M_\alpha(b(\alpha), \{\Psi^\mu(\theta_i,\alpha)\}_i)\right] + tB \\
=&(1+t)\mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}}\left[\frac{w^T\alpha}{1+t}\cdot \mathbb{1}(b(\alpha) \geq \max\left(r(\alpha), \{\Psi^\mu(\theta_i,\alpha)\}_i)\right) - M_\alpha\left(\psi_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right), \{\Psi^\mu(\theta_i,\alpha)\}_i\right)\right] + tB \\
=&(1+t)\mathbb{E}_\alpha\mathbb{E}_{\{X_i\}_{i=1}^{n-1}\sim\lambda_\alpha^\mu}\left[\frac{w^T\alpha}{1+t}\cdot \mathbb{1}\left\{\psi_\alpha^\mu(x) \geq \max(r(\alpha), \{\psi_\alpha^\mu(X_i)\}_i)\right\} - M_\alpha\left(\psi_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right), \{\psi_\alpha^\mu(X_i)\}_i\right)\right] + tB \\
=&(1+t)\mathbb{E}_\alpha\left[U_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right)\right] + tB \\
=&(1+t)\mathbb{E}_\alpha\left[\mathbb{1}\left\{\frac{w^T\alpha}{1+t} \geq r(\alpha)\right\}\int_{r(\alpha)}^{\frac{w^T\alpha}{1+t}} H_\alpha^\mu(s)ds\right] + tB
\end{aligned}$$

$=q^{\mu}(w, B, t)$

Hence, we have shown that, for every buyer type, all anonymous standard auctions have identical dual optimization problems. In light of this, to prove Theorem 9, it suffices to prove strong duality for $R^{\mu}(w, B)$, where μ is a fixed-point which is guaranteed to exist by Proposition 7. We give the full argument below.

Proof of Theorem 9 .

By Lemma 15, we know that if $\mu \in C_0^*(\mu)$, then $\mu(w, B) \in \operatorname{argmin}_{t\in[0,\omega/B]} q^{\mu}(w, B, t)$ almost surely w.r.t. $(w, B) \sim G$. Moroeover, by part (b) of Lemma 36, we have $\mu(w, B) \in \operatorname{argmin}_{t\in[0,\infty)} q^{\mu}(w, B, t)$. Consider a $\theta = (w, B) \in \Theta'$ (see Definition 19) for which $\mu(w, B) \in \operatorname{argmin}_{t\in[0,\infty)} q^{\mu}(w, B, t)$. Observe that such θ form a subset which has measure one under G. According to Theorem 5.1.5 from [BHM98], in order to prove that $\Psi^{\mu}(w, B, \alpha)$ (as a function of α) is an optimal solution for the optimization problem $R^{\mu}(w, B)$, it suffices to show the following conditions:

(i) Primal feasibility:

$$\mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}} \left[M_\alpha\left(\Psi^{\mu}(w, B, \alpha), \{\Psi^{\mu}(\theta_i, \alpha)\}_i\right)\right] \leq B$$

(ii) Dual feasibility: $\mu(w, B) \geq 0$

(iii) Lagrangian Optimality: $\Psi^{\mu}(w, B)$ is an optimal solution for

$$\max_{b:A\to\mathbb{R}_{\geq 0}} \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}} \left[w^T\alpha \cdot \mathbb{1}\{b(\alpha) \geq \max(r(\alpha), \{\Psi^{\mu}(\theta_i, \alpha)\}_i)\} - (1 + \mu(w, B))M_\alpha(b(\alpha), \{\Psi^{\mu}(\theta_i, \alpha)\}_i)\right] \\ + \mu(w, B)B$$

(iv) Complementary slackness:

$$\mu(w, B). \left\{B - \mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}} \left[M_\alpha\left(\Psi^{\mu}(w, B, \alpha), \{\Psi^{\mu}(\theta_i, \alpha)\}_i\right)\right]\right\} = 0$$

First, we simplify the expression for the expected expenditure used in the sufficient conditions (i)-(iv) stated above to show that it is equal to the expected payment made by buyer type (w, B) in the SFPE determined by pacing function μ:

$$
\begin{aligned}
&\mathbb{E}_{\alpha,\{\theta_i\}_{i=1}^{n-1}}\left[M_\alpha\left(\Psi^\mu(\theta,\alpha),\{\Psi^\mu(\theta_i,\alpha)\}_i\right)\right]\\
=&\mathbb{E}_\alpha\mathbb{E}_{\{X_i\}_{i=1}^{n-1}\sim\lambda_\alpha^\mu}\left[M_\alpha\left(\psi_\alpha^\mu\left(\frac{w^T\alpha}{1+\mu(w,B)}\right),\{\psi_\alpha^\mu(X_i)\}_i\right)\right]\\
=&\mathbb{E}_\alpha\left[m_\alpha^\mu\left(\frac{w^T\alpha}{1+\mu(w,B)}\right)\right]\\
=&\mathbb{E}_\alpha\left[\left(\frac{w^T\alpha}{1+\mu(w,B)}H_\alpha^\mu\left(\frac{w^T\alpha}{1+\mu(w,B)}\right)-\int_{r(\alpha)}^{\frac{w^T\alpha}{1+\mu(w,B)}}H_\alpha^\mu(s)ds\right)\mathbb{1}\left\{\frac{w^T\alpha}{1+\mu(w,B)}\geq r(\alpha)\right\}\right]\\
=&\mathbb{E}_\alpha\left[\sigma_\alpha^\mu\left(\frac{w^T\alpha}{1+\mu(w,B)}\right)H_\alpha^\mu\left(\frac{w^T\alpha}{1+\mu(w,B)}\right)\mathbb{1}\left\{\frac{w^T\alpha}{1+\mu(w,B)}\geq r(\alpha)\right\}\right]\\
=&\mathbb{E}_{\alpha,\{(\theta_i)\}_{i=1}^{n-1}}\left[\beta^\mu(\theta,\alpha)\,\mathbb{1}\{\beta^\mu(\theta,\alpha)\geq\max(r(\alpha),\{\beta^\mu(\theta_i,\alpha)\}_i)\}\right]
\end{aligned}
$$

Hence, Theorem 9 will follow if we establish the aforementioned sufficient conditions (i)-(iv). Note that $\mu(w, B)$ satisfies the following first order conditions of optimality

$$\frac{\partial q^\mu(w,B,\mu(w,B))}{\partial t}\geq 0\qquad \mu(w,B)\geq 0\qquad \mu(w,B)\cdot\frac{\partial q^\mu(w,B,\mu(w,B))}{\partial t}=0 \tag{C.2}$$

Using Lemma 38, we can write

$$\frac{\partial q^\mu(w,B,\mu(w,B))}{\partial t}=B-\mathbb{E}_\alpha\left[\sigma_\alpha^\mu\left(\frac{w^T\alpha}{1+\mu(w,B)}\right)H_\alpha^\mu\left(\frac{w^T\alpha}{1+\mu(w,B)}\right)\mathbb{1}\left\{\frac{w^T\alpha}{1+\mu(w,B)}\geq r(\alpha)\right\}\right]$$

To establish the sufficient conditions (i)-(iv), observe that (after simplification) conditions (i), (ii) and (iv) are the same as (C.2), and condition (iii) is a direct consequence of Lemma 39, thereby completing the proof of Theorem 9. □

C.3.1 Revenue Equivalence under Ex-Ante Budget Constraints

The argument developed in this section also applies to the setting with non-contextual i.i.d. values and ex-ante budget constraints, which is the symmetric special case of the models studied in [BBW15] and [Bal+21]. More precisely, consider a single-item auction setting with n buyers, and assume that the value of each buyer is drawn i.i.d. from a common atom-less distribution $\mathcal{F}$ over the space of all possible values $[0, \overline{V}] \subset \mathbb{R}_{\geq 0}$. Moreover, assume that every buyer has an ex-ante budget of B, i.e., she is constrained to spend at most B in expectation, where the expectation is taken over her own value and the values of other buyers. Let $\mathbb{A} = (r, M)$ be the anonymous standard auction with reserve price r and payment rule M that the seller uses to sell the item.

In this simpler setting, a strategy $\beta^* : [0, \overline{V}] \to \mathbb{R}$ is a symmetric equilibrium if β^* is the optimal bidding strategy for a buyer when all other buyers employ β^* to bid. Concretely, $\beta^* : [\underline{V}, \overline{V}] \to \mathbb{R}$ is a symmetric equilibrium if it is an optimal solution to the following optimization problem:

$$\begin{aligned} \max_{b:[0,\overline{V}]\to\mathbb{R}_{\geq 0}} \quad & \mathbb{E}_{v,\{v_i\}_{i=1}^{n-1}} \left[v \cdot \mathbb{1}\{b(v) \geq \max(r, \{\beta^*(v_i)\}_i)\} - M(b(v), \{\beta^*(v_i)\}_i) \right] \\ \text{s.t.} \quad & \mathbb{E}_{v,\{v_i\}_{i=1}^{n-1}} \left[M(b(v), \{\beta^*(v_i)\}_i) \right] \leq B \end{aligned} \tag{C.3}$$

When $\mathbb{A}$ is a second-price auction, the results of both [BBW15] and [Bal+21] imply that strong duality holds for the optimization problem given in (C.3), and there exists a dual solution $\mu^* \geq 0$ such that $\beta^*(v) = v/(1+\mu^*)$ is a symmetric equilibrium. With this existence result for second-price auctions in hand, we can leverage the argument developed earlier to establish the existence of a value-pacing-based equilibrium for all standard auctions and revenue equivalence.

Let $\mathcal{H}$ be the distribution of $v/(1+\mu^*)$ when $v \sim \mathcal{F}$ and $\psi^{\mathcal{H}}$ be the single-auction equilibrium for distribution $\mathcal{H}$ and auction $\mathbb{A} = (r, M)$, as defined at the beginning of Section 4.3. Then, we claim that the value-pacing-based strategy given by

$$\Psi(v) = \psi^{\mathcal{H}} \left(\frac{v}{1+\mu^*} \right)$$

is a symmetric equilibrium (as defined in equation (C.3)). To see this, first observe that, when all of the other buyers use $\beta^* = \Psi$ to bid, the dual of the optimization problem (C.3) is given by

$$\begin{aligned}
&\min_{\mu \geq 0} \max_{b:[0,\overline{V}]\to\mathbb{R}_{\geq 0}} \mathbb{E}_{v,\{v_i\}_{i=1}^{n-1}} \left[v \cdot \mathbb{1}\{b(v) \geq \max(r, \{\Psi(v_i)\}_i)\} - (1+\mu) M(b(v), \{\Psi(v_i)\}_i)\right] + \mu \cdot B \\
=&\min_{\mu \geq 0} (1+\mu)\, \mathbb{E}_v \left[\max_{b \in \mathbb{R}_{\geq 0}} \mathbb{E}_{\{v_i\}_{i=1}^{n-1}} \left[\frac{v}{1+\mu} \cdot \mathbb{1}\{b \geq \max(r, \{\Psi(v_i)\}_i)\} - M(b, \{\Psi(v_i)\}_i)\right]\right] + \mu \cdot B
\end{aligned}$$

Next, observe that the inner optimization problem over $b \in \mathbb{R}_{\geq 0}$ is exactly the bidding problem faced by a buyer with value $v/(1+\mu^*)$ who aims to maximize her utility in the single-auction setting when the values of the other buyers are drawn from the distribution $\mathcal{H}$. Since $\psi^{\mathcal{H}}(\cdot)$ is the equilibrium strategy in the single-auction setting, $\Psi(v) = \psi^{\mathcal{H}}(v/(1+\mu^*))$ is an optimal solution to this bidding problem. Moreover, we know from [Mye81] that the interim expected utility of a buyer under equilibrium strategies is independent of payment rule of the standard auction. Hence, the dual optimization problem is the same for all standard auctions. In particular, μ^* is an optimal solution for this common dual problem. Finally, using a proof similar to the one we provide for Theorem 9 in Appendix C.3, it is possible to show that strong duality holds for the optimization problem stated in (C.3) when $\beta^* = \Psi$ and $\Psi(v/(1+\mu^*))$ is an optimal solution of (C.3) as required.

C.4 Worst-Case Efficiency Guarantees

The following example demonstrates that the Price of Anarchy of social welfare can be arbitrarily small for value-pacing-based equilibria.

Example 8. *Fix the number of buyers to $n = 2$ and consider the second-price auction format. Let the distribution of feature vectors F be the uniform distribution over $A = [1,2] \times [1,2]$. Moreover, assume that the buyer weight vectors are distributed uniformly over $[1,2] \times [1,2] \cup [y^4, y^4 + 1/y] \times [y^4, y^4 + 1/y]$ for some large $y \geq 1$. Also, suppose the budget of all buyer types with weight vector $w \in [1,2] \times [1,2]$ is 10 and the budget of all buyer types with weight vector $w \in [y^4, y^4 + 1/y] \times [y^4, y^4 + 1/y]$ is $1/y^2$. By Theorem 8, we get that there exists a value-pacing-based equilibrium for this instance. Let μ be the pacing function associated with it and x^μ be the corresponding* *allocation. First, observe that all of the buyer types with weight vectors in* $[1,2]\times[1,2]$ *are not paced in equilibrium and bid their value on each item, i.e.,* $\mu(w,10)=1$ *for all* $w\in[1,2]\times[1,2]$*. This is because their budget far exceeds their expected value: even if they win every item, their payments is as most 8, which is smaller than their budget of 10. Next, consider a buyer* $i\in\{1,2\}$ *with type* $\theta_i=(w,1/y^2)$ *for some* $w\in[y^4,y^4+1/y]\times[y^4,y^4+1/y]$*. Then, her expected payment (expectation over competing buyer type and item type) is at least*

$$P(w_{-i}\in[1,2]^2)\cdot\mathbb{E}_{\alpha,\theta_{-i}}[x_i^\mu(\alpha,\theta_i,\theta_{-i})\cdot 1\mid w_{-i}\in[1,2]^2]=\frac{1}{1+y^{-2}}\cdot\mathbb{E}_\alpha[x_i^\mu(\alpha,\theta_i,\theta_{-i})]$$

because, when $w_{-i}\in[1,2]\times[1,2]$*, buyer* $-i$ *bids her value on each item and her value is always at least 1. Moreover, the budget of the buyer with type* θ_i *is* $1/y^2$*. Therefore, we get*

$$\frac{1}{1+y^{-2}}\cdot\mathbb{E}_\alpha[x_i^\mu(\alpha,\theta_i,\theta_{-i})]\le\frac{1}{y^2}$$

Let x be the allocation that always gives the item to a buyer with weight vector $w\in[y^4,y^4+1/y]\times[y^4,y^4+1/y]$ *when such a buyer type is present. We partition the space of buyer-type profiles into 4 regions, and bound the expected social welfare (expectation taken only over* $\alpha\sim F$*) of* x^μ *and x:*

1. $\theta_i=(w_i,1)$ *with* $w_i\in[1,2]\times[1,2]$ *for both buyers* $i\in\{1,2\}$*. This occurs with probability at most 1 and the expected social welfare under* x^μ *when* $\alpha\sim F$ *is bounded above by 8 for each type profile in this region.*

2. $\theta_i=(w_i,1/y^2)$ *with* $w_i\in[y^4,y^4+1/y]\times[y^4,y^4+1/y]$ *for both buyers* $i\in\{1,2\}$*. This occurs with probability at most* $1/y^4$ *and the expected social welfare under* x^μ *when* $\alpha\sim F$ *is bounded above by* $8y^4$ *for each type profile in this region.*

3. $\theta_1=(w_1,1/y^2)$ *with* $w_1\in[y^4,y^4+1/y]\times[y^4,y^4+1/y]$ *and* $\theta_2=(w_2,10)$ *with* $w_2\in[1,2]\times[1,2]$*. This occurs with probability* $y^{-2}/(1+y^{-2})$*. As we argued earlier,* $\mathbb{E}_\alpha[x_1^\mu(\alpha,\theta_1,\theta_2)]\le(1+y^{-2})/y^2$ *in this case. Therefore, the expected social welfare under* x^μ *when* $\alpha\sim F$ *is bounded above by* $8y^4\cdot\{(1+y^{-2})/y^2\}+8\le 24y^2$*. On the other hand, the expected social*

welfare under x when $\alpha \sim F$ is at least y^4 in this region since buyer 1 always gets the item.

4. $\theta_2 = (w_2, 1/y^2)$ *with* $w_2 \in [y^4, y^4 + 1/y] \times [y^4, y^4 + 1/y]$ *and* $\theta_1 = (w_1, 10)$ *with* $w_1 \in [1, 2]\times[1, 2]$. *This is the same as region 3 with the roles of buyer 1 and buyer 2 interchanged.*

Combining the bounds for the different regions, we get that the total expected social welfare under x^μ is bounded above by

$$8 + 8y^4 \cdot \frac{1}{y^4} + 24y^2 \cdot \frac{y^{-2}}{1 + y^{-2}} + 24y^2 \cdot \frac{y^{-2}}{1 + y^{-2}} \leq 64 \,,$$

and the total expected social welfare under x is bounded below by

$$0 + 0 + y^4 \cdot \frac{y^{-2}}{1 + y^{-2}} + y^4 \cdot \frac{y^{-2}}{1 + y^{-2}} \geq y^2.$$

Hence, the Price of Anarchy of social welfare is at most $64/y^2$*, which tends to zero as* $y \to \infty$.

Proof of Theorem 10. We will focus on second-price auctions. Consider an allocation x, an equilibrium pacing function μ with $\mu \in C_0^*(\mu)$ and the associated allocation x^μ. Since x and μ are arbitrary, it suffices to show that $\mathrm{LW}(x^\mu) \geq \mathrm{LW}(x)/2$.

Let $p(\alpha, \vec{\theta})$ denote the second-highest bid on item α in the equilibrium parameterized by μ when the buyer-type profile is given by $\vec{\theta}$, i.e., it is the second largest element in the set $\{w_i^T \alpha/(1 + \mu(w_i, B_i)) \mid i \in [n]\}$. The following lemma is a key step in the proof of the theorem.

Lemma 41. *For all $i \in [n]$ and $\theta_i \in \Theta$, we have*

$$\begin{aligned}
&\min\left\{\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T \alpha \cdot x_i^\mu(\alpha, \theta_i, \theta_{-i})], B_i\right\} \\
&\quad \geq \min\left\{\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T \alpha \cdot x_i(\alpha, \theta_i, \theta_{-i})], B_i\right\} - \mathbb{E}_{\alpha,\theta_{-i}}[p(\alpha, \theta_i, \theta_{-i}) \cdot x_i(\alpha, \theta_i, \theta_{-i})] \,.
\end{aligned}$$

Proof. Fix $i \in [n]$ and $\theta_i \in \Theta$. We will prove the lemma separately for paced and unpaced buyer types. First, consider the case when θ_i is paced in equilibrium, i.e., $\mu(\theta_i) > 0$. Then, since $\mu(w_i, B_i) \in \operatorname{argmin}_{t \geq 0} q^\mu(w_i, B_i, t)$, complementary slackness (see proof of Theorem 9) implies that:

$$\mathbb{E}_{\alpha,\theta_{-i}}\left[p(\alpha,\theta_i,\theta_{-i})\cdot x_i^{\mu}(\alpha,\theta_i,\theta_{-i})\right] = B_i\,.$$

Moreover, note that $w_i^T\alpha/(1+\mu(w_i,B_i)) \geq p(\alpha,\theta_i,\theta_{-i})$ whenever $x_i^{\mu}(\alpha,\theta_i,\theta_{-i}) > 0$ because only the highest bidder(s) win the item in a second-price auction. This allows us to establish the lemma for paced buyers:

$$\begin{aligned}
&\min\left\{\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T\alpha\cdot x_i^{\mu}(\alpha,\theta_i,\theta_{-i})], B_i\right\}\\
&\quad\geq \min\left\{\mathbb{E}_{\alpha,\theta_{-i}}\left[p(\alpha,\theta_i,\theta_{-i})\cdot x_i^{\mu}(\alpha,\theta_i,\theta_{-i})\right], B_i\right\}\\
&\quad= B_i\\
&\quad\geq \min\left\{\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T\alpha\cdot x_i(\alpha,\theta_i,\theta_{-i})], B_i\right\}\\
&\quad\geq \min\left\{\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T\alpha\cdot x_i(\alpha,\theta_i,\theta_{-i})], B_i\right\} - \mathbb{E}_{\alpha,\theta_{-i}}[p(\alpha,\theta_i,\theta_{-i})\cdot x_i(\alpha,\theta_i,\theta_{-i})]\,,
\end{aligned}$$

where the first inequality follows because $w_i^T\alpha \geq w_i^T\alpha/(1+\mu(w_i,B_i))$ since $\mu(w_i,B_i)\geq 0$, the first equality because budgets binds, the second inequality because $B_i \geq \min(a,B_i)$ for every $a\in\mathbb{R}$, and the last inequality because payments are non-negative.

Next, consider the case when θ_i is unpaced in equilibrium, i.e., $\mu(\theta_i)=0$. Then, by definition of a pacing-based strategy for second-price auctions, buyer type θ_i bids her value $w_i^T\alpha$ on item α in equilibrium, for all items $\alpha\in A$. As a consequence, if $x_i^{\mu}(\alpha,\theta_i,\theta_{-i}) < 1$, then we have $w_i^T\alpha \leq p(\alpha,\theta_i,\theta_{-i})$. In other words,

$$\mathbb{E}_{\alpha,\theta_{-i}}[(w_i^T\alpha - p(\alpha,\theta_i,\theta_{-i}))\cdot(1-x_i^{\mu}(\alpha,\theta_i,\theta_{-i}))\cdot x_i(\alpha,\theta_i,\theta_{-i})] \leq 0\,. \tag{C.4}$$

Moreover, observe that

$$\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T\alpha\cdot x_i^{\mu}(\alpha,\theta_i,\theta_{-i})] \geq \mathbb{E}_{\alpha,\theta_{-i}}[(w_i^T\alpha - p(\alpha,\theta_i,\theta_{-i}))\cdot x_i^{\mu}(\alpha,\theta_i,\theta_{-i})\cdot x_i(\alpha,\theta_i,\theta_{-i})]\,, \tag{C.5}$$

because payments are non-negative and $x_i \in [0,1]$. Combining (C.4) and (C.5) yields

$$\begin{aligned}
&\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T\alpha \cdot x_i^{\mu}(\alpha,\theta_i,\theta_{-i})] \\
&\quad \geq \mathbb{E}_{\alpha,\theta_{-i}}[(w_i^T\alpha - p(\alpha,\theta_i,\theta_{-i})) \cdot x_i(\alpha,\theta_i,\theta_{-i})] \\
&\quad \geq \min\left\{\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T\alpha \cdot x_i(\alpha,\theta_i,\theta_{-i})], B_i\right\} - \mathbb{E}_{\alpha,\theta_{-i}}[p(\alpha,\theta_i,\theta_{-i}) \cdot x_i(\alpha,\theta_i,\theta_{-i})]\,, \qquad \text{(C.6)}
\end{aligned}$$

where the last inequality follows because $a \geq \min(a, B_i)$ for every $a \in \mathbb{R}$. Furthermore, note the trivial inequality

$$\begin{aligned}
B_i &\geq \min\left\{\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T\alpha \cdot x_i(\alpha,\theta_i,\theta_{-i})], B_i\right\} \\
&\geq \min\left\{\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T\alpha \cdot x_i(\alpha,\theta_i,\theta_{-i})], B_i\right\} - \mathbb{E}_{\alpha,\theta_{-i}}[p(\alpha,\theta_i,\theta_{-i}) \cdot x_i(\alpha,\theta_i,\theta_{-i})]\,, \qquad \text{(C.7)}
\end{aligned}$$

where we used again that $B_i \geq \min(a, B_i)$ for every $a \in \mathbb{R}$ and that payments are non-negative. Finally, combining (C.6) and (C.7) yields the lemma for unpaced buyers

$$\begin{aligned}
&\min\left\{\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T\alpha \cdot x_i^{\mu}(\alpha,\theta_i,\theta_{-i})], B_i\right\} \\
\geq\quad &\min\left\{\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T\alpha \cdot x_i(\alpha,\theta_i,\theta_{-i})], B_i\right\} - \mathbb{E}_{\alpha,\theta_{-i}}[p(\alpha,\theta_i,\theta_{-i}) \cdot x_i(\alpha,\theta_i,\theta_{-i})], \qquad \text{(C.8)}
\end{aligned}$$

since (C.6) and (C.7) show the inequality separately for each of the two terms in the minimum on the left-hand side of (C.8). This establishes the lemma for all $i \in [n]$ and $\theta_i \in \Theta$. □

Continuing the proof of Theorem 10, next, we sum over $i \in [n]$ and take expectation w.r.t. θ_i for the inequality in Lemma 41. First, we study the effect of summing and taking expectations on the second term in the RHS. We have

$$\begin{aligned}
\sum_{i=1}^{n} \mathbb{E}_{\theta_i}\left[\mathbb{E}_{\alpha,\theta_{-i}}[p(\alpha,\theta_i,\theta_{-i}) \cdot x_i(\alpha,\theta_i,\theta_{-i})]\right] &= \mathbb{E}_{\alpha,\vec{\theta}}\left[p(\alpha,\vec{\theta}) \cdot \sum_{i=1}^{n} x_i(\alpha,\theta_i,\theta_{-i})\right] \\
&= \mathbb{E}_{\alpha,\vec{\theta}}\left[p(\alpha,\vec{\theta})\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{\alpha,\vec{\theta}}\left[p(\alpha,\vec{\theta}) \cdot \sum_{i=1}^{n} x_i^{\mu}(\alpha,\theta_i,\theta_{-i})\right] \\
&= \sum_{i=1}^{n} \mathbb{E}_{\theta_i}\left[\mathbb{E}_{\alpha,\theta_{-i}}[p(\alpha,\theta_i,\theta_{-i}) \cdot x_i^{\mu}(\alpha,\theta_i,\theta_{-i})]\right] \\
&\leq \sum_{i=1}^{n} \mathbb{E}_{\theta_i}\left[\min\left\{\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T\alpha \cdot x_i^{\mu}(\alpha,\theta_i,\theta_{-i})], B_i\right\}\right] \\
&= \mathrm{LW}(x^{\mu}), \qquad \text{(C.9)}
\end{aligned}
$$

where the first and fourth equalities follow from Fubini's theorem, the second and third because allocations sum up to one (i.e., there no reserve prices), and the last inequality follows from the budget-feasibility of the pacing-based equilibrium strategy given by μ for buyer type θ_i, which implies

$$
\mathbb{E}_{\alpha,\theta_{-i}}[p(\alpha,\theta_i,\theta_{-i}) \cdot x_i^{\mu}(\alpha,\theta_i,\theta_{-i})] \leq B_i
$$

and the winning criteria of second-price auctions, which implies

$$
p(\alpha,\theta_i,\theta_{-i}) \leq \frac{w_i^T\alpha}{1+\mu(w_i,B_i)} \leq w_i^T\alpha
$$

whenever $x_i^{\mu}(\alpha,\theta_i,\theta_{-i}) > 0$.

Using (C.9), we obtain by summing over $i \in [n]$ and integrate over θ_i the inequality in the statement of Lemma 41:

$$
\begin{aligned}
\mathrm{LW}(x^{\mu}) &= \sum_{i=1}^{n} \mathbb{E}_{\theta_i}\left[\min\left\{\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T\alpha \cdot x_i^{\mu}(\alpha,\theta_i,\theta_{-i})], B_i\right\}\right] \\
&\geq \sum_{i=1}^{n} \mathbb{E}_{\theta_i}\left[\min\left\{\mathbb{E}_{\alpha,\theta_{-i}}[w_i^T\alpha \cdot x_i(\alpha,\theta_i,\theta_{-i})], B_i\right\}\right] - \sum_{i=1}^{n} \mathbb{E}_{\theta_i}\left[\mathbb{E}_{\alpha,\theta_{-i}}[p(\alpha,\theta_i,\theta_{-i}) \cdot x_i(\alpha,\theta_i,\theta_{-i})]\right] \\
&= \mathrm{LW}(x) - \sum_{i=1}^{n} \mathbb{E}_{\theta_i}\left[\mathbb{E}_{\alpha,\theta_{-i}}[p(\alpha,\theta_i,\theta_{-i}) \cdot x_i(\alpha,\theta_i,\theta_{-i})]\right] \\
&\geq \mathrm{LW}(x) - \mathrm{LW}(x^{\mu}).
\end{aligned}
$$

Therefore, we have shown that $\mathrm{LW}(x^{\mu}) \geq \mathrm{LW}(x)/2$ as required. □

C.5 Structural Properties

Before proceeding with the proof of Proposition 8, we establish the following Lemma, which is informative in its own right.

Lemma 42. *The pacing function* $\mu : \Theta \to [0, \omega/B_{\min}]$ *is continuous.*

Proof. We start by observing that the following function is continuous for all $\alpha \in A$:

$$(w, B, t) \mapsto \int_0^{\frac{w^T \alpha}{1+t}} H_\alpha^\mu(s) ds$$

Therefore, Dominated Convergence Theorem implies $(w, B, t) \mapsto q^\mu(w, B, t)$ is continuous. Finally, applying Berge Maximum Theorem (Theorem 17.31 of [AB06]) yields the continuity of $(w, B) \mapsto \mu(w, B)$ because of our assumption that $\mu(w, B)$ is the unique minimizer of $q^\mu(w, B, t)$.

□

We now state the proof of Proposition 8.

Proof of Proposition 8. Consider a unit vector $\hat{w} \in \mathbb{R}_+^d$ and budget $B > 0$ such that $w/\|w\| = \hat{w}$, for some $(w, B) \in \delta(X)$. If $\mu(w, B) = 0$ for all buyers $(w, B) \in \delta(X)$ with $w/\|w\| = \hat{w}$, then the theorem statement holds trivially. So assume that there exists $x > 0$ such that $x\hat{w} \in \delta(X)$ and $\mu(x\hat{w}, B) > 0$. Define $x_0 := \inf\{x \in (0, \infty) \mid (x\hat{w}, B) \in \delta(X);\ \mu(x\hat{w}, B) > 0\}$. Then, as a consequence of the complementary slackness condition established in Proposition 6, for $x > x_0$, we have

$$E_\alpha \left[\sigma_\alpha^\mu \left(\frac{x\hat{w}^T \alpha}{1 + \mu(x\hat{w}, B)} \right) H_\alpha^\mu \left(\frac{x\hat{w}^T \alpha}{1 + \mu(x\hat{w}, B)} \right) \right] = B.$$

Recall that, in Lemma 34, we established the continuity of σ_α^μ and H_α^μ almost surely w.r.t. $\alpha \sim F$. Combining this with the continuity of μ established in Lemma 42, we can apply the Dominated Convergence Theorem to establish

$$E_\alpha \left[\sigma_\alpha^\mu \left(\frac{x_0\hat{w}^T \alpha}{1 + \mu(x_0\hat{w}, B)} \right) H_\alpha^\mu \left(\frac{x_0\hat{w}^T \alpha}{1 + \mu(x_0\hat{w}, B)} \right) \right] = B.$$

As $B > 0$, we get $x_0 > 0$. Next, observe that if $t^* \geq 0$ satisfies $x_0(1 + t^*) = x(1 + \mu(x_0\hat{w}, B)$, then

$$\frac{\partial q^{\mu}(w, B, t^*)}{\partial t} = B - E_{\alpha}\left[\sigma_{\alpha}^{\mu}\left(\frac{x\hat{w}^T\alpha}{1 + t^*}\right) H_{\alpha}^{\mu}\left(\frac{x\hat{w}^T\alpha}{1 + t^*}\right)\right] = 0$$

Therefore, by our uniqueness assumption on μ, we get $1 + \mu(x\hat{w}, B) = (x/x_0)(1 + \mu(x_0\hat{w}, B))$ for all $x \geq x_0$. Hence, for all $x \geq x_0$, we get

$$\frac{x\hat{w}^T\alpha}{1 + \mu(x\hat{w}, B)} = \frac{x_0\hat{w}^T\alpha}{1 + \mu(x_0\hat{w}, B)}$$

Part (1) of Proposition 8 follows directly. Part (2) considers the case when there exists $y \geq 0$ such that $(y\hat{w}, B) \in \delta(X)$ and $\mu(y\hat{w}, B) = 0$. In this case, Lemma 42 and the connectedness of $\delta(X)$ imply that $\mu(x_0\hat{w}, B) = 0$, with part (2) of Proposition 8 following as a direct consequence. □

Next, we state the proof of Proposition 9.

Proof of Proposition 9. First, note that

$$\begin{aligned} q^{\mu}(w, B, \mu(w, B)) &= \min_{t \geq 0} q^{\mu}(w, B, t) \\ &= \min_{t \geq 0} (1 + t)\, \mathbb{E}_{\alpha}\left[\int_0^{\frac{w^T\alpha}{1+t}} H_{\alpha}^{\mu}(s) ds\right] + tB. \end{aligned}$$

Next, define $g : \Theta \times A \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ as

$$g(w, B, \alpha, t) = (1 + t)\int_0^{\frac{w^T\alpha}{1+t}} H_{\alpha}^{\mu}(s) ds + tB.$$

Since H^{μ} is continuous (Lemma 34), we get that g is differentiable w.r.t. w and the derivative satisfies

$$\|\nabla_w g(w, B, \alpha, t)\| = \left\|\alpha \cdot H_{\alpha}^{\mu}\left(\frac{w^T\alpha}{1 + t}\right)\right\| \leq \max_{\alpha \in A} \|\alpha\|.$$

Therefore, dominated convergence theorem implies that

$$\nabla_w \mathbb{E}_\alpha[g(w, B, \alpha, t)] = \mathbb{E}_\alpha\left[\nabla_w g(w, B, \alpha, t)\right] = \mathbb{E}_\alpha\left[\alpha \cdot H_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right)\right].$$

Note that the dual function q^μ is convex in t and at least one dual optimal solution always lies in the compact set $[0, \omega/B_{\min}]$ (Lemma 36). Moreover, if $t_1, t_2 \in \operatorname{argmin}_{t\geq 0} q^\mu(w, B, t)$, then the optimality conditions discussed in the proof of Proposition 6 imply

$$\frac{\partial q^\mu(w, B, t_1)}{\partial t} = \frac{\partial q^\mu(w, B, t_2)}{\partial t} = 0.$$

Without loss of generality, assume $t_1 < t_2$. Moreover, suppose

$$H_\alpha^\mu\left(\frac{w^T\alpha}{1+t_1}\right) > H_\alpha^\mu\left(\frac{w^T\alpha}{1+t_2}\right).$$

In the proof of Lemma 34, we showed that the above equation implies

$$\sigma_\alpha^\mu\left(\frac{w^T\alpha}{1+t_1}\right) H_\alpha^\mu\left(\frac{w^T\alpha}{1+t_1}\right) > \sigma_\alpha^\mu\left(\frac{w^T\alpha}{1+t_2}\right) H_\alpha^\mu\left(\frac{w^T\alpha}{1+t_2}\right) \quad \forall\, \alpha \in A.$$

This contradicts Lemma 38 because

$$\frac{\partial q^\mu(w, B, t)}{\partial t} = \mathbb{E}_\alpha\left[\sigma_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right) H_\alpha^\mu\left(\frac{w^T\alpha}{1+t}\right)\right]$$

Hence, we have shown that

$$H_\alpha^\mu\left(\frac{w^T\alpha}{1+t_1}\right) = H_\alpha^\mu\left(\frac{w^T\alpha}{1+t_2}\right) \quad \forall\, t_1, t_2 \in \operatorname{argmin}_{t\geq 0} q^\mu(w, B, t)$$

This allows us to invoke Danskin's Theorem, which yields

$$\nabla_w q^\mu(w, B, \mu(w, B)) = B - \nabla_w \mathbb{E}_\alpha[g(w, B, \alpha, t)]\Big|_{t=\mu(w,B)} = \mathbb{E}_\alpha\left[\alpha \cdot H_\alpha^\mu\left(\frac{w^T\alpha}{1+\mu(w, B)}\right)\right],$$

thereby completing the proof. □

C.6 Analytical and Numerical Examples

Proof of Claim 1. Note that $w/(1+\mu(w,B)) = w/\|w\|$ for all $(w,B) \in \Theta$. Therefore, $w/(1+\mu(w,B))$ is distributed uniformly on the unit ring restricted to the positive quadrant $\{(x,y) \in \mathbb{R}^2_{\geq 0} \mid x^2+y^2 = 1\}$. Hence,

$$H^\mu_\alpha(s) = P_{(w,B)}\left(\frac{w^T\alpha}{1+\mu(w,B)} \leq s\right) = \frac{\arcsin(s)}{\pi/2} \qquad \text{for } \alpha \in A = \{e_1, e_2\}$$

Observe that H^μ_α is continuous for all $\alpha \in A$. This implies that, for all $(w,B) \in \Theta$, strong duality holds for the optimization problem $Q^\mu(w,B)$, because the proof of the results given in Section 4.2.2 only relied on continuity of H^μ_α. Therefore, to prove the claim, it suffices to show that each buyer (w,B) exactly spends her budget. The total payment made by buyer $(w,B) \in \Theta$, when everyone uses β^μ, is given by

$$\begin{aligned}
\mathbb{E}_\alpha\left[\tilde{\beta}^\mu_\alpha\left(\frac{w^T\alpha}{1+\mu(w,B)}\right) H^\mu_\alpha\left(\frac{w^T\alpha}{1+\mu(w,B)}\right)\right] &= \sum_{i=1}^{2}\frac{1}{2}\left[\hat{w}_i H^\mu_{e_i}(\hat{w}_i) - \int_0^{\hat{w}_i} H^\mu_{e_i}(s)ds\right] \\
&= \frac{2-\hat{w}_1-\hat{w}_2}{\pi} \\
&= \frac{2\|w\| - w_1 - w_2}{\pi\|w\|}\,.
\end{aligned}$$

Hence, the claim holds. □

C.7 Extension to Non-linear Response Functions

In this section, we discuss extensions of our results beyond linear valuation functions. Let $f : \mathbb{R} \to \mathbb{R}$ be a (potentially non-linear) monotonically increasing function. We assume that the value a buyer with weight vector w has for item with feature vector α is given by $f(w^T\alpha)$. Moreover, we relax the assumption that $\Theta, A \subset \mathbb{R}_+$ and only require that $f(w^T\alpha)$ is non-negative for all $w \in \Theta, \alpha \in A$. For example, the logistic function $f(t) = e^t/(1+e^t)$ is a non-linear increasing response function commonly used in practice which satisfies the above assumptions. Moreover, the linear function $f(t) = t$ yields our original linear model. Before proceeding further, we appropriately modify the terms defined earlier to accommodate this more general valuation model given by f.

Consider a pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$. We define the *paced value* of a buyer type (w, B) for item α as $f(w^T\alpha)/(1 + \mu(w, B))$. For item $\alpha \in A$, let λ^μ_α denote the distribution of paced values $f(w^T\alpha)/(1 + \mu(w, B))$ when $(w, B) \sim G$. Let H^μ_α denote the distribution of the highest value $Y := \max\{X_1, \ldots, X_{n-1}\}$ among $n - 1$ buyers, when each $X_i \sim \lambda^\mu_\alpha$ is drawn independently for $i \in \{1, \ldots, n - 1\}$. Observe that $H^\mu_\alpha((-\infty, x]) = \lambda^\mu_\alpha((-\infty, x])^{n-1}$ for all $\alpha \in A$ because the random variables are i.i.d.

To better understand how our results can be extended to this more general valuation model, it is important to understand how the linearity assumption was employed in our derivations. A careful analysis of the derivations would reveal that the linearity was only employed exactly once, and that was to prove part (a) of Lemma 34. In the following lemma, we prove the analogue of part (a) of Lemma 34. The analysis for the rest of our results remains the same for this more general non-linear valuation model.

Lemma 43. *λ^μ_α and H^μ_α (as defined above) have a continuous CDF for every pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$.*

Proof. Consider a pacing function $\mu : \Theta \to \mathbb{R}_{\geq 0}$. Let $\alpha_1, \alpha_2 \in A$ be linearly independent feature vectors and $x_1, x_2 \in [0, \omega]$ be two possible item values. We consider the set of buyer types which have paced value x_1 for α_1 and paced value x_2 for α_2. Define

$$S := \left\{(w, B) \in \Theta \,\middle|\, \frac{f(w^T\alpha_1)}{1 + \mu(w, B)} = x_1;\ \frac{f(w^T\alpha_2)}{1 + \mu(w, B)} = x_2 \right\}$$

Observe that, for $(w, B) \in S$ and $c := x_1/x_2$, we have $f(w^T\alpha_1) = cf(w^T\alpha_2)$. Therefore, the set $T = \{w \mid f(w^T\alpha_1) = cf(w^T\alpha_2)\}$ is a superset of the set S_w. Next, define $T(s) = \{w \mid w^T\alpha_1 = f^{-1}(cf(s));\ w^T\alpha_2 = s\}$. Then, it immediately follows that $T = \cup_{s: f(s) \geq 0} T(s)$. Due to their linear independence, we can find a basis that contains α_1, α_2, call it $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Let M be the invertible matrix whose rows are given by $\alpha_1, \alpha_2, \ldots, \alpha_n$. Now, note that the set $U \coloneqq \{(f^{-1}(cf(s)), s) \mid s \in \mathbb{R}, f(s) \geq 0\} \subseteq \mathbb{R}^2$ has Lebesgue measure zero because it is the graph of a monotonic continuous real-valued function. As a consequence, the set $U \times \mathbb{R}^{n-2}$ also has zero Lebesgue measure, which further implies that $M^{-1}(U \times \mathbb{R}^{n-2})$ has zero Lebesgue measure because M is an invertible linear transformation.

Observe that, if $w \in T = \cup_{s: f(s) \geq 0} T(s)$, then there exists s such that $f(s) \geq 0$, $\alpha_1^T w = f^{-1}(f(s))$ and $\alpha_2^T w = s$. Hence, the first two components of Mw are $f^{-1}(f(s))$ and s respectively, thereby implying $w \in M^{-1}(U \times \mathbb{R}^{n-2})$. Therefore, we get that $T \subset M^{-1}(U \times \mathbb{R}^{n-2})$ and, as a consequence, T has zero Lebesgue measure. Finally, this implies that $G(S) = 0$ because G has a density. The rest of the analysis is analogous to the one given in the proof of Lemma 34. □

Appendix D: Appendix to Chapter 5

D.1 Proof of Theorem 13

Consider a $\{0,1\}$-cost $n \times n$ bimatrix game (A, B) and let $\epsilon = 1/n$. Recall that an ϵ-well-supported Nash equilibrium is a pair $(x, y) \in \Delta_n \times \Delta_n$ such that $x_i > 0$ for any $i \in [n]$ implies that $\sum_j A_{ij} y_j \leq \sum_j A_{kj} y_j + \epsilon$ for all k and $y_j > 0$ for any $j \in [n]$ implies $\sum_i x_i B_{ij} \leq \sum_i x_i B_{ik} + \epsilon$ for all $k \in [n]$.

In this section we show how to construct an SPP game G with $4n + 1$ buyers from the bimatrix game (A, B) in time polynomial in n such that every (δ, γ)-approximate PE of G, where $\delta = \gamma = \epsilon/n^6$, can be mapped back to an ϵ-well-supported Nash equilibrium of (A, B) in polynomial time. Theorem 11 follows from the PPAD-completeness of the problem of finding an ϵ-well-supported Nash equilibrium in a $\{0,1\}$-cost bimatrix game with $\epsilon = 1/n$ [CTV07].

The SPP game G contains the following goods:

- Normalization goods: n goods $\{N(p, s)_1, \ldots, N(p, s)_n\}$ for each $p \in \{1, 2\}$ and $s \in [n]$.
- Expenditure goods: n goods $\{E(p, s)_1, \ldots, E(p, s)_n\}$ for each $p \in \{1, 2\}$ and $s \in [n]$.
- Threshold goods $T(p, s)$ for each $p \in \{1, 2\}$ and $s \in [n]$.

Set $\nu = 1/(16n)$. The set of buyers in G is defined as follows:

- Buyer $\mathbb{C}(p, s)$, $p \in \{1, 2\}$ and $s \in [n]$: $\mathbb{C}(p, s)$ has positive values for the following goods:
  - Normalization goods: $V(\mathbb{C}(p, s), N(p, s)_i) = 16$ for all $i \in [n] \setminus \{s\}$;
    $V(\mathbb{C}(p, s), N(p, s)_s) = 1$; and $V(\mathbb{C}(p, s), N(p, t)_s) = 1$ for all $t \in [n] \setminus \{s\}$.

- Threshold good $T(p, s)$: $V(\mathbb{C}(p, s), T(p, s)) = 2n^4$.
- Expenditure goods: $V(\mathbb{C}(p, s), E(p, s)_i) = 1$ for all $i \in [n]$.

  For $p = 1$: $V(\mathbb{C}(1, s), E(2, t)_s) = \nu B_{st}$ for all $t \in [n]$.

  For $p = 2$: $V(\mathbb{C}(2, s), E(1, t)_s) = \nu A_{ts}$ for all $t \in [n]$.

For $p = 1$, the budget of buyer $\mathbb{C}(1, s)$ is $n/2 + n^4 + 1/4 + \sum_{t\in[n]} \nu A_{st}/2$;
For $p = 2$, the budget of buyer $\mathbb{C}(2, s)$ is $n/2 + n^4 + 1/4 + \sum_{t\in[n]} \nu B_{ts}/2$.

- Threshold Buyer $\mathbb{T}$: $\mathbb{T}$ has positive values only for the following goods:
  - Threshold goods: $V(\mathbb{T}, T(p, s)) = (1 - \delta)n^4$ for each $p \in \{1, 2\}$ and $s \in [n]$.
  - Expenditure goods: $V(\mathbb{T}, E(1, s)_t) = \nu A_{st}/2$ and $V(\mathbb{T}, E(2, s)_t) = \nu B_{ts}/2$ for all $s, t \in [n]$.

  Buyer $\mathbb{T}$ has budget n^7.

- Dummy buyer $\mathbb{D}(p, s)$, $p \in \{1, 2\}$ and $s \in [n]$: The budget of $\mathbb{D}(p, s)$ is ν and she only values the normalization good $N(p, s)_s$: $V(\mathbb{D}(p, s), N(p, s)_s) = 1$.

Let $\mathcal{E}$ be a (δ, γ)-approximate PE of the game G. We will use $\alpha(\cdot)$ to denote pacing multipliers of buyers in $\mathcal{E}$. Observe that, from the definition of approximate pacing equilibria, we must have $\alpha(\mathbb{T}) \in [1 - \gamma, 1]$. The following lemma establishes bounds on pacing multipliers of other buyers.

Lemma 44. *For each $p \in \{1, 2\}$ and $s \in [n]$, we have*

$$\frac{(1-\delta)^2}{2} \le \alpha(\mathbb{C}(p, s)) \le \frac{7}{8} \quad \textit{and} \quad (1 - \delta) \cdot \alpha(\mathbb{C}(p, s)) \le \alpha(\mathbb{D}(p, s)) \le \frac{\alpha(\mathbb{C}(p, s))}{1 - \delta}.$$

Proof. Suppose for some $p \in \{1, 2\}$ and $s \in [n]$, we have $\alpha(\mathbb{C}(p, s)) < (1 - \delta)^2/2$. Then $\mathbb{C}(p, s)$ doesn't win any part of the threshold good $T(p, s)$. Observe that she has value at most 16 for every other good. Given that there are only $O(n^2)$ goods in G, she can not possibly spend all her budget (which is $\Omega(n^4)$). This contradicts the assumption that $\mathcal{E}$ is an approximate PE. Therefore, we have $\alpha(\mathbb{C}(p,s)) \geq (1-\delta)^2/2$ for each $p \in \{1,2\}$ and $s \in [n]$.

Next we prove the inequality about $\alpha(\mathbb{D}(p,s))$. Suppose $(1-\delta)\alpha(\mathbb{D}(p,s)) > \alpha(\mathbb{C}(p,s))$ for some $p \in \{1,2\}$ and $s \in [n]$. Then, $\mathbb{D}(p,s)$ wins all of good $N(p,s)_s$ at price $\alpha(\mathbb{C}(p,s)) \geq (1-\delta)^2/2$. This violates her budget constraint and leads to a contradiction. Hence $\alpha(\mathbb{D}(p,s)) \leq \alpha(\mathbb{C}(p,s))/(1-\delta)$. Moreover, if $\alpha(\mathbb{D}(p,s)) < (1-\delta)\alpha(\mathbb{C}(p,s))$ (which implies $\alpha(\mathbb{D}(p,s)) < 1-\delta = 1-\gamma$) then her expenditure is zero. This violates the no unnecessary pacing condition. Hence the inequality about $\alpha(\mathbb{D}(p,s))$ must hold. Observe that, in particular, this means that the price of $N(p,s)_s$ is between $(1-\delta)\alpha(\mathbb{C}(p,s))$ and $\alpha(\mathbb{C}(p,s))/(1-\delta)$.

Finally suppose $\alpha(\mathbb{C}(p,s)) > 7/8$ for some $p \in \{1,2\}$, $s \in [n]$. Then she wins:

- All of normalization good $N(p,s)_t$, for each $t \neq s$, by spending at least $(1-\delta)^2/2$ on each of them because $\alpha(\mathbb{C}(p,t)) \geq (1-\delta)^2/2$ by the first part of the proof.
- Part of normalization good $N(p,s)_s$ by spending at least $(1-\delta)(7/8) - \nu$. This is because $N(p,s)_s$ has price at least $(1-\delta)(7/8)$ and buyer $\mathbb{D}(p,s)$ only has budget ν.
- All of threshold good $T(p,s)$ by spending $\alpha(\mathbb{T})(1-\delta)n^4 \geq (1-\delta)^2 n^4$ (using $\gamma = \delta$).
- All of expenditure good $E(p,s)_t$, for each $t \in [n]$, by spending at least $\alpha(\mathbb{T})\nu A_{st}/2$ if $p = 1$ and $\alpha(\mathbb{T})\nu B_{ts}/2$ if $p = 2$.

Hence, the total expenditure of $\mathbb{C}(p,s)$ when $p = 1$ is at least

$$(1-\delta)^2 \cdot \frac{n-1}{2} + (1-\delta) \cdot \frac{7}{8} - \nu + (1-\delta)^2 n^4 + (1-\delta) \sum_{t \in [n]} \nu A_{st}/2$$

which is strictly higher the budget (using $\delta = 1/n^7$). The same also holds for $p = 2$. In both cases, the budget constraint is violated, leading to a contradiction. Therefore, the lemma holds. □

In particular, the above lemma implies that the total expenditure of each buyer $\mathbb{C}(p,s)$ is at least $(1-\gamma)$-fraction of her budget (and of course is also bounded from above by her budget). We also get the following corollary:

Corollary 5. *For each* $p \in \{1,2\}$ *and* $s \in [n]$, *the expenditure of* $\mathbb{C}(p,s)$ *on* $N(p,s)_s$ *lies in the following interval* $[(1-\delta)\alpha(\mathbb{C}(p,s)) - \nu, \alpha(\mathbb{C}(p,s)) - (1-\delta)\nu]$

Next, we define two vectors x' and y' with

$$x'_s = \{\alpha(1,s) - (\alpha(\mathbb{T})/2)\}^+ \quad \text{and} \quad y'_s = \{\alpha(2,s) - (\alpha(\mathbb{T})/2)\}^+$$

for each $s \in [n]$, where a^+ ddenotes $\max\{a, 0\}$. The following lemma will allow us to normalize x' and y' to obtain valid probability distributions.

Lemma 45. *The following inequalities hold:* $\sum_s x'_s > 1/8$ *and* $\sum_s y'_s > 1/8$.

Proof. We prove $\sum_s x'_s > 1/8$. The proof of $\sum_s y'_s > 1/8$ is analogous. Suppose that $\sum_s x'_s \leq 1/8$. Then, buyer $\mathbb{B}(1,1)$ only wins a non-zero fraction of the following goods, and spends:

- At most $\alpha(\mathbb{C}(1,t))$ on each normalization good $N(1,1)_t$ for each $t \in [n]$. The total expenditure is

$$\sum_{t\in[n]} \alpha(\mathbb{C}(1,t)) \leq n\alpha(\mathbb{T})/2 + \sum_{t\in[n]} x'_t \leq n/2 + 1/8.$$

- At most $(1-\delta)n^4$ on the threshold good $T(1,1)$.

- At most νA_{1t} on each expenditure good $E(1,1)_t$, $t \in [n]$.

Hence, the total expenditure of buyer $\mathbb{C}(1,1)$ is at most $n/2 + 1/8 + (1-\delta)n^4 + \sum_t \nu A_{1t}$, which is strictly less than her budget, a contradiction. □

Now, we are ready to define the mixed strategies (x, y) for the bimatrix game (A, B). Set player 1's mixed strategy x to be $x_s = x'_s/\sum_i x'_i$ and player 2's mixed strategy y to be $y_s = y'_s/\sum_i y'_i$. These are valid mixed strategies because of Lemma 16 and Lemma 17. The next lemma shows that (x, y) is indeed an ϵ-well-supported Nash equilibrium of (A, B).

Lemma 46. *(x, y) is an ϵ-well-supported Nash equilibrium of the bimatrix game (A, B).*

Proof. Assume there are $s, s^* \in [n]$ such that $x_s > 0$ but $\sum_t A_{st} y_t > \sum_t A_{s^*t} y_t + \epsilon$; the proof for y is analogous. Using $x_s > 0$, buyer $\mathbb{C}(1, s)$ spends non-zero amounts on the following goods:

- $\alpha(\mathbb{C}(1, t))$ on the normalization good $N(1, s)_t$ for each $t \neq s$.
- at least $(1 - \delta) \cdot \alpha(\mathbb{C}(1, s)) - \nu$ on the normalization good $N(1, s)_s$.
- $\alpha(\mathbb{T}) \cdot (1 - \delta)n^4$ on the threshold good $T(1, s)$.
- $\max\{\alpha(\mathbb{C}(2, t)), \alpha(\mathbb{T})/2\} \cdot \nu A_{st}$ on the expenditure good $E(1, s)_t$ for each $t \in [n]$.

Therefore, the total expenditure of buyer $\mathbb{C}(1, s)$ is at least

$$\begin{aligned}
&\sum_{t\in[n]} \alpha(\mathbb{C}(1, t)) - \delta \cdot \alpha(\mathbb{C}(1, s)) - \nu + \alpha(\mathbb{T}) \cdot (1 - \delta)n^4 + \sum_{t\in[n]} \max\{\alpha(\mathbb{C}(2, t)), \alpha(\mathbb{T})/2\} \cdot \nu A_{st} \\
&\quad = \sum_{t\in[n]} \alpha(\mathbb{C}(1, t)) - \delta \cdot \alpha(\mathbb{C}(1, s)) - \nu + \alpha(\mathbb{T}) \cdot (1 - \delta)n^4 + \alpha(\mathbb{T}) \sum_{t\in[n]} \nu A_{st}/2 + \nu \sum_{t\in[n]} y_t A_{st}.
\end{aligned}$$

On the other hand, buyer $\mathbb{C}(1, s^*)$ spends (without assuming $x_{s^*} > 0$):

- $\alpha(\mathbb{C}(1, t))$ on the normalization good $N(1, s^*)_t$ for each $t \neq s^*$.
- at most $\alpha(\mathbb{C}(1, s^*)) - (1 - \delta)\nu$ on the normalization good $N(1, s^*)_{s^*}$.
- at most $\alpha(\mathbb{T}) \cdot (1 - \delta)n^4$ on the threshold good $T(1, s^*)$.
- $\max\{\alpha(\mathbb{C}(2, t)), \alpha(\mathbb{T})/2\} \cdot \nu A_{s^*t}$ on the expenditure good $E(1, s^*)_t$ for each $t \in [n]$.

Therefore, the total expenditure of buyer $\mathbb{C}(1, s^*)$ is at most

$$\sum_{t\in[n]} \alpha(\mathbb{C}(1, t)) - (1 - \delta)\nu + \alpha(\mathbb{T}) \cdot (1 - \delta)n^4 + \alpha(\mathbb{T}) \sum_{t\in[n]} \nu A_{s^*t}/2 + \nu \sum_{t\in[n]} y_t A_{s^*t}.$$

Using the assumption that $\sum_t A_{st}y_t > \sum_t A_{s^*t}y_t + \epsilon$, we have that the total expenditure of $\mathbb{C}(1, s)$ minus that of $\mathbb{C}(1, s^*)$, denoted by (‡1), is at least

$$-\delta \cdot \alpha(\mathbb{C}(1, s)) - \delta\nu + \alpha(\mathbb{T}) \cdot \left(\sum_{t\in[n]} \nu A_{st}/2 - \sum_{t\in[n]} \nu A_{s^*t}/2 \right) + \epsilon\nu$$
$$\geq \alpha(\mathbb{T}) \cdot \left(\sum_{t\in[n]} \nu A_{st}/2 - \sum_{t\in[n]} \nu A_{s^*t}/2 \right) + \epsilon\nu/2$$

using $\epsilon\nu \gg \delta$. On the other hand, the budget of $\mathbb{C}(1, s)$ minus that of $\mathbb{C}(1, s^*)$, denoted (‡2), is

$$\sum_{t\in[n]} \nu A_{st}/2 - \sum_{t\in[n]} \nu A_{s^*t}/2.$$

Using $\alpha(\mathbb{T}) \geq 1 - \gamma$ and $\gamma = 1/n^7$, we have (‡1) $\geq$ (‡2) $+ \epsilon\nu/3$. However, the total expenditure of $\mathbb{C}(1, s)$ is at most her budget and the total expenditure of $\mathbb{C}(1, s^*)$ is at least $(1 - \gamma)$-fraction of her budget. Given that the budget of $\mathbb{C}(1, s^*)$ is $O(n^4)$, we also have

$$(‡1) \leq (‡2) + \gamma \cdot O(n^4) = (‡2) + O(1/n^3),$$

a contradiction because $\epsilon\nu = \Omega(1/n^2)$. □

Theorem 13 follows from the PPAD-hardness of finding an ϵ-well-supported Nash equilibrium in a $\{0, 1\}$-cost bimatrix game [CTV07].

D.2 Proof of Claim 2

Before stating the proof of Claim 2, we state and prove the following useful lemma.

Lemma 47. *If $\beta \in S$ is labelled i, then $\beta_i \geq \min\left\{\frac{1}{n}, \frac{B_{\min}}{2n\nu_{\max}}\right\}$.*

Proof. Without loss of generality, we will prove the lemma for $i = 1$. Suppose $\beta \in S$ is labelled 1 according to the above procedure. First, $\beta_1 > 0$ follows as a direct consequence. Furthermore, as $\max_i \beta_i \geq 1/n$ and $\sum_i \beta_i = 1$, we get $t^*(\beta) = t_1 \leq n$. We consider the two possible binding cases which can define t_1. If $t_1 = 1/\beta_1$, then $\beta_1 \geq 1/n$, and thus the lemma holds. On the other hand, if $t_1 = \frac{B_1}{\sum_j x_{1j} p_j(\beta)}$, then $\sum_j x_{ij} p_j(\beta) > 0$ and

$$B_1 = t_1 \sum_j x_{1j} p_j(\beta) \leq n \sum_{j:\, x_{ij}>0} \max_i \beta_i v_{ij} \leq n \sum_{j:\, x_{ij}>0} \frac{\beta_1 v_{1j}}{(1-\delta)} \leq n \sum_j \frac{\beta_1 v_{1j}}{(1-\delta)}$$

where the second inequality follows from the definition of (δ, γ)-approximate pacing equilibrium. Therefore, $\beta_1 \geq \min\left\{\frac{1}{n}, \frac{(1-\delta)B_1}{\sum_j v_{1j}}\right\}$. □

Proof of Claim 2. Let $B_{\min} = \min_{i\in[n]} B_i$, $B_{\max} = \max_{i\in[n]} B_i$, $v_{\max} = \max_{i,j} v_{ij}$ and $v_{\min} = \min_{i,j:v_{ij}>0} v_{ij}$. In this proof, we will use the following facts: if f, g are Lipschitz functions with Lipschitz constants L_f, L_g, then

(a) $f + g$ is Lipschitz with constant $L_f + L_g$

(b) $\max\{f, g\}$ is Lipschitz with constant $\max\{L_f, L_g\}$.

(c) If $|f|, |g| \leq M$, then fg is Lipschitz with constant $M(L_f + L_g)$.

Define $y_{ij} : S \longrightarrow \mathbb{R}$ as $y_{ij}(\beta) = [\beta_i v_{ij} - (1-\delta)\max_k \beta_k v_{kj}]^+$. Using facts (a) and (b), we can write

$$|y_{ij}(\beta) - y_{ij}(\beta')| \leq 2v_{\max} \|\beta - \beta'\|_\infty$$

Consider $\beta \in S_0$ and $i \in [n]$. As S_0 is panchromatic, there exists $\beta' \in S_0$ such that $T(\beta') = i$. By Lemma 47, we get

$$\beta'_i \geq \min\left\{\frac{1}{n}, \frac{B_{\min}}{2nv_{\max}}\right\}$$

Then, using the definition of ω, we get the following equivalent statements:

$$\beta_i \geq \frac{1}{2}\min\left\{\frac{1}{n}, \frac{B_{\min}}{2nv_{\max}}\right\} \iff \frac{1}{\beta_i} \leq U := 2\max\left\{n, \frac{2nv_{\max}}{B_{\min}}\right\}$$

Hence, for $\beta, \beta' \in S_0$, we have

$$\left|\frac{1}{\sum_r y_{rj}(\beta)} - \frac{1}{\sum_r y_{rj}(\beta')}\right| = \left|\frac{\sum_r y_{rj}(\beta') - \sum_r y_{rj}(\beta)}{\sum_r y_{rj}(\beta) \sum_r y_{rj}(\beta')}\right| \\ \leq \frac{2nv_{\max}U^2}{\delta^2 v_{\min}^2} \cdot \|\beta - \beta'\|_\infty$$

Using fact (c), for $\beta, \beta' \in S_0$, we can write

$$|x_{ij}(\beta) - x_{ij}(\beta')| \leq \max\left\{v_{\max}, \frac{U}{\delta v_{\min}}\right\}\left[2v_{\max} + \frac{2nv_{\max}U^2}{\delta^2 v_{\min}^2}\right] \cdot \|\beta - \beta'\|_\infty$$

Set $\bar{U} = \max\left\{v_{\max}, \frac{U}{\delta v_{\min}}\right\}\left[2v_{\max} + \frac{2nv_{\max}U^2}{\delta^2 v_{\min}^2}\right]$. Also, note that for $\beta, \beta' \in S$,

$$|p_j(\beta) - p_j(\beta')| \leq v_{\max}\|\beta - \beta'\|_\infty$$

For $\beta, \beta' \in S_0$, combining the above Lipschitz conditions using facts (a) and (c) yields

$$\left|\sum_j x_{ij}(\beta)p_j(\beta) - \sum_j x_{ij}(\beta')p_j(\beta')\right| \leq mv_{\max}(\bar{U} + v_{\max})\|\beta - \beta'\|_\infty$$

Set $W := mv_{\max}(\bar{U} + v_{\max})$. Define

$$P^* := \left\{i \in [n] \middle| \exists \beta \in T \text{ s.t. } \frac{B_i}{\sum_j x_{ij}(\beta)p_j(\beta)} < \frac{1}{\beta_i}\right\}$$

For $i \in P^*$ and $\beta \in S_0$, we can write $\frac{B_i}{\sum_j x_{ij}(\beta)p_j(\beta)} < \frac{1}{\beta_i} \leq U$, which implies $\frac{1}{\sum_j x_{ij}(\beta)p_j(\beta)} \leq \frac{U}{B_{\min}}$. Therefore, for $\beta, \beta' \in S_0$ and $i \in P^*$, we have

$$\left|\frac{B_i}{\sum_j x_{ij}(\beta)p_j(\beta)} - \frac{B_i}{\sum_j x_{ij}(\beta')p_j(\beta')}\right| \leq B_{\max}\frac{U^2}{B_{\min}^2}W\|\beta - \beta'\|_\infty \leq \frac{B_{\max}U^2 W}{B_{\min}^2}\|\beta - \beta'\|_\infty$$

Also, for $\beta, \beta' \in S_0$ and $i \in [n]$, we have

$$\left|\frac{1}{\beta_i} - \frac{1}{\beta'_i}\right| \leq U^2 \|\beta - \beta'\|_\infty$$

Note that for $\beta \in T$, we can rewrite $t^*(\beta)$ as follows

$$t^*(\beta) = \min\left\{\min_{i\in[n]} \frac{1}{\beta_i}, \min_{i\in P^*} \min\left\{\frac{1}{\beta_i}, \frac{B_i}{\sum_j x_{ij}(\beta)p_j(\beta)}\right\}\right\}$$

Using fact (b), for $\beta, \beta' \in T$,

$$|t^*(\beta) - t^*(\beta')| \leq 2n \max\left\{U^2, \frac{B_{\max}U^2 L}{B_{\min}^2}\right\} \|\beta - \beta'\|_\infty$$

Therefore, for $i \in [n]$, total payment made by buyer i is Lipschitz for $\beta \in S_0$:

$$\left|\sum_j x_{ij}(\beta)t^*(\beta)p_j(\beta) - \sum_j x_{ij}(\beta')t^*(\beta')p_j(\beta')\right| \\ \leq \max\{nv_{\max}, n\}\left(W + 2n\max\left\{U^2, \frac{B_{\max}U^2 W}{B_{\min}^2}\right\}\right)\|\beta - \beta'\|_\infty$$

Hence, the claim holds because

$$\max\{nv_{\max}, n\}\left(W + 2n\max\left\{U^2, \frac{B_{\max}U^2 W}{B_{\min}^2}\right\}\right) \leq L = \left(\frac{2^{|G|}}{\delta}\right)^{10{,}000}$$

□

D.3 Incorporating Reserve Prices

Consider the setting in which each item j has a reserve price r_j. Now, a buyer wins a good j only if her bid is the highest bid $h_j(\alpha)$ and it is greater than or equal to the reserve r_j. Moreover, the price of good j is the maximum of the second highest bid $p_j(\alpha)$ and its reserve price r_j. In the presence of reserve prices, we will use $H_j(\alpha) := \max\{h_j(\alpha), r_j\}$ to denote the winning threshold of good j and $P_j(\alpha) := \max\{p_j(\alpha), r_j\}$ to denote the price of good j. The next example illustrates that one needs to be careful in the way one extends the definition of pacing equilibrium (Definition 9) to model the presence of reserves.

Example 9. *There is one buyer and one good. The buyer values the good at 4 and has a budget of 1. The goods has a reserve price of 2. If she bids strictly less than* $1/2$*, then she does not win any part of the good. On the other hand, if we assume that she wins the entire good upon bidding* $1/2$ *or higher, then she violates her budget upon doing so. This suggests that a pacing equilibrium might not even exist if we extend it naively to the setting with reserves. Instead, we will take the approach that, in a pacing equilibrium, the seller may decide to not sell a fraction of a good if the highest bid is equal to the reserve price of that good. With this new definition, we can see that a pacing equilibrium does in fact exist, namely, when the buyer has a pacing multiplier of* $1/2$ *and wins* $1/2$ *of the item.*

Inspired by the above example, we define pacing equilibrium for the setting with reserves.

Definition 20 (Pacing Equilibria with reserves)**.** *Given an SPP game with reserves* $G = (n, m, (v_{ij}), (B_i), (r_j))$*, we say* (α, x) *with* $\alpha = (\alpha_i) \in [0,1]^n$*,* $x = (x_{ij}) \in [0,1]^{nm}$ *and* $\sum_{i\in[n]} x_{ij} \leq 1$ *for all* $j \in [m]$ *is a* pacing equilibrium *if*

(a) Only buyers above the winning threshold win the good: $x_{ij} > 0$ *implies* $\alpha_i v_{ij} = H_j(\alpha)$*.*

(b) Full allocation of each good for which the highest bid exceeds the reserve price: $h_j(\alpha) > r_j$ *implies* $\sum_{i\in[n]} x_{ij} = 1$*.*

(c) Budgets are satisfied: $\sum_{j\in[m]} x_{ij} P_j(\alpha) \leq B_i$*.*

(d) No unnecessary pacing: $\sum_{j\in[m]} x_{ij} P_j(\alpha) < B_i$ *implies* $\alpha_i = 1$*.*

Next, we extend our PPAD-membership result to the setting with reserves.

Theorem 26. *Finding a pacing equilibrium in a SPP game with reserves is in PPAD.*

Proof. Consider a pacing game with reserve prices G and the corresponding pacing game without reserve prices G'. Add an auxiliary buyer a to G' who values good j at r_j for all $j \in [m]$ and has a budget large enough to ensure that her pacing multiplier is always 1 in every pacing equilibrium (this can be achieved by setting her budget to be the sum of all values $\{v_{ij}\}$ and reserve prices $\{r_j\}$). We will call this updated game G'_+. The theorem follows from the simple observation that if we find a pacing equilibrium (α, x) for G'_+ and disregard the terms corresponding to the auxiliary buyer, then we get a pacing equilibrium (α_{-a}, x_{-a}) for G. This is because, in any pacing equilibrium of G'_+, the auxiliary buyer has a multiplier of 1 and hence bids r_j on good j for all $j \in [m]$. Moreover, any amount that the auxiliary buyer wins in (α, x) can be thought of as being not sold by the seller. As (α, x) satisfies Definition 9, it is straightforward to check that (α_{-a}, x_{-a}) satisfies Definition 20. □

We conclude this section by noting that our hardness results extend directly to the setting with reserves because it reduces to the setting without reserves when $r_j = 0$ for all goods $j \in [m]$.

D.4 Perturbed Second-Price Pacing Games

Before stating and proving the results, we define the relevant equilibrium notions. For a perturbed pacing game $(n, m, (v_{ij}), (B_i), \delta)$, let $p'_{ij}(\alpha)$ denote the expected payment made by buyer i on good j when the buyers use multipliers $\alpha \in [0, 1]^n$. Moreover, let $x_{ij}(\alpha)$ be the probability of buyer i winning good j when the buyers use the multipliers α.

Definition 21. *Consider a perturbed SPP game $(n, m, (v_{ij}), (B_i), \delta)$. Then, $\alpha \in [0, 1]^n$ is a pacing equilibrium of the perturbed SPP if:*

- *Budgets are satisfied: $\sum_{j=1}^{m} p'_{ij}(\alpha) \leq B_i$*
- *No unnecessary pacing: If $\sum_{j=1}^{m} p'_{ij}(\alpha) < B_i$, then $\alpha_i = 1$*

Moreover, $\alpha \in [0, 1]^n$ is an γ-approximate pacing equilibrium of the perturbed SPP if:

- *Budgets are satisfied: $\sum_{j=1}^{m} p'_{ij}(\alpha) \leq B_i$*

- *Not too much unnecessary pacing: If $\sum_{j=1}^{m} p'_{ij}(\alpha) < (1-\gamma)B_i$, then $\alpha_i \geq (1-\gamma)v_{ij}$*

Theorem 27. *Computing a γ-approximate pacing equilibrium of a perturbed SPP game $(n, m, (v_{ij}), (B_i), \delta)$ is PPAD-hard when $\delta = \gamma = 1/n^8$.*

Proof. First observe that

$$(1-\gamma)(1-\delta) = (1-n^{-8})^2 = 1 + n^{-16} - 2n^{-8} \geq 1 - n^{-7}$$

We will prove the theorem by reducing from the problem of computing approximate pacing equilibria of SPP games. Consider an SPP game $G = (n, m, (v_{ij}), (B_i))$. Define a perturbed SPP game $G' = (n, m, (v_{ij}), (B'_i), \delta)$ such that $B'_i = (1-\delta)B_i$. Let α be a γ-approximate pacing equilibrium of the perturbed SPP game G'. Then, as $\epsilon_{ij} \in [1-\delta, 1]$, we get that

$$(1-\delta)x_{ij}(\alpha)p_j(\alpha) \leq p'_{ij}(\alpha) \leq x_{ij}(\alpha)p_j(\alpha) \quad \forall\, i \in [n], j \in [m] \tag{D.1}$$

where, as earlier, $p_j(\alpha)$ denotes the second highest bid in an SPP game when the buyers use multipliers α). To complete the proof, it suffices to show that $(\alpha, x(\alpha))$ is a (δ, γ')-approximate pacing equilibrium of the SPP game G for $\gamma' = 1/n^7$. We establish the required properties below:

(a) As $\epsilon_{ij} \in [1-\delta, 1]$, $x_{ij}(\alpha) > 0$ only if $\alpha_i v_{ij} \geq (1-\delta)\max_{k\in[n]} \alpha_k v_{kj}$

(b) Full allocation of each good with positive bid: This follows directly from the allocation rules of a second-price auction.

(c) Budgets are satisfied: α being bugdet feasible for the perturbed SPP game G implies

$$\sum_{j=1}^{m} p'_{ij}(\alpha) \leq B'_i = (1-\delta)B_i$$

for all $i \in [n]$. As $p'_{ij}(\alpha) \geq (1-\delta)x_{ij}(\alpha)p_j(\alpha)$, we get $\sum_{j=1}^{m} x_{ij}(\alpha)p_j(\alpha) \leq B_i$ as required.

(d) Not too much unnecessary pacing: Suppose $\sum_{j=1}^{m} x_{ij}(\alpha)p_j(\alpha) < (1-\gamma')B_i$ for some buyer

$i \in [n]$. Then, using (D.1), we get

$$\sum_{j=1}^{m} p'_{ij}(\alpha) < \frac{(1-\gamma')}{(1-\delta)} \cdot (1-\delta)B_i = \frac{(1-\gamma')}{(1-\delta)} B'_i \leq (1-\gamma)B'_i$$

where we have used $(1-\gamma)(1-\delta) \geq (1-n^{-7}) = (1-\gamma')$. Now, as α is a γ-approximate equilibrium of the perturbed SPP game G', we get $\alpha_i \geq 1-\gamma \geq 1-n^{-7} = 1-\gamma'$.

Hence, we have shown that $(\alpha, x(\alpha))$ is a (δ, γ')-approximate pacing equilibrium for the SPP game G, where $\delta \leq n^{-7}$ and $\gamma' = n^{-7}$. As the perturbed SPP game G' can be constructed from the SPP game G in polynomial time, the theorem follows from Theorem 13. □

Let the expected utility of buyer i in a perturbed SPP game under multipliers α be denoted by $u_i(\alpha)$, i.e.,

$$u_i(\alpha) = \mathbb{E}_{\{\epsilon_{ij}\}_{i,j}} \left[\sum_{j=1}^{m} (v_{ij}\epsilon_{ij} - \max_{k \neq i} \alpha_k v_{kj}\epsilon_{kj}) \mathbf{1}(\alpha_i v_{ij}\epsilon_{ij} \geq \max_{k \neq i} \alpha_k v_{kj}\epsilon_{kj}) \right]$$

Definition 22. *Consider a perturbed SPP game $(n, m, (v_{ij}), (B_i), \delta)$. A vector of pacing multipliers α is called a Nash equilibrium of this game if for each $i \in [n]$ and α'_i such that $\sum_{j=1}^{m} p'_{ij}(\alpha'_i, \alpha_{-i}) \leq B_i$, we have $u_i(\alpha_i, \alpha_{-i}) \geq u_i(\alpha'_i, \alpha_{-i})$.*

Lemma 48. *Consider a perturbed SPP game $(n, m, (v_{ij}), (B_i), \delta)$ and let α be a Nash equilibrium of this game. If $\sum_{j=1}^{m} p'_{ij}(\alpha) < B_i$ and $\alpha_i < 1$, then $\sum_{j=1}^{m} p'_{ij}(\alpha) = \sum_{j=1}^{m} p'_{ij}(1, \alpha_{-i})$.*

Proof. Suppose α is a Nash equilibrium of the game but not a pacing equilibrium, and buyer i satisfies $\sum_{j=1}^{m} p'_{ij}(\alpha) < B_i$ and $\alpha_i < 1$. For contradiction, suppose $\sum_{j=1}^{m} p'_{ij}(\alpha) < \sum_{j=1}^{m} p'_{ij}(1, \alpha_{-i})$. Now, as the distribution of ϵ_{ij} is continuous, $x \mapsto p_{ij}(x, \alpha_{-i})$ is a continuous non-decreasing function. By the Intermediate Value Theorem, there exists $\alpha^*_i \in (\alpha_i, 1)$ such that

$$\sum_{j=1}^{m} p'_{ij}(\alpha^*_i, \alpha_{-i}) \leq B_i \,.$$

Now, observe that buyer i wins good j if and only if

$$\alpha_i^* v_{ij}\epsilon_{ij} \geq \max_{k \neq i} \alpha_k v_{kj}\epsilon_{kj}$$

Therefore, $v_{ij}\epsilon_{ij} \geq p'_{ij}(\alpha_i^*, \alpha_{-i})/\alpha_i^*$. As $\alpha_i^* < 1$, we get that

$$u_i(\alpha_i^*, \alpha_{-i}) - u_i(\alpha_i, \alpha_{-i}) \geq \frac{1}{\alpha_i^*} \cdot \left[\sum_{j=1}^{m} p'_{ij}(\alpha_i^*, \alpha_{-i}) - \sum_{j=1}^{m} p'_{ij}(\alpha_i, \alpha_{-i})\right] > 0$$

This contradicts the fact that α is a Nash equilibrium. Hence, the Lemma holds. □

Corollary 6. *Consider a perturbed SPP game $(n, m, (v_{ij}), (B_i), \delta)$ and let α be a Nash equilibrium of this game. If $\sum_{j=1}^{m} p'_{ij}(1, \alpha_{-i}) > \sum_{j=1}^{m} p'_{ij}(\alpha)$, then we have $\sum_{j=1}^{m} p'_{ij}(\alpha) = B_i$. Furthermore, as a consequence, if $\sum_{j=1}^{m} p'_{ij}(1, \alpha_{-i}) > B_i$, then $\sum_{j=1}^{m} p'_{ij}(\alpha) = B_i$.*

Theorem 28. *Computing a Nash equilibrium of a perturbed SPP game $(n, m, (v_{ij}), (B_i), \delta)$ is PPAD-hard when $\delta = 1/n^8$.*

Proof. Let G be the SPP game constructed in Appendix D.1 for the proof of Theorem 13. Like the proof of Theorem 27, define a perturbed SPP game $G' = (n, m, (v_{ij}), (B'_i), \delta)$ such that $B'_i = (1 - \delta)B_i$. Moreover, define an auxiliary perturbed SPP game $G'' = (n + 1, m + 1, (v_{ij}), (B'_i), \delta)$ by adding one more buyer and one more good to G'. We denote the new buyer by $\mathbb{T}^*$ and the new good by S. Buyer $\mathbb{T}^*$ has value 1 for good S, i.e., $V(\mathbb{T}^*, S) = 1$ and does not value any other good. She has a budget of n^7 (large enough to never be binding). The only other buyer who has a non-zero value for S is the Threshold buyer $\mathbb{T}$, who has a value of 1, i.e, $V(\mathbb{T}, S) = 1$.

We begin by showing that every Nash equilibrium of G'' is also a pacing equilibrium. Let α be a Nash equilibrium of G''. As a first step, we show that $\alpha(\mathbb{T}) = \alpha(\mathbb{T}^*) = 1$. We do so by ruling out the other cases:

1. If $\alpha(\mathbb{T}) < \alpha(\mathbb{T}^*)$, then buyer $\mathbb{T}$ can strictly increase her utility by setting $\alpha(\mathbb{T}) = 1$ as this allows her to win a strictly larger fraction of good S.

2. Similarly, if $\alpha(\mathbb{T}^*) < \alpha(\mathbb{T})$, then buyer $\mathbb{T}^*$ can strictly increase her utility by setting $\alpha(\mathbb{T}^*) = 1$ as this allows her to win a strictly larger fraction of good S.

3. If $\alpha(\mathbb{T}) = \alpha(\mathbb{T}^*) < 1$, then buyer $\mathbb{T}$ can strictly increase her utility by setting $\alpha(\mathbb{T}) = 1$ as this allows her to win a strictly larger fraction of good S.

For every other buyer in G'', we use Corollary 6 to show that they exactly spend their budget.

If $\alpha(\mathbb{C}(p, s)) \leq (1 - \delta)/2$, then the buyer $\mathbb{C}(p, s)$ wins no part of the threshold good $T(p, s)$ and spends strictly less than her budget because she has value at most 16 for all of the other goods and there are at most $O(n^2)$ such goods compared to her budget which is $\Omega(n^2)$. On the other hand, she can win all of the threshold good $T(p, s)$ by setting $\alpha(\mathbb{C}(p, s)) = 1$ and spend strictly more. Hence, by Corollary 6, we get that she exactly spends her budget, which is a contradiction. Therefore, $\alpha(\mathbb{C}(p, s)) \geq (1 - \delta)/2$.

Consider a dummy buyer $\mathbb{D}(p, s)$. If we set $\alpha(\mathbb{D}(p, s)) = 1$, then she wins at least half of the normalization good $N(p, s)_s$ at a price of at least $\alpha(\mathbb{C}(p, s))$ which violates her budget of $1/(16n)$. Thus, Corollary 6 implies that she exactly spends her budget under the Nash equilibrium α.

Consider buyer $\mathbb{C}(p, s)$. If we set $\alpha(\mathbb{C}(p, s)) = 1$, she she wins:

- All of normalization good $N(p, s)_t$, for each $t \neq s$, by spending at least $(1 - \delta)/2$ on each of them because $\alpha(\mathbb{C}(p, t)) \geq (1 - \delta)/2$ by the earlier part of the proof.

- Part of normalization good $N(p, s)_s$ by spending at least $(1 - \delta) - \nu$. This is because $N(p, s)_s$ has price at least $(1 - \delta)$ and buyer $\mathbb{D}(p, s)$ only has budget ν.

- All of threshold good $T(p, s)$ by spending at least $\alpha(\mathbb{T})(1 - \delta)n^4 = (1 - \delta)n^4$.

- All of expenditure good $E(p, s)_t$, for each $t \in [n]$, by spending at least $\alpha(\mathbb{T})\nu A_{st}/2$ if $p = 1$ and $\alpha(\mathbb{T})\nu B_{ts}/2$ if $p = 2$.

Hence, the total expenditure of $\mathbb{C}(p, s)$ when $p = 1$ is at least

$$(1-\delta) \cdot \frac{n-1}{2} + (1-\delta) \cdot -\nu + (1-\delta)n^4 + \sum_{t \in [n]} \nu A_{st}/2$$

which is strictly higher than her budget. Similar statement holds for $p = 2$. Therefore, Corollary 6 implies that buyer $\mathbb{C}(p, s)$ exactly spends her budget.

Hence, we have shown that every buyer either has her multiplier equal to 1 or exactly spends her budget, which means that α is a pacing equilibrium. Moreover, from our construction of G'' from G', we get that the restriction of α to the buyers other than $\mathbb{T}^*$ is a pacing equilibrium for the game G'. This is because only the Threshold buyer $\mathbb{T}$ is affected by this change and her multipliers satisfies $\alpha(\mathbb{T}) = 1$ and she spends strictly less than her budget. Finally, as we showed in the proof of Theorem 27, $(\alpha, x(\alpha))$ is a (δ, γ)-approximate pacing equilibrium of the SPP game G where $\delta = \gamma = 1/n^7$. Invoking Theorem 13 completes the proof. □

Appendix E: Appendix to Chapter 6

E.1 Appendix: Examples of Irrational Throttling Equilibria

First-Price Auctions: First, we give an example for which the unique *first-price* throttling equilibrium is irrational.

Example 10. *Define a throttling game as follows: There are 2 goods and 2 buyers, i.e.,* $m = 2$ *and* $n = 2$*;* $b_{11} = b_{12} = 2$ *and* $b_{21} = 1, b_{22} = 3$*;* $B_1 = 2$ *and* $B_2 = 1$*. Suppose, in equilibrium, the buyers use the throttling parameters* θ_1 *and* θ_2*. Then the payment of buyer 1 and buyer 2 are given by* $2\theta_1 + 2(1 - \theta_2)\theta_1$ *and* $3\theta_2 + (1 - \theta_1)\theta_2$ *respectively. Therefore, for this game, in any throttling equilibrium, we have* $0 < \theta_1, \theta_2 < 1$ *and* $\theta_3 = 1$*, which implies* $2\theta_1 + 2(1 - \theta_2)\theta_1 = 2$ *and* $3\theta_2 + (1 - \theta_1)\theta_2 = 1$*. Substituting* $\theta_1 = 1/(2 - \theta_2)$ *from the first equation into the second yields*

$$3\theta_2 + \theta_2 \cdot \frac{1 - \theta_2}{2 - \theta_2} = 1$$

which implies $4\theta_2^2 - 7\theta_2 + 1 = 0$*. As* $\theta_2 < 1$*, Solving the quadratic gives* $\theta_2 = (7 - \sqrt{33})/8$*.*

Second-Price Auctions: Next, we give an example for which all *second-price* throttling equilibria are irrational.

Example 11. *Define a throttling game as follows:*

- *There are 4 goods and 3 buyers, i.e.,* $m = 4$ *and* $n = 3$
- $b_{11} = b_{12} = 2$*,* $b_{14} = 1$*,* $b_{23} = b_{24} = 4$*,* $b_{22} = 1$*,* $b_{31} = 1$ *and* $b_{33} = 2$
- $B_1 = B_2 = 1$ *and* $B_3 = \infty$

For this game, in any throttling equilibrium, we have $0 < \theta_1, \theta_2 < 1$ *and* $\theta_3 = 1$. *Hence, if* θ *is a throttling equilibrium, then it satisfies* $\theta_1 + \theta_1\theta_2 = 1$ *and* $2\theta_2 + \theta_2\theta_1 = 1$. *Substituting* $\theta_1 = 1/(1+\theta_2)$ *from the first equation into the second equation yields*

$$2\theta_2 + \theta_2 \cdot \frac{1}{1+\theta_2} = 1$$

which further implies $2\theta_2^2 + 2\theta_2 - 1 = 0$. *As* $\theta_2 > 0$, *solving the quadratic gives* $\theta_2 = (\sqrt{3} - 1)/2$.

E.2 Appendix: Missing Proofs

E.2.1 Proof of Theorem 20

Consider a throttling game $\big(n, m, (b_{ij}), (B_i)\big)$ and an approximation parameter $\delta \in (0, 1/2)$. Define $f : [0,1]^n \to [0,1]^n$ as

$$f_i(\theta) = \min\left\{\frac{(1-\delta/2)B_i}{\sum_j p(1,\theta_{-i})_{ij}}, 1\right\} = \min\left\{\frac{(1-\delta/2)B_i}{\max\{\sum_j p(1,\theta_{-i})_{ij}, B_i/2\}}, 1\right\} \quad \forall \theta \in [0,1]^n$$

First, we prove that f is L-Lipschitz continuous with Lipschitz constant $L = 2mn\overline{B}\underline{B}^{-2}\overline{b}$, where $\overline{b} = \max_{i,j} b_{ij}$, $\overline{B} = \max_i B_i$. To achieve this, we will repeatedly use the following facts about Lipschitz functions. For Lipschitz continuous functions f and g with Lipschitz constants L_1 and L_2 respectively,

- $f + g$ is $L_1 + L_2$-Lipschitz continuous
- If f and g are bounded above by M, then $f \cdot g$ is $M(L_1 + L_2)$-Lipschitz continuous
- If f is bounded below by c, then $1/f$ is L_1/c^2-Lipschitz continuous
- For a constant C, $\max\{f, C\}$ and $\min\{f, C\}$ are both L_1-Lipschitz continuous

Observe that

$$p(1,\theta_{-i})_{ij} = \sum_{\ell : b_{\ell j} < b_{ij}} b_{\ell j}\theta_\ell \prod_{k \neq i : b_{kj} > b_{\ell j}} (1 - \theta_k)$$

Therefore, for all $i \in [n]$, $\theta \mapsto p(1, \theta_{-i})_{ij}$ is $(2n\overline{b})$-Lipschitz continuous, which further implies that $\theta \mapsto \sum_j p(1, \theta_{-i})_{ij}$ is $2mn\overline{b}$-Lipschitz continuous. Finally, due to the second equality in the definition of f, we get that f is $(2mn\overline{B}\underline{B}^{-2}\overline{b})$-Lipschitz continuous.

Since BROUWER is in PPAD [CD06], to complete the proof, it suffices to show that a $(\delta\underline{B}/4m\overline{b})$-approximate fixed point θ^* of f, i.e, θ^* such that $\|f(\theta^*) - \theta^*\|_\infty \leq \delta\underline{B}/4m\overline{b}$, is a δ-approximate throttling equilibrium. First, note that $p(1, \theta_{-i})_{ij} \leq \overline{b}$ for all $i \in [n], j \in [m]$. Therefore, $f(\theta)_i \geq \underline{B}/2m\overline{b}$ for all $i \in [n]$. Hence, for $i \in [n]$, we have

$$\left|1 - \frac{\theta_i^*}{f_i(\theta^*)}\right| \leq \frac{\delta\underline{B}}{f_i(\theta^*) \cdot 4m\overline{b}} \leq \frac{\delta}{2}$$

As a consequence, we get $\theta_i^* \leq (1+\delta/2)f_i(\theta^*)$ and $\theta^* \geq (1-\delta/2)f_i(\theta^*)$. The first inequality implies which in turn implies

$$\sum_j p(\theta^*)_{ij} = \theta_i^* \cdot \sum_j p(1, \theta^*)_{ij} \leq (1 + \delta/2)(1 - \delta/2)B_i \leq B_i$$

and the second one implies that if $\theta_i^* < 1 - \delta/2$, then

$$\sum_j p(\theta^*)_{ij} = \theta_i^* \cdot \sum_j p(1, \theta^*)_{ij} \geq (1 - \delta/2)^2 B_i \geq (1 - \delta)B_i$$

Hence, θ^* is a δ-approximate throttling equilibrium, thereby completing the proof. □

E.2.2 Proof of Theorem 21

Consider an instance of 3-SAT with variables $\{x_1, \ldots, x_n\}$ and clauses $\{C_1, \ldots, C_m\}$. Our goal is to define an instance $\mathcal{I}$ of REV (a throttling game G and a target revenue R) which always has the same solution (Yes or No) as the 3-SAT instance, and has a size of the order poly(n, m). We do so next, starting with an informal description to build intuition. To better understand the core motivations behind the gadgets, we will restrict our attention to exact throttling equilibria ($\delta = 0$) in the informal discussion that follows. As we will see in the formal proof, the target revenue R

can be chosen carefully to ensure that only exact throttling equilibria can achieve the revenue R.

Reciprocal Gadget: Fix $i \in [n]$. Corresponding to variable x_i, there are two goods $\mathbb{A}_i$ and $\mathbb{B}_i$, and two buyers V_i^+ and V_i^- in the throttling game G. Each buyer bids 1 for one of the goods and bids 2 for the other, with both buyers bidding differently on each good. Furthermore, we set the budgets of both buyers to be 1/2, and ensure that they do not spend any non-zero amount on goods other than $\mathbb{A}_i$ and $\mathbb{B}_i$. In equilibrium, this forces the throttling parameter of V_i^+ (which we denote by θ_i^+) to be half of the reciprocal of the throttling parameter of V_i^- (which we denote by θ_i^-) and vice-versa. As a consequence, both throttling parameters lie in the interval $[1/2, 1]$.

Binary Gadget: For each variable x_i, there are two additional goods $\mathbb{S}_i$ and $\mathbb{T}_i$, which receive a bid of 1 from buyers V_i^+ and V_i^- respectively. The throttling game G also has one unbounded buyer U who has an infinite budget, and bids 2 on both goods $\mathbb{S}_i$ and $\mathbb{T}_i$. By the definition of throttling equilibria (Definition 12), the throttling parameter of U is always 1 in equilibrium. Therefore, buyer U wins both $\mathbb{S}_i$ and $\mathbb{T}_i$ with probability one, and pays $\theta_i^+ + \theta_i^- = \theta_i^+ + 1/2\theta_i^+$ for it. Finally, observe that $t \mapsto t + 1/2t$, when restricted to $t \in [1/2, 1]$, is maximized at $t = 1$ or $t = 1/2$. Therefore, by appropriately choosing the target revenue R, we can ensure that revenue R is only achieved by throttling equilibria in which exactly one of the following holds: $(\theta_i^+ = 1, \theta_i^- = 1/2)$ or $(\theta_i^+ = 1/2, \theta_i^- = 1)$. This allows us to interpret $\theta_i^+ = 1$ as setting $x_i = 1$ and $\theta_i^- = 1$ as setting $x_i = 0$.

Clause Gadget: For each clause C_j, there is a good $\mathbb{C}_j$. If C_j contains a non-negated literal x_i, then buyer V_i^+ bids 1 on good $\mathbb{C}_j$, and if it contains a negated literal $\neg x_i$, then buyer V_i^- bids 1 on good $\mathbb{C}_j$. Furthermore, the unbounded buyer U bids 2 on good $\mathbb{C}_j$, thereby always winning it. Hence, the total payment on good $\mathbb{C}_j$ is 1 if some literal is satisfied (corresponding throttling parameter is 1), and is 1/2 if no literal is satisfied (corresponding throttling parameters are 1/2). The rest of the reduction boils down to choosing R appropriately.

Proof of Theorem 21. Guided by the informal intuition described above, we proceed with the formal definition of the instance $\mathcal{I}$, which involves specifying the throttling game G and the target revenue R. The throttling game G consists of the following goods:

- **Reciprocal Gadget:** For each variable x_i, there are two goods $\mathbb{A}_i$ and $\mathbb{B}_i$.
- **Binary Gadget:** For each variable x_i, there are two binary goods $\mathbb{S}_i$ and $\mathbb{T}_i$.
- **Clause Gadget:** For each clause C_j, there is a good $\mathbb{C}_j$.

Moreover, G has the following set of buyers:

- Corresponding to each variable x_i, there are two buyers V_i^+ and V_i^- with non-zero bids only for the following goods:
  - $b(V_i^+, \mathbb{A}_i) = 2$ and $b(V_i^+, \mathbb{B}_i) = 1$
  - $b(V_i^-, \mathbb{A}_i) = 1$ and $b(V_i^-, \mathbb{B}_i) = 2$
  - $b(V_i^+, \mathbb{S}_i) = 1$
  - $b(V_i^-, \mathbb{T}_i) = 1$
  - $b(V_i^+, \mathbb{C}_j) = 1$ if x_i is a literal in C_j
  - $b(V_i^-, \mathbb{C}_j) = 1$ if $\neg x_i$ is a literal in C_j

  Moreover, the budget of both V_i^+ and V_i^- is $1/2$ for all $i \in [n]$.
- There is one unbounded buyer U with $b(U, \mathbb{C}_j) = 2$ for all $j \in [m]$ and $b(U, \mathbb{S}_i) = b(U, \mathbb{T}_i) = 2$ for all $i \in [n]$. Moreover, U has a budget of ∞.

Set the target revenue to be $R = n + m + (3n/2)$. Suppose there exists a δ-approximate throttling equilibrium Θ, for some $\delta \in [0, 1)$, with revenue greater than or equal to R. Let θ_i^+ and θ_i^- denote the throttling parameters of V_i^+ and V_i^- in Θ. Then, $\theta_i^+\theta_i^- \leq 1/2$ by virtue of the budget constraints. Therefore, the revenue from goods $\{\mathbb{A}_i\}_{i=1}^n \cup \{\mathbb{B}_i\}_{i=1}^n$ is at most n. Furthermore, it is easy to see that the revenue from goods $\{\mathbb{C}_j\}_{j=1}^m$ is at most m. Additionally, the total payment by buyer U on goods $\mathbb{S}_i$ and $\mathbb{T}_i$ is at most $\theta_i^+ + \theta_i^- \leq \theta_i^+ + (1/2\theta_i^+)$. Note that $\theta_i^+ + (1/2\theta_i^+)$ is maximized at $\theta_i^+ = 1/2$ or $\theta_i^+ = 1$, with a value of $\theta_i^+ + (1/2\theta_i^+) = 3/2$. Therefore, the revenue from goods $\{\mathbb{S}_i\}_{i=1}^n \cup \{\mathbb{T}_i\}_{i=1}^n$ is at most $3n/2$. Hence, the total payment made on all the goods is at most R.

For the total revenue under Θ to be greater than or equal to R, the revenue from $\{\mathbb{S}_i\}_{i=1}^n \cup \{\mathbb{T}_i\}_{i=1}^n$ must be at least $3n/2$ and the revenue from $\{\mathbb{C}_j\}_{j=1}^m$ must be at least m. Hence, under Θ, buyer U has a throttling parameter of 1, and for each $i \in [n]$, either $(\theta_i^+ = 1, \theta_i^- = 1/2)$ or $(\theta_i^+ = 1/2, \theta_i^- = 1)$. Furthermore, the payment made by buyer U on $\mathbb{C}_j$ is 1 for every $j \in [m]$. This allows us to assign values to the variables as follows: set $x_i = 1$ if $\theta_i^+ = 1$ and $x_i = 0$ if $\theta_i^- = 1$. With this assignment of the variables, each clause is satisfied since the payment made by buyer U on $\mathbb{C}_j$ is 1 for all $j \in [m]$. Hence, we have shown that if there exists a δ-approximate throttling equilibrium with revenue R or greater, then there exists a satisfying assignment for the 3-SAT instance.

Conversely, note that if there exists a satisfying assignment for the 3-SAT instance, then setting $\theta_i^+ = 1, \theta_i^- = 1/2$ if $x_i = 1$ and $\theta_i^+ = 1/2, \theta_i^- = 1$ if $x_i = 0$ yields a throttling equilibrium with revenue equal to R. To complete the proof, observe that the size of the instance $|\mathcal{I}| = \text{poly}(n, m)$. □

E.2.3 Proof of Theorem 22

In this appendix, we analyze the correctness and runtime of Algorithm 7. To do so, we will make repeated use of the following crucial observation:

$$p(\theta)_{ij} = \begin{cases} \theta_i \theta_k b_{kj} & \text{if } b_{ij} > b_{kj} > 0 \text{ for some } k \in [n] \\ 0 & \text{otherwise} \end{cases} \tag{E.1}$$

In particular, this observation implies that $p(1, \theta_i)$ is a linear function of θ.

The following lemma makes a step towards the proof of correctness of the algorithm by showing that the budget constraints are always satisfied.

Lemma 49. *At the start of each iteration of the while loop, we have $\sum_j p(\theta)_{ij} \leq B_i$ for all $i \in [n]$.*

Proof. We will use induction on the number of iterations of the while loop to prove this lemma. By our choice of initialization of θ, the budget constraints are satisfied before the first iteration of the while loop. Suppose the constraints are satisfied before the start of the t-th iteration and the value of θ at that stage is $\theta^{(0)}$. We will use $\theta^{(1)}$ and $\theta^{(2)}$ to the denote the value of θ after step 1 and step 2 of the t-th iteration respectively. Consider a buyer i such that $\sum_j p(\theta^{(1)})_{ij} > B_i$. By equation E.1, we get

$$B_i < \sum_j p(\theta^{(1)})_{ij} \le \left(\sum_j p(\theta^{(0)})_{ij}\right) / (1-\gamma)^2$$

which further implies $\sum_j p(\theta^{(0)})_{ij} > (1-\gamma)^2 B_i$. Therefore, the throttling parameter of buyer i was not changed in step 1 of the t-th iteration, i.e., $\theta_i^{(0)} = \theta_i^{(1)}$. As a consequence, we get

$$\sum_j p(\theta^{(1)})_{ij} \le \left(\sum_j p(\theta^{(0)})_{ij}\right) / (1-\gamma)$$

After step 2 of the t-th iteration, we get $\theta^{(2)} = (1-\gamma)\theta^{(1)}$. Hence,

$$\sum_j p(\theta^{(2)})_{ij} \le (1-\gamma) \sum_j p(\theta^{(1)})_{ij} \le \left(\sum_j p(\theta^{(0)})_{ij}\right) \le B_i$$

where the last inequality follows from our inductive hypothesis. As $\theta^{(2)}$ is the value of θ after the t-th iteration, the lemma follows by induction. □

The next lemma establishes that the algorithm never loses any progress, i.e., any buyer who satisfies the 'Not too much unnecessary throttling condition' of Definition 13 at the beginning of some iteration of the while loop continues to do so at the end of it.

Lemma 50. *If $\sum_j p(\theta)_{ij} \ge (1-\gamma)^3 B_i$ or $\theta_i \ge 1-\gamma$ at the start of some iteration of the while loop, then $\sum_j p(\theta)_{ij} \ge (1-\gamma)^3 B_i$ or $\theta_i \ge 1-\gamma$ at the end of that iteration.*

Proof. Consider an iteration of while loop which starts with $\theta = \theta^{(0)}$. We will use $\theta^{(1)}$ and $\theta^{(2)}$ to the denote the value of θ after step 1 and step 2 of this iteration. If $\sum_j p(\theta^{(0)})_{ij} \ge (1-\gamma)^2 B_i$ at the beginning of the iteration, then $\sum_j p(\theta^{(2)})_{ij} \ge (1-\gamma)^3 B_i$ because

$$\sum_j p(\theta^{(1)})_{ij} \ge (1-\gamma)^2 B_i \quad \text{and} \quad \sum_j p(\theta^{(2)})_{ij} \ge (1-\gamma) \sum_j p(\theta^{(1)})_{ij}$$

Suppose $(1-\gamma)^3 B_i \le \sum_j p(\theta^{(0)})_{ij} < (1-\gamma)^2 B_i$ and $\theta_i^{(0)} < 1-\gamma$ at the start of the iteration. Then, after step 1, we have $(1-\gamma)^2 B_i \leq \sum_j p(\theta^{(1)})_{ij} \leq B_i$. Hence, after step 2, we get $(1-\gamma)^3 B_i \leq \sum_j p(\theta^{(3)})_{ij}$.

Finally, suppose $(1-\gamma)^3 B_i \leq \sum_j p(\theta^{(0)})_{ij} < (1-\gamma)^2 B_i$ and $\theta_i^{(0)} \geq 1-\gamma$ at the start of the iteration. Then, after step 1, we have $\sum_j p(\theta^{(1)})_{ij} \leq B_i$. Hence, after step 2, we still have $\theta_i^{(2)} \geq (1-\gamma)$. This completes the proof of the lemma. □

Finally, we combine the above lemmas to establish the correctness and polynomial-runtime of the algorithm.

Proof of Theorem 22. Let θ^* be the vector of throttling parameters returned by the algorithm. Lemma 49 implies that θ^* satisfies the budget constraints of every buyer. Furthermore, upon combining $(1-\gamma)^3 \geq 1-3\gamma$ with the termination condition of the while loop, we get that either $\theta_i^* \geq 1-\gamma$ or $\sum_j p(\theta^*)_{ij} \geq (1-3\gamma)B_i$ for all $i \in [n]$, which makes θ^* a $(1-3\gamma)$-approximate throttling equilibrium.

Next, we bound the running time of the algorithm. Define $c = \min_i \min\{B_i/(2\sum_j b_{ij}), 1\}$. Note that $c \leq \theta_i \leq 1$ for all $i \in [n]$ for the entire run of the algorithm. Based on Lemma 50, we define

$$A(\theta) := \{i \in [n] \mid \sum_j p(\theta)_{ij} \geq (1-\gamma)^3 B_i \text{ or } \theta_i \geq 1-\gamma$$

Then Lemma 50 simply states that if $i \in A(\theta)$ at the start of iteration T of the while loop, then $i \in A(\theta)$ at the start of all future iterations $t \geq T$. Moreover, recall that the while loop terminates when $A(\theta) = [n]$.

Observe that, in each iteration of the while loop, $\theta_i \leftarrow \theta_i/(1-\gamma)$ for some $i \notin A(\theta)$. Hence, the total number of iterations of the while loop T satisfies the following equivalent statements:

$$\frac{c}{(1-\gamma)^{T/n}} \leq 1 \iff T \leq \frac{n\log(1/c)}{\log(1/(1-\gamma))} \leq \frac{n\log(1/c)}{\gamma}$$

This completes the proof because each iteration takes polynomially many steps. □

E.2.4 Proof of Theorem 24

Fix a throttling equilibrium $\theta \in \Theta$. Recall that we use $X = (X_1, \ldots, X_n)$ to capture the random profile of buyers who participate in the auctions, where $X_i = 1$ if and only if buyer i participates in the auctions, and $\Pr(X_i = 1) = \theta_i$. Let $y_{ij}(X)$ be the indicator random variable which equals 1 if and only if good j is allocated to buyer i under the participation profile $X = (X_1, \ldots, X_n)$, and is zero otherwise. Moreover, let $p_j(X)$ denote the price of item j under the participation profile $X = (X_1, \ldots, X_n)$. Here, the price is the highest/second-highest bid for first-price/second-price auctions respectively, and is interpreted to be 0 if no buyers bid in an auction. Observe that

$$p_{ij}(\theta) = \mathbb{E}\left[p_j(X) y_{ij}(X)\right] .$$

Fix a benchmark allocation $y = \{y_{ij}\}$. We begin by establishing the following lemma, which will play a critical role in the proof of the theorem.

Lemma 51. *For all $i \in [n]$, we have*

$$\min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \geq \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}\right] .$$

Proof. We consider two cases. First assume that $\theta_i < 1$. Then, the no-unnecessary-throttling condition implies that $\sum_{j=1}^{m} p_{ij}(\theta) = B_i$. Now, observe that $y_{ij}(X) > 0$ only if $b_{ij} \geq p_j(X)$. Consequently, we have

$$\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right] \geq \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}(X)\right] = \sum_{j=1}^{m} p_{ij}(\theta) = B_i .$$

Hence, we get

$$\begin{aligned}\min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} &= B_i \\ &\geq \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\}\end{aligned}$$

$$\geq \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}\right] ,$$

thereby establishing the required lemma statement for a buyer i such that $\theta_i < 1$.

Next, consider a buyer i such that $\theta_i = 1$, i.e., buyer i always participates. Since $p_j(X) > b_{ij}$ whenever $y_{ij}(X) < 1$, we have

$$0 \geq \mathbb{E}\left[(b_{ij} - p_j(X))(1 - y_{ij}(X)) y_{ij}\right] .$$

Moreover, we also have

$$\mathbb{E}\left[b_{ij} y_{ij}(X)\right] \geq \mathbb{E}\left[(b_{ij} - p_j(X)) y_{ij}(X) y_{ij}\right]$$

Adding the two inequalities, we get

$$\mathbb{E}\left[b_{ij} y_{ij}(X)\right] \geq \mathbb{E}\left[(b_{ij} - p_j(X))(1 - y_{ij}(X)) y_{ij}\right] + \mathbb{E}\left[(b_{ij} - p_j(X)) y_{ij}(X) y_{ij}\right] = \mathbb{E}\left[(b_{ij} - p_j(X)) y_{ij}\right] .$$

Summing over all goods $j \in [m]$ yields

$$\begin{aligned}\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right] &\geq \sum_{j=1}^{m} b_{ij} y_{ij} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}\right] \\ &\geq \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}\right]\end{aligned}$$

Additionally, we also have

$$B_i \geq \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} \geq \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}\right]$$

Therefore,

$$\min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \geq \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}\right] .$$

This concludes the lemma by establishing it for buyers i with $\theta_i = 1$. □

With Lemma 51 in hand, we are ready to prove the theorem. First, note that

$$\begin{aligned}
\sum_{i=1}^{m} \min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} &\geq \sum_{i=1}^{m} \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \sum_{i=1}^{m} \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}\right] \\
&= \sum_{i=1}^{m} \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) \sum_{i=1}^{m} y_{ij}\right] \\
&= \sum_{i=1}^{m} \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X)\right] \\
&\geq \sum_{i=1}^{m} \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) \sum_{i=1}^{m} y_{ij}(X)\right] \\
&= \sum_{i=1}^{m} \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \sum_{i=1}^{m} \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}(X)\right]
\end{aligned}$$

where the second inequality follows from the observation that a good is always allocated whenever it has a positive bid, i.e., $\sum_{i=1}^{m} y_{ij}(X) = 1$ whenever $p_j(X) > 0$. Hence, if we can show that

$$\sum_{i=1}^{m} \min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \geq \sum_{i=1}^{m} \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}(X)\right], \tag{E.2}$$

we will get

$$\begin{aligned}
&\sum_{i=1}^{m} \min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \geq \sum_{i=1}^{m} \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\} - \sum_{i=1}^{m} \min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \\
\iff &\sum_{i=1}^{m} \min\left\{2 \cdot \mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \geq \sum_{i=1}^{m} \min\left\{\sum_{j=1}^{m} b_{ij} y_{ij}, B_i\right\}
\end{aligned}$$

and thereby complete the proof, because the benchmark allocation y and the throttling equilibrium θ are both arbitrary. In the remainder, we establish (E.2).

Since $y_{ij}(X) > 0$ only when $b_{ij} \geq p_j(X)$, we have

$$\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right] \geq \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}(X)\right].$$

Moreover, the budget constraint of buyer i implies

$$B_i \geq \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}(X)\right].$$

Combining the two inequalities, we get:

$$\min\left\{\mathbb{E}\left[\sum_{j=1}^{m} b_{ij} y_{ij}(X)\right], B_i\right\} \geq \mathbb{E}\left[\sum_{j=1}^{m} p_j(X) y_{ij}(X)\right].$$

Summing over all buyers $i \in [n]$ yields (E.2), as required.

E.3 Appendix: Examples for Section 6.4

First, we provide an example to show that the inequality REV(TE) $\leq 2 \times$ REV(TE) is tight.

Example 12. *Consider the throttling game in which there is 1 good and 2 buyers. The bids are given by $b_{11} = 1/\epsilon$, $b_{21} = 1 - \epsilon$ for $\epsilon > 0$ and the budgets are given by $B_1 = 1$, $B_2 = \infty$. Then, in the unique pacing equilibrium, we have $\alpha_1 = \epsilon$ and $\alpha_2 = 1$, whereas in the unique throttling equilibrium, we have $\theta_1 = \epsilon$ and $\theta_2 = 1$. Hence, REV(PE) = 1 and REV(TE) = $1 + (1 - \epsilon)^2$. Since, this is true for arbitrarily small ϵ, we get that the inequality REV(TE) $\leq 2 \times$ REV(TE) is tight established in Theorem 23 is tight.*

Next, we give a family of examples for which REV(PE) is arbitrarily close to $(4/3) \times$REV(TE).

Example 13. *Consider a throttling game with 2 goods and 2 buyers. Fix $\epsilon > 0$. The bids are given by $b_{11} = 1 + \epsilon$, $b_{12} = 1$ and $b_{21} = 1$. Moreover, the budgets are given by $B_1 = 1 - \epsilon$ and $B_2 = \infty$. Then, the unique pacing equilibrium is given by $\alpha_1 = 1 - \epsilon$, $\alpha_2 = 1$, and the unique throttling equilibrium is given by $\theta_1 = (1 - \epsilon)/(2 + \epsilon)$, $\theta_2 = 1$. Since ϵ was arbitrary, we can take it to be arbitrarily small. In which case, we get REV(PE) $\simeq$ 2 and REV(PE) $\simeq$ 1.5, as desired.*