Leonhard Euler's Research on the Multiplication of the Human Race with Models of Population Growth[1]

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Abstract:
The renowned Swiss mathematician Leonhard Euler created three variations of a simple population projection model, including one stable model and two non-stable models, that consider a couple with different fertility behaviors and life-spans. While one of the models was published by a German demographer, Johann Peter Süßmilch, in his book "The Divine Order", the other two are not widely known in contemporary literature. This paper compares and reanalyzes the three variants of Euler's population projections using matrix algebra, providing diagrams and tables of the population time series and their growth rates, as well as age structures of selected years. It is demonstrated that the non-stable projection models can be explained in the long run by their geometric trend component, which is a special case of strong ergodicity in demography as described by Euler. Additionally, a continuous variant of Euler's stable model is introduced, allowing for the calculation of the age structure, intrinsic growth rate, and population momentum in a straightforward manner. The effect of immortality on population size and age structure at high growth rates is also examined.

1. Introduction

In the collected works of Leonhard Euler (1707–1783), there is a paper (found in his notes) that deals with two mathematical models for the evolution of a closed population with simple reproductive and mortality behavior. The paper begins in French and is completed in Latin[2].

The first model in Euler's paper assumes a stably growing population, while the second model describes the theoretical basis for what is known as Euler's population growth model. The results of this model were tabulated by Johann Peter Süßmilch in his book "Die Göttliche Ordnung in den Veränderungen des menschlichen Geschlechts" in § 160, but it was not accompanied by any formula.

Euler is considered one of the pioneers of mathematical demography, although this is not widely known among mathematicians. In 1760, he derived the stable age distribution for the first time using a discrete model, in order to construct a life table when the population was not stationary but increasing geometrically. This work can be found in Moser (1839) or Bacaër (2011, pp. 14 ff.).

Euler also took into account population statistics in his "Introduction to the Analysis of the Infinite", published in 1748, where he discussed four examples in chapter 6 on logarithms and exponentials using the geometric growth model[3] $P_T = P_0 \cdot (1+r)^T$; solutions are given for concrete examples involving the population size at time 0 (P_0) and at time T (P_T), the annual growth rate (r), and the time horizon (T), if three of the variables are known (for details see also Bacaër, 2011, pp. 12-13).

[1] Paper presented in the plenary session on June 9, 2023, at the ASMDA 2023 International Conference held in Heraklion, Crete, Greece, from June 6th to 9th, 2023.
[2] The work entitled "Sur la multiplication du genre humain" was first published in: Euler, L; Du Pasquier.L G: Leonhardi Euleri opera omnia, Serie I, Vol. 7, Leipzig 1923, pp. 545-552. The work is found in the Notitzbuch H6, probably written between 1750 and 1755 (cf. notes in Euler, Du Pasquier, 1923, p.534).
[3] See http://www.17centurymaths.com/contents/introductiontoanalysisvol1.htm, pp. 162-164 (English translation) or pp. 177-180 (the original in Latin language)

Euler worked closely with Johann Peter Süßmilch (1707–1767), a German Protestant pastor in Berlin and a statistician and demographer. Süßmilch's most significant publication, "Die göttliche Ordnung" of 1741, is considered a pioneering work in demography and the history of population statistics[4]. In the second edition of "Die göttliche Ordnung", published in 1761, Süßmilch wanted to prove that high population levels in antiquity were compatible with the Christian calendar, even in light of the Flood and the belief that the population descended from a single couple. To achieve this goal, he sought mathematical assistance from Euler, who calculated tables for the doubling times of populations as a function of the growth rate and performed a population projection with a time horizon of 300 years.

2. Euler´s Population Growth Models

The following presentation describes the Eulerian models with some modifications in symbolism.

Model I (Stable Model: pp. 545-548)

In this model, it is assumed that the population grows by a factor λ every two years, or biennia, i.e., $P_{x,2} = \lambda \cdot P_{x,0}$ and $P_{x+2,t} = \lambda \cdot P_{x,t}$. All people are expected to reach the age of 50, marry at 20, and each marriage is expected to produce 6 children (3 girls and 3 boys), two at the age of 22, two at 24, and two at 26. The task is to find the multiplication factor, that is, the number of individuals in each age group.

Using $\eta = 1$ for the number of individuals who died in year t=0, and $P_{x,t}$ to represent the number of individuals at age x, we obtain the following table, which shows the age structure of the population at time 0. In this presentation, we will only focus on the female population.

Table 1. Age structure of the female population at time t=0, if $P_{50,0} = 1$

Age interval	Class k	Females in the age class k	Remarks
0-2	1	$P_{0,0} = \lambda^{25}$	
2-4	2	$P_{2,0} = \lambda^{24}$	
4-6	3	$P_{4,0} = \lambda^{23}$	
..			
x-(x+2)	x/2	$P_{x,0} = \lambda^{26-\frac{x+2}{2}} = \lambda^{25-\frac{x}{2}}$	
…			
20-22	11	$P_{20,0} = \lambda^{15}$	
22-24	12	$P_{22,0} = \lambda^{14}$	1 daughter (2 children)
24-26	13	$P_{24,0} = \lambda^{13}$	1 daughter (2 children)
26-28	14	$P_{26,0} = \lambda^{12}$	1 daughter (2 children)
…			
46-48	24	$P_{46,0} = \lambda^{2}$	
48-50	25	$P_{48,0} = \lambda$	
50-52	26	$P_{50,0} = 1$	Number of deaths η

[4] See Girlich (2007).

The net reproduction rate is $R_0=3$; in the general model is the number of deaths $P_{50,0}=\eta$.

Euler calculates the number of births or the population in age class 1 after two years as follows:

$$P_{0,2} = P_{22,0} + P_{24,0} + P_{26,0} = P_{50,0} \cdot \lambda^{12} \cdot \left(1+\lambda+\lambda^2\right) = \lambda \cdot P_{0,0} = \lambda \cdot \lambda^{25} \cdot P_{50,0}$$

et partant (as expressed by Euler)

$$1+\lambda+\lambda^2 = \lambda^{14}$$

or

$$\lambda^{-14} + \lambda^{-13} + \lambda^{-12} = 1$$

where the solution is to be determined numerically (biennial growth factor).
The last form is a discrete special case of the Lotka equation, which is well known in demography (cf., e.g., Keyfitz, 1977)

$$\int_{\alpha}^{\beta} e^{-r\cdot x} l(x) m(x) dx = 1 \ ,$$

if l(x)=1, m(x)=1 and $e^{-rx} = \lambda^{-x}$.

α and β are the limits of women's reproductive age, r is the intrinsic rate of growth, and $l(x)m(x)$ is the net reproduction function.

On p. 547 Euler[5] shows how he found an approximate solution $\lambda = 1.0883$.

The annual growth factor is $\lambda_{annual} = \sqrt{1.08838} = 1.04326$.
Hence, the annual growth rate is $r = \sqrt{1.08838} - 1 = 0.04326$.
Euler calculates from $\lambda^{\nu} = 2$ the doubling time $\nu = \dfrac{\lg 2}{\lg \sqrt{1.0883}} = 2 \cdot \dfrac{\ln 2}{\ln(1.08838)} = 16.38$ years

and the total population size $P_0 = \dfrac{\lambda^{26}-1}{\lambda-1} \cdot \eta \approx 91 \cdot \eta$, where η is the number of individuals dying in 2 years.
The total population with annual number of deaths φ is $P_0 = \dfrac{\lambda^{26}-1}{\lambda-1} \cdot \dfrac{\lambda-1}{\sqrt{\lambda}-1} \cdot \varphi \approx 185.4 \cdot \varphi$.

Model II (High fertility model: pp. 548-552)
In Model I, Euler established relationships between the age structure and age-specific fertility rates for a stable population, but did not specify the conditions necessary for stability. In Model II, he addresses this issue while again using a two-year period or biennium as the unit of time.

[5] According to Euler (1748, Art. 249, p. 354 ff.) there exists a root which is greater in absolute value than the absolute value of all other roots (cf. Euler (1748), Art. 349, p. 290 ff. and also the calculations of Gumbel (1917) p. 261.

Euler considers a couple at the age of 20 and applies the following rules to them and all their descendants:

A1: Age of marriage is 20 years.
A2: Each couple should give birth to one daughter and one son at the ages of 22, 24, 26, 28, and 30.
A3: Each person reaches the age of 50 and then dies.

Using an age-structured table (see pp. 549-550), Euler calculates the number of living individuals (P); and from the first column of the table, the number of births (B). The results are summarized in Table 2.

Table 2. Population series

Apres ans	*Somme (P)*	*Ser. Nascentium (B)*
0	2	0
2	4	2
4	6	2
6	8	2
8	10	2
10	12	2
12	12	0
14	12	0
16	12	0
18	12	0
20	12	0
22	12	0
24	14	2
26	18	4
28	24	6
30	30 (32)	8
32	40	10
34	48	8
36	54	6
38	58	4
40	60	2
42	60	0
44	60	0
46	62	2
48	68	6
50	80	12
52	98 (100)	20
54	126 (128)	30
56	160 (162)	36
58	196 (198)	38
60	230 (232)	36
62	260	30
64	280	20
66	292	12
68	300	8

(In parentheses are the incorrect values given by Euler.)

For $n \geq 15$ the following recurrence equation holds (see also Girlich 2007, pp. 12-13):
$B_n = B_{n-11} + B_{n-12} + B_{n-13} + B_{n-14} + B_{n-15}$.
The corresponding characteristic equation is
$\lambda^{15} - \lambda^4 - \lambda^3 - \lambda^2 - \lambda - 1 = 0$,
which Euler simplified to
$\lambda^{16} - \lambda^{15} - \lambda^5 + 1 = 0$ [6].
This last equation was solved approximately by Euler. He writes that the recursion equation of the births is continued in the limiting case as a geometric progression with the factor $\lambda = 1.13315$. The biennial growth factor corresponds to a yearly growth rate of $r = \sqrt{1.13315} - 1 \approx 0.064$.

Model III (Euler-Süßmilch model)[7]
In Chapter 8 of his revised edition of "Die göttliche Ordnung" from 1761, Euler calculated the doubling time of the population for Süßmilch, assuming geometric growth in § 152 and § 156, and presented a population projection without disclosing the analytical methods used in § 160.

A1: The projection starts with one married couple, both 20 years old.
A2: Marriage age is set at 20 years.
A3: Each couple gives birth to one daughter and one son at the ages of 22, 24, and 26.
A4: Everyone reaches the age of 40 and then dies.

The projection is biennial, taking place every two years, which is crucial for the stability of the model, as noted in Pflaumer (2023).

Chapter 8 does not offer much insight into the mathematics behind this population model, but the mathematical background used by Euler can be found in his notes and described in Model II. The results of the population projection up to year 300 can be found in Süßmilch (1761) on pages 293-297, and are summarized in Table 6.

The use of the following recursion equation, which is a variant of the recursion equation in his high fertility model II (10 children) in Section 2 of his manuscript (see Bacaër (2011, pp. 16-20), Girlich (2007, pp. 7-9), and Gumbel (1917)),

$$B_n = B_{n-11} + B_{n-12} + B_{n-13}.$$

with the characteristic equation

$$\lambda^{13} - \lambda^2 - \lambda - 1 = 0$$

leads to the solution of $\lambda = 1.0961$ resulting in a tripling time of 23.94 years and a doubling time of 15.1 years. The annual growth rate is $r = \left(\sqrt{1.0961} - 1\right) \cdot 100\% \approx 4.7\%$.

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[6] See Euler, L. (1748): Introductio in analysin infinitorum, Chs. XIII und XVII.
http://www.17centurymaths.com/contents/introductiontoanalysisvol1.htm

[7] Model III has been extensively described and analyzed by Pflaumer (2023).

Gumbel (1917, p. 255) notes that Euler's derivation has similarities to the derivation of the earliest recurrent series by Leonardo Pisano, also known as Fibonacci, in 1202. The Fibonacci sequence is a mathematical series that describes the growth of a population of rabbits. It starts with one pair of rabbits and assumes that each pair produces a new pair of offspring every year, with each offspring also becoming a reproductive pair after one year. The result is a series of numbers that grows in a manner similar to exponential growth.

Süßmilch (1761) presents his own population projection in a table in § 159 (with some calculation errors), with the goal of proving his thesis that the population after the Flood could increase from a low to a very high number in a short time. The table starts with 2 people and initially doubles every 10 years. The doubling time for calculating the population then gradually increases to 15, 20, 25, 30, 40, and finally to 50 years. After 300 years, the population reaches 8,388,608, which is more than double the result of Euler's projection. The projection ends after 900 years, as shown in table 3.

Table 3. Summary of Süßmilch´s population projection

Year	t	Population	Annual growth rate from 0 to t	Annual growth rate from t to t+1
0	1	2		
100	2	2,048	0.072	0.072
205	3	262,144	0.059	0.047
300	4	8,388,608	0.052	0.037
400	5	134,217,748	0.046	0.028
500	6	2,147,487,968	0.042	0.028
560	7	8,589,951,872	0.040	0.023
900	8	1,099,513,839,616	0.030	0.014

Today's world population of 8 billion will be reached in 342 years according to Model II and in 466 years according to Model III. In Süßmilch's projection, it takes 560 years to reach this population, and in Euler's example, it takes 336 years if calculated using a growth rate of 6.25%. Euler likely realized that a growth rate of 6% or higher in a geometric model would produce unrealistic population figures very quickly (as seen in Table 4). Therefore, it is logical that Euler presented a projection with a lower growth rate that ends in 300 years in Süßmilch's book.

Table 4. Comparison of different calculations

Year	Euler III	Euler II	Süßmilch	Euler (1748)*
r	0.047	0.064	0.03	0.0625
0	2	2	2	6
200	46,280	1,132,582	< 262,144	1,000,000
300	3,994,314	610,696,900	8,388,608	
400	353,041,300	320,264,500,000	134,217,748	166,666,666,666
900	3.4656E+18	1.1968E+25	1.1E+12	

*Example 3 in Euler (1748, p. 163/179)

3. Matrix Representation of Euler´s Population Models

3.1 Matrix Model

The matrix representation of the female part of the three models is given by

$n_t = A \cdot n_{t-1}$, $t = 0,1,2,3,...$
or
$n_t = A^t \cdot n_0$, $t = 0,1,2,3,...$

with the following population vectors:

Model I: (Stable Model)

The vector n_0 has 26 rows. The transpose is
$n_0^T = (\lambda^{25}, \lambda^{24}, ..., ...\lambda^2, \lambda^1, 1)$
with $\lambda = 1{,}0883$

Model II: (High Fertility Model)

The vector n_0 has 25 rows. The transpose is
$n_0^T = (0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$

Model III: (Nonstable Euler-Süßmilch Model)

The vector n_0 has 20 rows. The transpose is

$n_0^T = (0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0)$

The projection matrix A is a kxk-Leslie-matrix (see Table 5).

$$A=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & 0 & \ldots & \ldots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 1 & 0 & \cdots & & & & & & & & & & & & & & & & 0 \\ 0 & 0 & 1 & 0 & \cdots & & & & & & & & & & & & & & & \\ 0 & 0 & 0 & 1 & 0 & \cdots & & & & & & & & & & & & & & \\ \vdots & & & & & & & & & & & & & & & & & & & \vdots \\ & & & & & & & & & & & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & & & & & 1 & 0 \end{pmatrix}.$$

Table 5. Model assumptions

Model	k	Maternity rates	Net Reproduction rate	Growth factor
I	26	$m_{11}=0;\ m_{12}=1; m_{13}=1;\ m_{14}=1;\ m_{15}=0$	$R_0=3$	$\lambda=1.0883$
II	25	$m_{11}=1;\ m_{12}=1; m_{13}=1;\ m_{14}=1;\ m_{15}=1$	$R_0=5$	$\lambda=1.1332$
III	20	$m_{11}=1;\ m_{12}=1; m_{13}=1;\ m_{14}=0;\ m_{15}=0$	$R_0=3$	$\lambda=1.0961$

In Model I, Euler assumes that fertility starts at the end of the 12th biennium, while in Models II and III, fertility starts at the beginning of the 12th biennium. This difference in the timing of fertility is the cause of the different stable growth rates in Models I and III, both of which have a net reproduction rate of 3. The varying maximum age in the two models does not impact the growth rate, it only affects the population level.

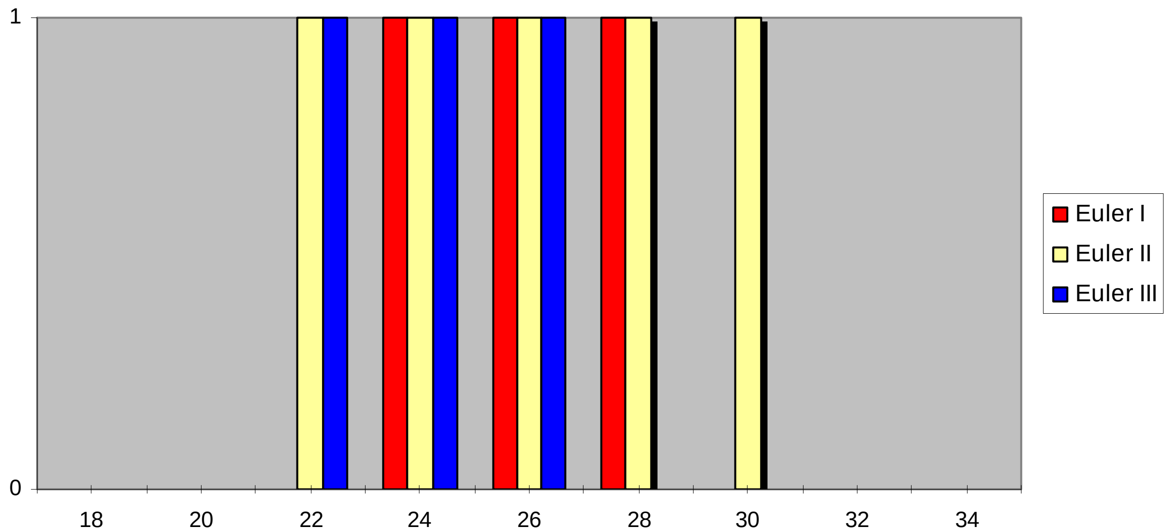

Fig. 1. Comparison of fertility assumptions

We describe the biennial projection step with t. The population vector of the total population is given by

$p_t = 2 \cdot n_t$..

The k×k projection matrix A is a special case of the Leslie matrix, widely used in demography (as discussed in Keyfitz, 1977). This allows us to use methods of matrix algebra to project the population and analyze the ergodic characteristics of the growth model. The results in Table 6 were calculated using these population models.

3.2 Results

The projection results in Figs. 2 to 4 show that the initial population in Model II and Model III is approaching a stable population. In fact, the oscillations decrease very slowly. Finally, the typical stable age structures of a growing population result, whereby the biennial growth rate (logarithmic difference) tends towards 9.6% (Model II) and 13.3% (Model III). The stable Model I has a constant biennial growth rate of 8.8%. The population size increases rapidly because of its high growth rate. Although both Model I and Model III have a net reproduction rate of 3, the growth rate of Model III is slightly higher because fertility starts at the beginning of the 12th biennium. As a result, the mean generation interval is somewhat lower here compared to Model I. The higher age at death in Model I has no effect on the stable growth rate. It only influences the population size. The biennial growth in Model III tends towards a lower stable growth rate compared to Model II with larger oscillations that decrease over time.

Table 6. Population sizes

Year	Model I .	Model II	Model III
	(stable)	(high fertility)	(E.-Süßmilch)
0	2.00	2	2
2	2.18	4	4
4	2.37	6	6
6	2.58	8	8
8	2.81	10	8
10	3.05	12	8
12	3.32	12	8
14	3.62	12	8
16	3.94	12	8
18	4.29	12	8
20	4.66	12	6
22	5.08	12	6
24	5.53	14	8
26	6.01	18	12
28	6.55	24	18
30	7.12	30	22
32	7.75	40	24
34	8.44	48	24
36	9.19	54	24
38	10.00	58	24
40	10.88	60	24
42	11.84	60	22
44	12.89	60	20

46	14.03	62	20
48	15.27	68	26
50	16.62	80	38
100	138	1,974	456
150	1,147	47,776	4,752
200	9,532	1,132,582	46,280
250	79,198	26,437,346	433,674
300	658,035	610,696,922	3,994,314

The total population of the model I at time 0 was assumed to be 2

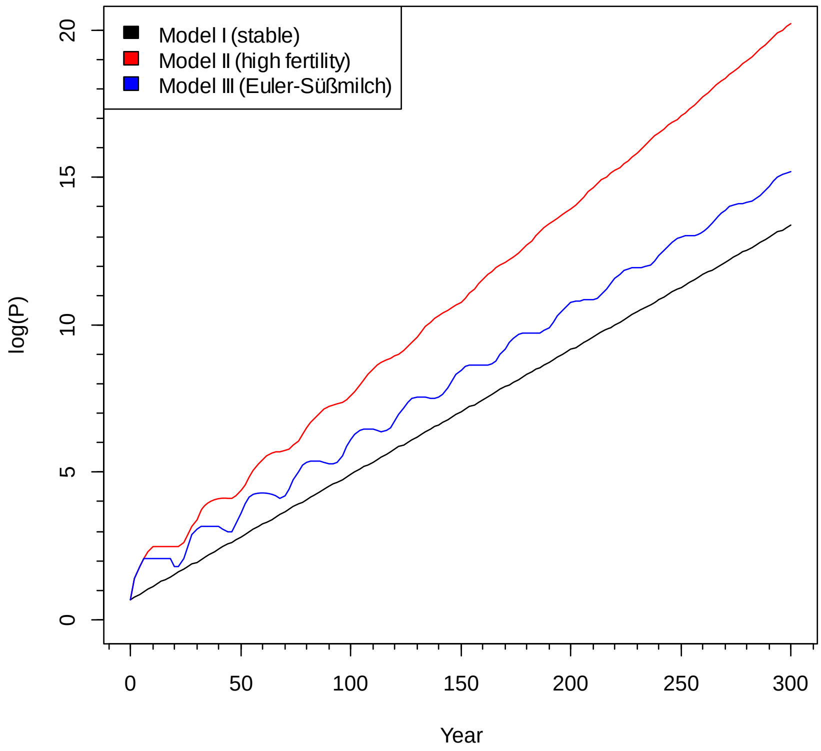

Fig. 2. Total population sizes up to the year 300

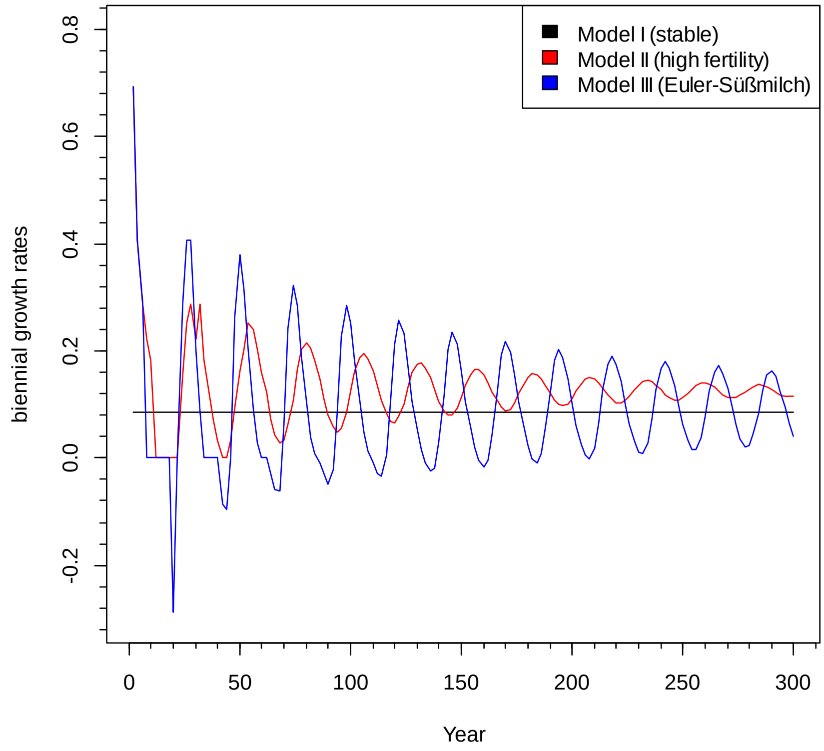

Fig. 3. Biennial growth rates up to the year 300

Table 7. Eigenvalues and moduli of complex numbers i=1,2,3

	Model I			Model II			Model III		
i	Re(λ_i)	Im(λ_i)	Modulus r_i	Re(λ_i)	Im(λ_i)	Modulus r_i	Re(λ_i)	Im(λ_i)	Modulus r_i
1	1.0884	0	1.0884	1.1331		1.1331	1.0961	0	1.0961
2	0.9568	0.5046	1.0817	0.9791	0.5250	1.1110	0.9404	0.5462	1.0875
3	0.9568	-0.5046	1.0817	0.9791	-0.5250	1.1110	0.9404	−0.5462	1.0875

For Model I and III, there is an eigenvalue of the matrix A with an absolute value greater than the absolute value of all other conjugate complex roots (as seen in Table 7). Although the difference between the largest real root and the absolute value of the next two largest conjugate complex roots is small in numerical terms, it is still very significant. If this difference were zero, two successive population numbers would be cyclic in the long run (as noted in Gumbel, 1917, p. 630). As a result, the age structure of the projected population approaches a stable age structure. Under the assumptions made, the progression of births, deaths, and total population will gradually tend towards a geometric series with a biennial growth rate of 9.61% for Model III and 13.3% for Model II, with decreasing cycles (see Fig. 3). The growth rate in Model III is equivalent to a tripling period of 24 years, as previously found by Süßmilch. Furthermore, the difference between the largest real root and the absolute value of the two next largest conjugate complex roots is larger in Model II than in Model III, leading to larger oscillations in Model III.

In the long run, the population time series in Models II and III can be explained well by the sum of two components: $ZR1 = Trend + S1$. The short-term fluctuations are only relevant at the beginning of the time series. In the very long run, the impact of the cycle S1 decreases, despite being an explosive cycle, as the ratio S1/Trend approaches zero. The series can be solely explained by its trend component.

The right eigenvector of the Leslie matrix, belonging to the dominant eigenvalue λ_1, contains only positive elements and represents the age structure of the stable population. It is proportional to the stable age distribution and can be scaled to show either the proportion or percentage of individuals in each age class. The left eigenvector of the Leslie matrix is referred to as the reproductive value of the population. This vector can be scaled so that its first element is one, as seen in Table 8. The reproductive value represents the total number of female offspring (discounted by the population growth rate) that can be expected from a woman at a given age. The reproductive value has its maximum at the beginning of the reproductive phase, but decreases to zero after the end of the reproductive phase. It is higher when the growth rate is higher, as shown in Table 8 and Figure 6.

In Model III, even after 500 years, the stable state has not yet been reached (as seen in Figure 4). The population size at that time is nearly five times larger than the current world population. The curves in Fig. 4 illustrate strong ergodicity in demography, where the final age structure depends only on mortality and fertility and not on the initial age structure. Due to higher fertility in Model II, the age structure of Model II is younger than that of Model III (as can be seen in Table 8 and Figure 5).

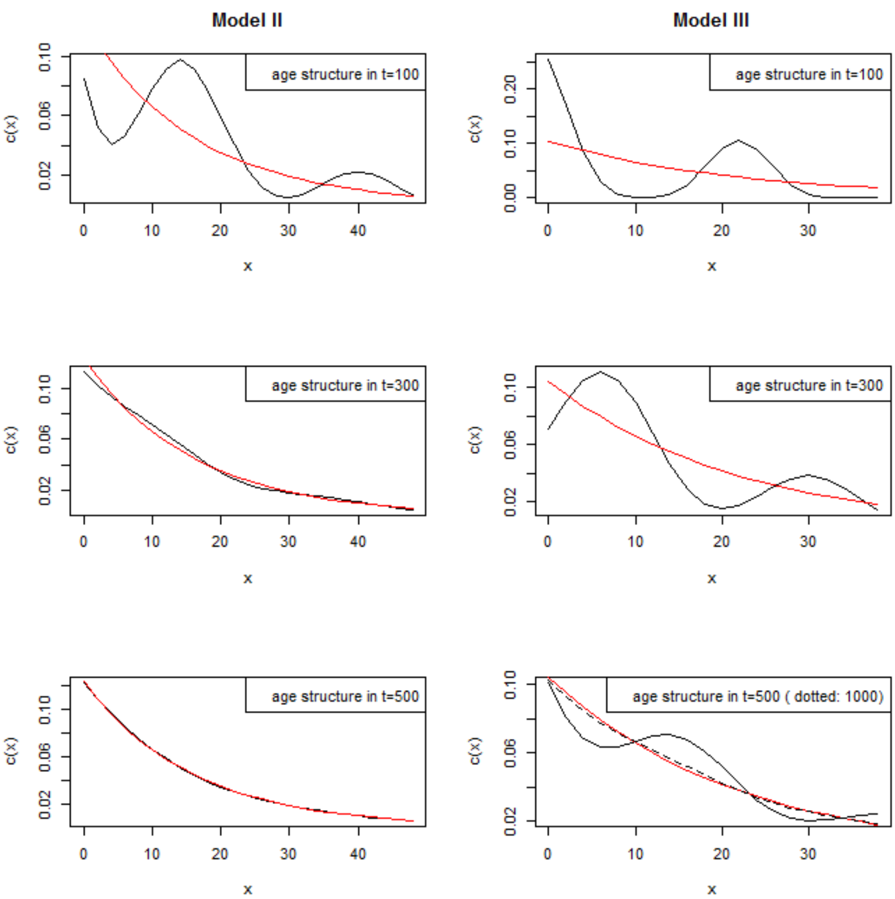

Fig. 4. Age structures after 100, 300, 500 and 1000 years (red: stable age structure, see also Tab. 8)

Table 8. Stable age structures c(x) and reproduction values

	Model I		Model II		Model III	
x	c(x)	$v(x)$	c(x)	$v(x)$	c(x)	$v(x)$
0-2	0.0913	1	0.1229	1	0.1043	1
2-4	0.0839	1.0884	0.1085	1.1331	0.0952	1.0961
4-6	0.0771	1.1846	0.0957	1.2840	0.0868	1.2015
6-8	0.0708	1.2893	0.0845	1.4550	0.0792	1.3170
8-10	0.0651	1.4032	0.0745	1.6487	0.0723	1.4436
10-12	0.0598	1.5272	0.0658	1.8683	0.0659	1.5824
12-14	0.0549	1.6622	0.0581	2.1170	0.0602	1.7345
14-16	0.0505	1.8091	0.0512	2.3989	0.0549	1.9012
16-18	0.0464	1.9690	0.0452	2.7183	0.0501	2.0840
18-20	0.0426	2.1431	0.0399	3.0802	0.0457	2.2843
20-22	0.0391	2.3325	0.0352	3.4904	0.0417	2.5039
22-24	0.0360	2.5386	0.0311	2.9551	0.0380	1.7446
24-26	0.0330	1.7630	0.0274	2.3486	0.0347	0.9123
26-28	0.0304	0.9188	0.0242	1.6613	0.0316	0
28-30	0.0279	0	0.0214	0.8825	0.0289	0
30-32	0.0256	0	0.0188	0	0.0263	0
32-34	0.0235	0	0.0166	0	0.0240	0
34-36	0.0216	0	0.0147	0	0.0219	0
36-38	0.0199	0	0.0130	0	0.0200	0
38-40	0.0183	0	0.0114	0	0.0182	0
40-42	0.0168	0	0.0101	0		
42-44	0.0154	0	0.0089	0		
44-46	0.0142	0	0.0079	0		
46-48	0.0130	0	0.0069	0		
48-50	0.0120	0	0.0061	0		
50-52	0.0110	0				

For example, a reproductive value of 3.49 in Table 8 (for age 22-24 in Model II) means that an immigration of a woman in that age class will have a 3.49-fold greater impact on the increase of the stable population size than an immigration at age 0-2. However, the long-term growth rate is not affected by this 3.49-fold greater effect.

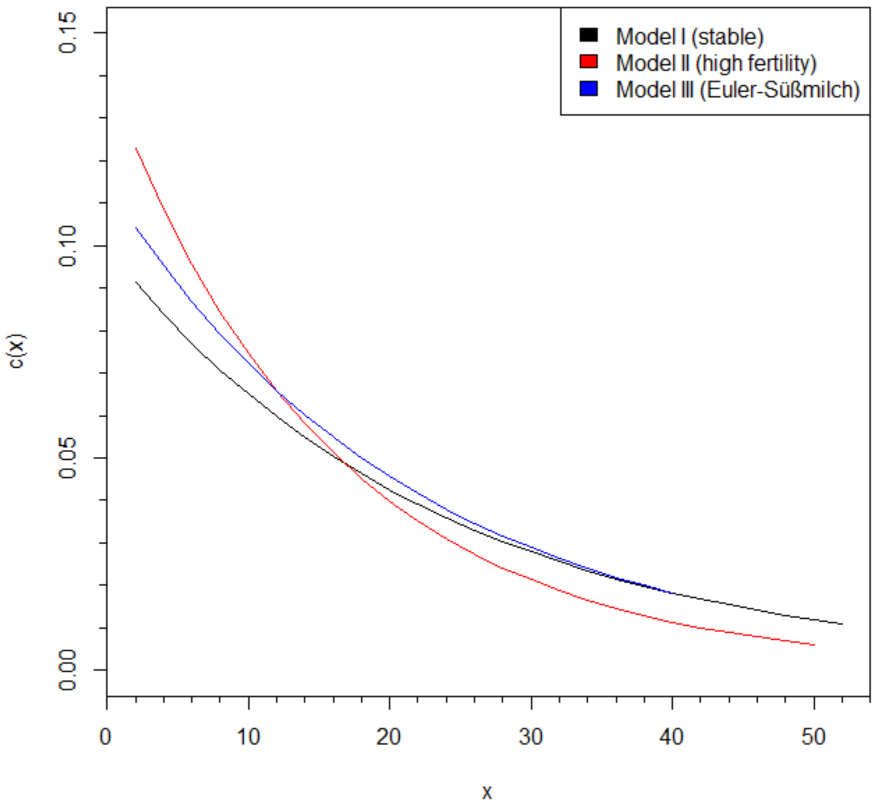

Fig. 5. Stable age structures

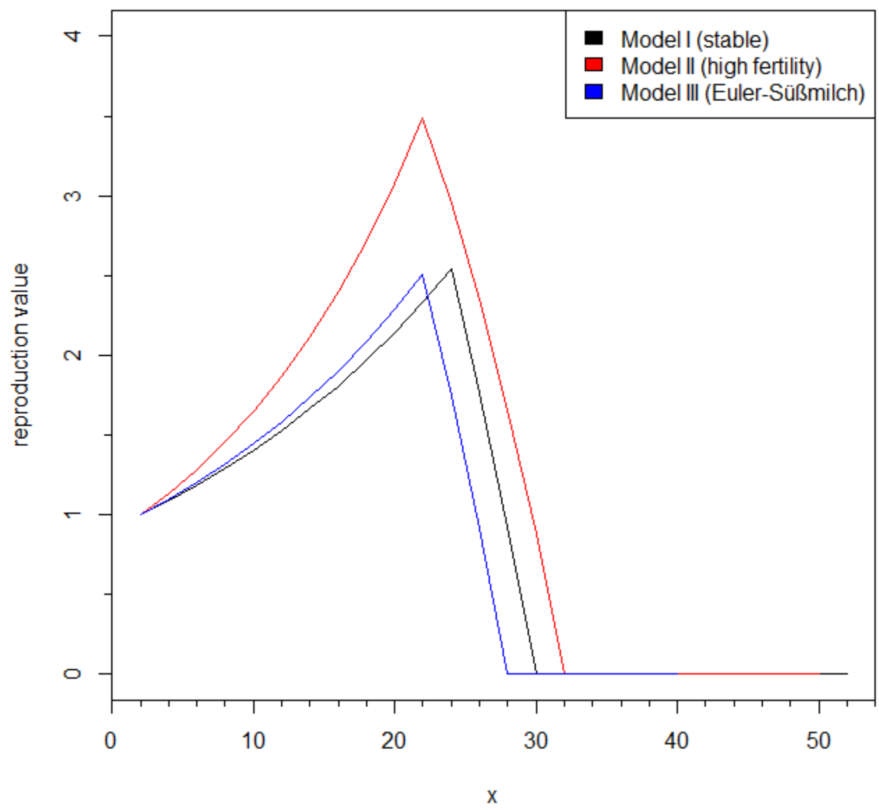

Fig. 6. Reproductive values at age x (Models I,II, and III)

4. A Continuous Variant of Euler's Stable Model I

In the following, we transform the discrete stable model I into a continuous form for easier analysis. This simple model is especially useful for educational purposes, as the integrals are straightforward to compute.

The International Union for the Scientific Study of Population[8] (IUSSP) defines a stable population as "theoretical models widely used by demographers to represent and understand the structure, growth, and evolution of human populations. By definition, stable populations have age-specific fertility and mortality rates that remain constant over time. Mathematically, it can be shown that populations with constant fertility and mortality patterns grow or shrink at a constant rate and attain a characteristic age structure that remains unchanged over time."

4.1 Stable Age Structure and Intrinsic Rate of Growth

The stable age structure is given by (see, e.g., Keyfitz, 1977)

$$c(x) = \frac{e^{-r \cdot x} \cdot l(x)}{\int_0^n e^{-r \cdot x} \cdot l(x) dx}$$

[8] https://papp.iussp.org/sessions/papp103_s07/PAPP103_s07_020_010.html

with the stable birth rate $b = \frac{1}{\int_0^n e^{-r\cdot x} \cdot l(x)dx}$.

The stable intrinsic growth rate r, can be determined by solving the characteristic equation

$$1 = \int_\alpha^\beta e^{-r\cdot x} l(x)m(x)dx,$$

where m(x) is the maternity function, and l(x)m(x) is the net maternity function. The limits of the integral are the youngest fertile age α and the highest β. The net reproduction rate is

$$R_0 = \int_\alpha^\beta l(x)m(x)dx.$$

Since in the Euler model I $l(x) = 1$ and $l(x)m(x)dx = 1$ we get[9]

$$c(x) = \frac{e^{-r\cdot x}}{\int_0^n e^{-r\cdot x}dx} = \frac{r \cdot e^{r\cdot(n-x)}}{e^{n\cdot r} - 1}, \text{ since } \int_0^n e^{-r\cdot x}dx = \frac{1-e^{-n\cdot r}}{r} = \frac{1}{r} - \frac{e^{-n\cdot r}}{r}$$

with $r = \ln \lambda$.

The continuous variant of Euler's characteristic equation is

$$\int_\alpha^\beta e^{-r\cdot x} l(x)m(x)dx = \int_\alpha^\beta \ln \lambda^{-x} \frac{R_0}{\beta - \alpha} dx = 1.$$

In our case with the given values:

$$\int_{12}^{14} \ln \lambda^{-x} \frac{3}{4} dx = \frac{3 \cdot \left(\lambda^2 - 1\right)}{2 \cdot \lambda^{14} \cdot \ln \lambda} = 1; \text{ the solution is } \lambda = 1.08828.$$

Approximation formulas (see, e.g., Keyfitz 1977. p.119) for λ or r, with $\lambda = e^r$ are:

$R_0 = 3,\ A_0 = \frac{12+14}{2} = 13$ (mean of the net maternity function) and $\sigma^2 = \frac{(14-12)^2}{12} = \frac{1}{3}$.

a) $R_0 = \exp\left(A_0 \cdot \ln \lambda - \frac{\sigma^2 \cdot (\ln \lambda)^2}{2} \right)$ and $\sigma^2 \cdot \frac{(\ln \lambda)^2}{2} - A_0 \cdot \ln \lambda + \ln R_0 = 0$; $\lambda = 1.08828$

b) $R_0 = \exp\left(A_0 \cdot \ln \lambda\right) \rightarrow \lambda = \exp\left(\frac{\ln R_0}{A_0}\right) = \sqrt[13]{R_0}$; $\lambda = 1.08818$

The approximate values hardly deviate from the exact value $\lambda = 1.0883$ of Model I.

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9 Euler's life table is a rectangular life table, i.e., the survival pattern shows a rectangular shape (see also Pflaumer, 2010).

4.2 Effect of an Infinite Life Expectancy

The maximum age has no effect on the dominant eigenvalue and the biennial growth rate, but it does affect the population size. According to Pflaumer (2023), who modeled the immortality of the Euler-Süßmilch model using the Lefkovitch matrix (which resembles the Leslie matrix but with the element in the kth row and the kth column being 1 instead of 0), in the long run, the population in which immortality occurs is only about 19% higher than the population in which all people die at the age of 40. This analysis is made easier with the continuous model. In the case of infinite life expectancy, the age structure can be represented by the following equation:

$$\lim_{n\to\infty}\frac{r\cdot e^{r\cdot(n-x)}}{e^{n\cdot r}-1}=r\cdot e^{-r\cdot x}.$$

If r=0 then the rectangular results.

$$\lim_{r\to 0}\frac{r\cdot e^{r\cdot(n-x)}}{e^{n\cdot r}-1}=\frac{1}{n}.$$

Important parameters can be represented by simple formulas, such as

Mean age and variance of the stable population:

$$\mu_S=\int_0^n x\cdot c(x)dx=\frac{e^{n\cdot r}-n\cdot r-1}{r\cdot\left(e^{n\cdot r}-1\right)}$$

$$\lim_{n\to\infty}\mu_S=\frac{1}{r}$$

$$\lim_{r\to 0}\mu_S=\frac{n}{2}$$

$$\sigma_S^2=\frac{e^{2\cdot n\cdot r}-e^{n\cdot r}\cdot\left(n^2\cdot r^2+2\right)+1}{r^2\cdot\left(e^{n\cdot r}-1\right)^2}$$

$$\lim_{n\to\infty}\sigma_S^2=\frac{1}{r^2}$$

$$\lim_{r\to 0}\sigma_S^2=\frac{n^2}{12}$$

Ratio of old and young people:

$$\alpha_A=\frac{\int_b^n c(x)dx}{\int_0^n c(x)dx}=\frac{e^{-b\cdot r}\left(e^{b\cdot r}-e^{n\cdot r}\right)}{1-e^{n\cdot r}}$$

$$\lim_{n\to\infty}\alpha_A=e^{-b\cdot r}$$

$$\alpha_J = \frac{\int_0^a c(x)dx}{\int_0^n c(x)dx} = \frac{e^{n\cdot r - a\cdot r}\left(e^{a\cdot r}-1\right)}{e^{n\cdot r}-1}$$

$$\lim_{n\to\infty}\alpha_A = e^{-a\cdot r}\cdot\left(e^{a\cdot r}-1\right).$$

The population size at time t is in the stable population with l(x)=1

$$P^n\left(t\right) = B(0)\cdot e^{r\cdot t}\cdot\int_0^n e^{-r\cdot x}\cdot l(x)dx = B(0)\cdot e^{r\cdot t}\cdot\frac{1-e^{-n\cdot r}}{r}.$$

$$\lim_{n\to\infty}P^n\left(t\right) = P^\infty\left(t\right) = B(0)\cdot e^{r\cdot t}\cdot\frac{1}{r}.$$

Thus, the ratio of a population with infinite to a population with finite life expectancy n (in the continuous Euler model) is

$$\frac{P_t^\infty}{P_t^n} = \frac{e^{n\cdot r}}{e^{n\cdot r}-1}$$

In the Euler model III, life expectancy is n=40 and the stable annual growth rate is $r = \ln\left(\sqrt{1.0961}\right) = 0.0459$ while in the Euler model II, n=50 and the growth rate is $r = \ln\left(\sqrt{1.1332}\right) = 0.0625$.

With these assumptions, the population increases by 18.99% in the first scenario and by only 4.59% in the second scenario. If we assume a rectangular life table up to the age of n=100, with a growth rate of 3%, the increase is 5.24%. The graph in Fig. 7 illustrates the relative increases for different growth rates. The simple model shows that at high population growth rates, the influence of life expectancy (mortality) is negligible on the age structure and population increase (see also Coale, 2003).

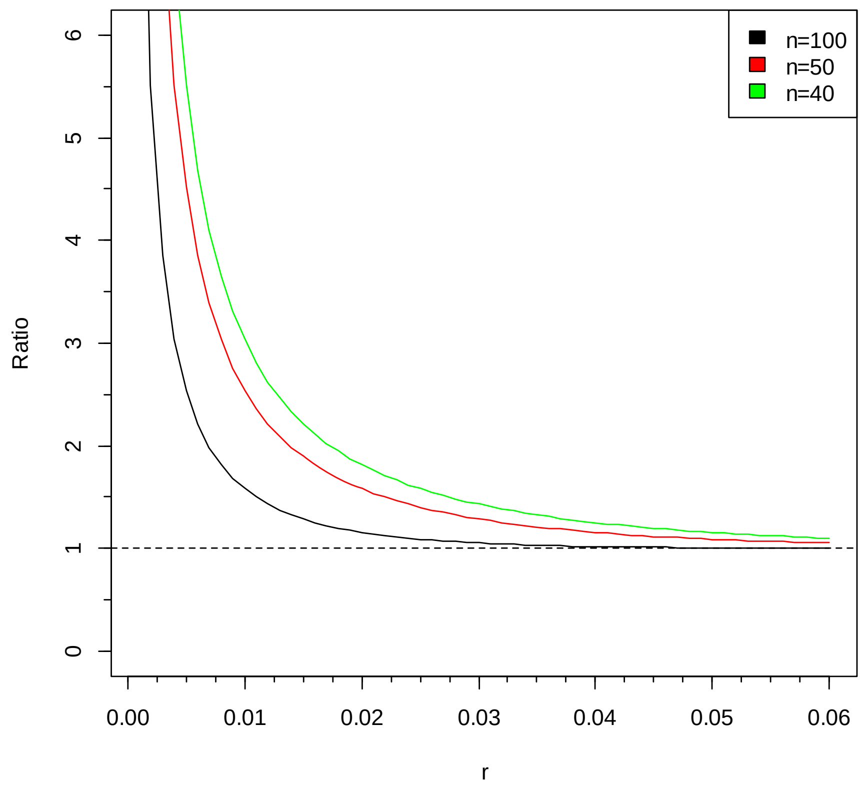

Fig. 7. Ratio of population size

4.3 Population Momentum

Keyfitz's population momentum measures the future growth of a population after an abrupt decrease (or increase) in fertility rates to replacement levels. In 2010, population momentum coefficients ranged from 0.83 in Germany to 1.69 in Guatemala[10].

Keyfitz´s formula for a stable population is

$$M = \frac{P_{stat}}{P_0} = \frac{b \cdot e_0}{r \cdot \mu} \cdot \left(\frac{R_0 - 1}{R_0} \right) \quad \text{(see Keyfitz 1977, p. 156)}$$

with P_0 = population size at t=0, P_{stat} = resulting stationary population, b= birth rate, e_0= life expectancy, r=stable growth rate, R_0= net reproduction rate, μ = mean age of childbearing in the stationary population.

Empirically, we will determine the population momentum by performing a long-term population projection for Model I (female population) using a Leslie matrix with a dominant eigenvalue of 1, or a net reproduction rate of 1, meaning the maternity rates are each 1/3. The results can be seen in Figure 8.

[10] https://papp.iussp.org/sessions/papp103_s08/PAPP103_s08_090_010.html

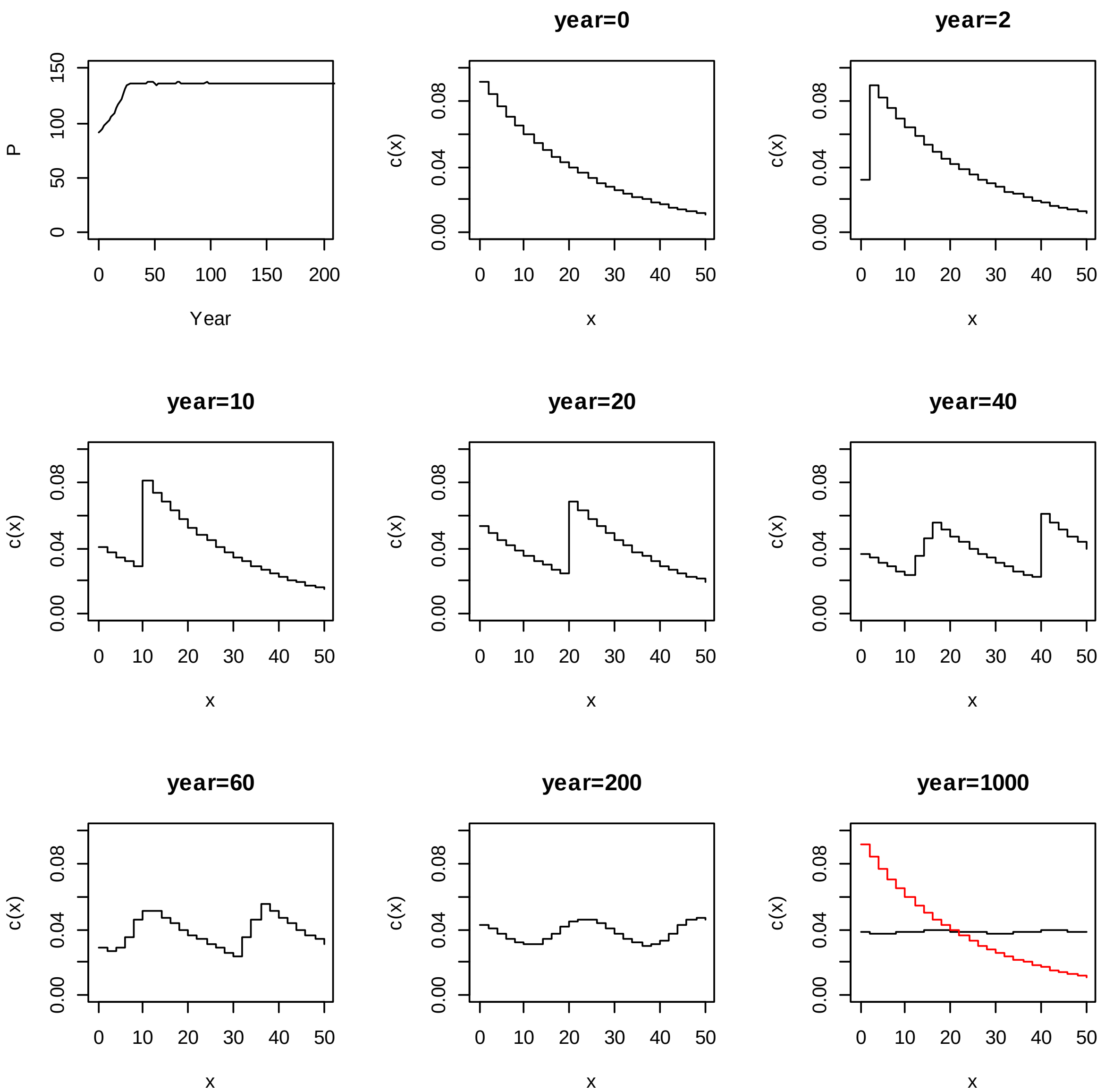

Fig. 8. Age structures in different years after reducing fertility to replacement level of model I (red: age structure of the population at time t=0)

The initial population of Model I with an assumed population size of 91 (see section 2) stabilizes relatively quickly at a level slightly above 136, which corresponds to a momentum of approximately 1.5. If fertility were to immediately decrease to the replacement level, the population would still increase by 50% due to age structure effects. It takes much longer for the rectangular stationary age structure to develop (as seen in Fig. 8). The discrete projection model allows for a dynamic analysis and shows how a stationary population gradually develops. On the other hand, the continuous model only provides a static comparative analysis, where the initial state is compared with the final state. Keyfitz's formula provides similar results as seen in Table 9.

Table 9. Calculations of the population momentum

Empirical population momentum after 1000 years (500 biennial projection steps)
$\frac{P(2000)}{P(0)}=\frac{136.42}{91}=1.5$
Keyfitz´s population momentum formula
$\frac{\frac{0.09130}{2}\cdot 52}{0.0423\cdot 25}\cdot\left(\frac{3-1}{3}\right)=1.495$ with $\mu=\frac{23+25+27}{3}$ (mids of the age classes)
$\frac{\frac{0.09130}{2}\cdot 52}{0.0423\cdot 26}\cdot\left(\frac{3-1}{3}\right)=1.437$ with $\mu=\frac{24+26+28}{3}$ (Upper limits of the age classes)
$\frac{\frac{0.09130}{2}\cdot 52}{0.0423\cdot 25}\cdot\left(\frac{3-1}{3}\right)=1.557$ with $\mu=\frac{22+24+26}{3}$ (Lower limits of the age classes)

Conclusion

Leonhard Euler, the renowned Swiss mathematician, established the foundation of mathematical demography. He developed a stable model that showed how to calculate the true growth rate based on fertility and mortality rates (which is a special case of Lotka's equation). Euler's projection models demonstrate that, in the long term, a constant fertility and mortality rate leads to geometric growth and a stable age structure of the population. This is the first numerical demonstration of the strong ergodic theorem of demography, which assumes fixed age-specific birth and death rates and indicates that the population will grow geometrically with a constant growth rate in the long run.

Euler's life table is a rectangular life table, where the process of rectangularization refers to transforming a life table, a statistical representation of the survival patterns of a population, into a rectangular shape. This type of life table is widely used in mathematical demography to demonstrate demographic relationships and the impact of changing factors. The rectangularized life table provides a very simple and intuitive representation of age-specific mortality rates, making it easier to study population growth and structure over time.

The increasing rectangularization of empirical life tables (see, e.g., Pflaumer, 2010) will lead to a growing significance of Euler's models as a tool for simple demonstrations of mathematical demography in the classroom, as demonstrated by Pflaumer in 2022. These models provide an easy way to illustrate dynamic processes, clarify demographic relationships, and study the impact of changing factors. Discrete Euler models can be conveniently created using statistical software such as R, while continuous Euler models are well-suited for comparative static analyses, as the important integral formulas of the stable model are easily solved in this case.

References

[1] N. Bacaër, A Short History of Mathematical Population Dynamics, Springer-Verlag, London, 2011.

[2] A.J. Coale, On Increases in Expectation of Life and Population Growth, Population and Development Review, March 2003, Vol. 29.1, 113-120 (first appeared in 1959).

[3] L. Euler and L.-G. Du Pasquier, Leonhardi Euleri Opera Omnia. Serie 1, Band 7, Leipzig, 1923.

[4] L. Euler, Sur la multiplication du genre humain. In: Leonhardi Euleri Opera Omnia, Serie 1, Band 7, 545–552, Leipzig, 1923.

[5] L. Euler, Introductio in analysin infinitorum 1748, (Introduction to the Analysis of the Infinite. Republished in English by Springer-Verlag, New York, Inc., 2000)

[6] H.-J.Girlich, Süßmilch und Euler — zwei kooperierende Stammväter der Demographie in Deutschland, Schriften der Universität Leipzig, Fakultät für Mathematik und Informatik, Mathematisches Institut, Leipzig, 2007.

[7] E. J. Gumbel, Eine Darstellung statistischer Reihen durch Euler. Jahresbericht der Deutschen Mathematiker-Vereinigung 25: 251–264, 1917.

[8] N. Keyfitz, Applied Mathematical Demography, Wiley, New York, 1977.

[9] L. Moser, Gesetze der Lebensdauer, Veit, Berlin, 1839.

[10] P. Pflaumer, Analyzing Euler and Süßmilch's Population Growth Model, in: C. H. Skiadas, C. Skiadas (eds.), Quantitative Demography and Health Estimates, The Springer Series on Demographic Methods and Population Analysis 55, 2023, 141-161. https://doi.org/10.1007/978-3-031-28697-1_12

[11] P. Pflaumer, Bevölkerungsstatistik und Demographie, Einführendes Vorlesungsskript, TU Dortmund, Feb. 2023. https://www.researchgate.net/publication/349552671

[12] P. Pflaumer, Measuring the Rectangularization of Life Tables Using the Gompertz Distribution, JSM Proceedings, Social Statistics Section. Alexandria, VA: American Statistical Association, 2010, 664-674.

[13] J. P. Süßmilch, Die göttliche Ordnung in den Veränderungen des menschlichen Geschlechts aus der Geburt, dem Tode und der Fortpflanzung desselben erwiesen, Verlag der Buchhandlung der Realschule, Berlin, 1761.