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On the complexity of the bilevel minimum spanning tree problem

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Abstract

We consider the bilevel minimum spanning tree (BMST) problem where the leader and the follower choose a spanning tree together, according to different objective functions. We show that this problem is NP-hard, even in the special case where the follower only controls a matching. Moreover, we give some evidence that BMST might even remain hard in case the follower controls only few edges. On the positive side, we present a $(|V|-1)$-approximation algorithm for BMST, where $|V|$ is the number of vertices. Moreover, we show that 2-approximating BMST is fixed-parameter tractable and that, in case of uniform costs on leader's edges, even solving BMST exactly is fixed-parameter tractable. We finally consider bottleneck variants of BMST and settle the complexity landscape of all combinations of sum or bottleneck objective functions for the leader and follower, for the optimistic as well as the pessimistic setting.

KEYWORDS

approximation algorithms, bilevel optimization, combinatorial optimization, complexity, spanning tree problem, Steiner tree

1 | INTRODUCTION

A bilevel optimization problem models the interplay between two decision makers, each of them having their own decision variables, objective function, and constraints. The two decisions can depend on each other and are made in a hierarchical way: the leader decides first and the follower second, already knowing what the leader has decided. However, the problem is usually viewed from the leader's perspective, who has perfect knowledge of the follower's problem and takes into account how the follower will react to her[1] decision. In other words, the optimality of the follower's decision can be viewed as a constraint in the leader's optimization problem. Several surveys and text books on bilevel optimization have been published [4-6]. Bilevel optimization problems turn out to be very hard in general. Even in the case where both objective functions and all constraints are linear, they are strongly NP-hard [11]. For more details concerning the complexity of bilevel linear optimization, see [7].

In this article, we investigate the complexity of a fundamental *combinatorial* bilevel optimization problem, namely, the bilevel minimum spanning tree problem. Here, each of the two decision makers controls a subset of the edges of a given graph and chooses some of them, such that all chosen edges together form a spanning tree in the graph.

In a possible application [21], the two decision makers can be imagined as a central and a local government whose common task is to design a transportation network connecting a given set of facilities. Hence, together they have to construct a spanning tree, where each of the decision makers may control a different set of potential links, for example, federal highways are

[1]Throughout this article, we will refer to the leader using *she/her* and to the follower using *he/him/his*.

controlled by the central government in contrast to local roads, or building grounds are owned by different actors. First, the central government constructs some of the connections, and then the local government decides how to complete the network. The central government, as the leader, pays for the construction of all connections, while the follower optimizes a different objective function, for example, respecting requests of the citizens. Another possible application is described in [9] and deals with a communication network between cities. A state is modeled as the leader and assumed to own some communication connections, while a private company, the follower, is permitted to build its own connections in specific places. The state subsidizes the connections built by the company, that is, the actual building costs are shared between the two actors. In turn, the company is required to ensure that each city can communicate with each other city using some activated state-owned and the new private connections, that is, that the result is a spanning tree. The decision process now has a hierarchical structure: first, the state decides which of its own connections are to be activated and thus paid for. Second, the company decides which new connections to build such that all cities are connected and the costs incurred for the company are minimized. However, the company's decision influences the state's costs as well, because of the subsidies. Hence, the state has to anticipate what the company will do already when making the first decision.

More formally, let $G = (V, E)$ be a connected, not necessarily simple graph with some edges E^ℓ being controlled by the leader and some edges E^f being controlled by the follower, such that $E = E^\ell \cup E^f$. Without loss of generality, we assume that E^ℓ and E^f are disjoint sets, using parallel edges otherwise. We are given cost functions $c : E \to \mathbb{R}_{\geq 0}$ and $d : E \to \mathbb{R}_{\geq 0}$ for the leader and follower, respectively, and define $c(Z) := \sum_{e \in Z} c(e)$ and $d(Z) := \sum_{e \in Z} d(e)$ for any edge set $Z \subseteq E$. With these definitions, the bilevel minimum spanning tree problem can be formulated as follows:

$$\begin{aligned} \min \quad & c(X \cup Y) \\ \text{s.t.} \quad & X \subseteq E^\ell \\ & Y \in \operatorname{argmin} \quad d(Y) \\ & \qquad \text{s.t.} \quad Y \subseteq E^f \\ & \qquad\qquad X \cup Y \text{ is a spanning tree in } G\,. \end{aligned} \tag{BMST}$$

Here and in the remainder of the article, we identify subgraphs of G, in particular trees and forests, with the corresponding subsets of E.

If the leader chooses some edge set X rendering the follower's problem infeasible, then by definition this choice is not valid for her. In particular, the leader must choose a cycle-free subset X of E^ℓ such that the graph $(V, X \cup E^f)$ is connected, and the follower will augment X to a spanning tree at minimum cost according to his own objective function d. The objective function minimized by the leader is the total cost of the resulting spanning tree with respect to the objective function c.

Given a feasible leader's choice X, the follower's problem can easily be solved in polynomial time, for example, by Kruskal's algorithm [15] applied to the graph resulting from G by contracting all edges in X and restricting to the edges in E^f. However, the follower's optimum solution might not be unique. In order to make the problem well-defined in this case, we will always assume that the follower chooses his solution greedily according to some given order of preference that is consistent with his cost function d. It is easy to verify that both the optimistic and the pessimistic version of bilevel minimum spanning tree (BMST) can be modeled in this way. These are the most common strategies to resolve nonuniqueness of follower's optimum solutions in bilevel optimization. In the former, the follower is assumed to decide in favor of the leader among his optimum solutions, that is, he uses the leader's objective function as a second criterion in his optimization. In contrast, the pessimistic view corresponds to the follower deciding worst possible for the leader. Note that the follower's feasible set is uniquely determined by the connected components of the graph (V, X), and therefore also his response Y when assuming any deterministic strategy to resolve nonuniqueness.

In Figure 1, we give an example of a BMST instance and its optimum solution. The cost of the leader's optimum solution is 9. In contrast to the follower, it is not optimum for the leader to choose edges in a greedy way since taking the edge $\{v_4, v_6\}$ into her solution would result in overall costs of at least 10. It is cheaper for her to let the follower connect the components $\{v_2, v_3, v_4\}$ and $\{v_5, v_6\}$ with each other. However, this strategy relies on the fact that the edge $\{v_3, v_6\}$ is cheaper than $\{v_3, v_5\}$ also for the follower.

Besides the problem with sum objective functions, we will also consider bottleneck versions of BMST, meaning that the leader and/or the follower only pay for the most expensive edge instead of the sum over the costs of all chosen edges. In case the follower has a bottleneck objective function, one has to distinguish between two possible models: either the follower pays for the most expensive edge he chooses himself, that is, his objective function is to minimize $\max_{e \in Y} d(e)$, or he pays for the most expensive edge chosen by any of the two actors, that is, he minimizes $\max_{e \in X \cup Y} d(e)$; the latter case is the only situation in which the follower's cost d of edges in E^ℓ is relevant. These two models are not equivalent, in contrast to the sum objective case, where $d(X \cup Y) = d(X) + d(Y)$ in any optimum solution and hence the two objectives only differ by $d(X)$, which is constant from the follower's perspective.

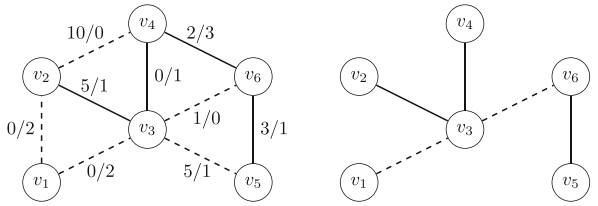

FIGURE 1 Example of a BMST instance and its optimum solution together with the corresponding response of the follower. The edge sets E^ℓ and E^f are represented as solid and dashed edges, respectively. The labels show the leader's and the follower's cost of an edge $e \in E$ in the form $c(e)/d(e)$.

Under the assumption that the leader's and the follower's edge sets are not disjoint, but that the follower controls all edges, that is, that $E^\ell \subseteq E^f = E$, it has recently been shown by Shi et al. [21] that BMST is tractable in case the leader or the follower (or both) optimize a bottleneck instead of the sum objective function, where the follower is assumed to minimize $\max_{e \in X \cup Y} d(e)$ in the bottleneck case. Related results have also been obtained by Gassner [9]. She considered the problem version in which E^ℓ and E^f are disjoint and the follower's objective is $\max_{e \in Y} d(e)$ in the bottleneck case. Polynomial-time algorithms are presented for the cases where the leader has a bottleneck objective and the follower either has a sum or a bottleneck objective, while restricting to the pessimistic problem version in the latter case. In [20], (single-level) mixed integer linear programming formulations for some variants of BMST are derived. For exact solution methods for general bilevel mixed integer programs, we refer to the survey [13].

Other variants of bilevel optimization problems dealing with minimum spanning trees are considered in the literature, but in contrast to the problem addressed here, they usually assume the leader to choose the prices of some edges, while the follower solves a minimum spanning tree problem on all edges according to these costs; see [16] and the references therein. Also the similar setting in which the lower level problem is a shortest path problem has been investigated several times; see the surveys [17,22]. Gassner and Klinz [10] studied a bilevel assignment problem in which leader and follower choose a perfect matching together, each of them having their own objective function on the edges, very similar to the BMST problem studied here. Sum and bottleneck objective functions are considered, and it is shown that in most cases, the problem is NP-hard. Only the optimistic problem version in which both decision makers have bottleneck objectives remains open.

The authors of [21] conjecture that the version of BMST in which both leader and follower have a sum objective is NP-hard. Our main result is a proof of this conjecture. More specifically, we show that BMST is at least as hard as the Steiner forest problem, hence it is not approximable to within a factor of $\frac{96}{95}$ unless P = NP. We can show the same result for the special case where the follower only controls a matching, and give some evidence that the problem might remain intractable even when the follower controls only a fixed number of edges. We also show that certain assumptions on the structure of the problem can be made without loss of generality, for example, that the follower controls a tree or that the leader controls a connected graph.

In view of the negative complexity results mentioned above, one can expect only very limited positive results. We are able to devise a $(|V| - 1)$-approximation algorithm for BMST and show that 2-approximating the optimum solution is fixed-parameter tractable in the number of edges controlled by the follower. For the same parameter, the decision whether a given follower's response can be enforced by the leader is fixed-parameter tractable, which implies that the variant of BMST with uniform costs $c(e)$ for all $e \in E^\ell$ is fixed-parameter tractable as well. For the bottleneck case, we show that the problem is tractable in case the leader has a bottleneck objective and the follower has a sum objective, while it is hard when the leader has a sum objective and the follower has a bottleneck objective. If both have a bottleneck objective, the problem turns out to be polynomial-time solvable in the pessimistic setting, while it is hard to solve in the optimistic case. An overview of our results for different objective functions can be found in Table 1. In this article, however, we consider a more general variant of BMST than Shi et al. [21] in terms of the edges controlled by the follower.

The remainder of this article is organized as follows. In Section 2, we consider different types of restrictions on the set of allowed instances and investigate their relations. In Section 3, we present an approximation-preserving reduction from Steiner forest to BMST and derive our main complexity results. Our results concerning fixed-parameter tractability are presented in Section 4, while in Section 5, we devise an approximation algorithm for BMST. Up to Section 5, we concentrate on the setting in which both leader and follower have a sum objective function. Finally, we review the case of bottleneck objective functions in Section 6. Section 7 concludes.

2 | RESTRICTED SETS OF INSTANCES

In this section, we show that, without loss of generality, we may restrict ourselves to instances of BMST with certain structural properties. Our aim is to simplify some of the proofs later on, but also to clarify the connections between different settings

TABLE 1 Results for all variants of BMST with sum or bottleneck objective functions, assuming an optimistic or pessimistic setting

Leader	Follower	Assumption	Results
Sum	Sum	Optimistic/Pessimistic	NP-hard (Theorem 12)
Sum	Bottleneck	Pessimistic	P for $\mathcal{I}_{E^f\text{ all}}$ and $\max_{e\in X\cup Y} d(e)$ ([21]) NP-hard (Corollary 28)
Sum	Bottleneck	Optimistic	P for $\mathcal{I}_{E^f\text{ all}}$ and $\max_{e\in X\cup Y} d(e)$ ([21]) NP-hard (Theorem 29)
Bottleneck	Sum	Optimistic/Pessimistic	P for $\mathcal{I}_{E^f\text{ all}}$ ([21]) P ([9] and Theorem 26)
Bottleneck	Bottleneck	Pessimistic	P for $\mathcal{I}_{E^f\text{ all}}$ and $\max_{e\in X\cup Y} d(e)$ ([21]) P for $\max_{e\in Y} d(e)$ ([9]) P (Theorem 27)
Bottleneck	Bottleneck	Optimistic	P for $\mathcal{I}_{E^f\text{ all}}$ and $\max_{e\in X\cup Y} d(e)$ ([21]) NP-hard (Theorem 30)

corresponding to reasonable restricted problem variants, which sometimes lead to different complexity results. All reductions are polynomial and approximation-preserving, that is, they can be used to transform an approximation algorithm for one problem to an approximation algorithm with the same guarantee for the other problem.

Let $\mathcal{I}$ be the set of all instances $I = (G, E^\ell, E^f, c, d)$ of BMST as described in Section 1. As already mentioned, we assume throughout that E^ℓ and E^f are disjoint sets. If this is not the case, we can replace any common edge $e \in E^\ell \cap E^f$ by two parallel edges, one belonging to E^ℓ and one to E^f, both having the same leader's and follower's costs as e. We now define the following subsets of $\mathcal{I}$, all corresponding to certain restrictions on the edge sets controlled by leader and follower:

- $\mathcal{I}_{E^\ell\text{ conn}}$, the set of instances for which the leader's graph (V, E^ℓ) is connected.
- $\mathcal{I}_{E^\ell\text{ forest}}$, the set of instances for which the leader's graph (V, E^ℓ) is cycle-free.
- $\mathcal{I}_{E^f\text{ conn}}$, the set of instances for which the follower's graph (V, E^f) is connected.
- $\mathcal{I}_{E^f\text{ forest}}$, the set of instances for which the follower's graph (V, E^f) is cycle-free.
- $\mathcal{I}_{E^f\text{ matching}}$, the set of instances for which the follower's graph (V, E^f) is a matching, that is, for which no vertex is incident to more than one edge.
- $\mathcal{I}_{E^f\text{ all}}$, the set of instances such that for each leader's edge in E^ℓ there exists a parallel follower's edge in E^f with the same leader's cost.

The instances in $\mathcal{I}_{E^f\text{ all}}$ exactly correspond to those considered by Shi et al. [21]. Since G is connected, we have $\mathcal{I}_{E^f\text{ all}} \subseteq \mathcal{I}_{E^f\text{ conn}}$, and $\mathcal{I}_{E^f\text{ conn}}$ is precisely the set of instances where any cycle-free choice of the leader is feasible, that is, for any cycle-free edge set $X \subseteq E^\ell$, there is at least one feasible response of the follower. Moreover, we have $\mathcal{I}_{E^f\text{ matching}} \subseteq \mathcal{I}_{E^f\text{ forest}}$.

Our first reduction shows that we may assume that the edges controlled by the follower connect all vertices of G.

Lemma 1. *BMST on $\mathcal{I}$ can be reduced to BMST on $\mathcal{I}_{E^f\text{ conn}}$. The reduction preserves $\mathcal{I}_{E^\ell\text{ conn}}$, $\mathcal{I}_{E^\ell\text{ forest}}$, and $\mathcal{I}_{E^f\text{ forest}}$, meaning that if we start with an instance in one of these sets, the reduction again results in an instance in this set.*

Proof. Let $I = (G, E^\ell, E^f, c, d) \in \mathcal{I}$. We construct an instance $I' \in \mathcal{I}_{E^f\text{ conn}}$ from I by adding arbitrary edges controlled by the follower in order to make (V, E^f) connected. The new edges e' have cost $c(e') := d(e') := M$ for some large enough number M, for example, one can set $M := \sum_{e\in E} \max\{c(e), d(e)\} + 1$.

Every solution of I is also a solution of I' of the same cost for both leader and follower. The follower's solution is still optimum because, by the choice of M, taking one of the new edges can only make the solution worse for him. Conversely, given a solution of I', it is also a solution of I of the same cost if it does not contain any new edges. Otherwise, the leader's solution of I' is not a feasible choice in I, as the follower will only take a new edge in I' if he cannot produce all necessary connections using only the original edges. In this case, any feasible solution of I is cheaper than the one of I', due to the choice of M.

Since E^ℓ is not changed, the reduction preserves all structural properties of E^ℓ, in particular (V, E^ℓ) being connected or cycle-free. By adding only a minimum number of edges necessary to make (V, E^f) connected, we may also assume that acyclicity of (V, E^f) is preserved. ∎

Using a similar construction, one can show the same result for the graph controlled by the leader:

Lemma 2. *BMST on $\mathcal{I}$ can be reduced to BMST on $\mathcal{I}_{E^\ell \text{ conn}}$. The reduction preserves $\mathcal{I}_{E^\ell \text{ forest}}$, $\mathcal{I}_{E^f \text{ conn}}$, $\mathcal{I}_{E^f \text{ forest}}$, and $\mathcal{I}_{E^f \text{ matching}}$.*

We next show an important structural result about BMST, from which we can conclude that we may assume without loss of generality that the follower controls a forest, but which will also be useful on its own. As stated in Section 1, we assume a fixed ordering of the edges in E^f that the follower will always, that is, for any choice of X, use in his greedy algorithm. This is important for the following proof. Moreover, we do not require X_1 or X_2 to be feasible leader's solutions in the following, that is, it might not be possible for the follower to complete them to a spanning tree. However, we assume that the follower applies his greedy algorithm anyway, leading to forests Y_1 and Y_2.

Lemma 3. *Given two cycle-free edge sets $X_1 \subseteq X_2 \subseteq E^\ell$, let $Y_1, Y_2 \subseteq E^f$ be the corresponding follower's responses. Then $Y_1 \supseteq Y_2$.*

Proof. Let $E^f = \{e_1, \dots, e_m\}$, where $e_1, \dots, e_m$ is the follower's order of preference, and let $Y_1^{(i)}$ and $Y_2^{(i)}$ be the partial solutions of the follower after considering edge e_i in his greedy algorithm, starting from the leader's choice X_1 or X_2, respectively. It then suffices to prove the following claim: for all $i = 0, \dots, m$, each pair of vertices that is connected in $X_1 \cup Y_1^{(i)}$ is also connected in $X_2 \cup Y_2^{(i)}$. This implies that if e_{i+1} is added to $Y_2^{(i)}$, it is also added to $Y_1^{(i)}$, so that the full follower's response $Y_2^{(m)} = Y_2$ to X_2 is contained in $Y_1^{(m)} = Y_1$. We show the claim by induction over i.

Since $Y_1^{(0)} = Y_2^{(0)} = \emptyset$ and $X_1 \subseteq X_2$, there is nothing to show for the case $i = 0$. For $i = 1, \dots, m$, consider two vertices $v, w \in V$ that are connected by $X_1 \cup Y_1^{(i)}$. If v and w are already connected by $X_1 \cup Y_1^{(i-1)}$, they are connected by $X_2 \cup Y_2^{(i-1)}$ as well, by the induction hypothesis, and thus also by the superset $X_2 \cup Y_2^{(i)}$. Otherwise, the connection has been established by adding $e_i = \{v_i, w_i\}$, implying that $X_1 \cup Y_1^{(i-1)}$ connects v to v_i and w to w_i (or vice versa). Again by the induction hypothesis, we derive that also $X_2 \cup Y_2^{(i-1)}$ connects v to v_i and w to w_i. Hence, either v and w are already connected by $X_2 \cup Y_2^{(i-1)}$, in which case we are done, or v_i and w_i are not connected by $X_2 \cup Y_2^{(i-1)}$. In the latter case, edge e_i will be contained in $Y_2^{(i)}$, so that v and w are connected by $X_2 \cup Y_2^{(i)}$ also in this case. ∎

Corollary 4. *BMST on $\mathcal{I}$ can be reduced to BMST on $\mathcal{I}_{E^f \text{ forest}}$. The reduction preserves $\mathcal{I}_{E^\ell \text{ conn}}$, $\mathcal{I}_{E^\ell \text{ forest}}$, and $\mathcal{I}_{E^f \text{ conn}}$.*

Proof. Let $I = (G, E^\ell, E^f, c, d) \in \mathcal{I}$ be an instance of BMST and let $Y^* \subseteq E^f$ be the result of Kruskal's algorithm applied to the graph (V, E^f), using the fixed order of edges defined by the follower's preferences. Note that Y^* is a forest in G, but not necessarily a spanning tree, since we do not require (V, E^f) to be connected. Let I' be the instance that arises from I by removing the edges in $E^f \setminus Y^*$ from E^f. Then $I' \in \mathcal{I}_{E^f \text{ forest}}$. By applying Lemma 3 for $X_1 := \emptyset$ and $X_2 := X$, it follows that for any leader's solution X in I, the follower's response Y lies in Y^*. Hence, X has the same objective value in I as in I'. As the leader's feasible set is not changed by the above transformation, we obtain the desired reduction result.

Since E^ℓ is not changed, the reduction preserves any specific structure of E^ℓ, in particular (V, E^ℓ) being cycle-free or connected. Connectedness of (V, E^f) is obviously preserved by the construction. ∎

It is worth mentioning that even if the follower's edge set E^f is cycle-free, the follower might have several feasible or even several optimum responses to some leader's choice X. Indeed, after the contraction of X, the follower's edges might form cycles again. For an example, consider the instance illustrated in Figure 1, in which the follower's edges form a tree. When the leader takes the edge $\{v_2, v_3\}$ into her solution, the vertices v_2 and v_3 can be thought of as being merged into a single vertex from the follower's perspective. This leads to the follower's edges $\{v_1, v_2\}$ and $\{v_1, v_3\}$ becoming parallel edges, of which the follower must choose one. In this example, the two edges even have the same leader's and follower's cost such that the follower will choose any of the two edges, depending on his preferences.

If we are not interested in the connectedness of the follower's edge set E^f, but rather in a simple combinatorial structure of the latter, we can even further restrict E^f to form a matching:

Lemma 5. *BMST on $\mathcal{I}_{E^f \text{ forest}}$ can be reduced to BMST on $\mathcal{I}_{E^f \text{ matching}}$. The reduction preserves $\mathcal{I}_{E^\ell \text{ conn}}$ and $\mathcal{I}_{E^\ell \text{ forest}}$.*

Proof. Let $I = (G, E^\ell, E^f, c, d) \in \mathcal{I}_{E^f \text{ forest}}$. From I, construct an instance $I' \in \mathcal{I}_{E^f \text{ matching}}$ by applying the following transformation to every connected component of the graph (V, E^f) containing more than one edge: define an arbitrary vertex in the connected component as its root. Replace every edge $e \in E^f$ in the connected component by a path of length two, with a new vertex in the middle. The new edge e' that is closer to the root, is added to E^ℓ and assigned $c(e') := 0$, while the other new edge e'' replaces e in E^f and is assigned $d(e'') := d(e)$ and $c(e'') := c(e)$; moreover, edge e'' takes the position of e in the follower's order of preference. This construction ensures that E^f forms a matching in I' because every new vertex has only one incident follower's edge, and for every vertex v that was already present in I, only the

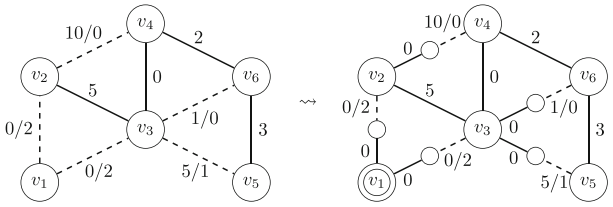

FIGURE 2 Illustration of the construction in the proof of Lemma 5 applied to the example instance of Figure 1, with vertex v_1 chosen as root.

follower's edge e'' arising from edge e which is contained in the unique path from v to the corresponding root is incident to v. See Figure 2 for an illustration.

A solution for I can be transformed to a solution for I' of the same cost by adding all newly introduced leader's edges to her solution, which does not change the cost. Indeed, the follower solves exactly the same problem after the leader's solution is contracted. For the opposite transformation, consider an optimum leader's solution X' for I'. Observe that we may assume all newly introduced edges to be in X' because otherwise, adding them would lead to the follower removing some of his edges from his response by Lemma 3, which cannot increase the leader's objective value. Now, remove all new edges from X' in order to get a solution X for I. Again, the follower has exactly the same choices responding to X and X', respectively. Thus, the objective value of X in I is at most the objective value of X' in I'.

Since the edges added to E^ℓ connect every new vertex by exactly one edge, the reduction preserves $\mathcal{I}_{E^\ell\,\text{conn}}$ and $\mathcal{I}_{E^\ell\,\text{forest}}$. ∎

Combining the reductions from Lemmas 1 and 2 and Corollary 4, we obtain the following result:

Corollary 6. *BMST on $\mathcal{I}$ can be reduced to BMST on $\mathcal{I}_{E^\ell\,\text{conn}} \cap \mathcal{I}_{E^f\,\text{conn}} \cap \mathcal{I}_{E^f\,\text{forest}}$. The reduction preserves $\mathcal{I}_{E^\ell\,\text{forest}}$.*

Dropping the connectedness of E^f, we can apply Lemma 5 to obtain:

Corollary 7. *BMST on $\mathcal{I}$ can be reduced to BMST on $\mathcal{I}_{E^\ell\,\text{conn}} \cap \mathcal{I}_{E^f\,\text{matching}}$. The reduction preserves $\mathcal{I}_{E^\ell\,\text{forest}}$.*

As mentioned above, the authors of [21] only consider instances from $\mathcal{I}_{E^f\,\text{all}}$, that is, the follower controlling many edges. This could be seen as an opposite assumption to instances being chosen from $\mathcal{I}_{E^f\,\text{forest}}$ or even $\mathcal{I}_{E^f\,\text{matching}}$. To show that our main complexity results still hold in the setting of [21], we use the following result:

Lemma 8. *BMST on $\mathcal{I}_{E^f\,\text{conn}}$ can be reduced to BMST on $\mathcal{I}_{E^f\,\text{all}}$. The reduction preserves $\mathcal{I}_{E^\ell\,\text{conn}}$ and $\mathcal{I}_{E^\ell\,\text{forest}}$.*

Proof. Let $I = (G, E^\ell, E^f, c, d) \in \mathcal{I}_{E^f\,\text{conn}}$. Construct an instance $I' \in \mathcal{I}_{E^f\,\text{all}}$ from I by creating a copy e' of each edge $e \in E^\ell$ that does not have a parallel follower's edge of the same leader's cost, adding e' to E^f and setting $c(e') := c(e)$ and $d(e') := M$, for some large M, for example, $M := \sum_{e\in E} d(e) + 1$. The construction is illustrated in Figure 3.

All cycle-free sets $X \subseteq E^\ell$ are feasible leader's solutions for both I and I' because we assume (V, E^f) to be connected. By construction, any feasible leader's solution X leads to the same follower's response in I and I', since the additional edges are the most expensive ones for the follower and will thus never be chosen because he can establish any desired connection using only the original edges.

As the reduction does not change the set E^ℓ, it clearly preserves all its structural properties, in particular (V, E^ℓ) being cycle-free or connected. ∎

From Lemmas 1 and 8, we derive

Corollary 9. *BMST on $\mathcal{I}$ can be reduced to BMST on $\mathcal{I}_{E^f\,\text{all}}$. The reduction preserves $\mathcal{I}_{E^\ell\,\text{conn}}$ and $\mathcal{I}_{E^\ell\,\text{forest}}$.*

In the following sections, we will also consider the case of uniform leader's costs on the leader's edges E^ℓ. For this, we show

Lemma 10. *BMST on $\mathcal{I}_{E^f\,\text{conn}}$ with polynomially bounded integer costs c on E^ℓ can be reduced to BMST with $c(e) = 1$ for all $e \in E^\ell$. The reduction preserves $\mathcal{I}_{E^\ell\,\text{conn}}$, $\mathcal{I}_{E^\ell\,\text{forest}}$, $\mathcal{I}_{E^f\,\text{conn}}$, and $\mathcal{I}_{E^f\,\text{forest}}$.*

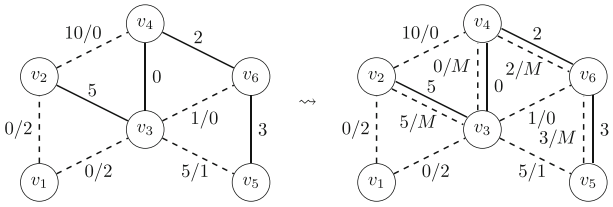

FIGURE 3 Illustration of the construction in the proof of Lemma 8 applied to the example instance of Figure 1.

Proof. Let $I = (G, E^\ell, E^f, c, d) \in \mathcal{I}_{Ef\ \text{conn}}$ with polynomially bounded integer costs c on E^ℓ. Construct an instance I' of BMST with uniform costs c on E^ℓ as follows: contract all edges $e \in E_0^\ell := \{e \in E^\ell | c(e) = 0\}$. Each edge $e = \{v, w\} \in E^\ell \setminus E_0^\ell$ is replaced by a path P_e of length $c(e)$, consisting of leader's edges again. Each interior vertex u of P_e is connected to v by a new edge e' added to E^f with $c(e') := 0$ and $d(e') := M$ for some large enough constant $M := \sum_{e \in E} d(e) + 1$. Note that for edges $e \in E^\ell$ with $c(e) = 1$ nothing changes.

We claim that the instances I and I' have the same optimum value. Given an optimum solution X to I, we first may assume that X contains a maximal forest in E_0^ℓ because otherwise, we could add an edge from E_0^ℓ to X, replacing some edge e with $c(e) \geq 0$ in the resulting spanning tree. A feasible solution X' to I' having the same objective value as X can be defined by setting $X' := \cup_{e \in X} P_e$. This is true since the follower has to connect all interior vertices of paths P_e with $e \notin X$ using the newly introduced follower's edges in order to ensure that the resulting graph is a tree. These edges have cost 0 for the leader. After adding these edges, the follower has exactly the same choices as in the instance I.

Conversely, given an optimum solution X' to I', we may assume that, for each edge $e \in E^\ell \setminus E_0^\ell$, either all edges in P_e belong to X' or none: assume this is not true and consider some solution X' to I' that contradicts this property. Let Y' be the follower's response to X'. We construct a solution $X'' \subset X'$ to I' with follower's response Y'' with $c(X'') < c(X')$ and $c(Y') = c(Y'')$ as follows: let

$$X'' := \bigcup \{P_e | e \in E^\ell \setminus E_0^\ell, P_e \subseteq X'\}$$

consist of all the paths that are entirely contained in X', that is, we simply leave out all edges of paths that were only taken partially in X'. As we assume $I \in \mathcal{I}_{Ef\ \text{conn}}$, which implies also $I' \in \mathcal{I}_{Ef\ \text{conn}}$, the leader's solution X'' is clearly feasible because the follower can complete any solution to a spanning tree. Moreover, observe that, since the edges connecting the inner vertices of the paths P_e have very high cost for the follower, they are only taken if absolutely necessary. Therefore, the response Y'' to X'' is the same as the response Y' to X' with some additional edges that connect the inner vertices of the paths P_e that are connected by X', but not by X''. This shows $c(Y') = c(Y'')$, since these additional edges have cost 0 for the leader; hence, X'' is the desired solution. Thus, we can assume that, for each $e \in E^\ell \setminus E_0^\ell$, either all edges in P_e belong to X' or none. Setting $X := \{e \in E^\ell \setminus E_0^\ell | P_e \subseteq X'\} \cup F$, where F is a maximal forest in E_0^ℓ, then yields a feasible solution to I with the same objective value as X' in I' because the follower's responses to X and X' have the same cost, by the same arguments as in the first part of the proof.

Acyclicity and connectedness of both E^ℓ and E^f are preserved because the construction ensures that the newly introduced vertices are all connected to the old vertices in an acyclic manner in both E^ℓ and E^f. ∎

The reduction described in the proof of Lemma 1 only introduces follower's edges. We can thus combine it with Lemma 10 to obtain

Corollary 11. *BMST on $\mathcal{I}$ with polynomially bounded integer costs c on E^ℓ can be reduced to BMST with $c(e) = 1$ for all $e \in E^\ell$. The reduction preserves $\mathcal{I}_{E^\ell\ \text{conn}}$, $\mathcal{I}_{E^\ell\ \text{forest}}$, $\mathcal{I}_{Ef\ \text{conn}}$, and $\mathcal{I}_{Ef\ \text{forest}}$.*

3 | MAIN COMPLEXITY RESULTS

In this section, we establish a first hardness result for BMST using a reduction from the well-known *Steiner forest problem*:

(SF) Given a connected graph $G = (V, E)$ with edge lengths $\ell : E \to \mathbb{R}_{\geq 0}$ and k disjoint sets $S_1, \ldots, S_k \subseteq V$, find a forest $F \subseteq E$ of minimum total length $\ell(F)$, such that for each terminal set S_i, all vertices in S_i are connected in the graph (V, F).

The best approximation ratio that is known for SF is 2 [12] and the problem is NP-hard to approximate within a factor of $\frac{96}{95}$ [3]. We will reduce SF to BMST in order to obtain the following result:

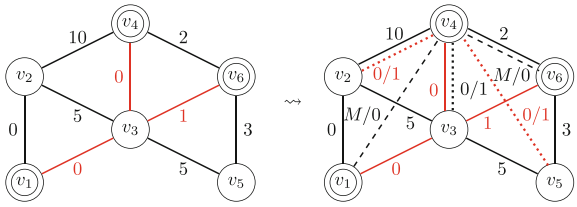

FIGURE 4 Illustration of the proof of Theorem 12 for $k = 1$ with E_f being a star. The marked vertices v_1, v_4, v_6 are the given terminals. Edges in E_0^f are represented as dashed edges, the remaining edges in E^f are drawn as dotted edges. Any vertex of S_1 can be chosen as the center of the star, here it is v_4. Red edges mark the optimum Steiner tree in the input graph and the optimum leader's solution and corresponding follower's response in the BMST instance.

Theorem 12. *BMST cannot be approximated to within a factor of* $\frac{96}{95}$ *in polynomial time, unless* $P = NP$*, even if* E^f *is a tree.*

Proof. Let I be an instance of SF, consisting of a graph $G = (V, E)$ with edge lengths $\ell : E \to \mathbb{R}_{\geq 0}$ and disjoint terminal sets $S_1, \ldots, S_k \subseteq V$. We construct an instance I' of BMST as follows. The graph in I' is $G' := (V, E^\ell \cup E^f)$, where $E^\ell := E$ and E^f is defined as follows: first introduce edges forming any forest with connected components having vertex sets $S_1, \ldots, S_k$ and call this edge set E_0^f. Then, add any further edges turning E_0^f into a spanning tree on V. All new edges together form the set E^f. The cost function for the leader is

$$c(e) := \begin{cases} \ell(e), & \text{if } e \in E^\ell, \\ M, & \text{if } e \in E_0^f, \\ 0, & \text{if } e \in E^f \setminus E_0^f, \end{cases}$$

where M is some large constant such as $\sum_{e \in E} \ell(e) + 1$. The cost function for the follower is given by

$$d(e) := \begin{cases} 0, & \text{if } e \in E_0^f, \\ 1, & \text{if } e \in E^f \setminus E_0^f. \end{cases}$$

This finishes the construction of I'. We now show that any optimum solution X to I corresponds to a feasible leader's solution X' to I' of the same cost, and vice versa. So let $X \subseteq E$ be any solution to I. Then $X' := X$ is a feasible leader's solution since X forms a forest and E^f connects all vertices, so that the follower can complete any leader's solution to a tree. Since X connects each terminal set, the follower's response to X' does not contain any edges from E_0^f as they would form a cycle together with X'. Hence, the follower's response only consists of edges having cost 0 for the leader. Therefore, the overall cost for the leader is simply $c(X') = \ell(X)$.

It remains to show that any optimum solution X' to I' corresponds to a feasible solution X to SF of the same cost. Clearly, there exists a leader's solution to I' of cost at most $M - 1$, for example, one could choose any spanning tree in $G = (V, E^\ell)$. By optimality of X', this implies that the follower's response to X' does not contain any of the edges in E_0^f. However, since the follower's cost for the edges in E_0^f is cheaper than the cost of the edges in $E^f \setminus E_0^f$, this implies that the leader's solution X' connects each terminal set. As X' is also cycle-free, it is a solution to I having cost $\ell(X')$. ■

Remark 13. The definition of E^f in the proof of Theorem 12 leaves a lot of freedom concerning the structure of the follower's tree. For example, it can always be chosen to form a path. Moreover, the reduction can be performed analogously from the Steiner tree problem instead of the Steiner forest problem, that is, where only one terminal set S_1 is given. Then the structure of the follower's tree is even less restricted, for example, the set E^f can be chosen to form a star; see Figure 4 for an illustration. Thus, the hardness of Theorem 12 still holds for restrictions of the follower's tree's structure such as E^f being a path or a star.

Theorem 12 and Corollary 9 together prove a conjecture stated by Shi et al. [21]:

Corollary 14. *BMST on* $\mathcal{I}_{E^f \text{ all}}$ *cannot be approximated to within a factor of* $\frac{96}{95}$ *in polynomial time, unless* $P = NP$*.*

Remark 15. If we allow negative costs in BMST, the proof of Theorem 12 works in the same way if we define $d(e) := -c(e)$ for all $e \in E^\ell \cup E^f$ instead. This shows that the special case of BMST in which the follower is adversarial to the leader, having the opposite objective function, is hard as well. This is in contrast to [21] where this special case (called MMST there) is shown to be polynomial-time solvable, for both sum and bottleneck objective. However, this is not a

contradiction because the authors of [21] only work with instances from $\mathcal{I}_{E^f \text{ all}}$. In fact, Corollary 14 does not carry over to the special case of MMST since the property of opposite objective functions is lost in the construction in Lemma 8.

Together with Corollary 7, we can conclude that BMST remains hard even if the follower controls a matching, and hence a very simple combinatorial structure.

Corollary 16. *BMST cannot be approximated to within a factor of* $\frac{96}{95}$ *in polynomial time, unless* $P = NP$*, even if* E^f *is a matching.*

From Corollary 11, it follows that the hardness of BMST is preserved even in the case of uniform leader's costs on her own edges. We emphasize that Theorem 12 still holds for polynomially bounded and integer leader's cost since Steiner forest is strongly NP-hard [1].

Corollary 17. *BMST cannot be approximated to within a factor of* $\frac{96}{95}$ *in polynomial time, unless* $P = NP$*, even if* E^f *is a tree and* $c(e) = 1$ *holds for all* $e \in E^\ell$.

To conclude this section, we consider a related question which could be asked in any bilevel optimization problem: can the leader enforce a given follower's response? More formally, we consider the following decision problem:

(BMST-R) Given an instance of BMST and a set $\overline{Y} \subseteq E^f$, does there exist some leader's choice $X \subseteq E^\ell$ such that $\overline{Y}$ is the follower's response to X?

For this problem to be well defined, as for BMST itself, it is essential to assume that the follower has a consistent strategy to select a follower's response in case his optimum solution is not unique. As discussed in Section 1, we ensure such a consistent strategy by assuming that the follower chooses edges greedily according to some deterministic order.

Apart from being an interesting structural question in its own right, we will see in Section 4 that BMST-R—or more precisely, the optimization version in which the cheapest solution X enforcing $\overline{Y}$ is desired—is related to the fixed-parameter tractability of BMST in terms of $|E^f|$. However, we will prove that BMST-R, even in the decision version, is NP-complete. For this, we use the so-called *vertex-disjoint Steiner trees* problem:

(VDST) Given a connected graph $G = (V, E)$ and k disjoint sets $S_1, \dots, S_k \subseteq V$, do there exist vertex-disjoint trees $T_1, \dots, T_k \subseteq E$ in G such that T_i spans S_i for all $i = 1, \dots, k$?

This problem is similar to the Steiner forest problem defined previously, but not the same. The important difference is that in the Steiner forest problem, no disjointness of the trees in the solution is required, that is, it is feasible to have several sets S_i lying in the same connected component of the solution. Moreover, we are considering the decision version of the vertex-disjoint Steiner trees problem here, without any edge costs. Such a decision version of Steiner forest would not be interesting because it is always feasible to select a spanning tree.

The problem VDST is known to be NP-complete even for $k = 2$ in so-called two-layer routing graphs [14]. We use this fact to prove the following result:

Theorem 18. *BMST-R is NP-complete, even if* $|\overline{Y}| = 1$ *and* E^f *forms a path on a subset of the vertex set.*

Proof. BMST-R clearly belongs to NP. To show completeness, we reduce VDST for $k = 2$ to BMST-R. Given an instance of VDST consisting of a connected graph $G = (V, E)$ and disjoint sets $S = \{s_1, \dots, s_r\}$ and $S' = \{s'_1, \dots, s'_{r'}\}$, we define an instance of BMST-R on V by setting $E^\ell := E$ and

$$E^f := \{\{s_i, s_{i+1}\} | i = 1, \dots, r-1\} \cup \left\{\left\{s'_i, s'_{i+1}\right\} | i = 1, \dots, r'-1\right\} \cup \left\{\left\{s_1, s'_1\right\}\right\},$$

where $d(\{s_1, s'_1\}) := 1$ and $d(e) := 0$ for all $e \in E^f \setminus \left\{\left\{s_1, s'_1\right\}\right\}$. Let $\overline{Y} := \left\{\left\{s_1, s'_1\right\}\right\}$. The leader's cost function c is irrelevant for the problem BMST-R. An illustration of this construction is given in Figure 5. We now show that the answer to this instance of BMST-R is yes if and only if the answer to the given VDST instance is yes.

Assume that $T, T' \subseteq E$ are vertex-disjoint trees such that T spans S and T' spans S'. Since G is connected, we may assume that $T \cup T'$ covers all vertices of G, by connecting all non-covered vertices to either T or T' arbitrarily. We claim that the leader's choice $X := T \cup T'$ forces the follower to respond with $\overline{Y}$. Indeed, the follower's preferred edges e with $d(e) = 0$ would all produce cycles, while $\{s_1, s'_1\}$ needs to be added to turn X into a spanning tree.

Now assume that there exists a leader's solution X forcing the follower to respond with exactly the set $\overline{Y}$. Since the latter prefers edges from $E^f \setminus \overline{Y}$, the leader must prevent him from adding any of those, that is, all vertices in S are connected by X and the same is true for the vertices in S'. On the other hand, since the follower chooses $\{s_1, s'_1\}$, the sets S and S' cannot be connected by X. Hence, X contains two vertex-disjoint trees spanning S and S', respectively. ∎

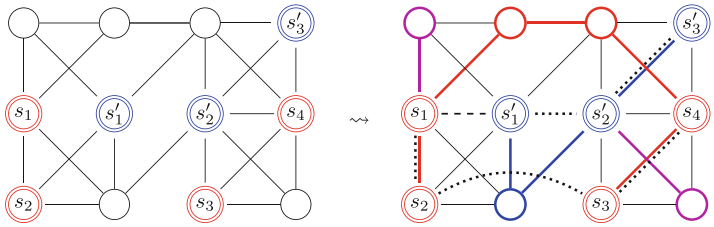

FIGURE 5 Illustration of the proof of Theorem 18. The two terminal sets in the instance of VDST are marked by red and blue vertices, respectively. Dotted lines represent follower's edges of cost 0, whereas the dashed line represents the follower's edge in $\overline{Y}$ having cost 1. One feasible solution to VDST is marked by red and blue edges; a corresponding leader's solution consists of the red, blue, and purple edges.

Note that, similar to the proof of Theorem 12, there is some freedom in the construction of the follower's edge set E^f in this proof; see Remark 13. Instead of the paths given by $\{\{s_i, s_{i+1}\}|i = 1, \dots, r_1\}$ and $\{\{s'_i, s'_{i+1}\}|i = 1, \dots, r' - 1\}$, one could choose any other graph structure spanning the vertices in S and S', respectively. Therefore, Theorem 18 does not only hold for sets E^f forming a path, but also for many other topologies.

Using the same construction as in Lemma 8, one can show that the result of Theorem 18 holds for instances in $\mathcal{I}_{E^f \text{ all}}$ as well. Moreover, since the leader's costs are not relevant in the problem BMST-R, Theorem 18 trivially remains true for any specific choice of leader's costs, in particular in the case of uniform leader's costs.

4 | FIXED-PARAMETER TRACTABILITY

It is easy to see that BMST is tractable when the number of edges controlled by the leader is bounded. In fact, we have

Theorem 19. *BMST is fixed-parameter tractable in the number of edges controlled by the leader.*

Proof. If $k = |E^\ell|$, the leader can choose between at most 2^k different solutions. Computing the follower's response and the corresponding objective function value is possible in polynomial time. ∎

We now turn to the question whether BMST is fixed-parameter tractable in the number of edges controlled by the *follower*, which is much more involved. In fact, we are not able to answer it in general. However, we will show some results related to this question. We start by considering the problem BMST-R introduced in the previous section. In the proof of Theorem 18, a connection between BMST-R and VDST was established in order to prove NP-completeness. It turns out that this relation is also useful for translating positive results from VDST to BMST-R. More precisely, the fact that VDST is fixed-parameter tractable in the total number $\sum_{i=1}^k |S_i|$ of terminals [18,19] can be used to prove the fixed-parameter tractability of BMST-R in terms of $|E^f|$.

Theorem 20. *BMST-R is fixed-parameter tractable in the number of edges controlled by the follower.*

Proof. Consider an instance of BMST-R with graph $G = (V, E^\ell \cup E^f)$. Let V^f be the set of all end vertices of edges in E^f. The algorithm proceeds as follows: all partitions of V^f into non-empty subsets are enumerated. For a given partition $S_1, \dots, S_k$, the problem VDST on (V, E^ℓ) is solved by the algorithm given in [19]. Note that the graph (V, E^ℓ) does not have to be connected, but the definition of VDST and the algorithm can be used anyway. If the result is negative, the partition is discarded. Otherwise, let $T_1, \dots, T_k$ be a corresponding solution of VDST and extend the sets T_i such that $X := \cup_{i=1}^k T_i$ covers all vertices, while the T_i must remain vertex-disjoint. This is possible, since we assume that G is connected. Next, compute the follower's response Y' to X. If it agrees with $\overline{Y}$, stop and return "yes" and, if desired, the set X. If the end of the enumeration is reached, return "no."

The correctness of the algorithm immediately follows from the fact that the follower's response only depends on whether two vertices in V^f are connected by the leader or not, and all possible situations are enumerated. For the running time, note that the number and size of the enumerated partitions only depend on $|V^f| \leq 2|E^f|$, but not on the size of the overall graph. ∎

The algorithm proposed in the proof of Theorem 20 can actually be used for enumerating all possible follower's responses, along with one inducing leader's choice for each response. Unfortunately, Theorem 20 does not imply that the problem of computing the *best* leader's choice enforcing a given response $\overline{Y}$ is fixed-parameter tractable; see the discussion below, so that

we cannot derive that BMST itself is fixed-parameter tractable in the number of edges controlled by the follower. However, all leader's choices enforcing a given follower's response consist of the same number of edges because every spanning tree in the overall graph has the same number of edges. Hence, if the leader has uniform costs on the edges in E^ℓ, this algorithm can be used to solve BMST, leading to the following result:

Corollary 21. *BMST with $c(e) = \bar{c}$ for all $e \in E^\ell$, for some constant $\bar{c} \geq 0$, is fixed-parameter tractable in the number of edges controlled by the follower.*

However, different costs on the edges in E^ℓ cannot be handled easily. In particular, we cannot use the reduction in Lemma 10 to make the costs uniform, since it increases the size of E^f by $\sum_{e \in E^\ell}(c(e)-1)$. The result only carries over to instances where the latter sum is bounded by some function in the original number of follower's edges. Unfortunately, we are not able to answer the question whether Corollary 21 also holds for arbitrary weights, but we conjecture that this is not the case. In fact, there is some evidence that BMST is not easy to solve even for a *fixed* number of edges controlled by the follower. To justify this conjecture, we will establish a relation between BMST and the optimization version of VDST, the *shortest vertex-disjoint Steiner trees* problem:

(SVDST) Given a connected graph $G = (V, E)$ with edge lengths $\ell : E \to \mathbb{R}_{\geq 0}$ and k disjoint sets $S_1, \ldots, S_k \subseteq V$, find vertex-disjoint trees $T_1, \ldots, T_k \subseteq E$ such that T_i spans S_i for $i = 1, \ldots, k$, minimizing their total length $\sum_{i=1}^k \ell(T_i)$, or decide that such trees do not exist.

Given that already the decision problem VDST is a very difficult problem, it can be expected that SVDST is very hard as well. In fact, even for the special case in which each set S_i consists of only two vertices, which is called the *shortest vertex-disjoint paths* (SVDP) problem, there are a lot of open complexity questions. Considerable research has been devoted to SVDP for $k = 2$. Very recently, a randomized polynomial-time algorithm for this case has been developed [2]. To the best of our knowledge, no deterministic polynomial-time algorithm for $k = 2$ nor the complexity of SVDP for any fixed $k \geq 3$ is known. According to the next result, presenting an efficient algorithm for BMST with a fixed number $|E^f| = 2k$ of edges controlled by the follower would settle these open questions for k, and even similar ones about the more general problem SVDST. In particular, an efficient algorithm for BMST with $|E^f| = 4$ would lead to an efficient algorithm for SVDP with $k = 2$.

Theorem 22. *SVDST with fixed number $K := \sum_{i=1}^k |S_i|$ can be polynomially reduced to BMST with K edges controlled by the follower.*

Proof. Given an instance of SVDST as defined above, we construct an instance of BMST as follows. We extend $G = (V, E)$ by one vertex s_0, that is, we set $V' := V \cup \{s_0\}$. The edges controlled by the leader are given by $E^\ell := E \cup E_0^\ell$, where

$$E_0^\ell := \left\{ \{s_0, v\} \mid v \in V \setminus \bigcup_{i=1}^k S_i \right\}.$$

For the follower's edges, we introduce an arbitrary spanning tree on each vertex set S_i and call the set of these edges E_0^f. Moreover, for each $i = 1, \ldots, k$, we select a vertex $s_i \in S_i$ arbitrarily and introduce a follower's edge $\{s_{i-1}, s_i\}$. Together with E_0^f, these edges form the set E^f. The cost function for the leader is defined as

$$c(e) := \begin{cases} \ell(e) + M, & \text{if } e \in E, \\ M, & \text{if } e \in E_0^\ell, \\ M|V|, & \text{if } e \in E_0^f, \\ 0, & \text{if } e \in E^f \setminus E_0^f, \end{cases}$$

where $M := \sum_{e \in E} \ell(e) + 1$. The cost function for the follower is given by

$$d(e) := \begin{cases} 0, & \text{if } e \in E_0^f, \\ 1, & \text{if } e \in E^f \setminus E_0^f. \end{cases}$$

Clearly, this construction is polynomial, with $|E^f| = k + \sum_{i=1}^k (|S_i| - 1) = K$; an illustration is given in Figure 6. We claim that the given instance of SVDST is feasible if and only if the optimum value of the constructed BMST instance is smaller than $M(|V| - k + 1)$, and that in this case the optimum values differ by exactly $M(|V| - k)$.

So first assume that vertex-disjoint trees T_i spanning S_i for $i = 1, \ldots, k$, exist. Then consider the leader's choice X consisting of all edges contained in any of the trees T_i and, for each vertex $v \in V$ not belonging to any tree, the edge $\{s_0, v\}$.

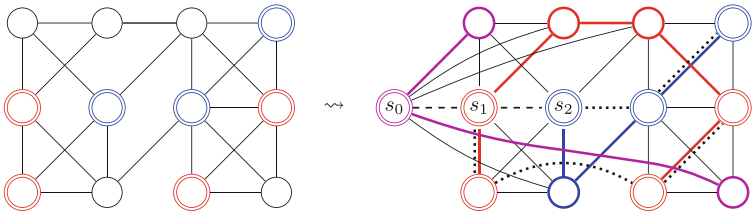

FIGURE 6 Illustration of the proof of Theorem 22 for $k = 2$ and $K = 7$. The two terminal sets S_1 and S_2 in the instance of SVDST are marked by red and blue vertices, respectively. Dotted lines represent follower's edges of cost 0, whereas the dashed lines represent the follower's edges having follower's cost 1. An optimum leader's solution is given by the colored edges, where red and blue edges correspond to an optimum solution of the original instance of SVDST and purple edges connect all vertices not covered by the latter with the auxiliary vertex s_0.

We have $|X| = |V| - k$ because X forms a forest with $k + 1$ connected components on $|V| + 1$ vertices. The follower's response to X is $Y := \{\{s_{i-1}, s_i\} | i = 1, \dots, k\}$ with $c(Y) = 0$. In summary, the objective value of X is

$$c(X) + c(Y) = \sum_{i=1}^{k} \ell(T_i) + M(|V| - k) + 0 < M(|V| - k + 1) .$$

For the other direction, consider any feasible leader's choice X in the constructed instance of BMST and assume that it has an objective value less than $M(|V| - k + 1)$. Then for all $i = 1, \dots, k$, all vertices in S_i must be connected in X, as otherwise the follower would choose an edge with leader's cost $M|V| \geq M(|V| - k + 1)$. Moreover, since each leader's edge costs at least M and the final tree must have $|V|$ edges, the only way to achieve a weight less than $M(|V| - k + 1)$ is to take exactly $|V| - k$ edges and make the follower choose all edges $\{s_{i-1}, s_i\}$ for $i = 1, \dots, k$. It follows that X has $k + 1$ components containing exactly one of the vertices $s_0, \dots, s_k$ each. Thus, X contains disjoint trees T_i spanning S_i with total weight

$$\sum_{i=1}^{k} \ell(T_i) = \sum_{e \in X} (c(e) - M) = \sum_{e \in X} c(e) - M(|V| - k) .$$

This concludes the proof. ∎

As in Theorems 12 and 18, also other topologies of the follower's edges are possible; see Remark 13. We emphasize that Theorem 22 gives a second proof for the NP-hardness of BMST, if we do not bound the number of edges controlled by the follower. In particular, it shows that BMST is at least as hard as SVDST. However, the reduction used in the proof of Theorem 22 is not approximation-preserving, so that the negative result of Theorem 12 concerning approximability does not follow from Theorem 22.

While Theorem 22 makes it unlikely that BMST is fixed-parameter tractable in the number of follower's edges, we will show next that at least approximating BMST within a factor of 2 is fixed-parameter tractable in the same parameter. As a first step, we show that a similar result holds for a variant of SF defined as follows:

(SF+) Given a connected graph $G = (V, E)$ with edge lengths $\ell : E \to \mathbb{R}_{\geq 0}$ and k disjoint sets $S_1, \dots, S_k \subseteq V$, find a forest $F \subseteq E$ of minimum total length $\ell(F)$, such that each terminal set S_i is connected in the graph (V, F) and every vertex in $V \setminus \bigcup_{i=1}^{k} S_i$ is connected to one of the sets S_i.

The difference from the usual Steiner forest problem is hence that in addition to connecting each terminal set S_i, all nonterminals need to be connected to one of the terminal sets.

Theorem 23. *The problem of approximating SF+ within a factor of 2 is fixed-parameter tractable in the total number of terminals.*

Proof. We use the fact that the Steiner forest problem is fixed-parameter tractable, which can be seen as follows: for the classical Steiner tree problem, an exact algorithm with running time $O(3^{|S|}|V|)$, where S is the set of terminals, is well known [8]. This can be extended to the Steiner forest problem in the following way: enumerate all partitions of the set $\{S_1, \dots, S_k\}$ of terminal sets, each resulting in a coarser partition $S'_1, \dots, S'_r$ of the set $\bigcup_{i=1}^{k} S_i$ of all terminals. Now solve the Steiner tree problem for each terminal set S'_i and merge the resulting r edge sets in order to obtain a feasible solution of the Steiner forest problem. Obviously, the best solution obtained in this way is optimum.

Now we compute a solution to the problem SF+ in the following way: first, compute an optimum solution F of the corresponding Steiner forest problem, for example using the algorithm described above. Second, merge the set $\bigcup_{i=1}^{k} S_i$ of all terminals, together with all nonterminals that are connected to a terminal by edges in F, into a single new vertex. Now

compute a minimum spanning tree T in the resulting graph and return the set $F \cup T$ as a solution to the given instance of SF+.

Clearly, the solution is feasible for SF+ and the running time is the same as the one of the applied Steiner forest algorithm because the running time for the computation of a minimum spanning tree is negligible. It remains to show that it is a 2-approximation. For this, observe that both F and T have at most the cost of an optimum solution to SF+, since every such solution must contain a Steiner forest having at least the cost of F, as well as a spanning tree in the graph in which T is a minimum spanning tree. ∎

Theorem 23 now allows us to show the desired result about BMST:

Theorem 24. *The problem of approximating BMST within a factor of 2 is fixed-parameter tractable in the number of edges controlled by the follower.*

Proof. We may assume that (V, E^ℓ) is connected, since the construction according to Lemma 2 does not increase the number of follower's edges. The idea of the algorithm is to enumerate all possible follower's solutions and apply Theorem 23 to each of them. More precisely, as in the proof of Theorem 20, we consider the set $V^f \subseteq V$ of vertices incident to a follower's edge and enumerate all possible partitions of V^f into non-empty disjoint sets. For a fixed partition $S_1, \dots, S_k$, we solve SF+ on the leader's graph (V, E^ℓ) with edge lengths defined by the leader's cost function c, using the algorithm given in Theorem 23, and obtain some forest $X \subseteq E^\ell$. Next, we compute the follower's response to X and, if there is a feasible response Y, store it together with X as a candidate for our final solution. Finally, we return the candidate solution minimizing the total weight $c(X) + c(Y)$.

Clearly, the running time of this algorithm is as desired. Moreover, it computes a feasible solution to BMST if there is one. It remains to prove that in this case the algorithm always computes a 2-approximate solution. For this, let X^* be an optimum leader's solution of the given BMST instance, together with the follower's response Y^*, and let $S_1^*, \dots, S_k^*$ be the partition of V^f corresponding to the connected components of (V, X^*). Then X^* is a (not necessarily optimum) solution for SF+ corresponding to this partition. Let X be the solution for SF+ computed by the algorithm presented above when considering this partition. Since we use a 2-approximation algorithm for SF+, we have $c(X) \leq 2c(X^*)$. Moreover, the partition of V^f induced by X is either $S_1^*, \dots, S_k^*$ or a coarser one, which implies that the follower's response Y to X is a subset of Y^* by Lemma 3, based on the deterministic behavior of the follower. Altogether, we now obtain

$$c(X) + c(Y) \leq 2c(X^*) + c(Y^*) \leq 2(c(X^*) + c(Y^*)) ,$$

which shows the desired result. ∎

5 | APPROXIMATION ALGORITHM FOR BMST

The previous section showed that already questions about fixed-parameter tractability of BMST and related problems can be hard to answer. In this section, we present a polynomial-time $(|V| - 1)$-approximation algorithm for BMST.

Theorem 25. *BMST admits a polynomial-time* $(|V| - 1)$*-approximation algorithm.*

Proof. The algorithm starts with an empty leader's solution $X := \emptyset$ and iteratively adds leader's edges to X. At the same time the graph $G = (V, E^\ell \cup E^f)$, initially given as part of the considered BMST instance, is modified in each iteration of the algorithm. More specifically, in each iteration, we first apply Corollary 4 in order to turn E^f into a forest. Then, in the current graph $G = (V, E^\ell \cup E^f)$, we compute a minimum spanning tree T according to the leader's cost function c. Let $T^\ell := T \cap E^\ell$ be the part of the spanning tree that is controlled by the leader, and add the edges in T^ℓ to X. If $T^\ell = T$ or $T^\ell = \emptyset$, we stop and output X as the leader's solution. Otherwise, we contract the edges in T^ℓ and start the next iteration.

The algorithm clearly runs in polynomial time, since we perform at most $|V| - 1$ iterations, and in each iteration we apply the polynomial reduction of Corollary 4 and compute a minimum spanning tree. It is also not hard to see that the algorithm computes a feasible solution: if it stops with $T^\ell = T$, the leader's solution X already forms a spanning tree in the original graph. Otherwise, it stops with $T^\ell = \emptyset$. In this case, the follower is able to complete X to a spanning tree, for example using the edges in T. It remains to show that the objective value of X is at most $|V| - 1$ times the optimum value.

We prove this by induction on the number $|V|$ of vertices. If $|V| = 2$, the statement is clearly true since we may assume that we only have two edges, one leader's and one follower's edge. So let us assume that for some arbitrary but fixed $n \in \mathbb{N}$

the statement is true for all graphs that have at most n vertices. Let I be an instance with $G = (V, E^\ell \cup E^f)$, where E^f is a forest and $|V| = n+1$. Let T be the spanning tree that is computed in the first iteration of the algorithm. If it stops after the first iteration, that is, if $T^\ell = T$ or $T^\ell = \emptyset$, either the leader or the follower chooses the whole tree T, while the other player chooses $\emptyset$; note that for $T^\ell = \emptyset$ the follower can only choose T as response, as E^f is cycle-free. In both cases, the leader's objective value is $c(T)$, which is clearly optimum. Otherwise, we have $c(T^\ell) \leq c(T) \leq OPT(I)$, where $OPT(I)$ denotes the value of an optimum solution to instance I. Let $\hat{I}$ with the graph $\hat{G} = (\hat{V}, \hat{E})$ be the instance that is considered in the second iteration, that is, after contracting T^ℓ. Observe that by Lemma 3, we have $OPT(\hat{I}) \leq OPT(I)$, since $\hat{I}$ arises from I by contracting certain edges of the graph. Furthermore, a solution $\hat{X}$ to $\hat{I}$ with follower's response $\hat{Y}$ can be augmented to a solution to I by simply adding T^ℓ, such that $\hat{Y}$ remains the follower's response. Finally, observe that $|\hat{V}| \leq n$ and hence the induction hypothesis holds, that is, the solution $\hat{X}$ to $\hat{I}$ produced by the algorithm is a $(|\hat{V}| - 1)$-approximation, where $|\hat{V}| - 1 \leq |V| - 2$. Putting things together, we derive that

$$\begin{aligned} c(X) + c(\hat{Y}) &= c(T^\ell) + c(\hat{X}) + c(\hat{Y}) \\ &\leq OPT(I) + (|\hat{V}| - 1)OPT(\hat{I}) \\ &\leq (|V| - 1)OPT(I) \end{aligned}$$

holds for the objective value of the leader's solution $X = T^\ell \cup \hat{X}$ returned by the algorithm. ∎

6 | BOTTLENECK OBJECTIVE

In this section, we consider variants of BMST in which one or both of the two decision makers have a bottleneck objective function instead of a sum objective, that is, they pay only for the most expensive edge in their solution. Recall that when the follower has a bottleneck objective, we have to distinguish two variants of this objective, namely minimizing either $\max_{e \in Y} d(e)$ or $\max_{e \in X \cup Y} d(e)$, that is, the follower either takes only his own edges into account or both the edges chosen by the leader and by himself. As already mentioned in Section 1, these variants are not equivalent, in contrast to the corresponding variants in the sum objective case. The problem version in which the follower considers only his own edges can be seen as a special case of the one in which he considers all edges by setting $d(e) := 0$ for all $e \in E^\ell$.

Consider the example depicted in Figure 1 and assume that the leader still has a sum objective, but the follower has a bottleneck objective. In his response to the leader's choice shown in Figure 1, the follower could now also choose the edge $\{v_3, v_5\}$ instead of the edge $\{v_3, v_6\}$. Both options are optimum from the follower's perspective. Under the optimistic assumption, the follower would choose $\{v_3, v_6\}$ because it is better for the leader. But under the pessimistic assumption, the follower would choose $\{v_3, v_5\}$ instead, increasing the leader's objective value by 4.

Shi et al. [21] showed that BMST is tractable as soon as the leader or the follower (or both) optimize a bottleneck objective. However, the general assumption in [21] is that the follower's and the leader's edge sets are not disjoint, but that the follower controls all edges, or, equivalently, that instances belong to $\mathcal{I}_{E^f \text{ all}}$. Note that, in the definition of $\mathcal{I}_{E^f \text{ all}}$, we have to require the parallel edges to have not only the same leader's, but also the same follower's cost now. Without this assumption, the tractability results do not hold anymore in general. In fact, we will see that most cases are NP-hard then. Gassner [9] developed two polynomial-time algorithms without the assumption that the follower controls all edges, namely for the cases in which the leader has a bottleneck objective and the follower either has a sum objective or a bottleneck objective, restricting to the pessimistic problem version in the latter case. In this case, however, she always assumes the follower to minimize $\max_{e \in Y} d(e)$. We generalize this result to the case of the follower's objective being $\max_{e \in X \cup Y} d(e)$ and slightly simplify her other algorithm. Moreover, our hardness results show that these are the only two cases which are polynomial-time solvable in general, unless P = NP. An overview of the different cases and results is given in Table 1.

Theorem 26. *The variant of BMST in which the leader has a bottleneck objective and the follower has a sum objective can be solved in polynomial time.*

Proof. We first present the algorithm: for each $\gamma \in C := \{c(e) | e \in E^\ell\} \cup \{0\}$, the leader considers the set $E_\gamma := \{e \in E^\ell | c(e) \leq \gamma\}$ and chooses any edge set $X_\gamma \subseteq E_\gamma$ consisting of a spanning tree in each connected component of $G_\gamma := (V, E_\gamma)$. Let Y_γ be the corresponding response of the follower and c_γ the resulting leader's objective value, where $c_\gamma := \infty$ in case the follower cannot extend X_γ to a spanning tree. Finally, choose $\gamma^* \in \operatorname{argmin}_{\gamma \in C} c_\gamma$ and return X_{γ^*} as optimum solution.

The algorithm clearly runs in polynomial time, so it remains to show that X_{γ^*} is indeed an optimum solution. For this, it suffices to show that, for any γ, choosing a solution $X \subseteq E_\gamma$ with the same bottleneck cost cannot yield a smaller objective

function value than c_γ. Since the objective function of the follower is a sum, Lemma 3 applies, thus his response Y to X is a superset of Y_γ. Now the cost of $X \cup Y$ (in the leader's bottleneck objective) is at least the cost of $X_\gamma \cup Y_\gamma$. ∎

We now turn to the case in which both leader and follower have a bottleneck objective. Then, the above algorithm does not work in general because Lemma 3 is not true in case the follower has a bottleneck objective. However, Gassner [9] showed that the same algorithm solves the problem version in which both leader and follower have a bottleneck objective, the pessimistic setting is assumed and the follower's objective is to minimize $\max_{e\in Y} d(e)$. We next prove that a generalized form of the algorithm can be used to solve the problem version with follower's objective $\max_{e\in X\cup Y} d(e)$, completing the investigation of all polynomial-time solvable cases.

Theorem 27. *The variant of BMST in which both leader and follower have a bottleneck objective and the pessimistic setting is assumed can be solved in polynomial time.*

Proof. The algorithm works as follows: for all $e_c, e_d \in E^\ell$ such that $c(e_c) \geq c(e_d)$ and $d(e_c) \leq d(e_d)$, and such that either $e_c = e_d$ or e_c and e_d are not parallel, consider the set

$$E_{e_c,e_d} := \{e \in E^\ell \,|\, c(e) \leq c(e_c) \text{ and } d(e) \leq d(e_d)\} .$$

The leader chooses any edge set $X_{e_c,e_d} \subseteq E_{e_c,e_d}$ with $e_c, e_d \in X_{e_c,e_d}$ that consists of a spanning tree in each connected component of $G_{e_c,e_d} := (V, E_{e_c,e_d})$. Let Y_{e_c,e_d} be the corresponding response of the follower and c_{e_c,e_d} the resulting objective value, where $c_{e_c,e_d} := \infty$ in case the follower cannot extend X_{e_c,e_d} to a spanning tree. If the case $c(e_c) = d(e_d) = 0$ does not occur, consider $X_0 := \emptyset$ as an additional candidate. Finally, choose (e_c^*, e_d^*) minimizing c_{e_c,e_d} and return $X_{e_c^*,e_d^*}$.

The algorithm clearly runs in polynomial time, so it remains to show that $X_{e_c^*,e_d^*}$ is indeed an optimum solution. For this, let $e_c, e_d \in E^\ell$ be such that $c(e_c)$ and $d(e_d)$ are the maximum leader's and follower's edge costs, respectively, among a leader's optimum solution $X \subseteq E^\ell$, assuming $X \neq \emptyset$. We show that X cannot have a smaller objective function value than c_{e_c,e_d}.

If X is a maximal forest in G_{e_c,e_d}, it leads to the same follower's response and hence the same objective function value as X_{e_c,e_d}. Otherwise, we may assume that $X \subset X_{e_c,e_d}$. The follower cannot achieve a better objective value when responding to X than to X_{e_c,e_d}. Hence, by the pessimistic assumption, the maximum leader's edge cost among the follower's response cannot be smaller in the former than in the latter case. Since the maximum leader's and follower's edge costs among X and X_{e_c,e_d}, respectively, are the same, it follows that X cannot lead to a smaller objective function value than X_{e_c,e_d}. ∎

Turning to the hardness results, we will reuse several ideas from Sections 3 and 4 that can be applied directly or need to be changed slightly for the bottleneck cases. First, note that in Theorem 12, the follower's sum objective can be easily replaced by a bottleneck objective, assuming the pessimistic setting:

Corollary 28. *The variant of BMST in which the leader has a sum objective, the follower has a bottleneck objective and the pessimistic setting is assumed, cannot be approximated to within a factor of* $\frac{96}{95}$ *in polynomial time, unless P = NP, even if* E^f *is a tree.*

Proof. We can use the same reduction from the Steiner forest problem as in the proof of Theorem 12. For sake of simplicity, the follower's cost function can now be defined as $d(e) := 0$ for all edges $e \in E \cup E^f$. Then the follower's objective value is always 0 and his decision is only guided by the pessimism. For this definition of d, both variants of the follower's bottleneck objective function, $\max_{e\in Y} d(e)$ and $\max_{e\in X\cup Y} d(e)$, are equivalent, hence this proof clearly holds for both of them. The pessimistic assumption about the follower's behavior here is equivalent to the behavior of a follower having a cost function of $d(e) := -c(e)$ for all $e \in E^f$ and a sum objective, which is equivalent to the problem variant which is reduced to in Theorem 12; see also Remark 15. ∎

Corollary 28 cannot be easily adapted to the optimistic assumption. However, the hardness of this case can be concluded using Theorem 18:

Theorem 29. *All variants of BMST where the leader has a sum objective and the follower has a bottleneck objective are NP-hard, even if* $c(e) = 1$ *for all* $e \in E^\ell$.

Proof. First, note that the proof of Theorem 18 works without modification if the follower has a bottleneck objective, for both the optimistic and pessimistic setting, no matter if the follower is taking only his own or all edges into account;

for the latter case, define $d(e) := 0$ for all $e \in E^\ell$. Hence, all corresponding modifications of the problem BMST-R are NP-complete as well, even if $|\overline{Y}| = 1$ and E^f forms a path on a subset of the vertex set.

We now show that BMST-R can be reduced to BMST in all these variants, assuming a sum objective for the leader, which proves the desired result. Consider an instance of BMST-R, consisting of a graph $G = (V, E^\ell \cup E^f)$, a follower's objective d and a set $\overline{Y} \subseteq E^f$. Define a leader's cost function $c : E \to \mathbb{R}_{\geq 0}$ by setting

$$c(e) := \begin{cases} 1, & \text{if } e \in E^\ell, \\ 0, & \text{if } e \in \overline{Y}, \\ |V|, & \text{if } e \in E^f \setminus \overline{Y}. \end{cases}$$

We claim that the answer to the given instance of BMST-R is yes if and only if the leader's optimum solution value in this BMST instance is at most $|V| - |\overline{Y}| - 1$.

Assume that $X \subseteq E^\ell$ is a leader's solution such that $\overline{Y}$ is the follower's response to X. Choosing X then yields a leader's objective value of $c(X) + c(\overline{Y}) = |V| - |\overline{Y}| - 1$, since X and $\overline{Y}$ form a tree and hence together have $|V| - 1$ edges. Conversely, assume that the leader can achieve an objective value of at most $|V| - |\overline{Y}| - 1$. By construction, this is only possible if the follower's response is exactly $\overline{Y}$ and the leader thus chooses $|V| - |\overline{Y}| - 1$ of her edges. Hence, the follower's response $\overline{Y}$ can be enforced. ∎

The proof of Theorem 29 does not carry over to cases in which the leader has a bottleneck objective, because the reduction from BMST-R to BMST does not work there. However, the case in which both leader and follower have a bottleneck objective, assuming the optimistic setting, is NP-hard as well, which can be shown using similar ideas as in the proof of Theorem 29.

Theorem 30. *The variant of BMST in which both the leader and the follower have a bottleneck objective and the optimistic setting is assumed is NP-hard.*

Proof. We show the result by reduction from VDST, restricted to $k = 2$. Given an instance of VDST consisting of a connected graph $G = (V, E)$ and disjoint vertex sets $S = \{s_1, \dots, s_r\}$ and $S' = \{s'_1, \dots, s'_{r'}\}$, we define an instance of BMST by adding a vertex s_0 to V, setting $E^\ell := E$,

$$E^f_0 := \{\{s_i, s_{i+1}\} | i = 1, \dots, r-1\} \cup \left\{\left\{s'_i, s'_{i+1}\right\} | i = 1, \dots, r'-1\right\},$$

and $E^f := E^f_0 \cup \left\{\left\{s_1, s'_1\right\}, \{s_0, s_1\}, \left\{s_0, s'_1\right\}\right\}$, where the leader's and follower's costs are defined as follows:

$$c(e) := \begin{cases} 0, & \text{if } e \in E^\ell \cup \left\{\left\{s_1, s'_1\right\}, \{s_0, s_1\}\right\}, \\ 1, & \text{if } e \in E^f_0 \cup \left\{\left\{s_0, s'_1\right\}\right\}, \end{cases}$$

$$d(e) := \begin{cases} 0, & \text{if } e \in E^\ell \cup E^f_0 \cup \left\{\left\{s_0, s'_1\right\}\right\}, \\ 1, & \text{if } e \in \left\{\left\{s_1, s'_1\right\}, \{s_0, s_1\}\right\}. \end{cases}$$

This construction is illustrated in Figure 7. We now show that the answer to the given instance of VDST is yes if and only if the leader's optimum value in the constructed instance of BMST is 0. Since $d(e) = 0$ for all $e \in E^\ell$, the following arguments hold for both types of follower's objective functions, $\max_{e \in Y} d(e)$ as well as $\max_{e \in X \cup Y} d(e)$.

Assume that $T, T' \subseteq E$ are vertex-disjoint trees such that T spans S and T' spans S'. Since G is connected, we may assume that $T \cup T'$ covers all vertices of G, by connecting all non-covered vertices to either T or T' arbitrarily. If the leader chooses $X := T \cup T'$ as her solution, the follower must take any two of the three edges $\{s_1, s'_1\}$, $\{s_0, s_1\}$ and $\{s_0, s'_1\}$ in order to complete X to a spanning tree. As the follower's objective value is 1 for any of these choices and we assume the optimistic setting, his response is $Y := \left\{\{s_1, s'_1\}, \{s_0, s_1\}\right\}$, resulting in a leader's objective value of 0.

For the other direction, assume that the leader can achieve an objective value of 0. This means that the follower uses the edge $\{s_0, s_1\}$ in order to connect the vertex s_0 to the original graph. Since this edge is more expensive than $\{s_0, s'_1\}$ for the follower, he will only do that if he is also forced to connect the vertices s_1 and s'_1, because otherwise, he can always achieve an objective value of 0. Hence, the leader must not connect s_1 and s'_1. Moreover, all vertices in S have to be connected by the leader, as well as all vertices in S', in order to prevent the follower from taking any edge from E^f_0. Thus, the leader's solution contains two vertex-disjoint trees spanning S and S', respectively. ∎

The proof of Theorem 30 shows that even computing any approximate solution is NP-hard, because the reduction only relies on distinguishing whether the optimum value is 0 or 1.

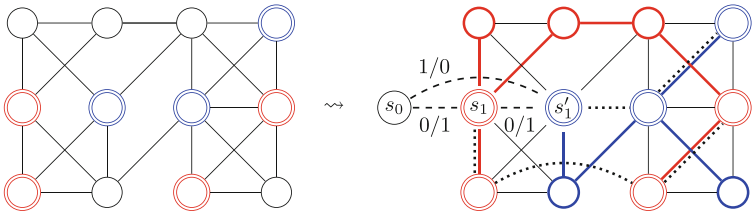

FIGURE 7 Illustration of the proof of Theorem 30. The two terminal sets S and S' in the instance of VDST are marked by red and blue vertices, respectively. Dotted lines represent edges in E_0^f having costs $1/0$, dashed lines have costs as specified. Solid edges are controlled by the leader and have costs $0/0$.

7 | CONCLUSION

In this article, we investigated the computational complexity of the BMST problem. After giving some structural insights about the problem, we proved that BMST is NP-hard, thus answering a conjecture stated by Shi et al. [21]. Furthermore, we considered the parameterized complexity of the problem in the number of edges controlled by the follower and showed that the problem is at least as hard as the shortest vertex-disjoint Steiner trees problem, parameterized by the number of terminal vertices, giving some evidence that the problem might be intractable even for a fixed number of follower's edges. Finally, we considered several variants of BMST in which at least one of the decision makers has a bottleneck objective function and gave a complete complexity classification of all these variants.

It is still open whether BMST is solvable for a fixed number of follower's edges or even fixed-parameter tractable in this parameter. Also given the close relation to the shortest vertex-disjoint paths problem, we consider this to be an interesting open question. Moreover, the approximability of BMST is an interesting question to study further, given that the best approximation ratio achieved is $|V| - 1$.

As a generalization of BMST, one could consider the bilevel minimum matroid basis (BMMB) problem, in which both decision makers together have to compute a basis of a given matroid. We think that some of our structural results can be generalized to the matroid setting. On the one hand, it would be interesting to see which of the positive results can be generalized to BMMB. Furthermore, we are curious if the negative results could be strengthened, in particular if the follower only controls a fixed number of elements.

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Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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