Convergence of an adaptive discontinuous Galerkin method for the Biharmonic problem

Dissertation

zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften
(Dr. rer. nat.)

der Fakultät für Mathematik
der Technischen Universität Dortmund

Vorgelegt von Alexander Dominicus
März 2020

Dissertation

Convergence of an adaptive discontinuous Galerkin method for the Biharmonic problem

Fakultät für Mathematik
Technische Universität Dortmund

Erster Gutachter: Prof. Dr. Christian Kreuzer

Zweiter Gutachter: Prof. Dr. Emmanuil H. Georgoulis

Mündliche Prüfung: 17.7.2020

Danksagung

Die vorliegende Dissertation wurde unter der Betreuung von Prof. Dr. Christian Kreuzer angefertigt, dem ich an dieser Stelle meinen herzlichen Dank aussprechen möchte. Er bot mir stets Unterstützung in theoretischen Fragestellungen, ließ mir dabei jedoch immer ausreichend Möglichkeiten zur Entwicklung eigener Ideen. Außerdem ermöglichte er mir das Knüpfen vieler wertvoller Kontakte. Insbesondere für den so geschaffenen fachlichen Austausch und die Diskussionen mit Prof. Dr. Emmanuil H. Georgoulis möchte ich mich an dieser Stelle bedanken.

Dr. Fernando Gaspoz hatte immer Zeit und Muße. Vielen Dank für die wertvollen Korrekturen, Anregungen und Gespräche. Außerdem möchte ich mich bei allen meinen Kollegen der TU Dortmund bedanken, insbesondere bei meiner Kollegin Dr. Korinna Rosin, die mich tatkräftig bei der Revision dieser Arbeit unterstützte und zur Not immer mit einem Kaffee zur Stelle war.

Schließlich gebührt ein großer Dank der DFG, die mit dem Projekt KR 3984/5-1 “Convergence Analysis for Adaptive Discontinuous Galerkin Methods” in hohem Maße die Verwirklichung dieser Arbeit ermöglichte.

Ein ganz besonderes Dankeschön gilt meiner Familie, die mir immer den nötigen Halt gab diese Dissertation zu verwirklichen und stets ein offenes Ohr für alle anfallenden Problemchen hatte.

Contents

List of Figures

1 Introduction

1.1 The Biharmonic equation

The Biharmonic equation is a fourth-order partial differential equation, which arises as a result of modeling phenomena encountered in problems in science and engineering. One of the earliest developments, concerning the Biharmonic equation, is the classical theory of flexure of elastic plates, which goes back, amongst others, to J. Bernoulli, Euler and Lagrange. Kirchhoff and Poisson continued the developments of the mathematical modelling of plates (compare e.g. [Poi38, Lov13, Kir50]). Their contributions have been extensively applied to the stress analysis of structural plates made of metallic and non-metallic materials. Additionally, the Biharmonic equation is heavily involved in the mathematical theory of elasticity, which is part of the mechanics of deformable media.

The Biharmonic equation is also used to model slow viscous flow problems involving Newtonian viscous flows. This theory is a particular simplification of the Navier-Stokes equation and reveals the relation between the Biharmonic equation and the Stokes equation. The developments in slow viscous flow problems are applied to many industrial problems, e.g. flow of molten metals, flow of particulate suspensions and to the modelling of bio fluid-dynamics. For a more general overview of the history and applications of the Biharmonic equation compare [Sel13].

For a bounded domain $\Omega \subset \mathbb{R}^2$ the classical formulation of the Biharmonic problem is given by

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ \nabla u \cdot \boldsymbol{n}_\Omega &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1.1}$$

where $\boldsymbol{n}_\Omega$ is the unit outward normal vector of Ω. In general, analytical solutions of the Biharmonic equation are not known explicitly. Therefore, numerical methods to approximate their solutions became important.

One branch of these numerical methods is the *conforming* finite element method (FEM) used to approximate partial differential equations stated in variational form over a function space $\mathbb{V}$. The idea is to replace the infinite dimensional function space $\mathbb{V}$ by some finite dimensional subspace $\mathbb{V}_N \subset \mathbb{V}$, $N = \dim(\mathbb{V}_N)$, in the variational formulation, leading to a discrete solution. This is called the *Ritz-Galerkin Ansatz*.

Considering the Biharmonic equation in variational form, the Ritz-Galerkin Ansatz requires C^1-conforming polynomial spaces [AFS68, Cia74, DDPS79] (so called C^1-conforming elements), which are typically very cumbersome to implement, since they require polynomial degree $\geqslant 5$ in $2d$ or constructions via macrotriangulations.

Another approach is to rewrite the Biharmonic problem into a system of second order problems and use a *mixed* finite element method (see e.g. [BBF13, dB74, Joh73] and the references therein). Moreover, non-conforming methods for the Biharmonic equation gained attraction (e.g. [BCI65, Mor68]).

One certain class of non-conforming methods for the Biharmonic equation, are the so-called C^0-interior penalty Galerkin methods ($\mathsf{C^0IPGM}$), which are based on standard continuous Lagrange finite elements of order $\geqslant 2$. These methods penalise jumps of the normal derivatives across element interfaces due to the lack of C^1-conformity; compare e.g. [BS05, EGH$^+$02, HL02].

Dropping also C^0-conformity, leads to *discontinuous Galerkin* finite element methods (DGFEM) (cf. [ABCM02, ABCM00]). These methods allow discontinuities in the trial and test space. Therefore, local element bases can be chosen independently of interelement continuity requirements, leading to very sparse stiffness matrices. Moreover, they provide a high amount of flexibility in mesh-design and adaptivity, i.e. they allow for meshes including hanging nodes and/or local varying polynomial degrees (even on polygonal, polyhedral or arbitrarily-shaped meshes); see [CDGH17, GHH06, CDG19, Don18].

In this thesis, we consider the so-called symmetric interior penalty discontinuous Galerkin discretisation (SIPDG) of (1.1.1), which goes back to Baker ([Bak77]). This method uses standard discontinuous Galerkin finite elements of order $r \geqslant 2$. Consistency is ensured and jumps of functions and normal derivatives, across element interfaces, are penalised. We refer to [GH09, SM03, SM07, MSB07, FK07, Don18] for a detailed introduction of (hp)-SIPDG methods for the Biharmonic equation.

A posteriori error estimators for the SIPDGM were developed in [GHV11] and can be used to design an adaptive SIPDGM (so-called ASIPDGM) based on the standard loop

$$\mathsf{SOLVE} \to \mathsf{ESTIMATE} \to \mathsf{MARK} \to \mathsf{REFINE}. \tag{1.1.2}$$

Convergence theory of (1.1.2), however, becomes a particular challenging problem for two reasons: First, the discontinuity penalisation terms include negative powers of the mesh-size h and thus are not necessarily monotone under refinement. Second, the lack of a conforming subspace with proper approximation properties, since a C^1-conforming subspace is only available, if the polynomial degree exceeds e.g. 4 in $2d$ (see [dBD83, GS02]).

The first issue is also present in adaptive discontinuous Galerkin methods for 2nd order problems. Here, Dörflers marking strategy typically ensures uniform error reduction [KP07, HKW09] and even optimal convergence rates [BN10]. All of these results are based on the observation that the penalty is dominated by the 'conforming parts' of the estimator, provided the penalisation parameter is chosen sufficiently large; see [Doe96, MNS00, CKNS08]. In a similar fashion the authors in [FHP15] attempt to prove convergence of $\mathsf{AC^0IPGM}$ for the biharmonic problem (1.1.1). However, the resulting argument is unclear to hold.

A different approach was taken in the convergence result for adaptive discontinuous Galerkin methods for 2nd order problems ([KG18], compare also [KG19]), motivated by the convergence results for conforming adaptive finite element methods [MSV08, Sie11]. The authors develop a new limit space of the non-conforming discrete spaces, created by the adaptive loop (1.1.2), and proof the existence of a generalised Galerkin solution in the limit space. Convergence of the sequence of discrete approximations to the generalised Galerkin solution is actually a consequence of a version of the medius analysis of Gudi [Gud10] and a local C^0-conforming reconstruction operator. The coincidence of the exact solution and the generalised Galerkin solution is finally a consequence of the marking strategy. The convergence result is not restricted to symmetric problems and holds for all penalty parameters ensuring discrete coercivity and all practically relevant marking strategies.

Very recently in [DGK19], the convergence result for adaptive discontinuous Galerkin methods for second order problems ([KG18]) has been extended to quadratic (polynomial degree $r = 2$) AC^{0}IPGM for the Biharmonic problem. The proof adresses the challenge that a conforming subspace of a Lagrange finite element space is prohibitive in AC^{0}IPGM unless the polynomial degree is chosen large enough. The convergence theory of AC^{0}IPGM uses essential new techniques based on the embedding properties of (broken) Sobolev and BV spaces. Similarly to the convergence result in [KG18], the convergence theory also holds for non-symmetric problems and, all practically relevant marking strategies and all values of the penalty parameter, for which the method is coercive. This has important consequences in practical computations: Since the condition number of the respective stiffness matrix grows as the penalty parameter grows, the magnitude of the penalisation affects the performance of iterative linear solvers. This fact becomes even more relevant for the here considered fourth order problem.

In this thesis, we extend the quadratic AC^{0}IPGM ([DGK19]) to an ASIPDGM for the Biharmonic problem (1.1.1) covering arbitrary polynomial degrees of the finite element spaces. For simplicity of the presentation, we restrict ourself to the SIPDG method. We emphasise, however, that other DG methods, e.g. semi-symmetric interior penalty Galerkin methods (cf. [SM07]) or Baker's method (cf. [Bak77]) can be treated analogously. As in the case of AC^{0}IPGM this convergence result holds for all marking strategies commonly used in praxis and all penalty parameters ensuring discrete coercivity. We stress, however, that this technique still not provides linear or even optimal convergence rates.

1.2 Overview

This thesis starts in Chapter 2 giving an analytical background from functional analysis, in order to give an overview of most notions, which are used in this work. This includes Lebesgue and Sobolev spaces and a review of the idea of distributional theory. This theory is important in the context of Sobolev spaces and in particular in view of spaces of functions of bounded variation. At the end of this chapter we introduce the fourth order model problem which is used in this thesis and we derive existence and uniqueness of the solution of this model problem.

In Chapter 3 we give some preliminaries of the discontinuous Galerkin discretisation of the model problem, including discrete function spaces, meshes and traces. Afterwards, we recall the discrete bilinear form from [GH09], leading us to the discrete problem. After that, we prove existence and uniqueness of this problem. In this chapter we also repeat the proof of an efficient and reliable a posteriori error estimator ([GHV11]) for the discrete problem. Finally, we introduce the space of functions of bounded variation and state some compactness properties of this space which will be used later on.

The following Chapter 4 introduces the model algorithm and therefore states precisely the loop (1.1.2) which produces a sequence of adaptively created discrete solutions. The rest of this chapter is therefore devoted to the proof that this sequence of discrete solutions is converging to the exact solution of the model problem and that the related a posteriori error estimators are vanishing in the limit. The latter is important in view of practical calculations.

Chapter 5 adresses numerical experiments. We examine two different model problems with different regularities of the exact solutions and analyse the related rates of convergence.

In Chapter 6 we conclude this thesis by a summary of the achieved results and consider future directions of research related to this work.

Finally, the Appendix states some results about measure theory and bubble functions which are important in our context. Furthermore, we present a counterexample to a previous version of [DGK19] which was brought to our attention by an anonymous referee to whom we wish to express our greatest gratitude.

2 Analytical background

2.1 Preliminaries

In this chapter, we recall some basics from functional analysis which is useful in this thesis. Moreover, we introduce function spaces as Lebesgue Spaces, Sobolev spaces and the space of functions of bounded variation. Furthermore, the aim of this chapter is to fix the notation and make the exposition self-contained.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ be the natural numbers including $\{0\}$. Moreover, let $\mathbb{R}$ be the set of real numbers. By $\mathbb{R}^d$ we denote the d-dimensional euclidean $\mathbb{R}$-vector space. The corresponding inner product is denoted by $v \cdot w = \sum_{i=1}^{d} v_i d_i$, for all $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$, $w = (w_1, \ldots, w_d) \in \mathbb{R}^d$ with induced norm $\|v\|_{\mathbb{R}^d} = (v \cdot v)^{1/2}$, for all $v \in \mathbb{R}^d$. If no confusion is possible we drop the subscript, i.e. we write $\| \cdot \|$. To avoid confusion we sometimes write vectors in boldface i.e. $\boldsymbol{v} \in \mathbb{R}^2$, if necessary.

By $\mathbb{R}^{d\times d}$ we denote d^2-dimensional vector space of $d \times d$-matrices. The inner product on a matrix space is denoted by $A : B = \sum_{i,j=1}^{d} A_{i_j} B_{ij}$, for all $A = (A_{ij})_{1\leqslant i,j\leqslant d} \in \mathbb{R}^{d\times d}$, $B = (B_{ij})_{1\leqslant i,j\leqslant d} \in \mathbb{R}^{d\times d}$. For $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ the tensor product is denoted by $v \otimes w \in \mathbb{R}^{d\times d}$ and defined by $(v \otimes w)_{ij} = v_i w_j$, $1 \leqslant i, j \leqslant d$. Here, we use the convention that a first order tensor can uniquely be represented by a $\mathbb{R}^d$-vector and a second order tensor can be uniquely represented by a $d \times d$ matrix if the vector space $\mathbb{R}^d$ is equipped with the Euclidean standard basis $\{e_1, \ldots, e_d\}$. Since no confusion is possible we use the same boldface notation for tensors of order two and vectors in the sequel, i.e. we simply write $\boldsymbol{T} \in \mathbb{R}^{d\times d}$, if necessary.

Let $\omega \subset \mathbb{R}^d$ be a bounded domain, then we denote by $\overline{\omega}$ the closure of ω and by $\partial\omega$ the boundary of ω. From here on, Ω denotes a bounded domain and we define the set of *continuous functions* on Ω as

$$C^0(\Omega) := \{f : \Omega \to \mathbb{R} : \ f \text{ is continuous on } \Omega\}$$

together with the supremum norm

$$\|f\|_{C^0(\Omega)} := \sup_{x\in\Omega} |f(x)| \,.$$

For $d \in \mathbb{N}$ consider a multi-index $\alpha = (\alpha_1, \ldots \alpha_d) \in \mathbb{N}_0^d$ with $|\alpha| =: \alpha_1 + \cdots + \alpha_d$. We denote the (pointwise) partial derivative by $\frac{\partial}{\partial x_i} = \partial_i$ and for $|\alpha| \leqslant k$ we write $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d}$. Moreover, for $m \in \mathbb{N}_0$ we define the space of (m-times) *differentiable functions*

$$\begin{aligned} C^m(\overline{\Omega}) := \{f : \Omega \to \mathbb{R} : \ &\partial^\alpha f \in C^0(\Omega) \text{ exists for } |\alpha| \leqslant m \\ &\text{and can be continuously extended to } \overline{\Omega}\}. \end{aligned}$$

We note that $C^m(\overline{\Omega})$ is a vector space and becomes a normed space by using the norm

$$\|f\|_{C^m(\overline{\Omega})} := \sum_{|\alpha| \leqslant m} \|\partial^\alpha f\|_{C^0(\overline{\Omega})} .$$

Note that the space of continuous functions and the space of m-times differentiable functions are *Banach spaces*, i.e. complete normed vector spaces (see [Alt16, 3.2]).

2.2 Background from functional analysis

In this section we only provide the results without proofs. For more information see, e.g. [Alt16] or [Rud91].

Let $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ and $(\mathbb{W}, \|\cdot\|_{\mathbb{W}})$ be Banach spaces. We denote the *space of all linear and continuous mappings* from $\mathbb{V}$ to $\mathbb{W}$ by $(\mathbb{V}, \mathbb{W})$. It is a Banach space with respect to the operator norm

$$\|B\|_{(\mathbb{V};\mathbb{W})} = \sup_{\|v\|_{\mathbb{V}}=1} \|Bv\|_{\mathbb{W}} .$$

The dual space of $\mathbb{V}$ is defined by

$$\mathbb{V}' := (\mathbb{V};\mathbb{R}) = \{f : \mathbb{V} \to \mathbb{R} : f \text{ is linear and continuous}\} \tag{2.2.1}$$

and we let $\langle f, v \rangle_{\mathbb{V}',\mathbb{V}} := f(v)$. It is a Banach space if it is equipped with the operator norm

$$\|f\|_{\mathbb{V}'} = \sup_{\|v\|_{\mathbb{V}}=1} \langle f, v \rangle_{\mathbb{V}' \times \mathbb{V}} =: \sup_{\|v\|_{\mathbb{V}}=1} f(v). \tag{2.2.2}$$

In the subsequent analysis we need to define dual spaces where $\mathbb{V}$ is not a normed space. To this end, we have to extend the definition (2.2.1) to topological vector spaces which we introduce now (see [Alt16, 2.11]).

Definition 2.1. A *topological vector space* is a pair (X, τ), where X is a set and τ is a system of subsets of X (the elements of τ are called *open sets*), with the following properties:

(i) $\emptyset \in \tau, X \in \tau$,

(ii) $\tilde{\tau} \subset \tau \implies \bigcup_{U \in \tilde{\tau}} U \in \tau$,

(iii) $U_1, U_2 \in \tau \implies U_1 \cap U_2 \in \tau$.

Definition 2.2. Let $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ be a Banach space.

1. We say that a sequence $\{v_k\}_{k\in\mathbb{N}} \subset \mathbb{V}$ *converges weakly* to $v \in \mathbb{V}$ (and write $v_k \rightharpoonup v$ as $k \to \infty$) if

$$\forall v' \in \mathbb{V}' : \quad \left\langle v', v_k \right\rangle_{\mathbb{V}',\mathbb{V}} = \left\langle v', v \right\rangle_{\mathbb{V}',\mathbb{V}} , \quad \text{as } k \to \infty.$$

2. We say that a sequence $\{v'_k\}_{k\in\mathbb{N}} \subset \mathbb{V}'$ *converges weakly** to $v' \in \mathbb{V}'$ (and write $v'_k \overset{*}{\rightharpoonup} v'$ as $k \to \infty$) if

$$\forall v \in \mathbb{V}\colon \quad \left\langle v'_k,\, v\right\rangle_{\mathbb{V}',\mathbb{V}} = \left\langle v',\, v\right\rangle_{\mathbb{V}',\mathbb{V}}\,, \quad \text{as } k \to \infty.$$

We state two basic properties of weakly convergent and weakly* convergent sequences.

Proposition 2.3. *Let $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ be a Banach space. Then we have that*

1. *the norm is sequentially lower-semicontinuous, i.e. for all $v_k \rightharpoonup v$ in $\mathbb{V}$ as $k \to \infty$, we have*

$$\|v\|_{\mathbb{V}} \leqslant \liminf_{k\to\infty} \|v_k\|_{\mathbb{V}}\,.$$

2. *Weakly convergent sequences and weakly* convergent sequences are bounded.*

Proof. Compare [Alt16, Chapter 8.2]. □

We emphasise that if $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ is a Banach space, then $(\mathbb{V}', \|\cdot\|_{\mathbb{V}'})$ is also Banach space and we define the *bidual* of $\mathbb{V}$ by

$$\mathbb{V}'' := (\mathbb{V}')' = (\mathbb{V}'; \mathbb{R}).$$

We note that each $v \in \mathbb{V}$ generates a function $J(v)\colon \mathbb{V}' \to \mathbb{R}$ via

$$J(v)(f) := \langle f,\, v\rangle_{\mathbb{V}'\times\mathbb{V}} = f(v), \quad f \in \mathbb{V}' \tag{2.2.3}$$

and $J(v)$ is a continuous linear functional on $\mathbb{V}'$, i.e. $J(v) \in \mathbb{V}''$. Writing J by using the dual pairing

$$\left\langle J(v),\, v'\right\rangle_{\mathbb{V}'',\mathbb{V}'} = \left\langle v',\, v\right\rangle_{\mathbb{V}',\mathbb{V}}$$

reveals that $J \in (\mathbb{V}; \mathbb{V}'')$ is an isometry and therefore injective.

The following definition deals with the case when J is also surjective, and therefore $J\colon \mathbb{V} \to \mathbb{V}''$ is an isometric isomorphism.

Definition 2.4. Let $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ be a normed space and let $J\colon \mathbb{V} \to \mathbb{V}''$ be the linear map defined in (2.2.3). Then we call

$$\mathbb{V} \text{ reflexive} \quad :\Longleftrightarrow \quad J \text{ is surjective.}$$

As a consequence we have the following weak-compactness property of reflexive Banach spaces.

Theorem 2.5 ([AF03, 1.18 Theorem])**.** *Let $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ be a reflexive Banach space. Then, its closed unit Ball*

$$\overline{B_1(0)} = \{v \in \mathbb{V}\colon\ \|v\|_{\mathbb{V}} \leqslant 1\}$$

is weakly sequentially compact, i.e. every sequence in $\overline{B_1(0)}$ has a subsequence converging weakly in $\mathbb{V}$ to a point in $\overline{B_1(0)}$.

Definition 2.6 (Bilinear form)**.** A *symmetric bilinear form* $\mathfrak{B}$ on a Banach space $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ is a mapping

$$\mathfrak{B}\colon \mathbb{V} \times \mathbb{V} \to \mathbb{R}, \tag{2.2.4}$$

which is symmetric, i.e. $\mathfrak{B}[v, w] = \mathfrak{B}[w, v]$, for all $v, w \in \mathbb{V}$, and linear in the first and second argument. The bilinear from is *continuous* if the exists a constant $C_{\mathsf{cont}} > 0$, such that

$$|\mathfrak{B}[v, w]| \leqslant C_{\mathsf{cont}} \|v\|_{\mathbb{V}} \|w\|_{\mathbb{V}}, \quad \forall v, w \in \mathbb{V}.$$

Moreover, we call the bilinear form *coercive* if there exist a constant $C_{\mathsf{coerc}} > 0$, satisfying

$$\mathfrak{B}[v, v] \geqslant C_{\mathsf{coerc}} \|v\|_{\mathbb{V}}^2 \quad \forall v \in \mathbb{V}.$$

Definition 2.7 (Scalar product and Hilbert Space)**.** A symmetric and positive definit bilinear form $\langle \cdot, \cdot \rangle_{\mathbb{V}} : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ on a vector space $\mathbb{V}$ is called *scalar product.*

A Banach space $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ is called *Hilbert space* if there exist a scalar product $\langle \cdot, \cdot \rangle_{\mathbb{V}} : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$, satisfying $\|v\|_{\mathbb{V}} = \langle v, v \rangle_{\Omega}^{1/2}$, for all $v \in \mathbb{V}$. We also use the notation $(\mathbb{V}, \langle \cdot, \cdot \rangle_{\mathbb{V}})$ to denote a Hilbert space.

The notion of a Hilbert space is crucial in the subsequent analysis. To see this let $(\mathbb{V}, \langle \cdot, \cdot \rangle_{\mathbb{V}})$ be a Hilbert space. Then the representation theorem of Riesz (compare [Alt16, p. 163]) implies that for every $\psi \in \mathbb{V}'$ there exist a unique $w \in \mathbb{V}$ such that

$$\psi(v) = \langle w, v \rangle_{\mathbb{V}', \mathbb{V}} \quad \forall v \in \mathbb{V}.$$

Hence, $\mathbb{V}$ is isomorphic to its dual space $\mathbb{V}'$ and we infer that the dual space is again a Hilbert space. Hence, by using again the representation theorem of Riesz every Hilbert space is also a reflexive space (compare[Alt16, 8.11(1)]).

Now, we are in a positon on introduce the notion of a variational problem: Let $(\mathbb{V}, \langle \cdot, \cdot \rangle_{\mathbb{V}})$ be a Hilbert space with dual $\mathbb{V}'$ and let $\mathfrak{B}[\cdot, \cdot]\colon \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ be a bilinear form. For a given $f \in \mathbb{V}'$ we want to solve the following general *variational problem*: Find $u \in \mathbb{V}$ such that

$$\mathfrak{B}[u, v] = \langle f, v \rangle \quad \forall v \in \mathbb{V}. \tag{2.2.5}$$

The following theorem provides the existence and uniqueness of a solution of (2.2.5).

Theorem 2.8 (Lax-Milgram)**.** *Let $(\mathbb{V}, \langle \cdot, \cdot \rangle_{\mathbb{V}})$ be a Hilbert space with dual $\mathbb{V}'$ and $\mathfrak{B}$ be a bilinear form on $\mathbb{V}$ which is continuous and coercive on $\mathbb{V}$ with coercivity constant $C_{\mathsf{coerc}} > 0$. Then, (2.2.5) admits for any $f \in \mathbb{V}'$ a unique solution $u \in \mathbb{V}$. Moreover, the solution is stable in the sense*

$$\|u\|_{\mathbb{V}} \leqslant \frac{1}{C_{\mathsf{coerc}}} \|f\|_{\mathbb{V}'}.$$

Proof. See [EG13, Lemma 2.2]. □

2.3 Lebesgue and Sobolev spaces

We briefly state the basic properties of two important classes of functions spaces, namely Lebesgue and Sobolev spaces. For a more detailed introduction into these function spaces the reader is referred to the books [Alt16], [Gri85], [AF03].

2.3.1 Boundary regularity

Nearly all properties of Sobolev spaces on a domain Ω depend on the regularity of the boundary $\partial\Omega =: \Gamma$. Consequently, the notions of boundary regularity has to be defined carefully. In this section we follow the lines of [GR86].

Definition 2.9. Let Ω be an bounded domain in $\mathbb{R}^d$. We say that its boundary Γ is Lipschitz-continuous (resp. of Class C^m, for some $m \in \mathbb{N}$) if for every $x \in \Gamma$ there exists a neighbourhood $U \subset \mathbb{R}^d$ of x and new coordinates $y = (y', y_d)$, where $y' = (y_1, \dots, y_{d-1})$ such that:

1. U is a hypercube in the new coordinates:
$$U = \{y \colon -a_j < y_j < a_j,\, 1 \leqslant j \leqslant d\} \subset \mathbb{R}^d.$$
2. There exists a Lipschitz-continuous function (resp. a C^m-function) ϕ defined in
$$U' = \left\{y' \colon -a_j < y_j < a_j,\, 1 \leqslant j \leqslant d-1\right\} \subset \mathbb{R}^{d-1}$$
satisfying
   a) $|\phi(y')| \leqslant \frac{a_d}{2}$ for all $\forall y' \in U'$;
   b) $\Omega \cap U = \{y \colon y_d < \phi(y')\}$ and
   c) $\Gamma \cap U = \{y \colon y_d = \phi(y')\}$.

This definition states that locally Ω is below the graph of some function ϕ, the boundary Γ is represented by the graph of ϕ and the regularity of Γ is determined by the regularity of the function ϕ. In particular, the continuity of ϕ implies that the Ω is never on both sides of Γ at any point of Γ (e.g. think of domains with a cuts or cusps). However, this definiton allows boundaries with corners. For example bounded polygons in $\mathbb{R}^2$ or bounded polyhedrons in $\mathbb{R}^3$. In the sequel we will say that Ω is a Lipschitz domain, meaning that the Ω is a bounded domain with Lipschitz-continuous boundary. Note that on a Lipschitz domain, a unit exterior normal vector, which we denote by $\boldsymbol{n}_\Omega$ or simply $\boldsymbol{n}$ is well defined for almost every $x \in \partial\Omega$ ([Gri85, Chapter 1.5]).

2.3.2 Definitions and basic properties

In this section we give some basic definitions and results from standard theory of partial differential equations. Let $\omega \subset \mathbb{R}^d$ be an Lebesgue-measurable set and let $f \colon \omega \to \mathbb{R}$ be a measurable function. We denote the Lebesgue integral of f over ω by $\int_\omega f \,\mathrm{d}x$ (compare [Bar14] for a detailed introduction of the Lebesgue integral).

We define $L^1_{\mathrm{loc}}(\Omega)$ to be the space of locally integrable functions, i.e the set of all measurable functions $f\colon \Omega \to \mathbb{R}$, such that

$$\int_K f \,\mathrm{d}x < \infty$$

for all compact subsets $K \subset \mathbb{R}^d$.

Definition 2.10 (Lebesgue space)**.** For $1 \leqslant p \leqslant \infty$, let

$$L^p(\Omega) := \left\{ f\colon \Omega \to \mathbb{R}\colon\ f \text{ is measurable and } :\ \|f\|_{L^p(\Omega)} < \infty \right\},$$

where

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p \,\mathrm{d}x \right)^{1/p}, \quad \text{for } 1 \leqslant p < +\infty,$$
$$\|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| := \inf\left\{ M \geqslant 0\colon\ |f(x)| \geqslant M \text{on } \Omega \right\}.$$

We note that Lebesgue spaces and the space $L^1_{\mathrm{loc}}(\Omega)$ are actually defined as equivalence classes of functions, whose values differ only on a set of Lebesgue measure zero.

For $1 \leqslant p \leqslant \infty$ we have that $L^p(\Omega)$ is a Banach space if its equipped with the $\|\cdot\|_{L^p(\Omega)}$-norm; compare [AF03, p. 29].

In the case $p = 2$, $L^2(\Omega)$ is a (real) Hilbert space when it is equipped with the inner product ([AF03, p. 31])

$$\langle v,\, w\rangle_{L^2(\Omega)} := \int_\Omega vw \,\mathrm{d}x$$

and the induced norm $\|\cdot\|_{L^2(\Omega)}$. In order to shorten the notation we also write $\langle v,\, w\rangle_\Omega$ and $\|\cdot\|_\Omega$ in the case $p = 2$.

For $1 \leqslant p \leqslant \infty$, we denote by p' the *conjugate* of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$, with $p' = 1$ if $p = +\infty$ and $p' = +\infty$ if $p = 1$. This leads us to Hölder's inequality (compare e.g. [Bar01, p. 404]), which states that for $v \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ there holds $vw \in L^1(\Omega)$ and

$$\int_\Omega vw \,\mathrm{d}x \leqslant \|v\|_{L^p(\Omega)} \|w\|_{L^{p'}(\Omega)}.$$

In particular, for $p = p' = 2$ we end up with the Cauchy-Schwarz inequality, namely for all $v, w \in L^2(\Omega)$ we have

$$\langle v,\, w\rangle_{L^2(\Omega)} \leqslant \|v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}.$$

We will also use *Young's inequality*: For $a, b \geqslant 0$ and $1 < p < \infty$ we have that

$$ab \leqslant \frac{1}{p} a^p + \frac{1}{p'} b^{p'}, \tag{2.3.1}$$

compare [Alt16, (3-11)].

for $1 \leqslant p < \infty$ the dual space of $L^p(\Omega)$ can be identified with $L^{p'}(\Omega)$; see [Alt16, 6.12]. As a consequence $L^p(\Omega)$ is reflexive if $1 < p < +\infty$. However, the spaces $L^1(\Omega)$ and $L^\infty(\Omega)$ are not reflexive. This is due to the fact that the dual of $L^1(\Omega)$ is $L^\infty(\Omega)$ but the dual of $L^\infty(\Omega)$ is the space of signed Borell measures, which is strictly larger than $L^1(\Omega)$ (compare [AF03, Chapter 2]).

For a function $f\colon \Omega \to \mathbb{R}$ we define the *support* of f by

$$\operatorname{supp}(f) := \overline{\{x \in \Omega\colon f(x) \neq 0\}}.$$

Moreover, we define the *space of continuos function with compact support by*

$$C_0(\Omega) := C_0^0(\Omega) := \left\{f \in C^0(\Omega)\colon \operatorname{supp}(f) \subset \Omega\right\}.$$

Let

$$C^\infty(\Omega) := \bigcap_{m \in N} C^m(\Omega)$$

be the function space of infinitely differentiable functions. The space of *test functions* on Ω is then defined by

$$C_0^\infty(\Omega) := \{\varphi \in C^\infty(\Omega)\colon \operatorname{supp}(\varphi) \subset \Omega\}.$$

Moreover, we set

$$C_0^\infty(\overline{\Omega}) = \left\{\varphi|_\Omega\colon \varphi \in C_0^\infty(\mathbb{R}^d)\right\}.$$

We remark that there exist a certain topology τ (which is called the *canonical LF topology*; compare [Alt16, 5.20, 5.21]) such that $(C_0^\infty(\Omega), \tau)$ is a topological vector space. Henceforth, the topological vector space $(C_0^\infty(\Omega), \tau)$ will simply denoted by $\mathcal{D}(\Omega)$.

The dual space of the topological vector space $\mathcal{D}(\Omega)$ is defined by

$$\mathcal{D}(\Omega)' = \{T\colon \mathcal{D}(\Omega) \to \mathbb{R}\colon T \text{ is linear and continuous}\} \tag{2.3.2}$$

and will play crucial role crucial role in the following definition; compare [Alt16, 5.17].

Definition 2.11 (Distributions)**.** Let $T\colon C_0^\infty(\Omega) \to \mathbb{R}$ be linear.

1. We call the map T a *distribution* on Ω, and use the notation $T \in \mathcal{D}'(\Omega)$, if for all open sets $D \subset \Omega$ there exist a constant C_D and a $k_D \in \mathbb{N}_0$ such that

   $$|T(\varphi)| \leqslant C_D \|\varphi\|_{C^{k_D}(\overline{D})} \quad \text{for all } \varphi \in C_0^\infty(\Omega) \text{ with } \operatorname{supp}(\varphi) \subset D.$$

   If $k = k_D$ can be chosen independently of D, then k (if chosen minimally) is called the *order* of T.

2. For all multi-indices s, the distributional-derivative $\partial^s T$ is the linear map $\partial^s T\colon C_0^\infty(\Omega) \to \mathbb{R}$ defined by

   $$(\partial^s T)(\varphi) := (-1)^{|s|} T(\partial^s \varphi), \qquad \varphi \in C_0^\infty(\Omega).$$

3. If T is a distribution, then so is $\partial^s T$ for all multi-indices s. If T is a distribution of order k, then $\partial^s T$ is a distribution of order $k + |s|$.

Recalling Definition 2.11 we have the space of distributions $\mathcal{D}'(\Omega)$ on the one hand and the dual space $\mathcal{D}(\Omega)'$ from (2.3.2) on the other hand. However, it is possible to proof that $\mathcal{D}(\Omega)' = \mathcal{D}'(\Omega)$ (see [Alt16, 5.23]), i.e. T is a distribution if and only if $T\colon \mathcal{D}(\Omega) \to \mathbb{R}$ is linear and continuous with respect to the topology chosen on $C_0^\infty(\Omega)$, (i.e. $T \in (\mathcal{D}(\Omega))'$).

Note that every function in $f \in L^1_{\mathrm{loc}}(\Omega)$ can be uniquely identified with the distribution of order zero

$$T_f\colon \mathcal{D}(\Omega) \ni \varphi \mapsto T_f(\varphi) = \int_\Omega f\varphi \,\mathrm{d}x,$$

see [Alt16, 4.22].

This observation is crucial since it leads us to the notion of the distributional derivative of a *function*: Regard the distribution T_f as defined above, then Definition 2.11 reveals that $T_f \in \mathcal{D}'(\Omega)$ is differentiable in the following sense: For $1 \leqslant i \leqslant 2$ the distributional derivative $D_i T_f \in \mathcal{D}(\Omega)'$ is defined by

$$D_i T_f\colon \mathcal{D}(\Omega) \ni \varphi \mapsto D_i T_f(\varphi) := -T_f(\frac{\partial \varphi}{\partial x_i})$$

and more generally for a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$, the distribution $D^\alpha T_f = D_1^{\alpha_1} \dots D_d^{\alpha_d} T_f$ is defined by

$$D^\alpha T_f\colon \mathcal{D}(\Omega) \ni \varphi \mapsto D^\alpha T_f(\varphi) = (-1)^{|\alpha|} T_f(\partial^\alpha \varphi).$$

In the sequel, we will ease the notation by identifying $f \in L^1_{\mathrm{loc}}(\Omega)$ with $T_f \in \mathcal{D}(\Omega)'$. Note that T_f is well defined since $f \mapsto T_f$ is injective, i.e. f can be reconstructed from T_f ([Alt16, 5.16(2)]). Moreover, we write $D^\alpha f$ for the distributional derivative of f.

Definition 2.12 (Sobolev Spaces)**.** Let $n \in \mathbb{N}_0$ and $1 \leqslant p \leqslant +\infty$. We define the Sobolev space $W^{n,p}(\Omega)$ by

$$W^{n,p}(\Omega) := \{u \in L^p(\Omega)\colon D^\alpha u \in L^p(\Omega), \,\forall\, |\alpha| \leqslant n\},$$

where we understand the derivatives in the distributional sense (and henceforth call them 'weak derivatives'). Moreover, we set $W^{0,p}(\Omega) := L^p(\Omega)$.

We equip the space $W^{n,p}(\Omega)$ with the norm

$$\|u\|_{W^{n,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leqslant n} \|D^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p} & \text{for } 1 \leqslant p < +\infty, \\ \max_{|\alpha| \leqslant n} \|D^\alpha u\|_{L^\infty(\Omega)} & \text{for } p = +\infty. \end{cases}$$

The space $(W^{n,p}(\Omega), \|\cdot\|_{W^{n,p}(\Omega)})$ is a Banach space. Moreover for $1 < p < +\infty$ the space $W^{n,p}(\Omega)$ is reflexive; see [Alt16, 8.11(3)].

For the case $p = 2$ we denote $W^{n,2}(\Omega) =: H^n(\Omega)$ and emphasise that $H^n(\Omega)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{n,\Omega} := \sum_{|\alpha|\leqslant n} \int_\Omega D^\alpha u D^\alpha v \, \mathrm{d}x,$$

and induced norm $\|\cdot\|_{H^n(\Omega)}$; see [AF03, 3.6 Theorem].

Moreover, we denote

$$H_0^n(\Omega) = \overline{\mathcal{D}(\Omega)}^{H^n(\Omega)},$$

i.e. the closure of $\mathcal{D}(\Omega)$ for the norm $\|\cdot\|_{H^n(\Omega)}$

The following estimate is crucial in the context of $H_0^n(\Omega)$-Sobolev spaces; see [Alt16, 6.7].

Lemma 2.13 (Poincaré-inequality)**.** *If Ω is open and bounded, then there exists a constant $C_0 > 0$, which depends on Ω, such that*

$$\|v\|_\Omega^2 \leqslant C_0 \int_\Omega |\nabla v|^2 \, \mathrm{d}x \quad \forall u \in H_0^1(\Omega).$$

Throughout this thesis we use the fact that on the space $H_0^n(\Omega)$ the following semi-norm

$$|u|_{H^n(\Omega)} := \left(\sum_{|\alpha|=n} \|D^\alpha u\|_{L^2(\Omega)}^2 \right)^{1/2}$$

is equivalent to the Sobolev norm $\|u\|_{H^n(\Omega)}$. Indeed, for the case $n = 1$ this is a consequence of the Poincaré inequality. Using an induction argument, we can therefore conclude that for any $n \in \mathbb{N}$ the semi-norm $|\cdot|_{H^n}$ is a norm on $H_0^n(\Omega)$ and in particular $(H^n(\Omega), |\cdot|_{H^n(\Omega)})$ is a Banach spaces.

For $n = 2$ the dual space of $H_0^2(\Omega)$ is denoted by $H^{-2}(\Omega)$. We define a norm on $H^{-2}(\Omega)$ by

$$\|f\|_{H^{-2}(\Omega)} := \sup_{\substack{v \in H_0^2(\Omega) \\ v \neq 0}} \frac{\langle f, v\rangle_{L^2(\Omega)}}{\|v\|_{H^2(\Omega)}}.$$

In the subsequent analysis we make use of several embedding Theorems wich we state now (compare [Alt16, 10.9]).

Theorem 2.14 (Embedding of Sobolev Spcaces)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $n_1, n_2 \in \mathbb{N}_0$. Moreover, let $1 \leqslant p_1 < \infty$ and $1 \leqslant p_2 < \infty$.*

1. *If*

$$n_1 - \frac{d}{p_1} \geqslant n_2 - \frac{d}{p_2}, \quad \textit{and} \quad n_1 \geqslant n_2,$$

then there exists a continuous embedding

$$\mathrm{id}\colon W^{n_1,p_1} \to W^{n_2,p_2}.$$

Hence, there exists a constant $C > 0$*, depending on* $d, \Omega, n_1, p_1, n_2, p_2$ *such that*

$$\|u\|_{W^{n_2,p_2}} \leqslant C \|u\|_{W^{n_1,p_1}} \quad \forall u \in W^{n_1,p_1}.$$

2. *If*

$$n_1 - \frac{d}{p_1} > n_2 - \frac{d}{p_2}, \quad \textit{and} \quad n_1 > n_2,$$

then the identity mapping

$$\mathrm{id}\colon W^{n_1,p_1} \to W^{n_2,p_2}.$$

is a compact operator. This means, that for every bounded sequence in W^{n_1,p_1}*, there exists a converging subsequence in* W^{n_2,p_2}*.*

Remark 2.15. *Note that all definitions of this section can be extended to vector-valued functions with the following convention: A function* $f\colon \Omega \to \mathbb{R}^m$*,* $m \in \mathbb{N}$ *is located in the space* $L^p(\Omega)^m$ *if every of its component functions is located in the space* $L^p(\Omega)$*. We obtain a Banach space by replacing*

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p \,\mathrm{d}x\right)^{1/p}, \quad \textit{for } 1 \leqslant p < +\infty,$$

by

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega \|f\|_{\mathbb{R}^m}^p \,\mathrm{d}x\right)^{1/p}, \quad \textit{for } 1 \leqslant p < +\infty,$$

in Definition 2.10, for a vector norm $\|\cdot\|_{\mathbb{R}^m}$ *on* $\mathbb{R}^m$*. The same holds for the case* $p = \infty$ *and Sobolev spaces of vector-valued functions are generalised the same way.*

2.4 The model problem

From here on $\Omega \subset \mathbb{R}^2$ denotes a bounded polygonal domain with Lipschitz boundary $\partial\Omega$. We consider the following Biharmonic problem

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{2.4.1}$$

with right-hand side $f \in L^2(\Omega)$ and Dirichlet boundary values (*essential boundary values*)

$$u = \frac{\partial u}{\partial \boldsymbol{n}_\Omega} = 0 \quad \text{on } \Gamma.$$

To deduce the weak formulation of (2.4.1) we multiply both sides with a function $v \in H_0^2(\Omega)$ and perform integration by parts twice. Now, taking into account the boundary values, the weak formulation reads: Find $u \in H_0^2(\Omega)$, such that

$$\mathfrak{B}[u, v] = F(v), \quad \forall v \in H_0^2(\Omega), \tag{2.4.2}$$

for the bilinear form

$$\mathfrak{B}[w, v] := \int_\Omega D^2 w \colon D^2 v \, \mathrm{d}x = \int_\Omega \sum_{i,j=1}^{2} \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \, \mathrm{d}x$$

and right hand side $F(v) := \int_\Omega f v \, \mathrm{d}x$. By using the Cauchy-Schwarz inequality we have that the bilinear form $\mathfrak{B}[\cdot, \cdot]$ is continuous on $H_0^2(\Omega)$, i.e. there exists a constant C_1 such that

$$|\mathfrak{B}[v, w]| \leqslant C_1 \|v\|_{H_0^2(\Omega)} \|w\|_{H_0^2(\Omega)}, \quad \forall v, w \in H_0^2(\Omega).$$

Moreover, the Poincaré inequality (Lemma 2.13) implies that $\mathfrak{B}(\cdot, \cdot)$ is also coercive on $H_0^2(\Omega)$, i.e. there exist a constant C_2 with

$$\mathfrak{B}[v, v] \geqslant C \|v\|_{H_0^2(\Omega)}^2 \quad \forall v \in H_0^2(\Omega).$$

Finally, we emphasise that the space $L^2(\Omega)$ is a subspace of $H^{-2}(\Omega)$ in the sense that for $f \in L^2(\Omega)$ the mapping

$$v \mapsto \int_\Omega f v \, \mathrm{d}x$$

belongs to $H^{-2}(\Omega)$ and we have $\|f\|_{H^{-2}(\Omega)} \leqslant \|f\|_\Omega$, due to the Cauchy-schwarz inequality. Whence, we infer from the Lax-Milgram Theorem 2.8 the existence of a unique solution $u \in H_0^2(\Omega)$, which solves (2.4.2) and the solution u is bounded by $\|u\|_{H_0^2(\Omega)} \leqslant C_3 \|f\|_\Omega$.

Note that the additional regularity $f \in L^2(\Omega)$ is necessary due to the a posteriori analysis below. However, despite additional regularity of f, the solution u does *not* belong to $H^4(\Omega)$ in general (e.g. compare the model problem in the numerical example 2, Chapter 5), due to our restriction to polygonal domains. For details of regularity theory for the Biharmonic problem compare e.g. [Gri85, Gri92, Dau06, BRL80]. Moreover, for the generalisation to various non-homogenous boundary conditions we refer to [Gri85, GR86, GGS10].

3 Discontinuous Galerkin Finite Element Methods

In the first part of this chapter we recall some common definitions, related to Discontinuous Galerkin finite element spaes which are used throughout the rest of this thesis. Afterwards, in Section 3.1.1 we derive a discontinuous Galerkin discretisation for the model problem, leading to the SIPDG-problem which is defined in Section 3.2. In this section, we also discuss existence and uniqueness of the SIPDG-problem. In the following Section 3.3 we introduce lifting operators, ensuring that the resulting discrete bilinear form can be applied to functions of lower regularity.

In Section 3.4 we recall the a posteriori error estimator developed in [GHV11] and state the proofs of upper and lower bounds. Finally, in Section 3.5 we precisely formulate the embedding of discontinuous Galerkin spaces into BV-spaces.

3.1 Discrete function spaces, meshes and traces

Let $\mathcal{T}$ be a conforming (i.e. not containing any hanging nodes) subdivision of $\Omega \subset \mathbb{R}^2$ into closed disjoint triangular elements $K \in \mathcal{T}$ such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}} K$. We assume that $\mathcal{T}$ is constructed by affine linear bijections $F_K \colon \hat{K} \to K$ (with non vanishing Jacobian), defined on the *reference element*

$$\hat{K} = \{(x,y)\colon 0 \leqslant x \leqslant 1,\, 1 \leqslant y \leqslant 1-x\} \subset \mathbb{R}^2.$$

Let $\mathcal{F}_\mathcal{T} := \mathcal{F}(\mathcal{T})$ be the set of one-dimensional faces F, associated with the subdivision $\mathcal{T}$ (including $\partial\Omega$), which are straight lines, due to the restriction to triangular elements. Moreover, we define $\mathring{\mathcal{F}}_\mathcal{T}$ to be the subset of interior sides only and $\mathcal{F}_\mathcal{T}^b := \mathcal{F}_\mathcal{T} \backslash \mathring{\mathcal{F}}_\mathcal{T}$ be the boundary faces. The corresponding *skeletons* are then defined by

$$\begin{aligned}\Gamma_\mathcal{T} &:= \bigcup\{F \in \mathcal{F}_\mathcal{T}\},\\ \mathring{\Gamma}_\mathcal{T} &:= \bigcup\{F \in \mathring{\mathcal{F}}_\mathcal{T}\} \quad \text{and}\\ \Gamma_\mathcal{T}^b &:= \bigcup\{F \in \mathcal{F}_\mathcal{T}^b\},\end{aligned}$$

respectively.

Moreover, let $h_\mathcal{T} : \Omega \to \mathbb{R}_{\geqslant 0}$ the piecewise constant *mesh-size* function which is defined by

$$h_\mathcal{T}(x) := \begin{cases} h_K := |K|^{1/d}, & x \in K\backslash\partial K,\\ h_F := |F|^{1/(d-1)}, & x \in F \in \mathcal{F}.\end{cases} \tag{3.1.1}$$

Additionally, let $\overline{h}_K = \operatorname{diam}(K)$ be the diameter of K and

$$\underline{h}_K = \sup\{r \colon B_r \subset K \text{ is a Ball of radius r}\}$$

the diameter of the largest inscribed ball in K.

We assume that $\mathcal{T}$ is derived by iterative or recursive newest vertex bisection of an initial conforming mesh $\mathcal{T}_0$; compare with [Bae91, Kos94, Mau95]. By $\mathbb{G}$ we denote the family of *shape-regular* triangulations consisting of such refinements of $\mathcal{T}_0$, i.e. there exists a constant $C_{\mathsf{reg}} > 0$

$$\frac{\overline{h}_K}{\underline{h}_K} \leqslant C_{\mathsf{reg}} \quad \forall K \in \mathcal{T}, \quad \forall \mathcal{T} \in \mathbb{G}.$$

Additionally, we note the following estimate of the Jacobian determinant of $F_K \colon \hat{K} \to K$

$$C_{J,1}\underline{h}_K \leqslant |\det DF_K| \leqslant C_{J,2}\overline{h}_K, \tag{3.1.2}$$

for constants $C_{J,1}, C_{J,2} > 0$, only depending on the $\mathcal{T}_0$ (resp. C_{reg}; see [Cia02a, Theorem 3.1.3]). For $\mathcal{T}, \mathcal{T}_\star \in \mathbb{G}$, we write $\mathcal{T}_\star \geqslant \mathcal{T}$ whenever $\mathcal{T}_\star$ is a refinement of $\mathcal{T}$. We recall that refinement by bisection has the following property: Let $\mathcal{T}_\star$ be a refinement of $\mathcal{T} \in \mathbb{G}$. Then, we have that the mesh-size function is monotone in the interior of elements, i.e.

$$\forall K \in \mathcal{T}_\star \backslash \mathcal{T}\colon \quad h_{\mathcal{T}_\star}|_{\mathring{K}} \leqslant 2^{-1/2} h_{\mathcal{T}}|_{\mathring{K}}, \tag{3.1.3}$$

whereas $h_{\mathcal{T}_\star}|_F = h_{\mathcal{T}}|_F$ is possible for $F \subset K$, compare the Definition of the mesh-size function (3.1.1).

For $r \geqslant 2$, we define the *Discontinuous Galerkin finite-element space* by

$$\mathbb{V}(\mathcal{T}) := \mathbb{P}_r(\mathcal{T}) \quad \text{with} \quad \mathbb{P}_r(\mathcal{T}) := \{v \in L^2(\Omega) \colon v|_K \in \mathbb{P}_r(K) \ \forall K \in \mathcal{T}\},$$

where we use the notation $\mathbb{P}_r(\mathcal{M})$ for a subset $\mathcal{M} \subset \mathcal{T}$. In the same vein, we define $\mathbb{V}(\mathcal{T})^2 := \left\{v \in L^2(\Omega)^2 \colon v|_K \in \mathbb{P}_r(K)^2\right\}$. We note that the dimension of $\mathbb{V}(\mathcal{T})$ equals the *global degrees of freedom* of $\mathbb{V}(\mathcal{T})$ and is given by $N = \dim(\mathbb{V}(\mathcal{T})) = (\#\mathcal{T}) \times \dim(\mathbb{P}_r(K))$, due to the fact that the restriction of a function $v \in \mathbb{V}(\mathcal{T})$ to each element can be chose independently of its restriction to other elements. Additionally, we define the L^2-projection onto $\mathbb{V}(\mathcal{T})$, i.e. $\Pi \colon L^2(\Omega) \to \mathbb{V}(\mathcal{T})$ for any $v \in L^2(\Omega)$ as

$$\langle \Pi v, w\rangle_\Omega = \langle v, w\rangle_\Omega, \quad \text{for all } v \in \mathbb{V}(\mathcal{T}). \tag{3.1.4}$$

Standard estimates reveal that the projection Π is stable in the sense that $\|\Pi v\|_\Omega \leqslant \|v\|_\Omega$, for all $v \in L^2(\Omega)$. Here $\|\cdot\|_\Omega$ denotes the L^2-norm; see Section 2.3.2.

In view of regularity we emphasise that in general have $\mathbb{V}(\mathcal{T}) \not\subset H_0^1(\Omega)$ and thus also $\mathbb{V}(\mathcal{T}) \not\subset H_0^2(\Omega)$. On the other hand, since each function $V \in \mathbb{V}(\mathcal{T})$ is locally a polynomial on each element $K \in \mathcal{T}$, we have, however

$$\mathbb{V}(\mathcal{T}) \subset H^n(\mathcal{T}) := \{v \in L^2(\Omega) \colon v|_K \in H^n(K), \ \forall K \in \mathcal{T}\},$$

for all $n \in \mathbb{N}_0$. For $v \in H^m(\mathcal{T})$, $m \geqslant 2$, we define the *piecewise gradient* $\nabla_{\mathtt{pw}} v$ and the *piecewise Hessian* $D^2_{\mathtt{pw}} v$ by

$$\begin{aligned}(\nabla_{\mathtt{pw}} u)|_K &:= \nabla(u|_K) \in L^2(K)^2 \qquad \forall K \in \mathcal{T},\\ (D^2_{\mathtt{pw}} u)|_K &:= D^2(u|_K) \in L^2(K)^{2\times 2} \qquad \forall K \in \mathcal{T}.\end{aligned}$$

Note that for $v \in H^m(\mathcal{T})$, $m \geqslant 4$, the function v as well as all relevant derivatives $\nabla v, D^2 v$ and $\nabla \cdot D^2 v$ are measurable on element boundaries ∂K, $K \in \mathcal{T}$ and the corresponding L^2-norms are defined.

Let $\mathcal{N}_\mathcal{T}$ be the nodal degrees of freedom of $\mathbb{V}(\mathcal{T})$ and be $\mathcal{Z}_\mathcal{T}$ be the set of nodes (*Lagrange nodes*) associated with the degrees of $\mathcal{N}_\mathcal{T}$, i.e. we identify a node $z \in \overline{\Omega}$ with its degree of freedom $N_z \in \mathcal{N}_\mathcal{T}$. For $z \in \overline{\Omega}$, we denote its neighbourhood by $N_\mathcal{T}(z) := \{K' \in \mathcal{T} \mid z \in K'\}$, and the corresponding domain is defined by $\omega_\mathcal{T}(z) := \Omega(N_\mathcal{T}(z))$. Hereafter, we use $\Omega(X) := \bigcup\{K \mid K \in X\}$ for a collection of elements X. With a little abuse of notation for an element $K \in \mathcal{T}$ we define its *jth neighbourhood* recursively by

$$N^j_\mathcal{T}(K) := \left\{K' \in \mathcal{T} \mid K' \cap N^{j-1}_\mathcal{T}(K) \neq \varnothing\right\},$$

where we set $N^0_\mathcal{T}(K) := K$, and the corresponding domain by $\omega^j_\mathcal{T}(K) := \Omega(N^j_\mathcal{T}(K))$. We shall skip the superindex if $j = 1$, e.g. we write $N_\mathcal{T}(K) = N^1_\mathcal{T}(K)$ and $\omega_\mathcal{T}(K) = \omega^1_\mathcal{T}(K)$ for simplicity. For a side $F \subset \mathcal{F}_\mathcal{T}$, we set $N_\mathcal{T}(F) := \{K \in \mathcal{T} \colon F \subset K\}$ with corresponding domain

$$\omega_\mathcal{T}(F) := \bigcup \{K \in N_\mathcal{T}(F)\}.$$

We extend the above definitions to subsets $\mathcal{M} \subset \mathcal{T}$ setting

$$N^j_\mathcal{T}(\mathcal{M}) := \{K \in \mathcal{T} \colon \exists K' \in \mathcal{M} \text{ such that } K \in N^j_\mathcal{T}(K')\}.$$

In the sequel we use the notation $a \lesssim b$ when $a \leqslant Cb$ for a constant $C > 0$ which is independent of the actual element K, but depends on given parameters, e.g. like the polynomial degree r, the dimension d or the parameter n of a Sobolev space $H^n(K)$.

Note that the shape regularity and conformity of $\mathbb{G}$ implies local quasi-uniformity, i.e.

$$\sup_{\mathcal{T}\in\mathbb{G}} \max_{K' \in N_\mathcal{T}(K)} \frac{|K|}{|K'|} \lesssim 1 \qquad \text{and} \qquad \sup_{\mathcal{T}\in\mathbb{G}} \max_{K\in\mathcal{T}} \# N_\mathcal{T}(K) \lesssim 1, \tag{3.1.5}$$

see [BNQ+12, Section 1.5].

In order to formulate the discrete bilinear form, we first need to introduce the so-called *jump* and *mean* of a function on the skeleton $\Gamma_\mathcal{T}$. In fact, for $v \in \mathbb{V}(\mathcal{T})$, we define

$$[\![v]\!]_F := \begin{cases} v|_{K_1} - v|_{K_2}, & F \in \mathring{\mathcal{F}}_\mathcal{T},\\ v|_K, & F \in \mathcal{F}^b_\mathcal{T}, F \subset K,\end{cases} \tag{3.1.6}$$

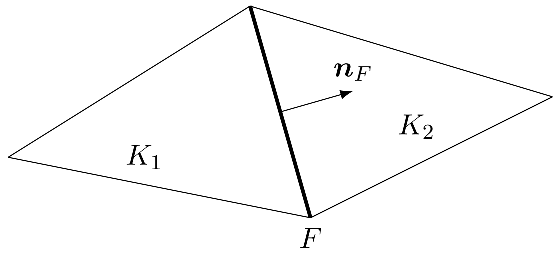


Figure 3.1: Normal direction with respect to a side F.

$$\{\!\{v\}\!\}_F := \begin{cases} \frac{1}{2}(v|_{K_1} + v|_{K_2}), & F \in \mathring{\mathcal{F}}_{\mathcal{T}}, \\ v|_K, & F \in \mathcal{F}^b_{\mathcal{T}}, F \subset K, \end{cases}$$

where $F \in \mathring{\mathcal{F}}_{\mathcal{T}}$ with $F = K_1 \cap K_2$ and $K_1, K_2 \in \mathcal{T}$ are the two adjacent elements of F (see Figure 3.1). Jump and mean across $F \in \mathcal{F}_{\mathcal{T}}$ are defined analogously for vector fields $\boldsymbol{w} \in \mathbb{V}(\mathcal{T})^2$ and tensorfields $\boldsymbol{T} \in \mathbb{V}(\mathcal{T})^{2\times 2}$, i.e. the above jump and average operators act component wise in these cases.

Remark 3.1. *We note that for $F \in \mathring{\mathcal{F}}_{\mathcal{T}}$ the definition of $[\![\cdot]\!]\,|_F$ in* (3.1.6) *in general depends on the choice of the ordering of the elements K_1, K_2. However, in combination with face normals the definition of jump terms become symmetric. To be precise, let $\boldsymbol{n}_{K_1}$ and $\boldsymbol{n}_{K_2}$ be the unit outward normal corresponding to ∂_{K_1} and ∂_{K_2} and define $\boldsymbol{n}_F := \boldsymbol{n}_{K_1} = -\boldsymbol{n}_{K_2}$ (compare Figure 3.1). Then, we have*

$$[\![v]\!]\,\boldsymbol{n}_F = v|_{K_1}\boldsymbol{n}_{K_1} + v|_{K_2}\boldsymbol{n}_{K_2},$$

i.e. K_1, K_2 play symmetric roles. The same holds true if v is replaced by a vector valued function $\boldsymbol{w} \in \mathbb{V}(\mathcal{T})^2$. In this case we have

$$[\![\boldsymbol{w}]\!] \cdot \boldsymbol{n}_F = \boldsymbol{w}|_{K_1} \cdot \boldsymbol{n}_{K_1} + \boldsymbol{w}|_{K_2} \cdot \boldsymbol{n}_{K_2}$$

which is again independent of the ordering of K_1 and K_2.

To simplify the notation, we sometimes drop the subscripts of the unit outward normal, i.e. when no confusion is possible we simple write $\boldsymbol{n}$ instead of $\boldsymbol{n}_K$ or $\boldsymbol{n}_F$.

3.1.1 Derivation of the discontinuous Galerkin finite element method

This section we loosely follow the same path as in [DPE12] (compare also [SM03, Section 3]) in order to derive our discontinuous Galerkin bilinear form $\mathfrak{B}_{\mathcal{T}}$. A different approach can be found in [GH09]. Here, we give a full derivation of the discrete bilinear for two reasons: The first reason is, that the presentation of this thesis should be self-contained. Second, we aim to derive slightly different *formulation* (related to a bilinear form including piecewise Hessians terms instead of piecewise Laplace terms) of the discrete bilinear form compared to [SM03] (compare Remark 3.5 below).

The derivation of the discrete bilinear $\mathfrak{B}_{\mathcal{T}}$ form hinges on consistency, i.e. $\mathfrak{B}_{\mathcal{T}}$ should satisfy

$$\mathfrak{B}_{\mathcal{T}}[u,\, w_h] = \int_\Omega f w_h \,\mathrm{d}x \quad \forall w_h \in \mathbb{V}(\mathcal{T}) \tag{3.1.7}$$

whenever the exact solution of (2.4.2) has extended regularity $u \in H_0^2(\Omega) \cap H^4(\Omega)$. This regularity assumption can be asserted for instance for convex domains Ω.

Before we start, we recall the following Lemma which is crucial in the subsequent analysis.

Lemma 3.2. *Let $\psi \in H^1(\Omega, \mathcal{T})$ and $\boldsymbol{\phi} \in H^1(\Omega, \mathcal{T})^2$, then we have*

$$\begin{aligned}&\sum_{K\in\mathcal{T}} \int_{\partial K} (\boldsymbol{\phi}\cdot \boldsymbol{n}_K)\psi \,\mathrm{d}s \\ &\quad = \sum_{F\in\mathcal{F}_{\mathcal{T}}} \int_F \{\!\{\boldsymbol{\phi}\}\!\} \cdot \boldsymbol{n}_F \,[\![\psi]\!] \,\mathrm{d}s + \sum_{F\in\mathring{\mathcal{F}}_{\mathcal{T}}} \int_F [\![\boldsymbol{\phi}]\!] \cdot \boldsymbol{n}_F \,\{\!\{\psi\}\!\} \,\mathrm{d}s.\end{aligned} \tag{3.1.8}$$

Moreover, for $\boldsymbol{T} \in H^1(\Omega, \mathcal{T})^{2\times 2}$ and $\boldsymbol{b} \in H^1(\Omega, \mathcal{T})^2$ we have

$$\begin{aligned}&\sum_{K\in\mathcal{T}} \int_{\partial K} \boldsymbol{T}\boldsymbol{b} \cdot \boldsymbol{n}_K \,\mathrm{d}s \\ &\quad = \sum_{F\in\mathcal{F}_{\mathcal{T}}} \int_F \{\!\{\boldsymbol{T}\}\!\} \,[\![\boldsymbol{b}]\!] \cdot \boldsymbol{n}_F \,\mathrm{d}s + \sum_{F\in\mathring{\mathcal{F}}_{\mathcal{T}}} \int_F [\![\boldsymbol{T}]\!] \,\{\!\{\boldsymbol{b}\}\!\} \cdot \boldsymbol{n}_F \,\mathrm{d}s\end{aligned} \tag{3.1.9}$$

Proof. Equation (3.1.8) directly follows from [DPE12, p. 123] and we restrict ourself to the proof of (3.1.9). Note that for all $F \in \mathring{\mathcal{F}}$ with $F = K_1 \cap K_2$, we have $\boldsymbol{n}_F = \boldsymbol{n}_{K_1} = -\boldsymbol{n}_{K_1}$ (compare figure 3.1) and therefore

$$\sum_{K\in\mathcal{T}} \int_{\partial K} \boldsymbol{T}\boldsymbol{b} \cdot \boldsymbol{n}_K \,\mathrm{d}s = \sum_{F\in\mathring{\mathcal{F}}} \int_F [\![\boldsymbol{T}\boldsymbol{b}]\!] \cdot \boldsymbol{n}_F \,\mathrm{d}s + \sum_{F\in\mathcal{F}^b} \int_F \boldsymbol{T}\boldsymbol{b} \cdot \boldsymbol{n}_F \,\mathrm{d}s. \tag{3.1.10}$$

Setting $C_i = \boldsymbol{T}|_{K_i}$, $D_i = \boldsymbol{b}|_{K_i}$, $1 \leqslant i \leqslant 2$, gives us

$$\begin{aligned}[\![\boldsymbol{T}\boldsymbol{b}]\!] &= C_1 D_1 - C_2 D_2 \\ &= \frac{1}{2}(C_1 + C_2)(D_1 - D_2) + (C_1 - C_2)\frac{1}{2}(D_1 + D_2) \\ &= \{\!\{\boldsymbol{T}\}\!\} \,[\![\boldsymbol{b}]\!] + [\![\boldsymbol{T}]\!] \,\{\!\{\boldsymbol{b}\}\!\},\end{aligned}$$

where we used the definitions of $[\![\cdot]\!]$ and $\{\!\{\cdot\}\!\}$. Hence, inserting the last equation into (3.1.10), yields

$$\begin{aligned}\sum_{K\in\mathcal{T}} \int_{\partial K} \boldsymbol{T}\boldsymbol{b} \cdot \boldsymbol{n}_K \,\mathrm{d}s = &\sum_{F\in\mathring{\mathcal{F}}_{\mathcal{T}}} \int_F \{\!\{\boldsymbol{T}\}\!\} \,[\![\boldsymbol{b}]\!] \cdot \boldsymbol{n}_F \,\mathrm{d}s + \sum_{F\in\mathring{\mathcal{F}}_{\mathcal{T}}} \int_F [\![\boldsymbol{T}]\!] \,\{\!\{\boldsymbol{b}\}\!\} \cdot \boldsymbol{n}_F \,\mathrm{d}s \\ &+ \sum_{F\in\mathcal{F}^b} \int_F \boldsymbol{T}\boldsymbol{b} \cdot \boldsymbol{n}_F \,\mathrm{d}s\end{aligned}$$

and the desired estimate (3.1.9) follows from the definition of mean- and jump terms on $\partial\Omega$. $\square$

Remark 3.3. *We note that for $F \in \mathring{\mathcal{F}}$, $F = K_1 \cap K_2$, the proof of Lemma 3.2 reveals*

$$\int_F [\![\boldsymbol{T}\boldsymbol{b}]\!] \cdot \boldsymbol{n}_F \,\mathrm{d}s = \int_F \{\!\{\boldsymbol{T}\}\!\} [\![\boldsymbol{b}]\!] \cdot \boldsymbol{n}_F \,\mathrm{d}s + \sum_{F \in \mathring{\mathcal{F}}_{\mathcal{T}}} \int_F [\![\boldsymbol{T}]\!] \{\!\{\boldsymbol{b}\}\!\} \cdot \boldsymbol{n}_F \,\mathrm{d}s.$$

We stress that the jump term on the left-hand side is symmetric with respect to the ordering of the elements K_1 and K_2 (compare Remark 3.1) and thus, the jump-terms on the right-hand side are also independent of this ordering.

In order to ease the notation we restrict ourself to homogeneous boundary values introduced in Section 2.4. The case of non-homogeneous boundary values can be handled as in [GH09].

In the subsequent analysis, we make use of a tensor-valued integration by parts formula. To this end, we note that a 2-tensor

$$\boldsymbol{T} = \sum_{i,j=1}^{2} T_{ij} e_i \otimes e_j \in \mathbb{R}^{2\times 2}$$

is represented by the matrix $\boldsymbol{T} = (T_{ij})_{1\leqslant i,j\leqslant 2} \in \mathbb{R}^{2\times 2}$, where the vector space $\mathbb{R}^2$ is equipped with the standard basis $\{e_1, e_2\}$. Consequently, for a sufficiently smooth 2-*tensor-valued* function $\boldsymbol{T} = (T_{i,j})_{1\leqslant i,j\leqslant 2}$ with column vectors $\boldsymbol{T}^{(i)} = (T_{1i}, T_{2i})^T$, $1 \leqslant i \leqslant 2$, the *divergence* is defined by

$$\nabla \cdot \boldsymbol{T} = \sum_{i,j=1}^{2} \frac{\partial T_{ij}}{\partial x_i} e_i = (\nabla \cdot \boldsymbol{T}^{(1)}, \nabla \cdot \boldsymbol{T}^{(2)})^T.$$

Let $K \in \mathcal{T}$ be an arbitrary element, $\boldsymbol{\phi} \in H^1(K)^2$ be a vector-valued function and $\boldsymbol{W} \in H(K)^{2\times 2}$ be a tensor-valued function, then we have

$$\int_K \nabla\boldsymbol{\phi} \colon \boldsymbol{W}^T \,\mathrm{d}x = -\int_K \boldsymbol{\phi} \cdot (\nabla \cdot \boldsymbol{W}) \,\mathrm{d}x + \int_{\partial K} \boldsymbol{W}\boldsymbol{\phi} \cdot \boldsymbol{n} \,\mathrm{d}s.$$

For $K \in \mathcal{T}, v \in H^4(K)$ and $w_h \in \mathbb{V}(\mathcal{T})$, the last equation implies the following integration by parts fromula

$$\begin{aligned} \int_K D^2 v : D^2 w_h \,\mathrm{d}x = \int_K (\Delta^2 v) w_h \,\mathrm{d}x + \int_{\partial K} D^2 v \nabla w_h \cdot \boldsymbol{n}_K \,\mathrm{d}s \\ - \int_{\partial K} (\nabla \cdot D^2 v \cdot \boldsymbol{n}_K). \end{aligned} \tag{3.1.11}$$

Summing (3.1.11) over all $K \in \mathcal{T}$ and using (3.1.8) and (3.1.9) leads us to

$$\begin{aligned} &\sum_{K\in\mathcal{T}} \int_K D^2 v : D^2 w_h \,\mathrm{d}x \\ &\quad = \sum_{K\in\mathcal{T}} \int_K (\Delta^2 v) w_h \,\mathrm{d}x - \int_{\mathcal{F}} \{\!\{\nabla \cdot D^2 v\}\!\} \cdot \boldsymbol{n} [\![w_h]\!] \,\mathrm{d}s \\ &\quad - \int_{\mathring{\mathcal{F}}} [\![\nabla \cdot D^2 v]\!] \cdot \boldsymbol{n} \{\!\{w_h\}\!\} \,\mathrm{d}s + \int_{\mathcal{F}} \{\!\{D^2 v\}\!\} [\![\nabla w_h]\!] \cdot \boldsymbol{n} \,\mathrm{d}s \\ &\quad + \int_{\mathring{\mathcal{F}}} [\![D^2 v]\!] \{\!\{\nabla w_h\}\!\} \cdot \boldsymbol{n} \,\mathrm{d}s. \end{aligned} \tag{3.1.12}$$

We observe that on the left-hand side of (3.1.12) we have localised the Hessians of the bilinear form $\mathfrak{B}$ (see (2.4.2)) to mesh-elements. Therefore, as a naive approach we choose $\mathfrak{B}_{\mathcal{T}}^{(nc)}[u, w_h] = \sum_{K\in\mathcal{T}} \int_K D^2u : D^2w_h \,\mathrm{d}x$ as our discrete bilinear form. In order to check the consistency requirement (3.1.7) we chose $v = u$ and assume $u \in H_0^2(\Omega) \cap H^4(\Omega)$. From this we obtain

$$
\begin{aligned}
\mathfrak{B}_{\mathcal{T}}^{(nc)}[u, w_h] &:= \sum_{K\in\mathcal{T}} \int_K D^2u : D^2w_h \,\mathrm{d}x \\
&= \int_\Omega f w_h \,\mathrm{d}x - \int_{\mathcal{F}} \{\!\{\nabla \cdot D^2u\}\!\} \cdot \boldsymbol{n} [\![w_h]\!] \,\mathrm{d}s \\
&\quad + \int_{\mathcal{F}} \{\!\{D^2u\}\!\} [\![\nabla w_h]\!] \cdot \boldsymbol{n} \,\mathrm{d}s.
\end{aligned}
$$

Hence, the last equation suggest that in order to satisfy the consistency assumption (3.1.7) we have to add *consistency-terms* to the discrete bilinear form $\mathfrak{B}_{\mathcal{T}}^{(nc)}$ i.e.

$$
\begin{aligned}
\mathfrak{B}_{\mathcal{T}}^{(c)}[u, w_h] &:= \mathfrak{B}_{\mathcal{T}}^{(nc)}[u, w_h] + \int_{\mathcal{F}} \{\!\{\nabla \cdot D^2u\}\!\} \cdot \boldsymbol{n} [\![w_h]\!] \,\mathrm{d}s \\
&\quad - \int_{\mathcal{F}} \{\!\{D^2u\}\!\} [\![\nabla w_h]\!] \cdot \boldsymbol{n} \,\mathrm{d}s \\
&= \sum_{K\in\mathcal{T}} \int_K D^2u : D^2w_h \,\mathrm{d}x + \int_{\mathcal{F}} \{\!\{\nabla \cdot D^2u\}\!\} \cdot \boldsymbol{n} [\![w_h]\!] \,\mathrm{d}s \\
&\quad - \int_{\mathcal{F}} \{\!\{D^2u\}\!\} [\![\nabla w_h]\!] \cdot \boldsymbol{n} \,\mathrm{d}s = \int_\Omega f w_h \,\mathrm{d}x.
\end{aligned}
\tag{3.1.13}
$$

Unfortunately, the resulting discrete bilinear form $\mathfrak{B}_{\mathcal{T}}^{(c)}$ in (3.1.13) is non-symmetric with respect to the two arguments. In order to recover symmetry of $\mathfrak{B}_{\mathcal{T}}$, we have to add *symmetry-terms*, i.e.

$$
\begin{aligned}
\mathfrak{B}_{\mathcal{T}}^{(sym)}[u, w_h] &:= \mathfrak{B}_{\mathcal{T}}^{(c)}[u, w_h] + \int_{\mathcal{F}} \{\!\{\nabla \cdot D^2w_h\}\!\} \cdot \boldsymbol{n} [\![u]\!] \,\mathrm{d}s \\
&\quad - \int_{\mathcal{F}} \{\!\{D^2u\}\!\} [\![\nabla w_h]\!] \cdot \boldsymbol{n} \,\mathrm{d}s \\
&= \sum_{K\in\mathcal{T}} \int_K D^2u : D^2w_h \,\mathrm{d}x \\
&\quad + \int_{\mathcal{F}} \{\!\{\nabla \cdot D^2u\}\!\} \cdot \boldsymbol{n} [\![w_h]\!] + \{\!\{\nabla \cdot D^2w_h\}\!\} \cdot \boldsymbol{n} [\![u]\!] \,\mathrm{d}s \\
&\quad - \int_{\mathcal{F}} \{\!\{D^2u\}\!\} [\![\nabla w_h]\!] \cdot \boldsymbol{n} + \{\!\{D^2w_h\}\!\} [\![\nabla u]\!] \cdot \boldsymbol{n} \,\mathrm{d}s \\
&= \int_\Omega f w_h \,\mathrm{d}x.
\end{aligned}
\tag{3.1.14}
$$

Here, we used that $[\![u]\!]_F = [\![\nabla u]\!]_F = 0$ for all $F \in \mathcal{F}$ since $u \in H_0^2(\Omega)$. Finally, we have to ensure the coercivity of the discrete bilinear form $\mathfrak{B}_{\mathcal{T}}^{(sym)}$. For $w_h \in$

$\mathbb{V}(\mathcal{T})$ we have that

$$\begin{aligned}\mathfrak{B}_{\mathcal{T}}^{(sym)}[w_h, w_h] = &\sum_{K\in\mathcal{T}}\int_K \left|D^2 w_h\right|^2 \,\mathrm{d}x \\ &+ 2\int_{\mathcal{F}} \left\{\!\!\left\{\nabla\cdot D^2 w_h\right\}\!\!\right\}\cdot \boldsymbol{n}\,[\![w_h]\!] - \left\{\!\!\left\{D^2 w_h\right\}\!\!\right\}[\![\nabla w_h]\!]\cdot\boldsymbol{n}\,\mathrm{d}s.\end{aligned}$$

We emphasise that the the desired estimate $\mathfrak{B}_{\mathcal{T}}^{(sym)}[w_h, v_h] \geqslant C\,|\!|\!|w_h|\!|\!|_{\mathcal{T}}^2$ is unclear since the face integrals $\int_{\mathcal{F}} \left\{\!\!\left\{\nabla\cdot D^2 w_h\right\}\!\!\right\}\cdot\boldsymbol{n}\,[\![w_h]\!] - \left\{\!\!\left\{D^2 w_h\right\}\!\!\right\}[\![\nabla w_h]\!]\cdot\boldsymbol{n}\,\mathrm{d}s$ do not have a positive sign in general. In order to cure this issue we add *penalty-terms* to the discrete bilinear form i.e.

$$\begin{aligned}\mathfrak{B}_{\mathcal{T}}^{(pen)}[w_h, v_h] :=\; &\mathfrak{B}_{\mathcal{T}}^{(sym)}[w_h, v_h] \\ &+ \int_{\mathcal{F}_{\mathcal{T}}} \frac{\alpha}{h_{\mathcal{T}}}[\![\nabla u]\!]\cdot\boldsymbol{n}\,[\![\nabla w_h]\!]\cdot\boldsymbol{n} + \frac{\beta}{h_{\mathcal{T}}^3}[\![u]\!]\,\boldsymbol{n}\cdot[\![w_h]\!]\,\boldsymbol{n}\,\mathrm{d}s \\ =\; &\int_{\mathcal{T}} D^2 u : D^2 w_h \,\mathrm{d}x \\ &+ \int_{\mathcal{F}_{\mathcal{T}}} \left\{\!\!\left\{\nabla\cdot D^2 w_h\right\}\!\!\right\}\cdot[\![u]\!]\,\boldsymbol{n} + \left\{\!\!\left\{\nabla\cdot D^2 u\right\}\!\!\right\}\cdot[\![w_h]\!]\,\boldsymbol{n}\,\mathrm{d}s \\ &- \int_{\mathcal{F}_{\mathcal{T}}} \left\{\!\!\left\{D^2 u\right\}\!\!\right\}[\![\nabla w_h]\!]\cdot\boldsymbol{n} + \left\{\!\!\left\{D^2 w_h\right\}\!\!\right\}[\![\nabla u]\!]\cdot\boldsymbol{n}\,\mathrm{d}s \\ &+ \int_{\mathcal{F}_{\mathcal{T}}} \frac{\alpha}{h_{\mathcal{T}}}[\![\nabla u]\!]\cdot\boldsymbol{n}\,[\![\nabla w_h]\!]\cdot\boldsymbol{n} + \frac{\beta}{h_{\mathcal{T}}^3}[\![u]\!]\,\boldsymbol{n}\cdot[\![w_h]\!]\,\boldsymbol{n}\,\mathrm{d}s \\ =\; &\int_{\mathcal{T}} f w_h \,\mathrm{d}x.\end{aligned} \tag{3.1.15}$$

for some $\alpha, \beta \geqslant 1$ (to be chosen later). In order to shorten the notation, for $v \in \mathbb{V}(\mathcal{T})$, we also write

$$[\![\partial_n v]\!]\,|_F := [\![\nabla v]\!]\,|_F \cdot \boldsymbol{n}_F = \nabla v|_{K_1}\boldsymbol{n}_{K_1} + \nabla v|_{K_2}\boldsymbol{n}_{K_2},$$

where we used $\boldsymbol{n}_F = \boldsymbol{n}_{K_1} = -\boldsymbol{n}_{K_2}$ for $F = K_1 \cap K_2$, compare Figure 3.1. We emphasise that in this definition K_1 and K_2 are allowed to play symmetric roles. Finally, the same holds true for the two remaining jump-terms, due to Remark 3.3.

Finally, in view of (3.1.15) we are in a position to define our discrete SIPDG bilinear form by

$$\begin{aligned}\mathfrak{B}_{\mathcal{T}}[v,\, w] :=\; &\int_{\mathcal{T}} D^2 v : D^2 w \,\mathrm{d}x \\ &+ \int_{\mathcal{F}_{\mathcal{T}}} \left\{\!\!\left\{\nabla\cdot D^2 v\right\}\!\!\right\}\cdot[\![w]\!]\,\boldsymbol{n} + \left\{\!\!\left\{\nabla\cdot D^2 w\right\}\!\!\right\}\cdot[\![v]\!]\,\boldsymbol{n}\,\mathrm{d}s \\ &- \int_{\mathcal{F}_{\mathcal{T}}} \left\{\!\!\left\{D^2 v\right\}\!\!\right\}[\![\nabla w]\!]\cdot\boldsymbol{n} + \left\{\!\!\left\{D^2 w\right\}\!\!\right\}[\![\nabla v]\!]\cdot\boldsymbol{n}\,\mathrm{d}s \\ &+ \int_{\mathcal{F}_{\mathcal{T}}} \frac{\alpha}{h_{\mathcal{T}}}[\![\partial_n v]\!]\,[\![\partial_n w]\!] + \frac{\beta}{h_{\mathcal{T}}^3}[\![v]\!]\,\boldsymbol{n}\cdot[\![w]\!]\,\boldsymbol{n}\,\mathrm{d}s,\end{aligned} \tag{3.1.16}$$

for all $v, w \in \mathbb{V}(\mathcal{T})$.

Remark 3.4. *For the ease of notation we restrict ourself to the SIPDG variant of the bilinear from, in (compare also [SM07]). However, we emphasise, that by modifying the above derivation we could also obtain the non-symmetric and the semi-symmetric variant of the interior penalty discontinuous Galerkin bilinear introduced in [SM07]. Even the classical method of Baker [Bak77] can be obtained in this way; see [GH09, Remark 3.1]).*

3.2 The discrete problem

We define the *symmetric interior penalty discontinuous Galerkin method* (SIPDGM):

$$\text{Find } u_{\mathcal{T}} \in \mathbb{V}(\mathcal{T}) \quad \text{such that} \quad \mathfrak{B}_{\mathcal{T}}[u_{\mathcal{T}}, v_{\mathcal{T}}] = \int_\Omega f v_{\mathcal{T}} \quad \forall v_{\mathcal{T}} \in \mathbb{V}(\mathcal{T}), \tag{3.2.1}$$

where $\mathfrak{B}_{\mathcal{T}}[\cdot, \cdot]\colon \mathbb{V}(\mathcal{T}) \times \mathbb{V}(\mathcal{T}) \to \mathbb{R}$ is defined in (3.1.16).

Remark 3.5. *Similar discontinuous Galerkin methods are derived in [SM03] and [GH09] although they use a slightly different 'divergence formulation' of the method instead of the 'plate formulation' used in* (3.1.16). *The bilinear from in divergence formulation is defined by*

$$\begin{aligned}\widetilde{\mathfrak{B}}_{\mathcal{T}}[v, w] := &\int_{\mathcal{T}} \Delta v \Delta w \,\mathrm{d}x \\ &+ \int_{\mathcal{F}} \Big([\![v]\!]\, \boldsymbol{n} \cdot \{\!\{\nabla \Delta w\}\!\} + [\![w]\!]\, \boldsymbol{n} \cdot \{\!\{\nabla \Delta v\}\!\} \\ &\quad - \{\!\{\Delta v\}\!\}\, [\![\nabla w]\!] \cdot \boldsymbol{n} - \{\!\{\Delta w\}\!\}\, [\![\nabla v]\!] \cdot \boldsymbol{n} \Big) \,\mathrm{d}s \\ &+ \int_{\mathcal{F}} \frac{\sigma}{h_{\mathcal{T}}^3} [\![v]\!]\, \boldsymbol{n} \cdot [\![w]\!]\, \boldsymbol{n} + \frac{\tau}{h_{\mathcal{T}}} [\![\nabla v]\!] \cdot \boldsymbol{n}\, [\![\nabla w]\!] \cdot \boldsymbol{n} \,\mathrm{d}s\end{aligned} \tag{3.2.2}$$

and follows from a slightly different integration by parts formula. In this definition all tensor-fields occurring in the method are of order $k \in \{0, 1\}$. *Hence, the definitions of the trace operators* $[\![\cdot]\!]$ *and* $\{\!\{\cdot\}\!\}$ *can be slightly simplified. However, an advantage of the plate formulation is that we can use more general boundary conditions of fourth order problems (see e.g.[EGH*$^+$ *02]).*

In order to prove continuity and coercivity of the discrete bilinear form $\mathfrak{B}_{\mathcal{T}}$ on $\mathbb{V}(\mathcal{T})$, we define the *energy norm*

$$|\!|\!| v |\!|\!|_{\mathcal{T}}^2 := \int_{\mathcal{T}} D^2 v \colon D^2 v \,\mathrm{d}x + \int_{\mathcal{F}_{\mathcal{T}}} \frac{\alpha}{h_{\mathcal{T}}} |[\![\partial_n v]\!]|^2 + \frac{\beta}{h_{\mathcal{T}}^3} \|[\![v]\!]\, \boldsymbol{n}\|^2 \,\mathrm{d}s,$$

where $v \in H_0^2(\mathcal{T})$ and $\alpha, \beta \geqslant 1$ are the penalty parameters (to be chosen later). Additionally, we define for some subset $\mathcal{M} \subset \mathcal{T}$

$$|\!|\!| v |\!|\!|_{\mathcal{M}}^2 := \int_{\mathcal{M}} D^2 v \colon D^2 v \,\mathrm{d}x + \alpha \left\| h^{-1/2}\, [\![\partial_n v]\!] \right\|_{\Gamma(\mathcal{M})}^2 + \beta \left\| h^{-3/2}\, [\![v]\!] \right\|_{\Gamma(\mathcal{M})}^2$$

In order to keep the presentation simple we will henceforth write $\int_\Omega \left|D^2_{\mathtt{pw}} v\right|^2 \,\mathrm{d}x$ instead of $\int_\Omega D^2_{\mathtt{pw}} v : D^2_{\mathtt{pw}} v \,\mathrm{d}x$. Moreover, we simply write $|\cdot|$ instead of the vector-norm $\|\cdot\|$, when no confusion is possible, i.e.

$$\int_F |[\![v]\!]\,\boldsymbol{n}|^2 \,\mathrm{d}s = \int_F \|\,[\![v]\!]\,\boldsymbol{n}\|^2,$$

for $v \in \mathbb{V}(\mathcal{T})$ and $F \in \mathcal{F}$.

The proof of coercivity and continuity of $\mathfrak{B}_\mathcal{T}$ is based on the following two crucial estimates.

Lemma 3.6 (Inverse estimate)**.** *Let $\mathcal{T} \in \mathbb{G}$. Then,*

$$\|\nabla v\|_K \leqslant C_{\mathsf{inv}} h_K^{-1} \|v\|_K\,, \quad \forall v \in \mathbb{V}(\mathcal{T}),\ K \in \mathcal{T}, \tag{3.2.3}$$

where C_{inv} only depends on the shape regularity and the polynomial degree r.

Proof. See [DPE12, Lemma 1.44]. □

Lemma 3.7 (Trace inequality)**.** *Let $\mathcal{T} \in \mathbb{G}$. Then, for all $F \in \mathcal{F}_\mathcal{T}$, such that $F \subset K \in \mathcal{T}$,*

$$\|v\|_F \leqslant C_{\mathsf{tr}} h_K^{-1/2} \|v\|_K\,, \quad \forall v \in \mathbb{V}(\mathcal{T}), \tag{3.2.4}$$

where C_{tr} only depends on the shape regularity and the polynomial degree r.

Proof. Compare [DPE12, Lemma 1.46]. □

In the following proposition we will write down constants related to coercivity of $\mathfrak{B}_\mathcal{T}$ explicitly, since we are interested on the dependence of the penalty parameters α, β with respect to the polynomial degree r.

Lemma 3.8 (Continuity and Coercivity)**.** *Let $\mathcal{T} \in \mathbb{G}$ and chose the penalty parameters α, β such that $\alpha > 6C_{tr}^2$ and $\beta > 6C_{tr}^2 C_{inv}^2$. Then, there exist positive constants $C_{\mathsf{cont}}, C_{\mathsf{coer}}$ such that*

$$\mathfrak{B}_\mathcal{T}[v,\, w] \leqslant C_{\mathsf{cont}} \|v\|_\mathcal{T} \|w\|_\mathcal{T} \qquad \text{and} \qquad C_{\mathsf{coer}} \|v\|_\mathcal{T}^2 \leqslant \mathfrak{B}_\mathcal{T}[v,\, v].$$

for all $v, w \in \mathbb{V}(\mathcal{T})$. The constants C_{cont}, and C_{coer} solely depend on α, β, the mesh parameters and the polynomial degree r.

Proof. Before we establish coercivity and continuity of $\mathfrak{B}_\mathcal{T}$ we consider the following estimates of face integrals: For all $v, w \in \mathbb{V}(\mathcal{T})$ we have

$$\begin{aligned} &\left| \int_\mathcal{F} \{\!\{\nabla \cdot D^2 v\}\!\} \cdot [\![w]\!]\,\boldsymbol{n} \,\mathrm{d}s \right| \\ &\quad \leqslant \left(\sum_{K \in \mathcal{T}} \sum_{F \subset \mathcal{T}} h_F^3 \left\|\nabla \cdot D^2 v|_K\right\|_{L^2(F)}^2 \right)^{1/2} \left(\int_\mathcal{F} h_\mathcal{T}^{-3} \,|[\![w]\!]\,\boldsymbol{n}|^2 \,\mathrm{d}s \right)^{1/2} \end{aligned} \tag{3.2.5}$$

and

$$
\begin{aligned}
&\left| \int_{\mathcal{F}} \{\!\{ D^2 v \}\!\} \, [\![\nabla w]\!] \cdot \boldsymbol{n} \, \mathrm{d}s \right| \\
&\quad \leqslant \left(\sum_{K \in \mathcal{T}} \sum_{F \subset \mathcal{T}} h_F \, \| D^2 v|_K \|_{L^2(F)}^2 \right)^{1/2} \left(\int_{\mathcal{F}} h_{\mathcal{T}}^{-1} \, | [\![\partial_n w]\!] |^2 \, \mathrm{d}s \right)^{1/2} .
\end{aligned} \tag{3.2.6}
$$

This is a consequence of the Cauchy-Schwarz inequality and a regrouping of the face contributions (compare [DPE12, Lemma 4.11] for details).

In order to proof coercivity, let $v \in \mathbb{V}(\mathcal{T})$ and write

$$
\begin{aligned}
\mathfrak{B}_{\mathcal{T}}[v,\, v] \geqslant \| D^2_{\mathtt{pw}} v \|_{\Omega}^2 &- 2 \left| \int_{\mathcal{F}_{\mathcal{T}}} \{\!\{ \nabla \cdot D^2 v \}\!\} \cdot \boldsymbol{n} \, [\![v]\!] \, \mathrm{d}s \right| \\
&- 2 \left| \int_{\mathcal{F}_{\mathcal{T}}} \{\!\{ D^2 v \}\!\} \, [\![\nabla v]\!] \cdot \boldsymbol{n} \, \mathrm{d}s \right| \\
&+ \int_{\mathcal{F}_{\mathcal{T}}} \frac{\alpha}{h_{\mathcal{T}}} \, [\![\partial_n v]\!]^2 + \frac{\beta}{h_{\mathcal{T}}^3} \, | [\![v]\!] \, \boldsymbol{n} |^2 \, \mathrm{d}s.
\end{aligned} \tag{3.2.7}
$$

In (3.2.5) we use the trace inequality (3.2.4) and the inverse inequality (3.2.3) in conjunction with $h_F|_F \leqslant h_K|_F$, for all $F \subset \partial K$ and all $K \in \mathcal{T}$, to obtain

$$
\begin{aligned}
&\left| \int_{\mathcal{F}} \{\!\{ \nabla \cdot D^2 v \}\!\} \cdot [\![w]\!] \, \boldsymbol{n} \, \mathrm{d}s \right| \\
&\quad \leqslant \left(\sum_{K \in \mathcal{T}} \sum_{F \subset \mathcal{T}} h_F^3 \, \| \nabla \cdot D^2 v|_K \|_{L^2(F)}^2 \right)^{1/2} \left(\int_{\mathcal{F}} h_{\mathcal{T}}^{-3} \, | [\![w]\!] \, \boldsymbol{n} |^2 \, \mathrm{d}s \right)^{1/2} \\
&\quad \leqslant \left(\sum_{K \in \mathcal{T}} h_K^3 \, \| \nabla \cdot D^2 v|_K \|_{L^2(\partial K)}^2 \right)^{1/2} \left(\int_{\mathcal{F}} h_{\mathcal{T}}^{-3} \, | [\![w]\!] \, \boldsymbol{n} |^2 \, \mathrm{d}s \right)^{1/2} \\
&\quad \leqslant C_{\mathsf{inv}} C_{\mathsf{tr}} \sqrt{3} \, \| D^2_{\mathtt{pw}} v \|_{\Omega} \left(\int_{\mathcal{F}} h_{\mathcal{T}}^{-3} \, | [\![w]\!] \, \boldsymbol{n} |^2 \, \mathrm{d}s \right)^{1/2} .
\end{aligned} \tag{3.2.8}
$$

In the same vein we obtain from (3.2.6) by the trace inequality and similar arguments

$$
\left| \int_{\mathcal{F}} \{\!\{ D^2 v \}\!\} \, [\![\nabla w]\!] \cdot \boldsymbol{n} \, \mathrm{d}s \right| \leqslant C_{\mathsf{tr}} \sqrt{3} \, \| D^2_{\mathtt{pw}} v \|_{\Omega} \left(\int_{\mathcal{F}} h_{\mathcal{T}}^{-1} \, | [\![\partial_n w]\!] |^2 \, \mathrm{d}s \right)^{1/2} . \tag{3.2.9}
$$

Using (3.2.8) and (3.2.9) in (3.2.7), in conjunction with Young's inequality, yields

$$
\begin{aligned}
\mathfrak{B}_{\mathcal{T}}[v,\, v] \geqslant{} & \frac{1}{2} \left\| D^2_{\mathsf{pw}} v \right\|^2_{\Omega} - 6 C^2_{\mathsf{inv}} C^2_{\mathsf{tr}} \int_{\mathcal{F}} h_{\mathcal{T}}^{-3} \left| [\![w]\!] \, \boldsymbol{n} \right|^2 \, \mathrm{d}s \\
& - 6 C^2_{\mathsf{tr}} \int_{\mathcal{F}} h_{\mathcal{T}}^{-1} \left| [\![\partial_n w]\!] \right|^2 \, \mathrm{d}s \\
& + \int_{\mathcal{F}_{\mathcal{T}}} \frac{\alpha}{h_{\mathcal{T}}} [\![\partial_n v]\!]^2 + \frac{\beta}{h^3_{\mathcal{T}}} [\![v]\!]^2 \, \mathrm{d}s \\
={} & \frac{1}{2} \left\| D^2_{\mathsf{pw}} v \right\|^2_{\Omega} \\
& + \int_{\mathcal{F}_{\mathcal{T}}} \frac{\alpha - 6 C^2_{\mathsf{tr}}}{h_{\mathcal{T}}} [\![\partial_n v]\!]^2 + \frac{\beta - 6 C^2_{\mathsf{inv}} C^2_{\mathsf{tr}}}{h^3_{\mathcal{T}}} \left| [\![v]\!] \, \boldsymbol{n} \right|^2 \, \mathrm{d}s.
\end{aligned}
\tag{3.2.10}
$$

As a result we obtain coercivity of $\mathfrak{B}_{\mathcal{T}}$ on $\mathbb{V}(\mathcal{T})$ since we assumed $\alpha > 6C^2_{\mathsf{tr}}$ and $\beta > 6C^2_{\mathsf{inv}}C^2_{\mathsf{tr}}$.

Finally, continuity of $\mathfrak{B}_{\mathcal{T}}$ follows from (3.2.8) and (3.2.9), since we obtain

$$
\begin{aligned}
& \mathfrak{B}_{\mathcal{T}}[v,\, w] \\
& \quad \leqslant \left\| D^2 v \right\|_{\Omega} \left\| D^2 w \right\|_{\Omega} + \left| \int_{\mathcal{F}_{\mathcal{T}}} \{\!\{ \nabla \cdot D^2 v \}\!\} \cdot \boldsymbol{n} \, [\![w]\!] \, \mathrm{d}s \right| \\
& \qquad + \left| \int_{\mathcal{F}_{\mathcal{T}}} \{\!\{ \nabla \cdot D^2 w \}\!\} \cdot \boldsymbol{n} \, [\![v]\!] \, \mathrm{d}s \right| + \left| \int_{\mathcal{F}_{\mathcal{T}}} \{\!\{ D^2 v \}\!\} \, [\![\nabla w]\!] \cdot \boldsymbol{n} \right| \\
& \qquad + \left| \int_{\mathcal{F}_{\mathcal{T}}} \{\!\{ D^2 w \}\!\} \, [\![\nabla v]\!] \cdot \boldsymbol{n} \, \mathrm{d}s \right| \\
& \qquad + \int_{\mathcal{F}_{\mathcal{T}}} \frac{\alpha}{h_{\mathcal{T}}} [\![\partial_n v]\!] \, [\![\partial_n w]\!] + \frac{\beta}{h^3_{\mathcal{T}}} [\![v]\!] \, \boldsymbol{n} \cdot [\![w]\!] \, \boldsymbol{n} \, \mathrm{d}s \\
& \quad \leqslant \left\| D^2 v \right\|_{\Omega} \left\| D^2 w \right\|_{\Omega} + C_{\mathsf{tr}} C_{\mathsf{inv}} \sqrt{3} \left\| D^2_{\mathsf{pw}} v \right\|_{\Omega} \left\| h_{\mathcal{T}}^{-3/2} [\![w]\!] \, \boldsymbol{n} \right\|_{\Gamma} \\
& \qquad + C_{\mathsf{tr}} C_{\mathsf{inv}} \sqrt{3} \left\| D^2_{\mathsf{pw}} w \right\|_{\Omega} \left\| h_{\mathcal{T}}^{-3/2} [\![v]\!] \, \boldsymbol{n} \right\|_{\Gamma} + C_{\mathsf{tr}} \sqrt{3} \left\| D^2_{\mathsf{pw}} w \right\|_{\Omega} \left\| h_{\mathcal{T}}^{-1/2} [\![\partial_n v]\!] \right\|_{\Gamma} \\
& \qquad + C_{\mathsf{tr}} \sqrt{3} \left\| D^2_{\mathsf{pw}} v \right\|_{\Omega} \left\| h_{\mathcal{T}}^{-1/2} [\![\partial_n w]\!] \right\|_{\Gamma} + \left\| \alpha^{1/2} h_{\mathcal{T}}^{-1/2} [\![\partial_n w]\!] \right\|_{\Gamma} \left\| \alpha^{1/2} h_{\mathcal{T}}^{-1/2} [\![\partial_n v]\!] \right\|_{\Gamma} \\
& \qquad + \left\| \beta^{1/2} h_{\mathcal{T}}^{-3/2} [\![w]\!] \, \boldsymbol{n} \right\|_{\Gamma} \left\| \beta^{1/2} h_{\mathcal{T}}^{-3/2} [\![v]\!] \, \boldsymbol{n} \right\|_{\Gamma} \\
& \quad \leqslant C_{\mathsf{cont}} \left\| v \right\|_{\mathcal{T}} \left\| w \right\|_{\mathcal{T}}.
\end{aligned}
$$

□

The following theorem states that the discrete problem (3.2.1) yields a unique solution, provided the penalty parameters where chosen large enough to ensure coercivity of the discrete bilinear form.

Theorem 3.9 (Discrete solution)**.** *Let $\mathcal{T} \in \mathbb{G}$ and chose the penalty parameters α, β such that $\alpha > 6C^2_{tr}$ and $\beta > 6C^2_{tr}C^2_{inv}$. Then, the SIPDG problem* (3.2.1) *yields a unique solution $u_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$.*

Proof. We already established coercivity and continuity of $\mathfrak{B}_{\mathcal{T}}$ on $\mathbb{V}(\mathcal{T})$ in Lemma 3.8. Hence, in order to apply the Lax-Milgram Theorem 2.8, we have to prove that $(\mathbb{V}(\mathcal{T}), |\!|\!|\cdot|\!|\!|_{\mathcal{T}})$ is a Banach space. Note that $|\!|\!|\cdot|\!|\!|_{\mathcal{T}}$ is obviously a semi-norm on $\mathbb{V}(\mathcal{T})$ and we only have to prove

$$|\!|\!|v|\!|\!|_{\mathcal{T}} = 0 \Longleftrightarrow v = 0, \quad \forall v \in \mathbb{V}(\mathcal{T}).$$

Whence, let $v \in \mathbb{V}$ and assume $|\!|\!|v|\!|\!|_{\mathcal{T}} = 0$, then $\left\| D^2_{\mathtt{pw}} v \right\|_{\Omega} = 0$ implies $D^2_{\mathtt{pw}} v = 0$ and therefore $\nabla_{\mathtt{pw}} v$ is a constant on every $K \in \mathcal{T}$. Moreover, $\left\| h_{\mathcal{T}}^{-1/2} [\![\partial_n v]\!] \right\|_{\Gamma} = 0$ implies that interface and boundary jumps of $\nabla_{\mathtt{pw}} v$ vanish and therefore $\nabla_{\mathtt{pw}} v \equiv 0$ on the whole domain Ω. Consequently, v is a constant on every $K \in \mathcal{T}$. Since also $\left\| h_{\mathcal{T}}^{-3/2} [\![v]\!]\, \boldsymbol{n} \right\|_{\Gamma} = 0$, we infer that $v = 0$ on the whole domain Ω. As a consequence, we have that $(\mathbb{V}(\mathcal{T}), |\!|\!|\cdot|\!|\!|_{\mathcal{T}})$ is a Banach space and the assertion follows from the Lax-Milgram Lemma 2.8. □

Remark 3.10 (r-dependency of the penalty parameters)**.** *We note, that C_{inv} and C_{tr} depend on the polynomial degree r,. i.e. C_{tr} scales as $\sqrt{r(r+2)}$ (compare [WH03] and C_{inv} scales as r^2 on triangles (see [SS98]). Hence Lemma 3.8 implies, that $\alpha = \mathcal{O}((r+1)^2)$ and $\beta = \mathcal{O}((r+1)^6)$. The specific choices of the penalty parameters in the numerical experiments (Chapter 5) follow from these estimates.*

Unfortunately, we can not apply the discrete bilinear form $\mathfrak{B}_{\mathcal{T}}$ to the exact solution u of (2.4.2) in cases of minimal regularity $u \in H^2_0(\Omega)$. The reason is that the discrete bilinear form requires traces for second and third order derivatives of u and therefore we need $u \in H^4(\Omega)$ which does not hold in general (compare Section 2.4 and the references therein). In order to solve this problem we have to introduce so called *lifting operators*, resp. *liftings*.

3.3 Liftings: Definition and stability

Lifting operators map scalar valued functions defined on mesh faces to tensor valued functions defined on mesh elements. In this way, the second and third order derivatives face integrals are replaced by volume terms.

In order to give a proper definitoon, we fix $F \in \mathcal{F}_{\mathcal{T}}$ and define a local lifting operator $\mathcal{L}^F_{\mathcal{T}} \colon \mathbb{V}(\mathcal{T}) + H^2_0(\Omega) \to \mathbb{P}_{r-2}(\mathcal{T})^{2\times 2}$, by

$$\int_{\Omega} \mathcal{L}^F_{\mathcal{T}}(\phi) \colon \boldsymbol{\psi} \,\mathrm{d}x = \int_F \{\!\{\nabla \cdot \boldsymbol{\psi}\}\!\} \cdot [\![\phi]\!]\, \boldsymbol{n} - \{\!\{\boldsymbol{\psi}\}\!\}\, [\![\nabla \phi]\!] \cdot \boldsymbol{n} \,\mathrm{d}s \tag{3.3.1}$$

for all $\boldsymbol{\psi} \in \mathbb{P}_{r-2}(\mathcal{T})^{2\times 2}$. A simple interpretation of the lifting operators is the following: For each $\phi \in \mathbb{V}(\mathcal{T}) + H^2_0(\Omega)$ the right-hand side of (3.3.1) is a linear operator over $\mathbb{P}_{r-2}(\mathcal{T})^{2\times 2}$. Consequently, the lifting operator $\mathcal{L}^F_{\mathcal{T}}(\phi)$ is the representative of this linear operator in $\mathbb{P}_{r-2}(\mathcal{T})^{2\times 2}$ under the $L^2(\Omega)$-scalar product in $\mathbb{P}_{r-2}(\mathcal{T})^{2\times 2}$. Note that by the definition of $\mathcal{L}^F_{\mathcal{T}}(\varphi)$ the support is given by $\omega_{\mathcal{T}}(F)$, i.e. the one or two mesh elements of which F is part of the boundary. We expand the local definition (3.3.1) to a *global lifting operator* $\mathcal{L}_{\mathcal{T}} \colon \mathbb{V}(\mathcal{T}) + H^2_0(\Omega) \to \mathbb{P}_{r-2}(\mathcal{T})^{2\times 2}$ by

$$\mathcal{L}_{\mathcal{T}}(\varphi) := \sum_{F \in \mathcal{F}_{\mathcal{T}}} \mathcal{L}^F_{\mathcal{T}}(\varphi).$$

We emphasise that $v, \partial_n v \in L^2(\Gamma_{\mathcal{T}})$, for all $v \in H^2(\mathcal{T})$ and therefore we can extend the bilinear form $\mathfrak{B}_{\mathcal{T}}$ from $\mathbb{V}(\mathcal{T})$ to $H^2(\mathcal{T})$ by

$$\begin{aligned}\mathfrak{B}_{\mathcal{T}}[v,\, w] := \int_{\mathcal{T}} D^2 v\colon D^2 w \,\mathrm{d}x + \int_{\Omega} \mathcal{L}_{\mathcal{T}}(w)\colon D^2_{\mathtt{pw}} v + \mathcal{L}_{\mathcal{T}}(v)\colon D^2_{\mathtt{pw}} w \,\mathrm{d}x \\ + \int_{\mathcal{F}_{\mathcal{T}}} \frac{\alpha}{h_{\mathcal{T}}} [\![\partial_n v]\!] \, [\![\partial_n w]\!] + \frac{\beta}{h_{\mathcal{T}}^3} [\![v]\!] \,\boldsymbol{n} \cdot [\![w]\!] \,\boldsymbol{n} \,\mathrm{d}s.\end{aligned} \tag{3.3.2}$$

We emphasise that the bilinear $\mathfrak{B}_{\mathcal{T}}$ defined in (3.1.16) is equivalent to (3.3.2) at the discrete level, i.e. for $v, w \in \mathbb{V}(\mathcal{T})$. In particular, coercivity and continuity (Lemma 3.8) holds true also (3.3.2). Differences occur on larger spaces, e.g. broken Sobolev spaces, since (3.1.16) is *not* defined on $H^2(\mathcal{T})$. For the rest of this thesis $\mathfrak{B}_{\mathcal{T}}$ always refers to (3.3.2) unless stated otherwise.

In view of consistency, let $u, v \in H^2_0(\Omega)$. Then, we have from (3.3.1) that $\mathcal{L}_{\mathcal{T}}(u) = \mathcal{L}_{\mathcal{T}}(v) = 0$ and therefore

$$\mathfrak{B}_{\mathcal{T}}[u,\, v] = \mathfrak{B}[u,\, v] = \int_{\Omega} D^2 u : D^2 v \,\mathrm{d}x = \int_{\Omega} f v \,\mathrm{d}x.$$

Hence, we infer consistency of the bilinear form $\mathfrak{B}_{\mathcal{T}}$.

The following lemma states crucial bounds of the lifting operators in the $L^2(\Omega)$-norm.

Lemma 3.11 (Bounds on liftings)**.** *Let $K \in \mathcal{T}$ and $F \in \mathcal{F}$ such that $F \subset K$. Then, For all $\phi \in H^2(\mathcal{T})$,*

$$\left\|\mathcal{L}^F_{\mathcal{T}}(\phi)\right\|_{\Omega} \leqslant C_{\mathit{tr}} h_F^{-1/2} \left\|[\![\partial_n \phi]\!]\right\|_F + C_{\mathit{tr}} C_{\mathit{inv}} h_F^{-3/2} \left\|[\![\phi]\!]\,\boldsymbol{n}\right\|_F . \tag{3.3.3}$$

In particular, this implies with

$$\left\|\mathcal{L}_{\mathcal{T}}(\phi)\right\|_{\Omega} \leqslant \sqrt{3} \left(\sum_{F \in \mathcal{F}} \left\|\mathcal{L}^F_{\mathcal{T}}(\phi)\right\|^2_{\Omega} \right)^{1/2}$$

that

$$\left\|\mathcal{L}_{\mathcal{T}}(\phi)\right\|_{\Omega} \leqslant \sqrt{3} \bigg(\int_{\mathcal{F}} \frac{C_{\mathit{tr}}}{h_{\mathcal{T}}} [\![\partial_n \phi]\!]^2 + \frac{C_{\mathit{tr}} C_{\mathit{inv}}}{h_{\mathcal{T}}^3} \left|[\![\phi]\!]\,\boldsymbol{n}\right|^2 \,\mathrm{d}s \bigg)^{1/2}. \tag{3.3.4}$$

Proof. Let $F \in \mathcal{F}$ and $K \in \mathcal{T}$ such that $F \subset K$. Note that $\phi \in H^2(K)$ implies that $\nabla\phi$ has a L^1-trace on $F \subset K$. Definition (3.3.1), together with the discrete trace and the inverse inequality (compare (3.2.4) and (3.2.3)) yields

$$\begin{aligned}
\left\|\mathcal{L}_{\mathcal{T}}^{F}(\phi)\right\|_{\Omega}^{2} &= \int_{\Omega} \mathcal{L}_{\mathcal{T}}^{F}(\phi) \colon \mathcal{L}_{\mathcal{T}}^{F}(\phi)\,\mathrm{d}x \\
&= \int_{F} \left\{\!\!\left\{\nabla\cdot\mathcal{L}_{\mathcal{T}}^{F}(\phi)\right\}\!\!\right\}\cdot \boldsymbol{n}\,[\![\phi]\!] - \left\{\!\!\left\{\mathcal{L}_{\mathcal{T}}^{F}(\phi)\right\}\!\!\right\}[\![\nabla\phi]\!]\cdot\boldsymbol{n}\,\mathrm{d}s \\
&\leqslant \left(h_F^3\int_F \left|\nabla\cdot\mathcal{L}_{\mathcal{T}}^{F}(\phi)\right|^2\,\mathrm{d}s\right)^{1/2}\left(h_F^{-3}\int_F \left|[\![\phi]\!]\,\boldsymbol{n}\right|^2\,\mathrm{d}s\right)^{1/2} \\
&\quad + \left(h_F\int_F \left|\mathcal{L}_{\mathcal{T}}^{F}(\phi)\right|^2\,\mathrm{d}s\right)^{1/2}\left(h_F^{-1}\int_F [\![\partial_n\phi]\!]^2\,\mathrm{d}s\right)^{1/2} \\
&\leqslant \left\|h_F^{-3/2}\,[\![\phi]\!]\,\boldsymbol{n}\right\|_F C_{\mathsf{tr}}C_{\mathsf{inv}}\left((\#N_{\mathcal{T}}(F))^{-1}\sum_{K\in\omega_{\mathcal{T}}(F)}\int_K\left|\mathcal{L}_{\mathcal{T}}^{F}(\phi)\right|^2\,\mathrm{d}x\right) \\
&\quad + \left\|h_F^{-1/2}\,[\![\partial_n\phi]\!]\right\|_F C_{\mathsf{tr}}\left((\#N_{\mathcal{T}}(F))^{-1}\sum_{K\in\omega_{\mathcal{T}}(F)}\int_K\left|\mathcal{L}_{\mathcal{T}}^{F}(\phi)\right|^2\,\mathrm{d}x\right).
\end{aligned}$$

Therefore, (3.3.3) follows from

$$\sum_{K\in\omega_{\mathcal{T}}(F)}\int_K\left|\mathcal{L}_{\mathcal{T}}^{F}(\phi)\right|^2\,\mathrm{d}x = \left\|\mathcal{L}_{\mathcal{T}}^{F}(\phi)\right\|_{\Omega}^{2} \quad \text{and} \quad \#N_{\mathcal{T}}(F)^{-1} \leqslant 1.$$

The bound of the global lifting operators $\|\mathcal{L}_{\mathcal{T}}(\phi)\|_{\Omega}$ follows from the local ones since $\|\mathcal{L}_{\mathcal{T}}(\phi)\|_{\Omega}^2 \leqslant 3\sum_{F\in\mathcal{F}}\left\|\mathcal{L}_{\mathcal{T}}^{F}(\phi)\right\|_{\Omega}^2$, in conjunction with the fact that the mesh consists of triangles. Finally, by the last estimate and (3.3.3) we obtain (3.3.4).

□

In view of the subsequent convergence analysis we are interested in the stability of the solution $u_{\mathcal{T}}$ with respect to the energy norm. To this end, we observe from coercivity and (3.2.1)

$$|\!|\!|u_{\mathcal{T}}|\!|\!|_{\mathcal{T}}^2 \lesssim \mathfrak{B}_{\mathcal{T}}[u_{\mathcal{T}},\, u_{\mathcal{T}}] = \int_{\Omega} f u_{\mathcal{T}}\,\mathrm{d}x \leqslant \|f\|_{\Omega}\,\|u_{\mathcal{T}}\|_{\Omega}\,.$$

As a consequence we have

$$|\!|\!|u_{\mathcal{T}}|\!|\!|_{\mathcal{T}} \lesssim \|f\|_{\Omega}, \tag{3.3.5}$$

due to the following broken Poincaré-Friedrichs inequality (Proposition 3.12) below.

Proposition 3.12 (Poincaré Inequality on $H^2(\mathcal{T})$)**.** *Let $\mathcal{T}\in\mathbb{G}$ and $\phi\in H^2(\mathcal{T})$. Then,*

$$\|\phi\|_{\Omega}^2 + \|\nabla_{pw}\phi\|_{\Omega}^2 \lesssim \left\|D_{pw}^2\phi\right\|_{\Omega}^2 + \left\|h_{\mathcal{T}}^{-3/2}\,[\![\phi]\!]\,\boldsymbol{n}\right\|_{\Gamma_{\mathcal{T}}}^2 + \left\|h_{\mathcal{T}}^{-1/2}\,[\![\partial_n\phi]\!]\right\|_{\Gamma_{\mathcal{T}}}^2,$$

where the constants in $'\lesssim'$ are independent of the mesh-size $h_{\mathcal{T}}$. In particular, for $v\in\mathbb{V}(\mathcal{T})$ this implies

$$\|v\|_{\Omega}^2 + \|\nabla_{pw}v\|_{\Omega}^2 \lesssim |\!|\!|v|\!|\!|_{\mathcal{T}}^2\,.$$

Proof. Compare [BWZ04]. □

In view of the numerical example 1 in Chapter 5, we are interested in the convergence rates in case of a arbitrary smooth exact solution.

The next Theorem can be found in [GH09, Theorem 5.5] (compare also [SM07]) and states that the solution converges upon h-refinement with optimal rates. As usual in a priori analysis, additional regularity beyond $u \in H_0^2(\Omega)$ leads to higher convergence rates. See [GH09] for hp a priori analysis of SIPDGM.

Theorem 3.13 (A priori error bound)**.** *Assume that for the solution u of* (2.4.2) *it holds that $u|_K \in H^{k_K+2}(K)$, $k_K \geqslant 2$, $K \in \mathcal{T}$. Then, the following error bound holds,*

$$\|u - u_{\mathcal{T}}\|_{\mathcal{T}}^2 \leqslant C \sum_{K \in \mathcal{T}} h_K^{2s_K} \, |u|_{H^{s_K+2}(K)}^2 \,, \tag{3.3.6}$$

where $1 \leqslant s_K \leqslant \min\{r-1, k_K\}$, and the constant $C > 0$ is independent of u and h.

This theorem implies that for a sufficient smooth solution, i.e. $u \in H_0^2(\Omega) \cap H^\ell(\Omega)$, $\ell \geqslant r+1$ (c.f. also [SM03], [SM07]) we have $\|u - u_{\mathcal{T}}\|_{\mathcal{T}} = \mathcal{O}(h_{\mathcal{T}}^{r-1}) = \mathcal{O}(N^{-(r-1)/2})$. In particular, we have $\mathcal{O}(N^{-1/2})$ for $r = 2$, $\mathcal{O}(N^{-1})$ for $r = 3$, $\mathcal{O}(N^{-3/2})$ for $r = 4$ and $\mathcal{O}(N^{-2})$ for $r = 5$. Here, N denotes the number of degrees of freedom. These are exactly the (asymptotical) rates we observe in Chapter 5, example 1.

3.4 A posteriori error bounds

a In this chapter we recall the residual-based a posteriori error indicator for SIPDGM from [GHV11]. For the sake of a complete presentation we state full proofs of reliability and efficiency.

In order to proof upper bounds of the a posteriori error estimator, we consider a recovery operator from (compare [GHV11, Section 3]), which maps $\mathbb{V}(\mathcal{T})$ onto a $H_0^2(\Omega)$-conforming space constructed by macro elements.

3.4.1 Smoothing operator

We start with the definition of the Hsieh-Clough Tocher (HCT) macro element (compare [DDPS79, BGS10, GHV11]).

Definition 3.14 (HCT element)**.** Let $\mathcal{T} \in \mathbb{G}$ and $K \in \mathcal{T}$. Then for $m \geqslant 4$ the HCT nodal macro finite element $(K, \hat{\mathbb{P}}_m(K), \mathcal{N}_m^{\texttt{HCT}}(K))$ is defined as follows.

a) The local space is given by

$$\hat{\mathbb{P}}_m(K) = \left\{ p \in C^1(K) \colon p|_{K_i} \in \mathbb{P}_m(K_i), \quad i = 1, 2, 3 \right\}.$$

Here, the three triangles K_1, K_2 and K_3 denote subtriangulation of K obtained by connecting the vertices of K with its barycenter; compare with Figure 3.2.

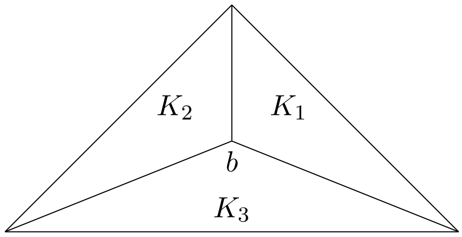


Figure 3.2: A macro triangle K subdivided into three small sub triangles which share a common point b.

b) The degrees of freedom $\mathcal{N}_m^{\mathtt{HCT}}(K)$ are given by (compare also with Figure 3.3)
   (i) the value of the functions and its gradient at the vertices of K;
   (ii) the function value at $(m-3)$ distinct points in the interior of each side $F \in \mathcal{F}_\mathcal{T}$, $F \subset \partial K$;
   (iii) the normal derivative at $(m-2)$ distinct points in the interior of each side $F \in \mathcal{F}_\mathcal{T}$, $F \subset \partial K$;
   (iv) the value of the function and its gradient at the barycentre of K;
   (v) the function value at $(m-4)$ distinct points in the interior of each edge $F \subset K_i$, $F \notin \mathcal{F}_\mathcal{T}$, $i = 1, 2, 3$;
   (vi) the normal derivative at $(m-4)$ distinct points in the interior of each edge $F \subset K_i$, $F \notin \mathcal{F}_\mathcal{T}$, $i = 1, 2, 3$;
   (vii) the function value at $(m-4)(m-5)/2$ distinct points in the interior of each K_i, $i = 1, 2, 3$ chosen so that if a polynomial of degree $(m-6)$ vanishes at those points, then it vanishes identically.

The corresponding finite element space is denoted by

$$\widetilde{\mathbb{V}}(\mathcal{T}) := \left\{ V \in C^1(\overline{\Omega}) \colon V|_K \in \hat{\mathbb{P}}_m(K) \text{ for all } K \in \mathcal{T} \right\}$$

and its global degrees of freedom are given by

$$\mathcal{N}_m^{\mathtt{HCT}}(\mathcal{T}) := \bigcup_{K \in \mathcal{T}} \mathcal{N}_m^{\mathtt{HCT}}(K),$$

which is well-posed thanks to conformity of $\widetilde{\mathbb{V}}(\mathcal{T}) \subset H^2(\Omega)$.

For $m = 4$ the degrees of freedom are depicted in Figure 3.3. Obviously, $\mathcal{N}_4^{\mathtt{HCT}}(K)$ contains the point evaluations in the vertices and edge midpoints of K (the Lagrange nodes $\mathcal{Z}_K$ of $\mathbb{P}_2(K)$). We emphasise that for a general polynomial degree $2 \leqslant r \leqslant 4$ the set of nodal points of the Lagrange basis is a subset of the set of the nodal points of the macro element of degree $r + 2$. This follows directly from the definition of the macro elements of the respective degree. Additionally, we have $\mathbb{P}_r(K) \subset \hat{\mathbb{P}}_{r+2}(K)$, and therefore we can apply $\mathcal{N}_{r+2}^{\mathtt{HCT}}(K)$ to $\mathbb{P}_r(K)$. Here, we are not interested in the case $r > 4$, since this would imply

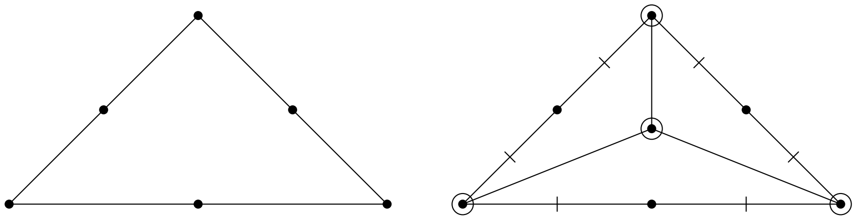


Figure 3.3: The Lagrange element of degree two and the corresponding macro element of degree four. Here point evaluations are denoted by small dots, (first) partial derivatives by circles and normal derivatives by lines. Compare [GHV11, DDPS79] for degrees of freedom related to macro elements of degree 5 and 6.

that $\mathbb{V}(\mathcal{T})$ contains a conforming discretisation of our fourth order problem, which would make HCT elements redundant. The reason is that for $r \geqslant 5$ it is possible to construct a basis for the space of C^1 piecewise polynomials which is parametrised by 'nodal variables', i.e. the values and derivatives of the basis functions at a discrete set of points (compare [MS75] and also [AFS68]). The case $m = 3$ corresponds to the classical HCT-Element ([Cia02b, Ch. 6]) which is not considered here.

We define the *recovery operator* $\mathcal{E}_{\mathcal{T}} \colon \mathbb{V}(\mathcal{T}) \to \widetilde{\mathbb{V}}(\mathcal{T}) \subset H_0^2(\Omega)$, by setting for all $K \in \mathcal{T}$ and all degrees of freedom $N_z^K \in \mathcal{N}_K^{\mathtt{HCT}}$, $z \in \mathcal{Z}_{\mathcal{T}}^{\mathtt{HCT}}$:

$$N_z(\mathcal{E}_{\mathcal{T}}(v)) = \begin{cases} \sum\limits_{K \in \omega_k(z)} \frac{|K|}{|\omega_k(z)|} N_z^K(v|_K) & z \notin \partial\Omega, \\ 0 & z \in \partial\Omega \end{cases} \tag{3.4.1}$$

Here, $\mathcal{Z}_{\mathcal{T}}^{\mathtt{HCT}}$ denotes the set of nodes z associated with some degree of freedom $N_z \in \mathcal{N}_{\mathcal{T}}^{\mathtt{HCT}}$ and corresponding local degree of freedom $N_z^K \in \mathcal{N}_K^{\mathtt{HCT}}$. Note that there may be different degrees of freedom associated with one node; compare with Figure 3.3.

Lemma 3.15. *Let $\mathcal{T} \in \mathbb{G}$. The operator $\mathcal{E}_{\mathcal{T}} \colon \mathbb{V}(\mathcal{T}) \to H_0^2(\Omega)$ defined in* (3.4.1) *satisfies*

$$\|D^\gamma(v - \mathcal{E}_{\mathcal{T}}(v))\|_K^2 \lesssim \int_{\mathcal{F}(N_{\mathcal{T}}(K))} \left| h_{\mathcal{T}}^{\frac{3}{2}-\gamma} \left[\!\left[\partial_n v\right]\!\right] \right|^2 + \left| h_{\mathcal{T}}^{1/2-\gamma} \left[\!\left[v\right]\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s, \tag{3.4.2}$$

with $\gamma = 0, 1, 2$, and the hidden constant depends only on the shape coefficient of $\mathcal{T}_0$. In particular this implies

$$\sum_{K \in \mathcal{T}} \left\|D^j(v - \mathcal{E}_{\mathcal{T}}(v))\right\|_K^2 \lesssim \left\| h_{\mathcal{T}}^{3/2-j} \left[\!\left[\partial_n v\right]\!\right] \right\|_{\Gamma_{\mathcal{T}}}^2 + \left\| h_{\mathcal{T}}^{1/2-j} \left[\!\left[v\right]\!\right] \boldsymbol{n} \right\|_{\Gamma_{\mathcal{T}}}^2. \tag{3.4.3}$$

Proof. The proof can be found in [GHV11, Lemma 3.1]. For the sake of completeness, we give a sketch of the proof. Let $K \in \mathcal{T}$ and $v \in \mathbb{V}(\mathcal{T})$, then an inverse estimate (Lemma 3.6) yields

$$\left\|D^j(v - \mathcal{E}_{\mathcal{T}}(v))\right\|_K^2 \lesssim \left\| h_{\mathcal{T}}^{-j}(v - \mathcal{E}_{\mathcal{T}}(v)) \right\|_K^2.$$

In order to keep the presentation readable we slightly modify the notation and write $\mathcal{N}(K)^{\texttt{HCT}}$ instead of $\mathcal{N}_K^{\texttt{HCT}}$. From equivalence of norms on a finite dimensional vector space we obtain

$$\left\|D^j(v - \mathcal{E}_{\mathcal{T}}(v))\right\|_K^2 \lesssim \sum_{i=0}^{1} \sum_{N_z \in \mathcal{N}(K,i)^{\texttt{HCT}}} h_K^{2(i-j+1)} (N_z(v - \mathcal{E}_{\mathcal{T}}(v)))^2,$$

where $\mathcal{N}(K,0)^{\texttt{HCT}}$ and $\mathcal{N}(K,1)^{\texttt{HCT}}$ are the nodal variables consisting of function evaluations, and those involving partial and normal derivatives (compare the Definition 3.14). Now, for each $N_z^K \in \mathcal{N}(K)^{\texttt{HCT}}$, which is not on the boundary $\partial\Omega$, we consider a local numbering $K_1, \ldots, K_{\#N_{\mathcal{T}}(z)-1}$ of the elements in $N_{\mathcal{T}}(z)$, such that each pair K_i, K_{i+1} share a common face $F = K_i \cap K_{i+1}$.

First, we regard the nodal variables $\mathcal{N}(K,0)^{\texttt{HCT}}$ only and use the arithmetic-geometric mean inequality (compare [KP03, Lemma 2.2]), to obtain

$$\begin{aligned}
&\sum_{N_z^K \in \mathcal{N}(K,0)^{\texttt{HCT}}} h_K^{2(1-j)} (N_z^K(v - \mathcal{E}_{\mathcal{T}}(v)))^2 \\
&= \sum_{\substack{N_z \in \mathcal{N}(K,0)^{\texttt{HCT}} \\ z \in K \cap \mathring{\Gamma}_{\mathcal{T}}}} h_K^{2(1-j)} \left(v(z)|_K - \frac{|K'|}{|\omega_k(z)|} \sum_{K' \in \omega_k(z)} v(z)|_{K'} \right)^2 \\
&\quad + \sum_{\substack{N_z \in \mathcal{N}(K,0)^{\texttt{HCT}} \\ z \in K \cap \Gamma_{\mathcal{T}}^b}} h_K^{2(1-j)} \left(v(z)|_K\right)^2 \\
&\lesssim \sum_{\substack{N_z \in \mathcal{N}(K,0)^{\texttt{HCT}} \\ z \in K \cap \mathring{\Gamma}_{\mathcal{T}}}} h_K^{2(1-j)} \left(\sum_{j=1}^{\#N_{\mathcal{T}}(z)-1} (v(z)|_{K_j} - v(z)|_{K_{j+1}})^2 \right) \\
&\quad + \sum_{\substack{N_z \in \mathcal{N}(K,0)^{\texttt{HCT}} \\ z \in K \cap \Gamma_{\mathcal{T}}^b}} h_K^{2(1-j)} \left(v(z)|_K\right)^2 \\
&\lesssim \max_{\substack{z \in F \\ F \in \mathcal{F}(N_{\mathcal{T}}(K))}} \left| h_{\mathcal{T}}^{1-j} [\![v(z)]\!] \boldsymbol{n} \right|^2 \\
&\lesssim \int_{\mathcal{F}(N_{\mathcal{T}}(K))} \left| h_{\mathcal{T}}^{1/2-j} [\![v]\!] \boldsymbol{n} \right|^2 \mathrm{d}s,
\end{aligned}$$

where we used a scaling argument in the last estimate.

The remaining proof of (3.4.2) follows analogously by splitting the nodal variables $\mathcal{N}(K,1)^{\texttt{HCT}} = \{\mathcal{N}(K,\boldsymbol{n})^{\texttt{HCT}} \cup \mathcal{N}(K,p)^{\texttt{HCT}}\}$, into the set of nodal variables evaluating only normal derivatives (Definition 3.14 b) (iii) and (vi)) and the remaining set of nodal variables (values of the gradient in Definition 3.14 b) (i) and (iv)); compare [GHV11, Lemma 3.1].

The second assertion follows from the local estimate (3.4.2) together with the finite overlap of the neighbourhoods $N_{\mathcal{T}}(K)$, $K \in \mathcal{T}$. □

3.4.2 Upper bounds

We introduce the a posteriori error estimators from [GHV11]. For $v \in \mathbb{V}(\mathcal{T})$ and $K \in \mathcal{T}$ let

$$\begin{aligned}\eta(v,K)^2 := &\int_K h_{\mathcal{T}}^4 \left|f - \Delta^2 v\right|^2 \,\mathrm{d}x \\ &+ \int_{\partial K \cap \Omega} h_{\mathcal{T}}^3 \left|[\![\nabla \cdot D^2_{\mathtt{pw}} v]\!] \cdot \boldsymbol{n}_K\right|^2 + h_{\mathcal{T}} \left|[\![D^2_{\mathtt{pw}} v]\!]\, \boldsymbol{n}_K\right|^2 \,\mathrm{d}s \\ &+ \int_{\partial K} \frac{\alpha^2}{h_{\mathcal{T}}} [\![\partial_n v]\!]^2 + \frac{\beta^2}{h_{\mathcal{T}}^3} \left|[\![v]\!]\, \boldsymbol{n}\right|^2 \,\mathrm{d}s.\end{aligned} \tag{3.4.4}$$

When $v = u_{\mathcal{T}}$ we simply write $\eta_{\mathcal{T}}(K) := \eta(u_{\mathcal{T}}, K)$. Moreover, for $\mathcal{M} \subset \mathcal{T}$, we set

$$\eta_{\mathcal{T}}(v, \mathcal{M}) := \left(\sum_{K \in \mathcal{M}} \eta(v,K)^2\right)^{1/2} \quad \text{and} \quad \eta_{\mathcal{T}}(\mathcal{M}) := \eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{M}).$$

From [GHV11, Theorem 4.1] we have that (3.4.4) defines a reliable estimator.

Proposition 3.16. *Let $u \in H_0^2(\Omega)$ be the solution of* (2.4.2) *and $u_{\mathcal{T}}$ the discrete solution of* (3.2.1). *Then,*

$$|\!|\!| u - u_{\mathcal{T}} |\!|\!|_{\mathcal{T}} \lesssim \eta_{\mathcal{T}}(\mathcal{T}),$$

where the constants in $\lesssim$ are independent of u, $u_{\mathcal{T}}$ and $h_{\mathcal{T}}$.

Proof. The statement follows by the same arguments as in [GHV11, Theorem 4.1]. In order to keep this thesis self-contained we give short proof. Let $v_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$ and $v \in H_0^2(\Omega)$ be arbitrary (to be defined later) and $\psi = v - v_{\mathcal{T}}$. Moreover, let $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}) \in \widetilde{\mathbb{V}}(\mathcal{T}) \cap H_0^2(\Omega)$ the smoothing operator as in (3.4.1). The error is decomposed in an $H_0^2(\Omega)$-conforming part and nonconforming part via

$$e := u - u_{\mathcal{T}} = (u - \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}})) + (\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}) - u_{\mathcal{T}}) \equiv e^c + e^{nc}.$$

For $u, v \in H_0^2(\Omega)$ we have $\mathcal{L}_{\mathcal{T}}(u) = \mathcal{L}_{\mathcal{T}}(v) = 0$ since all jump terms vanish. Consequently, since u is the solution of the weak problem we have $\mathfrak{B}_{\mathcal{T}}[u,\, v] = \int_\Omega f v \,\mathrm{d}x$. In conjunction with $\mathfrak{B}_{\mathcal{T}}[u_{\mathcal{T}},\, v_{\mathcal{T}}] = \int_\Omega f v_{\mathcal{T}} \,\mathrm{d}x$ this implies

$$\begin{aligned}\mathfrak{B}_{\mathcal{T}}[e,\, v] = \mathfrak{B}_{\mathcal{T}}[u,\, v] - \mathfrak{B}_{\mathcal{T}}[u_{\mathcal{T}},\, v] &= \int_\Omega f v \,\mathrm{d}x - \mathfrak{B}_{\mathcal{T}}[u_{\mathcal{T}},\, v - v_{\mathcal{T}}] - \mathfrak{B}_{\mathcal{T}}[u_{\mathcal{T}},\, v_{\mathcal{T}}] \\ &= \int_\Omega f \psi \,\mathrm{d}x - \mathfrak{B}_{\mathcal{T}}[u_{\mathcal{T}},\, \psi]\end{aligned}$$

and therefore by using $e^c = e - e^{nc}$

$$\mathfrak{B}_{\mathcal{T}}[e^c,\, v] = \int_\Omega f \psi \,\mathrm{d}x - \mathfrak{B}_{\mathcal{T}}[u_{\mathcal{T}},\, \psi] - \mathfrak{B}_{\mathcal{T}}[e^{nc},\, v].$$

Setting $v = e^c$ in the last equation, we obtain

$$\left\|D^2 e^c\right\|_\Omega^2 = \mathfrak{B}_{\mathcal{T}}[e^c,\, e^c] = \int_\Omega f \psi \,\mathrm{d}x - \mathfrak{B}_{\mathcal{T}}[u_{\mathcal{T}},\, \psi] - \mathfrak{B}_{\mathcal{T}}[e^{nc},\, e^c]. \tag{3.4.5}$$

By using the stability of the lifting operators (3.3.4) we deduce for the last term on the right-hand side of (3.4.5)

$$
\begin{aligned}
&|\mathfrak{B}_{\mathcal{T}}[e^{nc},\, e^{c}]| \\
&= \left| \int_\Omega (D^2_{\mathtt{pw}} e^{nc} + \mathcal{L}_{\mathcal{T}}(e^{nc}))\colon \mathcal{D}^2 e^c \,\mathrm{d}x \right| \\
&\lesssim \left(\left\| D^2_{\mathtt{pw}} e^{nc} \right\|_\Omega^2 + \left\| h_{\mathcal{T}}^{-1/2} \left[\!\left[\partial_n u_{\mathcal{T}} \right]\!\right] \right\|_{\Gamma_{\mathcal{T}}}^2 + \left\| h_{\mathcal{T}}^{-3/2} \left[\!\left[u_{\mathcal{T}} \right]\!\right] \boldsymbol{n} \right\|_{\Gamma_{\mathcal{T}}}^2 \right)^{1/2} \left\| D^2 e^c \right\|_\Omega . \\
&\lesssim \left(\left\| h_{\mathcal{T}}^{-1/2} \left[\!\left[\partial_n u_{\mathcal{T}} \right]\!\right] \right\|_{\Gamma_{\mathcal{T}}}^2 + \left\| h_{\mathcal{T}}^{-3/2} \left[\!\left[u_{\mathcal{T}} \right]\!\right] \boldsymbol{n} \right\|_{\Gamma_{\mathcal{T}}}^2 \right)^{1/2} \left\| D^2 e^c \right\|_\Omega ,
\end{aligned}
\tag{3.4.6}
$$

where we also used that the nonconforming part is bounded by

$$
\begin{aligned}
\left\| D^2_{\mathtt{pw}} e^{nc} \right\|_\Omega^2 &= \sum_{K \in \mathcal{T}} \int_K \left| D^2 (u_{\mathcal{T}} - \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}})) \right|^2 \,\mathrm{d}x \\
&\lesssim \left\| h_{\mathcal{T}}^{-1/2} \left[\!\left[\partial_n u_{\mathcal{T}} \right]\!\right] \right\|_{\Gamma}^2 + \left\| h_{\mathcal{T}}^{-3/2} \left[\!\left[u_{\mathcal{T}} \right]\!\right] \boldsymbol{n} \right\|_{\Gamma}^2 ,
\end{aligned}
$$

due to Lemma 3.15. For the remaining two terms on the right-hand side of (3.4.5) we use integration by parts, to obtain

$$
\begin{aligned}
&\int_\Omega f\psi \,\mathrm{d}x - \mathfrak{B}_{\mathcal{T}}[u_{\mathcal{T}},\, \psi] \\
&\quad = \int_\Omega (f - \Delta^2 u_{\mathcal{T}})\psi \,\mathrm{d}x - \int_\Omega (\mathcal{L}_{\mathcal{T}}(\psi)\colon D^2_{\mathtt{pw}} u_{\mathcal{T}} + \mathcal{L}_{\mathcal{T}}(u_{\mathcal{T}})\colon D^2_{\mathtt{pw}} \psi) \,\mathrm{d}x \\
&\qquad - \sum_{K \in \mathcal{T}} \left[\int_{\partial K} D^2 u_{\mathcal{T}} \nabla \psi \cdot \boldsymbol{n} - \psi \nabla \cdot D^2 u_{\mathcal{T}} \cdot \boldsymbol{n} \,\mathrm{d}s \right] \\
&\qquad - \int_{\mathcal{F}_{\mathcal{T}}} \frac{\alpha}{h_{\mathcal{T}}} \left[\!\left[\partial_n u_{\mathcal{T}} \right]\!\right] \left[\!\left[\partial_n \psi \right]\!\right] + \frac{\beta}{h_{\mathcal{T}}^3} \left[\!\left[u_{\mathcal{T}} \right]\!\right] \boldsymbol{n} \cdot \left[\!\left[\psi \right]\!\right] \boldsymbol{n} \,\mathrm{d}s.
\end{aligned}
\tag{3.4.7}
$$

On the one hand we have from $u_{\mathcal{T}}, v_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$ and $v \in H_0^2(\Omega)$ by the definition of the lifting operators

$$
\begin{aligned}
&\int_\Omega \mathcal{L}_{\mathcal{T}}(\psi)\colon D^2_{\mathtt{pw}} u_{\mathcal{T}} \\
&\quad = \int_{\mathcal{F}} \left\{\!\!\left\{ \nabla \cdot D^2_{\mathtt{pw}} u_{\mathcal{T}} \right\}\!\!\right\} \cdot \boldsymbol{n} \left[\!\left[\psi \right]\!\right] - \left\{\!\!\left\{ D^2_{\mathtt{pw}} u_{\mathcal{T}} \right\}\!\!\right\} \left[\!\left[\nabla \psi \right]\!\right] \cdot \boldsymbol{n} \,\mathrm{d}s
\end{aligned}
\tag{3.4.8}
$$

and on the other hand we have for the sum over element boundaries by (3.1.8) and (3.1.9)

$$
\begin{aligned}
&\sum_{K \in \mathcal{T}} \left[\int_{\partial K} D^2 u_{\mathcal{T}} \nabla \psi \cdot \boldsymbol{n} - \psi \nabla \cdot D^2 u_{\mathcal{T}} \cdot \boldsymbol{n} \,\mathrm{d}s \right] \\
&\quad = \int_{\mathcal{F}_{\mathcal{T}}} \left\{\!\!\left\{ D^2_{\mathtt{pw}} u_{\mathcal{T}} \right\}\!\!\right\} \left[\!\left[\nabla \psi \right]\!\right] \cdot \boldsymbol{n} - \left\{\!\!\left\{ \nabla \cdot D^2_{\mathtt{pw}} u_{\mathcal{T}} \right\}\!\!\right\} \cdot \boldsymbol{n} \left[\!\left[\psi \right]\!\right] \,\mathrm{d}s \\
&\qquad + \int_{\mathring{\mathcal{F}}_{\mathcal{T}}} \left[\!\left[D^2 u_{\mathcal{T}} \right]\!\right] \left\{\!\!\left\{ \nabla \psi \right\}\!\!\right\} \cdot \boldsymbol{n} - \left\{\!\!\left\{ \psi \right\}\!\!\right\} \left[\!\left[\nabla \cdot D^2_{\mathtt{pw}} u_{\mathcal{T}} \right]\!\right] \cdot \boldsymbol{n} \,\mathrm{d}s.
\end{aligned}
\tag{3.4.9}
$$

Consequently, using (3.4.8) and (3.4.9) in (3.4.7) we have

$$\begin{aligned}
&\int_\Omega f\psi \,\mathrm{d}x - \mathfrak{B}_{\mathcal{T}}[u_{\mathcal{T}},\, \psi] \\
&= \int_\Omega (f - \Delta^2 u_{\mathcal{T}})\psi - \mathcal{L}_{\mathcal{T}}(u_{\mathcal{T}})\colon D^2_{\mathtt{pw}}\psi \,\mathrm{d}x \\
&\quad - \int_{\mathring{\mathcal{F}}_{\mathcal{T}}} [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \,\{\!\{\nabla\psi\}\!\} \cdot \boldsymbol{n} - \{\!\{\psi\}\!\} \,[\![\nabla \cdot D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \cdot \boldsymbol{n} \,\mathrm{d}s \\
&\quad - \int_{\mathcal{F}_{\mathcal{T}}} \big(\frac{\alpha}{h_{\mathcal{T}}} [\![\partial_n u_{\mathcal{T}}]\!] \,[\![\partial_n \psi]\!] + \frac{\beta}{h^3_{\mathcal{T}}} [\![u_{\mathcal{T}}]\!] \,\boldsymbol{n} \cdot [\![\psi]\!] \,\boldsymbol{n}\big) \,\mathrm{d}s.
\end{aligned} \tag{3.4.10}$$

In order to bound the right-hand side of (3.4.10), we set $v_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$ to be the element-wise polynomial approximation to e^c such that

$$|e^c - v_{\mathcal{T}}|_{H^j(K)} \leqslant C h_K^{m-j} \,|e^c|_{H^m(K)} \quad 0 \leqslant j \leqslant m \leqslant 2, \quad K \in \mathcal{T}, \tag{3.4.11}$$

where, $C > 0$ is independent of the mesh-size (compare [Cia02a]). Note that Langrange interpolation is sufficient in this case since, functions in $H^2_0(\Omega)$ are continuous on two-dimensional domains ([Alt16, 10.13]).

Consequently, for the first term on the right-hand side of (3.4.10), we have by (3.4.11) and the stability of the lifting operators

$$\begin{aligned}
&\left| \int_\Omega (f - \Delta^2 u_{\mathcal{T}})\psi \,\mathrm{d}x - \mathcal{L}_{\mathcal{T}}(u_{\mathcal{T}})\colon D^2_{\mathtt{pw}}\psi) \,\mathrm{d}x \right| \\
&\lesssim \left(\left\| h^2_{\mathcal{T}}(f - \Delta^2 u_{\mathcal{T}}) \right\|^2_\Omega + \left\| h_{\mathcal{T}}^{-1/2} [\![\partial_n u_{\mathcal{T}}]\!] \right\|^2_{\Gamma_{\mathcal{T}}} + \left\| h_{\mathcal{T}}^{-3/2} [\![u_{\mathcal{T}}]\!] \,\boldsymbol{n} \right\|^2_{\Gamma_{\mathcal{T}}} \right)^{1/2} \left\| D^2 e^c \right\|_\Omega .
\end{aligned} \tag{3.4.12}$$

From (3.4.11) in conjuncotion with a scaled trace inequality we derive the following estimate:

$$\begin{aligned}
\int_{\mathring{\mathcal{F}}_{\mathcal{T}}} h_{\mathcal{T}}^{-1} \,|\{\!\{\nabla\psi\}\!\}|^2 \,\mathrm{d}s &= \sum_{F \in \mathring{\mathcal{F}}_{\mathcal{T}}} \int_F h_F^{-1} \,|\{\!\{\nabla\psi\}\!\}|^2 \,\mathrm{d}s \\
&\leqslant C \sum_{K \in \mathcal{T}} \int_{\partial K} h_K^{-1} \,|\{\!\{\nabla\psi\}\!\}|^2 \,\mathrm{d}s \\
&\leqslant C \sum_{K \in \mathcal{T}} \left(|h_{\mathcal{T}}^{-1}\psi|^2_{H^1(K)} + |\psi|^2_{H^2(K)} \right) \leqslant C \left\| D^2 e^c \right\|^2_\Omega ,
\end{aligned}$$

where the constant $C > 0$ is independent of the mesh-size. Using the last estimate for the second term on the right-hand side of (3.4.10) we obtain

$$\begin{aligned}
\left| \int_{\mathring{\mathcal{F}}_{\mathcal{T}}} [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \,\{\!\{\nabla\psi\}\!\} \cdot \boldsymbol{n} \,\mathrm{d}s \right| &= \left| \int_{\mathring{\mathcal{F}}_{\mathcal{T}}} [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \,\boldsymbol{n} \cdot \{\!\{\nabla\psi\}\!\} \,\mathrm{d}s \right| \\
&\lesssim \left\| h_{\mathcal{T}}^{1/2} [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \,\boldsymbol{n}_{\mathcal{F}} \right\|_{\mathring{\Gamma}_{\mathcal{T}}} \left\| D^2 e^c \right\|_\Omega .
\end{aligned} \tag{3.4.13}$$

By similar calculations we derive

$$\int_{\mathring{\mathcal{F}}_{\mathcal{T}}} h_{\mathcal{T}}^{-3} \,|\{\!\{\psi\}\!\}|^2 \,\mathrm{d}s \lesssim \sum_{K \in \mathcal{T}} \left(\left\| h_{\mathcal{T}}^{-2}\psi \right\|^2_K + |h_{\mathcal{T}}^{-1}\psi|^2_{H^1(K)} \right) \lesssim \left\| D^2 e^c \right\|^2_\Omega$$

to bound the third term of (3.4.10) by

$$\begin{aligned}\left|\int_{\mathring{\mathcal{F}}_{\mathcal{T}}} \{\!\{\psi\}\!\} \, [\![\nabla \cdot D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \cdot \boldsymbol{n} \, \mathrm{d}s\right| \\ \lesssim \left\| h_{\mathcal{T}}^{3/2} \, [\![\nabla \cdot D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \cdot \boldsymbol{n} \right\|_{\mathring{\Gamma}_{\mathcal{T}}} \left\| D^2 e^c \right\|_{\Omega} .\end{aligned} \tag{3.4.14}$$

Analogous arguments applied to the penalty terms leads us to

$$\begin{aligned}\left|\int_{\mathcal{F}_{\mathcal{T}}} \frac{\alpha}{h_{\mathcal{T}}} [\![\partial_n u_{\mathcal{T}}]\!] \, [\![\partial_n \psi]\!] + \frac{\beta}{h_{\mathcal{T}}^3} [\![u_{\mathcal{T}}]\!] \, \boldsymbol{n} \cdot [\![\psi]\!] \, \boldsymbol{n} \, \mathrm{d}s\right| \\ \lesssim \left(\alpha^2 \left\| h_{\mathcal{T}}^{-1/2} \, [\![\partial_n u_{\mathcal{T}}]\!] \right\|_{\Gamma}^2 + \beta^2 \left\| h_{\mathcal{T}}^{-3/2} \, [\![u_{\mathcal{T}}]\!] \, \boldsymbol{n} \right\|_{\Gamma}^2 \right) \left\| D^2 e^c \right\|_{\Omega} .\end{aligned} \tag{3.4.15}$$

Finally, for the conforming part of the error we obtain by (3.4.5), (3.4.6), (3.4.10) and (3.4.12) -(3.4.15)

$$\begin{aligned}\left\| D^2 e^c \right\|_{\Omega} \lesssim \Bigg(& \left\| h_{\mathcal{T}}^2 (f - \Delta^2 u_{\mathcal{T}}) \right\|_{\Omega}^2 + \left\| h_{\mathcal{T}}^{3/2} \, [\![\nabla \cdot D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \cdot \boldsymbol{n} \right\|_{\mathring{\Gamma}_{\mathcal{T}}} \\ & + \left\| h_{\mathcal{T}}^{1/2} \, [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \, \boldsymbol{n} \right\|_{\mathring{\Gamma}_{\mathcal{T}}} + \alpha^2 \left\| h_{\mathcal{T}}^{-1/2} \, [\![\partial_n u_{\mathcal{T}}]\!] \right\|_{\Gamma}^2 \beta^2 \left\| h_{\mathcal{T}}^{-3/2} \, [\![u_{\mathcal{T}}]\!] \, \boldsymbol{n} \right\|_{\Gamma}^2 \Bigg)^{1/2} .\end{aligned}$$

Finally the triangle inequality

$$\left\| D^2_{\mathtt{pw}} e \right\|_{\Omega} \leqslant \left\| D^2 e^c \right\|_{\Omega} + \left\| D^2_{\mathtt{pw}} e^{nc} \right\|_{\Omega}$$

concludes the proof. □

3.4.3 Lower Bounds

In this section we state [GHV11, Theorem4.4], providing the efficiency of the SIPDG error indicator. In the proofs of the so-called lower bounds we make heavily use of the fact, that our finite-element space is finite dimensional. Obviously, this is not true for $L^2(\Omega)$. Consequently, we need to project f onto the finite dimensional space $\mathbb{P}_r(\mathcal{T})$ by using the L^2-orthogonal projection $\Pi\colon L^2(\Omega) \to \mathbb{P}_r(\mathcal{T})$, defined in (3.1.4). Hence, for $v \in \mathbb{V}(\mathcal{T})$ the element residual in (3.4.4) is bounded by

$$\int_K h_{\mathcal{T}}^4 \left| f - \Delta^2 v \right|^2 \leqslant \int_K h_{\mathcal{T}}^4 \left| \Pi f - \Delta^2 v \right|^2 + \int_K h_{\mathcal{T}}^4 \left| f - \Pi f \right|^2 .$$

The term $\int_K h_{\mathcal{T}}^4 \left| f - \Pi f \right|^2 \mathrm{d}x := \operatorname{osc}(K, f)^2$ is called the local *data oscillation* and is bounded by the estimator. Indeed, for all $v \in \mathbb{V}(\mathcal{T})$ we have by the properties of the L^2-projection operator

$$\begin{aligned}\int_K h_{\mathcal{T}}^4 \left| f - \Pi f \right|^2 &= \int_K h_{\mathcal{T}}^4 \left| f - \Delta^2 v + \Pi(\Delta^2 v - f) \right|^2 \\ &\leqslant 2 \int_K h_{\mathcal{T}}^4 \left| f - \Delta^2 v \right|^2 .\end{aligned} \tag{3.4.16}$$

Proposition 3.17. *Let $u \in H_0^2(\Omega)$ be the solution of* (2.4.2) *and $u_\mathcal{T}$ the discrete solution of* (3.2.1)*. Then, for each $K \in \mathcal{T}$ we have*

$$\left\|h_K^2(f - \Delta^2 u_\mathcal{T})\right\|_K^2 \lesssim \left\|D^2(u - u_\mathcal{T})\right\|_K^2 + \left\|h_K^2(f - \Pi f)\right\|_K^2 \tag{3.4.17}$$

and for each $F \in \mathring{\mathcal{F}}_\mathcal{T}$ with $F = K_1 \cap K_2$ we have

$$\left\|h_F^{1/2} \left[\!\left[D_{pw}^2 u_\mathcal{T}\right]\!\right] \boldsymbol{n}_F\right\|_F \lesssim \left\|D_{pw}^2(u - u_\mathcal{T})\right\|_{K_1 \cup K_2}^2 + \left\|h_\mathcal{T}^2(f - \Pi f)\right\|_{K_1 \cup K_2}^2 \tag{3.4.18}$$

and

$$\begin{aligned}\left\|h_F^{3/2} \left[\!\left[\nabla \cdot D_{pw}^2 u_\mathcal{T}\right]\!\right] \cdot \boldsymbol{n}_F\right\|_F \lesssim& \left\|D_{pw}^2(u - u_\mathcal{T})\right\|_{K_1 \cup K_2}^2 \\ &+ \left\|h_\mathcal{T}^2(f - \Pi f)\right\|_{K_1 \cup K_2}^2.\end{aligned} \tag{3.4.19}$$

In particular, for all $\mathcal{M} \subset \mathcal{T} \in \mathbb{G}$ and for all $v \in \mathbb{V}(\mathcal{T})$ and $K \in \mathcal{T}$, we have

$$\eta_\mathcal{T}(\mathcal{M}) \lesssim \|u - v\|_{N_\mathcal{T}(\mathcal{M})} + \operatorname{osc}(N_\mathcal{T}(\mathcal{M}), f), \tag{3.4.20}$$

where all constants in $\lesssim$ are independent of $u_\mathcal{T}$ and $h_\mathcal{T}$. Here, the data-oscillation is defined on $\mathcal{M} \subset \mathcal{T}$ by

$$\operatorname{osc}(\mathcal{M}, f) := \left(\sum_{K \in \mathcal{M}} \operatorname{osc}(K, f)^2\right)^{1/2} \mathrm{d}x.$$

Proof. We follow the lines of [GHV11, Theorem 4.4].

1: For the element residual we fix $K \in \mathcal{T}$ and let $v \in H_0^2(\Omega) \cap H_0^2(K)$, with $v \equiv 0$ on $\Omega \backslash K$, be a polynomial which we define later. Since v is a test function in $H_0^2(\Omega)$ and vanishes outside the element K, we have

$$\begin{aligned}\int_K D^2(u - u_\mathcal{T})\colon D^2 v \,\mathrm{d}x &= \int_K D^2 u\colon D^2 v \,\mathrm{d}x - \int_K D^2 u_\mathcal{T}\colon D^2 v \,\mathrm{d}x \\ &= \int_K f v \,\mathrm{d}x - \int_K (\Delta^2 u_\mathcal{T}) v \,\mathrm{d}x,\end{aligned} \tag{3.4.21}$$

where we used integration by parts in conjunction with the fact that $v \in H_0^2(K)$. Hence, we have

$$\begin{aligned}\int_K D^2(u - u_\mathcal{T})\colon D^2 v \,\mathrm{d}x &= \int_K (f - \Delta^2 u_\mathcal{T}) v \,\mathrm{d}x \\ &= \int_K (\Pi f - \Delta^2 u_\mathcal{T}) v \,\mathrm{d}x + \int_K (f - \Pi f) v \,\mathrm{d}x\end{aligned}$$

and as a result

$$\begin{aligned}&\int_K (\Pi f - \Delta^2 u_\mathcal{T}) v \,\mathrm{d}x \\ &\quad \leqslant \left\|D^2(u - u_\mathcal{T})\right\|_K \left\|D^2 v\right\|_K + \|f - \Pi f\|_K \|v\|_K \\ &\quad \lesssim \left(\left\|D^2(u - u_\mathcal{T})\right\|_K + \left\|h_K^2(f - \Pi f)\right\|_K\right) \left\|h_K^{-2} v\right\|_K.\end{aligned} \tag{3.4.22}$$

Now, fix $v|_K = (\Pi f - \Delta^2 u_{\mathcal{T}})b_K^2$, where $\psi_K \colon K \to \mathbb{R}$ is the element bubble function on the element K (compare Appendix B), defined as $\psi_K := \hat{\psi}_{\hat{K}} \circ F_K$, where $\psi_{\hat{K}} := 27\hat{\lambda}_0\hat{\lambda}_1\hat{\lambda}_2$ is the bubble function on the reference element $\hat{K}$ with barycentric coordinates $\hat{\lambda}_0, \hat{\lambda}_1, \hat{\lambda}_2$ and the scaling is due to the normalisation, i.e. $\psi_{\hat{K}}$ attains the value 1 at the barycentre of $\hat{K}$. Using a scaling argument together with the fact that $\mathbb{P}_r(K)$ is finite dimensional, we obtain $\left\|\Pi f - \Delta^2 u_{\mathcal{T}}\right\|_K \approx \left\|(\Pi f - \Delta^2 u_{\mathcal{T}})\psi_K\right\|_K$ (compare Lemma B.2, Appendix B, p. 121). Hence, we deduce

$$\left\|\Pi f - \Delta^2 u_{\mathcal{T}}\right\|_K^2 \lesssim \int_K (\Pi f - \Delta^2 u_{\mathcal{T}})^2 \psi_K^2 \,\mathrm{d}x = \int_K (\Pi f - \Delta^2 u_{\mathcal{T}}) v \,\mathrm{d}x. \quad (3.4.23)$$

Finally, using the triangle inequality together with (3.4.22) and (3.4.23) yields

$$\begin{aligned}
\left\|f - \Delta^2 u_{\mathcal{T}}\right\|_K^2 &\leqslant \|f - \Pi f)\|_K^2 + \left\|\Pi f - \Delta^2 u_{\mathcal{T}}\right\|_K^2 \\
&\lesssim \|(f - \Pi f)\|_K^2 + \int_K (\Pi f - \Delta^2 u_{\mathcal{T}}) v \,\mathrm{d}x \\
&\lesssim \left(\left\|D^2(u - u_{\mathcal{T}})\right\|_K + \left\|h_K^2(f - \Pi f)\right\|_K\right) h_K^{-2} \left\|v\right\|_K \\
&\lesssim \left(\left\|D^2(u - u_{\mathcal{T}})\right\|_K + \left\|h_K^2(f - \Pi f)\right\|_K\right) h_K^{-2} \left\|f - \Delta^2 u_{\mathcal{T}}\right\|_K^2
\end{aligned}$$

which implies (3.4.17).

$\boxed{2}$: In order to proof (3.4.18) fix $F \in \mathring{\mathcal{F}}$, $F = K_1 \cap K_2$ and let $\tilde{K}$ be the largest rhombus contained in $K_1 \cup K_2$, compare figure B.2 (p. 119). Moreover, let $\psi_{\tilde{K}} \colon \tilde{K} \to \mathbb{R}$ be the bubble function on $\tilde{K}$ and b_ℓ be a linear polynomial defined on $\tilde{K}$ such that $b_\ell|_F = 0$ and $\nabla b_\ell = h_F^{-1} \left[\!\left[D^2_{\mathtt{pw}} u_{\mathcal{T}}\right]\!\right] \boldsymbol{n}_K$. We define

$$b_F := \begin{cases} b_\ell \psi_{\tilde{K}}^3 & \text{on } \tilde{K}, \\ 0 & \text{on } \Omega \backslash \tilde{K}, \end{cases}$$

satisfying:

(i) b_F vanishes on the boundary $\partial\tilde{K}$ together with its first and second derivatives,

(ii) $b_F \in C^2(\Omega) \cap H_0^2(\Omega)$,

(iii) $[\![b_F]\!]\,|_F = [\![\nabla b_F]\!]\,|_F = \{\!\{b_F\}\!\}\,|_F = 0$ for all $F \in \mathcal{F}$.

(iv) $(\{\!\{\nabla b_F\}\!\})|_F = (\psi_{\tilde{K}}^3 h_F^{-1} \left[\!\left[D^2_{\mathtt{pw}} u_{\mathcal{T}}\right]\!\right] \boldsymbol{n}_F)|_F$ and $(\{\!\{\nabla b_{F'}\}\!\})|_{F'} = 0$ for all $F' \in \mathcal{F}\backslash F$.

Here, the statements in (i) and (ii) follow from the construction of the smooth bubble function $b_{\tilde{K}}$ (compare Appendix B; p.119), (iii) follows from $b_F \in C^2(\Omega)$ (resp. $b_\ell|_F = 0$) and (iv) follows again from $b_\ell|_F = 0$ since we have

$$(\nabla b_F)|_F = (\psi_{\tilde{K}}^3 \nabla b_\ell + 0)|_F = (h_F^{-1} \left[\!\left[D^2_{\mathtt{pw}} u_{\mathcal{T}}\right]\!\right] \boldsymbol{n}_F)|_F$$

Now, let ϕ be a constant function on $\tilde{K}$, and set $v = \phi b_F$. Using this v and arguing as in (3.4.21), but on the domain $\tilde{K} \subset K_1 \cup K_2$ instead of K, we get

$$\begin{aligned}
&\int_{\tilde{K}} D^2(u - u_{\mathcal{T}})\colon D^2 v \, \mathrm{d}x \\
&= \int_{K_1 \cup K_2} D^2(u - u_{\mathcal{T}})\colon D^2 v \, \mathrm{d}x \\
&= \int_{K_1 \cup K_2} (f - \Delta^2 u_{\mathcal{T}}) v \, \mathrm{d}x \\
&\quad - \sum_{K' \in \{K_1, K_2\}} \int_{\partial K'} D^2 u_{\mathcal{T}} \nabla v \cdot \boldsymbol{n}_{K'} - v \nabla \cdot D^2 u_{\mathcal{T}} \cdot \boldsymbol{n}_{K'} \, \mathrm{d}x,
\end{aligned} \tag{3.4.24}$$

where we used that v vanishes to the second order on $\partial\tilde{K}$ and $v = 0$ on $\Omega \backslash \tilde{K}$. In equation (3.4.24) we reformulate the sum over element boundaries by using (3.1.8) and (3.1.9), to obtain

$$\begin{aligned}
\int_{K_1 \cup K_2} D^2(u - u_{\mathcal{T}})\colon D^2 v \, \mathrm{d}x = &\int_{K_1 \cup K_2} (f - \Delta^2 u_{\mathcal{T}}) v \, \mathrm{d}x \\
&- \int_F [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \, \{\!\{ \nabla v \}\!\} \cdot \boldsymbol{n} \, \mathrm{d}s.
\end{aligned} \tag{3.4.25}$$

Here, we used $[\![\nabla v]\!]|_F = [\![v]\!]|_F = \{\!\{ v \}\!\}|_F = 0$ for all $F \in \mathcal{F}$, thanks to property (iii) of the function b_F. Setting $\phi \equiv h_F^{-1}$ in (3.4.25) gives

$$\begin{aligned}
\int_F [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \, \{\!\{ \nabla v \}\!\} \cdot \boldsymbol{n} \, \mathrm{d}s &= \int_F h_F^{-1} [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \, \{\!\{ \nabla b_F \}\!\} \cdot \boldsymbol{n} \, \mathrm{d}s \\
&= \int_F h_F^{-1} [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \, b_{\tilde{K}}^3 h_F^{-1} [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \, \boldsymbol{n} \cdot \boldsymbol{n} \, \mathrm{d}s \\
&= \left\| \psi_{\tilde{K}}^{3/2} h_F^{-1} [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \, \boldsymbol{n} \right\|_F^2 \\
&\gtrsim \left\| h_F^{-1} [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \, \boldsymbol{n} \right\|_F^2,
\end{aligned} \tag{3.4.26}$$

where we used (B.1.2) (Appendix B, p. 121) in the last estimate. Now, in view of the Poincaré inequality, together with $h_{\tilde{K}} \lesssim h_{K_i} \lesssim h_F$, $1 \leqslant i \leqslant 2$ we have

$$\begin{aligned}
\|v\|_{K_1 \cup K_2}^2 &\lesssim h_{K_1}^2 \|\nabla v\|_{K_1 \cup K_2}^2 \lesssim \|\nabla b_F\|_{K_1 \cup K_2}^2 \lesssim \|\nabla b_\ell\|_{K_1 \cup K_2}^2 \\
&\lesssim \left|\tilde{K}\right| \left| h_F^{-1} [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \, \boldsymbol{n}_F \right|^2 \lesssim \left\| h_F^{-1/2} [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \, \boldsymbol{n}_F \right\|_F^2,
\end{aligned} \tag{3.4.27}$$

where we used (B.1.2) (Appendix B, p. 121). Combining (3.4.25) and (3.4.26) in conjunction with the Cauchy-Schwarz inequality and an inverse estimate we get

$$\begin{aligned}
&\left\| h_F^{-1} [\![D^2_{\mathtt{pw}} u_{\mathcal{T}}]\!] \, \boldsymbol{n} \right\|_F^2 \\
&\lesssim \left(h_{\tilde{K}}^{-3/2} \left\| D^2_{\mathtt{pw}}(u - u_{\mathcal{T}}) \right\|_{K_1 \cup K_2} + \left\| h_{\mathcal{T}}^{1/2} (f - \Delta^2 u_{\mathcal{T}}) \right\|_{K_1 \cup K_2} \right) \left\| h_F^{-1/2} v \right\|_{K_1 \cup K_2},
\end{aligned} \tag{3.4.28}$$

where we used also $h_{\tilde{K}} \approx h_F$. Multiplying both sides of (3.4.28) by $h_F^{1/2}$ and using (3.4.27) we end up with

$$\left\| h_F^{-1/2} \llbracket D^2_{\mathtt{pw}} u_{\mathcal{T}} \rrbracket \boldsymbol{n} \right\|_F \lesssim h_{\tilde{K}}^{-1} \left\| D^2_{\mathtt{pw}}(u - u_{\mathcal{T}}) \right\|_{K_1 \cup K_2} + \left\| h_{\mathcal{T}}(f - \Delta^2 u_{\mathcal{T}}) \right\|_{K_1 \cup K_2}. \tag{3.4.29}$$

Finally, (3.4.18) follows by multiplying both sides of (3.4.29) by $h_{\tilde{K}}$ and using $h_{\tilde{K}} \approx h_F$.

[3]: The estimate (3.4.19) follows by similar arguments. Nonetheless, we will sketch the proof in sake of completeness. Consider the function $\psi_{\tilde{K}}$ (continuous bubble function on the rhombus $\tilde{K}$) and recall from Appendix B:

(i) $\psi_{\tilde{K}}^3 \in C^2(\Omega) \cap H_0^2(\Omega)$,

(ii) $\left\llbracket \psi_{\tilde{K}}^3 \right\rrbracket |_F = \left\llbracket \nabla \psi_{\tilde{K}}^3 \right\rrbracket |_F = \left\{\!\!\left\{ \nabla \psi_{\tilde{K}}^3 \right\}\!\!\right\} |_F = 0$ for all $F \in \mathcal{F}$ and

(iii) $\left\{\!\!\left\{ \psi_{\tilde{K}} \right\}\!\!\right\} |_{F'} = 0$ for all $F' \in \mathcal{F} \backslash F$.

Now, let ξ be a constant function in the normal direction to F and set $v = \xi \psi_{\tilde{K}}^3$. We use integration by parts as in (3.4.24), reformulate the integral over element boundaries to face integrals and deduce

$$\int_{K_1 \cup K_2} D^2(u - u_{\mathcal{T}}) : D^2 v \,\mathrm{d}x = \int_{K_1 \cup K_2} (f - \Delta^2 u_{\mathcal{T}}) v \,\mathrm{d}x - \int_F \{\!\!\{ v \}\!\!\} \llbracket \nabla \cdot D^2_{\mathtt{pw}} u_{\mathcal{T}} \rrbracket \cdot \boldsymbol{n} \,\mathrm{d}s. \tag{3.4.30}$$

Note that on the right-hand side of (3.4.30) orientation of $\boldsymbol{n}$ is independent of the ordering of K_1 and K_2, compare Remark 3.1. Next, set $\xi|_F = \left\llbracket \nabla \cdot D^2_{\mathtt{pw}} u_{\mathcal{T}} \right\rrbracket \cdot \boldsymbol{n}_F$ in (3.4.30), use a norm-equivalence, and standard estimates to obtain

$$\begin{aligned} \left\| h_F^{-1/2} \llbracket \nabla \cdot D^2_{\mathtt{pw}} u_{\mathcal{T}} \rrbracket \cdot \boldsymbol{n}_F \right\|_F &\lesssim \int_F \{\!\!\{ v \}\!\!\} \llbracket \nabla \cdot D^2_{\mathtt{pw}} u_{\mathcal{T}} \rrbracket \cdot \boldsymbol{n}_F \,\mathrm{d}s \\ &\lesssim h_{\tilde{K}}^{-2} \left\| D^2_{\mathtt{pw}}(u - u_{\mathcal{T}}) \right\|_{K_1 \cup K_2} + \left\| (f - \Delta^2 u_{\mathcal{T}}) \right\|_{K_1 \cup K_2}. \end{aligned}$$

Consequently, (3.4.19) follows by multiplying both side of the last estimate by $h_{\tilde{K}}^2 \approx h_F^2$.

Note that $K \in \mathcal{T}$ in step [1] (resp. $F \in \mathring{\mathcal{F}}$ in steps [2] and [3]) were chosen arbitrarily and therefore the proof of the bounds in (3.4.17)-(3.4.19) is completed.

[4]: The global estimate in the last assertion of Proposition 3.17 follows from the local ones together with the definition of the error indicators on a subset $\mathcal{M} \subset \mathcal{T}$ and the definition of the global data-oscillation. □

3.5 Functions of Bounded Variation

3.5.1 Motivating example

The following basic example of convergence of *conforming* adaptive finite element methods gives an overview of the various theoretical concepts of our convergence theory (compare e.g. [MSV08, NV11, Sie11] for details).

Let $\Omega \subset \mathbb{R}^d$ be a polygonal domain with Lipschitz boundary. Consider a Hilbert space $((\mathbb{V}, \langle \cdot, \cdot \rangle_{\mathbb{V}})$ with underlying domain Ω and let $\mathfrak{B}\colon \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ be a symmetric bilinear form, which is coercive and continuous with respect to $\|\cdot\|_{\mathbb{V}} = \sqrt{\langle \cdot, \cdot \rangle_{\mathbb{V}}}$. Here, we chose

- $\mathbb{V} = H_0^2(\Omega)$,
- $\mathfrak{B}[v, w] = \int_\Omega D^2 v\colon D^2 w \,\mathrm{d}x$ and
- $\mathbb{V}(\mathcal{T})$ a $H^2(\Omega)$-conforming finite element space (e.g. Argyris finite element space [AFS68]).

For $f \in \mathbb{V}'$, consider the following abstract problem: Find $u \in \mathbb{V}$ such that

$$\mathfrak{B}[u, v] = \langle f, v \rangle_{\mathbb{V}', \mathbb{V}} \quad \forall v \in \mathbb{V}.$$

Due to the Lax-Milgram Theorem 2.8 there exists a unique solution $u \in \mathbb{V}$.

Let $\mathcal{T}$ be a conforming and shape regular subdivision of the domain Ω and $\mathbb{V}(\mathcal{T}) \subset \mathbb{V}$ be a *conforming* finite dimensional space. We consider the following discrete problem: Find $U_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$ such that

$$\mathfrak{B}[U_{\mathcal{T}}, V] = \langle f, V \rangle_{\mathbb{V}', \mathbb{V}} \quad \forall V \in \mathbb{V}(\mathcal{T}). \tag{3.5.1}$$

The Lax-MIlgram Theorem 2.8 implies the existence of a unique solution $U_{\mathcal{T}} \in \mathbb{V}(\mathcal{T})$ of the discrete problem. In this context we emphasise that continuity and coercivity of $\mathfrak{B}_{\mathcal{T}}$ on $\mathbb{V}(\mathcal{T})$ are inherited from continuity and coercivity of $\mathfrak{B}_{\mathcal{T}}$ on $\mathbb{V}$.

Now, let $\{\mathcal{T}_k\}_{k \geqslant 0}$ be a sequence of partitions of the domain Ω (e.g. think about the application of some adaptive algorithm for the discrete problem). We assume that the corresponding discrete spaces satisfy

- for each $k \in \mathbb{N}_0$ $\mathbb{V}_k := \mathbb{V}(\mathcal{T}_k)$ is conforming and shape regular and
- we have a *nested* sequence of discrete spaces, i.e. $\mathbb{V}_0 \subset \mathbb{V}_1 \subset \ldots \subset \mathbb{V}_\ell \subset \mathbb{V}_{\ell+1} \subset \ldots$ for all $\ell \in \mathbb{N}_0$.

Hence, we have a sequence of discrete solutions $\{U_{\mathcal{T}_k}\}_{k \in \mathbb{N}}$ (which we denote by $\{U_k\}_{k \in \mathbb{N}}$ for brevity.) corresponding to the sequence $\{\mathbb{V}_k\}_{k \in \mathbb{N}}$ of discrete spaces.

The aim of this section is to prove the convergence of the sequence $\{U_k\}_{k \in \mathbb{N}}$ to some *limit function*, located in some *limit space*. To this end we define the limit space as the completion $\mathbb{V}_\infty := \overline{\bigcup_{k \geqslant 0} \mathbb{V}_k}^{\|\cdot\|_{\mathbb{V}}} \subset \mathbb{V}$. Since $\mathbb{V}_\infty$ is closed in $\mathbb{V}$ we conclude that the following problem yields a unique solution: Find $u_\infty \in \mathbb{V}_\infty$ such that

$$\mathfrak{B}[u_\infty, v] = \langle f, v \rangle_{\mathbb{V}', \mathbb{V}} \quad \forall v \in \mathbb{V}_\infty. \tag{3.5.2}$$

On the one hand, we have a sequence of discrete solution $\{U_k\}_{k \geqslant 0}$ corresponding to the sequence of partitions and on the other hand we have some limit solution u_∞ located in the limit space $\mathbb{V}_\infty$. Consequently, we shall prove that $u_\infty \in \mathbb{V}_\infty$ is the limit of $\{U_k\}_{k \geqslant 0}$ with respect to the norm $\|\cdot\|_{\mathbb{V}}$, i.e.

$\|U_k - u_\infty\|_{\mathbb{V}} \to 0$ as $k \to \infty$ (neglecting the question if $u_\infty = u$ for the moment). We emphasise that for $k \in \mathbb{N}_0$ we have from $\mathbb{V}_k \subset \mathbb{V}_\infty$ and therefore we can use Cea's Lemma ([Cia02a, Theorem 2.4.1]) to obtain

$$\|u_\infty - U_k\|_{\mathbb{V}} \leqslant \inf_{V \in \mathbb{V}_k} \|u_\infty - V\|_{\mathbb{V}} \to 0$$

as $k \to \infty$ by the definition of $\mathbb{V}_\infty$.

Nonetheless, we want to give a proof of $\lim_{k\to\infty} \|U_k - u_\infty\|_{\mathbb{V}} = 0$ which is closer to the analysis we use in the sequel without using Cea's Lemma. Observe that $\|U_k\|_{\mathbb{V}} \leqslant C \|f\|_{\mathbb{V}'}$ from coercivity and continuity of the bilinear form $\mathfrak{B}$. Since $\mathbb{V}$ is a Hilbert space, Theorem 2.5 provides a weak limit $\overline{u}_\infty \in \mathbb{V}$ such that

$$U_k \rightharpoonup \overline{u}_\infty \in \mathbb{V}, \quad \text{as } k \to \infty.$$

Let $v \in \mathbb{V}_\infty$. From the definition of the limit space there exists a sequence $\{V_k\}_{k\geqslant 0}$, $V_k \in \mathbb{V}_k$, such that $\|V_k - v\|_{\mathbb{V}} \to 0$ as $k \to \infty$. Consequently

$$\mathfrak{B}[\bar{u}_\infty, v] \leftarrow \mathfrak{B}[U_k, V_k] = \langle f, V_k \rangle_{\mathbb{V}',\mathbb{V}} \to \langle f, v \rangle_{\mathbb{V}',\mathbb{V}}, \quad \text{as } k \to \infty$$

and therefore $\overline{u}_\infty = u_\infty$, thanks to the uniqueness of the solution u_∞. From the properties of the bilinear form $\mathfrak{B}$ we conclude

$$\begin{aligned}\frac{1}{C} \|U_k - u_\infty\|_{\mathbb{V}}^2 &\leqslant \mathfrak{B}[U_k - u_\infty, U_k - u_\infty] \\ &= \underbrace{\mathfrak{B}[U_k, U_k]}_{=\langle f, U_k \rangle_{\mathbb{V}',\mathbb{V}}} - 2\mathfrak{B}[u_\infty, U_k] + \underbrace{\mathfrak{B}[u_\infty, u_\infty]}_{\langle f, u_\infty \rangle_{\mathbb{V}',\mathbb{V}}} \to 0\end{aligned}$$

as $k \to \infty$. This is the desired convergence $U_k \to u_\infty$, as $k \to \infty$.

Coming back to original problem (3.5.1), we observe, that in order to proof $\lim_{k\to\infty} \|U_k - u\|_{\mathbb{V}} = 0$, we have to proof $u = u_\infty$. This, in turn, is equivalent to proof $u \in \mathbb{V}_\infty$, thanks to the conformity of the discrete spaces and uniqueness of the solution u_∞ of (3.5.2).

This basic examples yields that $\|U_k - u_\infty\|_{\mathbb{V}} \to 0$ as $k \to \infty$ heavily relies on the properties of the underlying *conforming* discrete spaces and the compactness properties of the Hilbert space $\mathbb{V}$.

If we replace the conforming finite element spaces $\mathbb{V}_k$ by nonconforming discontinuous Galerkin spaces $\mathbb{V}_k^{dg}$, we have $\mathbb{V}_k^{dg} \not\subset \mathbb{V}$ for all $k \in \mathbb{N}$ and we can not use compactness properties of $\mathbb{V}$ for a sequence $\{V_k\}_{k\in\mathbb{N}}$, $V_k \in \mathbb{V}_k^{dg}$. That means we have to find a space $\tilde{\mathbb{V}}$ with proper compactness properties such that the embedding $\mathbb{V}_k^{dg} \subset \tilde{\mathbb{V}}$ holds.

As it turns out, it is possible to embed the non-conforming discontinuous Galerkin finite element spaces *continuously* into the space of functions of bounded variation (i.e. $\tilde{\mathbb{V}} = BV(\Omega)$). This space provides several compactness properties as we will see in the following section

3.5.2 Space of functions of bounded variation

In This Section we introduce the space of functions of bounded variation (BV-spaces). For an introduction to the general concept of BV-spaces and all related definitions, compare Appendix A. By $\boldsymbol{MR}(\Omega, \mathbb{R}^d)$ we denote the space of regular Borel measures on the domain Ω with values in $\mathbb{R}^d$.

In order to introduce the BV-space we consider pairs (u, ν) with $u \in L^1(\Omega)$ and $\nu \in \boldsymbol{MR}(\Omega, \mathbb{R}^d)$ satisfying the following integration by parts formula

$$\int_\Omega \partial_i \varphi u \,\mathrm{d}x = -\int_\Omega \varphi \,\mathrm{d}\nu_i, \quad \forall \varphi \in C_0^\infty(\Omega), \quad i = 1, \dots d. \tag{3.5.3}$$

That means, we have in the distributional sense $D_i u := \nu_i \in \mathcal{D}(\Omega)'$ and $Du = (D_1 u, \dots D_d u)$. We emphasise that (3.5.3) holds true even for $\varphi \in C_0^1(\Omega)$, compare Remark A.3. The set

$$BV(\Omega) := \left\{ u \in L^1(\Omega) \colon \text{there exists } \nu \in \boldsymbol{MR}(\Omega, \mathbb{R}^d) \text{ satisfying (3.5.3)} \right\}$$

of *functions of bounded variation.* is a Banach space if equipped with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|,$$

where $\|Du\| = |Du|(\Omega) = |\nu|(\Omega)$ and $|Du|(\Omega)$ is the total variation of the measure $|Du|$.

Remark 3.18. *Let $u \in BV(\Omega)$ and assume that $Du = 0$. Then u is constant almost everywhere in Ω. This follows from a smoothing property together with a convolution argument (compare [AFP00, Proposition 3.2])*

In order to keep the notation simple, we write (3.5.3) in a single formula

$$\int_\Omega u \operatorname{div} \varphi = -\sum_{i=1}^d \int_\Omega \varphi_i \,\mathrm{d}D_i u = -\int_\Omega \varphi \cdot \mathrm{d}Du \qquad \forall \varphi \in C_0^\infty(\Omega)^d. \tag{3.5.4}$$

Note that we use the same notation also for functions in $BV(\Omega)^m$, $m \in \mathbb{N}$. In this case Du is a $m \times d$-matrix of meausres $D_i u^j$ in Ω satisfying

$$\int_\Omega u^j \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x = -\int_\Omega \varphi \,\mathrm{d}D_i u^j \qquad \forall \varphi \in \mathcal{D}(\Omega), \quad i = 1, \dots, d \quad j = 1, \dots, m$$

or equivalently

$$\sum_{j=1}^m \int_\Omega u^j \operatorname{div} \varphi_j \,\mathrm{d}x = -\sum_{j=1}^m \sum_{i=1}^d \int_\Omega \varphi_{ji} \,\mathrm{d}D_i u^j \qquad \forall \varphi \in C_0^\infty(\Omega)^{d \times m}.$$

For the last equation we also use the shorthand notation

$$\int_\Omega u \cdot \operatorname{div} \varphi \,\mathrm{d}x = -\int_\Omega \varphi^T \colon \mathrm{d}Du \qquad \forall \varphi \in C_0^\infty(\Omega)^{d \times m}. \tag{3.5.5}$$

The space $W^{1,1}(\Omega)$ is contained in $BV(\Omega)$. In particular, for every $u \in W^{1,1}(\Omega)$ the distributional derivative is given by $\nabla u \mathfrak{L}^d$, where $\mathfrak{L}^d$ denotes the Lebesgue-measure on $\mathbb{R}^d$. Note that this inclusion is strict as we will see from the following example.

Example 3.19. Let $\Omega = (-1, 1)$ and consider the Heavyside function defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Let $\phi \in C_0^\infty(-1, 1)$ then we have by piecewise integration by parts

$$\int_{-1}^{1} H(x)\varphi' \,\mathrm{d}x = \int_{0}^{1} H(x)\varphi' \,\mathrm{d}x = [\varphi(x)]_{x=0}^{x=1} = -\varphi(0).$$

Consequently, in the distributional sense we have $\frac{d}{dx}H(x) = \delta_{x=0}$, where $\delta_{x=0}$ denotes the *Dirac measure* supported at 0. Note that $\delta_{x=0} \notin L^1(\Omega)$, i.e. there is no function $f \in L^1(\Omega)$ such that $\varphi(0) = \int_\Omega f\varphi \,\mathrm{d}x$ for all $\phi \in D(\Omega)$. Otherwise we could take successively test functions $\varphi \in C_0^\infty(-1, 0)$ and $\varphi \in C_0^\infty(0, 1)$ to conclude that $f = 0$ almost everywhere in the domain Ω, which is a contradiction.

Definition 3.20 (Weak* convergence)**.** Let u, $u_k \in [BV(\Omega)]^m$. We say that $\{u_k\}_{k\in\mathbb{N}}$ weakly*-converges in $BV(\Omega)^m$ to u if $\{u_k\}_{k\in\mathbb{N}}$ converges to u in $L^1(\Omega)^m$ and $\{Du_k\}_{k\in\mathbb{N}}$ weakly*-converges to Du in Ω, i.e.

$$\lim_{k\to\infty} \int_\Omega \varphi\colon \,\mathrm{d}Du_k = \int_\Omega \varphi\colon \,\mathrm{d}Du \quad \forall \varphi \in C_0(\Omega)^{d\times m}.$$

Remark 3.21. *We emphasise that Definition 3.20 differs from the definition of weak* convergence given in the beginning of Chapter 2.3. This is due to the reason that the dual of $BV(\Omega)$ as a Banach space is hard to characterise (see e.g. [FS18, Section 2]). However, at least for sufficiently regular domains Ω the convergence of Definition 3.20 corresponds to the weak* convergence in the usual sense (compare [AFP00, Remark 3.12].)*

The following Theorem states a useful compactness property of $BV(\Omega)$-spaces (see [AFP00, Corollary 3.49]).

Theorem 3.22. *Let Ω be a Lipschitz domain. Then, the embedding $BV(\Omega) \hookrightarrow L^{\tilde{p}}(\Omega)$ is continuous and the embeddings $BV(\Omega) \hookrightarrow L^p(\Omega)$ are compact for $1 \leqslant p < \tilde{p}$. Here $\tilde{p} = \infty$ if $d = 1$ and $d/(d-1)$ otherwise.*

The following proposition is motivated by Theorem 3.22 and provides a simple criterion for weak* convergence in the space of bounded variation

Proposition 3.23 (Weak* convergence in $BV(\Omega)^m$)**.** *Let $\{u_k\}_{k\in\mathbb{N}} \subset BV(\Omega)^m$. Then $\{u_k\}_{k\in\mathbb{N}}$ weakly* converges to u in $BV(\Omega)^m$ if and only if $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $BV(\Omega)^m$ and converges to u in $L^1(\Omega)^m$.*

Proof. Assume first that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $BV(\Omega)^m$ and converges to some $u \in L^1(\Omega)^m$. The boundedness of the total variation implies by Theorem A.4 a weak*-limit $\lim_{k\to\infty} Du_k = \mu$ of a subsequence (not relabelled here). We have to prove that $Du = \mu$ in a distributional sense for any subsequence of $\{u_k\}_{k\in\mathbb{N}}$.

Note that $u_k \in BV(\Omega)^m$ and therefore for all $k \in \mathbb{N}$,

$$\int_\Omega u_k \cdot \operatorname{div} \varphi = -\int_\Omega \varphi \colon \, \mathrm{d}Du_k \quad \forall \varphi \in C_0^\infty(\Omega)^{d\times m}. \tag{3.5.6}$$

Using the convergence of $\{u_k\}_{k\in\mathbb{N}}$ in $L^1(\Omega)^m$ we obtain in (3.5.6) as $k \to \infty$

$$\int_\Omega u \cdot \operatorname{div} \varphi = -\int_\Omega \varphi \colon \, \mathrm{d}\mu \quad \forall \varphi \in C_0^\infty(\Omega)^{d\times m},$$

where $u \in BV(\Omega)^m$ is the weak* limit of $\{u_k\}_{k\in\mathbb{N}}$ in $BV(\Omega)^m$.

Next, let $\{u_k\}_{k\in\mathbb{N}}$ be a weak* convergent sequence to u in $BV(\Omega)^m$. Then we have the $L^1(\Omega)^m$-convergence by definition of weak*-convergence and as a consequence the boundedness of $\{u_k\}_{k\in\mathbb{N}}$ in $L^1(\Omega)^m$. The boundedness of $|Du_k|(\Omega)$ for all $k \in \mathbb{N}$ follows from Proposition 2.3 (2). □

The following useful compactness theorem for $BV(\Omega)$-functions can be found in [ABM14, Theorem 3.23].

Theorem 3.24 (Compactness in $BV(\Omega)$)**.** *Let Ω be a Lipschitz domain with boundary Γ and $(u_n)_{n\in\mathbb{N}_0} \subset BV(\Omega)$ with $\|u_n\|_{BV(\Omega)} < \infty$ for all $n \in \mathbb{N}$. Then, there exist a subsequence $(u_{n_k})_{k\in\mathbb{N}_0}$ weakly* converging to some $u \in BV(\Omega)$.*

We recall some facts about traces of functions of bounded variation. The following Theorem states $u \in BV(\Omega)$ has a measurable L^1-trace on the boundary $\partial\Omega$. In this context, we denote by $\mathcal{H}^{d-1}$ the $(d-1)$-dimensional Hausdorff.

Theorem 3.25 (Trace Theorem on $BV(\Omega)^m$)**.** *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and $u \in BV(\Omega)^m$. There exists a bounded, linear operator $T\colon BV(\Omega)^m \to L^1(\partial\Omega)^m$ (we write $Tu = u$) such that we have*

$$\int_\Omega u \cdot \operatorname{div} \varphi \, \mathrm{d}x = -\int_\Omega \varphi \colon \, \mathrm{d}Du + \int_\Gamma \varphi \colon (u \otimes \boldsymbol{n}) \, \mathrm{d}\mathcal{H}^{d-1}, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d)^{d\times m}.$$

Moreover, for $\mathcal{H}^{d-1}$-almost every $x \in \partial\Omega$ there exists $Tu(x) \in \mathbb{R}^m$ such that

$$\lim_{r\to 0} r^{-d} \int_{\Omega \cap B_r(x)} \|u(y) - Tu(x)\| \, \mathrm{d}y = 0.$$

Proof. Compare [EG15, Theorem 5.6 and Theorem 5.7]. □

In order to clarify the various embeddings in the following section we introduce the so-called *variation* of a function $u \in L^1_{loc}(\Omega)^m$, which is defined by

$$V(u, \Omega) := \sup \left\{ \int_\Omega u \cdot \operatorname{div} \varphi \, \mathrm{d}x \colon \; \varphi \in C_0^1(\Omega)^{d\times m}, \|\varphi\|_\infty \leqslant 1 \right\}.$$

Here, $C_0^1(\Omega)^{dm}$ denotes the space of continuously differentiable functions with compact support in Ω. The following Proposition states, that the variation of $u \in BV(\Omega)$ and the total variation of $|Du|(\Omega)$ coincide.

Proposition 3.26. *Let $u \in L^1_{loc}(\Omega)^m$. Then u belongs to $BV(\Omega)^m$ if and only if $V(u,\Omega) < \infty$. In addition, $V(u,\Omega)$ coincides with $|Du|(\Omega)$ for any $u \in BV(\Omega)^m$ and $u \mapsto |Du|(\Omega)$ is lower semicontinuous with respect to weak* convergence. In particular, this implies that the whole BV-norm is lower semi-continuos with respect to weak* convergence.*

Proof. The characterisation of a BV-function $u \in BV(\Omega)^m \iff V(u,\Omega) < \infty$ and the resulting equality $V(u,\Omega) = |Du|(\Omega)$ for all $u \in BV(\Omega)^m$ can be found in [AFP00, Proposition 3.6].

Now let $(u_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $BV(\Omega)^m$ with limit $u \in BV(\Omega)^m$. Then we have from the definiton of the BV-space that $(u_k)_{k\in\mathbb{N}}$ is also a Cauchy sequence in $L^1(\Omega)^m$. Hence, for arbitrary $\varphi \in C_0^1(\Omega)^{d\times m}$ with $\|\varphi\|_\infty \leqslant 1$ we have from the definition of the variation of a function

$$\begin{aligned}\liminf_{k\to\infty} V(u_k,\Omega) &\geqslant \liminf_{k\to\infty} \int_\Omega u_k \cdot \operatorname{div}\varphi \,\mathrm{d}x \geqslant \int_\Omega \liminf_{k\to\infty} u_k \cdot \operatorname{div}\varphi \,\mathrm{d}x \\ &= \int_\Omega u \cdot \operatorname{div}\varphi \,\mathrm{d}x.\end{aligned}$$

Since the last inequality also holds for the supremum over all $\varphi \in C_0^1(\Omega)^{d\times m}$ with $\|\varphi\|_\infty \leqslant 1$ we obtain the lower-semicontinuity of $u \mapsto V(u,\Omega)$, i.e.

$$\liminf_{k\to\infty} V(u_k,\Omega) \geqslant V(u,\Omega).$$

Consequently, the lower semicontinuity of the BV-norm follows by the equality of the variation and the total variation of a BV-function and the continuity of the norm $u \mapsto \|u\|_{L^1(\Omega)}$, $u \in L^1(\Omega)$. □

The following lemma can be found in [BO09, Lemma 6] and is a consequence of the compactness properties stated in Theorems 3.24 and 3.22.

Lemma 3.27 (Friedrichs inequality for $BV(\Omega)$)**.** *Let $u \in BV(\Omega)$ and let $\widetilde{\Gamma} \subset \partial\Omega$ with positive $(d-1)$-dimensional measure. Then, there exists a constant C_F such that*

$$\|u\|_{L^1(\Omega)} \leqslant C_F \left(|Du|(\Omega) + \int_{\widetilde{\Gamma}} |u| \,\mathrm{d}s \right) \quad \forall u \in BV(\Omega).$$

Proof. The proof uses various concepts of $BV(\Omega)$-spaces, stated in this section. Hence, for the sake of clarity we give a full proof. We use a proof by contradiction. Assume that no such constant C_F exists. Then, there exist a sequence $(u_n)_{n\in\mathbb{N}_0} \subset BV(\Omega)$ such that $\|u_n\|_{L^1(\Omega)} = 1$ for all $n \in \mathbb{N}_0$ and

$$|Du_n|(\Omega) + \int_{\widetilde{\Gamma}} |u_n| \,\mathrm{d}s \to 0 \tag{3.5.7}$$

as $n \to \infty$. The limit in (3.5.7) and $\|u_n\|_{L^1(\Omega)} = 1$ implies that there exist $C > 0$ such that $\|u\|_{BV(\Omega)} \leqslant C$. Hence, by Theorem 3.24 there exist a subsequence (not relabelled here) satisfying $u_n \overset{*}{\rightharpoonup} u$ in $BV(\Omega)$ as $n \to \infty$. Applying Theorem 3.22

we have also that $u_n \to u$ strongly in $L^1(\Omega)$ as $n \to \infty$ with $\|u\|_{L^1(\Omega)} = 1$. Note that the functional

$$v \mapsto |Dv|(\Omega) + \|u\|_{L^1(\tilde{\Gamma})}$$

is lower semi-continuous respect to weak* convergence (compare Proposition 3.26 and also [BC$^+$11, Chapter 9] for details). Whence, we infer $|Du|(\Omega) = 0$ and therefore, u is constant almost everywhere in Ω. Moreover, we have that $\|u\|_{L^1(\tilde{\Gamma})} = 0$ and therefore the trace of u vanishes on $\tilde{\Gamma}$. From this we conclude $u = 0$ almost everywhere in Ω, which contradicts the assumption $\|u\|_{L^1(\Omega)} = 1$. □

Using Theorem 3.25, we can give another example of $BV(\Omega)$-functions leading to a clearer picture of the $BV(\Omega)$-space (also compare [AFP00, Example 3.3]). Before we state the example, we have to declare the *restriction* of measures: Let $\mu \in \boldsymbol{MR}(\Omega)$ and $\mathcal{B}$ the underlying Borel sets. If $A \in \mathcal{B}$ we set $\mu|_A(B) = \mu(A \cap B)$ for all $B \in \mathcal{B}$.

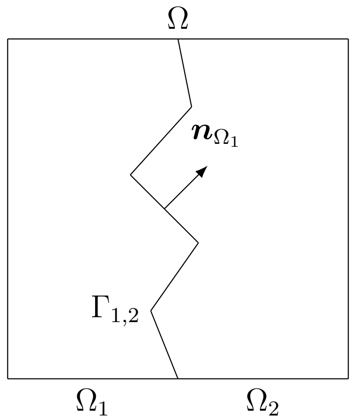


Figure 3.4: Domain Ω with subdomains Ω_1 and Ω_2.

Example 3.28. Assume that $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ are two disjoint bounded Lipschitz domains which are included into a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, compare Figure 3.4. Let $\Gamma_{1,2} := \partial\Omega_1 \cap \Omega_2$ the common Lipschitz boundary of the subdomains satisfying $\mathcal{H}^1(\Gamma_{1,2}) > 0$. We define a function on the domain Ω as

$$u = \begin{cases} u_1 & \text{in } \Omega_1, \\ u_2 & \text{in } \Omega_2, \end{cases}$$

where $u_1 \in BV(\Omega_1)$ and $u_2 \in BV(\Omega_2)$ are chosen arbitrarily. We claim $u \in BV(\Omega)$. To see this let $\varphi \in C_0^\infty(\Omega)^d$. Then we have in the distributional sense

$$\int_\Omega \varphi \cdot \mathrm{d}Du = -\int_\Omega u \operatorname{div} \varphi \, \mathrm{d}x = -\int_{\Omega_1} u_1 \operatorname{div} \varphi \, \mathrm{d}x - \int_{\Omega_2} u_2 \operatorname{div} \varphi \, \mathrm{d}x. \quad (3.5.8)$$

Note that $\Gamma_{1,2}$ is Lipschitz-continuous by construction and therefore we obtain by Theorem 3.25 on each subdomain Ω_1 and Ω_2

$$\int_{\Omega_1} u_1 \operatorname{div}\varphi \,\mathrm{d}x = -\int_{\Omega_1} \varphi \cdot \mathrm{d}Du_1 + \int_{\Gamma_{1,2}} u_1\varphi \cdot \boldsymbol{n}_{\Omega_1} \,\mathrm{d}\mathcal{H}^{d-1}$$

and

$$\int_{\Omega_2} u_2 \operatorname{div}\varphi \,\mathrm{d}x = -\int_{\Omega_2} \varphi \cdot \mathrm{d}Du_2 + \int_{\Gamma_{1,2}} u_2\varphi \cdot \boldsymbol{n}_{\Omega_2}\varphi \,\mathrm{d}\mathcal{H}^{d-1}.$$

Combining this with (3.5.8) we have

$$\begin{aligned}\int_{\Omega} \varphi \cdot \mathrm{d}Du &= \int_{\Omega} \varphi \cdot (\,\mathrm{d}Du_1|_{\Omega_1} + \mathrm{d}Du_2|_{\Omega_2}) \\ &\quad - \int_{\Omega} \varphi \cdot (u_1\boldsymbol{n}_{\Omega_1} + u_2\boldsymbol{n}_{\Omega_2})\mathcal{H}^{d-1}|_{\Gamma_{1,2}}.\end{aligned}$$

Using Theorem 3.25 we see that the $L^1(\Gamma_{1,2})$-trace of u_1 and u_2 is bounded by their $BV(\Omega_1)$ and $BV(\Omega_1)$ norms and we conclude $u \in BV(\Omega)$.

Regarding the last example, we emphasise that a main advantage of the BV-space is that it includes, unlike Sobolev spaces, piecewise smooth functions. This is crucial in the following embedding theorems of discontinuous Galerkin functions into BV-spaces.

3.6 Embeddings of discontinuous Galerkin spaces into BV-spaces

In the current section we prove the crucial fact that discontinuous Galerkin functions can be continuously embedded into the space of BV-functions. Starting point is the following formula of the variation of $u \in \mathbb{V}(\mathcal{T})$: Let $\varphi \in C_0^1(\Omega)^2$, then we have

$$\begin{aligned}-\int_{\Omega} u \cdot \operatorname{div}\varphi \,\mathrm{d}x &= -\sum_{K\in\mathcal{T}} \int_K u \cdot \operatorname{div}\varphi \,\mathrm{d}x \\ &= \sum_{K\in\mathcal{T}} \int_K \nabla u \cdot \varphi \,\mathrm{d}x - \int_{\partial K} \varphi u \cdot \boldsymbol{n} \,\mathrm{d}s \qquad (3.6.1) \\ &= \int_{\Omega} \nabla_{\mathtt{pw}} u \cdot \varphi \,\mathrm{d}x - \int_{\mathcal{F}} [\![u]\!]\, \boldsymbol{n} \cdot \varphi \,\mathrm{d}s,\end{aligned}$$

where we used the definition of the piecewise gradient and Lemma 3.2 for the boundary integrals.

Proposition 3.29. *For $v \in \mathbb{V}(\mathcal{T})$ we have that*

$$\|v\|_{L^1(\Omega)} \lesssim \int_{\mathcal{T}} |\nabla_{pw} v| \,\mathrm{d}x + \int_{\mathcal{F}_{\mathcal{T}}} |[\![v]\!]\,\boldsymbol{n}| \,\mathrm{d}s.$$

Proof. Let $v \in \mathbb{V}(\mathcal{T})$. By Lemma 3.27 the $L^1(\Omega)$-norm is bounded by

$$\|v\|_{L^1(\Omega)} \leqslant C_F \left(|Dv|(\Omega) + \int_{\tilde{\Gamma}} |v| \, \mathrm{d}s \right). \tag{3.6.2}$$

Moreover, Proposition 3.26 yields $|Dv|(\Omega) = V(v, \Omega)$ and therefore by (3.6.1), we obtain

$$\begin{aligned} |Dv|(\Omega) = V(v,\Omega) &= \sup \left\{ \int_\Omega u \cdot \operatorname{div} \varphi \, \mathrm{d}x \colon \ \varphi \in C_0^1(\Omega)^{dm}, \|\varphi\|_\infty \leqslant 1 \right\} \\ &\leqslant \int_{\mathcal{T}} |\nabla_{\mathrm{pw}} v| \, \mathrm{d}x + \int_{\mathcal{F}_\mathcal{T}} |[\![v]\!] \boldsymbol{n}| \, \mathrm{d}s, \end{aligned} \tag{3.6.3}$$

where we used Hölder's inequality in the last line in conjunction with $\|\varphi\|_\infty \leqslant 1$. Now, the assertion directly follows by inserting (3.6.3) into (3.6.2) and the definition of jump-terms on the boundary. □

Proposition 3.30. *Let $v \in \mathbb{V}(\mathcal{T})$. Then, we have*

$$|Dv|(\Omega) \lesssim \|v\|_\mathcal{T}.$$

Proof. Let $v \in \mathbb{V}(\mathcal{T})$. Then, we obtain from Proposition 3.26 and (3.6.3)

$$|Dv|(\Omega) \leqslant \int_\mathcal{T} |\nabla v| \, \mathrm{d}x + \int_{\mathcal{F}_\mathcal{T}} |[\![v]\!] \boldsymbol{n}| \, \mathrm{d}s. \tag{3.6.4}$$

Hence, applying Proposition 3.29 to the piecewise gradient, we obtain

$$|Dv|(\Omega) \lesssim \int_\mathcal{T} |D^2 v| \, \mathrm{d}x + \int_{\mathcal{F}_\mathcal{T}} |[\![\partial_n v]\!]| \, \mathrm{d}s + \int_{\mathcal{F}_\mathcal{T}} |[\![v]\!] \boldsymbol{n}| \, \mathrm{d}s. \tag{3.6.5}$$

Since $|\Omega| < \infty$, we have by Hölder's inequality

$$\|D^2_{\mathrm{pw}} v\|_{L^1(\Omega)} \lesssim \|D^2_{\mathrm{pw}} v\|_\Omega. \tag{3.6.6}$$

Another application of Hölder's inequality to the jump terms reveals

$$\begin{aligned} \int_{\mathcal{F}_\mathcal{T}} |[\![v]\!] \boldsymbol{n}| \, \mathrm{d}s &= \int_{\mathcal{F}_\mathcal{T}} h_\mathcal{T}^{3/2} h_\mathcal{T}^{-3/2} |[\![v]\!] \boldsymbol{n}| \, \mathrm{d}s \\ &\leqslant \left(\int_{\mathcal{F}_\mathcal{T}} h_\mathcal{T}^3 \, \mathrm{d}s \right)^{1/2} \left(\int_{\mathcal{F}_\mathcal{T}} h_\mathcal{T}^{-3} |[\![v]\!] \boldsymbol{n}|^2 \, \mathrm{d}s \right)^{1/2}. \end{aligned}$$

Moreover, we note that the sum over mesh-faces is bounded by

$$\int_{\mathcal{F}_\mathcal{T}} h_\mathcal{T}^3 \, \mathrm{d}s = \sum_{F \in \mathcal{F}} \int_F h_\mathcal{T}^3 \, \mathrm{d}s = \sum_{F \in \mathcal{F}} h_F^4 \lesssim \sum_{F \in \mathcal{F}} \sum_{\substack{K \in \mathcal{T} \\ F \subset K}} h_K^4 \lesssim \sum_{K \in \mathcal{T}} |K|^2 \lesssim |\Omega|^2,$$

where we used the definition of $h_\mathcal{T}$ and $h_F \approx h_K$. Hence, we have

$$\int_{\mathcal{F}_\mathcal{T}} |[\![v]\!] \boldsymbol{n}| \, \mathrm{d}s \lesssim \left(\int_{\mathcal{F}_\mathcal{T}} h_\mathcal{T}^{-3} |[\![v]\!] \boldsymbol{n}|^2 \, \mathrm{d}s \right)^{1/2}. \tag{3.6.7}$$

By analogous arguments we also obtain

$$\int_{\mathcal{F}_\mathcal{T}} |[\![\partial_n v]\!]| \,\mathrm{d}s \lesssim \left(\int_{\mathcal{F}_\mathcal{T}} h_\mathcal{T}^{-1} \,|[\![\partial_n v]\!]|^2 \,\mathrm{d}s \right)^{1/2}. \tag{3.6.8}$$

Finally, the assertion follows by inserting (3.6.6)-(3.6.8) into (3.6.5). □

In the context of SIPDG methods the embedding stated in Proposition 3.30 also transfers to an embedding of the piece-wiese gradient $\nabla_{\mathtt{pw}} v \in \mathbb{P}_{r-1}(\mathcal{T})^2$. In order to see this, we consider the variation of the piece-wise gradient

$$V(\nabla_{\mathtt{pw}} v, \Omega) = \sup \left\{ \int_\Omega \nabla_{\mathtt{pw}} u \cdot \operatorname{div} \varphi \,\mathrm{d}x \colon\ \varphi \in C_0^1(\Omega)^{2\times 2}, \|\varphi\|_\infty \leqslant 1 \right\}.$$

Moreover, for $v \in \mathbb{V}(\mathcal{T})$ and $\varphi \in C_0^1(\Omega)^{2\times 2}$ we obtain the following formula from piece-wise integration by parts

$$-\int_\Omega \nabla_{\mathtt{pw}} v \cdot \operatorname{div} \varphi \,\mathrm{d}x = \int_\Omega D^2_{\mathtt{pw}} v \colon \varphi \,\mathrm{d}x - \int_\mathcal{F} \varphi \,[\![\nabla_{\mathtt{pw}} v]\!] \cdot \boldsymbol{n}. \tag{3.6.9}$$

Note that the jump terms on the right-hand side of (3.6.9) are independent of the ordering of the related elements K_1 and K_2, compare Remark 3.1.

Proposition 3.31. *Let $v \in \mathbb{V}(\mathcal{T})$ and $|D(\nabla_{pw} v)|(\Omega)$ the total variation of $\nabla_{pw} v \in L^2(\Omega)$. Then, we have*

$$|D(\nabla_{pw} v)|(\Omega) \lesssim \int_\Omega \left|D^2_{pw} v\right| \,\mathrm{d}x + \int_{\mathcal{F}(\mathcal{T})} |[\![\partial_n v]\!]| \,ds \lesssim |\!|\!| v |\!|\!|_\mathcal{T}.$$

Proof. Using Proposition 3.26 and (3.6.9) we obtain

$$\begin{aligned} |D(\nabla_{\mathtt{pw}} v)|(\Omega) &= V(\nabla_{\mathtt{pw}} v, \Omega) \\ &\lesssim \int_\Omega \left|D^2_{\mathtt{pw}} v\right| \,\mathrm{d}x + \int_{\mathcal{F}(\mathcal{T})} |[\![\partial_n v]\!]|, \quad \forall v \in V(\mathcal{T}), \end{aligned}$$

where we also used Hölder's inequality in and $\|\varphi\|_\infty \leqslant 1$ in the last estimate. The remaining proof is follows by similar arguments as Proposition 3.30. □

4 Convergence of AFEM

In this chapter we give the main result of this thesis. We define an adaptive algorithm in Section 4.1 based on the a posteriori error indicators introduced in Section 3.4. The adaptive algorithm produces to a sequence of increasingly refined grids $\{\mathcal{T}_k\}_{k\in\mathbb{N}}$ with a corresponding sequence of adaptively created discrete solutions $\{U_{\mathcal{T}_k}\}_{k\in\mathbb{N}}$. The main result (Theorem 4.3) provides convergence of the sequence of discontinuous Galerkin solutions, produced by the adaptive algorithm, to the exact solution $u \in H_0^2(\Omega)$ of (2.4.2). In order to keep this presentation simple, Section 4.2 provides the general framework of the proof of the Main Theorem, whereas we postpone the details to Section 4.3.

In this context we have to deal with the problem, that the mesh-size h is in genereal not strictly monotone under refinement, due to the adaptive algorithm. In order to fix this, we introduce a domain $\Omega^- \subset \Omega$, where we still have $h \to 0$ (see Section 4.2.1). Therefore, on the domain Ω^- we will commonly use the fact that $h \to 0$ in our convergence analysis (comparable to a priori convergence analysis).

On the remaining domain Ω^+ we have $h \nrightarrow 0$ since it is related to elements, which are not refined anymore. Hence, the local error indicators here have to be 'small', compared to elements which are consecutively refined. This is the idea of the marking strategy introduced below, ensuring convergence also on this domain.

4.1 Model Algorithm

We start with a precise formulation the adaptive algorithm (1.1.2) based on the modules SOLVE, ESTIMATE, MARK and REFINE, which are described in more detail below.

Algorithm 4.1 (ASIPDGM)**.** Let $\mathcal{T}_0$ be an initial triangulation. The adaptive algorithm is an iteration of the following form:

1. $u_k = \mathsf{SOLVE}(\mathbb{V}(\mathcal{T}_k))$;
2. $\{\eta_k(K)\}_{K\in\mathcal{T}_k} = \mathsf{ESTIMATE}(u_k, \mathcal{T}_k)$;
3. $\mathcal{M}_k = \mathsf{MARK}\left(\{\eta_k(K)\}_{K\in\mathcal{T}_k}, \mathcal{T}_k\right)$;
4. $\mathcal{T}_{k+1} = \mathsf{REFINE}(\mathcal{T}_k, \mathcal{M}_k)$; increment k and go to Step 1.

Here, we have replaced the subscript triangulations $\{\mathcal{T}_k\}_{k\in\mathbb{N}_0}$ with the iteration counter k, i.e. $u_k = u_{\mathcal{T}_k}$ and $\eta_k(\mathcal{T}_k) = \eta_{\mathcal{T}_k}(\mathcal{T}_k)$ for brevity. Similar short hand notations will be frequently used below when no confusion can occur, e.g. we write also $N_k^j(K) = N_{\mathcal{T}_k}^j(K)$, $\Gamma_k = \Gamma_{\mathcal{T}_k}$ or $|||\cdot|||_k = |||\cdot|||_{\mathcal{T}_k}$. Next, we comment on the modules SOLVE, ESTIMATE, MARK and REFINE.

SOLVE. For a given mesh $\mathcal{T} \in \mathbb{G}$ we assume that

$$u_{\mathcal{T}} = \mathsf{SOLVE}(\mathbb{V}(\mathcal{T}))$$

is the exact SIPDG solution of problem (3.2.1).

ESTIMATE. We suppose that

$$\{\eta_{\mathcal{T}}(K)\}_{K\in\mathcal{T}} := \mathsf{ESTIMATE}(u_{\mathcal{T}}, K)$$

is the elementwise error indicator defined in (3.4.4).

MARK. We assume a fixed marking strategy

$$\mathcal{M} := \mathsf{MARK}(\{\eta_{\mathcal{T}}(K)\}_{K\in\mathcal{T}}, \mathcal{T}),$$

which satisfies

$$\max\{\eta_{\mathcal{T}}(K) \colon K \in \mathcal{T}\backslash\mathcal{M}\} \leqslant g(\max\{\eta_{\mathcal{T}}(K) \colon K \in \mathcal{M}\}), \tag{4.1.1}$$

where $g \colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a fixed function, which is continuous in 0, with $g(0) = 0$.

REFINE. We assume for $\mathcal{T} \in \mathbb{G}$ and $\mathcal{M} \subset \mathcal{T}$ that

$$\mathcal{T} \leqslant \widetilde{\mathcal{T}} := \mathsf{REFINE}(\mathcal{T}, \mathcal{M}) \in \mathbb{G},$$

such that

$$K \in \mathcal{M} \quad \Rightarrow \quad K \in \mathcal{T}\backslash\widetilde{\mathcal{T}}, \tag{4.1.2}$$

i.e., each marked element is at least refined once.

Obviously, the modules SOLVE and ESTIMATE depend on the data of the of the variational problem, e.g. the right-hand side f. The refinement module REFINE in contrast is problem independent which is in general also true for the modul MARK. Some popular marking strategies for Algorithm 4.1 are:

- Maximum Strategy: For a given parameter $\theta \in [0, 1]$ we let

$$\mathcal{M} = \{K \in \mathcal{T} \colon \eta_{\mathcal{T}}(K) \geqslant \theta\eta_{\mathcal{T},\max}\} \quad \text{with} \quad \eta_{\mathcal{T},\max} = \max_{K\in\mathcal{T}} \eta_{\mathcal{T}}(K).$$

- Equidistribution Strategy: For a given parameter $\theta \in [0, 1]$ we let

$$\mathcal{M} = \left\{K \in \mathcal{T} \colon \eta_{\mathcal{T}}(K) \geqslant \theta\eta_{\mathcal{T}}(K)/\sqrt{\#\mathcal{T}}\right\}.$$

- Dörfler's Strategy: For a given parameter $\theta \in [0, 1]$ we let $\mathcal{M} \subset \mathcal{T}$ such that

$$\eta_{\mathcal{T}}(\mathcal{M}) \geqslant \theta\eta_{\mathcal{T}}(\mathcal{T}).$$

Remark 4.2. *In the refinement strategy* (4.1.2)*, we only require minimal refinement, i.e. each marked element is at least refined once. Of course, in praxis marked elements may be refined more than once. The standard choice is d bisection refinements.*

4.1.1 The main result

The main result of this work states that the sequence of SIPDG finite element approximations produced by the ASIPDG method (Algorithm 4.1) converges to the exact solution $u \in H_0^2(\Omega)$ of (2.4.2) and also $\eta_k(\mathcal{T}_k) \to 0$ as $k \to \infty$.

Theorem 4.3 (Main Theorem)**.** *Let $u \in H_0^2(\Omega)$ be the solution of (2.4.2) and $\{\mathcal{T}_k\}_{k\in\mathbb{N}}$ be a sequence of triangulation of Ω created by Algorithm 4.1. Moreover, let $\{u_k\}_{k\in\mathbb{N}}$ be the corresponding sequence of discrete solutions, i.e. $u_k \in \mathbb{V}(\mathcal{T}_k)$ is the SIPDG solution of (3.2.1) in $\mathbb{V}(\mathcal{T}_k)$, for all $k \in \mathbb{N}$. Finally, let $\eta_k(\mathcal{T}_k)$ be the a posteriori error indicators from (3.4.4), related to $\mathcal{T}_k$, and assume that the assumptions on the modules SOLVE, ESTIMATE, MARK and REFINE, stated in Section 4.1, are satisfied. Then, we have*

$$\eta_k(\mathcal{T}_k) \to 0 \quad \textit{and} \quad |\!|\!| u - u_k |\!|\!|_k \to 0 \quad \textit{as } k \to \infty.$$

4.2 Proof of the main result Theorem 4.3

The proof of convergence of ASIPDGM is based on ideas of [MSV08, Sie11] for conforming elements and its generalisation [KG18] to adaptive discontinuous Galerkin methods for the Poisson problem. For the sake of clarity, in this section, we present the main ideas of the proof of Theorem 4.3 following the ideas of [KG18]. In contrast to the latter result here we are faced with the problem that $\mathbb{V}(\mathcal{T})$ contains no proper conforming subspace. This requires new techniques of proof for two key auxiliary results, Theorem 4.15 and Lemma 4.12, which proofs are postponed to Section 4.3.

4.2.1 Sequence of Partitions

Similar as in [MSV08, Sie11, KG18], we split the domain Ω into essentially two parts according to whether the mesh-size function $h_k := h_{\mathcal{T}_k}$ vanishes or not. In order to make this rigorous, we define the set of eventually never refined elements by

$$\mathcal{T}^+ := \bigcup_{k\geqslant 0} \bigcap_{l\geqslant k} \mathcal{T}_l \qquad \text{with corresponding domain} \qquad \Omega^+ := \Omega(\mathcal{T}^+). \tag{4.2.1}$$

Additionally, we denote the complementary domain $\Omega^- = \Omega \backslash \Omega^+$.
For $k \in \mathbb{N}_0$, we define $\mathcal{T}_k^+ := \mathcal{T}_k \cap \mathcal{T}^+$ as well as for $j \geqslant 1$

$$\begin{aligned} \mathcal{T}_k^{j+} &:= \{K \in \mathcal{T}_k \colon N_k^j(K) \subset \mathcal{T}_k^+\} = \{K \in \mathcal{T}_k \colon N_k(K) \subset \mathcal{T}_k^{(j-1)+}\}, \\ \mathcal{T}_k^{j-} &:= \mathcal{T}_k \backslash \mathcal{T}_k^{j+}, \end{aligned}$$

where we used $\mathcal{T}_k^{0+} := \mathcal{T}_k^+$ and $\mathcal{T}_k^{0-} := \mathcal{T}_k^-$ in the identities when $j = 0$; compare with Figure 4.1 for an example of an adaptive grid. For the corresponding domains we denote $\Omega_k^{j-} := \Omega(\mathcal{T}_k^{j-})$ and $\Omega_k^{j+} := \Omega(\mathcal{T}_k^{j+})$. Moreover, we adopt the above notations for the corresponding faces, e.g. $\mathcal{F}^{j-} := \mathcal{F}(\mathcal{T}_k^{j-})$, $\mathcal{F}^{j+} := \mathcal{F}(\mathcal{T}_k^{j+})$. Note that for all $j, k \in \mathbb{N}_0$ we have $\mathcal{T}_k^{j+} \subset \mathcal{T}^+$ and therefore $\Omega_k^{j+} \subset \Omega^+$. In view of the domain Ω^- this also implies $\Omega^- \subset \Omega_k^{j-}$.

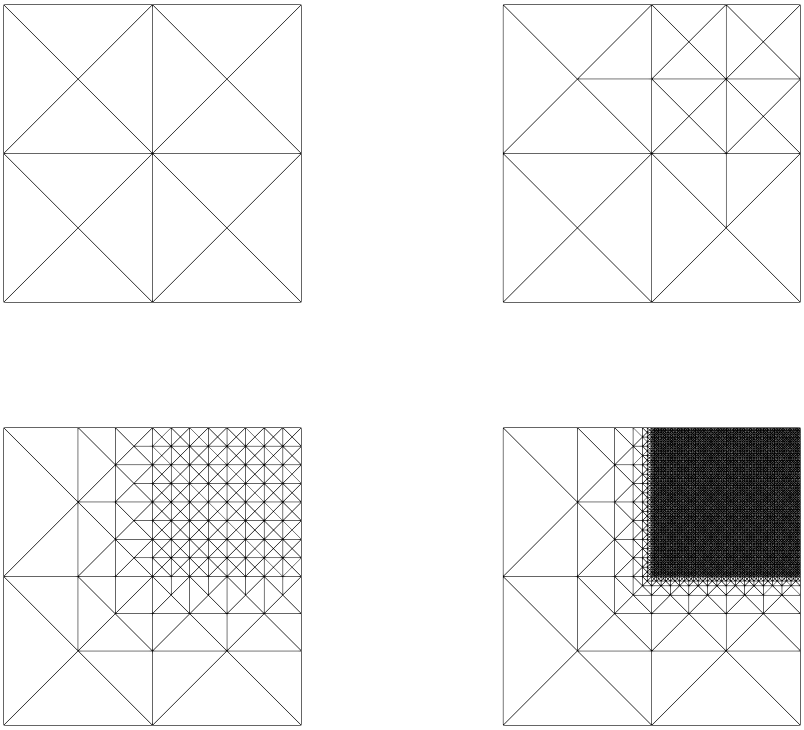


Figure 4.1: Example of a sequence of triangulations of $\Omega = (0,1)^2$. Here, in each iteration the elements in $\Omega^- = [0.5,1] \times [0.5,1]$ are refined. The remaining elements consisting to the grid $\mathcal{T}^+$ and build the domain $\Omega \backslash \Omega^-$. These elements are, after some iterations not refined anymore. Moreover, after some iterations, their whole neighbourhood is not refined anymore due to Lemma 4.4

We remark that we need the above definitions of $\mathcal{T}_k^{j-}$ and $\mathcal{T}_k^{j+}$ for technical reasons. In fact, our analysis involves interpolations based on local L^2-orthogonal projections for which local stability estimates involve neighbourhoods. However, for different but fixed js the above sets behave asymptotically similar for $k \to \infty$. This is a consequence of the following Lemma, which states that neighbours of never refined elements are also eventually never refined again.

Lemma 4.4. *For $K \in \mathcal{T}^+$ there exists a constant $L = L(K) \in \mathbb{N}_0$ such that*

$$N_k(K) = N_L(K)$$

for all $k \geqslant L$. In particular, we have $N_k(K) \subset \mathcal{T}^+$ for all $k \geqslant L$.

Proof. See [MSV08, Lemma 4.1]. □

The next Lemma essentially goes back to [MSV08, (4.15) and Corollary 4.1] and was proved for $j = 2$ in [KG18, Lemma 11].

Lemma 4.5. *For $j \in \mathbb{N}_0$ we have $\lim_{k\to\infty} \left\| h_k \chi_{\Omega_k^{j-}} \right\|_{L^\infty(\Omega)} = 0$, where $\chi_{\Omega_k^{j-}}$ denotes the characteristic function of Ω_k^{j-}. Moreover, $\left|\Omega_k^{j-} \backslash \Omega^-\right| = \left|\Omega^+ \backslash \Omega_k^{j+}\right| \to 0$ as $k \to \infty$.*

Proof. In order to see that $\left|\Omega^+ \backslash \Omega_k^{j+}\right| \to 0$ as $k \to \infty$, we observe from Lemma 4.4 that for $\ell \in \mathbb{N}$, there exists $L = L(\ell) \geqslant \ell$, such that $\mathcal{T}_\ell^+ \subset \mathcal{T}_L^{j+}$ since $\mathcal{T}_\ell^+$ contains only finitely many elements. Consequently, we have

$$|\Omega^+ \backslash \Omega_{L(\ell)}^{j+}| \leqslant |\Omega^+ \backslash \Omega_\ell^+| \to 0 \qquad \text{as } \ell \to \infty,$$

i.e. we have proved the claim for a subsequence. Since the sequence $\{|\Omega^+ \backslash \Omega_k^{j+}|\}_{k\in\mathbb{N}}$ is monotone, it must vanish as a whole.

The first claim follows for $j = 1$ from [Sie11, Corollary 3.3]. By shape regularity, we have for $j > 1$ that

$$h_K \eqsim |K|^{1/2} \leqslant |\Omega_k^{j-} \backslash \Omega_k^{1-}|^{1/2} \leqslant |\Omega_k^{j-} \backslash \Omega^-|^{1/2} \qquad \text{for all } K \in \mathcal{T}_k^{j-} \backslash \mathcal{T}_k^{1-}.$$

Consequently, we have

$$\begin{aligned} \left\| h_k \chi_{\Omega_k^{j-}} \right\|_{L^\infty(\Omega)} &\leqslant \left\| h_k \chi_{\Omega_k^{1-}} \right\|_{L^\infty(\Omega)} + \left\| h_k \chi_{\Omega_k^{j-} \backslash \Omega_k^{1-}} \right\|_{L^\infty(\Omega)} \\ &\leqslant \left\| h_k \chi_{\Omega_k^{1-}} \right\|_{L^\infty(\Omega)} + |\Omega_k^{j-} \backslash \Omega_k^{1-}|^{1/2} \to 0 \end{aligned}$$

as $k \to \infty$, which concludes the proof. □

Remark 4.6. *We note that Lemma 4.5 becomes important in the context of the absolute continuous dependence of an integral to the integration domain. To be precise: Let f be a function with a finite Lebesgue integral over Ω. Then, for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that for every measurable set E of Ω with $\mathfrak{L}^2(E) < \delta$, we have*

$$\left| \int_E f \,\mathrm{d}x \right| < \epsilon,$$

where $\mathfrak{L}^2$ *denotes the two-dimensional Lebesgue-measure. Compare to [PKJF12, Theorem 1.21.13] for this statement.*

In particular, for $f \in L^p(\Omega)$, $p \in [1,\infty)$, *we obtain that for arbitrary* $\epsilon' >$ *there exist* $\delta' > 0$, *such that for every* $E' \subset \Omega$ *satisfying* $\mathfrak{L}^2(E') < \delta'$

$$\left(\int_{E'} |f|^p \,\mathrm{d}x\right)^{1/p} < \epsilon'.$$

Consequently, for $p \in [1,\infty)$, *the* L^p*-norm is absolutely continuous with respect to the Lebesgue measure (compare with [PKJF12, Examples 6.3.5(i)]) and Lemma 4.5 implies*

$$\lim_{k\to\infty} \left\| f\chi_{\Omega_k^{j-}\backslash\Omega^-} \right\|_{L^p(\Omega)} = \lim_{k\to\infty} \left\| f\chi_{\Omega^+\backslash\Omega_k^{j+}} \right\|_{L^p(\Omega)} = 0,$$

due to the fact that $\left|\Omega_k^{j-}\backslash\Omega^-\right| = \left|\Omega^+\backslash\Omega_k^{j+}\right| \to 0$ *as* $k \to \infty$.

Remark 4.7. *A more intuitiv definition of the refined mesh would be to define* $\Omega^- = \operatorname{interior}(\Omega\backslash\Omega^+)$ *and*

$$\mathcal{T}_k^- = \left\{K \in \mathcal{T}_k \colon K \subset \overline{\Omega^-}\right\}$$

which would ease the theory significantly. In particular, in context of the limit space which is defined later. Defining the grid $\mathcal{T}_k^+$ *as above, we collect all remaining elements in* $\mathcal{T}_k^* = \mathcal{T}_k\backslash(\mathcal{T}_k^+ \cup \mathcal{T}_k^-)$, *which are not in one the two grids* $\mathcal{T}_k^+$ *and* $\mathcal{T}_k^-$ *(compare e.g. [MSV08, Section 4.2]).*

However, the definition of $\mathcal{T}_k^-$ *is based on the interior of the domain* Ω^- *and therefore problems arise, when* $\operatorname{interior}(\Omega^-) = \varnothing$ *but* $\Omega\backslash\Omega^+ \neq \varnothing$. *In Appendix C we give an example of a sequence of meshes, based on Cantor sets, with the properties* $\operatorname{interior}(\Omega^-) = \varnothing$ *and* $|\Omega\backslash\Omega^+| > 0$ *and analyse the resulting problems.*

4.2.2 The limit space

In this section we define the limit of the finite element spaces $\{\mathbb{V}_k\}_{k\in\mathbb{N}}$, based on [DGK19] and [KG18]. Before we state the definition of the limit space, we cite the following Lemma, proving a compactness propety of broken Sobolev spaces in case of vanishing mesh-size $h \to 0$ ([Pry14, Lemma 4.15], compare also [BO09, Theorem 5.2]) This will be a key property in the definition of the limit space.

Lemma 4.8. *Let* $\{\mathcal{T}_h\}_{h\geqslant 0}$ *be a sequence of grids with global mesh-size* $h = \max_{K\in\mathcal{T}_h} h_K$ *and* $h \in (0,1]$. *Moreover, let* $\{v_h\}_{h\geqslant 0}$ *be a sequence of finite element function with* $v_h \in \mathbb{V}(\mathcal{T}_h)$, *which is uniformly bounded in the* $|||\cdot|||_{\mathcal{T}_h}$*-norm. Then, there exist a subsequence* $h_\ell \to 0$ *as* $\ell \to \infty$ *and a function* $v \in H_0^2(\Omega)$ *such that* $v_{h_\ell} \rightharpoonup v$ *in* $L^2(\Omega)$ *as* $\ell \to \infty$. *Moreover,*

$$D_{pw}^2 v_{h_\ell} + \mathcal{L}_{\mathcal{T}_{h_\ell}}(v_{h_\ell}) \rightharpoonup D^2 v \quad \text{in } L^2(\Omega)^{2\times 2} \quad \text{as } \ell \to \infty.$$

The definition of the limit space is motivated by the following ideas: The space is generated from limits of discontinuous Galerkin functions in the sequence of discontinuous Galerkin spaces constructed by the adaptive algorithm. Therefore, there exists a sequence $\{v_k\}_{k\in\mathbb{N}}$, $v_k \in \mathbb{V}_k$, such that $\lim_{k\to 0} \|\!|v_k - v|\!\|_k = 0$, as $k \to \infty$ and $\limsup_{k\to\infty} \|\!|v_k|\!\|_k < \infty$ for all limit functions v located in the limit space.

We emphasise that the evaluation of the energy-norm $\|\!|v_k - v|\!\|_k$ requires traces of v and $\nabla_{\mathtt{pw}} v$ on skeletons Γ_k, $k \in \mathbb{N}$. This traces exist due to the following observation: Thanks to Propositions 3.12 and 3.30 and the uniform bound of $\{v_k\}_{k\in\mathbb{N}}$ in the energy-norm $\limsup_{k\to\infty} \|\!|v_k|\!\|_k < \infty$ we have that $\{v_k\}_{k\in\mathbb{N}}$ is also uniformly bounded in the BV-norm. Using the compactness property of the BV-space (Theorem 3.24) there exists $\tilde{v} \in BV(\Omega)$ such that $v_k \overset{*}{\rightharpoonup} \tilde{v}$ in $BV(\Omega)$ as $k \to \infty$. Unfortunately, it is a priori not clear if the limit $\tilde{v}$ coincides with v. Motivated by Proposition 3.12 we therefore assume additionally that $\{v_k\}_{k\in\mathbb{N}}$ satisfies the strong L^2-convergence $\lim_{k\to 0} \|v_k - v\|_\Omega = 0$, which implies $\tilde{v} = v \in BV(\Omega)$. Consequently, for $v \in BV(\Omega)$ there exist the L^1-trace on Γ_k, $k \in \mathbb{N}$; see [AFP00, Theorem 3.88] and the jump terms of v are measurable with respect to the 1-dimensional Hausdorff measure on $\mathcal{F}_k$. In the same vein we can use Proposition 3.31 in order to get $\nabla_{\mathtt{pw}} v_k \overset{*}{\rightharpoonup} \nabla_{\mathtt{pw}} v$ in $BV(\Omega)^2$ as $k \to \infty$. Hence, the L^1-trace of $\nabla_{\mathtt{pw}} v \in BV(\Omega)^2$ exists on Γ_k, $k \in \mathbb{N}$ and we conclude that the energy norm $\|\!|v|\!\|_k$, $k \in \mathbb{N}$ is measurable.

Next, we focus on the domain Ω^- . Using $\Omega^- \subset \Omega_k^-$ and Lemma 4.5 we have that the mesh-size vanishes in the limit. Consequently, Lemma 4.8 implies that a limit function v should ensure H^2-regularity on the domain Ω^-. To makes this precise: For a limit function v we require $v|_{\Omega^-} = w|_{\Omega^-}$, for some function $w \in H_0^2(\Omega)$. In this context we denote by $H^2_{\partial\Omega\cap\partial\Omega^-}(\Omega^-)$ the space of functions from $H_0^2(\Omega)$ restricted to the domain Ω^-.

Finally, we remark that the set $\mathcal{T}^+$, consists of all the elements, which are eventually no longer refined and therefore we have $v|_K \in \mathbb{P}_r(K)$ for all $K \in \mathcal{T}^+$, due to the definition of the finite-element space.

Motivated by the discussion above (compare also Proposition 4.10) we extend the definitions of the piece-wise gradient $\nabla_{\mathtt{pw}} v \in L^2(\Omega)^2$ and the piecewise Hessian $D^2_{\mathtt{pw}} v \in L^2(\Omega)^{2\times 2}$ to the limit case, i.e.

$$\nabla_{\mathtt{pw}} v|_{\Omega^-} := \nabla v|_{\Omega^-} \text{ on } \Omega^- \quad \text{and} \quad \nabla_{\mathtt{pw}} v|_K := \nabla v|_K \qquad \forall K \in \mathcal{T}^+, \tag{4.2.2}$$

and

$$D^2_{\mathtt{pw}} v|_{\Omega^-} := D^2 v|_{\Omega^-} \text{ on } \Omega^- \quad \text{and} \quad D^2_{\mathtt{pw}} v|_K := D^2 v|_K \qquad \forall K \in \mathcal{T}^+, \tag{4.2.3}$$

compare also Proposition 4.10 below.

Now, we are in a position to give the definition of the limit-space. Following the ideas in [KG18, Section 3.2] and [DGK19, Section 3.2] we define

$$\begin{aligned}\mathbb{V}_\infty := \big\{v \in BV(\Omega)\colon\ & \nabla_{\mathtt{pw}} v \in BV(\Omega)^2,\ v|_{\Omega^-} \in H^2_{\partial\Omega\cap\partial\Omega^-}(\Omega^-),\\ & v|_K \in \mathbb{P}_r(K),\ \forall K \in \mathcal{T}^+ \text{such that}\\ & \exists \{v_k\}_{k\in\mathbb{N}}, v_k \in \mathbb{V}_k \text{ with}\\ & \lim_{k\to\infty} \|\!|v - v_k|\!\|_k + \|v - v_k\|_\Omega = 0 \text{ and } \limsup_{k\to\infty} \|\!|v_k|\!\|_k < \infty\big\}.\end{aligned}$$

We will use the following bilinear form on $\mathbb{V}_\infty$: For $v, w \in \mathbb{V}_\infty$, we define

$$\langle v, w\rangle_\infty := \int_\Omega D^2_{\mathtt{pw}} v\colon D^2_{\mathtt{pw}} w \,\mathrm{d}x \\ + \int_{\mathcal{F}^+} \frac{\alpha}{h_+} \left[\!\left[\nabla_{\mathtt{pw}} v\right]\!\right] \cdot \boldsymbol{n} \left[\!\left[\nabla_{\mathtt{pw}} w\right]\!\right] \cdot \boldsymbol{n} + \frac{\beta}{h_+^3} [\![v]\!]\, \boldsymbol{n} \cdot [\![w]\!]\, \boldsymbol{n} \,\mathrm{d}s,$$

where we set $h_+ := h_{\mathcal{T}^+}$ and $\mathcal{F}^+ := \mathcal{F}(\mathcal{T}^+)$. The induced norm is denoted by $|\!|\!| v |\!|\!|_\infty = \langle v, v\rangle_\infty^{1/2}$. Note that we use the shorthand notation of the normal jumps also in the limit case i.e. for $F \in \mathcal{F}^+$ and $v \in \mathbb{V}_\infty$ we define

$$\left[\!\left[\nabla_{\mathtt{pw}} v\right]\!\right]|_F \cdot \boldsymbol{n}_F =: [\![\partial_n v]\!]\,|_F.$$

In the subsequent analysis we have to characterise the distributional derivatives of a limit function $v \in \mathbb{V}_\infty$. The following Proposition is a key tool to get this characterisation.

Proposition 4.9. *For $v \in \mathbb{V}_\infty$, we have*

$$|\!|\!| v |\!|\!|_k \nearrow |\!|\!| v |\!|\!|_\infty < \infty \quad \text{as } k \to \infty.$$

In particular, for fixed $\ell \in \mathbb{N}_0$, let $K \in \mathcal{T}_\ell$; then, we have

$$\int_{\{F\in\mathcal{F}_k\colon F\subset K\}} h_k^{-1} [\![\partial_n v]\!]^2 \,\mathrm{d}s \nearrow \int_{\{F\in\mathcal{F}^+\colon F\subset K\}} h_+^{-1} [\![\partial_n v]\!]^2 \,\mathrm{d}s \quad \text{as } k \to \infty$$

and

$$\int_{\{F\in\mathcal{F}_k\colon F\subset K\}} h_k^{-3} |[\![v]\!]\, \boldsymbol{n}|^2 \,\mathrm{d}s \nearrow \int_{\{F\in\mathcal{F}^+\colon F\subset K\}} h_+^{-3} |[\![v]\!]\, \boldsymbol{n}|^2 \,\mathrm{d}s \quad \text{as } k \to \infty.$$

Proof. For $v \in \mathbb{V}_\infty$ there exists a sequence $v_k \in \mathbb{V}_k$, $k \in \mathbb{N}$, such that $|\!|\!| v - v_k |\!|\!|_k + \|v - v_k\|_\Omega \to 0$ as $k \to \infty$ and $\limsup_{k\to\infty} |\!|\!| v_k |\!|\!|_k < \infty$. Therefore, $\{|\!|\!| v |\!|\!|_k\}_{k\in\mathbb{N}}$ is bounded, since $|\!|\!| v |\!|\!|_k \leqslant |\!|\!| v - v_k |\!|\!|_k + |\!|\!| v_k |\!|\!|_k < \infty$ uniformly in k. For $m \geqslant k$ we have, by inclusion $\bigcup_{F\in\mathcal{F}_k} F \subset \bigcup_{F\in\mathcal{F}_m} F$ and mesh-size reduction $h_k \geqslant h_m$, that

$$\int_{\mathcal{F}_k} h_k^{-1} [\![\partial_n v]\!]^2 + h_k^{-3} |[\![v]\!]\, \boldsymbol{n}|^2 \,\mathrm{d}s \leqslant \int_{\mathcal{F}_k} h_m^{-1} [\![\partial_n v]\!]^2 + h_m^{-3} |[\![v]\!]\, \boldsymbol{n}|^2 \,\mathrm{d}s \\ \leqslant \int_{\mathcal{F}_m} h_m^{-1} [\![\partial_n v]\!]^2 + h_m^{-3} |[\![v]\!]\, \boldsymbol{n}|^2 \,\mathrm{d}s.$$

Consequently, we have that $|\!|\!| v |\!|\!|_k \leqslant |\!|\!| v |\!|\!|_m$ and $\{|\!|\!| v |\!|\!|_k\}_{k\in\mathbb{N}}$ converges. In particular, for $\epsilon > 0$, the exists $L = L(\epsilon) \in \mathbb{N}$ such that for all $k \geqslant L$ and some sufficiently large $m > k$, we have

$$\epsilon > \left| |\!|\!| v |\!|\!|_m^2 - |\!|\!| v |\!|\!|_k^2 \right| = \int_{\mathcal{F}_m \setminus (\mathcal{F}_k \cap \mathcal{F}_m)} \frac{\alpha}{h_m} [\![\partial_n v]\!]^2 + \frac{\beta}{h_m^3} |[\![v]\!]\, \boldsymbol{n}|^2 \,\mathrm{d}s \\ - \int_{\mathcal{F}_k \setminus (\mathcal{F}_k \cap \mathcal{F}_m)} \frac{\alpha}{h_k} [\![\partial_n v]\!]^2 + \frac{\beta}{h_k^3} |[\![v]\!]\, \boldsymbol{n}|^2 \,\mathrm{d}s \\ \geqslant \int_{\mathcal{F}_k \setminus \mathcal{F}_k^+} \frac{\alpha}{h_k} [\![\partial_n v]\!]^2 + \frac{\beta}{h_k^3} |[\![v]\!]\, \boldsymbol{n}|^2 \,\mathrm{d}s.$$

This follows from the fact that $h_m|_F \leqslant 2^{-1}h_k|_F$ for all $F \in \mathcal{F}_m \backslash (\mathcal{F}_k \cap \mathcal{F}_m)$, and $\mathcal{F}_k^+ = \mathcal{F}_m \cap \mathcal{F}_k$ for sufficiently large $m > k$. Therefore,

$$\int_{\mathcal{F}_m \backslash \mathcal{F}_m^+} h_m^{-1} \left[\!\left[\partial_n v\right]\!\right]^2 + h_m^{-3} \left|\left[\!\left[v\right]\!\right] \boldsymbol{n}\right|^2 \,\mathrm{d}s \to 0$$

as $m \to \infty$ and thus

$$\begin{aligned} |\!|\!| v |\!|\!|_k^2 &= \int_\Omega \left|D_{\mathtt{pw}}^2 v\right|^2 \,\mathrm{d}x + \int_{\mathcal{F}_k^+} \frac{\alpha}{h_k} \left[\!\left[\partial_n v\right]\!\right]^2 + \frac{\beta}{h_k^3} \left|\left[\!\left[v\right]\!\right] \boldsymbol{n}\right|^2 \,\mathrm{d}s \\ &\quad + \int_{\mathcal{F}_k \backslash \mathcal{F}_k^+} \frac{\alpha}{h_k} \left[\!\left[\partial_n v\right]\!\right]^2 + \frac{\beta}{h_k^3} \left|\left[\!\left[v\right]\!\right] \boldsymbol{n}\right|^2 \,\mathrm{d}s \\ &\to |\!|\!| v |\!|\!|_\infty + 0 \qquad \text{as } k \to \infty. \end{aligned}$$

The second claim is a localised version and follows by analogous arguments. □

The following proposition characterises the distributional Hessian of a limit function $v \in \mathbb{V}_\infty$ and states, that the distributional Hessian is given by the piece-wise Hessian (4.2.3) with additional contributions of jump-terms since v is discontinuous. In the same vein, the distributional derivative is given by the piece-wise gradient (4.2.2) with additional contributions of jump-terms.

Proposition 4.10. *Let $v \in \mathbb{V}_\infty$. Then, for $\varphi \in C_0^\infty(\Omega)^{2\times 2}$ the distributional Hessian of v is given by*

$$\begin{aligned} \langle D^2 v, \boldsymbol{\varphi} \rangle &= -\langle Dv, \operatorname{div} \boldsymbol{\varphi} \rangle = \langle v, \operatorname{div}\operatorname{div} \boldsymbol{\varphi} \rangle \\ &= \int_\Omega D_{pw}^2 v : \boldsymbol{\varphi} \,\mathrm{d}x - \int_{\mathcal{F}^+} \boldsymbol{\varphi} \left[\!\left[\nabla_{pw} v\right]\!\right] \cdot \boldsymbol{n} \,\mathrm{d}s + \int_{\mathcal{F}^+} \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[v\right]\!\right] \boldsymbol{n} \,\mathrm{d}s \end{aligned}$$

and for $\varphi \in C_0^\infty(\Omega)^2$ the distributional derivative is given by

$$\langle Dv, \boldsymbol{\varphi} \rangle = \int_\Omega \nabla_{pw} v \cdot \boldsymbol{\varphi} \,\mathrm{d}x - \int_{\mathcal{F}^+} \boldsymbol{\varphi} \left[\!\left[v\right]\!\right] \cdot \boldsymbol{n} \,\mathrm{d}s.$$

Proof. Let $v \in \mathbb{V}_\infty$. Then, there exists a sequence $\{v_k\}_{k\in\mathbb{N}_0}$ with $|\!|\!| v - v_k |\!|\!|_k + \|v - v_k\|_\Omega \to 0$ as $k \to \infty$. For the distributional Hessian of v_k we have by element-wise integration by parts for $\varphi \in C_0^\infty(\Omega)$:

$$\begin{aligned} \langle D^2 v_k, \boldsymbol{\varphi} \rangle &= -\langle D v_k, \operatorname{div} \boldsymbol{\varphi} \rangle = \langle v_k, \operatorname{div}\operatorname{div} \boldsymbol{\varphi} \rangle = \int_\Omega v_k \operatorname{div}\operatorname{div} \boldsymbol{\varphi} \,\mathrm{d}x \\ &= \int_\Omega D_{\mathtt{pw}}^2 v_k : \boldsymbol{\varphi} \,\mathrm{d}x - \int_{\mathcal{F}_k} \boldsymbol{\varphi} \left[\!\left[\nabla_{\mathtt{pw}} v_k\right]\!\right] \cdot \boldsymbol{n} \,\mathrm{d}s + \int_{\mathcal{F}_k} \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[v_k\right]\!\right] \boldsymbol{n} \,\mathrm{d}s. \end{aligned}$$

We already now that $\langle D^2 v_k, \boldsymbol{\varphi} \rangle \to \langle D^2 v, \boldsymbol{\varphi} \rangle$ as $k \to \infty$ since $v_k \to v$ in $L^2(\Omega)$ as $k \to \infty$. Hence, we are left to analyse the limits of the jump terms. To this end consider

$$\begin{aligned} \int_{\mathcal{F}_k} \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[v_k\right]\!\right] \boldsymbol{n} \,\mathrm{d}s &= \int_{\mathcal{F}_k} \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[v_k - v\right]\!\right] \boldsymbol{n} \,\mathrm{d}s + \int_{\mathcal{F}_k} \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[v\right]\!\right] \boldsymbol{n} \,\mathrm{d}s \\ &= \int_{\mathcal{F}_k} \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[v_k - v\right]\!\right] \boldsymbol{n} \,\mathrm{d}s + \int_{\mathcal{F}_k^+} \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[v\right]\!\right] \boldsymbol{n} \,\mathrm{d}s \qquad (4.2.4) \\ &\quad + \int_{\mathcal{F}_k^-} \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[v\right]\!\right] \boldsymbol{n} \,\mathrm{d}s. \end{aligned}$$

Now, the first term on the right-hand side of (4.2.4) vanishes thanks $\left\lVert\!\left\lvert v - v_k \right\rvert\!\right\rVert_k \to 0$ as $k \to \infty$. For the second term we have from the definition of $\mathcal{F}^+$

$$\int_{\mathcal{F}_k^+} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v]\!]\, \boldsymbol{n} \,\mathrm{d}s = \int_{\mathcal{F}^+} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v]\!]\, \boldsymbol{n} \,\mathrm{d}s - \int_{\mathcal{F}^+ \setminus \mathcal{F}_k^+} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v]\!]\, \boldsymbol{n} \,\mathrm{d}s. \quad (4.2.5)$$

Moreover, Hölder's inequality in conjunction with Lemma 4.5 reveals

$$\int_{\mathcal{F}^+ \setminus \mathcal{F}_k^+} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v]\!]\, \boldsymbol{n} \,\mathrm{d}s \lesssim \left\| h_+^{3/2} \operatorname{div} \varphi \right\|_{\Gamma^+} \left(\int_{\mathcal{F}^+ \setminus \mathcal{F}_k^+} h_+^{-3} \left| [\![v]\!]\, \boldsymbol{n} \right|^2 \,\mathrm{d}s \right)^{1/2}$$
$$\to 0 \quad \text{as } k \to \infty.$$

Hence, in (4.2.5) we obtain

$$\int_{\mathcal{F}_k^+} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v]\!]\, \boldsymbol{n} \,\mathrm{d}s \to \int_{\mathcal{F}^+} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v]\!]\, \boldsymbol{n} \,\mathrm{d}s \quad \text{as } k \to \infty.$$

For the remaining term in (4.2.4), we have by Hölder's inequality in conjunction with the scaled trace inequality and the finite overlap of patches $\omega_k(F)$, $\mathcal{F} \in \mathcal{F}_k$

$$\begin{aligned}
\int_{\mathcal{F}_k^-} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v]\!]\, \boldsymbol{n} \,\mathrm{d}s &= \int_{\mathcal{F}_k \setminus \mathcal{F}_k^+} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v]\!]\, \boldsymbol{n} \,\mathrm{d}s \\
&\leqslant \left(\int_{\mathcal{F}_k \setminus \mathcal{F}_k^+} h_\mathcal{T}^3 \left| \operatorname{div} \boldsymbol{\varphi} \right|^2 \,\mathrm{d}s \right)^{1/2} \left(\int_{\mathcal{F}_k \setminus \mathcal{F}_k^+} h_\mathcal{T}^{-3} \left| [\![v]\!]\, \boldsymbol{n} \right|^2 \,\mathrm{d}s \right)^{1/2} \\
&\lesssim \|\varphi\|_{H^2(\Omega)} \left(\int_{\mathcal{F}_k \setminus \mathcal{F}_k^+} h_\mathcal{T}^{-3} \left| [\![v]\!]\, \boldsymbol{n} \right|^2 \,\mathrm{d}s \right)^{1/2} \\
&\to 0 \quad \text{as } k \to \infty,
\end{aligned}$$

where we used Proposition 4.9 in the last line. Consequently, in (4.2.4) we have

$$\int_{\mathcal{F}_k} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v_k]\!]\, \boldsymbol{n} \,\mathrm{d}s \to \int_{\mathcal{F}^+} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v_k]\!]\, \boldsymbol{n} \,\mathrm{d}s \quad \text{as } k \to \infty.$$

Hence, we obtain

$$\int_{\mathcal{F}_k^+} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v]\!]\, \boldsymbol{n} \,\mathrm{d}s + \int_{\mathcal{F}_k^-} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v]\!]\, \boldsymbol{n} \,\mathrm{d}s \to \int_{\mathcal{F}^+} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v]\!]\, \boldsymbol{n} \,\mathrm{d}s$$

as $k \to \infty$. By similar arguments we have for the jump of the normal piece-wise gradient

$$\int_{\mathcal{F}_k^+} \boldsymbol{\varphi} \left[\!\left[\nabla_{\mathtt{pw}} v_k \right]\!\right] \cdot \boldsymbol{n} \,\mathrm{d}s \to \int_{\mathcal{F}^+} \boldsymbol{\varphi} \left[\!\left[\nabla_{\mathtt{pw}} v \right]\!\right] \cdot \boldsymbol{n} \,\mathrm{d}s$$

as $k \to \infty$. The assertion finally follows since $D^2_{\mathtt{pw}} v_k \to D^2_{\mathtt{pw}} v$ as $k \to \infty$ in $L^2(\Omega)^{2\times 2}$, due to $\lim_{k\to\infty} \|\!|v - v_k|\!\|_k = 0$ and we we obtain

$$\begin{aligned} \langle D^2 v, \boldsymbol{\varphi}\rangle \leftarrow \langle D^2 v_k, \boldsymbol{\varphi}\rangle &= \int_\Omega D^2_{\mathtt{pw}} v_k \colon \boldsymbol{\varphi} \,\mathrm{d}x \\ &\quad - \int_{\mathcal{F}_k} \boldsymbol{\varphi} \,[\![\nabla v_k]\!] \cdot \boldsymbol{n} \,\mathrm{d}s + \int_{\mathcal{F}_k} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v_k]\!] \,\boldsymbol{n} \,\mathrm{d}s \\ &\to \int_\Omega D^2_{\mathtt{pw}} v \colon \boldsymbol{\varphi} \,\mathrm{d}x \\ &\quad - \int_{\mathcal{F}^+} \boldsymbol{\varphi} \,[\![\nabla v]\!] \cdot \boldsymbol{n} \,\mathrm{d}s + \int_{\mathcal{F}^+} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v]\!] \,\boldsymbol{n} \,\mathrm{d}s \quad \text{as } k \to \infty. \end{aligned}$$

In order to prove the second assertion, we use again that for $v \in \mathbb{V}_\infty$ there exists a sequence $\{v_k\}_{k\in\mathbb{N}_0}$ with $\|\!|v - v_k|\!\|_k + \|v - v_k\|_\Omega \to 0$ as $k \to \infty$. Hence, we consider the distributional derivative of v_k and use integration by parts to obtain

$$\langle D v_k, \boldsymbol{\varphi}\rangle = \int_\Omega \nabla_{\mathtt{pw}} v_k \cdot \boldsymbol{\varphi} \,\mathrm{d}x - \int_{\mathcal{F}_k} \boldsymbol{\varphi} \,[\![v_k]\!] \cdot \boldsymbol{n} \,\mathrm{d}s \quad \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega)^2.$$

The assertion now follows by completely analogous arguments as in the case above. □

The following corollary states that the estimates of Propositions 3.12, 3.30 and 3.31 holds true on the limit space $\mathbb{V}_\infty$.

Corollary 4.11. *Let $v \in \mathbb{V}_\infty$. Then, we have*

(a) $\|v\|_\Omega \lesssim \|\!|v|\!\|_\infty$*;*

(b) $\|\nabla_{pw} v\|_\Omega \lesssim \|\!|v|\!\|_\infty$*;*

(c) $|Dv|(\Omega) \lesssim \|\!|v|\!\|_\infty$ *and*

(d) $|D(\nabla_{pw} v)|(\Omega) \lesssim \|\!|v|\!\|_\infty$*, where* $|D(\nabla_{pw} v)|(\Omega)$ *denotes the total variation of* $\nabla_{pw} v \in L^2(\Omega)$*.*

Proof. Let $v \in \mathbb{V}_\infty$. Then there exists a sequence $\{v_k\}_{k\in\mathbb{N}_0}$ with $\|\!|v - v_k|\!\|_k + \|v - v_k\|_\Omega \to 0$ as $k \to \infty$. Consequently, Propositions 3.12 and 4.9 imply

$$\|v_k\|_\Omega \lesssim \|\!|v_k|\!\|_k \leqslant \|\!|v - v_k|\!\|_k + \|\!|v|\!\|_k \to \|\!|v|\!\|_\infty < \infty$$

as $k \to \infty$. We thus conclude that $\|v_k\|_\Omega$ is bounded uniformly and therefore $v_k \rightharpoonup v$ in $L^2(\Omega)$ as $k \to \infty$. As a result, we have from lower-semicontinuity of the L^2-norm in conjunction with Proposition 3.12

$$\|v\|_\Omega \leqslant \liminf_{k\to\infty} \|v_k\|_\Omega \lesssim \liminf_{k\to\infty} \|\!|v_k|\!\|_k = \lim_{k\to\infty} \|\!|v_k|\!\|_k \leqslant \|\!|v|\!\|_\infty .$$

Consequently, Proposition 3.12 holds for all $v \in \mathbb{V}_\infty$.

The statement (b) follows by similar arguments. In order to prove statement (c), we argue as above and obtain, that $\{v_k\}_{k\in\mathbb{N}}$ is uniformly bounded in the BV-norm, due to Propositions 3.12 and 3.30. Hence, by Theorem 3.24 we have $v_k \overset{*}{\rightharpoonup} v$ in $BV(\Omega)$ as $k \to \infty$. Consequently, the assertion follows by similar arguments as above, but here we use lower semicontinuity of the BV-norm with respect to weak* convergence (compare Proposition 3.26 and Propositions 3.12 and 3.30.

Finally, assertion (d) follows similarly to the proof of statement (c). □

The next Lemma is crucial for the existence of a generalised Galerkin solution in $\mathbb{V}_\infty$, its proof is postponed to Section 4.3.3.

Lemma 4.12. *The space* $(\mathbb{V}_\infty, \langle\cdot,\cdot\rangle_\infty)$ *is a Hilbert space.*

In order to extend the discrete problem (3.2.1) to the space $\mathbb{V}_\infty$, we have to extend the bilinear form $\mathfrak{B}_\mathcal{T}$ to the space $\mathbb{V}_\infty$. To this end, we define suitable liftings for the limit space. Thanks to Lemma 4.4, for each $F \in \mathcal{F}^+$, there exists $L = L(F)$ such that $F \in \mathcal{F}_\ell^{1+}$ for all $\ell \geqslant L$. We define the local lifting operators

$$\mathcal{L}_\infty^F := \mathcal{L}_L^F = \mathcal{L}_{\mathcal{T}_L}^F. \tag{4.2.6}$$

From the definition of the discrete local liftings (3.3.1), we see that $\mathcal{L}_\infty^F$ vanishes outside the two neighbouring element K', K, with $F = K \cap K'$. Consequently, we have $\mathcal{L}_\ell^F = \mathcal{L}_L^F$ for all $\ell \geqslant L$, and therefore this definition is unique. The global lifting operator is defined by

$$\mathcal{L}_\infty = \sum_{F\in\mathcal{F}^+} \mathcal{L}_\infty^F. \tag{4.2.7}$$

From estimate (3.3.4) we have that $v_\ell := \sum_{F\in\mathcal{F}_\ell^+} \mathcal{L}_\infty^F(v)$ is a Cauchy sequence in $L^2(\Omega)^{d\times d}$ with limit $\mathcal{L}_\infty(v) = \sum_{F\in\mathcal{F}^+} \mathcal{L}_\infty^F(v)$. Therefore, $\mathcal{L}_\infty(v) \in L^2(\Omega)^{d\times d}$ and the estimate

$$\|\mathcal{L}_\infty(v)\|_\Omega^2 \lesssim \left\| h_+^{-1/2} [\![\partial_n v]\!] \right\|_{\Gamma^+}^2 + \left\| h_+^{-3/2} [\![v]\!] \boldsymbol{n} \right\|_{\Gamma^+}^2. \tag{4.2.8}$$

holds. Here we used the notation $\Gamma^+ := \bigcup\{F \mid F \in \mathcal{F}^+\}$. Now we are in position to generalise the DG-bilinear form to $\mathbb{V}_\infty$ setting

$$\begin{aligned}\mathfrak{B}_\infty[v,\, w] := &\int_\Omega D_{\mathtt{pw}}^2 v \colon D_{\mathtt{pw}}^2 w \,\mathrm{d}x + \int_\Omega \mathcal{L}_\infty(w) \colon D_{\mathtt{pw}}^2 v + \mathcal{L}_\infty(v) \colon D_{\mathtt{pw}}^2 w \,\mathrm{d}x \\ &+ \int_{\mathcal{F}^+} \frac{\alpha}{h_+} [\![\partial_n v]\!] [\![\partial_n w]\!] + \frac{\beta}{h_+^3} [\![v]\!] \boldsymbol{n} \cdot [\![w]\!] \boldsymbol{n} \,\mathrm{d}s,\end{aligned}$$

for all $v, w \in \mathbb{V}_\infty$.

Corollary 4.13. *There exists a unique* $u_\infty \in \mathbb{V}_\infty$*, such that*

$$\mathfrak{B}_\infty[u_\infty,\, v] = \int_\Omega f v \,\mathrm{d}x \quad \forall v \in \mathbb{V}_\infty. \tag{4.2.9}$$

Proof. From Lemma 4.12 we have that $\mathbb{V}_\infty$ is a Hilbert space. Moreover, stability of the lifting operators (4.2.8) and the Cauchy-Schwarz inequality imply the continuity of $\mathfrak{B}_\infty[\cdot,\,\cdot]$ since for $v, w \in \mathbb{V}_\infty$ we have

$$\begin{aligned}\mathfrak{B}_\infty[v,\,w] &\lesssim \left\|D^2_{\mathtt{pw}}v\right\|_\Omega \left\|D^2_{\mathtt{pw}}w\right\|_\Omega + \|\mathcal{L}_\infty(v)\|_\Omega \left\|D^2_{\mathtt{pw}}w\right\|_\Omega \\ &\quad + \|\mathcal{L}_\infty(w)\|_\Omega \left\|D^2_{\mathtt{pw}}v\right\|_\Omega + \left\|h_+^{-1/2}\,[\![\partial_n v]\!]\right\|_{\Gamma^+} \left\|h_+^{-1/2}\,[\![\partial_n w]\!]\right\|_{\Gamma^+} \\ &\quad + \left\|h_+^{-3/2}\,[\![v]\!]\right\|_{\Gamma^+} \left\|h_+^{-3/2}\,[\![w]\!]\right\|_{\Gamma^+} \\ &\lesssim |\!|\!| v |\!|\!|_\infty \, |\!|\!| w |\!|\!|_\infty \, .\end{aligned}$$

In view of coercivity of $\mathfrak{B}_\infty[\cdot,\,\cdot]$ we obtain for $v \in \mathbb{V}_\infty$ by standard estimates (compare also with Lemma 3.8)

$$\begin{aligned}\mathfrak{B}_\infty[v,\,v] &\geqslant \left\|D^2_{\mathtt{pw}}v\right\|^2_\Omega - 2\,\|\mathcal{L}_\infty(v)\|_\Omega \left\|D^2_{\mathtt{pw}}v\right\|_\Omega \\ &\quad + \int_{\mathcal{F}^+} \frac{\alpha}{h_+}\,[\![\partial_n v]\!]^2 + \frac{\beta}{h_+^3}\,|[\![v]\!]\,\boldsymbol{n}|^2 \,\mathrm{d}s \\ &\geqslant \frac{1}{2}\left\|D^2_{\mathtt{pw}}v\right\|^2_\Omega - 2\,\|\mathcal{L}_\infty(v)\|^2_\Omega \\ &\quad + \int_{\mathcal{F}^+} \frac{\alpha}{h_+}\,[\![\partial_n v]\!]^2 + \frac{\beta}{h_+^3}\,|[\![v]\!]\,\boldsymbol{n}|^2 \,\mathrm{d}s.\end{aligned}$$

Hence, the stability of the lifting operators (4.2.8) implies coercivity of the limit bilinear form.

The assertion finally follows from the Lax-Milgram Theorem 2.8. □

Remark 4.14. *By analogous arguments as in* (3.3.5) *we observe that the solution $u_\infty \in \mathbb{V}_\infty$ is also stable in the sense that*

$$|\!|\!| u_\infty |\!|\!|_\infty \lesssim \|f\|_\Omega \, .$$

The following Theorem states that the solution of (4.2.9) is indeed the limit of the adaptive sequence produced by the SIPDG method. Its proof is postponed to Section 4.3.

Theorem 4.15. *Let u_∞ the solution of* (4.2.9) *and let $\{u_k\}_{k\in\mathbb{N}_0}$ be the sequence of SIPDG solutions produced by ASIPDG method. Then,*

$$|\!|\!| u_\infty - u_k |\!|\!|_k \to 0 \quad \textit{as } k \to \infty.$$

4.2.3 Proof of the Main Theorem 4.3

In this section the marking strategy (4.1.1) becomes important. In particular, it essentially forces the maximal indicator to vanish, which allows to control the error on the sequence $\{\mathcal{T}_k^+\}_{k\in\mathbb{N}_0}$. Moreover, this has implications on the regularity of the Galerkin solution $u_\infty \in \mathbb{V}_\infty$ from Corollary 4.13, which finally allow us to prove that $u = u_\infty$. Thanks to the lower bound, we can thus conclude the proof of Theorem 4.3 from Theorem 4.15 employing the lower bound in Proposition 3.17.

We start with proving that the maximal indicator vanishes.

Lemma 4.16. *We have that*

$$\max_{K \in \mathcal{T}_k} \eta_k(u_k, K) \to 0 \quad \text{as } k \to \infty$$

Proof. Let $k \in \mathbb{N}_0$, and $K_k \in \mathcal{T}_k^-$ such that $\eta_k(u_k, K_k) = \max_{K' \in \mathcal{T}_k^-} \eta_k(u_k, K')$. Then we have by standard scaled trace- and inverse estimates that

$$\begin{aligned}
\eta_k(u_k, K_k)^2 &= \int_{K_k} h_k^4 \left|f - \Delta^2 u_k\right|^2 \mathrm{d}x \\
&\quad + \int_{\partial K_k \cap \Omega} h_k^3 \left|[\![\nabla \cdot D^2_{\mathtt{pw}} u_k]\!] \cdot \boldsymbol{n}\right|^2 + h_k \left|[\![D^2_{\mathtt{pw}} u_k]\!]\, \boldsymbol{n}\right|^2 \mathrm{d}s \\
&\quad + \int_{\partial K_k} \frac{\alpha^2}{h_k} [\![\partial_n u_k]\!]^2 + \frac{\beta^2}{h_{\mathcal{T}}^3} \left|[\![u_k]\!]\, \boldsymbol{n}\right|^2 \mathrm{d}s \\
&\lesssim \int_{K_k} h_k^4 \left|f\right|^2 \mathrm{d}x + \int_{K_k} \left|\Delta u_k\right|^2 \mathrm{d}x \\
&\quad + \int_{\omega_k(K_k)} \left|D^2_{\mathtt{pw}} u_k\right|^2 \mathrm{d}x \\
&\quad + \int_{\partial K_k} \frac{\alpha^2}{h_k} [\![\partial_n u_k]\!]^2 + \frac{\beta^2}{h_k^3} \left|[\![u_k]\!]\, \boldsymbol{n}\right|^2 \mathrm{d}s, \\
&\lesssim \int_{K_k} h_k^4 \left|f\right|^2 \mathrm{d}x + \int_{\omega_k(K_k)} \left|D^2_{\mathtt{pw}} u_k\right|^2 \mathrm{d}x \\
&\quad + \int_{\partial K_k} \frac{\alpha^2}{h_k} [\![\partial_n u_k]\!]^2 + \frac{\beta^2}{h_k^3} \left|[\![u_k]\!]\, \boldsymbol{n}\right|^2 \mathrm{d}s,
\end{aligned} \tag{4.2.10}$$

where we used

$$\int_{K_k} \left|\Delta u_k\right|^2 \mathrm{d}x \lesssim \int_{\omega_k(K_k)} \left|D^2_{\mathtt{pw}} u_k\right|^2 \mathrm{d}x.$$

The first term on the right hand side of (4.2.10) converges to zero thanks to Lemma 4.5. For the remaining terms, we have from triangle inequalities that

$$\begin{aligned}
&\int_{\omega_k(K_k)} |D^2_{\mathtt{pw}} u_k|^2 \,\mathrm{d}x + \int_{\partial K_k} \frac{\alpha^2}{h_k} [\![\partial_n u_k]\!]^2 + \frac{\beta^2}{h_k^3} \left|[\![u_k]\!]\, \boldsymbol{n}\right|^2 \mathrm{d}s \\
&\qquad \lesssim \|u_\infty - u_k\|_k^2 + \int_{\omega_k(K_k)} |D^2_{\mathtt{pw}} u_\infty|^2 \,\mathrm{d}x + \int_{\partial K} \frac{\alpha^2}{h_k} [\![\partial_n u_\infty]\!]^2 \,\mathrm{d}s \\
&\qquad\quad + \int_{\partial K_k} \frac{\beta^2}{h_k^3} \left|[\![u_\infty]\!]\, \boldsymbol{n}\right|^2 \mathrm{d}s \\
&\qquad \leqslant \|u_\infty - u_k\|_k^2 + \int_{\omega_k(K_k)} |D^2_{\mathtt{pw}} u_\infty|^2 \,\mathrm{d}x \\
&\qquad\quad + \int_{\mathcal{F}^+ \setminus \mathcal{F}_k^+} \frac{\alpha^2}{h_+} [\![\partial_n u_\infty]\!]^2 + \frac{\beta^2}{h_+^3} \left|[\![u_\infty]\!]\, \boldsymbol{n}\right|^2 \mathrm{d}s.
\end{aligned}$$

We have that $\|u_\infty - u_k\|_k \to 0$ as $k \to \infty$ due to Theorem 4.15 and also the jump terms vanish as $k \to \infty$ by Proposition 4.9, since $u_\infty \in \mathbb{V}_\infty$. For the remaining volume term we infer from local quasi-uniformity (3.1.5)

$$|\omega_k(K_k)| \lesssim |K_k| \lesssim \left\|h^2_{\mathcal{T}_k}\chi_{K_k}\right\|_{L^\infty(\Omega)} \lesssim \left\|h^2_{\mathcal{T}_k}\chi_{\Omega_k^-}\right\|_{L^\infty(\Omega)} \to 0 \qquad (4.2.11)$$

as $k \to \infty$, due to Lemma 4.5. Whence, the absolute continuity of the L^2-norm, with respect to the Lebesgue measure implies

$$\int_{\omega_k(K_k)} |D^2_{\mathtt{pw}} u_\infty|^2 \, \mathrm{d}x \to 0 \quad \text{as } k \to \infty,$$

compare with Remark 4.6. As a consequence, we infer that the maximum error indicator on $\mathcal{T}_k^-$ vanishes, i.e.

$$\max_{K \in \mathcal{T}_k^-} \eta_k(u_k, K) \to 0 \qquad \text{as } k \to \infty.$$

Now using the refinement strategy (4.1.2) we observe that all elements $\mathcal{T}_k^+$ will not be subdivided, i.e. $\mathcal{T}_k^+ \subset \mathcal{T}_k \backslash \mathcal{M}_k$. As a consequence we obtain by the marking strategy (4.1.1)

$$\begin{aligned} \lim_{k\to\infty} \max\left\{\eta_k(u_k, K) \colon K \in \mathcal{T}_k^+\right\} &\leqslant \lim_{k\to\infty} \max\left\{\eta_k(u_k, K) \colon K \in \mathcal{T}_k \backslash \mathcal{M}_k\right\} \\ &\leqslant \lim_{k\to\infty} g(\max\left\{\eta_k(u_k, K) \colon K \in \mathcal{M}_k\right\}) \\ &\leqslant \lim_{k\to\infty} g(\max\left\{\eta_k(u_k, K) \colon K \in \mathcal{T}_k^-\right\}) = 0. \end{aligned}$$

Here, we used in the last inequality, that each element in $\mathcal{M}_k$ will be refined by (4.1.2) and therefore $\mathcal{M}_k \subset \mathcal{T}_k^-$. □

Lemma 4.17. *We have $\eta_k(\mathcal{T}_k^+) \to 0$ as $k \to \infty$.*

Proof. We follow the lines of [MSV08, Proposition 4.3].

Employing Lemma 4.16, for $K \in \mathcal{T}_k^+$ we that $\eta_k(K) \to 0$ as $k \to \infty$. In order to proof $\eta_k(\mathcal{T}_k^+) \to 0$ as $k \to \infty$, we reformulate the estimator in an integral framework and use a generalised Lebesgue dominated convergence theorem.

1 From the definition of $\mathcal{T}_k^+$ we have that

$$\omega_k(K) = \omega_\ell(K) =: \omega(K)$$

and

$$N_k(K) = N_\ell(K) =: N(K)$$

for all $K \in \mathcal{T}_k^+$ and all $\ell \geqslant k$. Moreover, from Proposition 3.17 we obtain for $K \in \mathcal{T}_k^+$

$$\begin{aligned} \eta_k^2(K) &\lesssim \left\|D^2_{\mathtt{pw}}(u - u_k)\right\|^2_{\omega(K)} + \sum_{F \subset \omega(K)} \int_F h_k^{-1} \left[\!\left[\partial_n u_k\right]\!\right]^2 + h_k^{-3} \left|\left[\!\left[u_k\right]\!\right] \boldsymbol{n}\right|^2 \mathrm{d}s \\ &\quad + \operatorname{osc}(N_k(K), f)^2 \\ &\lesssim \|u_k - u_\infty\|^2_{N(K)} + \left\|D^2_{\mathtt{pw}} u\right\|^2_{\omega(K)} + \|u_\infty\|^2_{N(K)} + \|f\|^2_{\omega(K)} \\ &=: \|u_k - u_\infty\|^2_{N(K)} + C_K^2. \end{aligned} \qquad (4.2.12)$$

Here, we used in the last estimate the stability of the data-oscillation. Note that C_K does not depend on the integer $k \in \mathbb{N}$, and therefore the right-hand side of (4.2.12) tends to C_K^2 as $k \to \infty$ by Theorem 4.15. Adding up over all $K \in \mathcal{T}_k^+$ and using the finite overlap of patches $\omega_k(K)$, $K \in \mathcal{T}_k$ leads us to

$$\begin{aligned}\sum_{K\in\mathcal{T}_k^{1+}} C_K^2 &= \sum_{K\in\mathcal{T}_k^{1+}} \left\|D^2_{\mathtt{pw}} u\right\|^2_{\omega(K)} + \|u_\infty\|^2_{N(K)} + \|f\|^2_{\omega(K)} \\ &\lesssim \left\|D^2 u\right\|^2_\Omega + \|u_\infty\|^2_\infty + \|f\|^2_\Omega \\ &\lesssim 1,\end{aligned} \tag{4.2.13}$$

where we used the stability of $u \in H_0^2(\Omega)$ as well as the stability of $u_\infty \in \mathbb{V}_\infty$.
2 Now we're able to give the integral formulation. From Lemma 4.4 we have

$$\mathcal{T}^+ = \bigcup_{k\in\mathbb{N}} \mathcal{T}_k^+,$$

where the sequence $\{\mathcal{T}_k^+\}_{k\in\mathbb{N}}$ is nested. Now for $x \in \Omega^+$ let

$$\ell = \ell(x) := \min\left\{k \in \mathbb{N} \colon \exists K \in \mathcal{T}_k^+, \text{ such that } x \in K\right\}.$$

We define for $x \in K$

$$\epsilon_k(x) := M_k(x) = 0, \quad \text{for } k < \ell$$

and

$$\epsilon_k(x) := \frac{1}{|K|}\eta_k^2(K), \quad M_k(x) := \frac{1}{|K|}\left(\|u_k - u_\infty\|^2_{N(K)} + C_K^2\right) \quad \text{for } k \geqslant \ell.$$

Consequently, for any integer $k \in \mathbb{N}$ we obtain

$$\eta_k^2(\mathcal{T}_k^+) = \int_{\Omega^+} \epsilon_k(x)\,\mathrm{d}x.$$

Moreover, the element-wise convergence of the estimator from step 1 implies pointwise convergence of ϵ_k in Ω^+, i.e.

$$\epsilon_k(x) = \frac{1}{|K|}\eta_k^2(K) \to 0 \quad \text{as } k \to \infty.$$

From (4.2.12) and the definition of M_k, we have that each M_k is a majorant of ϵ_k which is also integrable thanks to $\sum_{K\in\mathcal{T}_k^+} \|u_k - u_\infty\|^2_{N(K)} \lesssim \|u_k - u_\infty\|_k^2$ and (4.2.13).
3 The last step is to prove convergence of the majorants $\{M_k\}_{k\in\mathbb{N}}$ in $L^1(\Omega^+)$ to a function M, defined by

$$M(x) := \frac{1}{|K|}C_K^2, \quad x \in K, \quad K \in \mathcal{T}^+.$$

Thanks to the definiton of M_k, we observe

$$\|M_k - M\|_{L^1(\Omega^+)} = \sum_{K\in\mathcal{T}_k^+} \|M_k - M\|_{L^1(K)} + \sum_{K\in\mathcal{T}^+\setminus\mathcal{T}_k^+} \|M\|_{L^1(K)},$$

since M_k vanishes on $\mathcal{T}^+ \backslash \mathcal{T}_k^+$. The first term on the right-hand side satisfies

$$\sum_{K \in \mathcal{T}_k^+} \|M_k - M\|_{L^1(K)} = \sum_{K \in \mathcal{T}_k^+} \|u_k - u_\infty\|_{N(K)}^2 \lesssim \|u_k - u_\infty\|_k^2 \to 0$$

as $k \to \infty$ thanks to Theorem 4.15. The second term is a tail of the series $\sum_{K \in \mathcal{T}^+} \|M\|_{L^1(K)} = \sum_{K \in \mathcal{T}^+} C_K^2$, which is bounded thanks to (4.2.13). Consequently, we have $M_k \to M$ in $L^1(\Omega^+)$.

$\boxed{4}$ Finally, the application of the generalised majorised convergence Theorem (see [Zei90, Appendix (19a)]) with $\epsilon_k = f_k$, $M_k = g_k$ and $M = g$ leads us to

$$\lim_{k \to \infty} \eta_k^2(\mathcal{T}_k^+) = \lim_{k \to \infty} \int_{\Omega^+} \epsilon_k \,\mathrm{d}x = \int_{\Omega^+} 0 \,\mathrm{d}x = 0.$$

□

Remark 4.18. *We note, that Lemma 4.17 yields in particular that*

$$\int_{\mathcal{F}_k^+} h_k^{-1} [\![\partial_n u_k]\!]^2 + h_k^{-3} |[\![u_k]\!]\, \boldsymbol{n}|^2 \,\mathrm{d}s \to 0 \quad \text{as } k \to \infty, \tag{4.2.14}$$

i.e. the jump terms vanish on the non-refined domain Ω_k^+. This means we can conclude additional regularity of the limit function. This is reflected in the following lemma.

Lemma 4.19. *We have for $u_\infty \in \mathbb{V}_\infty$ from Corollary 4.13 that $u_\infty \in H_0^2(\Omega)$.*

Proof. From Theorem 4.15, we know that

$$D_{\mathrm{pw}}^2 u_k \to D_{\mathrm{pw}}^2 u_\infty \qquad \text{in } L^2(\Omega)^{2\times 2} \quad \text{as } k \to \infty.$$

Additionally, we have that $u_k \to u_\infty$ in $L^2(\Omega)$ as $k \to \infty$ thanks to $u_\infty \in \mathbb{V}_\infty$ and

$$\begin{aligned}\|u_k - u_\infty\|_\Omega &\leqslant \|u_k - v_k\|_\Omega + \|v_k - u_\infty\|_\Omega \\ &\lesssim \|u_k - v_k\|_k + \|v_k - u_\infty\|_\Omega \\ &\lesssim \|u_k - u_\infty\|_k + \|u_\infty - v_k\|_k + \|v_k - u_\infty\|_\Omega \\ &\to 0,\end{aligned}$$

as $k \to \infty$, for a sequence $\{v_k\}_{k\in\mathbb{N}}$, $v_k \in \mathbb{V}_k$ due to the defintion of the limit space and Theorem 4.15.

We have that for $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^{2\times 2}$ the distributional Hessian of u_k is given by

$$\langle D^2 u_k, \boldsymbol{\varphi}\rangle = \int_\Omega D_{\mathrm{pw}}^2 u_k : \boldsymbol{\varphi} \,\mathrm{d}x - \int_{\mathcal{F}_k} \boldsymbol{\varphi}\, [\![\partial_n u_k]\!] \,\mathrm{d}s + \int_{\mathcal{F}_k} \operatorname{div} \boldsymbol{\varphi} \cdot [\![u_k]\!]\, \boldsymbol{n} \,\mathrm{d}s.$$

Consequently, u_∞ has second weak derivatives $D_{\mathrm{pw}}^2 u_\infty$ if and only if the two jump terms vanish as $k \to \infty$. This follows from

$$\int_{\mathcal{F}_k} h_k^{-1} [\![\partial_n u_k]\!]^2 + h_k^{-3} |[\![u_k]\!]\, \boldsymbol{n}|^2 \,\mathrm{d}s \to 0 \quad \text{as } k \to \infty, \tag{4.2.15}$$

which implies $u_\infty \in H_0^2(\Omega)$ since $\mathcal{F}_k$ contains also boundary sides. In order to verify (4.2.15), we estimate

$$\begin{aligned}\int_{\mathcal{F}_k} h_k^{-1} [\![\partial_n u_k]\!]^2 + h_k^{-3} |[\![u_k]\!]\boldsymbol{n}|^2 \,\mathrm{d}s \\ &= \int_{\mathcal{F}_k^-} h_k^{-1} [\![\partial_n u_k]\!]^2 + h_k^{-3} |[\![u_k]\!]\boldsymbol{n}|^2 \,\mathrm{d}s \\ &\quad + \int_{\mathcal{F}_k^+} h_k^{-1} [\![\partial_n u_k]\!]^2 + h_k^{-3} |[\![u_k]\!]\boldsymbol{n}|^2 \,\mathrm{d}s \\ &\leqslant 2\int_{\mathcal{F}_k^-} h_k^{-1} [\![\partial_n u_\infty]\!]^2 + h_k^{-3} |[\![u_\infty]\!]\boldsymbol{n}|^2 \,\mathrm{d}s \\ &\quad + 2\,|\!|\!| u_\infty - u_k |\!|\!|_k^2 + \int_{\mathcal{F}_k^+} h_k^{-1} [\![\partial_n u_k]\!]^2 + h_k^{-3} |[\![u_k]\!]\boldsymbol{n}|^2 \,\mathrm{d}s\end{aligned}$$

Thanks to Proposition 4.9, Theorem 4.15 and (4.2.14), we have that all three terms tend to zero. This proves the assertion. □

Next, we have to prove that u_∞ coincides with the exact solution $u \in H_0^2(\Omega)$ of (2.4.2). This proof is based on the following Lemma.

Lemma 4.20. *Let $\varphi \in C_0^\infty(\Omega)$ and u_k the discrete solution generated by the ASIPDG method. Then we have*

$$|\langle f, \varphi\rangle - \mathfrak{B}_k[u_k, \varphi]| \lesssim \sum_{K\in\mathcal{T}} \eta_k(u_k, K) h_K^{s_K} |\varphi|_{H^{2+s_K}(\omega_k^2(K))},$$

with $s_K \in \{0, 1\}$, $K \in \mathcal{T}_k$.

Proof. Note that by the Galerkin orthogonality we have for all $v_k \in \mathbb{V}_k$, that

$$\langle f, \varphi\rangle - \mathfrak{B}_k[u_k, \varphi] = \langle f, \varphi - v_k\rangle - \mathfrak{B}_k[u_k, \varphi - v_k]. \tag{4.2.16}$$

Let $v_k := \mathcal{I}_k\varphi$, where $\mathcal{I}_k\varphi$ is the quasi-interpolant from (4.3.5), and define $\rho_k := \varphi - \mathcal{I}_k\varphi$. Integration by parts yields

$$\begin{aligned}&\langle f, \rho_k\rangle - \mathfrak{B}_k[u_k, \rho_k] \\ &\quad = \int_\Omega (f - \Delta^2 u_k)\rho_k - \int_\Omega \mathcal{L}_k(\rho_k) : D_{\mathrm{pw}}^2 u_k + \mathcal{L}_k(u_k) : D_{\mathrm{pw}}^2 \rho_k \,\mathrm{d}x \\ &\quad\quad - \sum_{K\in\mathcal{T}_k} \int_{\partial K} D^2 u_k \boldsymbol{n}_K \cdot \nabla\rho_k - \rho_k \nabla\cdot D^2 u_k \cdot \boldsymbol{n}_K \,\mathrm{d}s \\ &\quad\quad - \int_{\mathcal{F}_k} \frac{\alpha}{h_k} [\![\partial_n u_k]\!] [\![\partial_n \rho_k]\!] + \frac{\beta}{h_k^3} [\![u_k]\!]\boldsymbol{n}\cdot[\![\rho_k]\!]\boldsymbol{n}\,\mathrm{d}s.\end{aligned} \tag{4.2.17}$$

Thanks to $D_{\mathrm{pw}}^2 u_k \in \mathbb{P}_{r-2}(\mathcal{T}_k)^{2\times 2}$, we can use the definition of the local lifting operators

$$\int_\Omega \mathcal{L}_k(\rho_k) : D_{\mathrm{pw}}^2 u_k \,\mathrm{d}x = \sum_{F\in\mathcal{F}_k} \int_F \{\!\{\nabla\cdot D_{\mathrm{pw}}^2 u_k\}\!\} \cdot \boldsymbol{n}_F [\![\rho_k]\!] - \{\!\{D_{\mathrm{pw}}^2 u_k\}\!\} [\![\nabla\rho_k]\!]\cdot\boldsymbol{n}_F \,\mathrm{d}s.$$

Moreover, reformulation of the boundary integrals to face integrals ((3.1.8) and (3.1.9)) reveals

$$\begin{aligned}\sum_{K\in\mathcal{T}_k}\int_{\partial K} & D^2u_k\boldsymbol{n}_K\cdot\nabla\rho_k-\rho_k\nabla\cdot D^2u_k\cdot\boldsymbol{n}_K\,\mathrm{d}s\\ &=\int_{\mathcal{F}}\{\!\{D^2u_k\}\!\}\,[\![\nabla\rho_k]\!]\cdot\boldsymbol{n}_{\mathcal{F}}-[\![\rho_k]\!]\,\{\!\{\nabla\cdot D^2u_k\}\!\}\cdot\boldsymbol{n}_{\mathcal{F}}\,\mathrm{d}s\\ &\quad+\int_{\mathring{\mathcal{F}}_k}[\![D^2u_k]\!]\,\{\!\{\nabla\rho_k\}\!\}\cdot\boldsymbol{n}_{\mathcal{F}}-\{\!\{\rho_k\}\!\}\,[\![\nabla\cdot D^2u_k]\!]\cdot\boldsymbol{n}_{\mathcal{F}}\,\mathrm{d}s\end{aligned}$$

Consequently, inserting this in (4.2.17) yields

$$\begin{aligned}\langle f,\,\varphi-\mathcal{I}_k\varphi\rangle & -\mathfrak{B}_k[u_k,\,\varphi-\mathcal{I}_k\varphi]\\ &=\int_\Omega(f-\Delta^2_{\mathrm{pw}}u_k)\rho_k\,\mathrm{d}x-\int_\Omega\mathcal{L}_k(u_k)\colon D^2_{\mathrm{pw}}\rho_k\,\mathrm{d}x\\ &\quad-\int_{\mathring{\mathcal{F}}_k}[\![D^2u_k]\!]\,\{\!\{\nabla\rho_k\}\!\}\cdot\boldsymbol{n}_{\mathcal{F}}-\{\!\{\rho_k\}\!\}\,[\![\nabla\cdot D^2u_k]\!]\cdot\boldsymbol{n}_{\mathcal{F}}\,\mathrm{d}s\\ &\quad-\int_{\mathcal{F}_k}\frac{\alpha}{h_k}[\![\partial_nu_k]\!]\,[\![\partial_n\rho_k]\!]\,\mathrm{d}s+\frac{\beta}{h_k^3}[\![u_k]\!]\,\boldsymbol{n}\cdot[\![\rho_k]\!]\,\boldsymbol{n}\,\mathrm{d}s.\end{aligned}\tag{4.2.18}$$

Thanks to $\varphi\in C_0^\infty(\Omega)$ standard interpolation estimates provide for $j\in\{0,1,2\}$ and $s_K\in\{0,1\}$ that

$$\int_K\left|D^j(\varphi-\mathcal{I}_k\varphi)\right|^2\,\mathrm{d}x\lesssim\int_{\omega_k(K)}h_k^{2(2+s_K-j)}\left|D^{2+s_K}\varphi\right|\,\mathrm{d}x\quad K\in\mathcal{T};\tag{4.2.19}$$

compare e.g. with [Cle75]. For the first term on the right-hand side of (4.2.18), we have

$$\begin{aligned}\left|\int_\Omega(f-\Delta^2_{\mathrm{pw}}u_k)\rho_k\,\mathrm{d}x\right| &\leqslant\sum_{K\in\mathcal{T}_k}\left\|f-\Delta^2u_k\right\|_K\|\rho_k\|_K\\ &\lesssim\sum_{K\in\mathcal{T}_k}\left\|h_k^2(f-\Delta^2u_k)\right\|_K h_K^{s_K}\,|\varphi|_{H^{2+s_K}(\omega_k(K))}\,,\end{aligned}$$

Using stability of the lifting operator (3.3.4) (Lemma 3.11), we estimate the second term on the right-hand side of (4.2.18) by

$$\left|\int_\Omega\mathcal{L}_k(u_k)\colon D^2_{\mathrm{pw}}\rho_k\,\mathrm{d}x\right|\lesssim\left(\left\|h_{\mathcal{T}}^{-1/2}\,[\![\partial_nu_k]\!]\right\|_{\Gamma_k}+\left\|h_{\mathcal{T}}^{-3/2}\,[\![u_k]\!]\,\boldsymbol{n}\right\|_{\Gamma_k}\right)h_K^{s_K}\,|\varphi|_{H^{2+s_K}(\omega_k(K))}\,.$$

Combining a scaled trace inequality with (4.2.19), we obtain

$$\begin{aligned}\left|\int_{\mathring{\mathcal{F}}_k}[\![D^2u_k]\!]\,\{\!\{\nabla\rho_k\}\!\}\cdot\boldsymbol{n}_{\mathcal{F}}\right| &\lesssim\sum_{F\in\mathring{\mathcal{F}}_k}\left|\int_F h_k^{-1}\,\{\!\{\nabla\rho_k\}\!\}^2\,h_k\,[\![D^2u_k]\!]^2\,\mathrm{d}s\right|\\ &\lesssim\sum_{K\in\mathcal{T}_k}\eta_k(u_k,K)h_K^{s_K}\,|\varphi|_{H^{2+s_K}(\omega_k^2(K))}\,.\end{aligned}$$

Analogous arguments, yield

$$\left|\int_{\mathring{\mathcal{F}}_k} \{\!\{\rho_k\}\!\} [\![\nabla\cdot D^2 u_k]\!]\cdot \boldsymbol{n}_{\mathcal{F}}\right| \lesssim \sum_{F\in\mathring{\mathcal{F}}_k} \left|\int_F h_k^{-3}\{\!\{\rho_k\}\!\}^2 h_k^3 \left|[\![\nabla\cdot D^2u_k]\!]\cdot\boldsymbol{n}_F\right|^2 \,\mathrm{d}s\right|$$
$$\lesssim \sum_{K\in\mathcal{T}_k} \eta_k(u_k,K) h_K^{s_K} |\varphi|_{H^{2+s_K}(\omega_k^2(K))}\,.$$

Finally, for the last term on the right-hand side of (4.2.18) we deduce in the same fashion

$$\left|\int_{\mathcal{F}_k} \frac{\alpha}{h_k} [\![\partial_n u_k]\!] [\![\partial_n \rho_k]\!] \,\mathrm{d}s + \frac{\beta}{h_k^3} [\![u_k]\!]\,\boldsymbol{n}\cdot[\![\rho_k]\!]\,\boldsymbol{n}\,\mathrm{d}s\right|$$
$$\lesssim \sum_{K\in\mathcal{T}_k} \eta_k(u_k,K) h_K^{s_K} |\varphi|_{H^{2+s_K}(\omega_k^2(K))}\,.$$

Inserting the above estimates in (4.2.18) proves the claim. □

Lemma 4.21. *Let* $u \in H_0^2(\Omega)$ *and* $u_\infty \in \mathbb{V}_\infty$ *be the solutions of* (2.4.2) *and* (4.2.9) *respectively. Then* $u = u_\infty$.

Proof. We recall that for $v, w \in H_0^2(\Omega)$ we have $\mathfrak{B}[v,\, w] = \mathfrak{B}_k[v,\, w] = \mathfrak{B}_\infty[v,\, w]$. Therefore, we obtain from $u_\infty \in H_0^2(\Omega)$ and (4.2.9) that

$$\begin{aligned}
|\!|\!| u - u_\infty |\!|\!|^2 &\lesssim \mathfrak{B}[u-u_\infty,\, u-u_\infty]\\
&= \mathfrak{B}[u,\, u-u_\infty] - \mathfrak{B}_\infty[u_\infty,\, u] + \mathfrak{B}_\infty[u_\infty,\, u_\infty]\\
&= \langle f, u-u_\infty\rangle - \mathfrak{B}_\infty[u_\infty,\, u] + \langle f, u_\infty\rangle\\
&= \langle f, u\rangle - \mathfrak{B}_k[u_\infty,\, u] = \langle f, u\rangle - \mathfrak{B}_k[u_k,\, u] + \mathfrak{B}_k[u_\infty - u_k,\, u]\\
&\leqslant \langle f, u\rangle - \mathfrak{B}_k[u_k,\, u] + |\!|\!| u |\!|\!|\, |\!|\!| u_\infty - u_k |\!|\!|_k\,.
\end{aligned}$$

The last product vanishes thanks to Theorem 4.15 and we are left with the remaining parts. By the density of $H_0^3(\Omega)$ in $H_0^2(\Omega)$, for $\epsilon > 0$ we choose $u_\epsilon \in H_0^3(\Omega)$ such that $|\!|\!| u - u_\epsilon |\!|\!| \leqslant \epsilon$. Recalling that $\langle f, v_k\rangle - \mathfrak{B}_k[u_k,\, v_k] = 0$ for all $v_k \in \mathbb{V}_k$, we employ Lemma 4.20 to obtain

$$\begin{aligned}
&|\langle f, u\rangle - \mathfrak{B}_k[u_k,\, u]|\\
&\qquad \leqslant |\langle f, u_\epsilon\rangle - \mathfrak{B}_k[u_k,\, u_\epsilon]| + |\langle f, u-u_\epsilon\rangle - \mathfrak{B}_k[u_k,\, u-u_\epsilon]|\\
&\qquad \lesssim \sum_{K\in\mathcal{T}_k^-} \eta_k(u_k,K) \|h_k\|_{L^\infty(\Omega^{1-})} |u_\epsilon|_{H^3(\omega_k^2(K))}\\
&\qquad\quad + \sum_{K\in\mathcal{T}_k^+} \eta_k(u_k,K)\, |\!|\!| u_\epsilon |\!|\!|_{N_k^2(K)} + \epsilon \|f\|_{L^2(\Omega)}\\
&\qquad \lesssim \|h_k\|_{L^\infty(\Omega^{1-})} \eta_k(u_k,\mathcal{T}_k^-) |u_\epsilon|_{H^3(\omega_k^2(K))} + \eta_k(u_k,\mathcal{T}_k^+)\, |\!|\!| u_\epsilon |\!|\!| + \epsilon \|f\|_{L^2(\Omega)}.
\end{aligned}$$

Here, we have used interpolation estimates in H^3 for the first term and stability of the interpolation for the second term as well as (3.3.5) and the finite overlap of the neighbourhoods. The first term on the right hand side vanishes thanks to Lemma 4.5 and since the estimator stays bounded (Proposition 3.17). The second term vanishes thanks to Lemma 4.17. Combining the above findings, we obtain by letting $k \to \infty$ that

$$|||u - u_\infty|||^2 \lesssim \epsilon \|f\|_{L^2(\Omega)}.$$

Since ϵ was arbitrary, this proves the assertion. □

Proof of Theorem 4.3. Thanks to Lemma 4.21 and Theorem 4.15, we have that $|||u - u_k|||_k \to 0$ as $k \to \infty$.

Combining the lower bound Proposition 3.17 with Lemmas 4.17, 4.21 and 4.5, we obtain

$$\begin{aligned}\eta_k(\mathcal{T}_k)^2 &\lesssim |||u - u_k|||_k^2 + \operatorname{osc}(\mathcal{T}_k, f)^2 \\ &= |||u - u_k|||_k^2 + \sum_{K \in \mathcal{T}_k^-} \int_K h_k^4 |f - \Pi f|^2 \,\mathrm{d}x + \sum_{K \in \mathcal{T}_k^+} \int_K h_k^4 |f - \Pi f|^2 \,\mathrm{d}x \\ &\leqslant |||u - u_k|||_k^2 + \left\| h_k \chi_{\Omega_k^-} \right\|_{L^\infty(\Omega)}^4 \|f\|_\Omega^2 + \eta_k(u_k, \mathcal{T}_k^+)^2 \\ &\to 0\end{aligned}$$

as $k \to \infty$. Here we have used $\int_K h_k^4 |f - \Pi_0 f|^2 \leqslant \eta_k(u_k, K)^2$ thanks to (3.4.16). □

4.3 Proofs of Lemma 4.12 and Theorem 4.15

In this Section we use the ideas of [DGK19, Section 4] to close the proof of the main result, Theorem 4.3. We emphasise that we still have to verify Lemma 4.12 and Theorem 4.15. The primer states that $\mathbb{V}_\infty$ is a Hilbert space with norm $|||\cdot|||_\infty$, and thus a unique solution $u_\infty \in \mathbb{V}_\infty$ of (4.2.9) exists; see Corollary 4.13. The latter proves that u_∞ is indeed the limit of the SIPDG approximations $\{u_k\}_{k \in \mathbb{N}_0}$ produced by the ASIPDG method.

We emphasise that in contrast to [KG18], the lack of proper H^2-conforming subspaces of SIPDG spaces, does not allow for a straight forward generalisation: For example, in order to prove $|||u_\infty - u_k|||_k \to 0$, in [KG18] the best-approximation property for inf-sup stable conforming elements [MSV08, Sie11] is replaced by a variant of Gudi's medius analysis [Gud10]. However, this required a discrete smoothing operator into $\mathbb{V}_\infty$, whose construction is heavily based on the existence of a proper conforming subspace of $\mathbb{V}_k$.

After recalling a Poincaré-type inequality we introduce a interpolation operator $\mathcal{I}_k \colon L^2(\Omega) \to \mathbb{V}_k$ and prove the crucial approximation property $|||\mathcal{I}_k v - v|||_k \to 0$ as $k \to \infty$ for $v \in \mathbb{V}_\infty$. Finally, we conclude the section with the proofs of Lemma 4.12 and Theorem 4.15.

4.3.1 Preliminary results

In order to prove the Poincaré and Friedrichs estimates below, we state a useful result from [KG18].

Proposition 4.22. *Let $\mathcal{T}$ be a triangulation of Ω and $\mathcal{T}_\star$ be some refinement of $\mathcal{T}$. Then, for $v \in \mathbb{V}(\mathcal{T}_\star)$, $K \in \mathcal{T}$ and $v_K := |\omega_\mathcal{T}(K)|^{-1} \int_{\omega_\mathcal{T}(K)} v \,\mathrm{d}x$, we have*

$$\|v - v_K\|^2_{\omega_\mathcal{T}(K)} \lesssim \int_{\omega_\mathcal{T}(K)} h_\mathcal{T}^2 \left|\nabla_{pw} v\right|^2 \,\mathrm{d}x + \int_{F\in\mathcal{F}_\star,\, F\subset\omega_\mathcal{T}(K)} h_\mathcal{T}^2 h_{\mathcal{T}_\star}^{-1} \left|[\![v]\!]\,\boldsymbol{n}\right|^2 \,\mathrm{d}s,$$

where $\mathcal{F}_\star = \mathcal{F}_{\mathcal{T}_\star}$ and the hidden constant depends on d and the shape regularity of $N_\mathcal{T}(K)$.

Proof. See [KG18, Proposition 1]. □

The following Poincaré estimate is subsequently used to prove stability of the smoothing and quasi-interpolation operators defined later.

Lemma 4.23. *Let $\mathcal{T}, \mathcal{T}_\star$ be two triangulations of Ω with $\mathcal{T} \leqslant \mathcal{T}_\star$ and let $v \in \mathbb{V}(\mathcal{T}_\star)$. Then, there exists a linear polynomial Q, defined on $\omega_\mathcal{T}(K)$ such that we have*

$$\begin{aligned}\|v - Q\|^2_{\omega_\mathcal{T}(K)} \lesssim & \int_{\omega_\mathcal{T}(K)} h_\mathcal{T}^4 \left|D^2_{pw} v\right|^2 \,\mathrm{d}x \\ & + \int_{\substack{F\in\mathcal{F}(\mathcal{T}_\star)\\ F\subset\omega_\mathcal{T}(K)}} h_\mathcal{T}^4 \left(h_{\mathcal{T}_\star}^{-1} [\![\partial_n v]\!]^2 + h_{\mathcal{T}_\star}^{-3} \left|[\![v]\!]\,\boldsymbol{n}\right|^2\right) \,\mathrm{d}s.\end{aligned} \tag{4.3.1}$$

Proof. Let $Q \in \mathbb{P}_1(\omega_\mathcal{T}(K))$ uniquely defined by

$$\begin{aligned}\int_{\omega_\mathcal{T}(K)} \partial_{\mathtt{pw},x_i} v \,\mathrm{d}x &= \int_{\omega_\mathcal{T}(K)} \partial_{x_i} Q \,\mathrm{d}x, \quad 1 \leqslant i \leqslant 2 \quad \text{and} \\ \int_{\omega_\mathcal{T}(K)} v \,\mathrm{d}x &= \int_{\omega_\mathcal{T}(K)} Q \,\mathrm{d}x.\end{aligned}$$

As a consequence from Proposition 4.22, together with $h_{\mathcal{T}_\star} \leqslant h_\mathcal{T}$ we get the following estimate

$$\begin{aligned}\|v - Q\|^2_{\omega_\mathcal{T}(K)} \lesssim & \int_{\omega_\mathcal{T}(K)} h_\mathcal{T}^2 \left|\nabla_{\mathtt{pw}}(v - Q)\right|^2 \,\mathrm{d}x \\ & + \int_{\substack{F\in\mathcal{F}(\mathcal{T}_\star)\\ F\subset\omega_\mathcal{T}(K)}} h_\mathcal{T}^2 h_{\mathcal{T}_\star}^{-1} \left|[\![v]\!]\,\boldsymbol{n}\right|^2 \,\mathrm{d}s.\end{aligned} \tag{4.3.2}$$

Finally, the proof of (4.3.1) follows from a second application of [KG18, Proposition 1]. Indeed, for the first term on the right-hand side of (4.3.2) we have

$$\int_{\omega_\mathcal{T}(K)} h_\mathcal{T}^2 \left|\nabla_{\mathtt{pw}}(v - Q)\right|^2 \,\mathrm{d}x \lesssim \int_{\omega_\mathcal{T}(K)} h_\mathcal{T}^4 \left|D^2_{\mathtt{pw}} v\right|^2 \,\mathrm{d}x + \int_{\substack{F\in\mathcal{F}(\mathcal{T}_\star)\\ F\subset\omega_\mathcal{T}(K)}} h_\mathcal{T}^4 h_{\mathcal{T}_\star}^{-1} [\![\partial_n v]\!]^2 \,\mathrm{d}s.$$

□

The following Lemma extends the previous result to the limit space $\mathbb{V}_\infty$.

Lemma 4.24 (Poincaré-Friedrichs $\mathbb{V}_\infty$). *For $v \in \mathbb{V}_\infty$, there exists $Q \in \mathbb{P}_1(\omega_k(K))$, such that*

$$\|v - Q\|^2_{\omega_k(K)} \lesssim \int_{\omega_k(K)} h_k^4 \left|D^2_{pw} v\right|^2 \mathrm{d}x + \int_{\substack{F \in \mathcal{F}^+ \\ F \subset \omega_k(K)}} h_k^4 \left(h_+^{-1} [\![\partial_n v]\!]^2 + h_+^{-3} |[\![v]\!]\,\boldsymbol{n}|^2\right) \mathrm{d}s.$$

Proof. Let $Q \in \mathbb{P}_1(\omega_k(K))$ be the L^2-orthogonal projection of v defined by

$$\int_{\omega_k(K)} (Q - v) P \,\mathrm{d}x = 0 \quad \forall P \in \mathbb{P}_1(\omega_k(K)). \tag{4.3.3}$$

Moreover, we define another linear polynomial $\tilde{Q} \in \mathbb{P}_1(\omega_k(K))$ by

$$\int_{\omega_k(K)} \partial_{\mathtt{pw},x_i} v \,\mathrm{d}x = \int_{\omega_k(K)} \partial_{x_i} \tilde{Q} \,\mathrm{d}x, \quad 1 \leqslant i \leqslant 2 \quad \text{and}$$
$$\int_{\omega_k(K)} v \,\mathrm{d}x = \int_{\omega_k(K)} \tilde{Q} \,\mathrm{d}x.$$

Now for $v \in \mathbb{V}_\infty$, there exists a sequence $v_\ell \in \mathbb{V}_\ell$, $\ell \in \mathbb{N}$, with $\lim_{\ell\to\infty} \|\!|v - v_\ell|\!\|_\ell + \|v - v_\ell\|_\Omega = 0$ and $\limsup_{\ell\to\infty} \|\!|v_\ell|\!\|_\ell < \infty$ and Proposition 4.9 implies

$$\int_{\omega_k(K)} |D^2_{\mathtt{pw}} v_\ell|^2 \,\mathrm{d}x + \sum_{\substack{F \in \mathcal{F}_\ell \\ F \subset \omega_k(K)}} \int_F h_\ell^{-1} [\![\partial_n v_\ell]\!]^2 + h_\ell^{-3} |[\![v_\ell]\!]\,\boldsymbol{n}|^2 \,\mathrm{d}s$$
$$\to \int_{\omega_k(K)} \left|D^2_{\mathtt{pw}} v\right|^2 \mathrm{d}x + \sum_{\substack{F \in \mathcal{F}^+ \\ F \subset \omega_k(K)}} \int_F h_+^{-1} [\![\partial_n v]\!]^2 + h_+^{-3} |[\![v]\!]\,\boldsymbol{n}|^2 \,\mathrm{d}s$$

as $\ell \to \infty$. Let $\ell \geqslant k$. Thanks to Lemma 4.23 there exists $\tilde{Q}_\ell \in \mathbb{P}_1(\omega_k(K))$ with

$$\begin{aligned} \left\|v_\ell - \tilde{Q}_\ell\right\|^2_{\omega_k(K)} &\lesssim \int_{\omega_k(K)} h_k^4 |D^2_{\mathtt{pw}} v_\ell|^2 \,\mathrm{d}x \\ &+ \sum_{\substack{F \in \mathcal{F}_\ell \\ F \subset \omega_k(K)}} \int_F h_k^4 \left(h_\ell^{-1} [\![\partial_n v_\ell]\!]^2 + h_\ell^{-3} |[\![v_\ell]\!]\,\boldsymbol{n}|^2\right) \mathrm{d}s \\ &\to \int_{\omega_k(K)} h_k^4 |D^2_{\mathtt{pw}} v|^2 \,\mathrm{d}x + \sum_{\substack{F \in \mathcal{F}^+ \\ F \subset \omega_k(K)}} \int_F h_k^4 \left(h_+^{-1} [\![\partial_n v]\!]^2 + h_+^{-3} |[\![v]\!]\,\boldsymbol{n}|^2\right) \mathrm{d}s, \end{aligned} \tag{4.3.4}$$

as $\ell \to \infty$; compare also with Proposition 4.9.

Next, let $Q_\ell \in P_1(\omega_k(K))$ be the L^2-orthogonal projection of v_ℓ. Then, we have by the definitions of Q, Q_ℓ and $\mathbb{V}_\infty$ that

$$\|Q_\ell - Q\|^2_{\omega_k(K)} \leqslant \|v_\ell - v\|^2_{\omega_k(K)} \leqslant \|v_\ell - v\|^2_\Omega \to 0$$

as $\ell \to \infty$. Hence, we have that $\|v_\ell - Q_\ell\|_{\omega_k(K)} \to \|v - Q\|_{\omega_k(K)}$ as $\ell \to \infty$. Finally, the definitions of Q_ℓ and $\tilde{Q}_\ell$ in conjunction with standard properties of the L^2-orthogonal projection imply $\|v_\ell - Q_\ell\|_{\omega_k(K)} \leqslant \left\|v_\ell - \tilde{Q}_\ell\right\|_{\omega_k(K)}$ and we conclude the statement of the Lemma in view of (4.3.4) for Q defined in (4.3.3). □

4.3.2 Polynomial Approximation

We fix $k \geqslant 0$ and define an interpolation operator $\mathcal{I}_k \colon L^2(\Omega) \to \mathbb{V}_k$ by

$$\int_\Omega (\mathcal{I}_k v - v) w \,\mathrm{d}x = 0 \quad \forall w \in \mathbb{V}_k, \tag{4.3.5}$$

that means $\mathcal{I}_k v$ is the L^2-orthogonal projection of $v \in L^2(\Omega)$ onto $\mathbb{V}_k$. We emphasise that the defintion of $\mathbb{V}_k$ implies that for a single element $K \in \mathcal{T}_k$ the restriction $\mathcal{I}_k v|_K \in \mathbb{P}_r(K)$ is defined analogously, i.e.

$$\int_K (\mathcal{I}_k v|_K - v) P \,\mathrm{d}x = 0 \quad \forall P \in \mathbb{P}_r(K).$$

Lemma 4.25 (Polynomial interpolation onto $\mathbb{V}_k$)**.** *For $k \geqslant 0$ let $\mathcal{I}_k \colon L^2(\Omega) \to \mathbb{V}_k$ be defined as in* (4.3.5)*. Then we have that*

(1) $\mathcal{I}_k \colon L^p(\Omega) \to L^p(\Omega)$ is a linear and bounded projection for all $1 \leqslant p \leqslant \infty$ and is stable in the following sense: If $v \in L^2(\Omega)$, then

$$\int_K |\mathcal{I}_k v|^2 \,\mathrm{d}x \lesssim \int_K |v|^2 \,\mathrm{d}x \qquad \text{for all } K \in \mathcal{T}_k,$$

where the constants in '$\lesssim$' are independent of the mesh-size h_k.

(2) $\mathcal{I}_k v \in \mathbb{V}_k$ for all $v \in L^2(\Omega)$,

(3) $\mathcal{I}_k v|_K = v|_K$ if $K \in \mathcal{T}_k$ and $v|_K \in \mathbb{P}_r(K)$.

Proof. Assertions (1) and (2) follow directly by the definition of the L^2-orthogonal projection (cf. [EG13, DPE12]). Claim (3) follows from definition (4.3.5) restricted to a single element $K \in \mathcal{T}$. Indeed, we have that $\mathbb{P}_r(K)$ is a finite dimensional space with L^2-inner product. Hence, if $v \in \mathbb{P}_r(K)$ then $\mathcal{I}_k v|_K - v \equiv 0 \in \mathbb{P}_r(K)$. □

We are interested on the projections of limit functions $v \in \mathbb{V}_\infty$ onto the finite element space $\mathbb{V}_k$. In this context we emphasise that we have $v \in L^2(\Omega)$ for all $v \in \mathbb{V}_\infty$. Indeed, we have from the continuous embedding of Theorem 3.22 that $BV(\Omega) \hookrightarrow L^2(\Omega)$ and our limit space is a subset of the space $BV(\Omega)$. In particular we are interested on the interplay of different refinement levels related to the sequence $\{\mathcal{T}_k\}_{k \in \mathbb{N}_0}$ of meshes produced by the ASIPDG method.

Lemma 4.26 (Stability of $\mathcal{I}_k$)**.** *Let $v \in \mathbb{V}_\ell$ for some $\ell \in \mathbb{N}_0 \cup \{\infty\}$. Then, for all $K \in \mathcal{T}_k$, $k \leqslant \ell$, we have*

$$\begin{aligned} &\int_K \left|D^2 \mathcal{I}_k v\right|^2 \mathrm{d}x + \int_{\partial K} h_k^{-1} \left[\!\left[\partial_n \mathcal{I}_k v\right]\!\right]^2 + h_k^{-3} \left|\left[\!\left[\mathcal{I}_k v\right]\!\right] \boldsymbol{n}\right|^2 \mathrm{d}s \\ &\quad\lesssim \int_{\omega_k(K)} \left|D^2_{pw} v\right|^2 \mathrm{d}x + \sum_{\substack{F \in \mathcal{F}_\ell \\ F \subset \omega_k(K)}} \int_F h_\ell^{-1} \left[\!\left[\partial_n v\right]\!\right]^2 + h_\ell^{-3} \left|\left[\!\left[v\right]\!\right] \boldsymbol{n}\right|^2 \mathrm{d}s, \end{aligned}$$

where $\mathcal{F}_\ell := \mathcal{F}^+$ and $h_\ell := h_+$, when $\ell = \infty$. In particular, we have $|\!|\!|\mathcal{I}_k v|\!|\!|_k \lesssim |\!|\!|v|\!|\!|_\ell$.

Proof. Let $\ell < \infty$ and assume that $K \in \mathcal{T}_k$. Let Q be the linear polynomial from Lemma 4.23 defined on $\omega_k(K)$. Then, the inverse estimate (3.2.3) and Lemma 4.25(1) and (3) reveal

$$\int_K \left|D^2 \mathcal{I}_k v\right|^2 \mathrm{d}x = \int_K \left|D^2 \mathcal{I}_k (v-Q)\right|^2 \mathrm{d}x \lesssim \int_K h_k^{-4} \left|\mathcal{I}_k(v-Q)\right|^2 \mathrm{d}x$$
$$\lesssim \int_K h_k^{-4} \left|(v-Q)\right|^2 \mathrm{d}x \leqslant \int_{\omega_k(K)} h_k^{-4} \left|(v-Q)\right|^2 \mathrm{d}x..$$

In order to bound the jump terms, we use again the linear polynomial from Lemma 4.23 defined on $\omega_k(K)$. We observe that $\nabla Q \equiv const$ and hence does not jump across interelement boundaries. Consequently, using Lemma 4.25(1) and (3), together with the trace estimate (3.2.4) and the inverse estimate 3.2.3, we obtain

$$\int_{\partial K} h_k^{-1} \left[\!\left[\partial_n \mathcal{I}_k v\right]\!\right]^2 \mathrm{d}s = \int_{\partial K} h_k^{-1} \left[\!\left[\partial_n \mathcal{I}_k (v-Q)\right]\!\right]^2 \mathrm{d}s$$
$$\lesssim \int_{\omega_k(K)} h_k^{-4} \left|\mathcal{I}_k(v-Q)\right|^2 \mathrm{d}x \lesssim h_K^{-4} \int_{\omega_k(K)} \left|v-Q\right|^2 \mathrm{d}x,$$

where we also used $\bigcup \{\omega_k(F) \colon F \subset \partial K\} \subset \omega_k(K)$.

In the same vein, using the continuity of the polynomial Q, we obtain

$$\int_{\partial K} h_k^{-3} \left|\left[\!\left[\mathcal{I}_k v\right]\!\right] \boldsymbol{n}\right|^2 \mathrm{d}s = \int_{\partial K} h_k^{-3} \left|\left[\!\left[\mathcal{I}_k (v-Q)\right]\!\right] \boldsymbol{n}\right|^2 \mathrm{d}s$$
$$\lesssim \int_{\omega_k(K)} h_k^{-4} \left|\mathcal{I}_k(v-Q)\right|^2 \mathrm{d}s \lesssim \int_{\omega_k(K)} h_k^{-4} \left|v-Q\right|^2 \mathrm{d}s.$$

Consequently, we proved

$$\int_K \left|D^2 \mathcal{I}_k v\right|^2 \mathrm{d}x + \int_{\partial K} h_k^{-1} \left[\!\left[\partial_n \mathcal{I}_k v\right]\!\right]^2 + h_k^{-3} \left|\left[\!\left[\mathcal{I}_k v\right]\!\right] \boldsymbol{n}\right|^2 \mathrm{d}s$$
$$\lesssim \int_{\omega_k(K)} h_k^{-4} \left|v-Q\right|^2 \mathrm{d}s$$

and the desired estimate is a direct consequence from Lemma 4.23.

For the case $\ell = \infty$ we replace Lemma 4.23 by Lemma 4.24 and proceed as before. ☐

In view of the proof of Lemma 4.12 below, we need a stability estimate comparable to Lemma 4.26 for $w \in H_0^2(\Omega)$.

Corollary 4.27. *Let $w \in H_0^2(\Omega)$. Then, we have for all $k \in \mathbb{N}$*

$$\left\|\!\left|\mathcal{I}_k w\right|\!\right\|_k \lesssim \left\|D^2 w\right\|_\Omega$$

Proof. This estimate follows by analogous arguments as in the proof of Lemma 4.26 but replacing Lemma 4.23 by the classical Poincaré-Friedrichs inequality for functions in $H_0^2(\Omega)$. ☐

The next corollary states the convergence of the interpolation operator

Corollary 4.28. *Let $v \in \mathbb{V}_\infty$, then $|||\mathcal{I}_k v - v|||_k + \|\mathcal{I}_k v - v\|_\Omega \to 0$ as $k \to \infty$.*

Proof. Thanks to the definition of $\mathbb{V}_\infty$ there exist a sequence $\{v_k\}_{k\in\mathbb{N}_0}$, $v_k \in \mathbb{V}_k$ with $|||v - v_k|||_k + \|v - v_k\|_\Omega \to 0$ as $k \to \infty$. Consequently, the claim follows from the stability and invariance of the interpolation operator $\mathcal{I}_k$. □

4.3.3 Proof of Lemma 4.12

Remark 4.29. *In the sequel, we use the following fact: For $k \geqslant \ell$ and $j \in \mathbb{N}_0$ we have $\mathcal{T}_\ell^{j+} \subset \mathcal{T}_k^{j+} \subset \mathcal{T}_k$. Moreover, the triangulation $\mathcal{T}_k \backslash \mathcal{T}_\ell^{+}$ covers the domain Ω_ℓ^{j-} and any refinement of $\mathcal{T}_k$ will not affect any element in $\mathcal{T}_\ell^{j+}$. Therefore, we obtain for all $k \geqslant \ell$:*

$$\Omega_\ell^{j-} = \Omega(\mathcal{T}_\ell \backslash \mathcal{T}_\ell^{j+}) = \Omega(\mathcal{T}_k \backslash \mathcal{T}_\ell^{j+}).$$

Consequently, Lemma 4.5 reveals that $\lim_{\ell\to\infty} \left\| h_\ell \chi_{\Omega_\ell^{j-}} \right\|_{L^\infty(\Omega)} = 0$ holds true on the domain $\Omega(\mathcal{T}_k \backslash \mathcal{T}_\ell^{j+})$.

Proof of Lemma 4.12. Recall, that we need to prove that $(\mathbb{V}_\infty, \langle\cdot,\cdot\rangle_\infty)$ is a Hilbert space. Thanks to Corollary 4.11 we have that $\|v\|_{BV(\Omega)} \lesssim |||v|||_\infty$. Hence, $|||\cdot|||_\infty$ is a norm on $\mathbb{V}_\infty$ and $\langle\cdot,\cdot\rangle_\infty$ is a scalar product. Therefore, it remains to show that $\mathbb{V}_\infty$ is complete with respect to $|||\cdot|||_\infty$, i.e. we have to prove that an arbitrary Cauchy sequence in $\mathbb{V}_\infty$ has a limit in $\mathbb{V}_\infty$.

Let $\{v^\ell\}_{\ell\in\mathbb{N}_0}$ be a Cauchy sequence in $(\mathbb{V}_\infty, |||\cdot|||_\infty)$. Corollary 4.11(a) and (c) imply $\|v^\ell - v^j\|_{BV(\Omega)} \lesssim |||v^\ell - v^j|||_\infty$. Consequently, there exists $v \in BV(\Omega)$ such that $v^\ell \to v \in BV(\Omega)$ as $\ell \to \infty$, due to the fact that $BV(\Omega)$ is a Banach space. We thus have to prove that $v \in \mathbb{V}_\infty$ in order to conclude the assertion of Lemma 4.12. Using norm equivalence on finite dimensional spaces, we readily conclude that $v|_K \in \mathbb{P}_r(K)$ for all $K \in \mathcal{T}^+$.

[1] In the first step of this proof we analyse the jump terms of the sequence $\{v^\ell\}_{\ell\in\mathbb{N}_0}$. In order to do so we recall that $v \in BV(\Omega)$ has L^1-traces on ∂K, $K \in \mathcal{T}_k$ $k \in \mathbb{N}_0$; see e.g. [AFP00, Theorem 3.88]. In view of Proposition 4.9, we shall therefore deal first with the jump terms of the function v and prove

$$\int_{\mathcal{F}_k} h_k^{-3} |[\![v]\!]\,\boldsymbol{n}|^2 \,\mathrm{d}s \to \int_{\mathcal{F}^+} h_+^{-3} |[\![v]\!]\,\boldsymbol{n}|^2 \,\mathrm{d}s \quad (k \to \infty). \tag{4.3.6}$$

To this end, we first observe that for $k \in \mathbb{N}_0$, $\{v^\ell\}_{\ell\in\mathbb{N}_0}$ is also a Cauchy sequence with respect to the $|||\cdot|||_k$-norm (Proposition 4.9), and thus $\left\| h_k^{-3/2} [\![v^\ell - v^j]\!]\,\boldsymbol{n} \right\|_{\Gamma_k} \to 0$ as $\ell, j \to \infty$. Hence, uniqueness of limits on Γ_k imply that

$$\left\| h_k^{-3/2} \left[\!\!\left[v^\ell - v \right]\!\!\right] \boldsymbol{n} \right\|_{\Gamma_k} \to 0 \tag{4.3.7}$$

as $\ell \to \infty$. Now, let $\epsilon > 0$ arbitrary fixed and consider

$$\begin{aligned} \int_{\mathcal{F}_k} h_k^{-3} |[\![v]\!]\,\boldsymbol{n}|^2 \,\mathrm{d}s \leqslant & \int_{\mathcal{F}_k} h_k^{-3} \left| \left[\!\!\left[v - v^\ell \right]\!\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s \\ & + \int_{\mathcal{F}_k} h_k^{-3} \left| \left[\!\!\left[v^\ell \right]\!\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s < \infty. \end{aligned} \tag{4.3.8}$$

For the first term on the right-hand side there exists $M = M(\epsilon)$ such that $\int_{\mathcal{F}_k} h_k^{-3} \left|[\![v - v^\ell]\!]\,\boldsymbol{n}\right|^2 \,\mathrm{d}s < \epsilon$, provided $\ell \geqslant M$. Additionally, the second term on the right-hand side is converging to $\int_{\mathcal{F}^+} h_+^{-3} \left|[\![v^\ell]\!]\,\boldsymbol{n}\right|^2 \,\mathrm{d}s$ as $k \to \infty$ (Proposition 4.9) and consequently we have that $\int_{\mathcal{F}_k} h_k^{-3} \left|[\![v]\!]\,\boldsymbol{n}\right|^2 \,\mathrm{d}s$ is uniformly bounded.

Next, there exists $L = L(\epsilon)$, such that $\|v^\ell - v^j\|_k \leqslant \|v^\ell - v^j\|_\infty \leqslant \epsilon$ for all $j, \ell \geqslant L$. Thanks to Proposition 4.9, there exists $K = K(\epsilon, L)$ such that for all $m \geqslant k \geqslant K$, we have

$$\int_{\mathcal{F}_m \setminus \mathcal{F}_k^+} h_k^{-3} \left|[\![v^L]\!]\,\boldsymbol{n}\right|^2 \,\mathrm{d}s \leqslant \epsilon^2. \tag{4.3.9}$$

In particular, for $m = k \geqslant K$, we have

$$\begin{aligned}\int_{\mathcal{F}_k} h_k^{-3} \left|[\![v]\!]\right|^2 \,\mathrm{d}s &= \int_{\mathcal{F}_k \setminus \mathcal{F}_k^+} h_k^{-3} \left|[\![v]\!]\,\boldsymbol{n}\right|^2 \,\mathrm{d}s + \int_{\mathcal{F}_k^+} h_k^{-3} \left|[\![v]\!]\,\boldsymbol{n}\right|^2 \,\mathrm{d}s \\ &= \lim_{\ell \to \infty} \int_{\mathcal{F}_k \setminus \mathcal{F}_k^+} h_k^{-3} \left|\left[\!\left[v^\ell\right]\!\right]\boldsymbol{n}\right|^2 \,\mathrm{d}s + \int_{\mathcal{F}_k^+} h_k^{-3} \left|[\![v]\!]\,\boldsymbol{n}\right|^2 \,\mathrm{d}s\end{aligned}$$

and

$$\int_{\mathcal{F}_k \setminus \mathcal{F}_k^+} h_k^{-3} \left|\left[\!\left[v^\ell\right]\!\right]\boldsymbol{n}\right|^2 \,\mathrm{d}s \leqslant 2\left\|v^\ell - v^L\right\|_k^2 + 2\int_{\mathcal{F}_k \setminus \mathcal{F}_k^+} h_k^{-3} \left|[\![v^L]\!]\,\boldsymbol{n}\right|^2 \,\mathrm{d}s \leqslant 4\epsilon^2$$

provided $\ell \geqslant L$. Due to (4.3.8), the reduction of mesh-size and the inclusion of skeletons, this proves the convergence stated in (4.3.6), since $\epsilon > 0$ was arbitrary (compare Proposition 4.9).

2 Next, we have to prove $v|_{\Omega^-} \in H^2_{\partial\Omega \cap \partial\Omega^-}(\Omega^-)$, i.e. we need to show that v is a restriction of a $H^2_0(\Omega)$-function. Thanks to Corollary 4.28, there exists $\{m_\ell\}_{\ell \in \mathbb{N}_0} \subset \mathbb{N}_0$ such that $\|v^\ell - v^\ell_{m_\ell}\|_{m_\ell} + \|v^\ell - v^\ell_{m_\ell}\|_\Omega \leqslant \frac{1}{\ell}$ for $v_{m_\ell} := \mathcal{I}_{m_\ell} v^\ell \in \mathbb{V}_{m_\ell}$, where $\mathcal{I}_{m_\ell} v^\ell$ is the interpolant from (4.3.5) with respect to $\mathcal{T}_{m_\ell}$. Consequently, since $\{v^\ell\}$ is a Cauchy sequence and thus bounded, we infer from Proposition 4.9 that

$$\left\|v^\ell_{m_\ell}\right\|_{m_\ell} \leqslant \left\|v^\ell_{m_\ell} - v^\ell\right\|_{m_\ell} + \left\|v^\ell\right\|_\infty \leqslant \frac{1}{\ell} + \left\|v^\ell\right\|_\infty,$$

i.e., the uniform boundedness of $\|v^\ell_{m_\ell}\|_{m_\ell}$. We now apply the smoothing operator defined in (3.4.1) to $v^\ell_{m_\ell} \in \mathbb{V}_{m_\ell}$, i.e. we consider the sequence $\left\{\mathcal{E}_{m_\ell}(v^\ell_{m_\ell})\right\}_{\ell \in \mathbb{N}} \subset H^2_0(\Omega)$. From Lemma 3.15 (with $\gamma = 2$) we obtain

$$\left\|D^2 \mathcal{E}_{m_\ell}(v^\ell_{m_\ell})\right\|_\Omega \lesssim \left\|D^2_{\mathtt{pw}}(\mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - v^\ell_{m_\ell})\right\|_\Omega + \left\|D^2_{\mathtt{pw}} v^\ell_{m_\ell}\right\|_\Omega \lesssim \left\|v^\ell_{m_\ell}\right\|_{m_\ell}.$$

Hence, there exists $w \in H^2_0(\Omega)$ such that, for a not relabelled subsequence

$$\mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) \rightharpoonup w \quad \text{weakly in } H^2_0(\Omega), \qquad \text{as } \ell \to \infty. \tag{4.3.10}$$

In order to prove $v|_{\Omega^-} = w|_{\Omega^-}$, we emphasise that for all $j \in \mathbb{N}$, we have $\Omega^- \subset \Omega^{j-}_{m_\ell}$ (recall that $\Omega^{j+}_{m_\ell} \subset \Omega^+$) and consider

$$\left\|\mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - v\right\|_{\Omega^-} \leqslant \left\|\mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - v^\ell_{m_\ell}\right\|_{\Omega^-_{m_\ell}} + \left\|v - v^\ell_{m_\ell}\right\|_\Omega. \tag{4.3.11}$$

For the first term on the right-hand side we have from Lemma 3.15 (with $\gamma = 0$) and the scaled trace inequality (3.2.4)

$$\begin{aligned} \left\| \mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - v^\ell_{m_\ell} \right\|^2_{\Omega^-_{m_\ell}} &\lesssim \int_{\mathcal{F}^{2-}_{m_\ell}} h^3_{m_\ell} \left[\!\left[\partial_n v^\ell_{m_\ell} \right]\!\right]^2 + h_{m_\ell} \left| \left[\!\left[v^\ell_{m_\ell} \right]\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s \\ &\lesssim \left\| h_{m_\ell} \chi_{\Omega^{2-}_{m_\ell}} \right\|^4_{L^\infty(\Omega)} \left\| v^\ell_{m_\ell} \right\|^2_{m_\ell}, \end{aligned} \tag{4.3.12}$$

where we used $\|h_{m_\ell}\|_{L^\infty(\mathcal{F}^{2-}_{m_\ell})} \lesssim \|h_{m_\ell}\chi_{\Omega^{2-}_{m_\ell}}\|_{L^\infty(\Omega)}$. Applying Lemma 4.5, the last term vanishes as $\ell \to \infty$. Consequently, we have $\lim_{\ell\to\infty} \mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) = w$ in $L^2(\Omega^-)$ due to $\Omega^- \subset \Omega^{2-}_{m_\ell}$. Additionally we use

$$\left\| v - v^\ell_{m_\ell} \right\|_\Omega \leqslant \left\| v - v^\ell \right\|_\Omega + \left\| v^\ell - v^\ell_{m_\ell} \right\|_\Omega \to 0 \qquad \text{as } \ell \to \infty,$$

to conclude $v|_{\Omega^-} = w|_{\Omega^-}$, i.e., $v|_{\Omega^-} \in H^2_{\partial\Omega\cap\partial\Omega^-}(\Omega^-)$. Here where we also used that for $d = 2$ the embedding $BV(\Omega) \hookrightarrow L^2(\Omega)$ is continuous (cf. Theorem 3.22) i.e.$\|v - v^\ell\|_\Omega \lesssim \|v - v^\ell\|_{BV(\Omega)} \to 0$ as $\ell \to \infty$. The equality $v|_{\Omega^-} = w|_{\Omega^-}$ implies that we can use the definitions (4.2.3) and (4.2.2) of the piecewise Hessian and the piecewise gradient also for v, i.e. on the domain Ω^- we have $\nabla_{\mathtt{pw}} v|_{\Omega^-} = \nabla w|_{\Omega^-}$ and $D^2_{\mathtt{pw}} v|_{\Omega^-} = D^2 w|_{\Omega^-}$. Note, that we already have the piece-wise gradient and piece-wise Hessian on $\mathcal{T}^+$ since $v|_K \in \mathbb{P}_r(K)$ for all $K \in \mathcal{T}^+$.

$\boxed{3}$ In order to deal with the jumps of the normal derivatives, we have to prove that $\nabla_{\mathtt{pw}} v \in BV(\Omega)^2$. To this end we recall from Corollary 4.11(b) and (d)

$$\left\| \nabla_{\mathtt{pw}} v^\ell - \nabla_{\mathtt{pw}} v^j \right\|_{BV(\Omega)} \lesssim \left\| v^\ell - v^j \right\|_\infty \to 0 \qquad (j, \ell \to \infty).$$

Hence, there exist $\boldsymbol{D} \in BV(\Omega)^d$ such that $\lim_{\ell\to\infty} \left\| \nabla_{\mathtt{pw}} v^\ell - \boldsymbol{D} \right\|_{BV(\Omega)} = 0$ and we have to prove that $\boldsymbol{D} = \nabla_{\mathtt{pw}} v$. To this end, we aim to use the representation of the distributional gradient of v^ℓ (Proposition 4.10), i.e.

$$\langle D v^\ell, \boldsymbol{\varphi} \rangle = \int_\Omega \nabla_{\mathtt{pw}} v^\ell \cdot \boldsymbol{\varphi} \,\mathrm{d}x - \int_{\mathcal{F}^+} \boldsymbol{\varphi} \cdot \left[\!\left[v^\ell \right]\!\right] \boldsymbol{n}. \tag{4.3.13}$$

Since $\nabla_{\mathtt{pw}} v^\ell \to \boldsymbol{D}$ as $\ell \to \infty$ we only have to investigate the limit of the jump-terms, i.e.

$$\int_{\mathcal{F}^+} h_+^{-3} \left| \left[\!\left[v^\ell - v \right]\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s \to 0 \quad \text{as } \ell \to \infty. \tag{4.3.14}$$

Thanks to (4.3.6) and Proposition 4.9 we have that

$$\int_{\mathcal{F}^+} h_+^{-3} \left| \left[\!\left[v^\ell - v \right]\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s = \lim_{k\to\infty} \int_{\mathcal{F}^+_k} h_k^{-3} \left| \left[\!\left[v^\ell - v \right]\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s$$

and

$$\int_{\mathcal{F}^+_k} h_k^{-3} \left| \left[\!\left[v^\ell - v \right]\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s = \lim_{j\to\infty} \int_{\mathcal{F}^+_k} h_k^{-3} \left| \left[\!\left[v^\ell - v^j \right]\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s.$$

Now, for $\epsilon > 0$ there exists $L = L(\epsilon)$ such that $\int_{\mathcal{F}_k^+} h_k^{-3} \left| [\![v^\ell - v^j]\!] \boldsymbol{n} \right|^2 \mathrm{d}s < \epsilon^2$, for $j, \ell \geqslant L$. Consequently, the convergence stated in (4.3.14) holds true.

Next, we conclude as for (4.3.12) from Lemma 3.15 (but this time with $\gamma = 1$) for $m_\ell \geqslant k$ that

$$\begin{aligned} &\left\| \nabla \mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - \nabla_{\mathtt{pw}} v^\ell_{m_\ell} \right\|^2_{\Omega_k^-} \\ &\quad \lesssim \int_{\mathcal{F}_{m_\ell} \setminus \mathcal{F}_k^{2+}} h_{m_\ell} \left[\!\left[\partial_n v^\ell_{m_\ell} \right]\!\right]^2 + h_{m_\ell}^{-1} \left| \left[\!\left[v^\ell_{m_\ell} \right]\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s \qquad (4.3.15) \\ &\quad \lesssim \left\| h_{m_\ell} \chi_{\Omega_k^{2-}} \right\|^2_{L^\infty(\Omega)} \left\| v^\ell_{m_\ell} \right\|^2_{m_\ell} \end{aligned}$$

where we used $\Omega_k^{2-} = \Omega(\mathcal{T}_{m_\ell} \setminus \mathcal{T}_k^{2+})$ (compare Remark 4.29), the uniform boundedness of $\| v^\ell_{m_\ell} \|_{m_\ell}$ and $h_k \geqslant h_{m_\ell}$. Consequently, there exists $K' \geqslant K$ such that (4.3.15) and $\| v^\ell - v^\ell_{m_\ell} \|_{m_\ell} \leqslant \frac{1}{\ell}$ imply

$$\lim_{\ell \to \infty} \left| \int_{\Omega_k^-} (\nabla_{\mathtt{pw}} \mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - \nabla_{\mathtt{pw}} v^\ell) \cdot \boldsymbol{\varphi} \,\mathrm{d}x \right| \lesssim \epsilon \, \|\boldsymbol{\varphi}\|_{L^2(\Omega)},$$

provided $k \geqslant K'$. Additionally, (4.3.10) implies $\mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) \to w$ in $H^1_0(\Omega)$ as $\ell \to \infty$ and therefore

$$\lim_{\ell \to \infty} \int_{\Omega_k^-} \nabla \mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) \cdot \boldsymbol{\varphi} \,\mathrm{d}x = \int_{\Omega_k^-} \nabla w \cdot \boldsymbol{\varphi} \,\mathrm{d}x.$$

Moreover, strong covergence $v^\ell|_{\Omega_k^+} \to v|_{\Omega_k^+}$ in $\mathbb{P}_r(\mathcal{T}_k^+)$ and (4.3.14) yield

$$\begin{aligned} &\int_{\Omega_k^+} \nabla_{\mathtt{pw}} v^\ell \cdot \boldsymbol{\varphi} \,\mathrm{d}x - \int_{\mathcal{F}^+} \boldsymbol{\varphi} \cdot \left[\!\left[v^\ell \right]\!\right] \boldsymbol{n} \,\mathrm{d}s \\ &\to \int_{\Omega_k^+} \nabla_{\mathtt{pw}} v \cdot \boldsymbol{\varphi} \,\mathrm{d}x - \int_{\mathcal{F}^+} \boldsymbol{\varphi} \cdot [\![v]\!] \boldsymbol{n} \,\mathrm{d}s \end{aligned}$$

as $\ell \to \infty$.

Now, fix $k \geqslant K'$ and apply the above findings to the distributional gradient of $v^\ell \in \mathbb{V}_\infty$:

$$\begin{aligned} \langle D v^\ell, \boldsymbol{\varphi} \rangle &= \int_\Omega \nabla_{\mathtt{pw}} v^\ell \cdot \boldsymbol{\varphi} \,\mathrm{d}x - \int_{\mathcal{F}^+} \boldsymbol{\varphi} \cdot \left[\!\left[v^\ell \right]\!\right] \boldsymbol{n} \\ &= \int_{\Omega_k^-} \nabla_{\mathtt{pw}} v^\ell \cdot \boldsymbol{\varphi} \,\mathrm{d}x + \int_{\Omega_k^+} \nabla_{\mathtt{pw}} v^\ell \cdot \boldsymbol{\varphi} \,\mathrm{d}x \\ &\quad - \int_{\mathcal{F}^+} \boldsymbol{\varphi} \cdot \left[\!\left[v^\ell \right]\!\right] \boldsymbol{n} \\ &= \int_{\Omega_k^-} \nabla \mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) \cdot \boldsymbol{\varphi} \,\mathrm{d}x \qquad (4.3.16) \\ &\quad + \int_{\Omega_k^-} (\nabla \mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - \nabla_{\mathtt{pw}} v^\ell) \cdot \boldsymbol{\varphi} \,\mathrm{d}x \\ &\quad + \int_{\Omega_k^+} \nabla_{\mathtt{pw}} v^\ell \cdot \boldsymbol{\varphi} \,\mathrm{d}x - \int_{\mathcal{F}^+} \boldsymbol{\varphi} \cdot \left[\!\left[v^\ell \right]\!\right] \boldsymbol{n} \,\mathrm{d}s, \end{aligned}$$

where $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^2$. A comparison of (4.3.13) and (4.3.16) in conjunction with the above findings we thus have for all $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^2$ and $k \geqslant K'$

$$\left| \int_\Omega (\chi_{\Omega_k^-} \nabla w + \chi_{\Omega_k^+} \nabla_{\mathtt{pw}} v - \boldsymbol{D}) \cdot \boldsymbol{\varphi} \, \mathrm{d}x \right| \lesssim \epsilon \, \|\boldsymbol{\varphi}\|_{L^2(\Omega)} . \tag{4.3.17}$$

Now using the absolute continuous dependence of the integral on the integration domain (Remark 4.6) as $k \to \infty$ and recalling that $\epsilon > 0$ was arbitrary, we conclude the assertion since $\nabla_{\mathtt{pw}} v|_{\Omega^-} = \nabla w|_{\Omega^-}$.

$\boxed{4}$ In this step we use the construction of $\boxed{2}$ in order to prove that $\|v - v^\ell\|_\infty \to 0$ as $\ell \to \infty$. To this end we remark, that $\nabla_{\mathtt{pw}} v \in BV(\Omega)^2$ has L^1-traces on ∂K, for all $K \in \mathcal{T}_k$, $k \in \mathbb{N}_0$; see e.g. [AFP00, Theorem 3.88] and therefore $[\![\nabla_{\mathtt{pw}} v]\!] \cdot \boldsymbol{n}$ is measurable on Γ_k. By using similar arguments as (4.3.14) we finally obtain

$$\int_{\mathcal{F}_k} h_k^{-1} [\![\partial_n v]\!]^2 \, \mathrm{d}s \to \int_{\mathcal{F}^+} h_+^{-1} [\![\partial_n v]\!]^2 \, \mathrm{d}s \tag{4.3.18}$$

as $k \to \infty$ and as a consequence

$$\int_{\mathcal{F}^+} h_+^{-1} \left[\!\!\left[\partial_n (v - v^\ell)\right]\!\!\right]^2 \mathrm{d}s \to 0 \tag{4.3.19}$$

as $\ell \to \infty$. In conjunction with (4.3.14) it therefore remains to prove that $\|D^2_{\mathtt{pw}} v - D^2_{\mathtt{pw}} v^\ell\|_\Omega \to 0$ as $\ell \to 0$. Since the Cauchy sequence property implies that $\|D^2_{\mathtt{pw}} v^\ell - \boldsymbol{H}\|_\Omega \to 0$ for some $\boldsymbol{H} \in L^2(\Omega)^{2\times 2}$ as $\ell \to \infty$, it thus suffices to prove $D^2_{\mathtt{pw}} v = \boldsymbol{H}$ and we argue similar as in step $\boxed{3}$. To this end we use Lemma 3.15 (with $\gamma = 0$) and $m_\ell \geqslant k$ to observe

$$\left\| D^2 \mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - D^2_{\mathtt{pw}} v^\ell_{m_\ell} \right\|_{\Omega_k^-} \lesssim \int_{\mathcal{F}^2_{m_\ell} \setminus \mathcal{F}^{2+}_k} h^{-1}_{m_\ell} \left[\!\!\left[\partial_n v^\ell_{m_\ell}\right]\!\!\right]^2 + h^{-3}_{m_\ell} \left| \left[\!\!\left[v^\ell_{m_\ell}\right]\!\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s.$$

Hence, arguing as in step $\boxed{1}$ of this proof, for $\epsilon > 0$ there exists $L = L(\epsilon)$ and $K = K(\epsilon, L)$, such that

$$\begin{aligned} \left\| D^2 \mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - D^2_{\mathtt{pw}} v^\ell_{m_\ell} \right\|_{\Omega_k^-} &\lesssim \int_{\mathcal{F}^2_{m_\ell} \setminus \mathcal{F}^{2+}_k} h^{-1}_{m_\ell} [\![\partial_n v^L]\!]^2 + h^{-3}_{m_\ell} \left| [\![v^L]\!] \boldsymbol{n} \right|^2 \mathrm{d}s \\ &\quad + \left\| v^L - v^\ell \right\|^2_\infty + \left\| v^\ell - v^\ell_{m_\ell} \right\|^2_{m_\ell} \\ &\leqslant 2\epsilon^2 + \frac{1}{\ell^2} \end{aligned}$$

for all $m_\ell \geqslant k \geqslant K$ and $\ell \geqslant L$. From this we infer that for $k \geqslant K$ we have that

$$\lim_{\ell \to \infty} \left| \int_{\Omega_k^-} (D^2 \mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - D^2_{\mathtt{pw}} v^\ell) : \boldsymbol{\varphi} \, \mathrm{d}x \right| \lesssim \epsilon \, \|\boldsymbol{\varphi}\|_\Omega .$$

Using (4.3.10) we have strong convergence $\mathcal{E}_{m_\ell} v^\ell_{m_\ell}|_{\Omega_k^+} \to w|_{\Omega_k^+}$ as $\ell \to \infty$ due to the fact that $\mathbb{P}_r(\mathcal{T}_k^+)$ is finite dimensional for fixed k and in the same vein we infer that $v^\ell|_{\Omega_k^+} \to v|_{\Omega_k^+}$ as $\ell \to \infty$. Additionally, (4.3.10) implies that for $|s| \leqslant 2$

we have $\partial^s \mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) \rightharpoonup \partial^s w$ in $L^2(\Omega)$ as $\ell \to \infty$ ([Alt16, 8.4 Examples(3)]) and therefore by (4.3.14) and (4.3.19) we obtain

$$\lim_{\ell\to\infty} \int_{\Omega_k^-} D^2 \mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) : \boldsymbol{\varphi} \, \mathrm{d}x = \int_{\Omega_k^-} D^2 w : \boldsymbol{\varphi} \, \mathrm{d}x$$

and

$$\begin{aligned}
&\int_{\Omega_k^+} D^2_{\mathtt{pw}} v^\ell : \boldsymbol{\varphi} \, \mathrm{d}x + \int_{\mathcal{F}^+} \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[v^\ell \right]\!\right] \boldsymbol{n} - \boldsymbol{\varphi} \left[\!\left[\nabla_{\mathtt{pw}} v^\ell \right]\!\right] \cdot \boldsymbol{n} \, \mathrm{d}s \\
&\to \int_{\Omega_k^+} D^2_{\mathtt{pw}} v : \boldsymbol{\varphi} \, \mathrm{d}x + \int_{\mathcal{F}^+} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v]\!] \, \boldsymbol{n} - \boldsymbol{\varphi} \left[\!\left[\nabla_{\mathtt{pw}} v \right]\!\right] \cdot \boldsymbol{n} \, \mathrm{d}s,
\end{aligned}$$

as $\ell \to \infty$.

We apply this to the distributional Hessian of $v^\ell \in \mathbb{V}_\infty$ (compare Proposition 4.10)

$$\begin{aligned}
\langle D^2 v^\ell, \boldsymbol{\varphi} \rangle &= \int_{\Omega_k^-} D^2_{\mathtt{pw}} v^\ell : \boldsymbol{\varphi} \, \mathrm{d}x + \int_{\Omega_k^+} D^2_{\mathtt{pw}} v^\ell : \boldsymbol{\varphi} \, \mathrm{d}x \\
&\quad + \int_{\mathcal{F}^+} \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[v^\ell \right]\!\right] \boldsymbol{n} - \boldsymbol{\varphi} \left[\!\left[\nabla_{\mathtt{pw}} v^\ell \right]\!\right] \cdot \boldsymbol{n} \, \mathrm{d}s \\
&= \int_{\Omega_k^-} D^2_{\mathtt{pw}} \mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) : \boldsymbol{\varphi} \, \mathrm{d}x \\
&\qquad + \int_{\Omega_k^-} (D^2_{\mathtt{pw}} \mathcal{E}_{m_\ell}(v^\ell_{m_\ell}) - D^2_{\mathtt{pw}} v^\ell) : \boldsymbol{\varphi} \, \mathrm{d}x \\
&\qquad + \int_{\Omega_k^+} D^2_{\mathtt{pw}} v^\ell : \boldsymbol{\varphi} \, \mathrm{d}x \\
&\qquad + \int_{\mathcal{F}^+} \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[v^\ell \right]\!\right] \boldsymbol{n} - \boldsymbol{\varphi} \left[\!\left[\nabla_{\mathtt{pw}} v^\ell \right]\!\right] \cdot \boldsymbol{n} \, \mathrm{d}s,
\end{aligned} \tag{4.3.20}$$

with $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^{2\times 2}$. The Cauchy property implies

$$\begin{aligned}
\langle D^2 v^\ell, \boldsymbol{\varphi} \rangle \to &\int_{\Omega_k^-} \boldsymbol{H} : \boldsymbol{\varphi} \, \mathrm{d}x + \int_{\Omega_k^+} D^2_{\mathtt{pw}} v : \boldsymbol{\varphi} \, \mathrm{d}x \\
&+ \int_{\mathcal{F}^+} \operatorname{div} \boldsymbol{\varphi} \cdot [\![v]\!] \, \boldsymbol{n} - \boldsymbol{\varphi} \left[\!\left[\nabla_{\mathtt{pw}} v \right]\!\right] \cdot \boldsymbol{n} \, \mathrm{d}s
\end{aligned}$$

as $\ell \to \infty$. In conjunction with the above findings we therefore conclude that for all $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^{2\times 2}$

$$\left| \int_\Omega (\chi_{\Omega_k^-} D^2 w + \chi_{\Omega_k^+} D^2_{\mathtt{pw}} v - \boldsymbol{H}) : \boldsymbol{\varphi} \, \mathrm{d}x \right| \lesssim \epsilon \, \| \boldsymbol{\varphi} \|_{L^2(\Omega)} .$$

Now using the absolute continuous dependence of the integral on the integration domain (Remark 4.6) as $k \to \infty$ and recalling that $\epsilon > 0$ was arbitrary, we conclude the assertion since $D^2_{\mathtt{pw}} v|_{\Omega^-} = D^2_{\mathtt{pw}} w|_{\Omega^-}$.

5 We conclude by showing that that there exists a sequence $\{v_k\}_{k\in\mathbb{N}_0}$, $v_k \in \mathbb{V}_k$, $k \in \mathbb{N}_0$, such that we have $|||v - v_k|||_k + \|v - v_k\|_\Omega \to 0$ as $k \to \infty$, and $\limsup_{k\to\infty} |||v_k|||_k < \infty$. To this end, we define v_k as

$$v_k|_K = \begin{cases} \mathcal{I}_k w|_K, & K \in \mathcal{T}_k^{2-} \\ \mathcal{I}_k v|_K, & K \in \mathcal{T}_k^{2+}. \end{cases}$$

Here, $w \in H_0^2(\Omega)$ is the function defined in (4.3.10) and $v|_K = \lim_{\ell\to\infty} v^\ell|_K \in \mathbb{P}_r(K)$ for all $K \in \mathcal{T}_k^{2+}$. From Lemma 4.26 and Corollary 4.27 we deduce the uniform boundedness of the sequence

$$\begin{aligned} |||v_k|||_k^2 &\lesssim \sum_{K\in\mathcal{T}_k} \left[\int_K \left|D^2 v_k\right|^2 \,\mathrm{d}x + \int_{\partial K} \left(h_K^{-1} \left[\!\left[\partial_n v_k\right]\!\right]^2 + h_k^{-3} \left|\left[\!\left[v_k\right]\!\right] \boldsymbol{n}\right|^2 \right) \mathrm{d}s \right] \\ &\lesssim \sum_{K\in\mathcal{T}_k^{2-}} \int_{\omega_k(K)} \left|D^2 w\right|^2 \,\mathrm{d}x \\ &\quad + \sum_{K\in\mathcal{T}_k^{2+}} \left[\int_{\omega_k(K)} \left|D^2_{\mathtt{pw}} v\right|^2 \,\mathrm{d}x + \int_{\partial K} (h_+^{-1} \left[\!\left[\partial_n v\right]\!\right]^2 + h_+^{-3} \left|\left[\!\left[v\right]\!\right] \boldsymbol{n}\right|^2) \,\mathrm{d}s \right] \\ &\lesssim \|w\|_{H_0^2(\Omega)}^2 + |||v|||_\infty^2 < \infty. \end{aligned}$$

We split $|||v - v_k|||_k^2$ according to $\mathcal{T}_k = \mathcal{T}_k^{2-} \cup \mathcal{T}_k^{2+}$, i.e.

$$\begin{aligned} |||v - v_k|||_k^2 \lesssim &\sum_{K\in\mathcal{T}_k^{2-}} \Bigg[\int_K \left|D^2 \mathcal{I}_k w - D^2_{\mathtt{pw}} v\right|^2 \,\mathrm{d}x \\ &\qquad + \int_{\partial K} h_k^{-1} \left[\!\left[\partial_n (\mathcal{I}_k w - v)\right]\!\right]^2 + h_k^{-3} \left|\left[\!\left[\mathcal{I}_k w - v\right]\!\right] \boldsymbol{n}\right|^2 \,\mathrm{d}s \Bigg] \\ &\sum_{K\in\mathcal{T}_k^{2+}} \Bigg[\int_K \left|D^2 \mathcal{I}_k v - D^2_{\mathtt{pw}} v\right|^2 \,\mathrm{d}x \\ &\qquad + \int_{\partial K} h_k^{-1} \left[\!\left[\partial_n (\mathcal{I}_k v - v)\right]\!\right]^2 + h_k^{-3} \left|\left[\!\left[\mathcal{I}_k v - v\right]\!\right] \boldsymbol{n}\right|^2 \,\mathrm{d}s \Bigg] \end{aligned}$$

and consider the corresponding terms separately. On the set $\mathcal{T}_k^{2-}$ we use the density of $H_0^3(\Omega)$ in $H_0^2(\Omega)$ and choose for arbitrarily fixed $\epsilon > 0$ some $w_\epsilon \in H_0^3(\Omega)$ such that $\|w - w_\epsilon\|_{H^2(\Omega^-)} \leqslant \|w - w_\epsilon\|_{H^2(\Omega)} < \epsilon$. Thanks to the triangle inequality and the stability of $\mathcal{I}_k$ (Lemma 4.26 and Corollary 4.27), we have

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_k^{2-}} \Bigg[\int_K \left|D^2\mathcal{I}_k w - D^2_{\mathtt{pw}} v\right|^2 \,\mathrm{d}x \\
&\quad + \int_{\partial K} h_k^{-1} [\![\partial_n(\mathcal{I}_k w - v)]\!]^2 + h_k^{-3} \left|[\![\mathcal{I}_k w - v]\!]\, \boldsymbol{n}\right|^2 \,\mathrm{d}s \Bigg] \\
&\quad \lesssim \sum_{K\in\mathcal{T}_k^{2-}} \Bigg[\int_K \left|D^2\mathcal{I}_k(w - w_\epsilon)\right|^2 + \left|D^2(\mathcal{I}_k w_\epsilon - w_\epsilon)\right|^2 + \left|D^2(w_\epsilon - v)\right|^2 \,\mathrm{d}x \\
&\qquad + \int_{\partial K} h_k^{-1} [\![\partial_n \mathcal{I}_k(w - w_\epsilon)]\!]^2 + h_k^{-1} [\![\partial_n \mathcal{I}_k w_\epsilon]\!]^2 + h_k^{-1} [\![\partial_n v]\!]^2 \,\mathrm{d}s \\
&\qquad + \int_{\partial K} h_k^{-3} \left|[\![\mathcal{I}_k(w - w_\epsilon)]\!]\, \boldsymbol{n}\right|^2 + h_k^{-3} \left|[\![\mathcal{I}_k w_\epsilon]\!]\, \boldsymbol{n}\right|^2 + h_k^{-3} \left|[\![v]\!]\, \boldsymbol{n}\right|^2 \,\mathrm{d}s \Bigg] \\
&\quad \lesssim \int_{N_k(\mathcal{T}_k^{2-})} \left|D^2(w - w_\epsilon)\right|^2 \,\mathrm{d}x + \int_{\mathcal{T}_k^{2-}} \left|D^2_{\mathtt{pw}}(w_\epsilon - v)\right|^2 + \left|D^2_{\mathtt{pw}}(\mathcal{I}_k w_\epsilon - w_\epsilon)\right|^2 \,\mathrm{d}x \\
&\qquad + \sum_{K\in\mathcal{T}_k^{2-}} \Bigg[\int_{\partial K} h_k^{-1} \left([\![\partial_n \mathcal{I}_k w_\epsilon]\!]^2 + [\![\partial_n v]\!]^2 \right) \\
&\qquad + h_k^{-3} \left(\left|[\![\mathcal{I}_k w_\epsilon]\!]\, \boldsymbol{n}\right|^2 + \left|[\![v]\!]\, \boldsymbol{n}\right|^2 \right) \Bigg] \,\mathrm{d}s.
\end{aligned} \tag{4.3.21}$$

In order to bound the terms concerning the interpolation operator, we employ a scaled trace theorem together with Lemma 4.25(1) and (3) to obtain

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_k^{2-}} \int_K \left|D^2(\mathcal{I}_k w_\epsilon - w_\epsilon)\right|^2 + \int_{\partial K} h_k^{-1} [\![\partial_n \mathcal{I}_k w_\epsilon]\!]^2 + h_k^{-3} \left|[\![\mathcal{I}_k w_\epsilon]\!]\, \boldsymbol{n}\right|^2 \,\mathrm{d}s \\
&\qquad \leqslant 2 \sum_{K\in\mathcal{T}_k^{2-}} \int_K \left|D^2\mathcal{I}_k(w_\epsilon - Q_K)\right|^2 + \left|D^2(w_\epsilon - Q_K)\right|^2 \,\mathrm{d}x \\
&\qquad\quad + \sum_{K\in\mathcal{T}_k^{2-}} \int_{\partial K} h_k^{-1} [\![\partial_n \mathcal{I}_k(w_\epsilon - Q_K)]\!]^2 + h_k^{-3} \left|[\![\mathcal{I}_k(w_\epsilon - Q_K)]\!]\, \boldsymbol{n}\right|^2 \,\mathrm{d}s \\
&\qquad \lesssim \sum_{K\in\mathcal{T}_k^{2-}} \int_{\omega_k(K)} h_k^{-4} \left|w_\epsilon - Q_K\right|^2 + h_k^{-2} \left|\nabla_{\mathtt{pw}}(w_\epsilon - Q_K)\right|^2 \\
&\qquad\quad + \left|D^2_{\mathtt{pw}}(w_\epsilon - Q_K)\right|^2 \,\mathrm{d}x \\
&\qquad \lesssim \int_{N_k(\mathcal{T}_k^{2-})} h_k^2 \sum_{|\alpha|=3} \left|D^\alpha w_\epsilon\right|^2 \lesssim \left\| h_k \chi_{\Omega_k^{3-}} \right\|^2_{L^\infty(\Omega)} \int_\Omega \sum_{|\alpha|=3} \left|D^\alpha w_\epsilon\right|^2 \,\mathrm{d}x.
\end{aligned} \tag{4.3.22}$$

Here, we have used the Bramble-Hilbert Lemma ([DS80]) for suitable chosen $Q_K \in \mathbb{P}_1(\omega_k(K))$, $K \in \mathcal{T}_k$ in the penultimate estimate as well as $\Omega(N_k(\mathcal{T}_k^{2-})) \subset \Omega_k^{3-}$ and the finite overlap of neighbourhoods in the last step. Thanks to Lemma 4.5 the last term vanishes as $k \to \infty$.

For the remaining volume terms on the right-hand side of (4.3.21), we recall $v|_{\Omega^-} = w|_{\Omega^-}$ and conclude from Lemma 4.5 in conjunction with the absolute continuous dependence of the integral on the integration domain

$$\begin{aligned}
&\lim_{k\to\infty}\left[\sum_{K\in\mathcal{T}_k^{2-}}\int_K \left|D^2 w_\epsilon - D^2_{\mathtt{pw}} v\right|^2 \,\mathrm{d}x + \int_{N_k(\mathcal{T}_k^{2-})} \left|D^2(w-w_\epsilon)\right|^2 \,\mathrm{d}x\right]\\
&\quad\lesssim \lim_{k\to\infty}\int_{\Omega_k^{2-}} \left|D^2 w_\epsilon - D^2_{\mathtt{pw}} v\right|^2 \,\mathrm{d}x + \|w_\epsilon - w\|^2_{H^2(\Omega)} \,\mathrm{d}x\\
&\quad\lesssim \int_{\Omega^-} \left|D^2 w_\epsilon - D^2 w\right|^2 \,\mathrm{d}x + \lim_{k\to\infty}\int_{\Omega^-\setminus\Omega_k^{2-}} \left|D^2 w_\epsilon - D^2_{\mathtt{pw}} v\right|^2 \,\mathrm{d}x\\
&\qquad + \|w_\epsilon - w\|^2_{H^2(\Omega)} \,\mathrm{d}x\\
&\quad\lesssim \|w_\epsilon - w\|^2_{H^2(\Omega)} \leqslant \epsilon^2,
\end{aligned} \tag{4.3.23}$$

where we also used $\Omega(N_k(\mathcal{T}_k^{2-})) \subset \Omega_k^{3-} \subset \Omega$ in the first estimate and $\Omega_k^{2-} = \Omega^- \cup \Omega_k^{2-}\setminus\Omega^-$ in the second estimate.

For the remaining jump terms in (4.3.21), we infer from the definition of $\mathcal{T}_k^{2-}$ that

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_k^{2-}}\int_{\partial K} (h_k^{-1}\,[\![\partial_n v]\!]^2 + h_k^{-3}\,|[\![v]\!]\,\boldsymbol{n}|^2)\,\mathrm{d}s\\
&\quad= \sum_{K\in\mathcal{T}_k\setminus\mathcal{T}_k^{2+}}\int_{\partial K} (h_k^{-1}\,[\![\partial_n v]\!]^2 + h_k^{-3}\,|[\![v]\!]\,\boldsymbol{n}|^2)\,\mathrm{d}s\\
&\quad= \sum_{K\in\mathcal{T}_k}\int_{\partial K} (h_k^{-1}\,[\![\partial_n v]\!]^2 + h_k^{-3}\,|[\![v]\!]\,\boldsymbol{n}|^2)\,\mathrm{d}s\\
&\qquad - \sum_{K\in\mathcal{T}_k^{2+}}\int_{\partial K} (h_+^{-1}\,[\![\partial_n v]\!]^2 + h_+^{-3}\,|[\![v]\!]\,\boldsymbol{n}|^2)\,\mathrm{d}s\\
&\quad\to \sum_{K\in\mathcal{T}^+}\int_{\partial K} (h_+^{-1}\,[\![\partial_n v]\!]^2 + h_+^{-3}\,|[\![v]\!]\,\boldsymbol{n}|^2)\,\mathrm{d}s\\
&\qquad - \sum_{K\in\mathcal{T}^+}\int_{\partial K} (h_+^{-1}\,[\![\partial_n v]\!]^2 + h_+^{-3}\,|[\![v]\!]\,\boldsymbol{n}|^2)\,\mathrm{d}s\\
&\quad= 0
\end{aligned} \tag{4.3.24}$$

as $k \to \infty$, thanks to (4.3.6), (4.3.18) and Lemma 4.5. Inserting this, (4.3.22) and (4.3.23) into (4.3.21), and recalling that that $\epsilon > 0$ was chosen arbitrary, we have proved

$$\begin{aligned}
&\lim_{k\to\infty}\sum_{K\in\mathcal{T}_k^{2-}}\Bigg[\int_K \left|D^2\mathcal{I}_k w - D^2_{\mathtt{pw}} v\right|^2 \,\mathrm{d}x\\
&\qquad + \int_{\partial K} h_k^{-1}\,[\![\partial_n(\mathcal{I}_k w - v)]\!]^2 + h_k^{-3}\,|[\![(\mathcal{I}_k w - v)]\!]\,\boldsymbol{n}|^2 \,\mathrm{d}s\Bigg] = 0.
\end{aligned} \tag{4.3.25}$$

Let now $K \in \mathcal{T}_k^{2+}$. Then, Lemma 4.25(3) and $v^\ell \to v$ in $\mathbb{P}_r(K)$ infers

$$v_k = \mathcal{I}_k v \leftarrow \mathcal{I}_k v^\ell = v^\ell \to v \quad \text{in } \mathbb{P}_r(K)$$

as $\ell \to \infty$. Consequently, for all $k \in \mathbb{N}_0$, we have

$$\sum_{K \in \mathcal{T}_k^{2+}} \int_K \left|D^2 v_k - D^2 v\right|^2 \,\mathrm{d}x + \int_{\partial K} h_k^{-1} \left[\!\left[\partial_n (v_k - v)\right]\!\right]^2 + h_k^{-3} \left|\left[\!\left[(v_k - v)\right]\!\right] \boldsymbol{n}\right|^2 \,\mathrm{d}s = 0.$$

Combining this with (4.3.25) we have constructed a sequence $\{v_k\}_{k \in \mathbb{N}_0}$ with $v_k \in \mathbb{V}_k$ such that that $\|v_k - v\|_k^2 \to 0$ as $k \to \infty$. The convergence $\|v_k - v\|_\Omega \to 0$ follows by similar arguments.

Overall, we have thus showed that $\lim_{\ell \to \infty} v^\ell = v \in \mathbb{V}_\infty$, which concludes the proof. □

4.3.4 Proof of Theorem 4.15

To identify a candidate for the limit of the sequence $\{u_k\}_{k \in \mathbb{N}_0}$ of discrete approximations computed by the ASIPDG method, we conclude from the boundedness of $\{u_k\}_{k \in \mathbb{N}}$ in $L^2(\Omega)$ (cf. Proposition 3.12 and (3.3.5))

$$u_{k_j} \rightharpoonup \overline{u}_\infty \qquad \text{weakly in } L^2(\Omega) \quad \text{as } j \to \infty \tag{4.3.26}$$

for some subsequence $\{k_j\}_{j \in \mathbb{N}_0} \subset \{k\}_{k \in \mathbb{N}_0}$ and $\overline{u}_\infty \in L^2(\Omega)$. In the following we shall see that $u_\infty = \overline{u}_\infty \in \mathbb{V}_\infty$ and thus $\{u_k\}_{k \in \mathbb{N}_0}$ has only one weak accumulation point and the whole sequence converges. Finally we shall conclude the section with proving the strong convergence $\lim_{k \to \infty} \|u_k - u_\infty\|_k = 0$ claimed in Theorem 4.15.

Lemma 4.30. *We have* $\overline{u}_\infty \in \mathbb{V}_\infty$.

Proof. $\boxed{1}$ We want to use the weak* convergence criterion of Proposition 3.23. To this end, we note that from the uniform boundedness (3.3.5) of $\|u_{k_j}\|_{k_j}$, and Propositions 3.12 and 3.30, we have that $\|u_{k_j}\|_{BV(\Omega)}$ is bounded uniformly. Consequently, we infer from (4.3.26) that

$$u_{k_j} \rightharpoonup^* \overline{u}_\infty \qquad \text{weakly* in } BV(\Omega) \quad \text{as } j \to \infty. \tag{4.3.27}$$

$\boxed{2}$ Next, we prove that $\overline{u}_\infty|_{\Omega^-} \in H^2_{\partial\Omega \cap \partial\Omega^-}(\Omega^-)$. Lemma 3.15 (with $\gamma = 2$) yields for the smoothing operator from (3.4.1) that

$$\left\|D^2 \mathcal{E}_{k_j}(u_{k_j})\right\|_\Omega \leqslant \left\|D^2_{\mathtt{pw}}(\mathcal{E}_{k_j}(u_{k_j}) - u_{k_j})\right\|_\Omega + \left\|D^2_{\mathtt{pw}} u_{k_j}\right\|_\Omega \lesssim \|u_{k_j}\|_{k_j}.$$

We thus have

$$\mathcal{E}_{k_j}(u_{k_j}) \rightharpoonup w \quad \text{weakly in } H^2_0(\Omega) \quad \text{as } j \to \infty \tag{4.3.28}$$

for a not relabelled subsequence. Arguing as in step $\boxed{4}$ in the proof of Lemma 4.12, we obtain, that $\left\|\mathcal{E}_{k_j}(u_{k_j}) - u_{k_j}\right\|_{\Omega^{2-}_{k_j}} \to 0$ as $j \to \infty$ and thus (4.3.26) implies

$$\overline{u}_\infty|_{\Omega^-} = w|_{\Omega^-} \in H^2_{\partial\Omega \cap \partial\Omega^-}(\Omega^-).$$

3 Now, we shall prove that $\nabla_{\mathtt{pw}}\overline{u}_\infty \in BV(\Omega)^2$. To this end, the combination of Propositions 3.12 and 3.31 with (3.3.5) yields that both $\|\nabla_{\mathtt{pw}} u_{k_j}\|_\Omega$ and $|D(\nabla_{\mathtt{pw}} u_{k_j})|(\Omega)$ are bounded uniformly. Hence, there exists $\boldsymbol{W} \in BV(\Omega)^2$ such that

$$\nabla_{\mathtt{pw}} u_{k_j} \rightharpoonup^* \boldsymbol{W} \qquad \text{weakly* in } BV(\Omega)^2 \quad \text{as } j \to \infty \tag{4.3.29}$$

Consequently, $\nabla_{\mathtt{pw}}\overline{u}_\infty$ is well defined since we have $\nabla_{\mathtt{pw}}\overline{u}_\infty|_{\Omega^-} := \nabla w|_{\Omega^-}$ and $\overline{u}_\infty$ is a piecewise polynomial on $\mathcal{T}^+$. The last statement is a consequence of the weak*-convergence (4.3.27) and the fact that $u_{k_j}|_K \in \mathbb{P}_r(K)$ for all $K \in \mathcal{T}^+$. It remains to prove $\nabla_{\mathtt{pw}}\overline{u}_\infty = \boldsymbol{W} \in BV(\Omega)^2$. To this end, we argue similar as in step 3 in the proof of Lemma 4.3.3:

$$\begin{aligned}\int_\Omega \nabla_{\mathtt{pw}} u_{k_j} \cdot \boldsymbol{\varphi}\,\mathrm{d}x &= \int_{\Omega_\ell^-} \nabla_{\mathtt{pw}} u_{k_j} \cdot \boldsymbol{\varphi}\,\mathrm{d}x + \int_{\Omega_\ell^+} \nabla_{\mathtt{pw}} u_{k_j} \cdot \boldsymbol{\varphi}\,\mathrm{d}x \\ &= \int_{\Omega_\ell^-} (\nabla_{\mathtt{pw}} u_{k_j} - \nabla_{\mathtt{pw}}\mathcal{E}_{k_j}(u_{k_j})) \cdot \boldsymbol{\varphi}\,\mathrm{d}x \\ &\quad + \int_{\Omega_\ell^-} \nabla_{\mathtt{pw}}\mathcal{E}_{k_j}(u_{k_j}) \cdot \boldsymbol{\varphi}\,\mathrm{d}x + \int_{\Omega_\ell^+} \nabla_{\mathtt{pw}} u_{k_j} \cdot \boldsymbol{\varphi}\,\mathrm{d}x,\end{aligned} \tag{4.3.30}$$

for $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^2$. Let $\epsilon > 0$ be chosen arbitrary, since $\ell \leqslant k_j$, we obtain from Lemma 3.15 (with $\gamma = 1$) for the first term on the right-hand side of (4.3.30)

$$\begin{aligned}&\|\nabla_{\mathtt{pw}}\mathcal{E}_{k_j}(u_{k_j}) - \nabla_{\mathtt{pw}} u_{k_j}\|_{\Omega_\ell^-}^2 \\ &\lesssim \int_{\mathcal{F}_{k_j}\setminus\mathcal{F}_\ell^{2+}} h_{k_j} [\![\partial_n u_{k_j}]\!]^2 + h_{k_j}^{-1} |[\![u_{k_j}]\!]\,\boldsymbol{n}|^2\,\mathrm{d}s \\ &\leqslant \left\|h_{k_j}\chi_{\Omega_\ell^{2-}}\right\|_{L^\infty(\Omega)}^2 |\!|\!|u_{k_j}|\!|\!|_{k_j}^2 \\ &\leqslant \left\|h_\ell\chi_{\Omega_\ell^{2-}}\right\|_{L^\infty(\Omega)}^2 \|f\|_\Omega^2.\end{aligned} \tag{4.3.31}$$

Hence, in view of Lemma 4.5, we obtain

$$\|\nabla_{\mathtt{pw}}\mathcal{E}_{k_j}(u_{k_j}) - \nabla_{\mathtt{pw}} u_{k_j}\|_{\Omega_\ell^-}^2 \lesssim \epsilon,$$

where $k_j \geqslant \ell \geqslant K_2(\epsilon, f)$. Whence, we infer from (4.3.28) and (4.3.31)

$$\lim_{j\to\infty}\left|\int_{\Omega_\ell^-} (\nabla_{\mathtt{pw}} u_{k_j} - \nabla_{\mathtt{pw}}\mathcal{E}_{k_j}(u_{k_j})) \cdot \boldsymbol{\varphi}\,\mathrm{d}x\right| \lesssim \epsilon\,\|\boldsymbol{\varphi}\|_\Omega^2 \tag{4.3.32}$$

and

$$\lim_{j\to\infty}\int_{\Omega_\ell^-} \nabla_{\mathtt{pw}}\mathcal{E}_{k_j}(u_{k_j}) \cdot \boldsymbol{\varphi}\,\mathrm{d}x = \int_{\Omega_\ell^-} \nabla w \cdot \boldsymbol{\varphi}\,\mathrm{d}x, \tag{4.3.33}$$

where we used $\mathcal{E}_{k_j}(u_{k_j}) \to w$ as $j \to \infty$ in $H_0^1(\Omega)$ in the last line, thanks to (4.3.28) and the fact, that the embedding $H_0^2(\Omega) \hookrightarrow H_0^1(\Omega)$ is compact.

Moreover, since weak convergence implies strong convergence on finite dimensional spaces, we have $u_{k_j}|_{\Omega_\ell^+} \to \overline{u}_\infty|_{\Omega_\ell^+}$ in $\mathbb{P}_r(\mathcal{T}_\ell^+)$ and additionally

$$\int_{\Omega_\ell^+} \nabla_{\mathtt{pw}} u_{k_j} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \to \int_{\Omega_\ell^+} \nabla_{\mathtt{pw}} \overline{u}_\infty \cdot \boldsymbol{\varphi} \, \mathrm{d}x,$$

as $j \to \infty$, where we also used norm equivalence of finite-dimensional spaces. Hence, in view of (4.3.30), (4.3.32) and (4.3.33) we obtain for all $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^2$

$$\left| \int_\Omega (\chi_{\Omega_\ell^-} \nabla w + \chi_{\Omega_\ell^+} \nabla_{\mathtt{pw}} \overline{u}_\infty - \boldsymbol{W}) \cdot \boldsymbol{\varphi} \, \mathrm{d}x \right| \lesssim \epsilon \, \|\boldsymbol{\varphi}\|_\Omega \, .$$

From absolute continuous dependence of the integral on the integration domain (Remark 4.6), as $\ell \to \infty$, in conjunction with $\overline{u}_\infty|_{\Omega^-} = w|_{\Omega^-}$ and the fact that $\epsilon > 0$ was chosen arbitrary we infer $\boldsymbol{W} = \nabla_{\mathtt{pw}} \overline{u}_\infty \in BV(\Omega)^2$ (compare also [3] in the proof of Lemma 4.12). In particular, we can apply the piece-wise gradient (4.2.2) also on $\overline{u}_\infty$.

[4] Next, we have to prove that the energy norm of $\overline{u}_\infty$ is bounded, i.e. $|||\overline{u}_\infty|||_\infty < \infty$. We analyse the jump terms first. Chose $k \leqslant k_j$, then (4.3.26) implies strong convergence $u_{k_j}|_{\Omega_k^+} \to \overline{u}_\infty|_{\Omega_k^+}$ as $j \to \infty$ on the finite-dimensional spaces $\mathbb{P}_r(\mathcal{T}_k^+)$. In particular equivalence of norms on finite-dimensional spaces imply $D^2_{\mathtt{pw}} u_{k_j}|_{\Omega_k^+} \to D^2_{\mathtt{pw}} \overline{u}_\infty|_{\Omega_k^+}$ as $j \to \infty$ in $\mathbb{P}_{r-2}(\Omega_k^+)^{2\times 2}$ and we note that we can apply the piece-wise Hessian 4.2.3 also on $\overline{u}_\infty$ (compare also step [2]). Note that we already have $\nabla_{\mathtt{pw}} u_{k_j}|_{\Omega_k^+} \to \nabla_{\mathtt{pw}} \overline{u}_\infty|_{\Omega_k^+}$ as $j \to \infty$ in $\mathbb{P}_{r-1}(\Omega_k^+)^2$ from step [3]. The uniform stability of the discrete solution (3.3.5) implies

$$\begin{aligned} C &\geqslant \int_{\mathcal{F}_{k_j}} h_{k_j}^{-1} \left[\!\left[\partial_n u_{k_j} \right]\!\right]^2 + h_{k_j}^{-3} \left| \left[\!\left[u_{k_j} \right]\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s \\ &\geqslant \int_{\mathcal{F}_{k_j}^+} h_{k_j}^{-1} \left[\!\left[\partial_n u_{k_j} \right]\!\right]^2 + h_{k_j}^{-3} \left| \left[\!\left[u_{k_j} \right]\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s \\ &\geqslant \int_{\mathcal{F}_k^+} h_k^{-1} \left[\!\left[\partial_n u_{k_j} \right]\!\right]^2 + h_k^{-3} \left| \left[\!\left[u_{k_j} \right]\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s, \end{aligned}$$

thanks to $\mathcal{F}_{k_j} \supset \mathcal{F}_{k_j}^+ \supset \mathcal{F}_k^+$ and $k \leqslant k_j$. Note, that the last estimate holds for arbitrary $k_j \geqslant k$ and that the constant is independet of k and k_j. Consequently, we have

$$\begin{aligned} &\int_{\mathcal{F}_k^+} h_k^{-1} \left[\!\left[\partial_n u_{k_j} \right]\!\right]^2 + h_k^{-3} \left| \left[\!\left[u_{k_j} \right]\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s \\ &\quad \to \int_{\mathcal{F}_k^+} h_k^{-1} \left[\!\left[\partial_n \overline{u}_\infty \right]\!\right]^2 + h_k^{-3} \left| \left[\!\left[\overline{u}_\infty \right]\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s \leqslant C \end{aligned} \tag{4.3.34}$$

as $j \to \infty$ and we note that the constant on the right-hand side is independent of k. We emphasise that

$$\int_{\mathcal{F}_k^+} h_k^{-1} \left[\!\left[\partial_n \overline{u}_\infty \right]\!\right]^2 + h_k^{-3} \left| \left[\!\left[\overline{u}_\infty \right]\!\right] \boldsymbol{n} \right|^2 \mathrm{d}s$$

increases monotonically, when increasing k, due to the positivity of the jump-norms, $\mathcal{F}_k^+ \subset \mathcal{F}_K^+$ for $k \leqslant K$ and the decrease of mesh-sizes. Therefore, the uniform bound (4.3.34) implies that the limit

$$\int_{\mathcal{F}^+} h_+^{-1} \left[\!\left[\partial_n \overline{u}_\infty\right]\!\right]^2 + h_+^{-3} \left|\left[\!\left[\overline{u}_\infty\right]\!\right] \boldsymbol{n}\right|^2 \,\mathrm{d}s$$

$$= \lim_{k\to\infty} \int_{\mathcal{F}_k^+} h_k^{-1} \left[\!\left[\partial_n \overline{u}_\infty\right]\!\right]^2 + h_k^{-3} \left|\left[\!\left[\overline{u}_\infty\right]\!\right] \boldsymbol{n}\right|^2 \,\mathrm{d}s \leqslant C$$

exists and is bounded.

Regarding the volume terms we obtain similarly for some $k \leqslant k_j$ the uniform bound $C \geqslant \int_{\Omega_k^+} \left|D^2_{\mathtt{pw}} u_{k_j}\right|^2 \,\mathrm{d}x$ and therefore

$$C \geqslant \int_{\Omega_k^{1+}} \left|D^2_{\mathtt{pw}} \overline{u}_\infty\right|^2 \,\mathrm{d}x,$$

where the constant $C > 0$ independet of k and k_j. Consequently, the volume terms are bounded by

$$\int_{\Omega^+} \left|D^2_{\mathtt{pw}} \overline{u}_\infty\right|^2 \,\mathrm{d}x = \lim_{k\to\infty} \int_{\Omega_k^+} \left|D^2_{\mathtt{pw}} \overline{u}_\infty\right|^2 \,\mathrm{d}x \leqslant C.$$

On the domain Ω^- we have $\overline{u}_\infty|_{\Omega^-} = w|_{\Omega^-}$ with $w \in H^2_0(\Omega)$ from (4.3.28). Whence, by combining the above arguments we proved $\|\!|\overline{u}_\infty|\!\|_\infty < \infty$.

[5] Finally, we have to prove that there exists a sequence $\{v_k\}_{k\in\mathbb{N}}$, $v_k \in \mathbb{V}_k$ such that $\lim_{k\to\infty} \|\!|v_k - \overline{u}_\infty|\!\|_k^2 + \|v_k - \overline{u}_\infty\|_\Omega^2 = 0$ and $\limsup_{k\to\infty} \|\!|v_k|\!\|_k < \infty$. We aim to argue similar as in step [5] in the proof of Lemma 4.12. In contrast to the proof of Lemma 4.12, we have for the sequence $\{u_k\}_{k\in\mathbb{N}}$ determining $\overline{u}_\infty$ via (4.3.26) that $\sup_{k\in\mathbb{N}} \|\!|u_k|\!\|_k < \infty$ (see (3.3.5)), i.e. it is not a Cauchy sequence in $\mathbb{V}_\infty$. This Cauchy property was used to prove (4.3.6) and (4.3.18). Thus, in order to proceed as in step [5] of Lemma 4.12 we need to show that the jump terms of $\overline{u}_\infty$ are stable on $\mathcal{F}_k^{2-}$ in the sense that

$$\int_{\mathcal{F}_k^{2-}} h_k^{-1} \left[\!\left[\partial_n \overline{u}_\infty\right]\!\right]^2 + h_k^{-3} \left[\!\left[\overline{u}_\infty\right]\!\right]^2 \,\mathrm{d}s \to 0, \tag{4.3.35}$$

as $k \to \infty$ (compare also Proposition 4.9). The convergence $\lim_{k\to\infty} \|\!|v_k - \overline{u}_\infty|\!\|_k^2 + \|v_k - \overline{u}_\infty\|_\Omega^2 = 0$ and $\limsup_{k\to\infty} \|\!|v_k|\!\|_k < \infty$ follows than along the same arguments.

We define a sequence $\{v_k\}_{k\in\mathbb{N}}$, $v_k \in \mathbb{V}_k$ by

$$v_k|_K = \begin{cases} \mathcal{I}_k w|_K, & K \in \mathcal{T}_k^{2-} \\ \mathcal{I}_k \overline{u}_\infty|_K, & K \in \mathcal{T}_k^{2+}. \end{cases} \tag{4.3.36}$$

where $w \in H^2_0(\Omega)$ is the limit of (4.3.28) and $\overline{u}_\infty|_K = \lim_{j\to\infty} u_{k_j}|_K \in \mathbb{P}_r(K)$ for all $K \in \mathcal{T}_k^{2+}$. In order to prove (4.3.35), we show that $\left\{\nabla_{\mathtt{pw}} v_k\right\}_{k\in\mathbb{N}_0}$ (resp. $\{v_k\}_{k\in\mathbb{N}_0}$) are Cauchy-sequences in $BV(\Omega)^2$ (resp. $BV(\Omega)$) with limit $\nabla_{\mathtt{pw}} \overline{u}_\infty \in$

$BV(\Omega)^2$ (resp. $\overline{u}_\infty \in BV(\Omega)$). To make this precise let $\ell \leqslant k$ and observe that for $\mathbb{V}_\ell \subset \mathbb{V}_k$ we have from Proposition 3.31

$$\begin{aligned} &\left|D(\nabla_{\mathtt{pw}} v_\ell - \nabla_{\mathtt{pw}} v_k)\right|(\Omega) \\ &\quad\lesssim \int_\Omega \left|D^2_{\mathtt{pw}} v_\ell - D^2_{\mathtt{pw}} v_k\right| \,\mathrm{d}x + \int_{\mathcal{F}_k} |[\![\partial_n v_\ell - \partial_n v_k]\!]| \,\mathrm{d}s. \end{aligned} \tag{4.3.37}$$

We define

$$\tilde{v}_\ell|_K := \begin{cases} w|_K, & K \in \mathcal{T}_\ell^{2-} \\ \overline{u}_\infty|_K, & K \in \mathcal{T}_\ell^{2+}. \end{cases} \tag{4.3.38}$$

For the volume-terms on the right-hand side of (4.3.37) we have that

$$\begin{aligned} &\int_\Omega \left|D^2_{\mathtt{pw}} v_\ell - D^2_{\mathtt{pw}} v_k\right| \,\mathrm{d}x \\ &\quad\leqslant \int_\Omega \left|D^2_{\mathtt{pw}} v_\ell - D^2_{\mathtt{pw}} \tilde{v}_\ell\right| \,\mathrm{d}x + \int_\Omega \left|D^2_{\mathtt{pw}} v_k - D^2_{\mathtt{pw}} \tilde{v}_\ell\right| \,\mathrm{d}x \end{aligned} \tag{4.3.39}$$

The first term vanishes on $\mathcal{T}_\ell^{2+}$ since we have for $\ell \leqslant k \leqslant k_j$, by the definition of $\tilde{v}_\ell$, that

$$v_\ell = \mathcal{I}_\ell \overline{u}_\infty \leftarrow \mathcal{I}_\ell u_{k_j} = \mathcal{I}_{k_j} u_{k_j} = u_{k_j} \to \overline{u}_\infty \quad \text{on } K \in \mathcal{T}_\ell^{2+}, \tag{4.3.40}$$

as $j \to \infty$. On $\mathcal{T}_\ell^{2-}$ we use (4.3.38) in conjunction with the density of $H^3_0(\Omega)$ in $H^2_0(\Omega)$ and choose for arbitrarily fixed $\epsilon > 0$ some $w_\epsilon \in H^3_0(\Omega)$ such that $\|w - w_\epsilon\|_{H^2(\Omega^-)} \leqslant \|w - w_\epsilon\|_{H^2(\Omega)} < \epsilon$. Consequently, we obtain

$$\begin{aligned} &\int_{\mathcal{T}_\ell^{2-}} \left|D^2 \mathcal{I}_\ell w - D^2_{\mathtt{pw}} w\right| \,\mathrm{d}x \\ &\quad\leqslant \int_{\mathcal{T}_\ell^{2-}} \left|D^2 \mathcal{I}_\ell (w - w_\epsilon)\right| + \left|D^2 \mathcal{I}_\ell w_\epsilon - D^2 w_\epsilon\right| + \left|D^2 w_\epsilon - D^2 w\right| \,\mathrm{d}x. \end{aligned}$$

By similar arguments as in step 5 in the proof of Lemma 4.12 we obtain that the first two terms on the right-hand side of the last estimate vanish as $\ell \to \infty$. Since $\epsilon > 0$ was chosen arbitrarily we therefore obtain in (4.3.39)

$$\int_\Omega \left|D^2_{\mathtt{pw}} v_\ell - D^2_{\mathtt{pw}} \tilde{v}_\ell\right| \,\mathrm{d}x < \epsilon$$

for $\ell \geqslant L_1 = L_1(\epsilon)$ (enlarge k if necessary).

For the second term on the right-hand side of (4.3.39) we use the definiton of $\tilde{v}_\ell$ and obtain

$$\int_\Omega \left|D^2_{\mathtt{pw}} v_k - D^2_{\mathtt{pw}} \tilde{v}_\ell\right| \,\mathrm{d}x = \int_{\mathcal{T}_k \setminus \mathcal{T}_k^{2+}} \left|D^2_{\mathtt{pw}} v_k - D^2_{\mathtt{pw}} \tilde{v}_\ell\right| \,\mathrm{d}x + \int_{\mathcal{T}_k^{2+}} \left|D^2_{\mathtt{pw}} v_k - D^2_{\mathtt{pw}} \tilde{v}_\ell\right| \,\mathrm{d}x.$$

The first term vanishes since $\mathcal{T}_k^{2+} \subset \mathcal{T}_\ell^{2+}$:

$$\int_{\mathcal{T}_k^{2-}} \left|D^2_{\mathtt{pw}} v_k - D^2_{\mathtt{pw}} \tilde{v}_\ell\right| \,\mathrm{d}x = \int_{\mathcal{T}_k^{2-}} \left|D^2_{\mathtt{pw}} \mathcal{I}_k w - D^2 w\right| \,\mathrm{d}x < \epsilon$$

for $k \geqslant K_1(\epsilon) \geqslant L_1$, as proved above. For the second term we use $\mathcal{T}_\ell^{2+} \subset \mathcal{T}_k^{2+}$ to obtain

$$\int_{\mathcal{T}_k^{2+}} \left|D^2_{\mathtt{pw}} v_k - D^2_{\mathtt{pw}} \tilde{v}_\ell\right| \,\mathrm{d}x = \int_{\mathcal{T}_\ell^{2+}} \left|D^2_{\mathtt{pw}} \mathcal{I}_k \overline{u}_\infty - D^2_{\mathtt{pw}} \overline{u}_\infty\right| \,\mathrm{d}x + \int_{\mathcal{T}_k^{2+}\setminus\mathcal{T}_\ell^{2+}} \left|D^2_{\mathtt{pw}} \mathcal{I}_k \overline{u}_\infty - D^2_{\mathtt{pw}} \overline{u}_\infty\right| \,\mathrm{d}x.$$

Now, the right-hand side of the last estimate vanishes as as $k, \ell \to \infty$ due to (4.3.40) and step [4] of this proof. Hence, we have that

$$\int_{\mathcal{T}_k^{2+}} \left|D^2_{\mathtt{pw}} v_k - D^2_{\mathtt{pw}} \tilde{v}_\ell\right| \,\mathrm{d}x < \epsilon$$

for $k \geqslant \ell$ with $k \geqslant K_2 = K_2(\epsilon)$ and $\ell \geqslant L_2 = L_2(\epsilon)$.

Regarding the jump-terms on the right-hand side of (4.3.37) we observe

$$\int_{\mathcal{F}_k} |[\![\partial_n v_\ell - \partial_n v_k]\!]| \,\mathrm{d}s \leqslant \int_{\mathcal{F}_k} |[\![\partial_n v_\ell - \partial_n \tilde{v}_\ell]\!]| \,\mathrm{d}s + \int_{\mathcal{F}_k} |[\![\partial_n v_k - \partial_n \tilde{v}_\ell]\!]| \,\mathrm{d}s. \tag{4.3.41}$$

For the first-term on the right-hand side we have from the definition of $\tilde{v}_\ell$ and $\mathcal{F}_\ell^{2+} \subset \mathcal{F}_k^{2+}$

$$\int_{\mathcal{F}_k} |[\![\partial_n v_\ell - \partial_n \tilde{v}_\ell]\!]| \,\mathrm{d}s = \int_{\mathcal{F}_k\setminus\mathcal{F}_\ell^{2+}} |[\![\partial_n v_\ell]\!]| \,\mathrm{d}s + \int_{\mathcal{F}_\ell^{2+}} |[\![\partial_n v_\ell - \partial_n \overline{u}_\infty]\!]| \,\mathrm{d}s,$$

where we used that $[\![\partial_n \tilde{v}_\ell]\!]\,|_F = [\![\partial_n w]\!]\,|_F = 0$ for $F \notin \mathcal{F}_\ell^{2+}$. For the first term on the right-hand side we observe that $v_\ell \in \mathbb{V}_\ell \subset H^2(\mathcal{T}_\ell)$ and therefore from $\mathcal{F}_\ell \subset \mathcal{F}_k$ we have that the jump-terms are only non-zero on the faces related to $\mathcal{T}_\ell$. Now, standard trace inequalities reveal (compare e.g. the proof of Proposition 3.30)

$$\begin{aligned}
\int_{\mathcal{F}_k\setminus\mathcal{F}_\ell^{2+}} |[\![\partial_n v_\ell]\!]| \,\mathrm{d}s &= \int_{\mathcal{F}_\ell\setminus\mathcal{F}_\ell^{2+}} |[\![\partial_n \mathcal{I}_\ell w]\!]| \,\mathrm{d}s \\
&= \int_{\mathcal{F}_\ell\setminus\mathcal{F}_\ell^{2+}} h_\ell^{-1} \,|[\![\partial_n \mathcal{I}_\ell w - w]\!]| \,\mathrm{d}s \\
&\lesssim \sum_{F\in\mathcal{F}_\ell^{1-}} \int_{\omega_\ell(F)} h_\ell^{-1} \left|\nabla_{\mathtt{pw}} \mathcal{I}_\ell w - \nabla w\right| dx \\
&\quad + \int_{\omega_\ell(F)} \left|D^2_{\mathtt{pw}} \mathcal{I}_\ell w - D^2 w\right| \,\mathrm{d}x \\
&\lesssim \int_{\omega_\ell(F)} \left|D^2_{\mathtt{pw}} \mathcal{I}_\ell w - D^2 w\right| \,\mathrm{d}x,
\end{aligned} \tag{4.3.42}$$

where we also used standard interpolation estimates and $w \in H^2(\Omega)$ in the last step. Following the same ideas as above we have that the term on the right-hand side vanishes as $\ell \to \infty$. Hence, for all $\ell \geqslant L_3(\epsilon)$, with $\ell \leqslant k$ (enlarge k if necessary) in (4.3.41) we have that

$$\int_{\mathcal{F}_k} |[\![\partial_n v_\ell - \partial_n \tilde{v}_\ell]\!]| \, \mathrm{d}s < \epsilon.$$

The remaining jump-term of (4.3.41) can be bounded by similar arguments, since we have

$$\begin{aligned}\int_{\mathcal{F}_k} |[\![\partial_n v_k - \partial_n \tilde{v}_\ell]\!]| \, \mathrm{d}s = & \int_{\mathcal{F}_\ell^{2+}} |[\![\partial_n \mathcal{I}_k \overline{u}_\infty - \partial_n \overline{u}_\infty]\!]| \, \mathrm{d}s \\ & + \int_{\mathcal{F}_k^{2+} \setminus \mathcal{F}_\ell^{2+}} |[\![\partial_n \mathcal{I}_k \overline{u}_\infty]\!]| \, \mathrm{d}s \\ & + \int_{\mathcal{F}_k^{2-}} |[\![\partial_n \mathcal{I}_k w - \partial_n w]\!]| \, \mathrm{d}s.\end{aligned}$$

The first term on the right hand side is zero due to (4.3.40). The last term on the right-hand side vanishes as $k \to \infty$ similar as in (4.3.42). For the penultimate term on the right-hand side we use Hölder's inequality in conjunction Lemma 4.25(3) to obtain

$$\begin{aligned}\int_{\mathcal{F}_k^{2+} \setminus \mathcal{F}_\ell^{2+}} |[\![\partial_n \mathcal{I}_k \overline{u}_\infty]\!]| \, \mathrm{d}s &= \int_{\mathcal{F}_k^{2+} \setminus \mathcal{F}_\ell^{2+}} |[\![\partial_n \overline{u}_\infty]\!]| \, \mathrm{d}s \\ &\lesssim \left(\int_{\mathcal{F}_k^{2+} \setminus \mathcal{F}_\ell^{2+}} h_k^{-1} \left|[\![\partial_n \overline{u}_\infty]\!]\right|^2 \mathrm{d}s \right)^{1/2} \\ &\leqslant \left(\int_{\mathcal{F}_k^{2+}} h_k^{-1} \left|[\![\partial_n \overline{u}_\infty]\!]\right|^2 \mathrm{d}s - \int_{\mathcal{F}_\ell^{2+}} h_\ell^{-1} \left|[\![\partial_n \overline{u}_\infty]\!]\right|^2 \mathrm{d}s \right)^{1/2} \\ &\to 0\end{aligned}$$

as $k, \ell \to \infty$, where we also used $h_\ell \geqslant h_k$ and the ideas of step 4 of this proof. Hence, we can chose $\ell \geqslant L_4(\epsilon)$ and $k \geqslant K_3(\epsilon)$ with ($\ell \leqslant k$; enlarge k if necessary) such that

$$\int_{\mathcal{F}_k} |[\![\partial_n v_k - \partial_n \tilde{v}_\ell]\!]| \, \mathrm{d}s < \epsilon$$

Overall, we thus proved the following: First, we can chose $\ell \geqslant \max\{L_1, L_2, L_3, L_4\}$ and $k \geqslant \max\{\ell, K_1, K_2, K_3\}$ yielding

$$\begin{aligned}&\left|D(\nabla_{\mathtt{pw}} v_\ell - \nabla_{\mathtt{pw}} v_k)\right|(\Omega) \\ &\quad \lesssim \int_\Omega \left|D^2_{\mathtt{pw}} v_\ell - D^2_{\mathtt{pw}} v_k\right| \mathrm{d}x + \int_{\mathcal{F}_k} |[\![\partial_n v_\ell - \partial_n v_k]\!]| \, \mathrm{d}s < \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was chosen arbitrarily we have that $\{\nabla_{\mathtt{pw}} v_k\}_{k \in \mathbb{N}}$ is a Cauchy-Sequence in $BV(\Omega)^2$.

In order to identify the limit of this Cauchy sequence, we observe for $k \leqslant k_j$

$$v_k = \mathcal{I}_k \overline{u}_\infty \leftarrow \mathcal{I}_k u_{k_j} = \mathcal{I}_{k_j} u_{k_j} = u_{k_j} \to \overline{u}_\infty \quad \text{on } K \in \mathcal{T}_k^{2+},$$

as $j \to \infty$. Consequently, $v_k \to \overline{u}_\infty$ in $\mathbb{P}_r(\mathcal{T}_k^{2+})$. On the remaining domain Ω_k^{2-} interpolation properties infer

$$\begin{aligned} \|v_k - \overline{u}_\infty\|_{\Omega_k^{2-}} &\leqslant \|v_k - w\|_{\Omega_k^{2-}} + \|w - \overline{u}_\infty\|_{\Omega_k^{2-}} \\ &= \|\mathcal{I}_k w - w\|_{\Omega_k^{2-}} + \|w - \overline{u}_\infty\|_{\Omega_k^{2-}\setminus\Omega^-}, \end{aligned}$$

where we used $\overline{u}_\infty|_{\Omega^-} = w|_{\Omega^-}$ in the last line. The first term on the right-hand side vanishes as $k \to \infty$ due to Lemma 4.5. For the remaining term we have

$$\|w - \overline{u}_\infty\|_{\Omega_k^{2-}\setminus\Omega^-} \leqslant \|w\|_{\Omega_k^{2-}\setminus\Omega^-} + \|\overline{u}_\infty\|_{\Omega_k^{2-}\setminus\Omega^-} \to 0$$

as $k \to \infty$ thanks to Lemma 4.5 and the absolute continuous dependence of the integral to the integration domain. Hence, the limit of the sequence $\{v_k\}_{k\in\mathbb{N}}$ is given by $v_k \to \overline{u}_\infty$ in $L^2(\Omega)$ as $k \to \infty$ and consequently $\nabla_{\mathtt{pw}} v_k \to \nabla_{\mathtt{pw}} \overline{u}_\infty \in BV(\Omega)^2$.

Now, let $\ell \leqslant k$ arbitrary but fixed. Then, from the properties of the trace operator in $BV(\Omega)^2$ (see Theorem 3.25) in conjunction with the fact that that $\left\{\nabla_{\mathtt{pw}} v_k\right\}_{k\in\mathbb{N}}$ is a Cauchy-sequence, we have that the jumps of $\{\nabla_{\mathtt{pw}} v_k\}_{k\in\mathbb{N}}$ on $\mathcal{T}_\ell$ have a unique limit in $L^1(\Gamma_\ell)^2$ (as $k \to \infty$). From this we conclude for arbitrary $\epsilon > 0$

$$\int_{\mathcal{F}_\ell} h_\ell^{-1} \left[\!\left[\partial_n v_k - \partial_n v_m\right]\!\right]^2 < \epsilon, \tag{4.3.43}$$

provided $k, m \geqslant \ell$ with k, m sufficiently large. Finally, we conclude

$$\begin{aligned} &\int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} \left[\!\left[\partial_n \overline{u}_\infty\right]\!\right]^2 \,\mathrm{d}s \\ &\leqslant \int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} \left[\!\left[\partial_n \overline{u}_\infty - \partial_n v_k\right]\!\right]^2 \,\mathrm{d}s + \int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} \left[\!\left[\partial_n v_k - \partial_n v_\ell\right]\!\right]^2 \,\mathrm{d}s \\ &\quad + \int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} \left[\!\left[\partial_n v_\ell\right]\!\right]^2 \,\mathrm{d}s. \end{aligned} \tag{4.3.44}$$

For the last term in (4.3.44) we observe

$$\int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} \left[\!\left[\partial_n v_\ell\right]\!\right]^2 \,\mathrm{d}s = \int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} \left[\!\left[\partial_n \mathcal{I}_\ell w - \partial_n w\right]\!\right]^2 \,\mathrm{d}s,$$

since $w \in H_0^2(\Omega)$. Thus, we are able to chose $\ell \geqslant L_1 = L_1(\epsilon)$ such that

$$\int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} \left[\!\left[\partial_n \mathcal{I}_\ell w - \partial_n w\right]\!\right]^2 \,\mathrm{d}s < \epsilon,$$

as in step 5 in the proof of Lemma 4.12. For the second term on the right-hand side of (4.3.44) we use $h_\ell \geqslant h_k$ for $\ell \leqslant k$ and inclusion of skeletons to obtain

$$\begin{aligned}\int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} [\![\partial_n v_k - \partial_n v_\ell]\!]^2 \,\mathrm{d}s &\leqslant 2\int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} [\![\partial_n v_k]\!]^2 \,\mathrm{d}s + 2\int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} [\![\partial_n v_\ell]\!]^2 \,\mathrm{d}s \\ &\lesssim \int_{\mathcal{F}_k^{2-}} h_k^{-1} [\![\partial_n v_k]\!]^2 \,\mathrm{d}s + \int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} [\![\partial_n v_\ell]\!]^2 \,\mathrm{d}s\end{aligned}$$

and argue similarly as in the case above. Finally, we are able to chose $k \geqslant K_2 = K_2(\epsilon)$ such

$$\int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} [\![\partial_n \overline{u}_\infty - \partial_n v_k]\!]^2 \,\mathrm{d}s < \epsilon,$$

due to the fact that $\{\nabla_{\mathtt{pw}} v_k\}_{k\in\mathbb{N}}$ is a Cauchy-sequence with limit $\nabla_{\mathtt{pw}}\overline{u}_\infty$.

Hence, we have proved that we can chose $\ell \geqslant L_1$ and then $k \geqslant \max\{\ell, K_1, K_2\}$ such that

$$\begin{aligned}&\int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} [\![\partial_n \overline{u}_\infty]\!]^2 \,\mathrm{d}s \\ &\leqslant \int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} [\![\partial_n \overline{u}_\infty - \partial_n v_k]\!]^2 \,\mathrm{d}s + \int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} [\![\partial_n v_k - \partial_n v_\ell]\!]^2 \,\mathrm{d}s \\ &\quad + \int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} [\![\partial_n v_\ell]\!]^2 \,\mathrm{d}s < \epsilon\end{aligned}$$

Consequently, since $\epsilon > 0$ was chosen arbitrarily, in view of (4.3.44) we have

$$\lim_{\ell\to\infty} \int_{\mathcal{F}_\ell^{2-}} h_\ell^{-1} [\![\partial_n \overline{u}_\infty]\!]^2 \,\mathrm{d}s = 0,$$

By analogous arguments (but now using (3.6.4) instead of Proposition 3.31) we are able to prove additionally that

$$\lim_{\ell\to\infty} \int_{\mathcal{F}_\ell^{2-}} h_\ell^{-3} [\![\overline{u}_\infty]\!]^2 \,\mathrm{d}s = 0$$

holds. Summing up the various arguments we conclude that indeed the limit in (4.3.35) holds true.

Finally, we consider the sequence $\{v_k\}_{k\in\mathbb{N}}$, defined in (4.3.36) and obtain from Lemma 4.26 and step 4 of this proof that $|\!|\!| v_k |\!|\!|_k \lesssim \|D^2 w\|_\Omega + \|\overline{u}_\infty\|_\infty < \infty$. The desired convergence $\lim_{k\to\infty} |\!|\!| v_k - \overline{u}_\infty |\!|\!|_k^2 + \|v_k - \overline{u}_\infty\|_\Omega^2 = 0$ follows by splitting the error according to $\mathcal{T}_k = \mathcal{T}_k^{2-} \cup \mathcal{T}_k^{2+}$ and treating the resulting terms separately similar to step 5 in the proof of Lemma 4.12. □

In order to prove that $\overline{u}_\infty$ solves (4.2.9), we need to identify the limit of its distributional derivatives. To this end, we note that by (3.3.4) and (3.3.5) we have $\|D^2_{\mathtt{pw}} u_{k_j}\|_\Omega \lesssim 1$ and $\|\mathcal{L}_{k_j}(u_{k_j})\|_\Omega \lesssim 1$. Consequently, there exist $\boldsymbol{T_r}, \boldsymbol{T_s} \in L^2(\Omega)^{2\times 2}$ such that for a not relabelled subsequence we obtain

$$D^2_{\mathtt{pw}} u_{k_j} \rightharpoonup \boldsymbol{T_r} \qquad \text{and} \qquad \mathcal{L}_{k_j}([\![\partial_n u_{k_j}]\!]) \rightharpoonup \boldsymbol{T_s} \tag{4.3.45}$$

weakly in $L^2(\Omega)^{2\times 2}$ as $j \to \infty$.

Lemma 4.31. *Let* $\{u_{k_j}\}_{j\in\mathbb{N}_0}$ *be the subsequence of discrete solutions with weak* $L^2(\Omega)$ *limit* $\overline{u}_\infty \in \mathbb{V}_\infty$ *from* (4.3.26)*. Then, we have for* $\boldsymbol{T_s}, \boldsymbol{T_r} \in L^2(\Omega)^{2\times 2}$ *from* (4.3.45) *that*

$$(\boldsymbol{T_r} + \boldsymbol{T_s})|_{\Omega^-} = D^2\overline{u}_\infty|_{\Omega^-} \qquad \textit{a.e. in } \Omega^-.$$

Proof. Propositions 3.12 and 3.30 in conjunction with (3.3.5) imply that $\{u_{k_j}\}_{j\in\mathbb{N}_0}$ is uniformly bounded in $BV(\Omega)$. Hence, as in the proof of Lemma 4.30 step $\boxed{1}$, we have that $u_{k_j} \overset{*}{\rightharpoonup} \overline{u}_\infty$ in $BV(\Omega)$ as $j \to \infty$. In particular this implies $u_{k_j} \to \overline{u}_\infty$ in $L^1(\Omega)$ as $j \to \infty$ (compare Proposition 3.23). Hence, for $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^{2\times 2}$, we have

$$\begin{aligned} \left\langle D^2 u_{k_j}, \boldsymbol{\varphi} \right\rangle &= \int_\Omega (\operatorname{div}\operatorname{div}\boldsymbol{\varphi}) u_{k_j}\,\mathrm{d}x \\ &\to \int_\Omega (\operatorname{div}\operatorname{div}\boldsymbol{\varphi}) \overline{u}_\infty\,\mathrm{d}x = \left\langle D^2\overline{u}_\infty, \boldsymbol{\varphi} \right\rangle, \end{aligned} \tag{4.3.46}$$

as $j \to \infty$ and therefore the distributional Hessian $D^2 u_{k_j}$ convergens to $D^2\overline{u}_\infty$ as $j \to \infty$ in the sense of distributions.

Using the fact that $\overline{u}_\infty \in \mathbb{V}_\infty$, we have that there exists a sequence $\{v_k\}_{k\in\mathbb{N}_0}$ with $v_k \in \mathbb{V}_k$, $k \in \mathbb{N}_0$, and $|||\overline{u}_\infty - v_k|||_k + \|\overline{u}_\infty - v_k\|_\Omega \to 0$ as $k \to \infty$. On the one hand, Proposition 4.10 reveals

$$\begin{aligned} \int_\Omega (\operatorname{div}\operatorname{div}\boldsymbol{\varphi}) \overline{u}_\infty &= \left\langle D^2\overline{u}_\infty, \boldsymbol{\varphi} \right\rangle \\ &= \int_\Omega D^2_{\mathtt{pw}}\overline{u}_\infty \cdot \boldsymbol{\varphi}\,\mathrm{d}x - \int_{\mathcal{F}^+} \boldsymbol{\varphi}\, [\![\nabla_{\mathtt{pw}}\overline{u}_\infty]\!] \cdot \boldsymbol{n}\,\mathrm{d}s \\ &\quad + \int_{\mathcal{F}^+} \operatorname{div}\boldsymbol{\varphi} \cdot [\![\overline{u}_\infty]\!]\,\boldsymbol{n}\,\mathrm{d}s, \end{aligned} \tag{4.3.47}$$

for $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^{2\times 2}$. On the other hand, we have in (4.3.46)

$$\begin{aligned} &\int_\Omega (\operatorname{div}\operatorname{div}\boldsymbol{\varphi})\ u_{k_j}\,\mathrm{d}x \\ &= \int_\Omega D^2_{\mathtt{pw}} u_{k_j} : \boldsymbol{\varphi}\,\mathrm{d}x \\ &\quad - \int_{\mathcal{F}_{k_j}} \boldsymbol{\varphi}\, [\![\nabla_{\mathtt{pw}} u_{k_j}]\!] \cdot \boldsymbol{n} - \operatorname{div}\boldsymbol{\varphi} \cdot [\![u_{k_j}]\!]\,\boldsymbol{n}\,\mathrm{d}s. \end{aligned} \tag{4.3.48}$$

From the left-hand side of (4.3.46), we know that $\int_\Omega (\operatorname{div}\operatorname{div}\boldsymbol{\varphi}) u_{k_j}\,\mathrm{d}x$ converges to the distributional Hessian of $\overline{u}_\infty$ as $j \to \infty$ and (4.3.47) states a formula for this distributional Hessian. Regarding (4.3.48), we already know that $D^2_{\mathtt{pw}} u_{k_j} \rightharpoonup \boldsymbol{T_r}$ in $L^2(\Omega)^{2\times 2}$ as $j \to \infty$. Hence, in view of (4.3.45) the statement is proved if on Ω^- the jump-terms in (4.3.48) generate the liftings in the limit.

In order to prove this we investigate the limit of jump terms in (4.3.48): Fix $\ell \in \mathbb{N}_0$, and let $\boldsymbol{\pi}_{k_j} = \boldsymbol{\pi}_{k_j}(\boldsymbol{\varphi})$ be the L^2-projection of $\boldsymbol{\varphi}$ onto $\mathbb{P}_r(\mathcal{T}_{k_j})^{2\times 2}$, then by using the definition of the lifting (3.3.1) and $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^{2\times 2}$ we have

$$\begin{aligned}
&-\int_{\mathcal{F}_{k_j}} \boldsymbol{\varphi} \left[\!\left[\nabla_{\mathtt{pw}} u_{k_j}\right]\!\right] \cdot \boldsymbol{n} - \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[u_{k_j}\right]\!\right] \boldsymbol{n} \, \mathrm{d}s \\
&= -\int_{\mathcal{F}_\ell^+} \boldsymbol{\varphi} \left[\!\left[\nabla_{\mathtt{pw}} u_{k_j}\right]\!\right] \cdot \boldsymbol{n} - \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[u_{k_j}\right]\!\right] \boldsymbol{n} \, \mathrm{d}s \\
&\quad - \int_{\mathcal{F}_{k_j} \setminus \mathcal{F}_\ell^+} \{\!\{\boldsymbol{\varphi}\}\!\} \left[\!\left[\nabla_{\mathtt{pw}} u_{k_j}\right]\!\right] \cdot \boldsymbol{n} - \{\!\{\operatorname{div} \boldsymbol{\varphi}\}\!\} \cdot \left[\!\left[u_{k_j}\right]\!\right] \boldsymbol{n} \, \mathrm{d}s \\
&= -\int_{\mathcal{F}_\ell^+} \boldsymbol{\varphi} \left[\!\left[\nabla_{\mathtt{pw}} u_{k_j}\right]\!\right] \cdot \boldsymbol{n} - \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[u_{k_j}\right]\!\right] \boldsymbol{n} \, \mathrm{d}s \\
&\quad - \int_{\mathcal{F}_{k_j} \setminus \mathcal{F}_\ell^+} \left\{\!\!\left\{\boldsymbol{\varphi} - \boldsymbol{\pi}_{k_j}\right\}\!\!\right\} \left[\!\left[\nabla_{\mathtt{pw}} u_{k_j}\right]\!\right] \cdot \boldsymbol{n} - \left\{\!\!\left\{\operatorname{div}(\boldsymbol{\varphi} - \boldsymbol{\pi}_{k_j})\right\}\!\!\right\} \cdot \left[\!\left[u_{k_j}\right]\!\right] \boldsymbol{n} \, \mathrm{d}s \\
&\quad - \int_{\mathcal{F}_{k_j} \setminus \mathcal{F}_\ell^+} \left\{\!\!\left\{\boldsymbol{\pi}_{k_j}\right\}\!\!\right\} \left[\!\left[\nabla_{\mathtt{pw}} u_{k_j}\right]\!\right] \cdot \boldsymbol{n} - \left\{\!\!\left\{\operatorname{div} \boldsymbol{\pi}_{k_j}\right\}\!\!\right\} \cdot \left[\!\left[u_{k_j}\right]\!\right] \boldsymbol{n} \, \mathrm{d}s \\
&= -\int_{\mathcal{F}_\ell^+} \boldsymbol{\varphi} \left[\!\left[\nabla_{\mathtt{pw}} u_{k_j}\right]\!\right] \cdot \boldsymbol{n} - \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[u_{k_j}\right]\!\right] \boldsymbol{n} \, \mathrm{d}s \\
&\quad - \int_{\mathcal{F}_{k_j} \setminus \mathcal{F}_\ell^+} \left\{\!\!\left\{\boldsymbol{\varphi} - \boldsymbol{\pi}_{k_j}\right\}\!\!\right\} \left[\!\left[\nabla_{\mathtt{pw}} u_{k_j}\right]\!\right] \cdot \boldsymbol{n} - \left\{\!\!\left\{\operatorname{div}(\boldsymbol{\varphi} - \boldsymbol{\pi}_{k_j})\right\}\!\!\right\} \cdot \left[\!\left[u_{k_j}\right]\!\right] \boldsymbol{n} \, \mathrm{d}s \\
&\quad - \int_{\Omega_\ell^-} \mathcal{L}_{k_j}(u_{k_j}) \colon (\boldsymbol{\varphi} - \boldsymbol{\pi}_{k_j}) \, \mathrm{d}x + \int_{\Omega_\ell^-} \mathcal{L}_{k_j}(u_{k_j}) \colon \boldsymbol{\varphi} \, \mathrm{d}x,
\end{aligned} \tag{4.3.49}$$

for all $\ell \leqslant k_j$. Thanks to Lemma 4.5, for $\epsilon > 0$, we have

$$\left\|\boldsymbol{\varphi} - \boldsymbol{\pi}_{k_j}\right\|_{L^\infty(\Omega_\ell^-)} \leqslant \left\|h_{k_j} \chi_{\Omega_\ell^-}\right\|_{L^\infty(\Omega)} \|\nabla \boldsymbol{\varphi}\|_{L^\infty(\Omega)} \leqslant \left\|h_\ell \chi_{\Omega_\ell^-}\right\|_{L^\infty(\Omega)} \|\nabla \boldsymbol{\varphi}\|_{L^\infty(\Omega)} < \epsilon$$

for sufficiently large $\ell = \ell(\epsilon, \boldsymbol{\varphi}) \leqslant k_j$ and thus

$$\begin{aligned}
\Bigg| - \int_{\mathcal{F}_{k_j} \setminus \mathcal{F}_\ell^+} &\left\{\!\!\left\{\boldsymbol{\varphi} - \boldsymbol{\pi}_{k_j}\right\}\!\!\right\} \left[\!\left[\nabla_{\mathtt{pw}} u_{k_j}\right]\!\right] \cdot \boldsymbol{n} - \left\{\!\!\left\{\operatorname{div}(\boldsymbol{\varphi} - \boldsymbol{\pi}_{k_j})\right\}\!\!\right\} \cdot \left[\!\left[u_{k_j}\right]\!\right] \boldsymbol{n} \, \mathrm{d}s \\
&- \int_{\Omega_\ell^-} \mathcal{L}_{k_j}(u_{k_j}) \colon (\boldsymbol{\varphi} - \boldsymbol{\pi}_{k_j}) \, \mathrm{d}x \Bigg| \lesssim \epsilon \, \|f\|_\Omega \, \|\nabla \boldsymbol{\varphi}\|_{L^\infty(\Omega)} .
\end{aligned}$$

As a consequence of (4.3.27), (4.3.29) and (4.3.45) in conjunction with the fact, that $u_{k_j}|_{\Omega_\ell^+} \in \mathbb{P}_2(\mathcal{T}_\ell^+)$ is finite dimensional, we have that

$$\begin{aligned}
- \int_{\mathcal{F}_\ell^+} &\boldsymbol{\varphi} \left[\!\left[\nabla_{\mathtt{pw}} u_{k_j}\right]\!\right] \cdot \boldsymbol{n} - \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[u_{k_j}\right]\!\right] \boldsymbol{n} \, \mathrm{d}s + \int_{\Omega_\ell^-} \mathcal{L}_{k_j}(u_{k_j}) \colon \boldsymbol{\varphi} \, \mathrm{d}x \\
&\to - \int_{\mathcal{F}_\ell^+} \boldsymbol{\varphi} \left[\!\left[\nabla_{\mathtt{pw}} \overline{u}_\infty\right]\!\right] \cdot \boldsymbol{n} - \operatorname{div} \boldsymbol{\varphi} \cdot \left[\!\left[\overline{u}_\infty\right]\!\right] \boldsymbol{n} \, \mathrm{d}s + \int_{\Omega_\ell^-} \boldsymbol{T_s} \colon \boldsymbol{\varphi} \, \mathrm{d}x
\end{aligned}$$

as $j \to \infty$.

Hence, we have that

$$
\begin{aligned}
\lim_{j\to\infty}\bigg|\int_\Omega (\operatorname{div}\operatorname{div}\boldsymbol{\varphi})u_{k_j}\,\mathrm{d}x - \int_\Omega D^2_{\mathtt{pw}}u_{k_j} : \boldsymbol{\varphi}\,\mathrm{d}x \\
+ \int_{\mathcal{F}_\ell^+} \boldsymbol{\varphi}\,[\![\nabla_{\mathtt{pw}}u_{k_j}]\!]\cdot\boldsymbol{n} - \operatorname{div}\boldsymbol{\varphi}\cdot[\![u_{k_j}]\!]\,\boldsymbol{n}\,\mathrm{d}s \\
- \int_{\Omega_\ell^-} \mathcal{L}_{k_j}(u_{k_j}) : \boldsymbol{\varphi}\,\mathrm{d}x\bigg| \\
= \bigg|\int_\Omega (\operatorname{div}\operatorname{div}\boldsymbol{\varphi})\overline{u}_\infty\,\mathrm{d}x - \int_\Omega \boldsymbol{T_r} : \boldsymbol{\varphi}\,\mathrm{d}x \\
+ \int_{\mathcal{F}_\ell^+} \boldsymbol{\varphi}\,[\![\nabla_{\mathtt{pw}}\overline{u}_\infty]\!]\cdot\boldsymbol{n} - \operatorname{div}\boldsymbol{\varphi}\cdot[\![\overline{u}_\infty]\!]\,\boldsymbol{n}\,\mathrm{d}s \\
- \int_{\Omega_\ell^-} \boldsymbol{T_s} : \boldsymbol{\varphi}\,\mathrm{d}x\bigg| \lesssim \epsilon\,\|f\|_\Omega\,\|\nabla\boldsymbol{\varphi}\|_{L^\infty(\Omega)}\,.
\end{aligned}
$$

Upon choosing ℓ even larger, we have also from the absolute continuous dependence of the integral on the integration domain (Remark 4.6) in conjunction with Lemma 4.5

$$
\bigg|\int_{\mathcal{F}^+\setminus\mathcal{F}_\ell^+} \boldsymbol{\varphi}\,[\![\nabla_{\mathtt{pw}}u_{k_j}]\!]\cdot\boldsymbol{n} - \operatorname{div}\boldsymbol{\varphi}\cdot[\![u_{k_j}]\!]\,\boldsymbol{n}\,\mathrm{d}s - \int_{\Omega_\ell^-\setminus\Omega^-} \boldsymbol{T_s} : \boldsymbol{\varphi}\,\mathrm{d}x\bigg| < \epsilon.
$$

Inserting this in (4.3.49), we have thanks to the fact that $\epsilon > 0$ was arbitrary, that

$$
\begin{aligned}
&\int_\Omega (\operatorname{div}\operatorname{div}\boldsymbol{\varphi})\cdot\nabla_{\mathtt{pw}}u_{k_j}\,\mathrm{d}x \\
&\quad\to \int_\Omega \boldsymbol{T_r} : \boldsymbol{\varphi}\,\mathrm{d}x + \int_{\Omega^-} \boldsymbol{T_s} : \boldsymbol{\varphi}\,\mathrm{d}x - \int_{\mathcal{F}^+} \boldsymbol{\varphi}\,[\![\nabla_{\mathtt{pw}}\overline{u}_\infty]\!]\cdot\boldsymbol{n} - \operatorname{div}\boldsymbol{\varphi}\cdot[\![\overline{u}_\infty]\!]\,\boldsymbol{n}\,\mathrm{d}s
\end{aligned}
$$

as $j \to \infty$. In view of (4.3.46) and (4.3.47), this thus implies that

$$
0 = \lim_{j\to\infty}\int_\Omega (\operatorname{div}\operatorname{div}\boldsymbol{\varphi})(u_{k_j} - \overline{u}_\infty)\,\mathrm{d}x = \int_\Omega (D^2_{\mathtt{pw}}\overline{u}_\infty - \boldsymbol{T_r} - \boldsymbol{T_s}\chi_{\Omega^-}) : \boldsymbol{\varphi}\,\mathrm{d}x
$$

for all $\boldsymbol{\varphi} \in C_0^\infty(\Omega)^{2\times 2}$. The desired assertion follows from the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$. □

Now, we are able to proof that $\overline{u}_\infty$ and u_∞ coincide.

Lemma 4.32. *We have that $\overline{u}_\infty \in \mathbb{V}_\infty$ solves* (4.2.9) *and thus $\overline{u}_\infty = u_\infty$. In particular, the limit in* (4.3.26) *is unique and the full sequence $\{u_k\}_{k\in\mathbb{N}_0}$ converges to u_∞ weakly in $L^2(\Omega)$.*

Proof. Let $v \in \mathbb{V}_\infty$ and $\{v_k\}_{k\in\mathbb{N}_0}$, $v_k \in \mathbb{V}_k$ such that $|\!|\!|v_k - v|\!|\!|_k + \|v_k - v\|_\Omega \to 0$ as $k \to \infty$. Consequently, for the subsequence (4.3.26) of discrete solutions $\{u_{k_j}\}_{j\in\mathbb{N}_0}$, we have

$$
\mathfrak{B}_{k_j}[u_{k_j}, v_{k_j}] = \langle f, v_{k_j}\rangle_\Omega \to \langle f, v\rangle_\Omega \quad \text{as } j \to \infty. \tag{4.3.50}
$$

Using $\|v_k - v\|_k \to 0$ as $k \to \infty$ again, it suffices to prove $\mathfrak{B}_{k_j}[u_{k_j}, v] \to \mathfrak{B}_\infty[\overline{u}_\infty, v]$ as $j \to \infty$.

To see this, we split the bilinear form according to

$$\begin{aligned}\mathfrak{B}_{k_j}[u_{k_j}, v] &= \int_\Omega (D^2_{\mathtt{pw}} u_{k_j} + \mathcal{L}_{k_j}(u_{k_j}))\colon D^2_{\mathtt{pw}} v \,\mathrm{d}x + \int_\Omega \mathcal{L}_{k_j}(v)\colon D^2_{\mathtt{pw}} u_{k_j} \,\mathrm{d}x \\ &\quad + \int_{\mathcal{F}_{k_j}} \frac{\alpha}{h_{k_j}} [\![\partial_n u_{k_j}]\!] [\![\partial_n v]\!] + \frac{\beta}{h^3_{k_j}} [\![u_{k_j}]\!] \boldsymbol{n} \cdot [\![v]\!] \boldsymbol{n} \,\mathrm{d}s \\ &=: I_j + II_j + III_j.\end{aligned}$$

and consider the limit of each term separately.

1 Here, we consider the limit of I_j. From (4.3.45) and Lemma 4.31 we have

$$\begin{aligned}&\int_{\Omega^-} \big(D^2_{\mathtt{pw}} u_{k_j} + \mathcal{L}_{k_j}(u_{k_j})\big)\colon D^2_{\mathtt{pw}} v \,\mathrm{d}x \\ &\qquad \to \int_{\Omega^-} \big(\boldsymbol{T_r} + \boldsymbol{T_s}\big)\colon D^2_{\mathtt{pw}} v \,\mathrm{d}x \quad \text{as } j \to \infty \\ &\qquad = \int_{\Omega^-} D^2 \overline{u}_\infty \colon D^2_{\mathtt{pw}} v \,\mathrm{d}x.\end{aligned} \tag{4.3.51}$$

For $\ell \leqslant k_j$ we split the domain Ω according to

$$\overline{\Omega} = \overline{\Omega^-} \cup \overline{\Omega^{1-}_\ell \backslash \Omega^-} \cup \overline{\Omega^{1+}_\ell}.$$

On $\Omega^-_\ell \backslash \Omega^-$, by uniform integrability of $D^2_{\mathtt{pw}} v$, Lemma 4.5 and the stability of liftings (3.3.4), for $\epsilon > 0$ there exists $K(\epsilon)$ such that for all $\ell \geqslant K(\epsilon)$, we have

$$\begin{aligned}&\Big| \int_{\Omega^{1-}_\ell \backslash \Omega^-} \Big(D^2_{\mathtt{pw}} u_{k_j} + \mathcal{L}_{k_j}(u_{k_j}) - D^2_{\mathtt{pw}} \overline{u}_\infty - \mathcal{L}_\infty(\overline{u}_\infty)\Big)\colon D^2_{\mathtt{pw}} v \,\mathrm{d}x\Big| \\ &\qquad\qquad \lesssim \Big(\|u_{k_j}\|_{k_j} + \|\overline{u}_\infty\|_\infty \Big) \|D^2_{\mathtt{pw}} v\|_{\Omega^{1-}_\ell \backslash \Omega^-} \leqslant \epsilon.\end{aligned}$$

From (4.3.26) (compare step 4 of the proof of Lemma 4.30) we observe on Ω^+_ℓ that $D^2_{\mathtt{pw}} u_{k_j}|_{\Omega^{1+}_\ell} \to D^2_{\mathtt{pw}} \overline{u}_\infty|_{\Omega^{1+}_\ell}$ strongly in $L^2(\Omega^{1+}_\ell)$ as $j \to \infty$ since $\mathbb{P}_{r-2}(\mathcal{T}^{1+}_\ell)^{2\times 2}$ is finite dimensional for fixed ℓ. Therefore, we have

$$\int_{\Omega^{1+}_\ell} D^2_{\mathtt{pw}} u_{k_j} \colon D^2_{\mathtt{pw}} v \,\mathrm{d}x \to \int_{\Omega^{1+}_\ell} D^2_{\mathtt{pw}} \overline{u}_\infty \colon D^2_{\mathtt{pw}} v \,\mathrm{d}x \quad \text{as } j \to \infty.$$

Similar arguments prove

$$[\![\nabla_{\mathtt{pw}} u_{k_j}]\!]\,|_{\mathcal{F}^{1+}_\ell} \to [\![\nabla_{\mathtt{pw}} \overline{u}_\infty]\!]\,|_{\mathcal{F}^{1+}_\ell} \quad \text{and} \quad [\![u_{k_j}]\!]\,|_{\mathcal{F}^{1+}_\ell} \to [\![\overline{u}_\infty]\!]\,|_{\mathcal{F}^{1+}_\ell}$$

strongly in $L^2(\mathcal{F}^{1+}_\ell)$ as $j \to \infty$ and, thanks to the fact that the local definition (3.3.1) of the liftings eventually does not change on $\mathcal{T}^{1+}_\ell$, we have

$$\begin{aligned}\int_{\Omega^{1+}_\ell} \mathcal{L}_{k_j}(u_{k_j})\colon D^2_{\mathtt{pw}} v \,\mathrm{d}x &= \int_{\Omega^{1+}_\ell} \mathcal{L}_\infty(u_{k_j})\colon D^2_{\mathtt{pw}} v \,\mathrm{d}x \\ &\to \int_{\Omega^{1+}_\ell} \mathcal{L}_\infty(\overline{u}_\infty)\colon D^2_{\mathtt{pw}} v \qquad \text{as } j \to \infty.\end{aligned}$$

From the estimate

$$\begin{aligned}
&\Big|\int_\Omega \Big(D^2_{\mathtt{pw}} u_{k_j} + \mathcal{L}_{k_j}(u_{k_j}) - D^2_{\mathtt{pw}}\overline{u}_\infty - \mathcal{L}_\infty(\overline{u}_\infty)\Big) : D^2_{\mathtt{pw}} v \,\mathrm{d}x\Big| \\
&\qquad \leqslant \Big|\int_{\Omega^-} \Big(D^2_{\mathtt{pw}} u_{k_j} + \mathcal{L}_{k_j}(u_{k_j}) - D^2_{\mathtt{pw}}\overline{u}_\infty\Big) : D^2_{\mathtt{pw}} v \,\mathrm{d}x\Big| \\
&\qquad\quad + \Big|\int_{\Omega^{1-}_\ell\backslash\Omega^-} \Big(D^2_{\mathtt{pw}} u_{k_j} + \mathcal{L}_{k_j}(u_{k_j}) - D^2_{\mathtt{pw}}\overline{u}_\infty - \mathcal{L}_\infty(\overline{u}_\infty)\Big) : D^2_{\mathtt{pw}} v \,\mathrm{d}x\Big| \\
&\qquad\quad + \Big|\int_{\Omega^{1+}_\ell} \Big(D^2_{\mathtt{pw}} u_{k_j} + \mathcal{L}_{k_j}(u_{k_j}) - D^2_{\mathtt{pw}}\overline{u}_\infty - \mathcal{L}_\infty(\overline{u}_\infty)\Big) : D^2_{\mathtt{pw}} v \,\mathrm{d}x\Big|,
\end{aligned}$$

we finally observe that the first and third term on the right-hand side vanish as $j \to \infty$, and arrive at

$$\lim_{j\to\infty}\Big|\int_\Omega \Big(D^2_{\mathtt{pw}} u_{k_j} + \mathcal{L}_{k_j}(u_{k_j}) - D^2_{\mathtt{pw}}\overline{u}_\infty - \mathcal{L}_\infty(\overline{u}_\infty)\Big) : D^2_{\mathtt{pw}} v \,\mathrm{d}x\Big| < \epsilon.$$

Since $\epsilon > 0$ was chosen arbitrarily, for $j \to \infty$, we conclude

$$\int_\Omega (D^2_{\mathtt{pw}} u_{k_j} + \mathcal{L}_{k_j}(u_{k_j})) : D^2_{\mathtt{pw}} v \,\mathrm{d}x \to \int_\Omega (D^2_{\mathtt{pw}}\overline{u}_\infty + \mathcal{L}_\infty(\overline{u}_\infty)) : D^2_{\mathtt{pw}} v \,\mathrm{d}x \tag{4.3.52}$$

$\boxed{2}$ In order to identify the limit of II_j, we split the domain Ω according to

$$\Omega = (\Omega\backslash\Omega^{1+}_\ell) \cup \Omega^{1+}_\ell$$

for some $\ell \leqslant k_j$. Thanks to uniform boundedness $|\!|\!| u_k |\!|\!|_k \lesssim \|f\|_\Omega$, for $\epsilon > 0$, we have

$$\Big|\int_{\Omega\backslash\Omega^{1+}_\ell} \mathcal{L}_{k_j}(v) : D^2_{\mathtt{pw}} u_{k_j} \,\mathrm{d}x\Big| \lesssim \|\mathcal{L}_{k_j}(v)\|_{\Omega\backslash\Omega^{1+}_\ell} \|f\|_\Omega < \epsilon \tag{4.3.53}$$

for all $k_j \geqslant \ell \geqslant K(\epsilon)$. Indeed, the stability of the lifting operator (3.3.3) together with Proposition 4.9 yields

$$\|\mathcal{L}_{k_j}(v)\|_{\Omega\backslash\Omega^{1+}_\ell} \lesssim \left(\int_{\mathcal{F}_{k_j}\backslash\mathcal{F}^{2+}_\ell} h^{-1}_{k_j} [\![\partial_n v]\!]^2 + h^{-3}_{k_j} |[\![v]\!]\,\boldsymbol{n}|^2 \,\mathrm{d}s\right)^{1/2} \to 0,$$

as $k_j \geqslant \ell \to \infty$. Similar as in $\boxed{1}$, on Ω^{1+}_ℓ we employ the strong convergence $D^2_{\mathtt{pw}} u_{k_j}|_{\Omega^{1+}_\ell} \to D^2_{\mathtt{pw}}\overline{u}_\infty|_{\Omega^{1+}_\ell} \in \mathbb{P}_{r-2}(\mathcal{T}^{1+}_\ell)^{2\times 2}$ in $L^2(\Omega^{1+}_\ell)$ as $j \to \infty$, in order to obtain from the local definitions of the liftings (3.3.1) and (4.2.6) that

$$\begin{aligned}
\int_{\Omega^{1+}_\ell} \mathcal{L}_{k_j}(v) : D^2_{\mathtt{pw}} u_{k_j} \,\mathrm{d}x &= \int_{\Omega^{1+}_\ell} \mathcal{L}_\infty(v) : D^2_{\mathtt{pw}} u_{k_j} \,\mathrm{d}x \\
&\to \int_{\Omega^{1+}_\ell} \mathcal{L}_\infty(v) : D^2_{\mathtt{pw}}\overline{u}_\infty \,\mathrm{d}x \qquad \text{as } j \to \infty.
\end{aligned}$$

Combining this with (4.3.53) yields

$$\int_\Omega \mathcal{L}_{k_j}(v) : D^2_{\mathtt{pw}} u_{k_j} \,\mathrm{d}x \to \int_\Omega \mathcal{L}_\infty(v) : D^2_{\mathtt{pw}}\overline{u}_\infty \,\mathrm{d}x \quad \text{as } k \to \infty. \tag{4.3.54}$$

3 For the last term III_j, we observe from $\mathcal{F}_\ell^+ \subset \mathcal{F}_{k_j}^+$, $\ell \leqslant k_j$, that

$$\begin{aligned}\int_{\mathcal{F}_{k_j}} \frac{\alpha}{h_{k_j}} [\![\partial_n u_{k_j}]\!] [\![\partial_n v]\!] + \frac{\beta}{h_{k_j}^3} [\![u_{k_j}]\!] \boldsymbol{n} \cdot [\![v]\!] \boldsymbol{n} \, \mathrm{d}s \\ = \int_{\mathcal{F}_\ell^+} \frac{\alpha}{h_{k_j}} [\![\partial_n u_{k_j}]\!] [\![\partial_n v]\!] + \frac{\beta}{h_{k_j}^3} [\![u_{k_j}]\!] \boldsymbol{n} \cdot [\![v]\!] \boldsymbol{n} \, \mathrm{d}s \\ + \int_{\mathcal{F}_{k_j} \setminus \mathcal{F}_\ell^+} \frac{\alpha}{h_{k_j}} [\![\partial_n u_{k_j}]\!] [\![\partial_n v]\!] + \frac{\beta}{h_{k_j}^3} [\![u_{k_j}]\!] \boldsymbol{n} \cdot [\![v]\!] \boldsymbol{n} \, \mathrm{d}s.\end{aligned}$$

For the second term on the right-hand side, we conclude from Proposition 4.9 that for arbitrary fixed $\epsilon > 0$ there exists $K(\epsilon) > 0$ such that

$$\begin{aligned}&\int_{\mathcal{F}_{k_j} \setminus \mathcal{F}_\ell^+} \frac{\alpha}{h_{k_j}} [\![\partial_n u_{k_j}]\!] [\![\partial_n v]\!] + \frac{\beta}{h_{k_j}^3} [\![u_{k_j}]\!] \boldsymbol{n} \cdot [\![v]\!] \boldsymbol{n} \, \mathrm{d}s \\ &\leqslant \left(\int_{\mathcal{F}_{k_j} \setminus \mathcal{F}_\ell^+} \frac{\alpha}{h_{k_j}} [\![\partial_n u_{k_j}]\!]^2 \, \mathrm{d}s \right)^{1/2} \left(\int_{\mathcal{F}_{k_j} \setminus \mathcal{F}_\ell^+} \frac{\alpha}{h_{k_j}} [\![\partial_n v]\!]^2 \, \mathrm{d}s \right)^{1/2} \\ &\quad + \left(\int_{\mathcal{F}_{k_j} \setminus \mathcal{F}_\ell^+} \frac{\beta}{h_{k_j}^3} |[\![u_{k_j}]\!] \boldsymbol{n}|^2 \, \mathrm{d}s \right)^{1/2} \left(\int_{\mathcal{F}_{k_j} \setminus \mathcal{F}_\ell^+} \frac{\beta}{h_{k_j}^3} |[\![v]\!] \boldsymbol{n}|^2 \, \mathrm{d}s \right)^{1/2} \\ &\lesssim |\!|\!| u_{k_j} |\!|\!|_{k_j} \left(\int_{\mathcal{F}_{k_j} \setminus \mathcal{F}_\ell^+} \frac{\alpha}{h_{k_j}} [\![\partial_n v]\!]^2 + \frac{\beta}{h_{k_j}^3} |[\![v]\!] \boldsymbol{n}|^2 \, \mathrm{d}s \right)^{1/2} \\ &\lesssim \|f\|_\Omega \left(\int_{\mathcal{F}^+ \setminus \mathcal{F}_\ell^+} \frac{\alpha}{h_+} [\![\partial_n v]\!]^2 + \frac{\beta}{h_+^3} |[\![v]\!] \boldsymbol{n}|^2 \, \mathrm{d}s \right)^{1/2} \leqslant \epsilon\end{aligned}$$

whenever $k_j \geqslant \ell \geqslant K(\epsilon)$. As in 1, we use for fixed ℓ that

$$[\![\partial_n u_{k_j}]\!] |_{\mathcal{F}_\ell^{1+}} \to [\![\partial_n \overline{u}_\infty]\!] |_{\mathcal{F}_\ell^{1+}} \quad \text{and} \quad [\![u_{k_j}]\!] |_{\mathcal{F}_\ell^{1+}} \to [\![\overline{u}_\infty]\!] |_{\mathcal{F}_\ell^{1+}}$$

as $j \to \infty$ strongly in $L^2(\mathcal{F}_\ell^{1+})$ and consequently

$$\begin{aligned}\int_{\mathcal{F}_\ell^+} \frac{\alpha}{h_{k_j}} [\![\partial_n u_{k_j}]\!] [\![\partial_n v]\!] + \frac{\beta}{h_{k_j}^3} [\![u_{k_j}]\!] \boldsymbol{n} \cdot [\![v]\!] \boldsymbol{n} \, \mathrm{d}s \\ \to \int_{\mathcal{F}_\ell^+} \frac{\alpha}{h_+} [\![\partial_n \overline{u}_\infty]\!] [\![\partial_n v]\!] + \frac{\beta}{h_+^3} [\![\overline{u}_\infty]\!] \boldsymbol{n} \cdot [\![v]\!] \boldsymbol{n} \, \mathrm{d}s\end{aligned}$$

as $j \to \infty$. Since $\epsilon > 0$ was arbitrary, the desired convergence

$$\begin{aligned}\int_{\mathcal{F}_{k_j}} \frac{\alpha}{h_{k_j}} [\![\partial_n u_{k_j}]\!] [\![\partial_n v]\!] + \frac{\beta}{h_{k_j}^3} [\![u_{k_j}]\!] \boldsymbol{n} \cdot [\![v]\!] \boldsymbol{n} \, \mathrm{d}s \\ \to \int_{\mathcal{F}_+} \frac{\alpha}{h_+} [\![\partial_n \overline{u}_\infty]\!] [\![\partial_n v]\!] + \frac{\beta}{h_+^3} [\![\overline{u}_\infty]\!] \boldsymbol{n} \cdot [\![v]\!] \boldsymbol{n} \, \mathrm{d}s \qquad \text{as } j \to \infty\end{aligned} \tag{4.3.55}$$

follows from

$$\int_{\mathcal{F}_+ \setminus \mathcal{F}_\ell^+} \frac{\alpha}{h_+} [\![\partial_n \overline{u}_\infty]\!] [\![\partial_n v]\!] + \frac{\beta}{h_+^3} [\![\overline{u}_\infty]\!] \boldsymbol{n} \cdot [\![v]\!] \boldsymbol{n} \, \mathrm{d}s \to 0 \qquad \text{as } \ell \to \infty.$$

[4] Finally, combining (4.3.52), (4.3.54) and (4.3.55), we have proved

$$\begin{aligned}\mathfrak{B}_{k_j}[u_{k_j},\, v] \to &\int_{\Omega^-} D^2\overline{u}_\infty \colon D^2 v \,\mathrm{d}x + \int_{\Omega^+} (D^2_{\mathtt{pw}}\overline{u}_\infty + \mathcal{L}_\infty(\overline{u}_\infty) \colon D^2_{\mathtt{pw}} v \,\mathrm{d}x \\ &+ \int_{\Omega^+} \mathcal{L}_\infty(v) \colon D^2_{\mathtt{pw}}\overline{u}_\infty \,\mathrm{d}x \\ &+ \int_{\mathcal{F}^+} \frac{\alpha}{h_+} [\![\partial_n \overline{u}_\infty]\!]\, [\![\partial_n v]\!] + \frac{\beta}{h_+^3} [\![\overline{u}_\infty]\!]\, \boldsymbol{n} \cdot [\![v]\!]\, \boldsymbol{n} \,\mathrm{d}s \\ = &\ \mathfrak{B}_\infty[\overline{u}_\infty,\, v] \qquad \text{as } j \to \infty.\end{aligned}$$

Hence, by (4.3.50) we have $\overline{u}_\infty = u_\infty$, thanks to $\overline{u}_\infty \in \mathbb{V}_\infty$ and the uniqueness of the generalised Galerkin solution (4.2.9). □

We conclude the section by finally proving Theorem 4.15.

Proof of Theorem 4.15. Using the coercivity of the bilinear form, Corollary 4.28, Lemma 4.32, the interpolation operator $\mathcal{I}_k u_\infty \in \mathbb{V}_k$ and (3.2.1), we observe

$$\begin{aligned}C_{coer} \, |\!|\!| \mathcal{I}_k u_\infty - u_k |\!|\!|_k^2 &\leqslant \mathfrak{B}_k[\mathcal{I}_k u_\infty - u_k,\, \mathcal{I}_k u_\infty - u_k] \\ &= \mathfrak{B}_k[\mathcal{I}_k u_\infty,\, \mathcal{I}_k u_\infty] - 2\mathfrak{B}_k[\mathcal{I}_k u_\infty,\, u_k] + \mathfrak{B}_k[u_k,\, u_k] \\ &= \mathfrak{B}_k[\mathcal{I}_k u_\infty,\, \mathcal{I}_k u_\infty] - 2\,\langle f,\, \mathcal{I}_k u_\infty\rangle_\Omega + \langle f,\, u_k\rangle_\Omega \\ &\to \mathfrak{B}_\infty[u_\infty,\, u_\infty] - \langle f,\, u_\infty\rangle_\Omega = 0 \qquad \text{as } k \to \infty.\end{aligned}$$

Hence, again with Corollary 4.28, we conclude

$$|\!|\!| u_\infty - u_k |\!|\!|_k^2 \leqslant |\!|\!| \mathcal{I}_k u_\infty - u_\infty |\!|\!|_k^2 + |\!|\!| \mathcal{I}_k u_\infty - u_k |\!|\!|_k^2 \to 0$$

as $k \to \infty$. □

5 Numerical Experiments

In the last chapter we proved the convergence of the adaptive Algorithm 4.1. However, this convergence result says nothing about the rates of convergence. Therefore, it is a priori not clear if the adaptive Algorithm 4.1 has any numerical advantages compared to uniform refinement strategies or is even competitive to them.

Based on a numerical example the current chapter adresses this issue. Theorem 3.13 reveals that a lacking Sobolev regularity leads to suboptimal rates of convergence (in view of the polynomial degree) in the case of uniform refinement. However, the optimal rates of convergence can be recovered by using the adaptive Algorithm 4.1 instead of a uniform refinement strategy.

In this regard the following numerical example suggest the advantage of adaptive SIPDGM compared to a non-adaptive method.

5.1 The exact solution

We analyse the perfomance of ASIPDGM for a non-smooth solution u (c.f. [GHV11, Section 5.2]). To this end, let Ω be the L-shaped domain $(-1,1)^2\backslash[0,1)\times(-1,0]$ and set $f=0$. By (r,φ) we denote a system of polar coordinates of $\mathbb{R}^2$ and set

$$u = r^{5/3}\sin(5\varphi/3).$$

For appropriate inhomogeneous Dirichlet boundary conditions u solves the non-homogeneous version of (2.4.2) with right-hand side f (compare [GR86, Section 1.5] for a treatment of the Biharmonic problem with non-homogenous boundary values). We emphasise that $u \in H^{8/3-\epsilon}$, $\epsilon > 0$ due to a corner singularity at the origin of Ω; see [Gri85, Chapter 7].

We apply the ASIPDGM with polynomial degree between $2 \leqslant r \leqslant 5$ and penalty parameters $\alpha = 12.5(r+1)^2$ and $\beta = 2.5(r+1)^6$.

5.2 Uniform refinement.

We use uniform refinements of the mesh. Regarding the specific Sobolev regularity of the solution u, we expect from Theorem 3.13 that (asymptotically) the error $\|u-u_k\|_k$ and the estimator η_k tend to zero with rate $\mathcal{O}(h^{2/3}) = \mathcal{O}(N^{-1/3})$ independent of the polynomial degree r. Here, $N = \#$DOFs is the total number of degrees of freedom. Figure 5.1 confirms the expected (suboptimal) rates of convergence and therefore show the sharpness of the a priori estimate in Theorem 3.13. Moreover, Table 5.1 lists the results of the computations for $r=2$ and $r=3$ in detail. Hence, this numerical example verifies that in the case of uniform refinement the best possible rate of convergence is restricted by 2/3 irrespective of the polynomial degree r. However, in the next section we will see that ASIPDGM improve the rate with respect to $\#\mathcal{T}$.

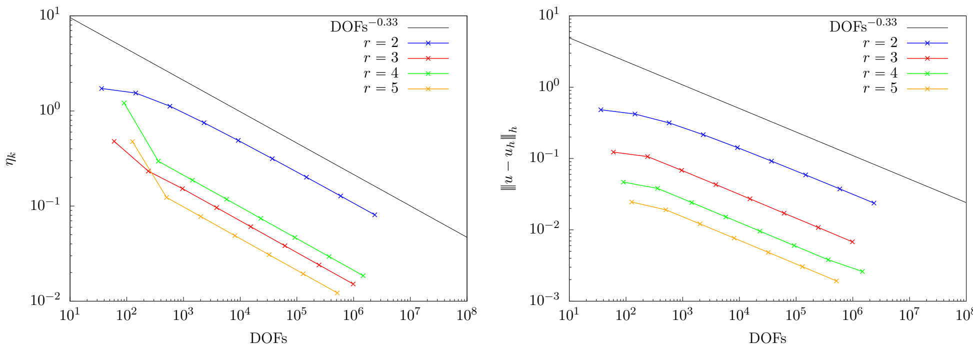


Figure 5.1: Error estimator and error in case of uniform refinements with polynomial degree $2 \leqslant r \leqslant 5$.

h	$r = 2$	$r = 3$
7.07×10^{-1}	0.20	0.21
3.53×10^{-1}	0.41	0.64
1.77×10^{-1}	0.55	0.66
8.84×10^{-2}	0.61	0.66
4.42×10^{-2}	0.63	0.66
2.21×10^{-2}	0.64	0.66
1.10×10^{-2}	0.66	0.66

Table 5.1: Rate of convergence of the error $\|u - u_h\|_h$ in the case of uniform refinements for polynomial degree $r = 2$ and $r = 3$.

5.3 Adaptive refinement.

The adaptive meshes are created by using the *Dörfler Strategy*

$$\eta_{\mathcal{T}}(\mathcal{M}) \geqslant \theta \eta_{\mathcal{T}}(\mathcal{T}) \quad \text{and} \quad \min_{K \in \mathcal{M}} \eta_{\mathcal{T}}(K) \geqslant \max_{K \in \mathcal{T} \setminus \mathcal{M}} \eta_{\mathcal{T}}(K),$$

with $\theta = 0.3$.

- the global error estimator η_k and the error $\|u - u_k\|_k$ as functions of the total number of degrees of freedom (#DOFs) on a log-log scale (*top left and top right*);
- the associated effectivity index $\eta_k / \|u - u_k\|_k$ (*bottom left*);
- an adaptive generated mesh for some iteration levels. (*bottom right*).

Additionally, in Table 5.2 we compare the EOCs (*Experimental orders of convergence*) for different polynomial degrees, which is defined by

$$\mathsf{EOC}_k := -\log\left(\frac{\eta_k(\mathcal{T}_k)}{\eta_{k+1}(\mathcal{T}_{k+1})}\right)/\log\left(\frac{\mathsf{DOFs}_k}{\mathsf{DOFs}_{k+1}}\right), \tag{5.3.1}$$

where DOFs_k are the degrees of freedom related $\mathcal{T}_k$; compare [BNQ$^+$12, Section 5.2]. Table 5.2 lists also the corresponding effectivity indices.

In contrast to uniform refinement we see the optimal rates of convergence (in view of polynomial degree), of the error estimator and the error $\|\!|u - u_{\mathcal{T}}|\!\|_{\mathcal{T}} = \eta_{\mathcal{T}} = \mathcal{O}(N^{-(r-1)/2})$, i.e. $\mathcal{O}(N^{-1/2})$ for $r = 2$, $\mathcal{O}(N^{-1})$ for $r = 3$, $\mathcal{O}(N^{-3/2})$ for $r = 4$ and $\mathcal{O}(N^{-2})$ for $r = 5$, where $N = \#\mathsf{DOFs}$ denotes the total number of degrees of freedom.

For polynomial degree $2 \leqslant r \leqslant 4$ the advantage of adaptive refinements is apparent for $\mathsf{DOFs} > 10^3$ (compare also Table 5.2). The calculations with polynomial degrees $r = 5$ show this beneficial effect for $\mathsf{DOFs} > 10^4$.

The exemplary meshes in Figures 5.2-5.5 show significant refinements in a vicinity of the reentrant corner, due to the singularity of the exact solution, which can be traced back to this reentrant corner. Moreover, we observe that the local refinements near the reentrant corner a much more pronounced for higher polynomial degrees compared to lower polynomial degrees.

The effectivity indices are between 1.0 and 5.0 for all polynomial degrees. Figures 5.2-5.5 only display the results before round-off errors have influences on the numerical results.

level	#DOFs	EOC	effectivity
1	9×10^1	0.26	3.07
3	1.88×10^2	0.28	3.61
5	8.22×10^2	0.33	2.67
7	2.33×10^3	0.49	2.61
9	6.35×10^3	0.48	2.68
11	1.64×10^4	0.49	2.75
13	4.05×10^4	0.49	2.79
15	9.52×10^4	0.49	2.78
17	2.19×10^5	0.50	2.84
19	5.16×10^5	0.50	2.84
21	1.17×10^6	0.49	2.87

Table 5.2: $\mathsf{EOC}s$ and effectivity indices for polynomial degree $r = 2$.

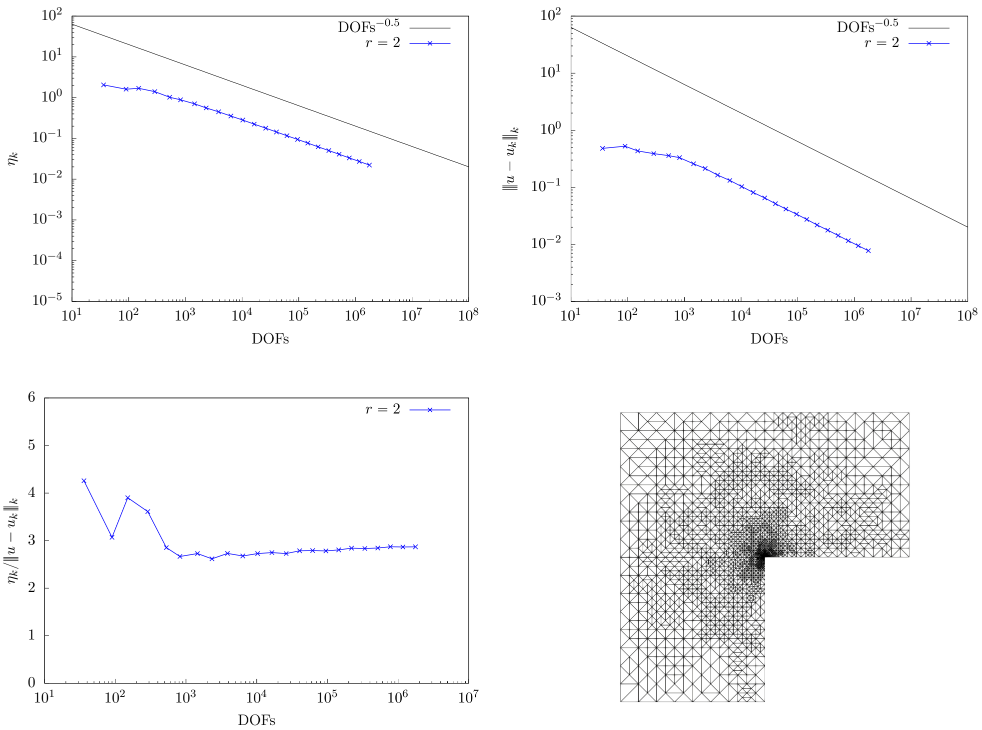


Figure 5.2: Error estimator, error, effectivity index and adaptively created mesh ($k = 12$) for $r = 2$.

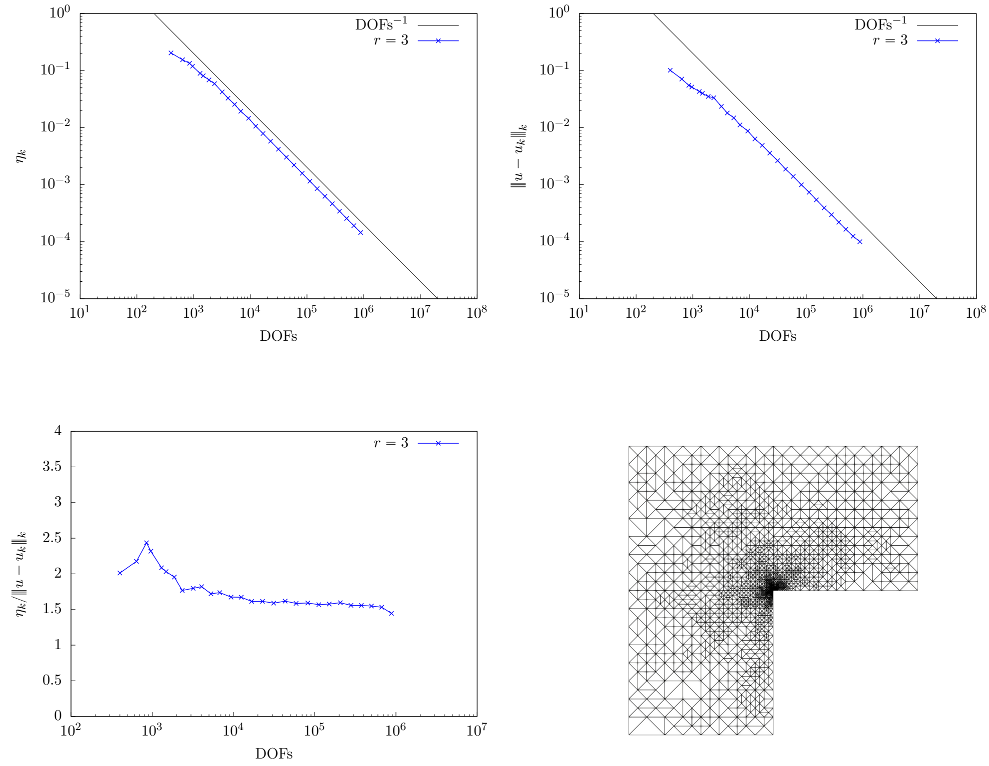
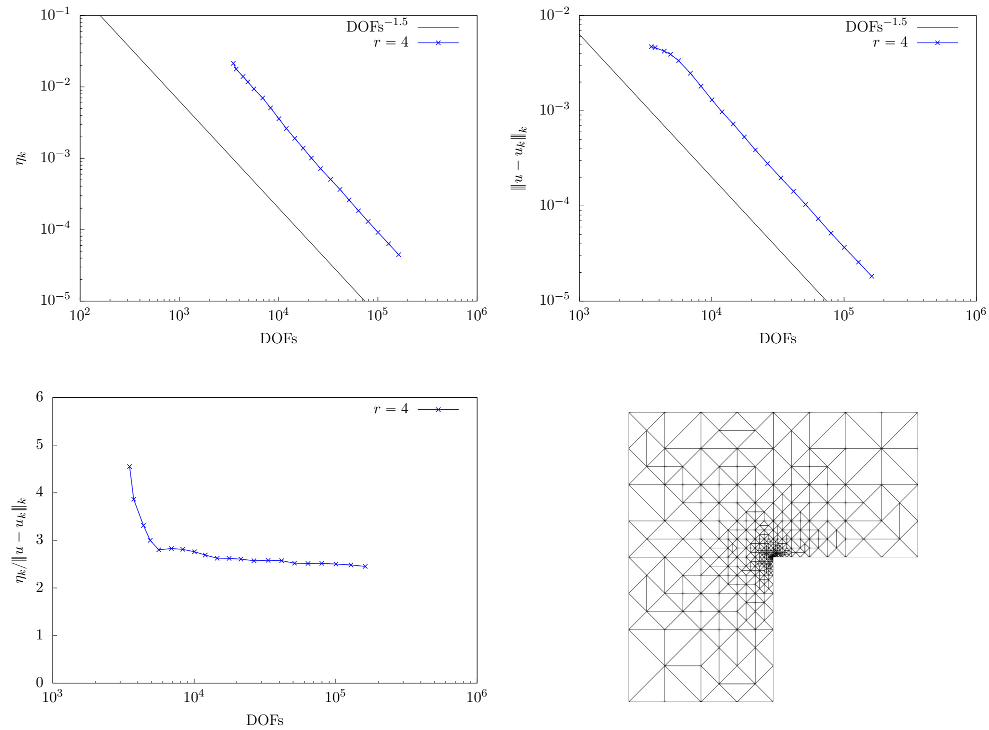


Figure 5.3: Error estimator, error, effectivity index and adaptively created mesh ($k = 20$) for $r = 3$.



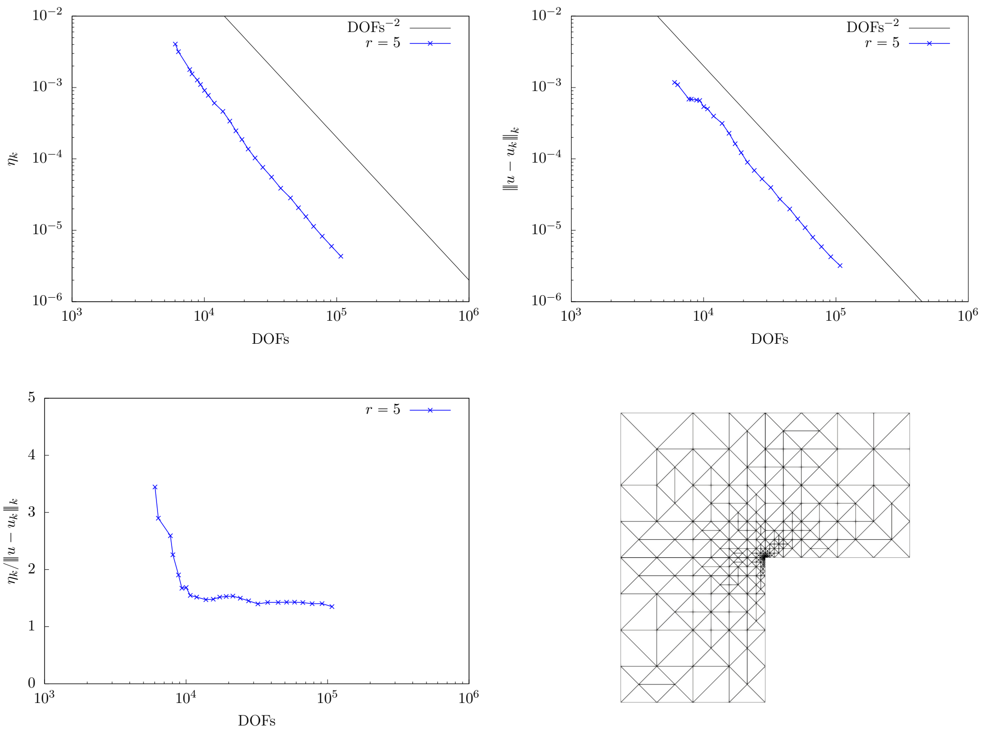


Figure 5.5: Error estimator, error, effectivity index and adaptively created mesh ($k = 30$) for $r = 5$.

6 Summary and outlook

6.1 Summary

In this thesis we generalised the convergence theory of $\mathsf{AC^0IPGM}$ ([DGK19]) to ASIPDGM covering arbitrary polynomial degrees of related discontinuous Galerkin spaces. We developed a new space limit of the discrete space sequences, created by the adaptive loop of (1.1.2). Based on embedding properties of (broken) Sobolev and BV spaces, we proved that the space limit possesses a Hilbert space structure and therefore yields a generalised (weak) Galerkin solution. The convergence of the sequence of DGFEM approximations to the generalised Galerkin solution is actually a consequence of the embedding properties mentioned above. Combing convergence of the sequence of DGFEM approximations to the generalised Galerkin solution with properties of the marking strategy finally yields coincidence of the generalised Galerkin solution and the exact solution. Moreover, numerical experiments confirm the theoretical result and suggest convergence rates as expected.

6.2 Summary and future work

We outline some possible future directions that could arise from the theory presented here:

- Generalisations to linear convergence or even optimal convergence rates of ASIPDGM.
- The convergence theory is not restricted to symmetric problems, which we used here. Therefore, generalisations to non-symmetric problems and different discontinuous Galerkin methods as proposed in [SH18] are possible.
- The development of a posteriori error estimator for non-homogenous problems could lead to ASIPDGM with non-homogenous boundary values. We note that in case of adaptive conforming Galerkin method for second order problems there are convergence results including non-homogenous problems ([MNS03, AFK$^+$13, FPP14]). However, for ASIPDGM including non-homogeneous boundary values this issue is far from solved. Compare also [GHV11, Remark 4.2].
- To ease the presentation we restricted ourself to conforming meshes without hanging nodes. We note that including hanging nodes would complicate the definition of the smoothing operator, defined in chapter 3.4, and we therefore avoided it. However, in view of practical computation, this

is quite restrictive, since discontinuous Galerkin methods naturally allow hanging nodes due to the fact that element base functions are independent of neighbouring elements. Hanging nodes of the mesh can be handled as in [BN10, KP07]. Moreover, generalisations of the convergence theory to polygonal, polyhedral or arbitary-shaped elements are conceivable (compare e.g. [GHH06, CDGH17, CDG19, Don18]).

- Extensions of the results to more general fourth order problems are also possible, compare e.g. [JB12, page 81 ff.], [HL02].

- Generalisations to arbitrary dimension $d > 2$. We note that various C^1-conforming elements for $d > 2$ are available in literature (compare [LS07]). Unfortunately, we used exhaustively that HCT-elements only contain first order derivatives and function evaluations as degrees of freedom and therefore leading to estimates (3.4.2). To the best of our knowledge, deriving equivalent estimates in the presence of second (or even higher) order derivaties as degrees of freedom is still an open question.

Appendices

Appendix A

Theory of measures and Riesz-Radon's Theorem

The theorem of Radon-Riesz states that the dual space of $C_0(\Omega)$ is isomorphic to some space of measures. An application of this theorem motivates the space of functions of bounded variation (Section 3.5), which plays an important role in the context of the convergence theory in Chapter 4. Before we state Riesz-Radon's theorem we give a short introduction of basic notions of measure theory, where we follow the lines of [Alt16, 6.20]. We provide the statements without proofs and refer to [Alt16] for proofs and are more detailed introduction.

Let $U \subset \mathbb{R}^d$ be equipped with the relative topology of $\mathbb{R}^d$, and $\mathcal{B}$ the Borel-σ algebra of U, i.e. the smallest σ-algebra containing the open sets of U. Consider the mappings $\mu\colon \mathcal{B} \to \mathbb{R}^m$ satisfying

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i), \tag{A.1.1}$$

for all sequences $(A_i)_{i=1}^{\infty}$, with $A_i \in \mathcal{B}$ pairwise disjoint. Note that in the case $m = 1$ the mappings are not assumed to be positive. For $\mu\colon \mathcal{B} \to \mathbb{R}^m$ we introduce a mapping $|\mu|\colon \mathcal{B} \to [0,\infty]$ via

$$|\mu|(A) := \sup\left\{\sum_{i=1}^{k} \|\mu(A_i)\|_{\mathbb{R}^m} : k \in \mathbb{N},\ A_i \in \mathcal{B} \text{ pairwise disjoint, } A_i \subset A\right\},$$

called the *variational measure* and we call $\|\mu\| := |\mu|(U)$ the *total variation* of μ. We define the following vector space of *Borel measures*

$$\boldsymbol{M}(U,\mathbb{R}^m) := \{\mu\colon \mathcal{B} \to \mathbb{R}^m\colon \mu \text{ satisfies (A.1.1)},\ \|\mu\| < \infty\},$$

and we simply write $\boldsymbol{M}(U)$ in the case $m = 1$. Moreover, $\boldsymbol{M}(U,\mathbb{R}^m)$ becomes a Banach space, if it is equipped with the variational norm (compare [Alt16, 6.20]).

Unfortunately, the space $\boldsymbol{M}(\Omega)$ is to rich to be isomorphic to $C_0(\Omega)$, therefore we have to restrict ourself to so called *regular* measures. We call $\mu \in \boldsymbol{M}(U,\mathbb{R}^m)$ *regular*, if for all $A \in \mathcal{B}$, we have for its variational measure

$$\begin{aligned}|\mu|(A) &= \sup\{|\mu|(C)\colon C \subset A,\ C \text{ compact}\}\\ &= \inf\{|\mu|(O)\colon A \subset O,\ O \text{ open}\}.\end{aligned}$$

Finally, we define the space of *regular Borel measures*

$$\boldsymbol{MR}(U,\mathbb{R}^m) := \{\mu \in \boldsymbol{M}(U,\mathbb{R}^m)\colon \mu \text{ is a regular measure}\},$$

and note that $\boldsymbol{MR}(U, \mathbb{R}^m)$ is also a Banach space if it is equipped with the total variation as a norm. For the case $m = 1$, we simply write $\boldsymbol{MR}(U)$ instead of $\boldsymbol{MR}(U, \mathbb{R})$. A measure $\mu \in \boldsymbol{MR}(U)$, is also called a *signed* measure, since we allow it to have negative values. For the case $m > 1$ we have $\mu = (\mu_1, \dots, \mu_m)$ and μ_i is a signed measure for all $i = 1, \dots, m$.

Remark A.1 (Radon measure)**.** *We emphasise that some notions of measure theory slightly differ in literature, e.g. in [EG15, Definition 1.9] a measure μ on a set X is regular if for each set $A \subseteq X$ there exists a μ-measurable set B such that $A \subseteq B$ abd $\mu(A) = \mu(B)$*

Observing that continuous functions are measurable with respect to regular Borel measure, we get the following theorem. Note that for simplicity we restrict ourself to real valued measures, i.e. $m = 1$.

Theorem A.2 (Riesz-Radon theorem (Dual space of C_0))**.** *Let $K \subset \mathbb{R}^d$ be compact. Then, every bounded linear functional $L\colon C_0(K) \to \mathbb{R}$ is represented uniquely by a regular Borel measure $\nu \in \boldsymbol{MR}(K)$ such that*

$$L(f) := \int_K f \, \mathrm{d}\nu \quad \forall f \in C_0(K).$$

Moreover, we have $\|L\| = \|\nu\|$.

Proof. Compare e.g. [Alt16, Section 6.23], [ABM14, Theorem 2.4.6] or[AFP00, Theorem 1.54]. □

By using the Riesz-Radon theorem we can provide a distributional characterisation of regular measures (compare e.g. [Alt16, 6.24 Corollary]). To this end we assume $\Omega \subset \mathbb{R}^d$ is open and bounded and let $C > 0$ such that the linear map $T\colon C_0(\Omega) \to \mathbb{R}$ satisfies

$$|T(f)| \leqslant C \|f\|_\infty \quad \forall f \in C_0(\Omega).$$

Then, there exists a unique $\nu \in \boldsymbol{MR}(\Omega)$ satisfying

$$\|\nu\| = \sup \{|T(f)| : f \in C_0(\Omega),\ \|f\|_\infty = 1\} \leqslant C$$

and

$$T(f) = \int_\Omega f \, \mathrm{d}\nu \quad \forall f \in C_0(\Omega).$$

Remark A.3. *Note that by a convolution argument it suffices to assume that*

$$T \in \mathcal{D}'(\Omega) \quad \text{with} \quad |T(\varphi)| \leqslant C \|\varphi\|_\infty \quad \forall \varphi \in C_0^\infty(\Omega),$$

since T can be uniquely extended to a linear map on $C_0(\Omega)$, which satisfies the above estimate.

We emphasise that $\boldsymbol{MR}(\Omega, \mathbb{R}^m)$ is isomorphic to the product space $[\boldsymbol{MR}(\Omega)]^m$ and that we have

$$\mu \in \boldsymbol{MR}(\Omega, \mathbb{R}^m) \iff \mu = (\mu_1, \ldots, \mu_m) \text{ and } \mu_i \in C_0(\Omega)' \quad \forall i = 1, \ldots, m.$$

Consequently, the dual of $C_0(\Omega, \mathbb{R}^m)$ can be identified with $\boldsymbol{MR}(\Omega, \mathbb{R}^m)$. From the definition of weak*-convergence we have, that $\{\mu_k\}_{k \in \mathbb{N}} \subset \boldsymbol{MR}(\Omega, \mathbb{R}^m)$ converges weakly* to μ ($\mu_k \overset{*}{\rightharpoonup} \mu$ in $C_0(\Omega, \mathbb{R}^m)'$) if

$$\lim_{k \to \infty} \int_\Omega \phi \cdot \mathrm{d}\mu_k = \int_\Omega \phi \cdot \mathrm{d}\mu \quad \forall \phi \in C_0(\Omega, \mathbb{R}^m).$$

Next, we prove that on the space $\boldsymbol{MR}(\Omega, \mathbb{R}^m)$ we have a weak*-compactness property, which is a consequence of Theorem of Alaoglu (see [Alt16, 8.7(3)] and the above identification of the dual of $[C_0(\Omega)]^m$ and $\boldsymbol{MR}(\Omega, \mathbb{R}^m)$. However, we give a proof, which is based on rather basic properties of measure spaces and gives a more detailed insight into this compactness property.

Theorem A.4 (Weak* compactness)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded and let $\{\mu_k\}_{k \in \mathbb{N}}$ be a sequence in $\boldsymbol{MR}(\Omega, \mathbb{R}^m)$ satisfying* $\sup\{|\mu_k|(\Omega) \colon k \in \mathbb{N}\} < \infty$. *Then, it has a weakly* converging subsequence in $\boldsymbol{MR}(\Omega, \mathbb{R}^m)$.*

Proof. Without loss of generality we assume that $|\mu_k|(\Omega) \leqslant 1$ for all $k \in \mathbb{N}$. Let $\{c_\ell\}_{\ell \in \mathbb{N}} \subset [C_0(\Omega)]^m$ be a sequence, from which we assume, that $\|c_\ell\|_\infty \leqslant 1$ for all $\ell \in \mathbb{N}$ and whose linear span L is dense $[C_0(\Omega)]^m$. We note that such a sequence exists, due to the separability of $C_0(\Omega)$. We will use the following notation: For a measure $\mu \in \boldsymbol{MR}(\Omega, \mathbb{R}^m)$ and a function $c \in [C_0(\Omega)]^m$ we write

$$\langle \mu, c \rangle = \int_\Omega c \cdot \mathrm{d}\mu = \mu(c)$$

as a shorthand notation for the duality bracket.

By the properties of $\{\mu_k\}_{k \in \mathbb{N}}$ and $\{c_\ell\}_{\ell \in \mathbb{N}}$, we have

$$\mu_k(c_1) = \int_\Omega c_1 \cdot \mathrm{d}\mu_k \leqslant |\mu_k|(\Omega)\, \|c_1\|_\infty \leqslant 1.$$

Hence, there exists $a_1 \in \mathbb{R}$ and a subsequence $\left\{k_j^1\right\}_{j \in \mathbb{N}} \subset \mathbb{N}$ such that $\mu_{k_j^1}(c_1) \to a_1$ as $j \to \infty$ and

$$\left|\mu_{k_j^1}(c_1) - a_1\right| < \frac{1}{j} \quad \forall j \geqslant 1.$$

Using an inductive argument we observe that for $\mu_{k_j^n} \subset \boldsymbol{MR}(\Omega, \mathbb{R}^m)$ there exists a subsequence such that $\mu_{k_j^{n+1}} \to a_{n+1}$ as $j \to \infty$ and

$$\left|\mu_{k_j^{n+1}}(c_{n_1}) - a_{n+1}\right| < \frac{1}{j}.$$

Consequently, for $\mu_{k_j^j}$ and $j > n$ we have

$$\left|\mu_{k_j^j}(c_n) - a_n\right| < \frac{1}{j}$$

and therefore for all $n \in \mathbb{N}$ we have

$$\lim_{j\to\infty} \mu_{k_j^j}(c_n) = a_n.$$

Note, that by the properties of $\{c_\ell\}_{\ell\in\mathbb{N}}$ this limit exists in the whole of L. We define μ on $\boldsymbol{MR}(\Omega, \mathbb{R}^m)$ by $\mu(c_n) = a_n$ and observe as above

$$\mu(c_n) = \lim_{j\to\infty} \mu_{k_j^j}(c_n) \leqslant |\mu_{k_j^j}|(\Omega) \, \|c_n\|_\infty \leqslant 1.$$

Hence, μ is a bounded linear functional and can be extended to the whole of $[C_0(\Omega)]^m$ by density.

To prove that μ is the weak*-limit of $\mu_{k^j(j)}$ we have to prove that for all $c \in [C_0(\Omega)]^m$ we have

$$\lim_{j\to\infty} \mu_{k_j^j}(c) = \mu(c).$$

To this end, let $\epsilon > 0$ be arbitrary and $d \in L$ such that $\|c - d\|_\infty < \epsilon$. Then, by the triangle inequality we conclude

$$\begin{aligned} \left|\mu_{k_j^j}(c) - \mu(c)\right| &\leqslant \left|\mu_{k_j^j}(c - d)\right| + \left|\mu_{k_j^j}(d) - \mu(d)\right| + |\mu(c - d)| \\ &\leqslant 2\,\|c - d\|_\infty + \left|\mu_{k_j^j}(d) - \mu(d)\right| < 3\epsilon, \end{aligned}$$

provided j large enough. Consequently, since ϵ was arbitrary, we proved that $\mu_{k_j^j}(c)$ weakly* converges to μ. □

Appendix B

Bubble functions

In order to unserstand the understand the various concepts of Chapter 3.4.3 (lower bounds of the error indicators), we recall some definitions of a posteriori analysis (compare also [Ver13]).

Let $\hat{K} = \{(x, y) \colon 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x\} \subset \mathbb{R}^2$ be the reference triangle with vertices $\hat{z}_0$, $\hat{z}_1$, $\hat{z}_2$. By $\hat{\lambda}_0$, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ we denote the barycentric coordinates on $\hat{K}$, i.e. $\hat{\lambda}_i(\hat{z}_j) = \delta_{ij}$ for $0 \leqslant i, j \leqslant 2$.

We note that the interior *bubble function* on $\hat{K}$ is defined by

$$\hat{\psi}_{\hat{K}} := 27\hat{\lambda}_0\hat{\lambda}_1\hat{\lambda}_2.$$

Now, let $K \in \mathcal{T}$ be an arbitrary element and $F_K \colon \hat{K} \to K$ be an invertible and affine linear mapping. Then, the associated bubble function on K is defined by

$$\psi_K := \hat{\psi}_{\hat{K}} \circ F_K^{-1}.$$

We extend ψ_K by zero to the whole domain Ω and obtain a piecewise polynomial which is globally continuous and therefore located in $W^{1,\infty}(\Omega)$ but *not* in $W^{2,1}(\Omega)$. In order to derive lower bounds of the local error indicators for fourth order problems, we need to construct local bubble functions of class $C^1(\Omega)$, i.e. bubble functions located in $W^{2,\infty}(\Omega)$. To this end, on K we define

$$b_K = (\psi_K)^2.$$

By construction we have that b_K als well as the first derivative of b_K vanishes on the boundary of K. Consequently, $b_K \in H_0^2(K)$ and by extending b_K by zero on $\Omega \backslash K$ we also have $b_K \in H_0^2(\Omega)$ (compare figure B.3(b)).

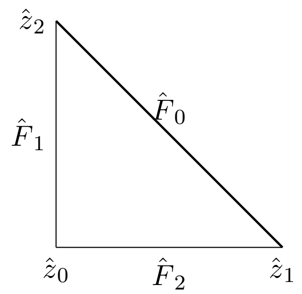


Figure B.1: Reference triangle $\hat{K}$.

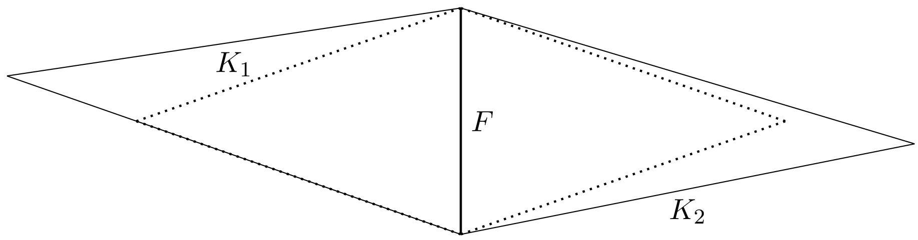


Figure B.2: Rhombus $\tilde{K}$ contained in $K_1 \cup K_2$ with common edge F.

In order to extend the above idea to face bubble functions we consider the reference triangle $\hat{K} \subset \mathbb{R}^2$ with one dimensional faces $\hat{F}_0, \hat{F}_1, \hat{F}_2$. Simple calculations reveal that the buble function on $\hat{F}_0$ (compare figure B.1) is defined by

$$\hat{\psi}_{\hat{F}_0} := 4\hat{\lambda}_1\hat{\lambda}_2$$

and the remaining bubble functions on $\hat{F}_1$ and $\hat{F}_2$ are then defined analogously. Now, let $K \in \mathcal{T}$ be arbitrary with $F \subset K$. Then, the associated face bubble function on $F \subset K$ is defined by

$$\widetilde{\psi}_F^K := \hat{\psi}_{\hat{F}} \circ F_K^{-1}.$$

Since $\widetilde{\psi}_F^K$ is a local polynomial, only defined on the element $K \supset F$, we extend the face bubble function to the domain $\omega_{\mathcal{T}}(F)$ via

$$\psi_F = \begin{cases} \widetilde{\psi}_F^K & \text{in } K \\ \widetilde{\psi}_F^{K'} & \text{in } K' \\ 0 & \text{in } \mathbb{R}^2 \backslash \omega_{\mathcal{T}}(F), \end{cases}$$

where the neighbouring element $K' \in \mathcal{T}$ is chosen such that $K' \cap K = F$.

Note that the face bubble functions are piecewise polynomials on each element and therefore the function ψ_F is not differentiable across inter-element faces (compare figure B.3(c)). Hence, we observe that the above simple device does not suffice in this case.

To overcome this issue, fix $F \in \mathring{\mathcal{F}}$, $F = K_1 \cap K_2$ and let $\tilde{K}$ be the largest rhombus contained in $K_1 \cup K_2$, that has F as one diagonal (compare figure B.2).

Regarding $\tilde{K}$ as a quadrilateral in $\mathbb{R}^2$ we are able to construct a bubble function $b_{\tilde{K}}$ on $\tilde{K}$ as follows: We define a reference quadrilateral

$$\hat{K} = \{(x, y) \colon -1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1\}$$

together with an invertible mapping $F_K \colon \hat{K} \to \tilde{K}$ and define an interior bubble function on $\hat{K}$ via

$$\hat{\psi} = (1 - x^2)(1 - y^2).$$

In an analogous fashion as above we define

$$\psi_{\tilde{K}} := \hat{\psi} \circ F_{\tilde{K}}^{-1}$$

to be the (continuous) bubble function on $\tilde{K} \subset \mathbb{R}^2$.

Finally, for $m \in \mathbb{N}_0$ we define a smooth bubble function on the rhombus $\tilde{K}$ via

$$b_{\tilde{K}} = (\psi_{\tilde{K}})^{m+1}.$$

Note that $b_{\tilde{K}}$ together with all its derivatives up to order m vanishes on the boundary of $\tilde{K}$. Consequently, $b_{\tilde{K}}$ is contained in $C^m(\Omega)$ and in particular in $W^{m+1,p}(\Omega)$ for every Lebesgue exponent p. Moreover, $b_{\tilde{K}}$ is positive on the interior of F and vanishes on $\Omega\backslash\tilde{K}$. Hence, we can use it as a smooth face bubble function in the proof of the lower bounds of the error indicators on element faces (see section 3.4.3)

Remark B.1. *It is also possbile to define face bubble function on the whole patch $\omega_{\mathcal{T}}(F)$ by extending the local polynomials $\widetilde{\psi}_F^K$ to global polynomials defined on the whole domain $\mathbb{R}^2$. Compare [Ver13, Section 3.2.5] for details and also figures B.3(c) and B.3(d) for an example.*

The following Lemma goes back to [Vir10, Lemma B.1].

Lemma B.2. *Let $\mathcal{T}$ be a triangulations of Ω and $K \subset \mathcal{T}$. Then, we have for any fixed $\ell \in \mathbb{N}_0$ and $m \in \mathbb{N}$*

$$\|v\|_K^2 \lesssim \int_K \psi_K^m v^2 \,\mathrm{d}x \lesssim \|v\|_K^2, \tag{B.1.1}$$

for all $v \in \mathbb{P}_\ell(K)$, $\ell \in \mathbb{N}_0$. Here, ψ_K denotes the (continuous) interior bubble function on K and the constants in $'\lesssim'$ are independent of K and the mesh-size $h_{\mathcal{T}}$.

Moreover, let $F \in \mathring{\mathcal{F}}$ such that $F = K_+ \cup K_-$ and $K_+, K_- \in \mathcal{T}$. Let $\tilde{K} \subset K_1 \cup K_2$ be an quadrilateral, contained in $K_1 \cup K_2$ having F as one diagonal. Then, for any fixed $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$ we have that

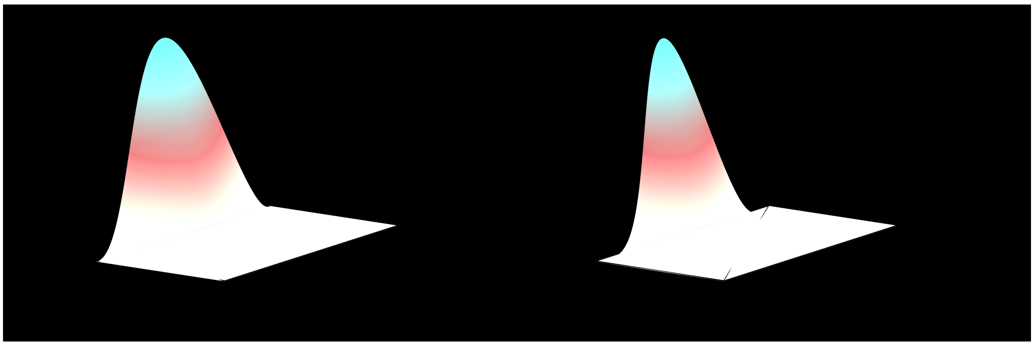
$$\|v\|_F^2 \lesssim \int_F \psi_{\tilde{K}}^n v^2 \,\mathrm{d}x \lesssim \|v\|_F^2, \tag{B.1.2}$$

for all $v \in \mathbb{P}_j(F)$, $j \in \mathbb{N}_0$. Here, $\psi_{\tilde{K}}$ denotes the (continuous) interior bubble function on $\tilde{K}$. and the constants in $'\lesssim'$ are independent of F the mesh-size $h_{\mathcal{T}}$.

Proof. Fix $\ell \in \mathbb{N}_0$, let $K \in \mathcal{T}$ and $u, v \in \mathbb{P}_\ell(K)$. Using the reference mapping $F_K \colon \hat{K} \to K$ and the definition of $P_\ell(K)$ we have, that $\hat{u}, \hat{v} \in \mathbb{P}_\ell(\hat{K})$ with $\hat{u}(\hat{x}) = u(F_K(\hat{x}))$ and $\hat{v}(\hat{x}) = v(F_K(\hat{x}))$ for all $\hat{x} \in \hat{K}$.

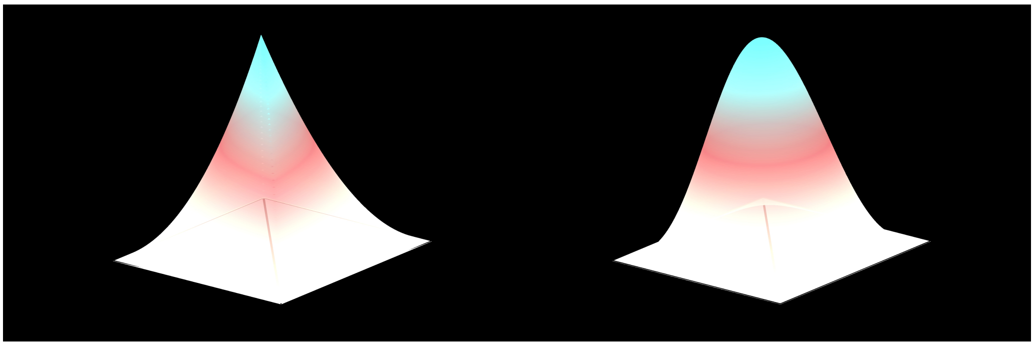
We fix $m \in \mathbb{N}$ and prove that

$$\hat{v} \mapsto \left(\int_{\hat{K}} \hat{\psi}_{\hat{K}}^m \hat{v}^2 \,\mathrm{d}\hat{x}\right)^{1/2} =: \varphi(\hat{v})$$



(a) Continuous element bubble function on the left element

(b) Smooth element bubble function on the left element



(c) Continuous face bubble function on an intersection of elements

(d) Smooth face bubble function on an intersection of elements

Figure B.3: Example of various bubble functions defines a morm on $\mathbb{P}_\ell(\hat{K})$. Since $\hat{\psi}_{\hat{K}}$ is non-negative on $\hat{K}$ we have also that $\varphi(\hat{v}) \geqslant 0$. Moreover, if $\varphi(\hat{v}) = 0$, then $\hat{\psi}_{\hat{K}} > 0$ in the interior of $\hat{K}$ implies $\hat{u} = 0$ almost everywhere in $\hat{K}$. Now let $\alpha \in \mathbb{R}$, then

$$\varphi(\alpha\hat{v}) = \left(\int_{\hat{K}} \hat{\psi}_{\hat{K}}^m \hat{\alpha}^2 v^2 \,\mathrm{d}\hat{x}\right)^{1/2} = |\alpha|\,\varphi(v)$$

and additionally we have

$$\begin{aligned}
\varphi(\hat{u}+\hat{v})^2 &= \int_{\hat{K}} \hat{\psi}_{\hat{K}}^m (\hat{u}+\hat{v})^2 \,\mathrm{d}\hat{x} \\
&= \int_{\hat{K}} \hat{\psi}_{\hat{K}}^m \hat{u}^2 \,\mathrm{d}\hat{x} + \int_{\hat{K}} \hat{\psi}_{\hat{K}}^m \hat{v}^2 \,\mathrm{d}\hat{x} + 2\int_{\hat{K}} \hat{\psi}_{\hat{K}}^m \hat{u}\hat{v} \,\mathrm{d}\hat{x} \\
&= \varphi(\hat{u})^2 + \varphi(\hat{v})^2 + 2\int_{\hat{K}} \hat{\psi}_{\hat{K}}^m \hat{u}\hat{v} \,\mathrm{d}\hat{x} \\
&\leqslant \varphi(\hat{u})^2 + \varphi(\hat{v})^2 + 2\varphi(\hat{u})\varphi(\hat{v}) = (\varphi(\hat{u}) + \varphi(\hat{v}))^2.
\end{aligned}$$

By taking the square root on both sides in the last estimate, we deduce that $\varphi(\cdot)$ also satisfies the triangle inequality and in particular defines a norm on the finite dimensional space $\mathbb{P}_\ell(\hat{K})$. Since $\|v\|_{\hat{K}}$ also defines a norm on $\mathbb{P}_\ell(\hat{K})$, we have from equivalence of norms on finite dimensional spaces, that there exist constants $\hat{C}_1, \hat{C}_2 > 0$ (depending on the refenrence element $\hat{K}$)

$$\hat{C}_1 \|\hat{v}\|_{\hat{K}}^2 \leqslant \int_{\hat{K}} \hat{\psi}_{\hat{K}}^m \hat{v}^2 \,\mathrm{d}\hat{x} \leqslant \hat{C}_2 \|\hat{v}\|_{\hat{K}}^2 .$$

Using the transformation formula for integrals we get

$$\|v\|_K^2 = \int_K v^2 \,\mathrm{d}x = \int_{\hat{K}} \hat{v}^2 \left|\det DF_K\right| \,\mathrm{d}\hat{x}$$

and

$$\varphi(v)^2 = \int_K \psi_K^m v^2 \,\mathrm{d}x = \int_{\hat{K}} \hat{\psi}_{\hat{K}}^m \hat{v}^2 \left|\det DF_K\right| \,\mathrm{d}\hat{x}.$$

Hence, assumption (3.1.2) together with the estimates above implies

$$\tilde{C}_1 \|v\|_K^2 \leqslant \int_K \psi_K^m v^2 \,\mathrm{d}x \leqslant \tilde{C}_2 \|v\|_K^2 ,$$

with $\tilde{C}_1 = \frac{C_2\hat{C}_1}{C_1 C_{\mathsf{reg}}}$ and $\tilde{C}_2 = \frac{C_1}{C_2}\hat{C}_2 C_{\mathsf{reg}}$. The proof of (B.1.2) follows by similar arguments. □

Appendix C

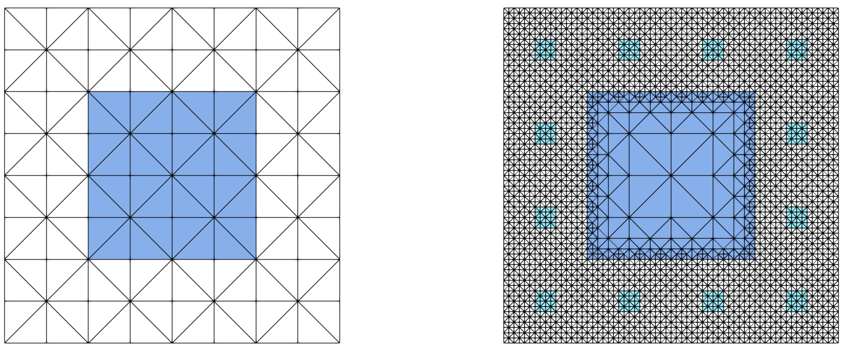
A Sequence of triangulations based on Cantor sets

In a previous version of [DGK19] (compare with [DGK19a]), we resorted to a simpler convergence proof for the adaptive method Algorithm 4.1 making use of the condition

$$\overline{\Omega} = \overline{\mathrm{interior}(\Omega^-) \cup \Omega^+}; \tag{C.1.1}$$

see also [KG17] and [KG20]. During the review process of [DGK19], one of the anonymous referees proved that (C.1.1) is wrong in genreal by means of an elaborate counterexample, which we present in full detail to the anonymous referee's credit.

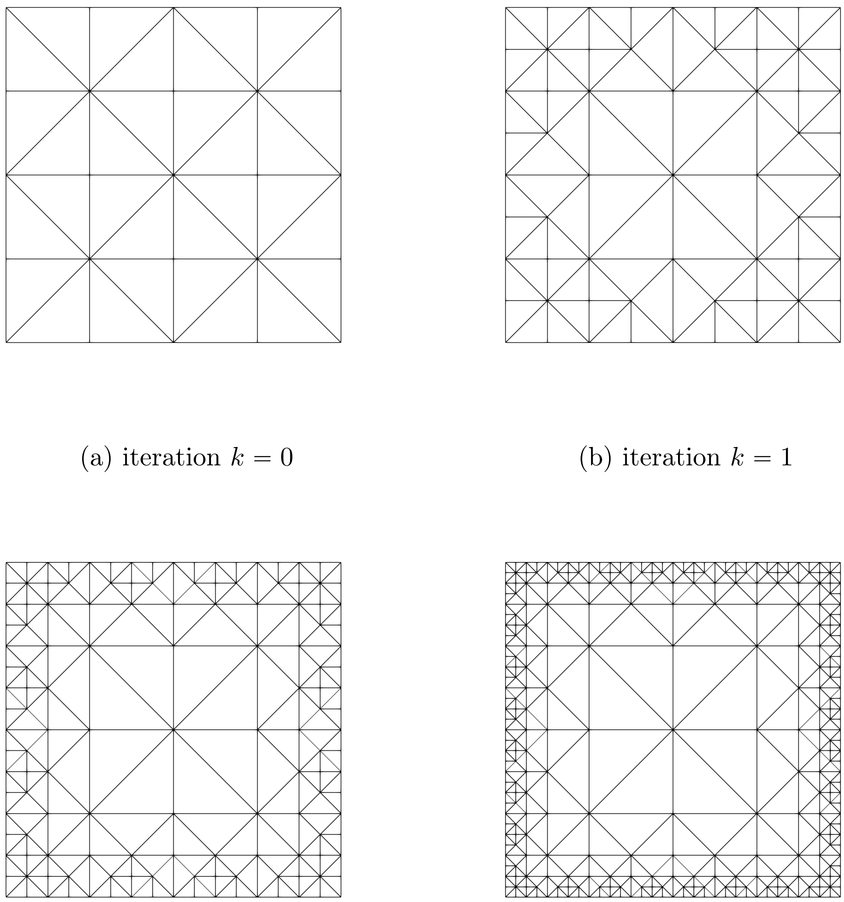
Based on the idea of Cantor sets, a sequence of refinements is constructed, such that $|\Omega^+| < |\Omega|$ and $\mathrm{interior}(\Omega^-) = \varnothing$, which clearly contradicts (C.1.1).



(a) $\mathcal{T}_0$ (b) $\mathcal{T}_1$

Figure C.1: Triangulations $\mathcal{T}_0$ and $\mathcal{T}_1$ with atoms of level 0 (resp. 1) which are shaded in dark blue (resp. light blue).

We partition the unit square $\Omega = (0,1)^2$ into 4^2 equal-sized squares, each of which is again meshed by a criss-cross triangulation. This is the initial triangulation as depicted in Figure C.1(a). The four criss-cross squares in the center of Ω (shaded in dark blue) will be called *atom*. Since we are exclusively dealing with right-angled isosceles triangles, newest vertex bisection corresponds to longest edge refinement. The mesh $\mathcal{T}_1$ is then constructed by partitioning each of the 12 non-atomic criss-cross squares into 8^2 equal-sized squares each of which is again meshed by a criss-cross triangulation. The atom will be only further refined in order to ensure conformity and it gets clear from Figure C.2, that eventually the whole interior of the atom will belong to Ω^+. Again, the center four criss-cross squares in each of the non-atomic criss-cross squares from $\mathcal{T}_0$ will be atoms of level 1 (shaded in light blue in Figure C.1(b)) and not marked for refinement any more.



(a) iteration $k = 0$ (b) iteration $k = 1$

(c) iteration $k = 2$ (d) iteration $k = 3$

Figure C.2: Atomic refinements.

This construction is now continued recursively, i.e. $\mathcal{T}_i$ is created from $\mathcal{T}_{i-1}$, by splitting each non-atomic criss-cross square into $(2^{2+i})^2$ criss-cross squares, performing necessary refinements due to conformity and taking the four center criss-cross squares in each criss-cross square of $\mathcal{T}_{i-1}$ as new atoms of level i.

In each triangulation $\mathcal{T}_i$ there are thus created less that

$$\prod_{j=0}^{i-1} \left(2^{2+j}\right)^2$$

new atoms (we neglect that no new atoms are created inside lower level atoms) of size

$$\frac{4}{\prod_{j=0}^{i} \left(2^{2+j}\right)^2}.$$

Therefore, the union of all atoms of a fixed level $i \geqslant 1$ occupy an area of size

$$\frac{4\prod_{j=0}^{i-1} \left(2^{2+j}\right)^2}{\prod_{j=0}^{i} \left(2^{2+j}\right)} = 4 \cdot 2^{-2(2+i)} = 2^{-2(i+1)}.$$

The set of Ω^+ of the constructed sequence of meshes consists of the union of the interiors of the atoms. Recalling, that the atom of level 0 has size $1/4 = 2^{-2(0+1)}$, we conclude that

$$\left|\Omega^+\right| \leqslant \sum_{i=0}^{\infty} 2^{-2(i+1)} = \frac{1}{3} < 1. \tag{C.1.2}$$

From any point $x \in \Omega$ the distance to the closest atom of level $\leqslant i$ is bounded by the diameter of the smallest criss-cross squares in $\mathcal{T}_{i-1}$, which is $\sqrt{2} \cdot 2^{-(i+1)}$. Therefore, we have that Ω^+ is dense in Ω and

$$\text{interior}(\Omega^-) = \text{interior}(\Omega \backslash \Omega^+) = \varnothing.$$

Together with (C.1.2) this contradicts (C.1.1).

Bibliography

[ABCM02] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal. **39** (2001/02), no. 5, 1749–1779. MR 1885715

[ABCM00] D. N Arnold, F. Brezzi, B. Cockburn, and D. Marini, *Discontinuous galerkin methods for elliptic problems*, Discontinuous Galerkin Methods, Springer, 2000, pp. 89–101.

[ABM14] H. Attouch, G. Buttazzo, and G. Michaille, *Variational analysis in Sobolev and BV spaces*, second ed., MOS-SIAM Series on Optimization, vol. 17, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, 2014, Applications to PDEs and optimization. MR 3288271

[AF03] R. A. Adams and J. J. F Fournier, *Sobolev spaces*, vol. 140, Elsevier, 2003.

[AFK$^+$13] M. Aurada, M. Feischl, J. Kemetmüller, M. Page, and D. Praetorius, *Each $H^{1/2}$-stable projection yields convergence and quasi–optimality of adaptive FEM with inhomogeneous dirichlet data in $\mathbb{R}^d$*, ESAIM: Mathematical Modelling and Numerical Analysis **47** (2013), no. 4, 1207–1235.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[AFS68] J. H Argyris, I. Fried, and D. W. Scharpf, *The TUBA family of plate elements for the matrix displacement method*, Aero J. Roy. Aero. Soc. **72** (1968), 701–709.

[Alt16] H. W. Alt, *Linear functional analysis: An application-oriented introduction*, Springer, 2016.

[Bae91] E. Bänsch, *Local mesh refinement in 2 and 3 dimensions*, Impact Comput. Sci. Engrg. **3** (1991), no. 3, 181–191. MR 1141298

[Bak77] G. A. Baker, *Finite element methods for elliptic equations using nonconforming elements*, Mathematics of Computation **31** (1977), no. 137, 45–59.

[Bar01] R. G. Bartle, *A modern theory of integration*, vol. 32, American Mathematical Soc., 2001.

[Bar14] R. G. Bartle, *The elements of integration and lebesgue measure*, John Wiley & Sons, 2014.

[BBF13] D. Boffi, F. Brezzi, and M. Fortin, *Mixed finite element methods and applications*, Springer Series in Computational Mathematics, vol. 44, Springer, Heidelberg, 2013. MR 3097958

[BC+11] H. H. Bauschke, P. L. Combettes, et al., *Convex analysis and monotone operator theory in Hilbert spaces*, vol. 408, Springer, 2011.

[BCI65] G. P. Bazeley, Y. K. Cheung, and B. M. Irons, *0. c. zienkiewicz. triangular elements in bending conforming and nonconforming solutions*, Proc, 1965.

[BGS10] S. C. Brenner, T. Gudi, and L.-Y. Sung, *An a posteriori error estimator for a quadratic C^0-interior penalty method for the Biharmonic problem*, IMA J. Numer. Anal. **30** (2010), no. 3, 777–798.

[BN10] A. Bonito and R. H. Nochetto, *Quasi-optimal convergence rate of an adaptive discontinuous Galerkin method*, SIAM J. Numer. Anal. **48** (2010), no. 2, 734–771.

[BNQ+12] S. Bertoluzza, R. H. Nochetto, A. Quarteroni, K. G. Siebert, and A. Veeser, *Multiscale and adaptivity: Modeling, numerics and applications: Cime summer school, cetraro, italy 2009*, vol. 2040, Springer, 2012.

[BO09] A. Buffa and C. Ortner, *Compact embeddings of broken Sobolev spaces and applications*, IMA J. Numer. Anal. **29** (2009), no. 4, 827–855.

[BRL80] H. Blum, R. Rannacher and R. Leis, *On the boundary value problem of the Biharmonic operator on domains with angular corners*, Mathematical Methods in the Applied Sciences **2** (1980), no. 4, 556–581.

[BS05] S. C. Brenner and L.-Y. Sung, *C^0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains*, J. Sci. Comput. **22/23** (2005), 83–118.

[BWZ04] S. C. Brenner, K. Wang, and J. Zhao, *Poincaré-Friedrichs inequalities for piecewise H^2 functions*, Numer. Funct. Anal. Optim. **25** (2004), no. 5-6, 463–478. MR 2106270

[CDG19] A. Cangiani, Z. Dong, and E. H Georgoulis, *hp-version discontinuous Galerkin methods on essentially arbitrarily-shaped elements*, arXiv preprint arXiv:1906.01715 (2019).

[CDGH17] A. Cangiani, Z. Dong, E. H. Georgoulis, and P. Houston, *hp-version discontinuous Galerkin methods on polygonal and polyhedral meshes*, Springer, 2017.

[Cia74] P. G. Ciarlet, *Sur l'élément de Clough et Tocher*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge **8** (1974), no. R-2, 19–27. MR 0381349

[Cia02a] P. G. Ciarlet, *The finite element method for elliptic problems*, vol. 40, Siam, 2002.

[Cia02b] P. G. Ciarlet, *The finite element method for elliptic problems*, Classics in Applied Mathematics, vol. 40, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002, Reprint of the 1978 original [North-Holland, Amsterdam; MR0520174 (58 #25001)]. MR 1930132

[CKNS08] J. M. Cascon, C. Kreuzer, R. H. Nochetto, and K. G. Siebert, *Quasi-optimal convergence rate for an adaptive finite element method*, SIAM J. Numer. Anal. **46** (2008), no. 5, 2524–2550. MR 2421046

[Cle75] Ph. Clement, *Approximation by finite element functions using local regularization*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. **9** (1975), no. R-2, 77–84. MR 0400739

[Doe96] W. Dörfler, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal. **33** (1996), no. 3, 1106–1124. MR 1393904

[Dau06] M. Dauge, *Elliptic boundary value problems on corner domains: smoothness and asymptotics of solutions*, vol. 1341, Springer, 2006.

[dB74] C. de Boor, *Mathematical aspects of finite elements in partial differential equations*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1974, Publication No. 33 of the Mathematics Research Center, The University of Wisconsin-Madison. MR 0349031

[dBD83] C. de Boor and R. DeVore, *Approximation by smooth multivariate splines*, Trans. Amer. Math. Soc. **276** (1983), no. 2, 775–788. MR 688977

[DDPS79] J. Douglas, Jr., T. Dupont, P. Percell, and R. Scott, *A family of C^1 finite elements with optimal approximation properties for various Galerkin methods for 2nd and 4th order problems*, RAIRO Anal. Numér. **13** (1979), no. 3, 227–255.

[DG12] A. Demlow and E. H. Georgoulis, *Pointwise a posteriori error control for discontinuous Galerkin methods for elliptic problems*, SIAM Journal on Numerical Analysis **50** (2012), no. 5, 2159–2181.

[DGK19] A. Dominicus, F. Gaspoz, and C. Kreuzer, *Convergence of an adaptive C^0-interior penalty galerkin method for the Biharmonic problem*, arXiv preprint arXiv:1910.12959 (2019).

[DGK19a] A. Dominicus, F. Gaspoz, and C. Kreuzer, *Convergence of an adaptive C^0-interior penalty galerkin method for the Biharmonic problem*, Ergebnisberichte des Instituts für Angewandte Mathematik Nr.593. http://hdl.handle.net/2003/37889

[DKS16] L. Diening, C. Kreuzer, and R. Stevenson, *Instance optimality of the adaptive maximum strategy*, Foundations of Computational Mathematics **16** (2016), no. 1, 33–68.

[Don18] Z. Dong, *Discontinuous Galerkin methods for the Biharmonic problem on polygonal and polyhedral meshes*, arXiv preprint arXiv:1807.07817 (2018).

[DPE12] D. A. Di Pietro and A. Ern, *Mathematical aspects of discontinuous Galerkin methods*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 69, Springer, Heidelberg, 2012. MR 2882148

[DS80] T. Dupont and R. Scott, *Polynomial approximation of functions in Sobolev spaces*, Math. Comp. **34** (1980), no. 150, 441–463. MR 559195

[EG13] A. Ern and J.-L. Guermond, *Theory and practice of finite elements*, vol. 159, Springer Science & Business Media, 2013.

[EG15] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Chapman and Hall/CRC, 2015.

[EGH+02] G. Engel, K. Garikipati, T. J. R. Hughes, M. G. Larson, L. Mazzei, and R. L. Taylor, *Continuous/discontinuous finite element approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity*, Comput. Methods Appl. Mech. Engrg. **191** (2002), no. 34, 3669–3750. MR 1915664

[FHP15] T. Fraunholz, R. H. W. Hoppe, and M. Peter, *Convergence analysis of an adaptive interior penalty discontinuous Galerkin method for the Biharmonic problem*, J. Numer. Math. **23** (2015), no. 4, 317–330. MR 3433030

[FK07] X. Feng and O. Karakashian, *Fully discrete dynamic mesh discontinuous Galerkin methods for the Cahn-Hilliard equation of phase transition*, Mathematics of computation **76** (2007), no. 259, 1093–1117.

[FPP14] M. Feischl, M. Page, and D. Praetorius, *Convergence and quasi-optimality of adaptive FEM with inhomogeneous Dirichlet data*, Journal of computational and applied mathematics **255** (2014), 481–501.

[FS18] N. Fusco and D. Spector, *A remark on an integral characterization of the dual of BV*, Journal of Mathematical Analysis and Applications **457** (2018), no. 2, 1370–1375.

[GGS10] F. Gazzola, H.-C. Grunau, and G. Sweers, *Polyharmonic boundary value problems: positivity preserving and nonlinear higher order elliptic equations in bounded domains*, Springer Science & Business Media, 2010.

[GH09] E. H. Georgoulis and P. Houston, *Discontinuous Galerkin methods for the Biharmonic problem*, IMA J. Numer. Anal. **29** (2009), no. 3, 573–594. MR 2520159

[GHH06] E. H. Georgoulis, E. Hall, and P. Houston, *Discontinuous Galerkin methods on hp-anisotropic meshes i: a priori error analysis.*

[GHV11] E. H. Georgoulis, P. Houston, and J. Virtanen, *An a posteriori error indicator for discontinuous Galerkin approximations of fourth-order elliptic problems*, IMA J. Numer. Anal. **31** (2011), no. 1, 281–298. MR 2755946

[GR86] V. Girault and P. Raviart, *Finite element methods for Navier-Stokes equations: Theory and algorithms((book))*, Berlin and New York, Springer-Verlag(Springer Series in Computational Mathematics. **5** (1986).

[Gri85] P. Grisvard, *Elliptic problems in nonsmooth domains*, vol. 69, SIAM, 1985.

[Gri92] P. Grisvard, , *Singularities in boundary value problems*, vol. 22, Springer, 1992.

[GS02] V. Girault and L. R. Scott, *Hermite interpolation of nonsmooth functions preserving boundary conditions*, Math. Comp. **71** (2002), no. 239, 1043–1074. MR 1898745

[Gud10] T. Gudi, *A new error analysis for discontinuous finite element methods for linear elliptic problems*, Math. Comp. **79** (2010), no. 272, 2169–2189. MR 2684360

[HKW09] R. H. W. Hoppe, G. Kanschat, and T. Warburton, *Convergence analysis of an adaptive interior penalty discontinuous Galerkin method*, SIAM journal on numerical analysis **47** (2009), no. 1, 534–550.

[HL02] P. Hansbo and M. G. Larson, *A discontinuous Galerkin method for the plate equation*, Calcolo **39** (2002), no. 1, 41–59. MR 1901201

[JB12] M. Jensen and J. F. Blowey, *Frontiers in numerical analysis: Durham 2010*, Springer, 2012.

[Joh73] C. Johnson, *On the convergence of a mixed finite-element method for plate bending problems*, Numer. Math. **21** (1973), 43–62. MR 0388807

[KG17] C. Kreuzer and E. H. Georgoulis, *Convergence of adaptive discontinuous Galerkin methods*, Ergebnisberichte des Instituts für Angewandte Mathematik Nr.575. http://hdl.handle.net/2003/36041

[KG18] C. Kreuzer and E. H. Georgoulis, *Convergence of adaptive discontinuous Galerkin methods*, Math. Comp. **87** (2018), no. 314, 2611–2640. MR 3834679

[KG19] C. Kreuzer and E. H. Georgoulis, *Convergence of adaptive discontinuous Galerkin methods (corrected version of [math. comp. 87 (2018), no. 314, 2611–2640])*, arXiv preprint arXiv:1909.12665 (2019).

[KG20] C. Kreuzer and E. H. Georgoulis, *Convergence of adaptive discontinuous Galerkin methods* (corrected version of [Math. Comp. **87** (2018), no. 314, 2611–2640]), Math. Comp. **90** (2021).

[Kir50] G. Kirchhoff, *On the equilibrium and motion of an elastic plate*, J. Math **40** (1850), 51–58.

[Kos94] I. Kossaczký, *A recursive approach to local mesh refinement in two and three dimensions*, J. Comput. Appl. Math. **55** (1994), no. 3, 275–288. MR 1329875

[KP03] O. A Karakashian and F. Pascal, *A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems*, SIAM J. Numer. Anal. **41** (2003), no. 6, 2374–2399. MR 2034620

[KP07] O. A Karakashian and F. Pascal, *Convergence of adaptive discontinuous Galerkin approximations of second-order elliptic problems*, SIAM J. Numer. Anal. **45** (2007), no. 2, 641–665. MR 2300291

[Lov13] A .E. H. Love, *A treatise on the mathematical theory of elasticity*, Cambridge university press, 2013.

[LS07] M.-J. Lai and L. L. Schumaker, *Spline functions on triangulations*, no. 110, Cambridge University Press, 2007.

[Mau95] J. M. Maubach, *Local bisection refinement for n-simplicial grids generated by reflection*, SIAM J. Sci. Comput. **16** (1995), no. 1, 210–227. MR 1311687

[MNS00] P. Morin, R. H. Nochetto, and K. G. Siebert, *Data oscillation and convergence of adaptive FEM*, SIAM J. Numer. Anal. **38** (2000), no. 2, 466–488. MR 1770058

[MS75] J. Morgan and R. Scott, *A nodal basis for C^1 piecewise polynomials of degree $n \geqslant 5$*, Math. Comp. **29** (1975), no. 131, 736–740.

[MNS03] P. Morin, R. Nochetto, and K. Siebert, *Local problems on stars: a posteriori error estimators, convergence, and performance*, Mathematics of Computation **72** (2003), no. 243, 1067–1097.

[Mor68] L. S. D. Morley, *The triangular equilibrium element in the solution of plate bending problems*, The Aeronautical Quarterly **19** (1968), no. 2, 149–169.

[MSB07] I. Mozolevski, E. Süli, and P. R. Bösing, *hp-version a priori error analysis of interior penalty discontinuous Galerkin finite element approximations to the Biharmonic equation*, Journal of Scientific Computing **30** (2007), no. 3, 465–491.

[MSV08] P. Morin, K. G. Siebert, and A. Veeser, *A basic convergence result for conforming adaptive finite elements*, Math. Models Methods Appl. Sci. **18** (2008), no. 5, 707–737. MR 2413035

[NV11] R. H. Nochetto and A. Veeser, *Primer of adaptive finite element methods*, Multiscale and adaptivity: modeling, numerics and applications, Springer, 2011, pp. 125–225.

[PKJF12] L. Pick, A. Kufner, O. John, and S. Fucík, *Function spaces, 1*, Walter de Gruyter, 2012.

[Poi38] Si. D. Poisson, *Traité de mécanique*, Société belge de librairie, 1838.

[Pry14] T. Pryer, *Discontinuous Galerkin methods for the p-biharmonic equation from a discrete variational perspective*, Electron. Trans. Numer. Anal. **41** (2014), 328–349. MR 3270092

[Rud91] W. Rudin, *Functional analysis. 1991*, Internat. Ser. Pure Appl. Math (1991).

[Sel13] A. P. S. Selvadurai, *Partial differential equations in mechanics 2: the Biharmonic equation, Poisson's equation*, Springer Science & Business Media, 2013.

[SH18] P. Sun and X. Huang, *Quasi-optimal convergence rate for an adaptive hybridizable C^0 discontinuous Galerkin method for Kirchhoff plates*, Numer. Math. **139** (2018), no. 4, 795–829. MR 3824867

[Sie11] K. G. Siebert, *A convergence proof for adaptive finite elements without lower bound*, IMA J. Numer. Anal. **31** (2011), no. 3, 947–970. MR 2832786

[SM03] E. Süli and I. Mozolevski, *A priori error analysis for the hp-version of the discontinuous Galerkin finite element method for the Biharmonic equation*, Computational Methods in Applied Mathematics **3** (2003), no. 4.

[SM07] E. Süli and I. Mozolevski, *hp-version interior penalty DGFEMs for the Biharmonic equation*, Comput. Methods Appl. Mech. Engrg. **196** (2007), no. 13-16, 1851–1863. MR 2298696

[SS98] C. Schwab, *p-and hp-finite element methods: Theory and applications in solid and fluid mechanics*, Oxford University Press, 1998.

[SZ90] L. R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp. **54** (1990), no. 190, 483–493. MR 1011446

[Ver13] R. Verfürth, *A posteriori error estimation techniques for finite element methods*, Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, 2013. MR 3059294

[Vir10] J. Virtanen, *Adaptive discontinuous Galerkin methods for fourth order problems*, 2010

[WH03] T. Warburton and J. S. Hesthaven, *On the constants in hp-finite element trace inverse inequalities*, Computer methods in applied mechanics and engineering **192** (2003), no. 25, 2765–2773.

[Zei90] E. Zeidler, *Nonlinear functional analysis and its applications. II/B*, Springer-Verlag, New York, 1990, Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron. MR 1033498

Notation Index