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Cutting Barnette graphs perfectly is hard [☆,☆☆]

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ABSTRACT

A *perfect matching cut* is a perfect matching that is also a cutset, or equivalently, a perfect matching containing an even number of edges on every cycle. The corresponding algorithmic problem, Perfect Matching Cut, is known to be NP-complete in subcubic bipartite graphs [Le & Telle, TCS '22], but its complexity was open in planar graphs and cubic graphs. We settle both questions simultaneously by showing that Perfect Matching Cut is NP-complete in 3-connected cubic bipartite planar graphs or *Barnette graphs*. Prior to our work, among problems whose input is solely an undirected graph, only Distance-2 4-Coloring was known to be NP-complete in Barnette graphs. Notably, Hamiltonian Cycle would only join this private club if Barnette's conjecture were refuted.

1. Introduction

Deciding if an input graph admits a perfect matching, i.e., a subset of its edges touching each of its vertices exactly once, notoriously is a tractable task. There is indeed a vast literature, starting arguably in 1947 with Tutte's characterization via determinants [41], of polynomial-time algorithms deciding Perfect Matching (or returning actual solutions) and its optimization generalization Maximum Matching.

In this paper, we are interested in another containment of a perfect matching (or, more generally, of a spanning subset of edges) than as a subgraph. As containing such a set of edges as an induced subgraph is a trivial property[1] (only shared by graphs that are themselves disjoint unions of edges), the meaningful other containment is as a *semi-induced subgraph*. By that, we mean that we look for a bipartition of the vertex set, called a *cut* such that the edges of the perfect matching are "induced" in the corresponding cutset (i.e., the set of edges going from one side of the bipartition to the other), while we do not set any requirement on the presence or absence of edges within each side of the bipartition.

This problem was introduced as the Perfect Matching Cut (PMC for short) problem[2] by Heggernes and Telle who show that it is NP-complete [17]. As the name Perfect Matching Cut suggests, we look for a perfect matching that is also a cutset. Le and Telle

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[1] Note however that the induced variant of Maximum Matching is an interesting problem that happens to be NP-complete [39].

[2] The authors consider the framework of (k, σ, ρ)-partition problem, where k is a positive integer, and σ, ρ are sets of non-negative integers, and one looks for a vertex-partition into k parts such that each vertex of each part has a number of neighbors in its own part in σ, and a number of other neighbors in ρ; hence, PMC is then the $(2, \mathbb{N}, \{1\})$-partition problem.

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further show that PMC remains NP-complete in subcubic bipartite graphs of arbitrarily large girth, whereas it is polynomial-time solvable in a superclass of chordal graphs, and in graphs without a particular subdivided claw as an induced subgraph [27]. An in-depth study of the complexity of PMC when forbidding a single induced subgraph or a finite set of subgraphs has been carried out [13,30].

We look at Le and Telle's hardness constructions and wonder what other properties could make PMC tractable (aside from chordality and forbidding a finite list of subgraphs or induced subgraphs). A simpler reduction for bipartite graphs is first presented below. Let us briefly sketch their reduction (without thinking about its correctness) from Monotone Not-All-Equal 3-SAT, where given a negation-free 3-CNF formula, one seeks a truth assignment that sets in each clause a variable to true and a variable to false. Every variable is represented by an edge, and each 3-clause, by a (3-dimensional) cube with three connector points at three pairwise non-adjacent vertices of the cube. One endpoint of the variable gadget is linked to the connector points corresponding to this variable among the clause gadgets. Note that this construction creates three vertices of degree 4 in each clause gadget and vertices of possibly large degrees in the variable gadgets. Le and Telle then reduce the maximum degree to at most 3, by appropriately subdividing the cubes and, tweaking the connector points, and replacing the variable gadgets by cycles.

Notably the edge subdivision of the clause gadgets creates degree-2 vertices, which are not easy to "pad" with a third neighbor (even more so while keeping the construction bipartite). Moreover, prior to our work, the complexity of PMC in cubic graphs was open. Let us observe that on cubic graphs, the problem becomes equivalent to partitioning the vertex set into two sets, each inducing a disjoint union of (independent) cycles. The close relative, Matching Cut, where one looks for a mere matching that is also a cutset, while NP-complete in general [5], is polynomial-time solvable in *subcubic* graphs [35,2]. The complexity of Matching Cut has further been examined in subclasses of planar graphs [37,2], when forbidding some (induced) subgraphs [13,31,30,12], on graphs of bounded diameter [31,26], and on graphs of large minimum degree [4]. Matching Cut has also been investigated with respect to parameterized complexity, exact exponential time algorithms [25,22], and enumeration [15].

It was also open if PMC is tractable on planar graphs. Note that Bouquet and Picouleau show that a related problem, Disconnected Perfect Matching, where one looks for a perfect matching that contains a cutset, is NP-complete on planar graphs of maximum degree 4, on planar graphs of girth 5, and on 5-regular bipartite graphs [3]. They incidentally call this related problem Perfect Matching Cut , but subsequent references [13,27] use the name Disconnected Perfect Matching to avoid confusion. We will observe that PMC is equivalent to asking for a perfect matching containing an even number of edges from every cycle of the input graph. The sum of even numbers being even, it is in fact sufficient that the perfect matching contains an even number of edges from every element of a cycle basis. There is a canonical cycle basis for planar graphs: the bounded faces. This gives rise to the following neat reformulation of PMC in planar graphs: is there a perfect matching containing an even number of edges along each face?

While Matching Cut is known to be NP-complete on planar graphs [37,2], it could have gone differently for PMC for the following "reasons." Not-All-Equal 3-SAT, which appears as the *right* starting point to reduce to PMC , is tractable on planar instances [34]. In planar graphs, perfect matchings are *simpler* than arbitrary matchings in that they alone [42] can be counted efficiently [40,21]. Let us finally observe that Maximum Cut can be solved in polynomial time in planar graphs [16].

In fact, we show that the reformulations for cubic and planar graphs cannot help algorithmically by simultaneously settling the complexity of PMC in cubic and planar graphs with the following stronger statement.

Theorem 1. Perfect Matching Cut *is NP-hard in 3-connected cubic bipartite planar graphs.*

Not very many problems are known to be NP-complete in cubic bipartite planar graphs. Of the seven problems defined on mere undirected graphs from Karp's list of 21 NP-complete problems [20], only Hamiltonian Path is known to remain NP-complete in this class, while the other six problems admit a polynomial-time algorithm. Restricting ourselves to problems where the input is purely an undirected graph,[3] besides Hamiltonian Path/Cycle [36,1], Minimum Independent Dominating Set was also shown NP-complete in cubic bipartite planar graphs [29], as well as P_3-Packing [24] (hence, an equivalent problem phrased in terms of disjoint dominating and 2-dominating sets [33]), and Distance-2 4-Coloring [10]. To our knowledge, Minimum Dominating Set is only known NP-complete in *subcubic* bipartite planar graphs [14,23].

It is interesting to note that the reductions for Hamiltonian Path, Hamiltonian Cycle, Minimum Independent Dominating Set, and P_3-Packing all produce cubic bipartite planar graphs that are *not* 3-connected. Notoriously, lifting the NP-hardness of Hamiltonian Cycle to the 3-connected case would require disproving Barnette's conjecture[4] (and that would be indeed sufficient [11]). Note that hamiltonicity in cubic graphs is equivalent to the existence of a perfect matching that is *not* an edge cut (i.e., whose removal is not disconnecting the graph). We wonder whether there is something inherently simpler about *3-connected* cubic bipartite planar graphs, which would go beyond hamiltonicity (assuming that Barnette's conjecture is true).

Let us call *Barnette* a 3-connected cubic bipartite planar graph. It appears that, prior to our work, Distance-2 4-Coloring was the only *vanilla* graph problem shown NP-complete in Barnette graphs [10]. Arguing that Distance-2 4-Coloring is a problem on *squares* of Barnette graphs more than it is on Barnette graphs, a case can be made for Perfect Matching Cut to be the first natural problem proven NP-complete in Barnette graphs.

[3] Among problems with edge orientations, vertex or edge weights, or prescribed subsets of vertices or edges, the list is significantly longer, and also includes Minimum Weighted Edge Coloring [7], List Edge Coloring and Precoloring Extension [32], k-In-A-Tree [8], etc.

[4] Which precisely states that every polyhedral (that is, 3-connected planar) cubic bipartite graphs admit a hamiltonian cycle.

Provably tight subexponential-time algorithm. Note that our reduction together with existing results and a known methodology give a fine-grained understanding under the Exponential-Time Hypothesis[5] (or ETH) [18], on solving Perfect Matching Cut in planar graphs.

On the algorithmic side, there is a $2^{O(\sqrt{n})}$-time algorithm for PMC in n-vertex planar graphs, as a consequence of a $2^{O(w)}n^{O(1)}$-time algorithm for n-vertex graphs given with a tree-decomposition of width w, and the fact that tree-decompositions of width $O(\sqrt{n})$ always exist in planar graphs and can be computed in polynomial-time [28]. The $2^{O(w)}n^{O(1)}$-time algorithm can be obtained directly or as a consequence of a result of Pilipczuk [38] that any problem expressible in Existential Counting Modal Logic (ECML) admits a single-exponential fixed-parameter algorithm in treewidth. ECML allows existential quantifications over vertex and edge sets followed by a counting modal formula to be satisfied *from every vertex*. Counting modal formulas enrich quantifier-free Boolean formulas with $\Diamond^S \varphi$, whose semantics is that the current vertex v has a number of neighbors satisfying φ in the ultimately periodic set S of non-negative integers. One can thus express Perfect Matching Cut in ECML as

$$\exists X \subseteq V(G), \forall v \in V(G),\ G, X, v \models X \to \Diamond^{\{1\}}(\neg X)\ \land\ \neg X \to \Diamond^{\{1\}} X,$$

which states that there is a set X such that every vertex in X has exactly one neighbor outside X, and vice versa.

On the complexity side, the Sparsification lemma [19], the folklore linear reductions from bounded-occurrence 3-SAT to bounded-occurrence Monotone Not-All-Equal 3-SAT and to Monotone Not-All-Equal 3-SAT-E4 [6], and finally our quadratic reduction, imply that $2^{\Omega(\sqrt{n})}$ time is required to solve PMC in n-vertex planar graphs. Our reduction (as we will see) indeed has a quadratic blow-up as it creates $O(1)$ vertices per variable and clause, and $O(1)$ vertices for each of the $O(n^2)$ crossings in a (non-planar) drawing of the variable-clause incidence graph.

Outline of the proof. We reduce the NP-complete problem Monotone Not-All-Equal 3-SAT with exactly 4 occurrences of each variable [6] to PMC . Observe that flipping the value of every variable of a satisfying assignment results in another satisfying assignment. We thus see a solution to Monotone Not-All-Equal 3-SAT simply as a bipartition of the set of variables.

As we already mentioned, Not-All-Equal 3-SAT restricted to planar instances (i.e., where the variable-clause incidence graph is planar) is in P. We thus have to design *crossing* gadgets in addition to *variable* and *clause* gadgets. Naturally, our gadgets are bipartite graphs with vertices of degree 3, except for some special *connector vertices*, that have degree 2 with one incident edge leaving the gadget.

The variable gadget is designed so that there is a unique way a perfect matching cut can intersect it. It might seem odd that no "binary choice" happens within it. The role of this gadget is only to serve as a baseline for which side of the bipartition the variable lands in, while the "truth assignments" take place in the clause gadgets. (Actually the same happens with Le and Telle's first reduction [27], where the variable gadget is a single edge, which has to be in any solution.)

Our variable gadget consists of 36 vertices, including 8 connector points; see Fig. 1. (We will later explain why we have 8 connector points and not simply 4, that is, one for each occurrence of the variable.) Note that in all the figures, we adopt the following convention:

- black edges cannot (or can no longer) be part of a perfect matching cut,
- red edges are in every perfect matching cut,
- each blue edge e is such that at least one perfect matching cut within its gadget includes e, and at least one excludes e, and
- brown edges are blue edges that were indeed chosen in the solution.

Let us recall that PMC consists of finding a perfect matching containing an even number of edges from each cycle. Thus we look for a perfect matching M such that every path (or walk) between v and w contains a number of edges of M whose parity only depends on v and w. If this parity is even v and w are on the *same side*, and if it is odd, v and w are on *opposite sides*. The 8 connector points of each variable gadget are forced on the same side. This is the *side of the variable*.

At the core of the clause gadget is a subdivided cube of blue edges; see Fig. 2. There are three vertices (u_1, u_8, u_{14} on the picture) of the subdivided cube that are forced on the same side as the corresponding three variables. Three perfect matching cuts are available in the clause gadget, each separating (i.e., putting on opposite sides) a different vertex of $\{u_1, u_8, u_{14}\}$ from the other two. Note that this is exactly the semantics of a not-all-equal 3-clause. We in fact need two copies of the subdivided cube, partly to increase the degree of some subdivided vertices, partly for the same reason we duplicated the connector vertices in the variable gadgets. (The latter will be explained when we present the crossing gadgets.) Increasing the degree of all the subdivided vertices complicate further the gadget and creates two odd faces. Fortunately, these two odd faces have a common neighboring even face. We can thus "fix" the parity of the two odd faces by plugging the sub-gadget D_j in the even face. We eventually need a total of 112 vertices, including 6 connector points.

Let us now describe the crossing gadgets. We want to replace every intersection point of two edges by a 4-vertex cycle. This indeed propagates black edges (those that cannot be in any solution). The issue is that going through such a crossing gadget flips one's side. As we cannot guarantee that a variable "wire" has the same parity of intersection points towards each clause gadget it is linked to, we duplicate these wires. At a previous intersection point, we now have two parallel wires crossing two other parallel

[5] The assumption that there is a $\lambda > 0$ such that no algorithm solves n-variable 3-SAT in time $\lambda^n n^{O(1)}$

wires, making four crossings. The gadget simply consists of four 4-vertex cycles; see Fig. 4. Check in Fig. 8 that the sides are indeed preserved. This explains why we have 8 connector points (not 4) in each variable gadget and 6 connector points (not 3) in each clause gadget.

2. Preliminaries

For a graph G, we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. If $U \subseteq V(G)$, the *subgraph of G induced by U*, denoted $G[U]$ is the graph obtained from G by removing the vertices not in U. $E_G(U)$ (or $E(U)$ when G is clear) is a shorthand for $E(G[U])$. For $M \subseteq E(G)$, $G - M$ is the subgraph of G obtained by removing the edges in M (while preserving their endpoints). A connected component of G is a maximal set $U \subseteq V(G)$ such that $G[U]$ is connected. A graph G is *cubic* if every vertex of G has exactly three neighbors. A graph is *bipartite* if it contains no odd cycles. We may use *k-cycle* as a short-hand for the k-vertex cycle.

Given two disjoint sets $X, Y \subseteq V(G)$ we denote by $E(X,Y)$ the set of edges between X and Y. A set $M \subseteq E(G)$ is a *cutset* of G if there is a partition $X \uplus Y = V(G)$, called *cut*, such that $M = E(X,Y)$. Note that a cut fully determines a cutset, and among connected graphs a cutset fully determines a cut. When dealing with connected graphs, we will write *the* cut of a cutset. For $X \subseteq V(G)$ the set of *outgoing edges of X* is $E(X, V(G) \setminus X)$. For a cutset M of a connected graph G, and $u, v \in V(G)$, we say that u and v are on the *same side* (resp. on *opposite sides*) of M if u and v are in the same part (resp. on different parts) of the cut of M.

A *matching* (resp. *perfect matching*) of G is a set $M \subseteq E(G)$ such that each vertex of G is incident to at most (resp. exactly) one edge of M. A *perfect matching cut* is a perfect matching that is also a cutset. For $M \subseteq E(G)$ and $U \subseteq V(G)$, we say that M is a *perfect matching cut of $G[U]$* if $M \cap E(U)$ is so.

A graph is *planar* if it can be embedded in the plane, i.e., drawn such that edges (simple curves) may only intersect at their endpoints (the vertices). A *plane graph* is a planar graph together with such an embedding. Given a plane graph G, a face of G is a path-connected subset of the plane after removing the embedding of G. A *facial cycle* of a plane graph G is a cycle of G that bounds a face of G. We say that two plane graphs G and H are *translates* if the embedding of G is obtained by applying a translation (i.e. a map of the form $x \to x + a$) to the embedding of H.

3. Proof of Theorem 1

Before we give our reduction, we start with a handful of useful lemmas and observations, which we will need later.

3.1. Preparatory lemmas

Lemma 2. *Let G be a graph, and $M \subseteq E(G)$. Then M is a cutset if and only if for every cycle C of G, $|E(C) \cap M|$ is even.*

Proof. Suppose that M is a cutset, and let (A, B) be a cut of M. Every closed walk (and in particular, cycle) contains an even number of edges of M, since for every edge between A and B, which we use in our walk to go from A to B, we have exactly one other edge that we use to go back.

Now assume that every cycle of G has an even number of edges in common with M. We build a cut (A, B). For each connected component H of G, we fix an arbitrary vertex $v \in V(H)$ and do the following. For each vertex $w \in V(H)$, put w in A if there is a path from v to w taking an even number of edges from M, and in B if there is a path from v to w taking an odd number of edges from M. It holds that $A \cup B = V(G)$. By our assumption on the cycles of G, $A \cap B = \emptyset$. Hence (A, B) is indeed a cut. The cutset of (A, B) is, by construction, M. □

Lemma 3. *Let G be a plane graph, and $M \subseteq E(G)$. Then M is a cutset if and only if for any facial cycle C of G, $|E(C) \cap M|$ is even.*

Proof. The forward implication is a direct consequence of Lemma 2. The converse comes from the known fact that the facial cycles form a cycle basis; see for instance [9]. For any subgraph H of G, denote by $\tilde{H}$ the vector of $\mathbb{F}_2^{E(G)}$ with 1 entries at the positions corresponding to edges of H.

Let C be a cycle of G. There is a decomposition of C into facial cycles: an integer k and facial cycles of $F_1, \ldots, F_k$ such that $\tilde{C} = \Sigma_{1 \leqslant i \leqslant k} \tilde{F}_i$. Consequently, $|M \cap E(C)|$ has the same parity as $\Sigma_{1 \leqslant i \leqslant k} |M \cap E(F_i)|$, a sum of even numbers. By application of Lemma 2, M is a cutset. □

Lemma 4. *Let M be a perfect matching cut of a cubic graph G. Let C be an induced 4-vertex cycle of G. Then, exactly one of the following cases holds:*

(a) *$E(C) \cap M = \emptyset$ and the four outgoing edges of $V(C)$ belong to M.*
(b) *$|E(C) \cap M| = 2$, the two edges of $E(C) \cap M$ are disjoint, and none of the outgoing edges of $V(C)$ belongs to M.*

Proof. The number of edges of M within $E(C)$ is even by Lemma 3. Thus $|E(C) \cap M| \in \{0, 2\}$, as all four edges of $E(C)$ do not make a matching.

Suppose that $E(C) \cap M = \emptyset$. As M is a perfect matching, for every $v \in V(C)$ there is an edge in M incident to v and not in $E(C)$. As G is cubic, every outgoing edge of $V(C)$ is in M.

Suppose instead that $|E(C) \cap M| = 2$. As M is a matching, the two edges of $E(C) \cap M$ do not share an endpoint. It implies that all four vertices of C are touched by these two edges. Thus, no outgoing edge of $V(C)$ can be in M. □

Corollary 5. *Let M be a perfect matching of a cubic graph G. Let C_1, C_2 be two vertex-disjoint induced 4-vertex cycles of G such that there is an edge between $V(C_1)$ and $V(C_2)$. Then $E(C_1) \cap M \neq \emptyset$ if and only if $E(C_2) \cap M \neq \emptyset$.*

Proof. Suppose $E(C_1) \cap M \neq \emptyset$. By Lemma 4 on C_1, no outgoing edge of $V(C_1)$ is in M. Thus, there is an outgoing edge of $V(C_2)$ that is not in M. Applying Lemma 4 on C_2, we have $E(C_2) \cap M \neq \emptyset$. We get the converse symmetrically. □

Lemma 6. *Let M be a perfect matching cut of a cubic graph G. If a 6-cycle has three outgoing edges in M, then all six outgoing edges are in M.*

Proof. Let C be our 6-cycle. Remember that, as M is a perfect matching cut, $|E(C) \cap M|$ is even. This means that $|E(C) \cap M|$ is either 0 or 2. If $|E(C) \cap M| = 2$, four vertices of C are touched by $E(C) \cap M$, which rules out that three outgoing edges of $V(C)$ are in M. Thus, $E(C) \cap M = \emptyset$ and, since G is cubic, every outgoing edge of $V(C)$ is in M. □

Lemma 7. *Let M be a perfect matching cut of a cubic bipartite graph G. Suppose C is a 6-cycle $v_1 v_2 \dots v_6$ of G, such that $v_2 v_3$, $v_3 v_4$, $v_5 v_6$ and $v_6 v_1$ are in some induced 4-cycles. Then $M \cap E(C) = \emptyset$.*

Proof. By applying Lemma 4 on the 4-cycle containing $v_2 v_3$, and the one containing $v_6 v_1$, it holds that $v_1 v_2 \in M \Leftrightarrow v_3 v_4 \in M \Leftrightarrow v_5 v_6 \in M$. Thus none of these three edges can be in M, because C would have an odd number of edges in M. Symmetrically, no edge among $v_2 v_3$, $v_4 v_5$ and $v_6 v_1$ can be in M. Thus, no edge of C is in M. □

Observation 8. *Let G be a graph and M be a perfect matching cut of G. Let u, v be two vertices of G. Then for any path P between u and v, $|E(P) \cap M|$ is even if and only if u and v are on the same side of M. Note that this implies that for any paths P, Q from u to v, $|E(P) \cap M|$ and $|E(Q) \cap M|$ have the same parity.*

3.2. Reduction

We will prove Theorem 1 by reduction from the NP-complete Monotone Not-All-Equal 3SAT-E4 [6]. In Monotone Not-All-Equal 3SAT-E4 , the input is a 3-CNF formula where each variable occurs exactly four times, each clause contains exactly three distinct literals, and no clause contains a negated literal. Here we say that a truth assignment on the variables *satisfies* a clause C if at least one literal of C is true and at least one literal of C is false. The objective is to decide whether there is a truth assignment that satisfies all clauses. We can safely assume (and we will) that the variable-clause incidence graph[6] inc(I) of I has no cutvertex[7] among its "variable" vertices. Indeed the reduction from Monotone Not-All-Equal 3-SAT to its four-occurrence variant [6] does not create such cutvertices if they do not exist originally. Now if there is a "variable" cutvertex v in a Monotone Not-All-Equal 3-SAT-instance J, one can split J into J_1 made of one connected component X of inc$(J) - \{v\}$ plus v, and J_2 made of inc$(J) \setminus X$. One can observe that J is positive if and only if J_1 and J_2 are positive. As inc(J_1) and inc(J_2) sum up to one more vertex than inc(J), such a scheme is a polynomial-time Turing reduction to subinstances without "variable" cutvertices.

Let I be an instance of Monotone Not-All-Equal 3SAT-E4 with variables $x_1, x_2, \dots, x_n$ and $m = 4n/3$ clauses $C_1, C_2, \dots, C_m$. We shall construct, in polynomial time, an equivalent PMC -instance $G(I)$ that is Barnette.

Our reduction consists of three steps. First we construct a cubic graph $H(I)$ by introducing *variable gadgets* and *clause gadgets*. Then we *draw* $H(I)$ on the plane, i.e., we map the vertices of $H(I)$ to a set of points on the plane, and the edges of $H(I)$ to a set of simple curves on the plane. We shall refer to this drawing as $\mathcal{R}$. Note that this drawing may not be planar, i.e., two simple curves (or analogously the corresponding edges) might intersect at a point which is not one of their endpoints. Finally, we eliminate the crossing points by introducing *crossing gadgets*. (Recall that if the variable-clause incidence graph of a Not-All-Equal 3-SAT instance is planar, then its satisfiability can be tested in polynomial time [34]; hence, we do need crossing gadgets.) The resulting graph $G(I)$ is Barnette, and we shall prove that $G(I)$ has a perfect matching if and only if I is a positive instance of Monotone Not-All-Equal 3SAT-E4 . We now describe the above steps.

1. For each variable x_i, let $\mathcal{X}_i$ denote a copy of the graph shown in Fig. 1. Note that the variable x_i appears in exactly four clauses, say C_j, C_k, C_p, C_q, with $j < k < p < q$. The *variable gadget* $\mathcal{X}_i$ contains the connector vertices $b_{i,j}$, $t_{i,j}$, $b_{i,k}$, $t_{i,k}$, $b_{i,p}$, $t_{i,p}$, $b_{i,q}$, $t_{i,q}$ as shown in the figure. Recall that edges that are contained in any perfect matching cut are colored red in the figure, while black

[6] I.e. the graph whose vertex set is the clause and variable, and a variable vertex x is adjacent to a clause vertex C if x or $\neg x$ is present in C.

[7] I.e. a vertex whose removal makes the graph disconnected.

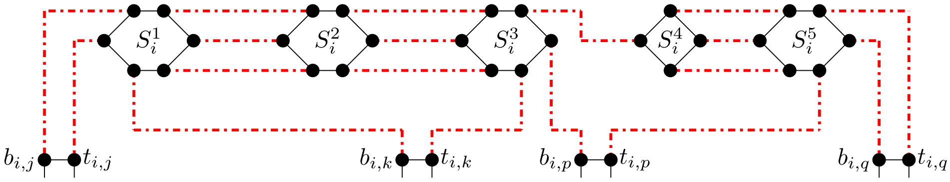


Fig. 1. The variable Gadget $\mathcal{X}_i$ corresponding to the variable x_i appearing in the clauses C_j, C_k, C_p, C_q with $j < k < p < q$.

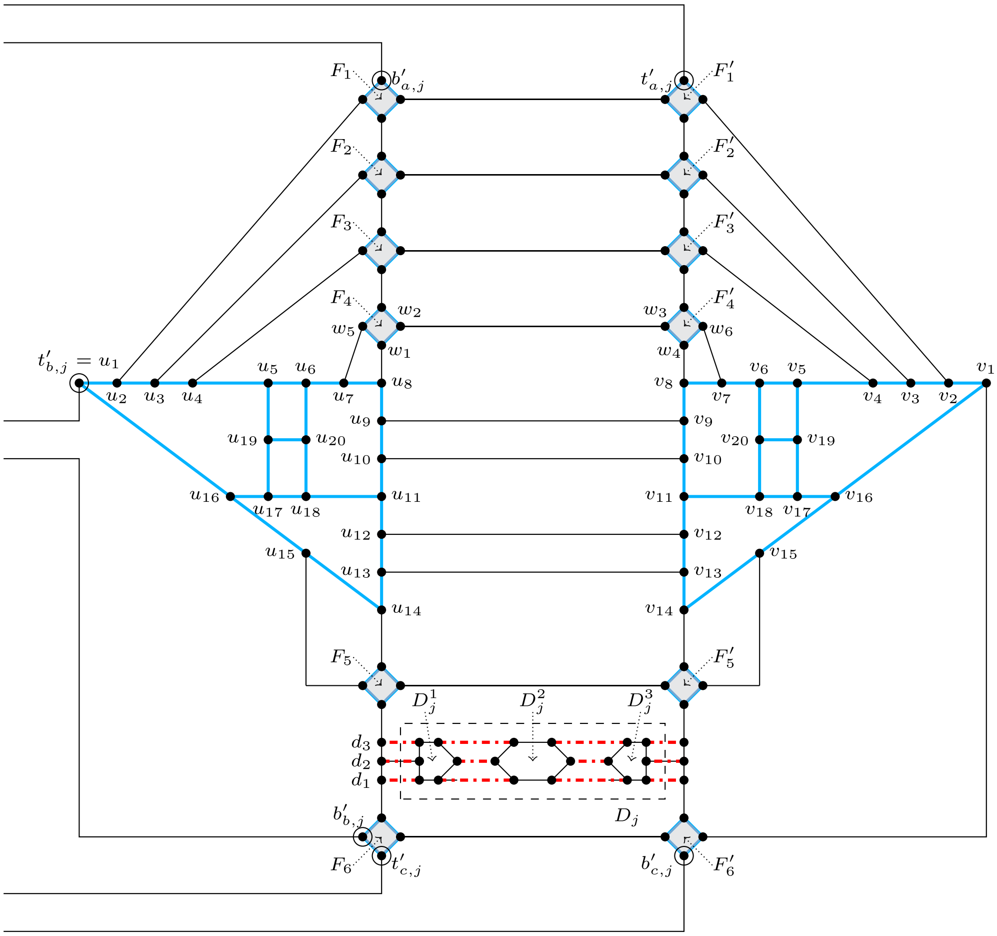


Fig. 2. Clause gadget $C_j = (x_a, x_b, x_c)$ with $a < b < c$. A red (dash-dotted) edge is selected in any perfect matching cut. A blue edge is selected in some perfect matching cut. A black edge is never selected in any perfect matching cut. The connector vertices are highlighted. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

edges are contained in no perfect matching cut. An essential part of the proof will consist of justifying the edge colors in our figures.

For each clause $C_j = (x_a, x_b, x_c)$ with $a < b < c$ let $\mathcal{C}_j$ denote a copy of the graph shown in Fig. 2. The *clause gadget* $\mathcal{C}_j$ contains the connector vertices $t'_{a,j}, b'_{a,j}, t'_{b,j}, b'_{b,j}, t'_{c,j}, b'_{c,j}$, as shown in the figure.

To connect the variable and clause gadgets, for each variable x_i that appears in the clause C_j, introduce two new edges $E_{ij} = \left\{t_{i,j}t'_{i,j}, b_{i,j}b'_{i,j}\right\}$ called *connector edges*. For a connector edge e incident to a variable gadget $\mathcal{X}_i$ we let $\text{var}(e) = x_i$, the associated variable. Now let $H(I)$ denote the graph obtained from the union of the variable gadgets, clause gadgets and the connector edges. Formally, $H(I)$ is defined as follows.

$$V(H(I)) = \bigcup_{i=1}^{n} V(\mathcal{X}_i) \cup \bigcup_{j=1}^{m} V(\mathcal{C}_j)$$

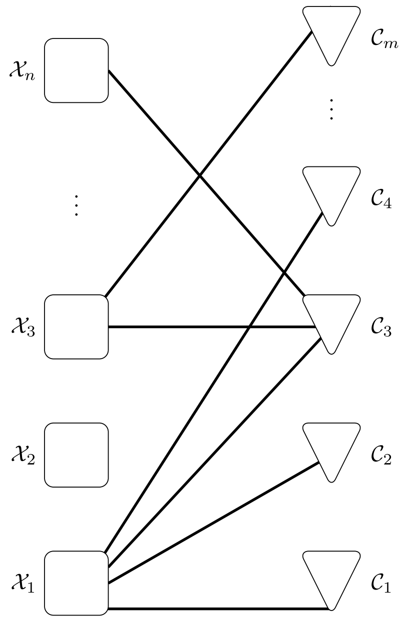


Fig. 3. A schematic diagram of $\mathcal{R}$. The boxes indicate variable gadgets, and the triangles indicate clause gadgets. The crossings of the edges will be replaced by crossing gadget.

$$E(H(I)) = \bigcup_{i=1}^{n} E(\mathcal{X}_i) \cup \bigcup_{j=1}^{m} E(C_j) \cup \bigcup_{x_i \in C_j} E_{ij}.$$

We assign to each edge $e \in E_{i,j}$ its variable as $\mathrm{var}(e) = i$. Note that, for a variable gadget $\mathcal{X}_i$, there are exactly eight outgoing edges of $V(\mathcal{X}_i)$.

2. In the next step, we generate a drawing $\mathcal{R}$ of $H(I)$ on the plane according to the following procedure. (See Fig. 3 for a schematic diagram of the diagram.)
   a. For each variable x_i, we embed the corresponding variable gadget $\mathcal{X}_i$ as a translate of the variable gadget of Fig. 1 into $[0,1] \times [2i, 2i+1]$.
   b. For each clause C_j, we embed the corresponding clause gadget C_j as a translate of the clause gadget of Fig. 2 into $[2,3] \times [2j, 2j+1]$.
   c. Two edges incident to vertices in the same variable gadget or the same clause gadget do not intersect in $\mathcal{R}$. For two variables $x_i, x_{i'}$ and clauses $C_j, C_{j'}$ with $x_i \in C_j, x_{i'} \in C_{j'}$, exactly one of the following holds:
      i. For each pair of edges $(e, e') \in E_{ij} \times E_{i'j'}$, e and e' intersect exactly once in $\mathcal{R}$. When this condition is satisfied, we call $(E_{ij}, E_{i'j'})$ a *crossing quadruple*. The name "quadruple" is due to the fact that each pair of crossing edges (e, e') actually produces four crossings in the drawing.
      Moreover, we ensure that the interior of the subsegment of $e \in E_{ij}$ between its two intersection points with edges of $E_{i'j'}$ is not crossed by any edge;
      ii. There is no pair of edges $(e, e') \in E_{ij} \times E_{i'j'}$ such that e and e' intersect in $\mathcal{R}$;
3. For each crossing quadruples $(E_{ij}, E_{i'j'})$, the four lines cross as shown in Fig. 4a. Modify the graph by adding a C_4 for each pair of crossing lines, as shown in Fig. 4b. The resulting *crossing gadget* is the union of the four C_4 added. For any edge e obtained by this operation, and any edge f deleted by this operation, if e and f have a common endpoint, we define $\mathrm{var}(e)$ to be $\mathrm{var}(f)$.

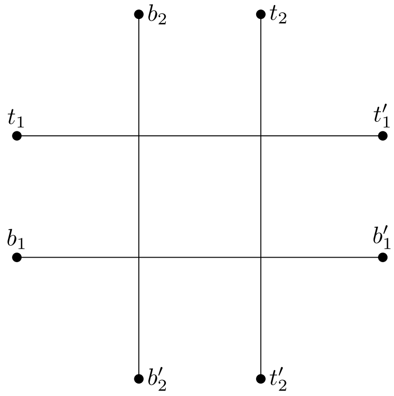
Let $G(I)$ denote the resulting graph. We shall need the following definitions.

Definition 9. Any edge of $G(I)$ whose both endpoints are not contained within the same gadget (variable, clause, or crossing) is a *connector edge*. Any endpoint of a connector edge is called a *connector vertex*. Note that for any connector edge e, $\mathrm{var}(e)$ is defined either in the creation of $H(I)$ or in Item 3.

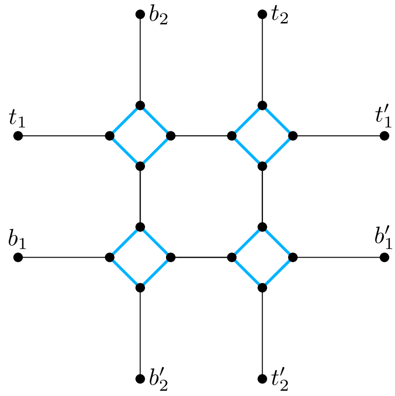
Now, we shall distinguish some 4-cycles of $G(I)$.

Definition 10. An (induced) 4-cycle C of $G(I)$ is a *crossover* 4-cycle if it belongs to some crossing gadget.

Definition 11. The induced 4-cycles F_i and F'_i, with $i \in [6]$ of some clause gadgets C_j, $j \in [m]$ are called *special*.



(a) Crossing quadruple.



(b) Replacement of step 3 (b).

Fig. 4. Replacement of crossing quadruple by a crossing gadget.

The special 4-cycles of a particular clause gadget C_j are highlighted in Fig. 2 by coloring the corresponding faces with gray. In the next section, we show that $G(I)$ is indeed a 3-connected cubic bipartite planar graph.

3.3. $G(I)$ is Barnette

We shall show that the constructed graph is Barnette.

Lemma 12. *The graph $G(I)$ is 3-connected.*

Proof. Two gadgets are *adjacent* if they are connected by at least one edge. Observe that, for any two adjacent gadgets $\mathcal{X}, \mathcal{Y}$, there are two disjoint connector edges from $\mathcal{X}$ to $\mathcal{Y}$. We consider $G(I)$ after the removal of two vertices u, v.

First assume that u and v are not both connector vertices. Then two gadgets $\mathcal{X}$ and $\mathcal{Y}$ are adjacent in $G(I)$ if and only if they are adjacent in $G(I) - \{u, v\}$. In particular, if Q denotes the partition of $(G(I) - \{u, v\})$ into gadgets, then the quotient graph[8] $(G(I) - \{u, v\})/Q$ is connected.

In this case, for $G(I) - \{u, v\}$ to be disconnected, we need $G(I)$ to contain a gadget, say $\mathcal{A}$, two vertices of which are disconnected after the removal of $\{u, v\}$. In particular, $G(I)[\mathcal{A}]$ is disconnected by the removal of u and v. This forces both u and v to be picked inside $\mathcal{A}$: indeed every gadget is 2-connected by construction. It remains to prove that $\mathcal{A} - \{u, v\}$ is still connected in $G(I) - \{u, v\}$. We go through the three kinds of gadgets for $\mathcal{A}$.

- Consider a variable gadget $\mathcal{X}_i$, and let x_i be the "variable" vertex of the incidence graph $\text{INC}(I)$ associated with $\mathcal{X}_i$. As not "variable" vertex of $\text{INC}(I)$ is a cutvertex, $\text{INC}(I) - x_i$ is connected. Let a and b be two connector vertices of $\mathcal{X}_i - \{u, v\}$. Then there is a path in $G(I) - \{u, v\}$ from a (resp. b) to a clause gadget, say $C(a)$ (resp. $C(b)$). As $\text{INC}(I) - x_i$ is connected, there is a path from $C(a)$ to $C(b)$ in $G(I) - \{u, v\}$, and thus a and b are in the same component.
  As each vertex of $\mathcal{X}_i$ is connected to the set of connector vertices of $\mathcal{X}_i$ by three vertex-disjoint paths, all the vertices of $\mathcal{X}_i - \{u, v\}$ are in the same component.
- Consider a clause gadget C_j. For l in $\{a, b, c\}$ we have that $b'_{l,j}$ and $t'_{l,j}$ are in the same component of $G(I) - \{u, v\}$: indeed there is a path from them to a same variable gadget. Consider now l and l' in $\{a, b, c\}$ with $l \neq l'$. Then their is three disjoint paths (in $G(I)$) from $\{b'_{l,j}, t'_{l,j}\}$ to $\{b'_{l',j}, t'_{l,j}\}$: two on the outer face of C_j and one going through the interior of C_j. Thus, all the connector vertices of $\mathcal{X}_j$ are in the same component of $G(I) - \{u, v\}$.
  As each vertex of C_j is connected to the set of connector vertices of C_j by three vertex-disjoint paths, all the vertices of $C_j - \{u, v\}$ are in the same component.
- Consider a crossing gadget Z, then the split separates Z in two connected components. Note that if two connector vertices of $Z - \{u, v\}$ are adjacent to the same other gadget Y, then they are in the same component because Y itself is connected in $G(I) - \{u, v\}$. As at most one of $\{u, v\}$ is a connector vertex, all the connector vertices of $Z - \{u, v\}$ are in the same component of $G(I) - \{u, v\}$. Every vertex of $Z - \{u, v\}$ being adjacent to a connector vertex, $Z - \{u, v\}$ is connected in $G(I) - \{u, v\}$.

We now deal with the case when both u and v are connector vertices. Observe that every gadget remains connected by removing two of its connector vertices. Therefore every gadget is connected in $G(I) - \{u, v\}$. By the first paragraph of this proof, the only

[8] Where vertices are gadgets and edges link all the pairs of adjacent gadgets

interesting case is when u and v are the endpoints of two distinct connector edges between the same pair of gadgets. Then, the effect of removing u, v is to remove the link between the two gadgets.

However, there is no edge of $\mathrm{inc}(I)$ whose removal disconnects $\mathrm{inc}(I)$: otherwise it would have a "variable" vertex that is a cutvertex. In turn, one can see that this implies that the gadget adjacency graph cannot be disconnected by the removal of a link between two gadgets. $\square$

Lemma 13. *The graph $G(I)$ is Barnette.*

Proof. By the plane embedding of the crossing gadgets, $G(I)$ is planar. One can check that $G(I)$ is cubic by observing that within each gadget (variable, clause, crossing), all the vertices have degree 3, except vertices of degree 2, which are exactly those with an incident edge leaving the gadget. By Lemma 12, $G(I)$ is 3-connected.

We shall thus prove the bipartiteness of $G(I)$. Recall that our construction had three main components: variable gadgets, clause gadgets and crossing gadgets. Observe that each gadget itself is bipartite. Therefore, we shall only concentrate on those faces that consist of vertices from different gadgets.

For a particular gadget H, observe that all the outgoing edges of H lie in the external face of H. Circularly order the outgoing edges of H by $e_1, \ldots, e_p$, when going, say, clockwise. Take any two consecutive outgoing edges e_i, e_{i+1}. Let a_i, a_{i+1} be the vertices of H that are also incident to e_i and e_{i+1}, respectively. From our construction it follows that a_i and a_{i+1} are distinct. We can also observe from our construction that the path from a_i to a_{i+1}, denoted as $P(H, a_i, a_{i+1})$ along the external face of H in clockwise order always has an even number of vertices.

We call the path $P(H, a_i, a_{i+1})$ an *exposed path* of H. (Observe that a particular gadget has several exposed paths.) Let F be a bounded face of $G(I)$. If F is a finite face of a variable, clause, or crossing gadget H, then $|V(F)|$ is even because H is bipartite. Otherwise, F is a union of exposed paths, and since all exposed paths have an even number of vertices, we have that $|V(F)|$ is even. $\square$

3.4. Properties of variable and crossing gadgets

Lemma 14. *Let M be a perfect matching cut of $G(I)$. Then for any variable gadget $\mathcal{X}_i$, $M \cap V(\mathcal{X}_i)$ is the matching formed by the red edges in Fig. 1. In particular, M does not contain any connector edge incident to a variable gadget.*

Proof. Consider the variable gadget $\mathcal{X}_i$. By applying Lemma 7 on the 6-cycle S_i^2 (which satisfies the requirement of having four particular edges in some 4-cycles), we get that all outgoing edges of $V(S_i^2)$ are in M. We can thus apply Lemma 6 on the 6-cycles S_i^1 and S_i^3, and obtain that all outgoing edges of these cycles are in M.

Now, there is an outgoing edge of the 4-cycle S_i^4 that is in M; hence, by Lemma 4, all of them are. We can finally apply Lemma 6 on the 6-cycle S_i^5, and get that all the red edges of Fig. 1 should indeed be in M. In particular, as all the vertices of $\mathcal{X}_i$ are touched by red edges, the connector edges incident to a variable gadget cannot be in M. $\square$

Now we prove a property of the crossover 4-cycles.

Lemma 15. *Let M be a perfect matching cut of $G(I)$ and F be a crossover 4-cycle. Then $|E(F) \cap M| = 2$.*

Proof. Say that a *path of 4-vertex cycles* is a sequence $C_1, \ldots, C_k$ of vertex-disjoint 4-cycles such that C_i is adjacent to C_{i+1}. Considering step 3 of the construction of the crossover gadgets, observe that for every crossover 4-cycle C, there is a path of 4-vertex cycles starting at C and ending at a crossover 4-cycle adjacent to a variable gadget corresponding to the variable whose incident edge was responsible for introducing the crossing gadget and the crossover four cycle C.

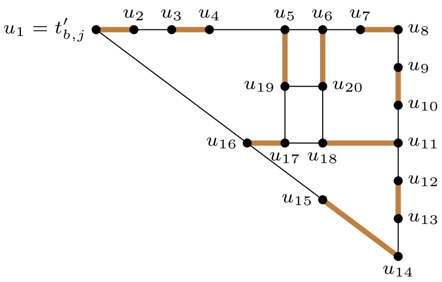
By Lemma 14, no edge incident to a variable gadget is in M. Thus any crossover 4-cycle adjacent to a variable gadget contains an edge of M. Repeated applications of Corollary 5 imply that C contains an edge of M, and we conclude with Lemma 4 applied on C. $\square$

Corollary 16. *For any perfect matching M of $G(I)$, M contains no connector edges.*

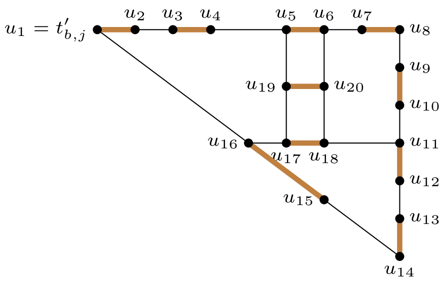
Proof. Observe that every connector edge is adjacent to at least one crossing or variable gadget. We conclude by Lemmas 14 and 15 which implies that any connector edge incident to a crossing gadget or to a variable gadget is not in M. $\square$

3.5. Properties of clause gadgets

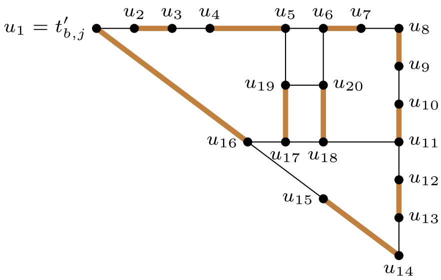
Consider the construction of the clause gadget, Fig. 2. For a clause gadget C_j, consider the induced cycles (each of order six) named as D_j^1, D_j^2, and D_j^3. These three induced cycles together are called D_j. Note that D_j is an induced subgraph of the clause gadget C_j.



(a) Edges of L_j^1 are in brown.



(b) Edges of L_j^2 are in brown.



(c) Edges of L_j^3 are in brown.

Fig. 5. The three types of perfect matching cuts of the graph induced by U_j of a clause gadget.

Lemma 17. *Any perfect matching cut of $G(I)$ contains the edges of D_j drawn in red in Fig. 2.*

Proof. The lemma follows from the application of the arguments used in the proof of Lemma 14. □

We prove a property of the special 4-cycles of a clause gadget.

Lemma 18. *Let M be a perfect matching cut of $G(I)$ and F be a special 4-cycle of C_j. Then $|E(F) \cap M| = 2$, and no outgoing edge of $V(F)$ is in M.*

Proof. We know from Corollary 16 that a connector edge is not in M, and from Lemma 17 that F_5 (see Fig. 2) has an incident edge not contained in M. For every $i \in [4]$, there is a path of 4-cycles between F_i (resp. F'_i) and F_1, which is incident to an edge that is not in M. Thus, every special 4-cycle is connected by a path of 4-cycles to a 4-cycle incident to an edge not in M. By application of Lemma 4 and Corollary 5, every special 4-cycle of C_j contains an edge of M. □

Lemma 19. *Let M be a perfect matching cut of $G(I)$ and C_j be a clause gadget. Let $U_j = \{u_1, \ldots, u_{20}\}$, and $V_j = \{v_1, \ldots, v_{20}\}$. Then, no outgoing edge of U_j or V_j is in M.*

Proof. By Lemma 18 and Corollary 16, the only edges that remain to be checked are $u_9v_9, u_{10}v_{10}, u_{12}v_{12}, u_{13}v_{13}$.

Suppose M contains u_9v_9 and therefore does not contain u_8u_9. As u_8u_9 is not available, by Lemma 4 on the cycle $u_7u_8w_1w_5$, we get that $w_1w_2 \notin M$. Symmetrically, we have that $w_3w_4 \notin M$. As by Lemma 18 u_8w_1, v_8w_4 and w_2w_3 are not in M, u_9v_9 would be the only edge of $u_9, u_8, w_1, w_2, w_3, w_4, v_8, v_9$ to be in M which is impossible by Lemma 2. A symmetric argument rules out that $u_{13}v_{13} \in M$. Thus we conclude applying Lemma 4 on the 4-cycles $u_9v_9v_{10}u_{10}$ and $u_{12}v_{12}v_{13}u_{13}$. □

From now on, we assume that for a clause gadget, the two sets $U_j = \{u_1, \ldots, u_{20}\}$, and $V_j = \{v_1, \ldots, v_{20}\}$ are defined. Now we shall prove that for every clause gadget C_j and perfect matching cut M, the set $M \cap E\left(U_j \cup V_j\right)$ can be of only three types. Before we prove the corresponding lemma, we introduce the following notations. Let H denote the subgraph of $G(I)$ induced by the vertices of $U_j \cup V_j$ of the clause gadget C_j. Let the vertices of H be named as shown in Fig. 2. We define the following sets, which are also illustrated in Fig. 5:

$$L_j^1 = \left\{u_1u_2, u_3u_4, u_5u_{19}, u_6u_{20}, u_7u_8, u_9u_{10}, u_{16}u_{17}, u_{18}u_{11}, u_{12}u_{13}, u_{15}u_{14}\right\},$$
$$L_j^2 = \left\{u_1u_2, u_3u_4, u_5u_6, u_7u_8, u_{19}u_{20}, u_9u_{10}, u_{16}u_{15}, u_{17}u_{18}, u_{11}u_{12}, u_{13}u_{14}\right\},$$
$$L_j^3 = \left\{u_2u_3, u_4u_5, u_6u_7, u_8u_9, u_1u_{16}, u_{19}u_{17}, u_{20}u_{18}, u_{10}u_{11}, u_{12}u_{13}, u_{15}u_{14}\right\}.$$

For $i \in \{1,2,3\}$, let R_j^i denote the set of edges $\{v_k v_l : u_k u_l \in L_j^i\}$.

Definition 20. We say that a perfect matching cut M of $G(I)$ is of *type i in C_j* with $i \in \{1,2,3\}$, if $M \cap E(U_j \cup V_j) = L_j^i \cup R_j^i$.

Lemma 21. *Let M be a perfect matching cut of $G(I)$ and C_j be a clause gadget. Then there exists exactly one integer $i \in \{1,2,3\}$ such that M is of type i in C_j.*

Proof. Let H denote the subgraph of $G(I)$ induced by the vertices of $U_j \cup V_j$ of the clause gadget C_j. Consider the 4-cycle C induced by $u_{17}, u_{18}, u_{19}, u_{20}$. Consider first the case when $M \cap E(C) = \emptyset$. In this case, applying Lemma 4 on C, we know that $\{u_{19}u_5, u_{20}u_6, u_{17}u_{16}, u_{18}u_{11}\} \subset M$; see Fig. 5a. Since, due to Lemma 19 no outgoing edge of U_j is in M, it is now easy to verify that $L_j^1 \subset M$. In the case where $M \cap E(C) = \{u_{19}u_{20}, u_{17}u_{18}\}$, applying Lemma 4 on the 4-cycle induced by u_5, u_6, u_{19}, u_{20}, we infer that $u_5u_6 \in M$, and once again it is now easy to verify that $L_j^2 \subset M$; see Fig. 5b. In the last case, $M \cap E(C) = \{u_{19}u_{17}, u_{18}u_{20}\}$. We again apply Lemma 4 on the 4-cycle induced by u_5, u_6, u_{19}, u_{20}, and infer this time that $u_5u_6 \notin M$. As no outgoing edge of U_j is in M, it is now easy to verify that $L_j^3 \subset M$; see Fig. 5c.

Observe that in L_j^1, $u_9u_{10} \in M$ and $u_{12}u_{13} \in M$, while for L_j^2 we have $u_9u_{10} \in M$ and $u_{12}u_{13} \notin M$ and for L_j^2 we have $u_9u_{10} \notin M$ and $u_{12}u_{13} \in M$. Thus $M \cap U_j$ is determined by the containment of u_9u_{10} and of $u_{12}u_{13}$ in M. This is also the fact, by symmetry, for $V_j \cap M$, when considering the edges v_9v_{10} and $v_{12}v_{13}$.

At this point, apply Lemma 4 to the two 4-cycles u_9, u_{10}, v_{10}, v_9 and $v_{17}, v_{18}, v_{19}, v_{20}$. We have that $u_9u_{10} \in M$ if and only if $v_9v_{10} \in M$, and $u_{12}u_{13} \in M$ if and only if $v_{12}v_{13} \in M$. Thus L_j^i propagates to $L_j^i \cup R_j^i$. □

As a direct consequence of Lemma 21, we get the following.

Lemma 22. *Let M be a perfect matching cut of $G(I)$ and let (A, B) be the cut of M. The vertices u_1, u_8, u_{14} of a clause gadget C_j cannot all be on the same side of M. More precisely:*

1. *L_j^1 sets u_1 to one side of M, and u_8, u_{14} to the other;*
2. *L_j^2 sets u_{14} to one side of M, and u_1, u_8 to the other;*
3. *L_j^3 sets u_8 to one side of M, and u_1, u_{14} to the other.*

Note that for a clause gadget C_j, if M is of type 1 (type 2, type 3, respectively) in C_j, then the edges in $M \cap E\left(C_j\right)$ are indicated in Fig. 6 (Fig. 7a, Fig. 7b, respectively) with brown colored edges.

3.6. Relation between variable and clause gadgets

Lemma 23. *Let M be a perfect matching cut of $G(I)$. Then for a variable x_i and a clause C_j with $x_i \in C_j$, $t_{i,j}, t'_{i,j}, b_{i,j}, b'_{i,j}$ are on the same side of M.*

Proof. Observe that $t_{i,j}$ and $b_{i,j}$ are connected by an edge that is not in M, hence they are on the same side of M.

Our construction of $G(I)$ ensures that there exists an even non-negative integer k (where $k = 0$ if $t_{i,j}$ and $t'_{i,j}$ are adjacent) such that all the following holds:

- there are k crossover 4-cycles $F_1, F_2, \ldots, F_k$ and a path P between $t_{i,j}$ and $t'_{i,j}$ where

$$V(P) \setminus \{t_{i,j}, t'_{i,j}\} \subset \bigcup_{l \in k} V(F_l)$$

- for each $1 \leqslant l \leqslant k$, $E(P) \cap E(F_l)$ is a 2-edge subpath.

Now due to Lemma 15 we know that for any $1 \leqslant l \leqslant k$, $|M \cap E(F_l)| = 2$. The above arguments further imply that $|M \cap E(F_l) \cap E(P)| = 1$. This implies that $|E(P) \cap M| = k$, which is even. Hence due to Observation 8 we have that $t_{i,j}$ and $t'_{i,j}$ are on the same side of M.

Using similar reasoning we can infer that $b'_{i,j}$ is on the same side as $b_{i,j}$. Hence $t_{i,j}, t'_{i,j}, b_{i,j}, b'_{i,j}$ are all on the same side. □

Lemma 24. *Let M be a perfect matching cut of $G(I)$. Then for any clause gadget C_j corresponding to the clause $C_j = (x_a, x_b, x_c)$ with $a < b < c$, the following hold:*

(a) *$t'_{c,j}$ and u_{14} are on the same side of M, and*
(b) *$b'_{a,j}$ and u_8 are on the same side of M.*

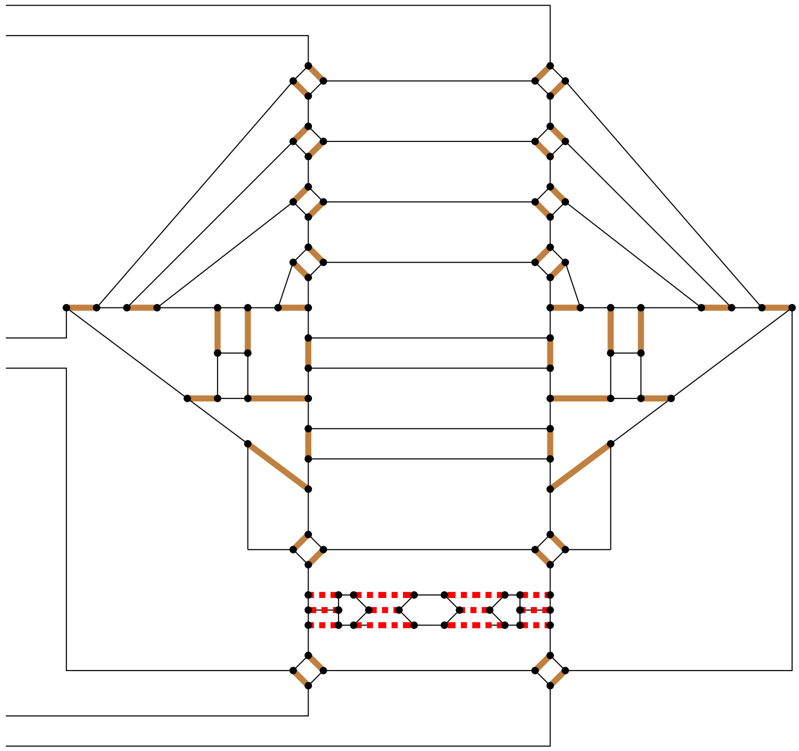
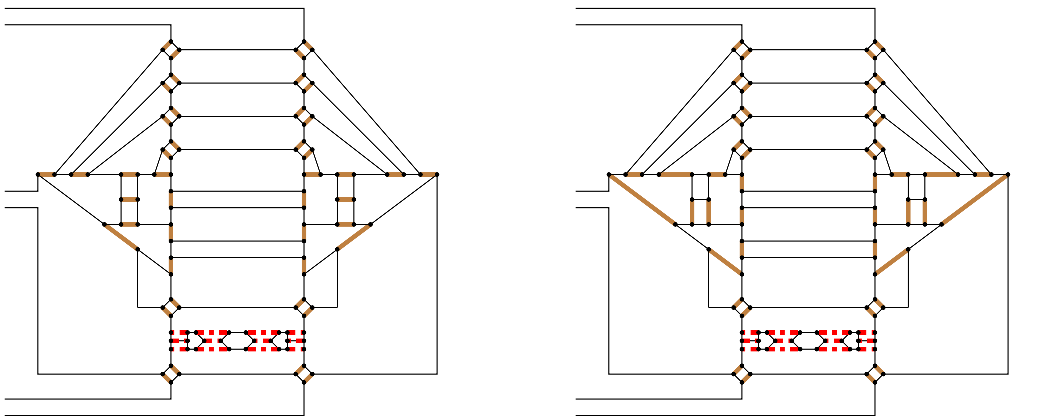


Fig. 6. The brown and red (dash-dotted) edges make the only intersection of a clause gadget with a perfect matching cut M such that $M \cap U_j = L_j^1$.



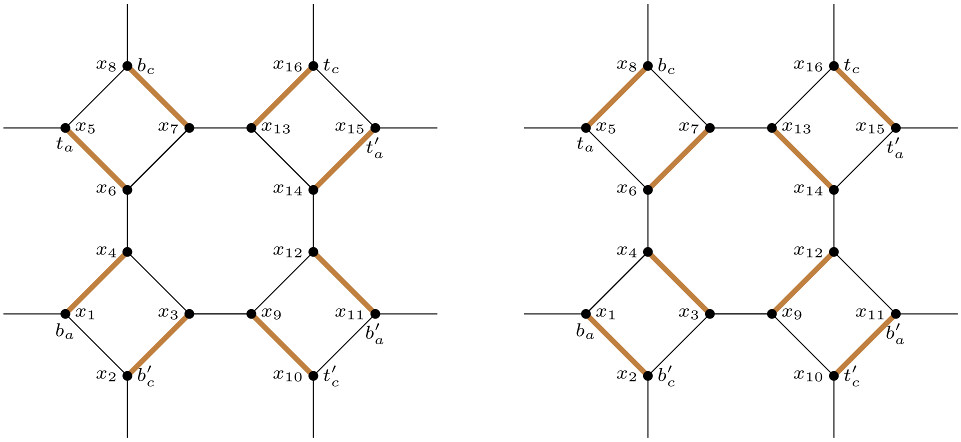
(a) Edges (brown and red (dash-dotted)) implied by L_j^2.

(b) Edges (brown and red (dash-dotted)) implied by L_j^3.

Fig. 7. Same as Fig. 6 for L_j^2 (left) and L_j^3 (right).

Proof. First we prove (a). Using Fig. 2 observe that there exists a path P between $t'_{c,j}$ and u_{14} such that P can be written as $t'_{c,j}\ z_1\ z_2\ d_1\ d_2\ d_3\ z_3\ z_4\ z_5\ u_{14}$ where $\{z_1, z_2\} \subset V(F_6)$ and $\{z_3, z_4, z_5\} \subset V(F_5)$. Note that F_5 and F_6 are special 4-cycles. Due to Lemma 18, we have that $|M \cap E(F_5)| = 2$ and $|M \cap E(F_6)| = 2$. This implies there exists exactly one edge $e \in \{t'_{c,j}z_1, z_1z_2\}$ such that $e \in M$. Similarly, there exists exactly one edge $e' \in \{z_3z_4, z_4z_5\}$ such that $e' \in M$. Moreover, from Lemma 17 it follows that none of $\{z_2d_1, d_1d_2, d_2d_3, d_3z_3\}$ belongs to M. Hence $M \cap E(P) = \{e, e'\}$, and $|M \cap E(P)|$ is even. Now applying Observation 8 we conclude that $t'_{c,j}$ and u_{14} are on the same side of M.

Now we prove (b). Using Fig. 2 observe that there exists a path P' between $b'_{a,j}$ and u_8 such that P' can be written as $b'_{c,j}\ z_1\ z_2\ z_3\ z_4\ z_5\ z_6\ z_7\ z_8\ z_9\ z_{10}\ z_{11}\ u_8$ where $\{z_1, z_2\} \subset V(F_1)$, $\{z_3, z_4, z_5\} \subset V(F_2)$, $\{z_6, z_7, z_8\} \subset V(F_3)$, and $\{z_9, z_{10}, z_{11}\} \subset V(F_4)$. Now arguing similarly as in (a) on the special 4-cycles F_1, F_2, F_3, F_4, we have that $|M \cap E(P)|$ is even. By Observation 8, we conclude that $b'_{a,j}$ and u_8 are on the same side of M. □



(a) Edges of P_j^1 are drawn in brown. **(b)** Edges of P_j^2 are drawn in brown.

Fig. 8. P_j^1 and P_j^2 are the only possible restrictions of M to a crossing gadget.

For any crossing gadget X_j as drawn in Fig. 8 we consider the two perfect matching cuts P_j^1 of Fig. 8a and P_j^2 of Fig. 8b on X_j. Any other partial solutions (e.g. $\{x_5x_8, x_6x_7, x_{13}x_{14}, x_{15}x_{16}, x_1x_4, x_2x_3, x_9x_{10}, x_{11}x_{12}\}$) would locally pick an odd number of edges in the face containing b_a and t_a (or in the one containing b_c or t_c). Indeed, no connector edge can be in the perfect matching cut by Corollary 16. Thus, each C_4 of the crossing gadget contains two edges of a perfect matching cut. Moreover, given a perfect matching cut of $\{x_5, x_6, x_7, x_8\}$, there is a single way of extending it to the whole gadget, if we want the perfect matching cut to contain a locally even number of edges on the face containing b_a and t_a and on the face containing b_c and t_c. This extension is either P_j^1 or P_j^2. Note that P_j^1 and P_j^2 contains a locally even number of edges on the face containing b'_a and t'_a and on the face containing b'_c and t'_c.

We conclude by induction: crossing gadgets can be partitioned in parts $P_1, P_2, \ldots$, where P_1 contains the crossing gadgets that share these two faces with variable gadgets, and P_{i+1} contains the crossing gadgets that share this two faces with variable gadgets or crossing gadgets in $\cup_{j \leqslant i} P_j$. The variable gadgets force a locally even number of picked edges on those faces, namely 0, and if an even number of edges of those faces is picked, outside the crossing gadget, then it is either P_j^1 or P_j^2.

Lemma 25. *Let X_j be a crossing gadget of $G(I)$ as shown in Fig. 8. For any $M \in \{P_j^1, P_j^2\}$, M is a perfect matching cut of X_j. The vertices t_a, b_a, t'_a, b'_a are always on the same side of M, and t_c, b_c, t'_c, b'_c are always on the same side of M. Moreover, if $M = P_j^1$, t_a and t_c are on the same side of M (in X_j), otherwise they are not.*

Proof. We refer to Fig. 8 for the notations on X_j. M is a perfect matching cut of X_j by Lemma 3. Let C be the external facial cycle of X_j. We conclude by Observation 8 on paths contained in C starting at t_a or t_c. $\square$

3.7. Existence of perfect matching cut implies satisfiability

In this section, we show that if $G(I)$ has a perfect matching cut then I has a satisfying assignment. Let M be a perfect matching cut of $G(I)$ and (A, B) be the cut of M. As we already observed, a potential assignment to I can be seen as a partition $\mathcal{P} = (\mathcal{V}_A, \mathcal{V}_B)$ of the variables, where $\mathcal{V}_A$ consists of all variables set to true and $\mathcal{V}_B$ consists of all variables set to false. We set x_i in $\mathcal{V}_A$ if and only if $V(S_i^2) \subset A$, and we show that $\mathcal{P}$ satisfies the Monotone Not-All-Equal 3SAT-E4-instance I.

Assume for a contradiction that there exists a clause C_j such that all variables, x_a, x_b, x_c, of C_j are on the same side of $\mathcal{P}$. Thus all the vertices in $\cup_{i \in \{a,b,c\}} V(S_i^2)$ are on the same side of M. Assume without loss of generality that this side is A. Let z_i be any vertex of S_i^2.

Now fix an integer $i \in \{a, b, c\}$. Observe, using Fig. 1, that there exists a path P between z_i and $t_{i,j}$ such that $|M \cap E(P)|$ is even. Hence, due to Observation 8, we infer that $t_{i,j}$ lies in A. Now due to Lemma 23 we have that $\left\{t_{i,j}, t'_{i,j}, b_{i,j}, b'_{i,j}\right\} \subset A$. The above discussion implies that

$$\bigcup_{i \in \{a,b,c\}} \left\{t_{i,j}, t'_{i,j}, b_{i,j}, b'_{i,j}\right\} \subset A.$$

Now applying Lemma 24, we have that all three vertices in $\{u_1, u_8, u_{14}\}$ lie in A. But this contradicts Lemma 22. Hence we get the following.

Lemma 26. *If $G(I)$ has a perfect matching cut then I is a satisfiable instance.*

3.8. Satisfiability implies the existence of a perfect matching cut

In this section, we show that given a Monotone Not-All-Equal 3SAT-E4-instance I and a partition $\mathcal{P} = (\mathcal{V}_A, \mathcal{V}_B)$ satisfying I, we can construct a perfect matching cut $M_{\mathcal{P}}$ of $G(I)$, such that:

- for each variable gadget $\mathcal{X}_i$, $M_{\mathcal{P}} \cap V(\mathcal{X}_i)$ is the matching imposed by Lemma 14,
- for each crossing gadget X_j, X_j has been introduced from a crossing quadruple. Hence there exists two edges e and f, incident to X_j such that $\text{var}(e) \neq var(f)$, we choose P_j^1 if $\text{var}(e)$ and $\text{var}(f)$ are on the same side of $\mathcal{P}$, and P_j^2 otherwise,
- for each clause gadget C_j over variables a, b, c, we choose the matching of Fig. 6 if b is not on the same side of $\mathcal{P}$ as a and c, the matching of Fig. 7a if c is not on the same side of $\mathcal{P}$ as a and b, and the matching of Fig. 7b in the last case.

As $M_{\mathcal{P}}$ is a perfect matching on each gadget, and as every vertex belongs to some gadget, $M_{\mathcal{P}}$ is a perfect matching of $G(I)$. By construction, $M_{\mathcal{P}}$ contains no connector edges. Recall that any edge that does not have both endpoints inside the same gadget is a connector edge. We call *connector vertex* a vertex v incident to a connector edge e, and such that $\text{var}(v) = \text{var}(e)$.

Lemma 27. *For any path Q between two connector vertices u and v, we have that $|Q \cap M_{\mathcal{P}}|$ even if and only if var(u) and var(v) are on the same side of $\mathcal{P}$.*

Proof. As $M_{\mathcal{P}}$ does not contain any connector edges, $|Q \cap M_{\mathcal{P}}|$ is defined by the parts of Q inside a gadget. Let $Q_1, \dots, Q_k$ be spanning vertex-disjoint subpaths of Q such that for any i, Q_i lies inside a gadget and, there is a connector edge from the last vertex of Q_i to the first vertex of Q_{i+1}, for every $1 \leqslant i < k$. We prove the lemma by induction on k.

If $k = 1$, then $Q = Q_1$ lies inside a gadget, and the property is true by Lemma 14 for variable gadgets, Lemma 22 for clause gadgets and Lemma 25 for crossing gadgets.

Assume that the property is true for $i \leqslant k-1$, let u' be the last vertex of Q_{k-1} and v' the first vertex of Q_k. By induction, $\text{var}(u)$ and $\text{var}(u')$ are on the same side of $\mathcal{P}$ if and only if $|\bigcup_{1 \leqslant i \leqslant k-1} E(Q_i) \cap M_{\mathcal{P}}|$ is even, and $\text{var}(v')$ and $\text{var}(v)$ are on the same side of $\mathcal{P}$ if and only if $|E(Q_k) \cap M_{\mathcal{P}}|$ is even. As $\text{var}(u') = \text{var}(v')$, we know that $\text{var}(u')$ is on the same side of $\mathcal{P}$ as $\text{var}(v')$, moreover $u'v' \notin M_{\mathcal{P}}$. Thus $\text{var}(u)$ and $\text{var}(v)$ are on the same side of $\mathcal{P}$ if and only if $|\bigcup_{1 \leqslant i \leqslant k} E(Q_i) \cap M_{\mathcal{P}}|$ and $|E(Q_k) \cap M_{\mathcal{P}}|$ have the same parity, thus $|\bigcup_{1 \leqslant i \leqslant k} E(Q_i) \cap M_{\mathcal{P}}|$ is even if and only if $\text{var}(u)$ and $\text{var}(v)$ are on the same side of $\mathcal{P}$. □

Lemma 28. *$M_{\mathcal{P}}$ is a perfect matching cut of $G(I)$.*

Proof. We already know that $M_{\mathcal{P}}$ is a perfect matching. We will show that $M_{\mathcal{P}}$ is a cutset by Lemma 3. Let C be any cycle in $G(I)$. If C is contained in a gadget then, as $M_{\mathcal{P}}$ is a cutset when restricted to a gadget, $|C \cap M_{\mathcal{P}}|$ is even. Otherwise, C contains a connector edge uv, so we can see C as the concatenation of the edge uv and a path Q from v to u. We know that $uv \notin M_{\mathcal{P}}$, and $\text{var}(u) = \text{var}(v)$. From Lemma 27 it follows that, $|E(C) \cap M_{\mathcal{P}}| = |E(Q) \cap M_{\mathcal{P}}|$ is even. □

We finally get Theorem 1, due to Lemmas 13, 26, and28.

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