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Separability in Morse local-to-global groups

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Abstract

We show that in a Morse local-to-global group where stable subgroups are separable, the product of any stable subgroups is separable. As an application, we show that the product of stable subgroups in virtually special groups is separable.

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1 | INTRODUCTION

Given a group G, we can equip it with the *profinite topology*, whose basic open subsets are cosets of finite index subgroups of G. A subset of G is said to be *separable* if it is closed in the profinite topology on G. The group G is called *residually finite* if the trivial subgroup is separable in G.

Knowing that particular subsets of groups are separable often gives useful information about the group. For example, in a finitely presented group, separability of a finitely generated subgroup gives a solution to the membership problem for that subgroup. In a geometric setting, separability properties of the fundamental group of a space correspond to desirable lifting properties of that space: immersed subcomplexes of a complex X may be promoted to embedded ones in a finite sheeted cover of X, provided that their corresponding subgroups are separable in $\pi_1 X$. For an example involving subsets rather than subgroups, it has been proven that if X is a nonpositively curved cube complex in which every double coset of hyperplane stabilisers is separable in $\pi_1 X$, then X has a finite sheeted special cover [9].

It is a difficult problem to show that a given subset of a group is separable, especially when one is only given some geometric data about the group. For instance, even the question of whether hyperbolic groups are residually finite is a long-standing open problem. It is known that all hyperbolic groups are residually finite if and only if every quasi-convex subgroup is separable in every hyperbolic group [1]. Minasyan showed that if G is a hyperbolic group in which every quasi-convex subgroup is separable, the setwise product of any finite number of quasi-convex subgroups is also separable in G [13], extending a result of Ribes and Zalesskii, who proved that the same result in the case G is free [18]. Recently, the first author and Minasyan provided generalisation of this result in the setting of relatively hyperbolic groups [14]. In this paper, we will provide another natural generalisation of this product separability result to the class of groups with the *Morse local-to-global* (MLTG) property.

Introduced in [16], the MLTG property roughly speaking requires that quasi-geodesics with hyperbolic-like properties behave similarly to quasi-geodesics in hyperbolic spaces. Consider the following two perspectives on hyperbolic spaces. The first involves Morse geodesics: we say that a quasi-geodesic is *Morse* if any other quasi-geodesic with the same endpoints stays uniformly finite Hausdorff-distance from it (see Definition 2.2). It is a well-known fact that every quasi-geodesic in a hyperbolic space satisfies the Morse property, and moreover, that a space is hyperbolic if and only if all of its geodesics are uniformly Morse [2] (for a discussion on uniformity, see Section 10 in [4]). This motivates the study of Morse quasi-geodesics in spaces that are not hyperbolic, an approach that has been successful in understanding the properties of spaces up to quasi-isometry [3, 5, 8, 10, 15]. On the other hand, Gromov showed that a space is hyperbolic if and only if all of it local quasi-geodesics, that is, paths that are quasi-geodesics on every subpath of a certain length (see Definition 2.1) are globally quasi-geodesics. The MLTG property puts these two perspectives together and prescribes that all paths that are locally Morse quasi-geodesics are globally Morse quasi-geodesics.

In groups with the MLTG property, elements acting on Morse geodesics behave 'as they should'. For instance, it is appealing to think that given two independent infinite order elements with Morse axes, then it is possible to use a ping-pong argument to generate a free subgroup using these elements. In general finitely generated groups this is not true, and we require the MLTG property in order to run such arguments. The above suggests that the failure of the MLTG property seems to imply some pathological behaviour. Indeed, the only known examples of groups without the MLTG property are not finitely presentable. On the other hand, many well-behaved classes of groups, such as 3-manifold groups, CAT(0) groups and mapping class groups are known to satisfy the MLTG property.

Our main theorem is concerned with separability of products of *stable subgroups*. Stable subgroups were introduced by Durham and Taylor, who showed that the convex cocompact subgroups of the mapping class groups are precisely the stable ones [6]. For infinite cyclic subgroups, the notion of stability and fixing a Morse quasi-geodesic agree, and in general stable subgroups present many properties akin to quasi-convex subgroups of hyperbolic groups.

Theorem 1.1. *Let G be a finitely generated group with the MLTG property, and suppose that any stable subgroup of G is separable. Then, the product of any stable subgroups of G is separable.*

Recall that a group is locally extended residually finite (*LERF*) if every finitely generated subgroup is separable. The following statement may be of more general interest, for instance, as a criterion for showing that a given group is not LERF. As stable subgroups are always finitely generated (see Lemma 2.8), the hypotheses are stronger than the above theorem.

Corollary 1.2. *Let G be a finitely generated LERF group with the MLTG property. Then, the product of any stable subgroups of G is separable.*

A group is *virtually special* if it is has a finite index subgroup that is the fundamental group of a special cube complex. Triple cosets of convex subgroups in virtually special groups are known to be separable [19]. We extend this result to arbitrary products of stable subgroups, which are quasi-convex.

Corollary 1.3. *Let G be a virtually special group. Then, the product of any stable subgroups of G is separable.*

Strongly quasi-convex subgroups, also known as *Morse* subgroups, were introduced independently by Genevois and Tran [7, 20], and Tran showed that a subgroup of a finitely generated group is stable if and only if it is strongly quasi-convex and hyperbolic [20, Proposition 4.3]. In the case of right-angled Artin groups, which contain many stable subgroups [11], we can use [17, Corollary 7.4] to deduce the following.

Corollary 1.4. *Suppose that Γ is a finite connected graph, and let A_Γ be the associated right-angled Artin group. Then, the product of any strongly quasi-convex subgroups of A_Γ is separable.*

2 | PRELIMINARIES

Let us establish some notational conventions. Given a group G and subgroup $H \leqslant G$, we will write $H \leqslant_f G$ when H has finite index in G. If $g \in G$, we will use H^g to denote the conjugate subgroup gHg^{-1}.

For a metric space X and points $x, y, z \in X$, we will write

$$\langle x, y\rangle_z = \frac{1}{2}(\mathrm{d}(x,z) + \mathrm{d}(y,z) - \mathrm{d}(x,y))$$

for the *Gromov product* of x and y with respect to z. When X is the Euclidean plane, $\langle x, y\rangle_z$ is exactly the distance between z and the points of tangency on an incircle for the triangle with vertices x, y and z. The Gromov product thus acts as a vague analogue for the notion of the angle spanned by two geodesics issuing from a single point in a metric space.

In this paper, we will restrict our attention to Cayley graphs of groups. Let G be a group and S a generating set for G. The *Cayley graph* of G with respect to S is the graph $\mathrm{Cay}(G, S)$ with vertex set G and elements g, h connected by an edge if either $gh^{-1} \in S$ or $hg^{-1} \in S$.

We equip the set of vertices of a graph with the metric induced by declaring all of its edges to have length one. For a Cayley graph $\mathrm{Cay}(G, S)$, we write d_S for this metric, which is exactly the word metric on G with respect to S. For $g \in G$, we will write $|g|_S = \mathrm{d}_S(1, g)$.

Given a path γ of a graph, we will denote its length (i.e. number of edges) by $\ell(\gamma)$. Given metric spaces (X, d_X) and (Y, d_Y), a (λ, c)-*quasi-isometric embedding* is a map $f : X \to Y$ such that the following holds for any pair $x, y \in X$.

$$\frac{1}{\lambda}\mathrm{d}_Y(f(x), f(y)) - c \leqslant \mathrm{d}_X(x, y) \leqslant \lambda \mathrm{d}_Y(f(x), f(y)) + c.$$

A (λ, c)-*quasi-geodesic* is a (λ, c)-quasi-isometric embedding of an interval $I \subset \mathbb{R}$.

The main geometric definition of the paper is the MLTG property. To define it, we need to define the Morse property and what is means for a given property of a path to be local.

Definition 2.1 (Local property). A path $\gamma : I \to X$ is said to L-*locally* satisfy a property P if each subpath of the form $\gamma|_{[t_1,t_2]}$ with $t_2 - t_1 \leqslant L$ satisfies P. When a path γ is L-locally a (λ, c)-quasi-geodesic, we say that γ is a $(L; \lambda, c)$-local quasi-geodesic.

Definition 2.2 (Morse quasi-geodesic). Let $M : \mathbb{R}_{\geqslant 1} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ be a non-decreasing function. A quasi-geodesic $\gamma : I \to X$ is *M-Morse* if for any (λ, c)-quasi-geodesic segment $\eta : [a, b] \to X$ such that $\eta(a) = \gamma(t_1), \eta(b) = \gamma(t_2)$, we have

$$\mathrm{d}_{Haus}(\gamma|_{[t_1,t_2]}, \eta) \leqslant M(\lambda, c),$$

where d_{Haus} denotes the Hausdorff distance. We say that γ is an $(M; \lambda, c)$-*Morse quasi-geodesic*, and M is its *Morse gauge*.

Morse geodesics in any geodesic space satisfy a thin triangles condition, similar to geodesics in a hyperbolic metric space.

Lemma 2.3 [12, Lemma 3.6]. *Let X be a geodesic metric space and suppose that p and q are M-Morse geodesics with $p_- = q_-$. There is a constant $\delta = \delta(M) \geqslant 0$ such that for any geodesic r with endpoints $r_- = p_+$ and $r_+ = q_+$, the geodesic triangle with sides p, q and r is δ-thin.*

Definition 2.4 (Local Morse quasi-geodesic). We say that a path is an $(L; M; \lambda, c)$-*local Morse quasi-geodesic* if it is L-locally an M-Morse (λ, c)-quasi-geodesic.

Definition 2.5 (MLTG property). We say that a metric space X satisfies the *MLTG property*, if for any choice of Morse gauge M and constants $\lambda \geqslant 1, c \geqslant 0$, there exist a scale $L > 0$, a Morse gauge M' and constants $\lambda' \geqslant 1, c' \geqslant 0$ such that every $(L; M; \lambda, c)$-local Morse quasi-geodesic is a $(M'; \lambda', c')$-Morse quasi-geodesic.

The strength of the MLTG property is that it allows us to draw global conclusions from local conditions, as the next lemma shows.

Lemma 2.6. *Let $p = p_1 * \cdots * p_n$ be a concatenation of M-Morse geodesics in space X with the MLTG property and let a_i and a_{i+1} be the ordered endpoints of p_i. For each $\varepsilon > 0$, there are constants $B \geqslant 0, \lambda \geqslant 1, c \geqslant 0$, and a gauge N (all depending only on M and ε) such that if we have $\ell(p_i) > B$ for all $i = 2, \dots, n-1$ and $\langle (a_{i-1}), (a_{i+1}) \rangle_{a_i} \leqslant \varepsilon$ for all $i = 2, \dots, n$, then p is a $(N; \lambda, c)$-Morse quasi-geodesic.*

Proof. We will show that there exists M' depending only on M and ε such that p is locally a $(M'; 1, 2\varepsilon)$-Morse quasi-geodesic, and then we will choose an appropriate B to use the MLTG property. Given the existence of such M', the MLTG property gives us a Morse gauge N and constants $\lambda \geqslant 1, c \geqslant 0, L \geqslant 0$ such that any $(L; M'; 1, 2\varepsilon)$-local Morse quasi-geodesic is also a $(N; \lambda, c)$-Morse quasi-geodesic.

We start with the quasi-geodesic claim. Take $B \geqslant L + \varepsilon$ and observe that if we consider two points x, y at parameterised distance at most L, they either lie on the same segment p_i (which is geodesic), or on two consecutive segments p_{i-1} and p_i. In the latter case, we have

$$\langle x, y \rangle_{a_i} \leqslant \langle a_{i-1}, a_{i+1} \rangle_{a_i} \leqslant \varepsilon,$$

which means

$$\mathrm{d}(x, y) + 2\varepsilon \geqslant \mathrm{d}(a_i, x) + \mathrm{d}(a_i, y) = \ell(p_{i-1}|_{[x,a_i]} * p_i|_{[a_i,y]}).$$

Thus, p is an $(L;1,2\varepsilon)$-local quasi-geodesic. A similar computation to that above (with $x = a_{i-1}, y = a_{i+1}$) shows that

$$\mathrm{d}(a_{i-1}, a_{i+1}) \geqslant 2B - 2\varepsilon > L$$

where the last inequality comes from the choice of B. Now, applying [16, Lemma 2.15] to each concatenation $p_i * p_{i+1}$, we obtain some M' (depending only on M and ε) such that p is an $(L;M';1,2\varepsilon)$-local Morse quasi-geodesic. Now applying the MLTG property shows that p is $(N;\lambda,c)$-Morse quasi-geodesic. □

The property of stability generalises the notion of having the Morse property from quasi-geodesics to arbitrary subgroups.

Definition 2.7 (Stable subgroup). Let G be a group with finite generating set S, and let M be a Morse gauge and $\mu \geqslant 0$ a constant. A subgroup $H \leqslant G$ is called (M,μ)-*stable* if any geodesic in $\mathrm{Cay}(G,S)$ with endpoints in H is M-Morse and lies in the μ-neighbourhood of H. A subgroup is called *stable* if it is (M,μ)-stable for some Morse gauge M and $\mu \geqslant 0$.

An immediate consequence of the definition is that a stable subgroup of a finitely generated group is undistorted. We note that while the gauge M and constant μ in the above depend on the choice of generating set S, the property of being stable does not (see, for example, [6, Lemma 3.4]). We recall some basic properties of stability, which follow from [6].

Lemma 2.8. *Let G be a group with finite generating set S and suppose $H \leqslant G$ is (M,μ)-stable. Then the following are true:*

(1) *if $K \leqslant_f H$, then K is (M,μ')-stable for some $\mu' \geqslant 0$;*
(2) *if $g \in G$, then gHg^{-1} is stable;*
(3) *H is finitely generated and undistorted in G;*
(4) *H is hyperbolic.*

The following lemma tells us that stable subgroups cannot be 'too parallel' away from their intersection. More precisely, that there is a uniform upper bound on the Gromov product of elements from one subgroup when taken with minimal length coset representatives of the other.

Lemma 2.9. *Let G be a group with finite generating set S, and suppose that H and K are (M,μ)-stable subgroups of G. There is a constant $\rho = \rho(M,\mu,S) \geqslant 0$ such that if $h \in H$ is a shortest (with respect to S) representative of its right coset $(H \cap K)h$, then for any $k \in K$, we have $\langle h,k\rangle_1 \leqslant \rho$.*

Proof. Suppose for a contradiction that we can find elements $h \in H, k \in K$ such that h is a shortest coset representative of $h(H \cap K)$ and $\langle h,k\rangle_1$ is arbitrarily large. Since H and K are stable, any choice of geodesics $p = [1,h]$ and $q = [1,k]$ is M-Morse and lies in a μ-neighbourhood of H and K, respectively.

Let $a_1,\dots,a_n$ be the vertices of p with $\mathrm{d}_S(1,a_i) \leqslant \langle h,k\rangle_1$. The assumption that $\langle h,k\rangle_1$ can be taken to be arbitrarily large means that n can be taken to be arbitrarily large. Corresponding to each vertex a_i, there is $v_i \in H$ such that $\mathrm{d}_S(a_i,v_i) \leqslant \mu$ by stability of H. Moreover, by Lemma 2.3, there is $\delta = \delta(M) \geqslant 0$ such that $\mathrm{d}_S(a_i,q) \leqslant \delta$ for each $i = 1,\dots,n$. By stability of K and the triangle inequality, therefore, we obtain that $d_S(v_i, K) \leqslant 2\mu + \delta$ for each $i = 1, \dots, n$. Note that since p is geodesic

$$d_S(1, v_i) \leqslant i - \mu \qquad \text{and} \qquad d_S(v_i, h) \leqslant d_S(1, h) - i - \mu \tag{1}$$

for each $i = 1, \dots, n$.

For each $i = 1, \dots, n$, let g_i be the shortest element of G with respect to S such that $v_i g_i \in K$, so $|g_i|_S \leqslant 2\mu + \delta$. Let N be the number of elements in the ball of radius $2\mu + \delta$ about the identity in $\mathrm{Cay}(G, S)$. Taking n to be sufficiently large with respect to N and μ, there must be some pair (i, j) with $g_i = g_j$ satisfying $j - i > 2\mu$. Then, Equation (1) gives

$$d_S(1, v_i) + d_S(v_j, h) < d_S(1, h). \tag{2}$$

But then $v_j g_j (v_i g_i)^{-1} = v_j v_i^{-1} \in H \cap K$, as $v_i, v_j \in H$ and $v_i g_i, v_j g_j \in K$. Moreover,

$$\begin{aligned} d_S(v_j v_i^{-1}, h) &\leqslant d_S(v_j v_i^{-1}, v_j) + d_S(v_j, h) \\ &= d_S(1, v_i) + d_S(v_j, h) < d_S(1, h), \end{aligned}$$

where the last inequality is an application of (2). It follows that $v_i v_j^{-1} h \in (H \cap K)h$ and

$$|v_i v_j^{-1} h|_S = d_S(v_j v_i^{-1}, h) < d_S(h, 1) = |h|_S,$$

which contradicts the fact that h is a minimal length representative of its $(H \cap K)$-coset. Thus, there must be an upper bound on the Gromov product. □

We finish this section by recalling the key property that we need for our proof, namely that the MLTG property allows a ping-pong-type argument for stable subgroups.

Proposition 2.10 [16, Theorem 3.1]. *Let G be a group with finite generating set S and suppose that G has the MLTG property. Let $Q, R \leqslant G$ be (M, μ)-stable subgroups of G. There is a constant $C = C(M, \mu, S) \geqslant 0$ such that the following is true.*

Let $Q' \leqslant Q$ and $R' \leqslant R$ be subgroups such that $Q' \cap R' = Q \cap R$ and $|g|_S \geqslant C$ for each $g \in (Q' \cup R') \setminus (Q' \cap R')$. Then $\langle Q', R' \rangle \cong Q' *_{Q' \cap R'} R'$. Moreover, if Q' and R' are finitely generated and undistorted in G, then $\langle Q', R' \rangle$ is stable.

3 | SEPARABILITY OF PRODUCTS

In this section, we will prove the main theorem. We start with some elementary observations about separable subsets.

Remark 3.1. If $U \subseteq G$ is a separable subset of G, then U^{-1}, gU and Ug are separable for any $g \in G$.

Remark 3.2. Let $H_1, \dots, H_n \leqslant G$ be subgroups of G and let $a_0, \dots, a_n \in G$. Observe that

$$a_0 H_1 a_1 \dots a_{n-1} H_n a_n = H_1^{a_0} H_2^{a_0 a_1} \dots H_n^{a_0 \dots a_{n-1}} a_0 \dots a_n,$$

which is a translate of a product of conjugates of the subgroups $H_1, \dots, H_n$. The set $a_0 H_1 a_1 \dots a_{n-1} H_n a_n$ is thus separable if and only if the product of subgroups $H_1^{a_0} \dots H_n^{a_0 \dots a_{n-1}}$ is separable.

In particular, suppose that there is $n \in \mathbb{N}$ such that any product of n stable subgroups is separable in G, and suppose $H_1, \dots, H_n$ are stable subgroups of G. Lemma 2.8(2) gives that $H_i^{a_0 \dots a_{i-1}}$ is stable for each $1 \leqslant i \leqslant n$, so that the set $H_1^{a_0} \dots H_n^{a_0 \dots a_{n-1}}$ is a product of n stable subgroups. By the observation above, we may conclude that the set $a_0 H_1 a_1 \dots a_{n-1} H_n a_n$ is separable in this situation.

In order to exploit the geometric properties afforded by the MLTG property, it is useful to choose coset representative that is geometrically meaningful, which we can do by the following remark.

Remark 3.3. Suppose that S is a generating set for group G and let $H_1, \dots, H_n \leqslant G$ be subgroups of G. Given elements $x_1 \in H_1, \dots, x_n \in H_n$, there are elements $y_1 \in H_1, \dots, y_n \in H_n$ such that $x_1 \dots x_n = y_1 \dots y_n$ and $|y_i|_S$ is minimal among elements of the coset $(H_{i-1} \cap H_i) y_i$ for each $1 < i \leqslant n$.

Indeed, there is $y_n \in H_n$ and $z_n \in H_n \cap H_{n-1}$ such that $x_n = z_n y_n$ and $|y_n|_S$ is minimal among elements of $(H_{n-1} \cap H_n) x_n = (H_{n-1} \cap H_n) y_n$. Similarly, there is $y_{n-1} \in H_{n_1}$ and $z_{n-1} \in H_{n-1} \cap H_{n-2}$ such that $x_{n-1} z_n = z_{n-1} y_{n-1}$ and y_{n-1} is a shortest representative of $(H_{n-2} \cap H_{n-1}) x_{n-1} z_n = (H_{n-2} \cap H_{n-1}) y_{n-1}$. We can proceed by finite induction to find elements $y_2 \in H_2, \dots, y_n \in H_n$ and $z_2 \in H_2 \cap H_3, \dots, z_n \in H_{n-1} \cap H_n$ with the properties described above. Setting $y_1 = x_1 z_2 \in H_1$ completes the observation.

We conclude the section by proving the main theorem of the paper and the related corollaries.

Proof of Theorem 1.1. We proceed by induction on the number n of stable subgroups. The case $n = 1$ is exactly the hypothesis that stable subgroups of G are separable, so let $H_1, \dots, H_n$ be stable subgroups of G with $n > 1$ and suppose that the product of any $n-1$ stable subgroups of G is separable.

Fix a finite generating set S of G. By taking maxima of gauges and constants, we may assume without loss of generality that $H_1, \dots, H_n$ are (M, μ)-stable. Let $\rho = \rho(M, \mu, S)$ be the constant obtained from Lemma 2.9, and let $C = C(M, \mu, S)$ be the constant of Proposition 2.10. Let $B, \lambda, c \geqslant 0$ be the constants and N the Morse gauge obtained from applying Lemma 2.6 with gauge M and constant ρ.

Suppose for a contradiction that the product $H_1 \dots H_n$ is not separable, so that there is some $g \notin H_1 \dots H_n$ belonging to the profinite closure of $H_1 \dots H_n$. This means that g is contained in every separable subset containing $H_1 \dots H_n$. For ease of reading, we will write $Q = H_1, R = H_2$ and $T_i = H_{i+2}$ whenever $1 \leqslant i \leqslant s = n-2$. By hypothesis, Q and R are separable, and thus, their intersection $I = Q \cap R$ is also. Let $\{N_i\}_{i \in \mathbb{N}}$ be an enumeration of the finite index subgroups of G containing I, and note that $I = \bigcap_{i \in \mathbb{N}} N_i$ as I is separable. For each i, we write

$$N_i' = \bigcap_{j=1}^{i} N_j$$

so that $\{N_i'\}_{i \in \mathbb{N}}$ is a sequence of nested finite index subgroups of G containing I whose intersection is equal to I.

For each $i \in \mathbb{N}$, we define the set

$$K_i = Q\langle Q'_i, R'_i\rangle RT_1 \dots T_s,$$

where $Q'_i = N'_i \cap Q \leqslant_f Q$ and $R'_i = N'_i \cap R \leqslant_f R$. Note that $I \subseteq N'_i$ for each $i \in \mathbb{N}$, so that $Q'_i \cap R'_i = I$. It is immediate from the definition that $K_i \supseteq QRT_1 \dots T_s$ for each $i \in \mathbb{N}$. Our aim is to show that for sufficiently large i, the set K_i is separable and excludes the element g.

Let us first show that K_i is separable when i is large. Indeed, for a given i, let $x_1, \dots, x_a$ be left coset representatives for Q'_i in Q and $y_1, \dots, y_b$ be right coset representatives for R'_i in R. Then, we have

$$K_i = \bigcup_{j=1}^{a} \bigcup_{k=1}^{b} x_j \langle Q'_i, R'_i\rangle y_k T_1 \dots T_s.$$

Since I is the intersection of all N'_i, there is an index $i_0 \in \mathbb{N}$ such that for any $i \geqslant i_0$, any element $n \in N'_i$ with $|n|_S \leqslant C$ belongs to I. By (1) and (3) of Lemma 2.8, Q'_i and R'_i are finitely generated and undistorted, so we may apply Proposition 2.10 to obtain that $\langle Q'_i, R'_i\rangle$ is stable. Thus, by Remark 3.2 and the induction hypothesis, K_i can be written as a finite union of separable subsets, and so K_i is separable whenever $i \geqslant i_0$.

We now show that there is $i \in \mathbb{N}$ such that $g \notin K_i$. As g belongs to the profinite closure of $QRT_1 \dots T_s$ and for each $i \geqslant i_0$, the set K_i is a profinitely closed subset of G containing $QRT_1 \dots T_s$, g belongs to K_i for each $i \geqslant i_0$. That is, for each $i \geqslant i_0$, we may write

$$g = q^{(i)} x_1^{(i)} \dots x_{m_i}^{(i)} r^{(i)} t_1^{(i)} \dots t_s^{(i)} \tag{3}$$

for some $m_i \in \mathbb{N}$ and $x_j^{(i)} \in Q'_i \cup R'_i$ for each $1 \leqslant j \leqslant m_i$, and where $q^{(i)} \in Q, r^{(i)} \in R, t_1^{(i)} \in T_1, \dots, t_s^{(i)} \in T_s$.

The remainder of the argument may be split into two essentially different cases based on the lengths of the elements obtained above: we summarise them here.

In one case, the lengths of infinitely many of the elements $r^{(i)}, t_1^{(i)}, \dots, t_{s-1}^{(i)}$ remain bounded as i tends to infinity. When this happens, we may pass to a subsequence where one these terms is constant in i. This reduces the number of subgroups we have to consider in the product and we may apply the induction hypothesis to obtain our contradiction. The other situation to consider is when the lengths of these elements increase without bound. In this case, for sufficiently large values of i, the products as in (3) define paths that are (arbitrarily long) local Morse quasi-geodesics. The MLTG property then shows that these are actually Morse quasi-geodesics, resulting in another contradiction.

<u>*Case 1:*</u> $\liminf_{i\to\infty} |r^{(i)}|_S < \infty$ or $\liminf_{i\to\infty} |t_j^{(i)}|_S < \infty$ for some $1 \leqslant j < s$.

We consider only the possibility that $\liminf_{i\to\infty} |r^{(i)}|_S < \infty$, for the other cases can be dealt with identically. It follows from this assumption that there is a subsequence of $(r^{(i)})_{i\in\mathbb{N}}$ whose terms have length bounded by some fixed constant. Since there are only finitely many elements of G with any given length with respect to S, we may pass to a further subsequence whose terms are all equal to a single element $r \in R$. Hence, we have

$$g \in Q\langle Q'_i, R'_i\rangle rT_1 \dots T_s \ \text{ for infinitely many } i \in \mathbb{N}. \tag{4}$$

Now by the induction hypothesis and Remark 3.2, the set $QrT_1 \dots T_s$ is separable in G. Since $g \notin QRT_1 \dots T_s$, we have $g \notin QrT_1 \dots T_s$, and so, there is $N \triangleleft_f G$ such that $g \notin NQrT_1 \dots T_s = QNrT_1 \dots T_s$. The subgroup $IN \leqslant_f G$ is a finite index subgroup of G containing I, so $N'_{i_1} \subseteq IN$ for some $i_1 \in \mathbb{N}$. Since the sequence of subgroups $\{N'_i\}_{i\in\mathbb{N}}$ is nested, we have thus shown that

$$Q\langle Q'_i, R'_i\rangle rT_1 \dots T_s \subseteq QN'_i rT_1 \dots T_s \subseteq QINrT_1 \dots T_s = QNrT_1 \dots T_s$$

for any $i \geqslant i_1$, where the last equality uses the fact that $QI = Q$. However, the fact that $g \notin QNrT_1 \dots T_s$ now contradicts the inclusions of (4), so this case is impossible.

Case 2: $\liminf_{i\to\infty} |r^{(i)}|_S = \infty$ and $\liminf_{i\to\infty} |t_j^{(i)}|_S = \infty$ for all $1 \leqslant j < s$.

Define $z_0 = 1, z_1 = q^{(i)}, z_2 = z_1 x_1^{(i)}, \dots, z_{m_i+1} = z_{m_i} x_{m_i}^{(i)}, z_{m_i+2} = z_{m_i+1} r^{(i)}$ and $z_{m_i+3} = z_{m_i+2} t_1^{(i)}, \dots, z_{m_i+2+s} = z_{m_i+s} t_s^{(i)}$. For each $0 \leqslant j \leqslant m_i + 1 + s$, we let p_j be a geodesic with $(p_j)_- = z_j$ and $(p_j)_+ = z_{j+1}$. Let p be the concatenation $p_0 * \cdots * p_{m_i+1+s}$ of these paths.

We will use Lemma 2.6 to conclude that the path p is a uniform quasi-geodesic. Assuming this, the fact that $\liminf_{i\to\infty} |r^{(i)}|_S = \infty$ and $\liminf_{i\to\infty} |t_j^{(i)}|_S = \infty$ means that for sufficiently large i, the distance between the endpoints of p is greater than $|g|_S$, contradicting the fact that p represents g.

Without loss of generality, we may assume $x_1^{(i)} \in R'_i \setminus Q$ and $x_{m_i}^{(i)} \in Q'_i \setminus R$, for otherwise we may replace $q^{(i)}$ with $q_1^{(i)} = q^{(i)} x_1^{(i)} \in Q$ and eliminate $x_1^{(i)}$ from the product (and likewise with $r^{(i)}$ and $x_{m_i}^{(i)}$). Further, we may assume by Remark 3.3 that $x_1^{(i)}, \dots, x_{m_i}^{(i)}$, and $r^{(i)}$ are shortest representatives of their right I-cosets, and, in particular, $x_1^{(i)}, \dots, x_{m_i}^{(i)}, r^{(i)} \notin I$. Similarly, we take $t_1^{(i)}$ to be a shortest representative of $(R \cap T_1)t_1^{(i)}$ and, for $1 < i \leqslant s$, the element $t_j^{(i)}$ to be a shortest representative of $(T_{j-1} \cap T_j)t_j^{(i)}$.

The above paragraph together with Lemma 2.9 shows that

$$\langle z_{j-1}, z_{j+1}\rangle_{z_j} \leqslant \rho \quad \text{for } j = 1, \dots, m_i + s. \tag{5}$$

We now verify the hypotheses of Lemma 2.6. Each of the geodesic segments p_i represents an element of a finite index subgroup of one of $Q, R, T_1, \dots, T_{s-1}$, or T_s. Therefore, by Lemma 2.8(1), we obtain that the geodesic segments p_i are M-Morse. For any given $B' \geqslant B$ (recall that B is the constant of Lemma 2.6 applied with M and ρ), we deduce the following. Since $\liminf_{i\to\infty} |r^{(i)}|_S = \infty$ and $\liminf_{i\to\infty} |t_j^{(i)}|_S = \infty$, we have that $|r^{(i)}|_S > B'$ and $|t_j^{(i)}|_S > B'$ for each $j = 1, \dots, s$ and sufficiently large i. Moreover, since $x_j^{(i)} \in (Q'_i \cup R'_i) \setminus I \subseteq N'_i \setminus I$ and since $\bigcap N'_i = I$, for i large enough, we have $|x_j^{(i)}|_S > B'$. Thus, Lemma 2.6 implies that p is $(N; \lambda, c)$-Morse quasi-geodesic. Finally, choosing B' sufficiently large with respect to λ, c, and $|g|_S$, gives us that the endpoints of p are a greater distance than $|g|_S$ apart, the desired contradiction.

From the above, there is some $i \in \mathbb{N}$ such that K_i is separable, contains the product $QRT_1 \dots T_s$ and excludes g. Therefore, the product $QRT_1 \dots T_s = H_1 \dots H_n$ is separable. ☐

Proof of Corollary 1.3. Stable subgroups are quasi-convex, and quasi-convex subgroups of virtually special groups are separable by [9, Corollary 7.9]. Moreover, CAT(0) groups have the MLTG property by [16, Theorem D]. Therefore, Theorem 1.1 applies to give the result. ☐

Proof of Corollary 1.4. Let $H_1, \dots, H_n$ be strongly quasi-convex subgroups of A_Γ. If there is some $1 \leqslant i \leqslant n$ such that H_i has finite index, then the product $H_1 \dots H_n$ is a union of finitely many cosets of H_i. Since H_i has finite index, it is separable in A_Γ, whence $H_1 \dots H_n$ is separable.

Now suppose that each of the subgroups $H_1, \dots, H_n$ has infinite index in A_Γ. By [17, Corollary 7.4], they must be stable subgroups. Noting that A_Γ is CAT(0) and hence has the MLTG property [16, Theorem D], the conclusion follows by applying Theorem 1.1. □

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REFERENCES

1. I. Agol, D. Groves, and J. F. Manning, *Residual finiteness, QCERF and fillings of hyperbolic groups*, Geom. Topol. **13** (2009), no. 2, 1043–1073.
2. M. Bonk, *Quasi-geodesic segments and Gromov hyperbolic spaces*, Geom. Dedicata **62** (1996), no. 3, 281–298.
3. M. Cordes and D. Hume, *Stability and the Morse boundary*, J. London Math. Soc. (2) **95** (2017), no. 3, 963–988.
4. C. H. Cashen and J. M. Mackay, *A metrizable topology on the contracting boundary of a group*, Trans. Amer. Math. Soc. **372** (2019), no. 3, 1555–1600.
5. M. Cordes, J. Russell, D. Spriano, and A. Zalloum, *Regularity of Morse geodesics and growth of stable subgroups*, J. Topol. **15** (2022), no. 3, 1217–1247.
6. M. G. Durham and S. J. Taylor, *Convex cocompactness and stability in mapping class groups*, Algebr. Geom. Topol. **15** (2015), no. 5, 2839–2859.
7. A. Genevois, *Hyperbolicities in* CAT(0) *cube complexes*, Enseign. Math. **65** (2019), no. 1–2, 33–100.
8. M. Graeber, A. Karrer, N. Lazarovich, and E. Stark, *Surprising circles in Morse boundaries of right-angled Coxeter groups*, Topology Appl. **294** (2021), Paper No. 107645, 3.
9. F. Haglund and D. T. Wise, *Special cube complexes*, Geom. Funct. Anal. **17** (2008), no. 5, 1551–1620.
10. M. Incerti-Medici, *Comparing topologies on the Morse boundary and quasi-isometry invariance*, Geom. Dedicata **212** (2021), 153–176.
11. T. Koberda, J. Mangahas, and S. J. Taylor, *The geometry of purely loxodromic subgroups of right-angled Artin groups*, Trans. Amer. Math. Soc. **369** (2017), no. 11, 8179–8208.
12. Q. Liu, *Dynamics on the Morse boundary*, Geom. Dedicata **214** (2021), 1–20.
13. A. Minasyan, *Separable subsets of GFERF negatively curved groups*, J. Algebra **304** (2006), no. 2, 1090–1100.
14. A. Minasyan and L. Mineh, *Quasiconvexity of virtual joins and separability of products in relatively hyperbolic groups*. arXiv preprint, 2022.
15. Y. Qing and K. Rafi, *Sublinearly Morse boundary I:* CAT(0) *spaces*, Adv. Math. **404** (2022), no. part B, Paper No. 108442, 51.
16. J. Russell, D. Spriano, and H. C. Tran, *The local-to-global property for Morse quasi-geodesics*, Math. Z. **300** (2022), no. 2, 1557–1602.

17. J. Russell, D. Spriano, and H. C. Tran, *Convexity in hierarchically hyperbolic spaces*, Algebr. Geom. Topol. **23** (2023), no. 3, 1167–1248.
18. L. Ribes and P. A. Zalesskii, *On the profinite topology on a free group*, Bull. London Math. Soc. **25** (1993), no. 1, 37–43.
19. S. Shepherd, *Imitator homomorphisms for special cube complexes*, Trans. Amer. Math. Soc. **376** (2023), no. 1, 599–641.
20. H. C. Tran, *On strongly quasiconvex subgroups*, Geom. Topol. **23** (2019), no. 3, 1173–1235.