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Existence Results for an *m*-Point Mixed Fractional-Order Problem at Resonance on the Half-Line

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Abstract: This work considers the existence of solutions for a mixed fractional-order boundary value problem at resonance on the half-line. The Mawhin's coincidence degree theory will be used to prove existence results when the dimension of the kernel of the linear fractional differential operator is equal to two. An example is given to demonstrate the main result obtained.

Keywords: coincidence degree; fractional-order; half-line; m-point; resonance

MSC: 34B40; 34B15

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1. Introduction

Fractional calculus has become increasingly popular lately as a result of some interesting properties of the fractional derivative. For instance, the fractional derivative has a memory property that enables its future state to be determined by the current state and all the previous states. This makes fractional differential equations applicable in various fields of science and engineering [1–3].

When the corresponding homogeneous equation of a fractional boundary value problem (FBVP) has a trivial solution then the FBVP is a non-resonance problem and its solution can be obtained using fixed point theorems, see [4–7] and the references cited therein. When the homogeneous equation of a FBVP has a non-trivial solution then the problem is a resonance problem and the solution can be obtained using topological degree methods [8–15].

In [16], the authors consider a higher-order fractional boundary value problem involving mixed fractional derivatives:

$$(-1)^{m\,C}D^{\alpha}_{1-}D^{\beta}_{1+} + f(t, u(t)) = 0, \quad 0 \le t \le 1,$$

$$u(0) = u^{(i)}(0) = 0, i = 1, \dots, m+n-2, \quad D^{\beta+m-1}_{0+}u(1) = 0,$$

where ${}^{C}D^{\alpha}_{1-}$ is the left Caputo fractional derivative of order $\alpha \in (m-1, m)$ and D^{β}_{1+} is the right Caputo fractional derivative of order $\beta \in (n-1, n)$, where $m, n \ge 2$ are integers.

Guezane Lakoud et al. [17] obtained existence results for a fractional boundary value problem at resonance on the half-line:

$$-{}^{C}D^{\alpha}_{0-}D^{\beta}_{0+}x(t) + f(t, x(t)) = 0, \quad t \in [0, 1],$$

$$u(0) = u'(0) = u(1) = 0,$$

where $-{}^{C}D^{\alpha}_{0-}$ is the left Caputo fractional derivative of order $\alpha \in (0, 1]$, and D^{β}_{0+} is the right Caputo fractional derivative of order $\beta \in (1, 2]$.

Zhang and Liu [15] considered the following FBVP

$$D^{\alpha}_{0+}x(t) = f(t, x(t), D^{\alpha-2}_{0+}x(t), D^{\alpha-1}_{0+}x(t)), \quad t \in (0, 1),$$

$$x(0)=0,\quad D_{0+}^{\alpha-1}x(0)=\sum_{i=1}^{+\infty}\alpha_i D_{0+}^{\alpha-1}x(\xi_i),\quad D_{0+}^{\alpha-1}x(1)=\sum_{i=1}^{+\infty}\alpha_i D_{0+}^{\alpha-1}x(\gamma_i),$$

where $2<\alpha\leq 3$, D_{0+}^{α} is the Riemann–Liouville derivative of order α, $f\in[0,1]\times\mathbb{R}^3\to\mathbb{R}$ is a Caratheodory function, $\xi_i,\ \gamma_i\in(0,1)$ and $\{\xi_i\}_{i=1}^{+\infty},\ \{\gamma_i\}_{i=1}^{+\infty}$ are two monotonic sequences with $\lim_{i\to+\infty}\xi_i=a,\ lim_{i\to+\infty}\gamma_i=b,\ a,b\in(0,1),\ \alpha_i,\beta_i f\in\mathbb{R}$.

Imaga et al. [18] obtained existence results for the following fractional-order boundary value problem at resonance on the half-line with integral boundary conditions:

$$D_{-}^{a}\phi_p(D_{0+}^{b}u(t))+e^{-t}w(t,u(t),D_{0+}^{b}u(t))=0,\ t\in(0,\infty), \tag{1}$$

$$I_{0+}^{1-b}u(0)=0,\ \phi_p(D_{0+}^{b}u(+\infty))=\phi_p(D_{0+}^{b}u(0)), \tag{2}$$

where D_{-}^{a} is the left Caputo fractional derivative on the half line and D_{0+}^{b} the right Riemann–Louville fractional derivative on the half-line, $0<a,b\leq 1$, $1<a+b\leq 2$, $\phi_p(r)=|r|^{p-2}$, $p>1$, with $\phi_q=\phi_p^{-1}$ and $1/q+1/p=1$. $w:[0,+\infty)\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function.

Chen and Tang [9] established existence of positive solutions for a FBVP at resonance in an unbounded domain:

$$D_{0+}^{\alpha}u(t)=f(t,u(t)),\quad t\in[0,+\infty),$$

$$u(0)=u'(0)=u''(0)=0,\quad D_{0+}^{\alpha-1}u(0)=\lim_{t\to+\infty}D_{0+}^{\alpha-1}u(t),$$

where D_{0+}^{α} is Riemann–Liouville fractional derivative, $3<\alpha<4$ and $f:[0,+\infty)\times\mathbb{R}\to\mathbb{R}$ is continuous.

Motivated by the results above, we will use the Mawhin coincidence degree theory [19] to study the solvability of the following mixed fractional-order m-point boundary value problem at resonance on the half-line:

$${}^{C}D_{0+}^{a}D_{0+}^{b}u(t)=f(t,u(t),D_{0+}^{b-1}u(t),D_{0+}^{b}u(t)),\quad t\in[0,+\infty) \tag{3}$$

$$I_{0+}^{2-b}u(0)=0,\ D_{0+}^{b-1}u(0)=\sum_{j=1}^{m}\alpha_j D_{0+}^{b-1}u(\xi_j),\ D_{0+}^{b}u(+\infty)=\sum_{k=1}^{n}\beta_k D_{0+}^{b}u(\eta_k) \tag{4}$$

where $f:[0,+\infty)\times\mathbb{R}^3\to\mathbb{R}$ is a continuous function, ${}^{C}D_{0+}^{a}$ is the Caputo fractional derivative, D_{0+}^{b} is the Riemann–Liouville fractional derivative, $0<a\leq 1$, $1<b\leq 2$, $0<a+b\leq 3$, $0<\xi_1<\xi_2<\cdots<\xi_m<+\infty$, $0<\eta_1<\eta_2<\cdots<\xi_m<+\infty$, $\alpha_j\in\mathbb{R}$, $j=1,2,\cdots,m$ and $\beta_k\in\mathbb{R}$, $k=1,2,\cdots,n$. The resonant conditions are $\sum_{k=1}^{n}\beta_k=\sum_{j=1}^{m}\alpha_j=1$ and $\sum_{k=1}^{n}\beta_k\eta_k^{-1}=\sum_{j=1}^{m}\alpha_j\xi_j^{-1}=0$.

In Section 2 of this work the required lemmas, theorem, and definitions will be given, while Section 3 is dedicated to stating and proving the main existence results. An example will be given in Section 4.

2. Materials and Methods

In this section, we will give some definitions and lemmas that will be used in this work.

Let U,Z be normed spaces, $L:dom\ L\subset U\to Z$ a Fredholm mapping of zero index and $A:U\to U,\ B:Z\to Z$ projectors that are continuous, such that:

$$Im\ A=\ker\ L,\ \ker\ B=Im\ L,\ U=\ker\ L\oplus\ker\ A,\ Z=Im\ L\oplus Im\ B.$$

Then,

$$L|_{dom\ L\cap\ker\ A}:dom\ L\cap\ker\ A\to Im\ L$$

is invertible. The inverse of the mapping L will be denoted by $K_A : Im\ L \to dom\ L \cap \ker\ A$ while the generalized inverse, $K_{A,B} : Z \to dom\ L \cap \ker\ A$ is defined as $K_{A,B} = K_A(I - B)$.

Definition 1. *Let $L : dom\ L \subset X \to Z$ be a Fredholm mapping, E a metric space and $N : E \to Z$ a non-linear mapping. N is said to be L-compact on E if $BN : E \to Z$ and $K_{A,B}N : E \to X$ are continuous and compact on E. Additionally, N is L-completely continuous if it is L-compact on every bounded $E \subset U$.*

Theorem 1 ([19]). *Let L be a Fredholm map of index zero and let N be L-compact on $\overline{\Omega}$ where $\Omega \subset U$ is an open and bounded. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(dom\ L\ \ker\ L \cap \partial\Omega] \times (0, 1)$;

(ii) $Nx \notin Im\ L$ *for every* $x \in \ker\ L \cap \partial\Omega$;

(iii) $deg(BN|_{\ker\ L}, \ker L, 0) \neq 0$, *where* $B : Z \to Z$ *is a projection with* $Im\ L = \ker\ B$.

Then, the abstract equation $Lu = Nu$ has at least one solution in $dom\ L \cap \overline{\Omega}$.

Definition 2 ([20]). *Let $\alpha > 0$, the Caputo and Riemann–Liouville fractional integral of a function x on $(0, +\infty)$ is defined by:*

$$I^{\alpha}_{0+}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(r)}{(r-t)^{1-\alpha}}dr, \quad t \in [0, 1]$$

Definition 3 ([20]). *Let $\alpha > 0$, the Caputo $({}^{C}D^{\alpha}_{0+}x(t))$ and Riemann–Liouville $(D^{\alpha}_{0+}x(t))$ fractional derivative of a function x on $(0, +\infty)$ is defined by:*

$${}^{C}D^{\alpha}_{0+}x(t) = D^{\alpha}_{0+}x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{x(r)}{(t-r)^{\alpha-n+1}}dr, \quad t \in (0, +\infty)$$

where $n = [a] + 1$.

Lemma 1 ([21]). *Let $a \in (0, +\infty)$. The general solution of the Riemman–Liouville fractional differential equation:*

$$D^{a}_{0+}g(t) = 0$$

is $g(t) = b_1t^{a-1} + b_2t^{a-2} + \cdots + b_nt^{a-n}$, where $b_j \in \mathbb{R}$, $j = 1, 2 \ldots, n$ while, the general solution of the Caputo fractional differential equation:

$$D^{a}_{0+}g(t) = 0$$

is $g(t) = d_0 + d_1t + \cdots + d_nt^n$, where $d_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$ and $n = [a] + 1$ is the smallest integer greater than or equal to a.

Lemma 2 ([21]). *Let $a \in (0, +\infty)$ and $i = 1, 2, \ldots, n$, $n = [a] + 1$ then*

$$(I^{a}_{0+}D^{a}_{0+}g)(t) = g(t) + d_1t^{a-1} + d_2t^{a-2} + \cdots + d_nt^{a-n}$$

holds almost everywhere on $[0, +\infty)$ for some $d_i \in \mathbb{R}$. Similarly,

$$(I^{a}_{0+}\,{}^{C}D^{a}_{0+}g)(t) = g(t) + d_0 + d_1t^1 + d_2t^2 + \cdots + d_nt^n$$

holds almost everywhere on $[0, +\infty)$ for some $d_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$.

Lemma 3 ([21]). *Let $a > 0$, $\rho > -1$, $t > 0$, $g(t) \in C[0, +\infty)$, then:*

(i) $I^{a}_{0+}t^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1+a)}t^{a+\rho}$;

(ii) $D^{a}_{0+}t^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-a)}t^{a-\rho}$, *for* $\rho > -1$, *in particular for* $D^{a}_{0+}t^{a-k} = 0$, $k = 1, 2, \ldots, N$, *where* N *is the smallest integer greater than or equal to* a;

(iii) $D^a_{0+}I^a_{0+}g(t) = g(t), g(t) \in C[0,+\infty)$;
(iv) $I^a_{0+}I^b_{0+}g(t) = I^{a+b}_{0+}g(t)$.

Let

$$U = \left\{ u \in C[0,+\infty) : \lim_{t\to+\infty} \frac{|u(t)|}{1+t^{a+b}}, \lim_{t\to+\infty} \frac{|D^{b-1}_{0+}u(t)|}{1+t^{a+1}} \text{ and } \lim_{t\to+\infty} \frac{|D^b_{0+}u(t)|}{1+t^a} \text{ exists} \right\}$$

with the norm $\|u\|_U = \max\{\|u\|_0, \|D^{b-1}_{0+}u\|_1, \|D^b_{0+}u\|_2\}$ defined on U where:

$$\|u\|_0 = \sup_{t\in[0,+\infty]} \frac{|u(t)|}{1+t^{a+b}}, \ \|D^{b-1}_{0+}u\|_1 = \sup_{t\in[0,+\infty]} \frac{|D^{b-1}_{0+}u(t)|}{1+t^{a+1}} \text{ and } \|D^b_{0+}u\|_2 = \sup_{t\in[0,+\infty]} \frac{|D^b_{0+}u(t)|}{1+t^a}.$$

Let $Z = \{z : C[0,+\infty) : \sup_{t\in[0,+\infty)} |z(t)| < +\infty\}$ equipped with the norm $\|z\|_Z = \sup_{t\in[0,+\infty)} |z(t)|$. The spaces $(U, \|\cdot\|_U)$ and $(Z, \|\cdot\|_Z)$ can be shown to be Banach Spaces. Additionally, define $Lu = {}^C D^a_{0+}D^b_{0+}u(t)$, with domain

$$\text{dom } L = \left\{ u \in U : {}^C D^a_{0+}D^b_{0+}u(t) \in Z, \text{ boundary conditions (4) is satisfied by } u \right\},$$

and the non-linear operator $N : U \to Z$ will be defined by

$$(Nu)t = f(t, u(t), D^{b-1}_{0+}u(t), D^b_{0+}u(t)), \quad t \in [0,+\infty),$$

hence, Equations (3) and (4) may be written as

$$Lu = Nu.$$

Definition 4. *The set $Y \subset U$ is said to be relatively compact if*

$$Y_1 = \left\{ \frac{u(t)}{1+t^{a+b}} : u \in Y \right\}, \quad Y_2 = \left\{ \frac{D^{b-1}_{0+}u(t)}{1+t^{a+1}} : u \in Y \right\}, \quad Y_3 = \left\{ \frac{D^b_{0+}u(t)}{1+t^a} : u \in Y \right\}$$

are uniformly bounded; equicontinuous on any compact subinterval of $[0,+\infty)$ and equiconvergent at: $+\infty$.

Definition 5. *The set $Y \subset U$ is said to be equiconvergent at $+\infty$ if given $\epsilon > 0$ there exists a $\tau(\epsilon) > 0$, such that:*

$$\left| \frac{u(t_1)}{1+t_1^{a+b}} - \frac{u(t_2)}{1+t_2^{a+b}} \right| < \epsilon, \left| \frac{D^{b-1}_{0+}u(t_1)}{1+t_1^{a+1}} - \frac{D^{b-1}_{0+}u(t_2)}{1+t_2^{a+1}} \right| < \epsilon \text{ and } \left| \frac{D^b_{0+}u(t_1)}{1+t_1^a} - \frac{D^b_{0+}u(t_2)}{1+t_2^a} \right| < \epsilon$$

where $t_1, t_2 > \tau$.

Lemma 4. $\ker L = \{c_1t^b + c_2t^{b-1} : c_1, c_2 \in \mathbb{R}, t \in [0,+\infty)\}$ *and Im* $L = \{z \in Z : B_1z = B_2z = 0\}$
where $B_1z = \sum_{k=1}^{n} \beta_k \int_0^{\eta_k} (\eta_k - r)^{a-1}z(r)dr$ *and* $B_2z = \sum_{j=1}^{m} \alpha_j \int_0^{\xi_j} (\xi_j - r)^a z(r)dr$.

Proof. Consider ${}^C D^a_{0+}D^b_{0+}u(t) = 0$ for $u \in \ker L$, then by Lemma 1

$$u(t) = c_1t^b + c_2t^{b-1} + c_3t^{b-2}, \quad c_1, c_2, c_3 \in \mathbb{R}.$$

Applying the boundary condition $I_{0+}^{2-b}u(0) = 0$, gives $c_3 = 0$. Thus, $u(t) = c_1 t^b + c_2 t^{b-1}$. Next, consider ${}^C D_{0+}^a D_{0+}^b u(t) = z(t)$ for $z(t) \in \text{Im } L$ and $u \in \text{dom } L$, then

$$u(t) = I_{0+}^{a+b} z(t) + c_1 t^b + c_2 t^{b-1} + c_3 t^{b-2}.$$

From $I_{0+}^{2-b}u(0) = 0$ we obtain $c_3 = 0$. Therefore,

$$D_{0+}^b u(t) = I_{0+}^a z(t) + c_1 + c_2 t^{-1} \tag{5}$$

By boundary condition $D_{0+}^b u(+\infty) = \sum_{k=1}^n \beta_k D_{0+}^b u(\eta_k)$ and the conditions $\sum_{k=1}^n \beta_k = 1$, $\sum_{k=1}^n \beta_k \eta_k^{-1} = 0$, (5) gives

$$B_1 z = \sum_{k=1}^n \beta_k \int_0^{\eta_k} (\eta_k - r)^{a-1} z(r) dr = 0,$$

Similarly,

$$D_{0+}^{b-1} u(t) = I_{0+}^{a+1} z(t) + c_1 t + c_2, \tag{6}$$

by boundary condition $D_{0^+}^{b-1} u(0) = \sum_{j=1}^m \alpha_j D_{0^+}^{b-1} u(\xi_j)$ and resonant conditions $\sum_{j=1}^m \alpha_j = 1$ and $\sum_{j=1}^m \alpha_j \xi_j^{-1} = 0$, (6) gives

$$B_2 z = \sum_{j=1}^m \alpha_j \int_0^{\xi_j} (\xi_j - r)^a z(r) dr.$$

□

Let $\Delta = (B_1 t^{b-1} e^{-t} \cdot B_2 t^b e^{-t}) - (B_2 t^{b-1} e^{-t} \cdot B_1 t^b e^{-t}) := (g_{11} \cdot g_{22}) - (g_{21} \cdot g_{12}) \neq 0$. Let the operator $B : Z \to Z$ be defined as

$$Bz = (\Delta_1 z) + (\Delta_2 z) \cdot t^b$$

where

$$\Delta_1 z = \frac{1}{\Delta}(\delta_{11} B_1 z + \delta_{12} B_2 z) e^{-t}, \ \Delta_2 z = \frac{1}{\Delta}(\delta_{21} B_1 z + \delta_{22} B_2 z) e^{-t},$$

and δ_{ij} is the algebraic cofactor of g_{ij}.

Lemma 5. *The following holds:*

(i) *$L : dom\ L \subset U$ is a Fredholm operator of index zero;*

(ii) *the generalized inverse $K_A : Im\ L \to dom\ L \cap \ker A$ may be written as*

$$K_A z = I_{0+}^{a+b} z(t).$$

Additionally,

$$\|K_A z\| = \|z\|_Z.$$

Proof. (i) For $z \in Z$, it is easily be seen that $\Delta_1((\Delta_1 z)) = (\Delta_1 z)$, $\Delta_1((\Delta_2 z)t^b) = 0$, $\Delta_2((\Delta_1 z)) = 0$, and $\Delta_2((\Delta_2 z)t^b) = (\Delta_2 y)$. Hence, $B^2 z = Bz$, thus Bz is a projector.

We now prove that $\ker B = \text{Im } L$. Let $z \in \ker B$, since $Bz = 0$ then $z \in \text{Im } L$. Conversely, if $z \in \text{Im } L$, then by $Bz = 0$, $z \in \ker B$. Therefore, $\ker B = \text{Im } L$.

Let $z \in Z$, then $z \in \text{Im } L$ and $z \in \ker B$, hence, $Z = \text{Im } L + \ker B$. Assuming $z = c_1 t^{b-1} + c_2 t^b$, then since $z \in \text{Im } L$, then from equation

$$\begin{cases} \Delta_1 c_1 t^{b-1} e^{-t} + \Delta_2 c_2 t^{b-1} e^{-t} = 0, \\ \Delta_1 c_1 t^b e^{-t} + \Delta_2 c_2 t^b e^{-t} = 0. \end{cases} \tag{7}$$

gives $c_1 = c_2 = 0$, since $\Delta \neq 0$. Therefore $\text{Im } L \cap \text{Im } B = \{0\}$ and $A = \text{Im } L \oplus \text{Im } B$. Thus $\dim \ker L = \text{codim Im } L = 2$ implying L is a Fredholm mapping of index zero.
(ii) Let $A : U \to U$ a continuous projector be defined as:

$$Au(t) = \frac{D_{0+}^{b} u(0)}{\Gamma(b)} t^{b-1} + \frac{D_{0+}^{b} u(0)}{\Gamma(b+1)} t^{b}$$

For $z \in \text{Im } L$, we have

$$(LK_A)z(t) = {}^{C}D_{0+}^{a} D_{0+}^{b} (K_A z) = {}^{C}D_{0+}^{a} D_{0+}^{b} I_{0+}^{b} I_{0+}^{a} z(t) = z(t).$$

Similarly, for $u \in \text{dom } L \cap \ker A$, we have

$$\begin{aligned}
(K_A L)u(t) &= (K_A)^{C} D_{0+}^{a} D_{0+}^{b} u(t) \\
&= I_{0+}^{b} I_{0+}^{a} \, {}^{C}D_{0+}^{a} D_{0+}^{b} u(t) \\
&= I_{0+}^{b} (D_{0+}^{b} u(t) + d_1) \\
&= u(t) - \frac{D_{0+}^{b-1} u(0)}{\Gamma(b)} t^{b-1} - \frac{I_{0+}^{2-b} u(0)}{\Gamma(b-1)} t^{b-2} - \frac{D_{0+}^{b} u(0)}{\Gamma(b+1)} t^{b}.
\end{aligned}$$

Since $u \in \text{dom } L \cap \ker A$, $Au(t) = 0$ and $I_{0+}^{2-b} u(0) = 0$, then $(K_A L)u(t) = u(t)$. Therefore, $K_A = (L|_{\text{dom } L \cap \ker A})^{-1}$. Furthermore,

$$\begin{aligned}
\|K_A z\|_0 &= \sup_{t \in [0,+\infty)} \frac{|I_{0+}^{a+b} z(t)|}{1 + t^{a+b}} = \sup_{t \in [0,+\infty)} \frac{1}{1 + t^{a+b}} \left| \frac{1}{\Gamma(a)\Gamma(b)} \int_0^t (t-r)^{a+b-1} z(r) dr \right| \\
&\leq \frac{1}{(a+b)\Gamma(a)\Gamma(b)} \|z\|_Z \leq \|z\|_Z,
\end{aligned}$$

$$\begin{aligned}
\|D_{0+}^{b-1} K_P z\|_1 &= \sup_{t \in [0,+\infty)} \frac{|I_{0+}^{a+1} z(t)|}{1 + t^{a+1}} = \sup_{t \in [0,+\infty)} \frac{1}{1 + t^{a+1}} \left| \frac{1}{\Gamma(a+1)} \int_0^t (t-r)^{a} z(r) dr \right| \\
&\leq \frac{1}{(a+1)\Gamma(a+1)} \|z\|_Z \leq \|z\|_Z
\end{aligned}$$

and

$$\begin{aligned}
\|D_{0+}^{b} K_A z\|_2 &= \sup_{t \in [0,+\infty)} \frac{|I_{0+}^{a} z(t)|}{1 + t^{a}} = \sup_{t \in [0,+\infty)} \frac{t^a}{1 + t^{a}} \frac{\|z\|_Z}{\Gamma(a+1)} \\
&\leq \frac{1}{\Gamma(a+1)} \|z\|_Z \leq \|z\|_Z.
\end{aligned}$$

Thus,

$$\|K_A z\| = \max\{\|K_A z\|_0, \|D_{0+}^{b-1} K_A z\|_1, \|D_{0+}^{b} K_A z\|_2\} \leq \|z\|_Z.$$

Proof of Lemma 5 is complete. □

Lemma 6. *The operator N is L-compact on $\overline{\Omega}$, where $\Omega \subset U$ is open and bounded with dom $L \cap \overline{\Omega} \neq \emptyset$.*

Proof. Let $u \in \overline{\Omega}$ then

$$\|Nu\|_Z = \sup_{t \in [0,+\infty)} |f(t, u(t), D_{0^+}^{b-1} u(t), D_{0^+}^{b} u(t))| < +\infty, \quad t \in [0, +\infty). \tag{8}$$

It follows that

$$|B_1Nu| = \left|\sum_{k=1}^{n} \beta_k \int_0^{\eta_k} (\eta_k - r)^{a-1} Nu(r)dr\right| \leq \frac{\|Nu\|_Z}{a} \sum_{k=1}^{n} |\beta_k| \eta_k^a < +\infty \tag{9}$$

and

$$|B_2Nu| = \left|\sum_{j=1}^{m} \alpha_j \int_0^{\xi_j} (\xi_j - r)^{a} Nu(r)drds\right| \leq \frac{\|Nu\|_Z}{(a+1)} \sum_{j=1}^{m} |\alpha_j| \xi_j^{a+1} < +\infty. \tag{10}$$

Then,

$$\begin{aligned}\|BNu\|_Z &= \sup_{t\in[0,+\infty)} |(\Delta_1 Nu(t)) + (\Delta_2 Nu(t))| \\ &\leq \frac{\|Nu\|_Z}{|\Delta|}\left[(|\delta_{11}| + |\delta_{21}|)\frac{1}{a}\sum_{k=1}^{n} |\beta_k|\eta_k^a + (|\delta_{12}| + |\delta_{22}|)\frac{1}{(a+1)}\sum_{j=1}^{m} |\alpha_j|\xi_j^{a+1}\right] < +\infty.\end{aligned} \tag{11}$$

Therefore, $BN(\overline{\Omega})$ is bounded. In addition, $\|Nu\|_Z + \|BNu\|_Z < +\infty$. In the following steps, we show that $K_A(I-B)N(\overline{\Omega})$ is compact. Let $u \in \overline{\Omega}$ and $m(t) = (I-B)Nu(t)$, then:

$$\begin{aligned}\frac{|K_A(I-B)Nu(t)|}{1+t^{a+b}} &= \frac{|I_{0+}^{a+b} m(t)|}{1+t^{a+b}} \leq \sup_{t\in[0,+\infty)} \frac{t^{a+b}}{1+t^{a+b}} \frac{\|m\|_Z}{(a+b)\Gamma(a)\Gamma(b)} \\ &\leq \frac{1}{(a+b)\Gamma(a)\Gamma(b)}\|m\|_Z,\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{|D_{0+}^{b-1} K_A(I-B)Nu(t)|}{1+t^{a+1}} &= \frac{|I_{0+}^{a+1} m(t)|}{1+t^{a+1}} \sup_{t\in[0,+\infty)} \frac{t^{a+1}}{1+t^{a+1}} \frac{\|m\|_Z}{(a+1)\Gamma(a+1)} \\ &\leq \frac{1}{\Gamma(a+2)}\|m\|_Z\end{aligned} \tag{13}$$

and

$$\begin{aligned}\frac{|D_{0+}^{b} K_A(I-B)Nu(t)|}{1+t^{a}} &= \frac{|I_{0+}^{a} m(t)|}{1+t^{a}} \leq \sup_{t\in[0,+\infty)} \frac{t^{a}}{1+t^{a}} \frac{\|m\|_Z}{\Gamma(a+1)} \\ &\leq \frac{1}{\Gamma(a+1)}\|m\|_Z.\end{aligned} \tag{14}$$

From (8), (11)–(14), we see that $K_A(I-B)N(\overline{\Omega})$ is bounded. Next, the equi-continuity of $K_A(I-B)N(\overline{\Omega})$ will be proved. For $u \in \overline{\Omega}$, t_1, $t_2 \in [0, M]$ with $t_1 < t_2$ and $M \in (0, +\infty)$, then:

$$\begin{aligned}&\left|\frac{K_A(I-B)Nu(t_1)}{1+t_1^{a+b}} - \frac{K_A(I-B)Nu(t_2)}{1+t_2^{a+b}}\right| \\ &\leq \frac{1}{\Gamma(a+b)}\left[\left|\int_0^{t_1} \frac{(t_1-r)^{a+b-1}}{1+t_1^{a+b}} m(r)dr - \int_0^{t_1} \frac{(t_1-r)^{a+b-1}}{1+t_1^{a+b}} m(r)dr\right|\right] \\ &\leq \frac{\|m\|_Z}{\Gamma(a+b)}\left[\int_0^{t_1}\left|\frac{(t_1-r)^{a+b-1}}{1+t_1^{a+b}} - \frac{(t_2-r)^{a+b-1}}{1+t_2^{a+b}}\right|dr + \frac{1}{a+b}\frac{(t_2-t_1)^{a+b}}{1+t_2^{a+b}}\right] \\ &\quad \to 0 \text{ as } t_1 \to t_2,\end{aligned} \tag{15}$$

$$\begin{aligned}
&\left|\frac{D_{0+}^{b-1}(K_A(I-B)Nu)(t_1)}{1+t_1^{a+1}}-\frac{D_{0+}^{b-1}(K_A(I-B)Nu)(t_2)}{1+t_2^{a+1}}\right| \\
&\leq \frac{\|m\|_Z}{\Gamma(a+1)}\left[\int_0^{t_1}\left|\frac{(t_1-r)^a}{1+t_1^{a+1}}-\frac{(t_2-r)^a}{1+t_2^{a+1}}\right|dr+\frac{1}{a+1}\frac{(t_2-t_1)^{a+1}}{1+t_2^{a-1}}\right] \\
&\quad \to 0 \text{ as } t_1 \to t_2,
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\left|\frac{D_{0+}^{b}(K_A(I-B)Nu)(t_1)}{1+t_1^{a}}-\frac{D_{0+}^{b}(K_A(I-B)Nu)(t_2)}{1+t_2^{a}}\right| \\
&\leq \frac{\|m\|_Z}{\Gamma(a)}\left[\int_0^{t_1}\left|\frac{(t_1-r)^{a-1}}{1+t_1^{a}}-\frac{(t_2-r)^{a-1}}{1+t_2^{a}}\right|dr+\frac{1}{a}\frac{(t_2-t_1)^{a}}{1+t_2^{a}}\right] \to 0 \text{ as } t_1 \to t_2.
\end{aligned} \tag{17}$$

Thus, (15)–(17) shows that $K_A(I-B)Nu(\overline{\Omega})$ is equi-continuous on the compact set $[0, M]$. Finally, we show equi-convergence at $+\infty$. Let $\tau > 0$ be a constant such that

$$|g(r)| = |(I-B)Nu(r)| \leq r, \quad u \in \overline{\Omega}.$$

In addition, since $\lim_{t\to+\infty}\frac{t^{a+b-1}}{1+t^{a+b}} = \lim_{t\to+\infty}\frac{t^{a}}{1+t^{a+1}} = \lim_{t\to+\infty}\frac{t^{a-1}}{1+t^{a}} = 0$, then for same $\epsilon > 0$, there exist $M > 0$, such that for $M < t_1 < t_2$, we have

$$\left|\frac{(t_1-r)^{a+b-1}}{1+t_1^{a+b}}-\frac{(t_2-r)^{a+b-1}}{1+t_2^{a+b}}\right| \leq \frac{t_1^{a+b-1}}{1+t_1^{a+b}}-\frac{t_2^{a+b-1}}{1+t_2^{a+b}} < \epsilon,$$

$$\left|\frac{(t_1-r)^{a}}{1+t_1^{a+1}}-\frac{(t_2-r)^{a}}{1+t_2^{a+1}}\right| \leq \frac{t_1^{a}}{1+t_1^{a+1}}-\frac{t_2^{a}}{1+t_2^{a+1}} < \epsilon,$$

and

$$\left|\frac{(t_1-r)^{a-1}}{1+t_1^{a}}-\frac{(t_2-r)^{a-1}}{1+t_2^{a}}\right| \leq \frac{t_1^{a-1}}{1+t_1^{a}}-\frac{t_2^{a-1}}{1+t_2^{a}} < \epsilon,$$

Hence,

$$\begin{aligned}
&\left|\frac{K_A(I-B)Nu(t_1)}{1+t_1^{a+b}}-\frac{K_A(I-B)Nu(t_2)}{1+t_2^{a+b}}\right| \\
&\leq \frac{1}{\Gamma(a)\Gamma(b)}\left[\left|\int_0^{t_1}\frac{(t_1-r)^{a+b-1}}{1+t_1^{a+b}}g(r)dr-\int_0^{t_1}\frac{(t_1-r)^{a+b-1}}{1+t_1^{a+b}}g(r)dr\right|\right] \\
&\leq \frac{1}{\Gamma(a)\Gamma(b)}\int_0^{M}\left|\frac{(t_1-r)^{a+b-1}}{1+t_1^{a+b}}-\frac{(t_2-r)^{a+b-1}}{1+t_2^{a+b}}\right||g(r)|dr \\
&\quad +\frac{1}{\Gamma(a)\Gamma(b)}\int_M^{t_1}\frac{(t_1-r)^{a+b-1}}{1+t_1^{a+b}}|g(r)|dr+\frac{1}{\Gamma(a)\Gamma(b)}\int_M^{t_2}\frac{(t_2-r)^{a+b-1}}{1+t_2^{a+b}}|g(r)|dr \\
&\leq \frac{M\tau\epsilon}{(a+b)\Gamma(a)\Gamma(b)}+\frac{2\tau\epsilon}{(a+b)\Gamma(a)\Gamma(b)},
\end{aligned} \tag{18}$$

$$\left|\frac{D_{0+}^{b-1}(K_A(I-B)Nu)(t_1)}{1+t_1^{a+1}}-\frac{D_{0+}^{b-1}(K_A(I-B)Nu)(t_2)}{1+t_2^{a+1}}\right| \tag{19}$$
$$\leq \frac{1}{\Gamma(a+1)}\left[\int_0^M\left|\frac{(t_1-r)^a}{1+t_1^{a+1}}-\frac{(t_2-r)^a}{1+t_2^{a+1}}\right||g(r)|dr\right.$$
$$\left.+\frac{1}{\Gamma(a+1)}\left[\int_M^{t_1}\frac{(t_1-r)^a}{1+t_1^{a+1}}g(r)dr+\int_M^{t_2}\frac{(t_2-r)^a}{1+t_2^{a+1}}g(r)dr\right]\right.$$
$$\leq \frac{M\tau\epsilon}{\Gamma(a+1)}+\frac{2\tau\epsilon}{\Gamma(a+2)}$$

and

$$\left|\frac{D_{0+}^{b}(K_A(I-B)Nu)(t_1)}{1+t_1^{a}}-\frac{D_{0+}^{b}(K_A(I-B)Nu)(t_2)}{1+t_2^{a}}\right| \tag{20}$$
$$\leq \frac{1}{\Gamma(a)}\left[\int_0^M\left|\frac{(t_1-r)^{a-1}}{1+t_1^{a}}-\frac{(t_2-r)^{a-1}}{1+t_2^{a}}\right||g(r)|dr\right.$$
$$\left.+\frac{1}{a\Gamma(a)}\left[\int_M^{t_1}\frac{(t_1-r)^{a-1}}{1+t_1^{a}}g(r)dr+\int_M^{t_2}\frac{(t_2-r)^{a-1}}{1+t_2^{a}}g(r)dr\right]\right.$$
$$\leq \frac{M\tau\epsilon}{\Gamma(a)}+\frac{2\tau\epsilon}{\Gamma(a+1)}.$$

Hence, $K_A(I-B)Nu(\overline{\Omega})$ is equi-convergent at $+\infty$. Therefore, by Definition 1, $K_A(I-B)Nu(\overline{\Omega})$ is compact, therefore, the non-linear operator N is L-compact on $\overline{\Omega}$. This concludes proof of Lemma 6. □

3. Results and Discussion

Here, the conditions for the existence of solutions to problem (1.1) subject to (1.2) is proved.

Theorem 2. *Let f be a continuous function. If (ϕ_1) and (ϕ_1) holds, then, the following conditions also hold:*

(H_1) There exists functions $\rho(t),\ \mu(t),\ \nu(t),\ \sigma\in C[0,+\infty)$, such that for all $(j,k,l)\in\mathbb{R}^3$ and $t\in[0,+\infty)$,

$$|f(t,u(t),D_{0+}^{b-1}u(t),D_{0+}^{b}u(t))|\leq\rho(t)\frac{|j|}{1+t^{a+b}}+\mu(t)\frac{|k|}{1+t^{a+1}}+\nu(t)\frac{|l|}{1+t^{a}}+\sigma(t). \tag{21}$$

(H_2) There exist constants $M>0$, such that for $u\in dom\ L$ if $|D_{0+}^{b}u(t)|>M$ for $t\in[0,+\infty)$, then either

$$B_1Nu(t)\neq 0\quad or\quad B_2Nu(t)\neq 0.$$

(H_3) There exists a constant $C>0$, such that if $|c_1|>C$ or $|c_2|>C$, then either

$$B_1N(c_1t^{b-1}+c_2t^b)+B_2N(c_1t^{b-1}+c_2t^b)<0 \tag{22}$$

or

$$B_1N(c_1t^{b-1}+c_2t^b)+B_2N(c_1t^{b-1}+c_2t^b)>0 \tag{23}$$

where $c_1,\ c_2\in\mathbb{R}$ satisfying $c_1^2+c_2^2>0$.

Then, the boundary value problem (3) and (4) has at least one solution provided:

$$\|\rho\|_Z+\|\mu\|_Z+\|\nu\|_Z<\frac{\Gamma(a+1)}{\Gamma(a+1)+2}.$$

Proof. The proof will be completed in four stages.

Stage 1. We will establish that $\Omega_1 = \{u \in \operatorname{dom} L \setminus \ker L : u = \lambda N u, \text{ for } \lambda \in [0,1]\}$ is bounded. Let $u \in \Omega_1$ then $u = (u - Au) + Au \in \operatorname{dom} L \setminus \ker L$. This means that $(I - A)u \in \operatorname{dom} L \cap \ker A$ and $Au \in \ker A$, hence, $LAu = 0$. By Lemma 5, we have

$$\|(I-A)u\| = \|K_A L(I-A)u\| \leq \|L(I-A)u\| = \|Lu\| = \|Nu\|_Z. \tag{24}$$

Since $u \in \Omega_1$, then $Lu = \lambda N u$. Additionally, by (H_2) there exists $t_1 \in [0, +\infty)$, such that $|D_{0+}^b u(t_1)| \leq M$, therefore

$$\begin{aligned} |D_{0+}^b u(0)| &\leq |D_{0+}^b u(t_1)| + \frac{\lambda}{\Gamma(a)} \int_0^{t_1} (t_1 - r)^{a-1} |f(r, u(r), D_{0+}^{b-1} u(r), D_{0+}^b u(r))| dr \\ &\leq M + \frac{1}{\Gamma(a+1)} \|Nu\|_Z. \end{aligned} \tag{25}$$

In addition,

$$\|Au\|_0 \leq |D_{0+}^b u(0)| \left(\frac{1}{\Gamma(b)} \sup_{t \in [0,+\infty)} \frac{t^{b-1}}{1 + t^{a+b}} + \frac{1}{\Gamma(b+1)} \sup_{t \in [0,+\infty)} \frac{t^b}{1 + t^{a+b}} \right) \leq 2|D_{0+}^b u(0)|,$$

$$\|D_{0+}^{b-1} Au\|_1 \leq |D_{0+}^b u(0)| \left(\frac{1}{\Gamma(b)} \sup_{t \in [0,+\infty)} \frac{1}{1 + t^{a+1}} + \frac{1}{\Gamma(b+1)} \sup_{t \in [0,+\infty)} \frac{t}{1 + t^{a+1}} \right) \leq 2|D_{0+}^b u(0)|$$

and

$$\|D_{0+}^b Au\|_2 \leq |D_{0+}^b u(0)| \left(\frac{1}{\Gamma(b)} \sup_{t \in [0,+\infty)} \frac{t^{-1}}{1 + t^a} + \frac{1}{\Gamma(b+1)} \sup_{t \in [0,+\infty)} \frac{1}{1 + t^a} \right) \leq 2|D_{0+}^b u(0)|.$$

Therefore, from (25), we have

$$\|Au\| \leq \max\{\|u\|_0, \|D_{0+}^{b-1} u\|_1, \|D_{0+}^b u\|_2\} \leq 2|D_{0+}^b u(0)| \leq 2M + \frac{2}{\Gamma(a+1)} \|Nu\|_Z \tag{26}$$

and from (24) and (26), we have

$$\begin{aligned} \|u\|_U &\leq \|Au\|_U + \|I - A\|_U \\ &\leq 2M + \left(1 + \frac{2}{\Gamma(a+1)}\right) \|Nu\|_Z \\ &\leq 2M + \left(1 + \frac{2}{\Gamma(a+1)}\right) \|u\|_U (\|\rho\|_Z + \|\mu\|_Z + \|\nu\|_Z) + \left(1 + \frac{2}{\Gamma(a+1)}\right) \|\sigma\|_Z. \end{aligned}$$

Hence,

$$\|u\|_U \leq \frac{2M + \left(1 + \frac{2}{\Gamma(a+1)}\right\| \|\sigma\|_Z)}{1 - \left(1 + \frac{2}{\Gamma(a+1)}\right) \|u\|_U (\|\rho\|_Z + \|\mu\|_Z + \|\nu\|_Z)}.$$

Thus, Ω_1 is bounded.

Step 2. Let $\Omega_2 = \{u \in \ker L : Nu \in \operatorname{Im} L\}$. For $u, Nu \in \Omega_2$, then $u(t) = c_1 t^{b-1} + c_2 t^b$. and $BNu = 0$. Thus, from (H_3), we have $|c_1| \leq C$ and $|c_2| \leq C$. Hence, Ω_2 is bounded.

Step 3. For $c_1, c_2 \in \mathbb{R}$, $t \in [0, +\infty)$, the isomorphism $J : \ker L \to \operatorname{Im} B$ is as

$$J(c_1 t^{b-1} + c_2 t^b) = \frac{1}{\Delta} \Big[(\delta_{11} c_1 + \delta_{12} c_2) + (\delta_{21} c_1 + \delta_{22} c_2) t \Big] e^{-t} \tag{27}$$

Suppose (22) holds, let

$$\Omega_3 = \{u \in \ker L : \lambda Ju + (1-\lambda)BNu = 0, \lambda \in [0,1]\}.$$

Let $u \in \Omega_3$, then $u(t) = c_1 t^{b-1} + c_2 t^b$. Since $\Delta \neq 0$, then

$$\begin{cases} c_1\lambda + (1-\lambda)B_1N(c_1t^{b-1} + c_2t^b) = 0, \\ c_2\lambda + (1-\lambda)B_2N(c_1t^{b-1} + c_2t^b) = 0. \end{cases} \tag{28}$$

When $\lambda = 1$, we obtain $c_1 = c_2 = 0$. When $\lambda = 0$, $B_1N(c_1t^{b-1} + c_2t^b) = B_2N(c_1t^{b-1} + c_2t^b) = 0$,which contradicts (22) and (23). Hence, from (H_3), we obtain $|c_1| \leq C$, and $|c_2| \leq C$. For $\lambda \in (0,1)$, if $|c_1| > C$ or $|C_2| > A$ by (22) and (28), we have

$$\lambda(c_1^2 + c_2^2) = -(1-\lambda)[B_1N(c_1t^{b-1} + c_2t^b) + B_2N(c_1t^{b-1} + c_2t^b)] < 0,$$

which is a contradiction. Hence, Ω_3 is bounded.

Similarly, if (23) holds and $\Omega_3 = \{u \in \ker L : \lambda Ju - (I-\lambda)BNu = 0, \lambda \in [0,1]\}$, Ω_3 can be shown to be bounded by similar argument.

Step 4. Let $\Omega \supset U_{i=1}^3 \overline{\Omega}_i$. Finally, we will show that a solution of (3) and (4) exists in $\operatorname{dom} L \cap \Omega$. We have shown in Steps 1 and 2 that (i) and (ii) of Theorem 1 hold. Finally, we show that (iii) also holds. Let $H(u,\lambda) = \pm\lambda Ju + (1-\lambda)BNu$, then following the arguments of Step 3, it follows that for every $(u,\lambda) \in (\ker L \cap \partial\Omega) \times [0,1]$, $H(u,\lambda) \neq 0$. Therefore, by the homotopy property of degree

$$\begin{aligned} \deg(BN|_{\ker L}, \Omega \cap \ker L, 0) &= \deg(\pm J, \Omega \cap \ker L, 0) \\ &= \pm 1 \neq 0. \end{aligned}$$

Therefore, by Theorem 1 at least one solution of (3) and (4) exists in U. □

4. Conclusions

This work considered a mixed fractional-order boundary value problem at resonance on the half-line. The Mawhin's coincidence degree theory was used to establish existence of solutions when the dimension of the kernel of the linear fractional differential operator is two. The result obtained is new and an example was used to demonstrate the result obtained.

5. Example

Example 1. *Consider the following boundary value problem:*

$$^C D_{0+}^{\frac{1}{2}} D_{0+}^{\frac{3}{2}} u(t) = \frac{e^{-5t}\sin D_{0+}^{\frac{1}{2}}u(t)}{17(1+t^2)} + \frac{e^{-t}D_{0+}^{\frac{3}{2}}u(t)}{9(1+t^{\frac{3}{2}})} + \frac{e^{-2t}}{15(1+t^{\frac{1}{2}})}, \quad t \in [0,+\infty) \tag{29}$$

$$\begin{aligned} &I_{0+}^{\frac{1}{2}}u(0) = 0,\ D_{0+}^{\frac{1}{2}}u(0) = \frac{2}{3}D_{0+}^{\frac{1}{2}}u\left(\frac{1}{4}\right) - \frac{1}{3}D_{0+}^{\frac{1}{2}}u\left(\frac{1}{2}\right), \\ &D_{0+}^{\frac{3}{2}}u(+\infty) = \frac{3}{4}D_{0+}^{\frac{1}{2}}u\left(\frac{1}{5}\right) + \frac{1}{4}D_{0+}^{\frac{1}{2}}u\left(\frac{3}{5}\right),. \end{aligned} \tag{30}$$

Here $a = \frac{1}{2}$, $b = \frac{3}{2}$ $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{5}{2}$, $\xi_1 = 4$, $\xi_2 = 2$, $\beta_1 = \frac{3}{4}$, $\beta_2 = \frac{1}{4}$, $\eta_1 = 5$, $\eta_2 = \frac{5}{3}$, $n = m = 2$. $\sum_{j=1}^2 \alpha_j \xi_j^{-1} = 0$, $\sum_{j=1}^2 \alpha_j = 1$, $\sum_{k=1}^2 \beta_k \eta_k^{-1} = 0$, $\sum_{k=1}^2 \beta_k = 1$.
$\|\rho\|_Z = \frac{1}{17}\sup_{t\in[0,+\infty)} |e^{-5t}| = \frac{1}{17}$, $\|\mu\|_Z = \frac{1}{9}\sup_{t\in[0,+\infty)} |e^{-t}| = \frac{1}{9}$,

$\|\nu\|_Z = \frac{1}{15}\sup_{t\in[0,+\infty)}|e^{-6t}| = \frac{1}{15}$. Then, $\|\rho\|_Z + \|\mu\|_Z + \|\nu\|_Z = \frac{1}{17} + \frac{1}{9} + \frac{1}{15} = 0.2367$ $\Gamma(a+1) = \Gamma(\frac{1}{2}+1) = 1$. Then, $\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2})+2} = 0.3071$. Hence,

$$\|\rho\|_Z + \|\mu\|_Z + \|\nu\|_Z < \frac{\Gamma(a+1)}{\Gamma(a+1)+2}.$$

Finally, conditions (H_1) - (H_3) can also be shown to hold. Therefore (29) and (30) has at least one solution.

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