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11 August 2023

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September 26, 2023

Abstract

Objective: Previous literature on optimal savings relies on specific utility and production technology functional forms which might not be able to produce robust results as different utility/production functions may lead to dramatically different or opposing results. This paper derives an optimal savings policy based on parameters (such as slopes of demand and supply curves) which can be empirically estimated and provides a unique and robust result irrespective of shape and form of individual utilities/production functions. Furthermore, existing literature does not consider welfare loss when savings market is adjusting to final equilibrium after a shock while deriving an optimal savings policy. In addition to that, number of savers (all public and private saving entities, including households, firms, etc.,) and saving rate are vital parameters for savings in an economy, and while deriving an optimal savings policy, it is necessary to take into consideration these parameters to ensure that quantum of savings due to interest rate movement gets adjusted in target time duration, without which there may be additional efficiency loss than that envisioned while deriving an optimal savings policy for an economy. **Methods:** This research project designs a dynamical model for savings market and extends that to a three-dimensional savings system in an economy by taking into consideration number of savers, saving rate, and interest rate; and based on that derives an optimal comprehensive savings policy while accounting for efficiency losses when savings market, saving rate, and number of savers are adjusting to final equilibrium, in addition to the welfare loss on account of equilibrium shift. **Results:** Without consideration of welfare loss/gain while savings market is adjusting after implementation of a savings policy, welfare picture remains incomplete, and the optimal savings policy based on partial welfare cannot be considered as optimal in true sense. Traditionally, welfare of only producer and consumer is taken into consideration without accounting for welfare of production factors. An expression of efficiency loss/gain as a result of savings policy based on welfare including those of production factors has been presented and optimal savings policies have been derived by minimizing efficiency losses and presented as a final result in the form of mathematical expressions. This paper demonstrates that both supply and demand shocks operate through a common channel, i.e., inventory of funds in savings market as both kinds of shock affect inventory of funds and hence can be categorized just as an inventory shock. **Conclusion:** For optimal welfare gains, practitioners/policy makers must estimate theoretically derived optimal savings policies based on the dynamic model developed

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in this paper and presented in the form of mathematical expressions from real world relevant data for implementation. (JEL E21, G10, G18)

1 Introduction

The national savings rate is the gross domestic product saved rather than consumed in an economy. It is the difference between the income and consumption of a country as a percentage of total income, and serves as an indicator of a nation's economic health because it provides information on saving trends which is a source of investment in a country. Savings by households serve as a borrowing source for the public sector to provide infrastructure and other public works for a nation, and also provide investment for the private sector. **Savings are important not just at the individual level, rather they contribute to national savings as well, making economic recovery faster at national level when personal savings rate is high.** With a higher saving level, consumers can more easily absorb the excessive spending shocks without having to rely too much on loans. Also, a higher saving rate implies that the consumers are able to allocate a substantial portion of their income to savings after paying for their living expenses. The ability of consumers to cope with their financial hardships is a positive sign for recovery of an economy.

However, savings are fraught with a lot of risk too. A government intervention can produce negative effects for savers; stimulus spending and inflation can both work against savers. A government typically finances an economic stimulus package to people through additional sovereign debt which is a liability on future generations. In a way, this implies that savers have to bail out non-savers at certain point in time. If government prints money for a federal stimulus, it could lead to higher inflation, which acts as a killer for savings. Each dollar loses its purchasing power with inflation. The value of currency is the amount of goods and services which can be bought with one unit of money, i.e., purchasing power, and with a high inflation rate, purchasing power of currency decreases. Although, government spending by borrowing also increases savings demand, and may result in a higher return for savers, however, the main point to emphasize is the risk of reduced welfare fraught with savings more than a certain level in an economy.

When personal savings are high, there is less need for governmental intervention in event of shocks. There is a risk involved in saving, but at the same time savings are vital for an economy, so there has to be an optimal level of savings. A lot of previous research sheds light on optimal level of savings using various macroeconomic models, however, previous literature on optimal savings relies on specific utility and production technology functional forms which might not be able to produce robust results as different utility/production functions may lead to dramatically different or opposing results. This paper derives an optimal savings policy based on parameters (such as slopes of demand and supply curves) which can be empirically estimated and provides a unique and robust result irrespective of shape and form of individual utilities/production functions. Furthermore, existing literature does not consider welfare loss when savings market is adjusting to final equilibrium after a shock while deriving an optimal savings policy. In addition to that, number of savers

(all public and private saving entities, including households, firms, etc.,) and saving rate are vital parameters for savings in an economy, and while deriving an optimal savings policy, it is necessary to take into consideration these parameters to ensure that quantum of savings due to interest rate movement gets adjusted in target time duration, without which there may be additional efficiency loss than that envisioned while deriving an optimal savings policy for an economy. This research project designs a dynamical model for savings market and extends that to a three-dimensional savings system in an economy by taking into consideration number of savers, saving rate, and interest rate; and based on that derives an optimal comprehensive savings policy while accounting for efficiency losses when savings market, saving rate, and number of savers are adjusting to final equilibrium, in addition to the welfare loss on account of equilibrium shift. Without consideration of welfare loss/gain while savings market is adjusting after implementation of a savings policy, welfare picture remains incomplete, and the optimal savings policy based on partial welfare cannot be considered as optimal in true sense. Traditionally, welfare of only producer and consumer is taken into consideration without accounting for welfare of production factors. An expression of efficiency loss/gain as a result of savings policy based on welfare including those of production factors has been presented. Finally, this paper demonstrates that both supply and demand shocks operate through a common channel, i.e., inventory of funds in graph B in figure 1 (cumulative number of savers in graph A) in savings market as both kinds of shock affect inventory of funds and hence can be categorized just as an inventory shock.

A lot of research has already been done on savings and their relationship with other macroeconomic variables. Sato and Davis (1971) is an attempt to study optimal savings policy in a world where the growth rate of labor responds to economic factors. Menezes and Auten (1978) extends the theory of interest by Irving Fisher (income risk increases current saving) by showing that the effect of income risk on the marginal rate of time preference is both a necessary and sufficient condition to characterize how income risk affects savings. With the help of a simple two-country version of the Solow steady-state model, Quibria (1986) shows that even without the presence of any visible market distortion, free mobility of capital is not necessarily beneficial for the capital-importing country. Deaton (1989) develops a model of households which cannot borrow but which accumulate assets over the long term, and have on average very small asset holdings. In Kimball (1989), the theory of precautionary saving is shown to be isomorphic to the Arrow-Pratt theory of risk aversion, making possible the application of a large body of knowledge about risk aversion to precautionary saving, and more generally, to the theory of optimal choice under risk. Chakravarty (1990) explains optimal savings and their significance. Carroll, Hall and Zeldes (1992) presents evidence that consumer pessimism about unemployment explains a substantial part of weakness in consumption. Gylfason (1993) have presented a simple model of optimal saving, interest rates, and economic growth. Thaler (1994) argues that it is important to incorporate more of the psychology

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of saving into economic theories. Arrow, Dasgupta and Mäler (2003) discusses the genuine savings criterion and the value of population. Bloom, Canning and Graham (2003) shows that increases in life expectancy lead to higher savings rates at every age, even when retirement is endogenous. Choi, Laibson, Madrian and Metrick (2003) provides a model of savings choices taking into account the costs which vary over time. Osili (2007) is an attempt to present a disaggregated view of international remittance flows using a matched sample of international migrants and their origin families. Benartzi and Thaler (2007) investigates both the heuristics and the biases that emerge in the area of retirement savings. Bodie, Treussard and Willen (2008) presents a theory of optimal life-cycle saving and investing **by examining employees decisions regarding joining savings plans, contributions toward savings and investment.** Bo, Kuo and Junlu (2013) investigates the relationship among savings rates, inflation rates and money policy parameters through an overlapping-generation model including a money supply reaction rule that is compatible with China's transitional reforming period. Baiardi, Magnani and Menegatti (2020) reviews recent developments in the literature analyzing precautionary saving. Brandstedt (2021) situates John Rawls' just savings principle in a discussion about how much a nation-state should save. Ma and Toda (2021) considers an income fluctuation problem with homothetic preferences and general shocks and prove that consumption functions are asymptotically linear, with an exact analytical characterization of asymptotic marginal propensities to consume.

A three-dimensional (in contrast with a traditional two-dimensional (price-quantity) market) dynamical savings market model has been developed to depict paths of interest rate, and saving rate, and to quantify welfare loss/social damage on those paths after an exogenous shock. The model also provides expressions for final equilibriums. The methodology presented in this paper is based on and a further extension of Nawaz (2017), Nawaz (2019)a, Nawaz (2019)b, Nawaz (2020), Nawaz (2021), and Nawaz (2022) to a three-dimensional market. This paper extends their methodology to a savings market to derive an optimal level of savings, interest rate, number of savers and saving rate. The results suggest that optimal policies are complex expressions which depend on various parameters, i.e., slopes of demand, supply, inventory (of funds), number of savers versus saving rate curves, and initial values of interest rate, inventory of funds, saving rate, number of savers, target revenue/cost, target increase/decrease in savings, etc., hence, all parameters need to be empirically estimated to have optimal savings policies. Expressions for paths of interest and saving rate depict how interest and saving paths change as one of the parameters varies while others stay put, which also determines how optimal savings policies vary as one of the paramaters gets changed while others stay the same.

We design a three-dimensional dynamical savings market (figure 1) and based on that derive optimal dynamical policies to keep savings at a level which minimize welfare loss and social damage. A set of optimal savings policies comprise of an optimal policy each for panels A and B in figure

1. A two-dimensional (traditional price-quantity market) dynamical savings market model has been developed for graph/panel B, based on which an optimal dynamical policy for quantity of savings and interest rate minimizing efficiency loss (supply/consumption of funds lost) on way to equilibrium plus that in equilibrium (total efficiency loss as compared to traditional dead-weight loss in final/post-policy equilibrium) subject to savings policy implementation cost constraint faced by government has been designed. Panel B policy decides constraint for an optimal dynamical policy for panel A, i.e., a change in savings per unit time. **Theory regarding panel A has been devised and a dynamical model has been developed for an optimal policy regarding number of savers and saving rate.** Optimality of policy is in the sense of minimizing social damage (excessive/fewer than optimal number of savers) on way to equilibrium plus that in equilibrium subject to change in quantity of savings per unit time (a constraint set by panel B policy). When a policy for panel A comes into play, number of savers and saving rate do not jump to final/post-policy equilibrium, and rather adopt a path with pairs of number of savers and saving rate to final equilibrium where economy's demand and supply (both public and private) of number of savers become equal.

In nature both panels, i.e., A and B and equilibria thereof occur simultaneously. A certain number of savers exist in an economy who save at a certain rate which forms the upward sloping supply (both private and public) curve for savers; with rectangular areas in plane A (formed by perpendiculars drawn from points on supply curve to x and y-axes) corresponding to horizontal coordinates or abscissas on supply curve in panel B. As government faces a cost constraint to be satisfied, it is natural to start deriving a policy for panel B (interest rate or quantity of funds chosen by government), which determines the constraint (change in quantity of savings per unit time) for optimal policy in panel A (saving rate or number of savers chosen by government), and then derive panel A policy. For panel B, existing literature does not account for welfare losses/gains on way to final equilibrium as well as those in final equilibrium in savings market to derive an optimal savings policy. When a savings policy comes into play, government's cost as supplier of funds jumps to pre-policy cost plus per unit funds cost incurred due to savings policy affects supply of funds in savings market and pushes it out of equilibrium. Savings market adjusts to final equilibrium and mechanism for adjustment is based on premise that there is no coordination between economic agents in savings market when it goes out of equilibium (after savings policy comes into play) at existing interest rate. For design of an optimal savings policy, there must be a consideration of minimizing social damage both during adjustment of number of savers and saving rate as well as in final equilibrium.

The basis of this research is a model for a perfectly competitive market of savings, where the quantity of savings and their price, i.e., the interest rate are presented in the traditional quantity-price picture of a market as shown in panel B. Panel A

determines the x-axis of panel B, i.e., the quantity of savings in an economy which depends on number of savers and their savings rate. The area in panel A determines the x-axis values on panel B.

The remainder of this paper is organized as follows: **In section 2, a model for panel B has been developed. In section 3, solution of the model regarding an expansionary savings policy has been presented. In section 4, an optimal dynamical expansionary savings policy has been derived for panel B. Section 5 provides a solution of the model for a contractionary savings policy. In section 6, an optimal dynamical contractionary savings policy has been derived for panel B. Section 7 presents a dynamical savings model regarding panel A, through joining individual components of panel A. In section 8, solution to model A with a savings policy has been provided. In section 9, an optimal dynamical savings policy has been derived for panel A. Section 10 concludes the paper by presenting summary of findings. Detailed mathematical steps regarding derivations in the main text have been provided in the appendix.**

2   Materials and Methods

2.1   The Model-Panel B

The model has a perfectly competitive market of savings in equilibrium, implying the initial conditions of model are equilibrium values. **In the market, there are five market agents,** i.e., the household as a supplier of savings, the public sector/government as a supplier of savings, consumer of savings (a producer/firm in need of money for investment), a financial intermediary/broker/commercial bank between suppliers and consumer, and government for exercising savings policy. Government influences interest rate through exercising savings policy, however, takes the interest rate as given in the role of a savings supplier. Household as savings supplier also takes the interest rate as given. As a result of an exogenous shock in savings market, interest rate cannot adjust on its own to lead savings market to final equilibrium. Financial broker as middleman varies interest rate in self-interest to lead market to attain final equilibrium after making interest rate follow a dynamic path. When final equilibrium arrives, it is in benefit of broker not to vary interest rate further. Suppliers of savings supply to financial broker/middleman, who maintains stock/inventory of savings (stock/inventory of savings can lie with a commercial bank or suppliers, which does not matter for mathematical treatment of inventory of funds as far as the financial broker has information regarding quantity of funds to be dealt with) and supplies it further to consumer (the producer/firm who borrows money from financial broker) at market interest rate. Suppliers of savings maximize their utility/benefit; financial broker maximizes profit as a difference of revenue earned through lending money to borrower/consumer, and cost of keeping savings

subject to constraints; consumer/producer/firm maximizes profit.

Mechanism of interest rate adjustment has the following explanation: When a shocks hits savings market, and puts it out of equilibrium, economic agents have no coordination regarding their economic decisions at existing interest rate. Following example illustrates savings market working: Consider a savings market in equilibrium, where the broker/commercial bank has an equilibrium quantity of stock of savings. An exogenous supply expansion shock happens to market, and increases inventory of savings/funds as savings output produced by producers no more equals demand by consumers at existing interest rate. In new equilibrium a higher quantity of savings and lower interest rate will result. Equilibrium in savings market has been defined as given below:

(i) Suppliers of savings maximize their utility/benefit; consumer/producer/firm maximizes profit; and broker/commercial bank maximizes profit as a difference of revenue earned through lending money to borrower/consumer, and cost of keeping money subject to respective economic constraints.

(ii) Amount of savings produced by producer equals the amount consumed by consumer, and size of funds inventory stays put during market equilibrium.

Conditions for equilibrium existence for a linear dynamical system, i.e., a necessary and sufficient condition for stability, based on Routh–Hurwitz stability criterion have been presented in Section 3. As it happens in a perfectly competitive savings market, financial intermediary/commercial bank **who lends funds to consumer at market rate is a price-taker when market is in equilibrium.** In absence of equilibrium, financial intermediary has an incentive to change interest rate on dynamic adjustment path until market attains final equilibrium and financial intermediary is again a price-taker. Government implements a savings policy, as a result of which market loses state of equilibrium, and interest rate adjusts to drift market toward final equilibrium. Adjustment of savings market is based on endogenous decisions taken by economic agents in self-interest, i.e., household, the public sector/government, the consumer of savings, and the financial intermediary. Suppose a producer of savings (household who saves) provides funds to financial broker/intermediary who supplies to consumer, i.e., a producer who uses funds for investment purposes, and savings market is in equilibrium. After government introduces and implements a savings policy, i.e., increases savings supply due to which interest rate goes down, some of the supply will not be borrowed by the consumers by the end of the time period in which the policy was exercised. If producers of savings and financial intermediary could have changed savings and market interest rate respectively without a delay, they would choose the quantity and interest rate maximizing their profit and clearing the savings market without a waste of savings had they been aware of exact pattern regarding demand of savings after implementation of savings policy by government. However, this information is not known to them, and commercial bank/financial intermediary reduces interest rate according to their best guess about post-policy demand of savings observing the increase in size of inventory of funds, which drives market closer

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to final post-policy equilibrium. After an interest rate reduction, producer decreases production of savings than before. If during next time period, commercial bank/financial intermediary continues with same size of funds inventory and interest rate, final equilibrium has been attained, however, if there is still a need for a variation in size of inventory, the financial broker will vary interest rate further to drift market toward final equilibrium. After some welfare loss, savings market will eventually attain final equilibrium. Welfare loss due to savings policy comprises of unutilized funds during time of adjustment of market. In final equilibrium there might be some loss in comparison with the initial equilibrium, so total welfare loss comprises of that during adjustment as well as in final equilibrium.

To derive mathematical results, objectives of market agents subject to their constraints are considered and maximized by taking first order derivatives. In order to get a complete picture of market, expressions **reflecting actions of individual agents are simultaneously solved.** We assume that final equilibrium after implementation of policy is not too off the initial equilibrium, which implies linearity of supply and demand curves is a reasonable assumption to make. Figure 1 depicts linearization **as apparently a reasonable approximation for moving market equilibrium from point a to b,** however, if market moves from point a to c, linearization does not seem to work, and a non-linear dynamical system must be designed to cater this scenario, which does not fall in the ambit of scope of this research.

2.1.1 Financial Intermediary/Broker/Commercial Bank

Financial intermediary/broker/commercial bank buys/rents money from suppliers to sell/lend to consumer of savings for earning profit. Financial intermediary holds an inventory of funds as quantity of funds bought and sold are unequal at various points in time. An inventory exists between supply and demand of money reflecting their difference in savings market. **The level of inventory stays the same if supply and demand rates do not change, and a change in inventory implies that there is either a change in supply, demand or both at a different rate.**

Figure 3 illustrates a connection between **funds' inventory**, production, demand (of savings) and interest rate. When supply/production curve for savings shifts to right whereas demand stays put, inventory in market increases at initial interest rate which reduces in final equilibrium. In the same manner, when demand shifts to right whereas production stays the same, inventory goes down at initial interest rate which rises to lead to final equilibrium. This reflects there exists an inverse relationship between a change in inventory of funds and a change in interest rate ceteris baribus. If both production and demand shift **such that** that there is no variation in inventoy, interest rate will also stay put. Both supply and demand shocks operate through a common channel, i.e., inventory of funds in savings market as both kind of shocks affect inventory and hence can be categorized

just as an inventory shock. Above discussion concludes an interest rate change is negatively related to a change in funds inventory. Mechanism which brings this result is as follows: Suppose there is a savings market where financial intermediaries hold inventory of funds at some cost and lend money to consumers. Cost of holding a funds inventory is positively related to its size, i.e., more funds are more costly to hold. If demand and supply rates are equal in absence of an exogenous shock, savings market is in equilibrium and interest rate stays the same.

Suppose a savings policy brings down the marginal cost of savings and households increase supply of savings, whereas demand does not change. As supply is no longer equal to demand, difference will get accumulated as an inventory of funds held by financial institutions. In order to lend additional funds to borrowers, financial institutions would be willing to decrease interest rate which could bring demand of funds up following demand schedule. Interest rate will eventually decrease to new marginal cost as savings market is perfectly competitive, however, how market adjusts to final equilibrium is contingent upon response of financial broker to change in funds inventory. After shock, marginal cost of production has gone down, however, **financial broker's marginal cost for holding an extra unit of funds has increased**. This phenomenon is consistent with profit maximization behavior of financial broker and producer. To illustrare mathematically, let us take a look at financial broker's profit maximization problem as follows:

One Time Period Problem   Short-term problem of financial intermediary/broker/commercial bank implies that financial broker is not doing dynamic optimization and his/her objective is short-term. For discrete time, it can be considered as an objective just for one time period without looking at future times for the sake of an intuitive explanation and to prepare reader for a much more **complex dynamic problem presented in next section**. **Financial intermediary's objective function** is as given below:

$$\Pi = rq(r) - \varsigma_B(m_B(r, e_B)), \tag{1}$$

where
Π = profit,
r = market interest rate,
$q(r)$ = quantity of funds sold (selling is in fact lending) at interest rate r,
m_B = inventory (quantity of funds *financial intermediary* holds),
e_B = other factors which influence inventory of funds other than market interest rate including financial broker's purchase price of funds from producer,
$\varsigma_B(m_B(r, e_B))$ = **cost of holding funds increasing in inventory**.
Taking derivative of eq. (1) with respect to interest rate, we obtain:

$$rq'(r) + q(r) - \varsigma'_B(m_B(r, e_B))m'_{B1}(r, e_B) = 0. \tag{2}$$

For financial intermediary/broker/commercial bank, there is an incentive to change interest rate only during adjustment of savings market. During equilibrium, financial broker will incur losses by changing interest rate to a value other than marginal cost. Supply does not equal demand during savings market adjustment and market drifts toward final equilibrium, this is why a change in interest rate during adjustment time which brings market closer to final equilibrium is in line with market forces. Financial intermediary loses business by changing interest rate during market equilibrium (which is not the case on the adjustment path) and commercial bank faces a demand which is infinitely elastic as shown below:

$$rq'(r) + q(r) = \varsigma'_B(m_B(r, e_B))m'_{B1}(r, e_B),$$

$$r\left[1 + \frac{1}{demand\ elasticity}\right] = \varsigma'_B(m_B(r, e_B))\frac{m'_{B1}(r, e_B)}{q'(r)}.$$

For an infinitely elastic demand, interest rate equals marginal cost in the above expression **where the expression on the right is the marginal cost**. Suppose a supply shock happens, which reduces marginal cost of production of funds, i.e., marginal cost of saving by households, and supply curve shifts downward. Supply is no longer equal to demand at interest rate in initial equilibrium when competitive market gets out of steady state. Eventually, interest rate decreases and brings the final equilibrium, however, there is no jump in interest rate, rather financial broker continues to charge an interest rate higher than the new marginal cost until there is a realization on his part through an accumulation of inventory of funds that the market supply is higher than before and the changed profit maximizing condition requires him to lower the interest rate. Similarly, a reverse supply shock eventually brings the interest rate up. In this scenario, financial intermediary continues to charge an interest rate lower than the new marginal cost until there is a realization on his part through a depletion of inventory of funds that the market supply is lower than before and the profit maximizing condition in the changed scenario requires him to increase the interest rate. In later case, consumer is the beneficiary during market adjustment period who will reap gains until interest rate is increased by financial broker. In equilibrium, market interest rate is equal to the sum of marginal costs of production and that of holding funds by the middleman in absence of a policy intervention, which implies neither the financial intermediary reaps an economic rent not does the consumer when there is an equilibrium in the competitive market.

In a mathematical context, suppose a supply shock happens, while demand does not change due to which the **production marginal cost of funds** decreases and supply increases. The financial intermediary's marginal cost for holding an extra unit i.e., $\varsigma'_B(m_B(r, e_B))\frac{m'_{B1}(r, e_B)}{q'(r)}$ is higher

at existing market interest rate on account of the term $\varsigma'_B(m_B(r, e_B))$ which is higher, i.e., the middleman has a higher cost for holding unutilized funds after supply shock. The other term, i.e., $\varsigma'_B(m_B(r, e_B))$ has not yet changed as the interest rate is the same as before. This is under the assumption that the financial intermediary's purchase rate has not changed **due to producer being** a price-taker during adjustment period of market as well and only charges a fraction of the market interest rate to financial intermediary. An analog in discrete time is that financial broker maximizes profit in each time period while not taking into account future time periods, taking the purchase interest rate as given and **just choosing the interest rate for lending**. On current interest rate, financial broker faces the following inequality:

$$\frac{\partial \Pi}{\partial r} = rq'(r) + q(r) - \varsigma'_B(m_B(r, e_B))m'_{B1}(r, e_B) < 0, \tag{3}$$

so to hold an extra unit of funds in inventory, financial intermediary must decrease interest rate after supply shock to satisfy profit maximization condition. **Short-term gains accrued from a reduced marginal cost will be reaped by** producer of funds, as the marginal cost of production has reduced but market interest rate is the same as before until changed by financial intermediary. A plot of profit maximizing pairs of inventory of funds and respective market interest rate is a downward sloping *inventory curve* with interest rate on y-axis and inventory on x-axis.

Dynamic Problem **For dynamic problem, present discounted value of future stream of profits of the financial intermediary are maximized, with zero time value being as given below:**

$$V(0) = \int_0^\infty \left[rq(r) - \varsigma_B(m_B(r, e_B)) \right] e^{-\sigma t} dt, \tag{4}$$

σ, $r(t)$, and $m_B(t)$ are discount rate, *control variable*, and *state variable* respectively. Maximization problem in mathematical notation is depicted below:

$$\underset{\{r(t)\}}{Max} V(0) = \int_0^\infty \left[rq(r) - \varsigma_B(m_B(r, e_B)) \right] e^{-\sigma t} dt,$$

subject to the constraints that
$\dot{m}_B(t) = m'_{B1}(r(t), e_B(r(t), z_B))\dot{r}(t) + m'_{B2}(r(t), e_B(r(t), z_B))e'_{B1}(r(t), z_B)\dot{r}(t)$ (state equation, **which describes the change in state variable with time; with z_B as exogenous factors**),
$m_B(0) = m_{Bs}$ (initial condition),
$m_B(t) \geq 0$ (non-negativity constraint on state variable),
$m_B(\infty)$ free (terminal condition).
Following is the expression for current-value Hamiltonian:

$$\widetilde{H} = r(t)q(r(t)) - \varsigma_B(m_B(r(t), e_B(r(t), z_B))) + \mu_B(t)\dot{r}(t)\left[\begin{array}{c} m'_{B1}(r(t), e_B(r(t), z_B)) + m'_{B2}(r(t), e_B(r(t), z_B))* \\ e'_{B1}(r(t), z_B) \end{array}\right].$$

(5)

Conditions for maximization are as given below:

(*i*) $r^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial r} = 0$,

(*ii*) $\dot{\mu}_B - \sigma\mu_B = -\frac{\partial \widetilde{H}}{\partial m_B}$,

(*iii*) $\dot{m_B}^* = \frac{\partial \widetilde{H}}{\partial \mu_B}$ (this just gives back the state equation),

(*iv*) $\lim\limits_{t \to \infty} \mu_B(t)m_B(t)e^{-\sigma t} = 0$ (the transversality condition).

Conditions (*i*) and (*ii*) are as follows:

$$\frac{\partial \widetilde{H}}{\partial r} = 0,$$

(6)

and

$$\dot{\mu}_B - \sigma\mu_B = -\frac{\partial \widetilde{H}}{\partial m_B} = \varsigma'_B(m_B(r(t), e_B(r(t), z_B))).$$

(7)

In equilibrium, $\dot{r}(t) = 0$, **substituting which in the expression for** $\frac{\partial \widetilde{H}}{\partial r}$, **it becomes the following** (see appendix):

$$r(t)\left[1 + \frac{1}{demand\ elasticity}\right] = \varsigma'_B(m_B(r(t), e_B(r(t), z_B)))\left\{\frac{m'_{B1}(r(t), e_B(r(t), z_B))}{q'(r(t))} + \frac{m'_{B2}(r(t), e_B(r(t), z_B))e'_{B1}(}{q'(r(t))}\right.$$

which implies that interest rate is equal to the marginal cost when demand is infinitely elastic. Marginal cost, i.e., the right hand side expression in the above equation is not the same as that in a myopic problem due to the reason that in dynamic consideration financial intermediary also **considers** impact of market interest rate on his purchase interest rate charged by producers. If a positive supply shock happens, the marginal cost of having another unit of funds in inventory is higher for financial intermediary as $\varsigma'_B(m_B(r(t), e_B(r(t), z_B)))$ term is higher at that point in time at existing interest rate. The term $\frac{m'_{B1}(r(t), e_B(r(t), z_B))}{q'(r(t))} + \frac{m'_{B2}(r(t), e_B(r(t), z_B))e'_{B1}(r(t), z_B)}{q'(r(t))}$ has not changed yet as the interest rate is still the same as before. The financial intermediary faces following inequality at contemporary interest rate:

$$\frac{\partial \widetilde{H}}{\partial r} < 0.$$

Financial intermediary must reduce interest rate in order to add another unit of funds in inventory for dynamic optimization condition to be satisfied. This depicts that a negative relationship exists between inventory and interest rate. Demand and supply concepts are unified through inventory;

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if their rates are equal, there is a state of equilibrium in market; whereas if a finite difference occurs between their rates, and the other agents do not react to that change, interest rate will get continuously changed by financial broker until the saturation of market takes place. This behavior of market can be expressed as follows:

Interest rate change $\propto$ *change in market inventory of funds.*

$$R = \textit{interest rate change.}$$

$$M_B = m_B - m_{Bs} = \textit{change in inventory of funds in the market,}$$

$$m_B = \textit{inventory of funds at time t,}$$

$$m_{Bs} = \textit{inventory of funds in steady state equilibrium.}$$

$$\textit{Input} - \textit{output} = \frac{dm_B}{dt} = \frac{d(m_B - m_{Bs})}{dt} = \frac{dM_B}{dt},$$

$$\textit{or } M_B = \int (\textit{input} - \textit{output}) \, dt.$$

Interest rate change $\propto \int (\textit{supply rate} - \textit{demand rate}) \, dt,$ or

$$R = -K_m \int (\textit{supply rate} - \textit{demand rate}) \, dt,$$

where K_m is a constant for proportionality. The sign (negative) reflects when (*supply rate* $-$ *demand rate*) is positive, R is negative, i.e., interest rate decreases. Above expression can also be written as:

$$\int (\textit{supply rate} - \textit{demand rate}) \, dt = -\frac{R}{K_m}, \text{ or}$$

$$\int (w_{Bi} - w_{B0}) \, dt = -\frac{R}{K_m}, \tag{8}$$

$$w_{Bi} = \textit{supply rate,}$$

$$w_{B0} = \textit{demand rate,}$$

$$K_m = \textit{dimensional constant.}$$

When $t = 0$, *supply rate* $=$ *demand rate*, and eq. (8) **changes into the following expression:**

$$\int (w_{Bis} - w_{B0s}) \, dt = 0. \tag{9}$$

Subscript s denotes a value in a steady state equilibrium and $R = 0$ in steady state. Subtracting eq. (9) from (8), we obtain:

$$\int (w_{Bi} - w_{Bis}) \, dt - \int (w_{B0} - w_{B0s}) \, dt = -\frac{R}{K_m}, \text{ or}$$

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$$\int \left(W_{Bi} - W_{B0}\right) dt = -\frac{R}{K_m}, \tag{10}$$

where $w_{Bi} - w_{Bis} = W_{Bi} = $ *change in supply rate,*

$w_{B0} - w_{B0s} = W_{B0} = $ *change in demand rate.*

R, W_{Bi} and W_{B0} are deviation variables, i.e., deviation from steady state equilibrium and have zero initial values. Eq. (10) can also be expressed as:

$$R = -K_m \int W_B dt = -K_m M_B, \tag{11}$$

where $W_B = W_{Bi} - W_{B0}$. **If R receives an input other than an inventory change, that can get added to eq. (11) as shown below:**

$$R = -K_m \int W_B dt + E_B = -K_m M_B + E_B. \tag{11a}$$

Inventory of funds can also undergo an exogenous shock in addition to feedback effect of interest rate.

2.1.2 Suppliers

There are various suppliers of funds. The dynamic problem of two important suppliers, i.e., the households, and the public sector/government has been discussed below:

Household The household as a supplier of funds has an objective of maximizing the present discounted value of series of utilities for all future times, and at time zero present value can be written:

$$V(0) = \int_0^\infty U(x(t))e^{-\rho t} dt, \tag{12}$$

ρ and $x(t)$ are discount rate and *control variable* respectively. Maximization problem has the following form:

$$\underset{\{x(t)\}}{Max} V(0) = \int_0^\infty U(x(t))e^{-\rho t} dt,$$

subject to following constraints

$\dot{a}(t) = r(t)a(t) + w(t) - p(t)x(t)$ (state equation, **which describes the change in state variable with time).** $a(t)$ **is a state variable representing quantum of asset holdings; and** $w(t)$ **(time path of wage) and** $p(t)$ **(time path of price of consumption** $x(t)$**) are exogenous.**

$a(0) = a_s$ (initial condition),

$a(t) \geq 0$ (non-negativity constraint on state variable), and

$a(\infty)$ free (terminal condition).

Current-value Hamiltonian is expressed as follows:

$$\widetilde{H} = U(x(t)) + \mu_p(t)\left[r\left(t\right)a\left(t\right) + w\left(t\right) - p\left(t\right)x\left(t\right)\right]. \tag{13}$$

The maximizing conditions can be written as:

(i) $x^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial x} = 0$,

(ii) $\dot{\mu}_p - \rho\mu_p = -\frac{\partial \widetilde{H}}{\partial a}$,

(iii) $\dot{a}^* = \frac{\partial \widetilde{H}}{\partial \mu_p}$ (this just gives back the state equation), and

(iv) $\lim_{t \to \infty} \mu_p(t)a(t)e^{-\rho t} = 0$ (the transversality condition).

Conditions (i) and (ii) can be written as:

$$\frac{\partial \widetilde{H}}{\partial x} = U'(x\left(t\right)) - \mu_p(t)p(t) = 0, \tag{14}$$

and

$$\dot{\mu}_p - \rho\mu_p = -\frac{\partial \widetilde{H}}{\partial a} = -\mu_p(t)r(t). \tag{15}$$

In a scenario where interest rate $r(t)$ rises, the above condition has the following form:

$$\dot{\mu}_p - \rho\mu_p + \mu_p(t)r(t) > 0.$$

After interest rate rises, the household's asset holdings must go up and hence the supply of funds in order to satisfy dynamic optimization conditions. Let r = market interest rate, c_p = **a feasible reference price for assets (for example, an asset yield including savings cost, and profits for household and financial intermediary)**. c_p is a reference point for the household's decision making, i.e., whether the supply of money should be increased or not as interest rate increases. It is a parameter which can either remain fixed for the time being or vary with time, such as the cost of savings can change or remain fixed depending on the exogenous economic conditions.

W_{mp} = *Change in funds supply by households/public due to change in interest rate,*

Household's incentive to supply more funds is the difference between r and c_p, i.e., $(r - c_p)$. Therefore,

$$W_{mp} \propto \alpha(r - c_p), \text{ or}$$

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αr is the interest rate the household gets from financial intermediaries, with $\alpha < 1$. We can write the above expression as:

$$W_{mp} = K_{sp}(r - c_p). \tag{16}$$

In a steady state equilbrium, $W_{mp} = 0$, or

$$0 = K_{sp}(r_s - c_{ps}). \tag{17}$$

K_{sp} is the **constant of proportionality**. r_s and c_{ps} **are steady state equilibrium values**. Subtracting eq. (17) from (16), we obtain:

$$W_{mp} = K_{sp}\left[(r - r_s) - (c_p - c_{ps})\right] = -K_{sp}\left(C_p - R\right) = -K_{sp}\varepsilon_p. \tag{18}$$

W_{mp}, C_p and R are deviation variables.

Public Sector/Government   Public sector/government maximizes present discounted value of series of all future net benefits, and **zero time value** is shown below:

$$V(0) = \int_0^\infty \left[Z_c\left(r(t)\right) - \varsigma_c(h\left(r(t)\right))\right] e^{-r_c t} dt. \tag{19}$$

$Z_c\left(r(t)\right)$ is the benefit for society, **a function decreasing in interest rate; a lower interest rate implies a higher benefit**. $\varsigma_c(h\left(r(t)\right))$ is the **public sector cost; a higher interest rate implies a higher cost**. **A plot of cost against interest rate is a concave downward curve,** i.e., decreasing in slope.

r_c and $r(t)$ are discount rate and *control variable*, and $h(t)$ is supply of public savings and a *state variable*. Maximization problem is as given below:

$$\underset{\{r(t)\}}{Max} V(0) = \int_0^\infty \left[Z_c\left(r(t)\right) - \varsigma_c(h\left(r(t)\right))\right] e^{-r_c t} dt,$$

subject to following constraints:
$\dot{h}(t) = h'(r\left(t\right))\dot{r}(t)$ (state equation, **which describes the change in state variable with time**),
$h(0) = h_s$ (initial condition),
$h(t) \geq 0$ (non-negativity constraint on state variable), and
$h(\infty)$ free (terminal condition).
Current-value Hamiltonian is expressed as follows:

$$\widetilde{H} = Z_c\left(r(t)\right) - \varsigma_c(h\left(r(t)\right)) + \mu_c(t)h'(r\left(t\right))\dot{r}(t). \tag{20}$$

Conditions for maximization are as given below:

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(i) $r^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial r} = 0$,

(ii) $\dot{\mu}_c - r_c \mu_c = -\frac{\partial \widetilde{H}}{\partial h}$,

(iii) $\dot{h}^* = \frac{\partial \widetilde{H}}{\partial \mu_c}$ (this just gives back the state equation), and

(iv) $\lim_{t \to \infty} \mu_c(t) h(t) e^{-r_c t} = 0$ (the transversality condition).

Conditions (i) and (ii) can be expressed as follows:

$$\frac{\partial \widetilde{H}}{\partial r} = Z_c'\left(r(t)\right) - \varsigma_c'(h\left(r\left(t\right)\right)) \; h'\left(r\left(t\right)\right) + \mu_c(t) \; h''(r(t)) \dot{r}\left(t\right) = 0. \tag{21}$$

and

$$\dot{\mu}_c - r_c \mu_c = -\frac{\partial \widetilde{H}}{\partial h} = \varsigma_c'(h\left(r\left(t\right)\right)). \tag{22}$$

If the interest rate rises, the term $\varsigma_c'(h\left(r\left(t\right)\right))$ **decreases, making public sector/government face the following expression**:

$$\dot{\mu}_c - r_c \mu_c - \varsigma_c'(h\left(r\left(t\right)\right)) > 0.$$

This implies that the funds supply by public sector/government should increase as interest rate rises to satisfy condition of dynamic optimization. If funds supply change is proportional to a change in interest rate, then following the logic for the household case, we can write:

$$W_{mc} = -K_{sc}\left(C_c - R\right). \tag{23}$$

2.1.3 Consumer

A consumer of funds, e.g., a firm **needing funds for investment purposes with an objective of maximization of** present discounted value **of series of future profits** where **zero time value** can be expressed as:

$$V(0) = \int_0^\infty \left[p(t) F\left(K\left(t\right), L\left(t\right)\right) - w(t) L\left(t\right) - r(t) I(t)\right] e^{-\varrho t} dt, \tag{24}$$

$p(t)$ is firm's output price; ϱ, $L(t)$ (labor), and $I(t)$ (level of investment) are discount rate, and *control variables respectively;* and $K(t)$ is *state variable*. Consumer's maximization problem is as follows:

$$\underset{\{L(t), I(t)\}}{Max} V(0) = \int_0^\infty \left[p(t) F\left(K\left(t\right), L\left(t\right)\right) - w(t) L\left(t\right) - r(t) I(t)\right] e^{-\varrho t} dt,$$

subject to following constraints:

$\dot{K}(t) = I(t) - \delta K(t)$ (state equation, **which describes the change in state variable with time**),

$K(0) = K_0$ (initial condition),

$K(t) \geq 0$ (non-negativity constraint on state variable), and

$K(\infty)$ free (terminal condition).

Current-value Hamiltonian is given below:

$$\widetilde{H} = p(t)F(K(t), L(t)) - w(t)L(t) - r(t)I(t) + \mu_f(t)[I(t) - \delta K(t)]. \tag{25}$$

Maximizing conditions can be written as:

(i) $L^*(t)$ and $I^*(t)$ maximize $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial L} = 0$ and $\frac{\partial \widetilde{H}}{\partial I} = 0$,

(ii) $\dot{\mu}_f - \varrho\mu_f = -\frac{\partial \widetilde{H}}{\partial K}$,

(iii) $\dot{K}^* = \frac{\partial \widetilde{H}}{\partial \mu_f}$ (this just gives back the state equation),

(iv) $\lim_{t \to \infty} \mu_f(t)K(t)e^{-\varrho t} = 0$ (the transversality condition).

Conditions (i) and (ii) can be written as:

$$\frac{\partial \widetilde{H}}{\partial L} = p(t)F_2'(K(t), L(t)) - w(t) = 0, \tag{26}$$

$$\frac{\partial \widetilde{H}}{\partial I} = -r(t) + \mu_f(t) = 0, \tag{27}$$

and

$$\dot{\mu}_f - \varrho\mu_f = -\frac{\partial \widetilde{H}}{\partial K} = -\left[p(t)F_1'(K(t), L(t)) - \delta\mu_f(t)\right]. \tag{28}$$

If $r(t)$ rises (with level of investment the same as before), the firm faces the following:

$$-r(t) + \mu_f(t) < 0.$$

After a rise in interest rate, consumer of funds must decrease level of investment to satisfy the dynamic optimization condition. Suppose change in demand of funds is proportional to interest rate change, i.e., R, then this change can be written as follows:

$$\textit{Change in money demand} \propto R, \text{ or}$$

$$W_d = -K_d R. \tag{29}$$

W_d is change in funds demand due to a change in interest rate, i.e., R. This shows that when R is positive, W_d is negative.

2.2 Solution of the Model with an Expansionary Savings Policy

Expressions from eqs. $(11a)$, (18), (23) and (29) respectively are as follows:

$$\frac{dR(t)}{dt} = -K_m W_B(t),$$
$$W_d(t) = -K_d R(t),$$
$$W_{mp} = -K_{sp}(C_p - R),$$
$$W_{mc} = -K_{sc}(C_c - R),$$

and

$$W_B(t) = W_m(t) - W_d(t),$$

in the absence of an exogenous demand/supply shock. $W_m(t)$ is the total funds supply including the supply from the public sector, households and firms, etc., and can be expressed as a sum of two sources of funds supply, i.e., the public sector, and the private sector (households, firms, etc.) as follows:

$$W_m(t) = -K_{sp}\left[C_p(t) - R(t)\right] - K_{sc}\left[C_c(t) - R(t)\right]. \tag{30}$$

Subscripts p and c denote private and public sector respectively. After combining above expressions, the following differential equation results:

$$\begin{aligned} \frac{dR(t)}{dt} &= -K_m\left[W_m(t) - W_d(t)\right] \\ &= -K_m\left[-K_{sp}\{C_p(t) - R(t)\} - K_{sc}\{C_c(t) - R(t)\} + K_d R(t)\right] \\ &= -K_m\left[-K_{sp}C_p(t) - K_{sc}C_c(t) + (K_{sp} + K_{sc} + K_d)R(t)\right]. \end{aligned}$$

Rearranging above expression gives:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) = K_m\left[K_{sp}C_p(t) + K_{sc}C_c(t)\right]. \tag{31}$$

Routh-Hurwitz stability criterion, **to ensure stability of above differential equation** requires $K_m(K_{sp} + K_{sc} + K_d) > 0$; and as K_m, K_{sp}, K_{sc}, and K_d are all positive, **the stability criterion holds**. This ensures that savings market settles at a new equilibrium for every adjustment mechanism once it gets out of initial equilibrium.

In savings market in comparison with a goods market, public sector has the role of government as well as that of a producer of funds. When government reduces interest rate on funds for investors

but not for depositors, public sector's supply curve shifts downward, and it incurs a cost in the role of government. We need to keep the role of public sector as producer separate from its role as government for clarity of what is exactly happening in savings/funds market when government exercises a savings policy. Suppose a change in the cost of public sector as a result of reducing interest rate on funds for investors is B, whereas the cost of private sector's supply of funds remains the same, the above equation can be expressed as:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) = -K_m K_{sc} B. \tag{32}$$

The solution is given by the following expression:

$$R(t) = -\frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}. \tag{33}$$

$R(0) = 0$ (initial condition), and $R(\infty) = -\frac{K_{sc}B}{(K_{sp}+K_{sc}+K_d)}$ (**final value**). As a result of a savings policy, interest rate dynamics depends on parameters K_{sp}, K_{sc}, K_d, K_m and B.

2.3 A Dynamic Optimal Expansionary Savings Policy-Panel B

Depending upon whether the savings policy is expansionary or contractionary, there could either be efficiency gains or losses in the **equilibrium (post policy) as compared to the one which existed before policy.** However, these are not the only efficiency gains or losses, **as some efficiency is lost during transient/adjustment to the final equilibrium**. **When savings policy is implemented, savings supply** either expands or contracts, **whereas at the initial interest rate, demand is still is the same as before, hence as a result of the policy, savings market gets out of equilibrium. Interest rate starts adjusting to equalize supply and demand to lead to new savings market equilibrium. Final equilibrium interest rate is a function of elasticities of supply and demand. As illustrated in the previous section, change in total supply as a result of a savings policy is as given below:**

$$W_m(0) = -K_{sp}\left[C_p(0) - R(0)\right] - K_{sc}\left[C_c(0) - R(0)\right] = K_{sc}B, \tag{34}$$
as $R(0) = 0$.

Due to savings policy, supply of savings either goes up or down by $K_{sc}B$. As demand **is the same as before**, therefore, the inventory of savings also changes by $K_{sc}B$, in the direction of the change in supply. **Savings market is out of equilibrium with market forces in action pushing savings market to post policy final equilibrium through an interest rate adjustment**. As interest rate changes, demand and supply of funds also change through feedback. If the savings inventory goes up, **it reflects supply is greater than demand** and vice versa. **If supply and**

demand are equal with savings market in equilibrium, there is no efficiency loss. For an out of equilibrium market, either supply and/or consumption of savings/funds is getting wasted at that point in time implying that total efficiency loss during market adjustment is sum of gaps in supply and demand at all points in time. After accounting for **a loss in efficiency** in post-policy equilibrium **in comparison with initial quilibrium** (if any), total efficiency loss for an expansionary and contractionary savings policy respectively can be expressed as:

$$EL \text{ (expansionary)} = \int_{-\infty}^{0} W_m(\infty)dt + \int_{0}^{\infty} [W_m(t) - W_d(t)] \, dt$$

$$= \int_{-\infty}^{0} W_m(\infty)dt + M_B(t). \tag{35}$$

$$EL \text{ (contractionary)} = \left| \int_{t}^{\infty} W_m(\infty)dt + \int_{0}^{t} W_d(t)dt + M_B(t) \right|$$

$$= \left| \int_{0}^{t} W_d(t)dt + \int_{t}^{\infty} W_d(\infty)dt + M_B(t) \right|$$

$$= \left| \int_{0}^{\infty} W_d(t)dt + \int_{0}^{\infty} [W_m(t) - W_d(t)] \, dt \right|$$

$$= \left| \int_{0}^{\infty} W_m(t)dt \right|. \tag{36}$$

With Expansionary Savings Policy Cost Constraint:

According to eq. (30), change **in funds supply** due to **a market interest rate change is as given below:**

$$W_m(t) = -K_{sp} [C_p(t) - R(t)] - K_{sc} [C_c(t) - R(t)] .$$

The part of supply coming from public sector is $-K_{sc} [C_c(t) - R(t)] ,$ and may be written as follows:

$$W_{mc}(t) = -K_{sc} [C_c(t) - R(t)] ,$$

$$w_{nmc}(t) - w_{imc}(0) = -K_{sc} [C_c(t) - R(t)] ,$$

where $w_{imc}(0)$ is the initial funds supply from the public sector, and $w_{nmc}(t)$ is **the supply after savings policy.** $W_{mc}(t) = w_{nmc}(t) - w_{imc}(0)$, which reflects **deviation from initial value. Cost to exercise savings policy is as given below:**

$$SPC = B\left[w_{imc}(0) + K_{sc}\left\{B + R(t)\right\}\right]. \tag{37}$$

For an optimal policy, efficiency loss needs to be minimized subject to the cost constraint as follows:

$$\min_{B} EL \quad \text{s.t.} \quad SPC \leq G_B.$$

G_B is **savings policy cost to the government. Control variable is savings policy** B, **and constraint is binding at** $t = 0$. **We can express Lagrangian for above problem as follows:**

$$\begin{aligned}
\mathcal{L} &= \int_{-\infty}^{0} W_m(\infty)dt + M_B(t) + \lambda\left[G_B - B\left[w_{imc}(0) + K_{sc}\left\{B + R(t)\right\}\right]\right] \\
&= \int_{-\infty}^{0} \left[K_{sc}B - \frac{K_{sc}(K_{sp} + K_{sc})B}{(K_{sp} + K_{sc} + K_d)}\right]dt \\
&\quad - \frac{1}{K_m}\left[-\frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}B\right] \\
&\quad + \lambda\left[G_B - B\left[w_{imc}(0) + K_{sc}\left\{B - \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right]\right] \\
&= \int_{-\infty}^{0} \frac{K_{sc}K_d B}{(K_{sp} + K_{sc} + K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}B\right] \\
&\quad + \lambda\left[G_B - B\left[w_{imc}(0) + K_{sc}\left\{B - \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right]\right].
\end{aligned}$$

First order derivative with respect to B leads to the following:

$$B = -\frac{\lambda w_{imc}(0) - \left[\int_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp}+K_{sc}+K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}\right]\right]}{2\lambda K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]}. \tag{38}$$

First order derivative with respect to λ is shown below:

$$G_B - B\left[w_{imc}(0) + K_{sc}\left\{B - \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right] = 0. \tag{39}$$

Putting eq. (38) into (39), we get:

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$$\lambda = \frac{J_B}{\sqrt{w_{imc}^2(0) + 4Q_B G_B}}.$$

$$\text{where } Q_B = K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right],$$

$$J_B = \int\limits_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp} + K_{sc} + K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}\right].$$

λ **is positive, as when** G_B **increases, minimum efficiency loss also increases**. From eq. (38):

$$B = -\frac{\lambda w_{imc}(0) - J_B}{2\lambda Q_B}. \tag{40}$$

Substituting λ **in above equation, following expression for** B **results:**

$$B = -\frac{w_{imc}(0) - \sqrt{w_{imc}^2(0) + 4Q_B G_B}}{2Q_B}. \tag{41}$$

Second order minimization condition has been checked in appendix. Suppose money available with government to be spent as savings policy cost is $1000. Initial value of the public sector's funds supply is $100 to financial broker, and values of all parameters, i.e., K_m**,** K_{sp}**,** K_{sc} **and** K_d **is equal to one. Substituting these values in eq. (41), gives:**

$$\mathbf{B} = -\frac{100 - \sqrt{10000 + 4000}}{2} = \mathbf{9.161},$$

where $Q_B = 1 - 0.333 + 0.333e^{-3t}$**, and at** $t = 0$**,** $Q_B = 1$**. Savings policy cost is** $SPC = B\left[w_{imc}(0) + Q_B B\right] = 1000$**. Therefore the optimal savings policy is that the government decreases interest rate on savings such that for each unit of savings, the government only bears a cost of $9.161 per unit.**

2.4 Solution of the Model with a Contractionary Savings Policy

Expressions from eqs. $(11a)$, (18), (23) and (29) respectively are as follows:

$$\frac{dR(t)}{dt} = -K_m W_B(t),$$
$$W_d(t) = -K_d R(t),$$
$$W_{mp} = -K_{sp}\left(C_p - R\right),$$
$$W_{mc} = -K_{sc}\left(C_c - R\right),$$

and

$$W_B(t) = W_m(t) - W_d(t),$$

in the absence of an exogenous demand/supply shock. $W_m(t)$ is the total funds supply including the supply from public sector, households and firms, etc., and can be expressed as a sum of two sources of supply, i.e., public sector; and private sector (households, firms, etc.) as follows:

$$W_m(t) = -K_{sp}\left[C_p(t) - R(t)\right] - K_{sc}\left[C_c(t) - R(t)\right], \tag{42}$$

Subscripts p and c denote private and public sector respectively. After combining above expressions, the following differential equation results:

$$\begin{aligned}
\frac{dR(t)}{dt} &= -K_m\left[W_m(t) - W_d(t)\right] \\
&= -K_m\left[-K_{sp}\left\{C_p(t) - R(t)\right\} - K_{sc}\left\{C_c(t) - R(t)\right\} + K_d R(t)\right] \\
&= -K_m\left[-K_{sp}C_p(t) - K_{sc}C_c(t) + (K_{sp} + K_{sc} + K_d)R(t)\right].
\end{aligned}$$

Rearranging the above expression gives:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) = K_m\left[K_{sp}C_p(t) + K_{sc}C_c(t)\right]. \tag{43}$$

When government increases interest rate (market interest rate which the investor pays) on savings, the public sector's supply curve shifts to left, and it gets some revenue in the role of government. Suppose a change in the cost of public sector as a supplier as a result of increasing interest rate on savings is B, whereas the cost of private supply of funds remains the same, the above equation can be written as:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) = K_m K_{sc} B. \tag{44}$$

The solution is given by the following expression:

$$R(t) = \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} - \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp} + K_{sc} + K_d)]t}. \tag{45}$$

$R(0) = 0$ (initial condition), and $R(\infty) = \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}$ (**final value**).

2.5 A Dynamic Optimal Contractionary Savings Policy-Panel B

Eq. ($11a$) states the following:

$$R(t) = -K_m M_B + E_B.$$

Value of E_B (for contractionary savings policy) can be determined through substituting the initial conditions as follows:

$$R(0) = -K_m M_B(0) + E_B,$$
$$0 = K_m K_{sc} B + E_B,$$
$$E_B = -K_m K_{sc} B.$$

Putting above expression in eq. $(11a)$, **the following equation results:**

$$R(t) = -K_m M_B(t) - K_m K_{sc} B, \text{ or}$$
$$M_B(t) = -\frac{1}{K_m} \left[R(t) + K_m K_{sc} B \right]. \tag{46}$$

With Contractionary Savings Policy Revenue Constraint:

According to eq. (30), change **in funds supply** due to **a market interest rate change is as given below:**

$$W_m(t) = -K_{sp} \left[C_p(t) - R(t) \right] - K_{sc} \left[C_c(t) - R(t) \right].$$

The part of the supply coming from the public sector is $-K_{sc} \left[C_c(t) - R(t) \right]$, and can be written as follows:

$$W_{mc}(t) = -K_{sc} \left[C_c(t) - R(t) \right],$$
$$w_{nmc}(t) - w_{imc}(0) = -K_{sc} \left[C_c(t) - R(t) \right],$$

where $w_{imc}(0)$ is the initial funds supply from the public sector, and $w_{nmc}(t)$ is **the supply after savings policy**. $W_{mc}(t) = w_{nmc}(t) - w_{imc}(0)$, which reflects **deviation from initial value.** Revenue received by government as a result of a contractionary savings policy can be expressed as:

$$SPR = B \left[w_{imc}(0) - K_{sc} \left\{ B - R(t) \right\} \right]. \tag{47}$$

For an optimal policy, efficiency loss needs to be minimized subject to the constraint as follows:

$$\min_T EL \quad \text{s.t.} \quad SPR \geq G_{BR}.$$

G_{BR} is the government revenue as a result of exercising the savings policy. **Control variable** is savings policy B, and constraint is binding at $t = 0$. **We can express Lagrangian for above problem as follows:**

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$$\mathcal{L} = -\int_0^\infty \left[K_{sp} \left\{ C_p(t) - R(t) \right\} + K_{sc} \left\{ C_c(t) - R(t) \right\} \right] dt + \lambda \left[G_{BR} - B \left[w_{imc}(0) - K_{sc} \left\{ B - R(t) \right\} \right] \right]$$

$$= \int_0^\infty \left[-K_{sc}B + \frac{K_{sc}(K_{sp}+K_{sc})B}{(K_{sp}+K_{sc}+K_d)} - \frac{K_{sc}(K_{sp}+K_{sc})B}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right] dt$$

$$+ \lambda \left[G_{BR} - B \left[w_{imc}(0) - K_{sc} \left\{ B - \frac{K_{sc}B}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}B}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right\} \right] \right].$$

Taking first order condition with respect to B, the following expression is obtained:

$$B = \frac{\lambda w_{imc}(0) - \int_0^\infty \left[-K_{sc} + \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} - \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right] dt}{2\lambda K_{sc} \left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right]}. \tag{48}$$

First order derivative with respect to λ is as given below:

$$G_{BR} - B \left[w_{imc}(0) - K_{sc} \left\{ B - \frac{K_{sc}B}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}B}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right\} \right] = 0. \tag{49}$$

Putting eq. (48) into (49), we obtain:

$$\lambda = \frac{J_B}{\sqrt{w_{imc}^2(0) - 4Q_B G_{BR}}},$$

$$\text{where } Q_B = K_{sc} \left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right],$$

$$J_B = \int_0^\infty \left[-K_{sc} + \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} - \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right] dt.$$

or

$$\left\{ w_{imc}^2(0) - 4Q_B G_{BR} \right\} \lambda^2 - J_B^2 = 0.$$

λ is positive, as when G_{BR} increases, minimum efficiency loss also increases. From eq. (48):

$$B = \frac{\lambda w_{imc}(0) - J_B}{2\lambda Q_B}. \tag{50}$$

Substituting λ in above equation, following expression for B results:

$$B = \frac{w_{imc}(0) - \sqrt{w_{imc}^2(0) - 4Q_B G_{BR}}}{2Q_B}.$$ (51)

Second order condition for minimization has been shown in appendix.

2.6 The Model-Panel A

Please refer to panel A in figure 1, where *number of savers per unit time* are plotted along *x*-axis, and saving *rate*, i.e., *quantum of savings per saver* is plotted along *y*-axis. **For number of savers, i.e., from public and private sector against each saving rate, supply curve is an upward sloping curve, Higher savings Higher savings by existing savers are due to some incentive, which also ecourages new savers. Demand curve based on public and private demand regarding number of savers in economy against each saving rate is a downward sloping curve. Demand for number of savers is low for a higher savings rate, and hence a negative relationship. The point where both curves cross is the equilibrium point. When demand is greater than supply, saving rate will increase to bring number of savers in equilibrium. Similarly, in case supply is greater than demand, saving rate will go down until number of savers equalize on both curves and brings equilibrium.**
Suppose number of savers are in equilibrium, i.e., the number on supply cruve equals that on demand curve. Following economic agents (infintely-lived) are modeled: public and private sector as demander for savers in economy, savers (a representative agent –or a unit mass of–) save money and make supplies to financial intermediary, and public and private sector as supplier of savers in economy against each saving rate. Mechanism regarding adjustment of saving rate has a basis of lack of coordination among agents after a shock/shift in demand/supply curve, as economic agents lack information on new demand and supply trends prevailing after shock. Suppose number of savers are in equilibrium, and due to an external shock, demand shifts to the right. The position of equilibrium is lost as now the demand of savers is greater than supply at the saving rate before shock. Due to a higher demand, saving rate will be increased by existing savers, and also new savers will enter saving industry. This will result in a higher number of savers and saving rate in final equilibrium. Equilibrium regarding number of savers has been defined as given below:
(i) Savers maximize profit, public and private sector maximizes utility in the role of having a certain demand for savings, and the public sector maximizes net benefit of public service for society, subject to their respective constraints mentioned in Section 7.

(ii) Number of savers supplied equals that demanded in economy and saving rate stays put during market equilibrium.

Conditions for equilibrium existence for a linear dynamical system, i.e., a necessary and sufficient condition for stability, based on Routh–Hurwitz stability criterion have been presented in Section 8. Public and private sector take saving rate as given. In absence of equilibrium, saver has an incentive to change saving rate on dynamic adjustment path until final equilibrium arrives. Government implements a savings policy (to increase or decrease number of savers), as a result of which market loses state of equilibrium, and saving rate adjusts to bring final equilibrium. Final equilibrium is achieved after number of savers and saving rate adjust to final values. Adjustment is based on endogenous decisions taken by economic agents in self-interest. Some social damage happens during adjustment process, which is defined as the sum of too many or too few savers before the new equilibrium arrives. Final equilibrium is more efficient than the initial one.

To derive mathematical results, objectives of market agents subject to their constraints are considered and maximized by taking first order derivatives. In order to get a complete picture of market, expressions reflecting actions of individual agents are simultaneously solved. We assume that final equilibrium after implementation of policy is not too off the initial equilibrium, which implies linearity of supply and demand curves is a reasonable assumption to make.

2.6.1 Saver

A saver saves money rather than consuming a part of it and deposits that in a financial institution which could be further supplied to consumer of funds. When demand of savers equals supply, there is an equilibrium regarding number of savers and saving rate. When either a supply or a demand shock happens, there is a change in number of savers. The link among supply (entry), demand, and cumulative number of savers is described as follows: When savers' demand in economy shifts to right while supply stays put, cumulative number of savers is lower than demand at contemporary saving rate, saving rate increases until supply equals demand in final equilibrium. If supply of savers shifts to right while demand remains the same, cumulative number of savers goes up at existing saving rate,down to equalize demand and supply to bring final equilibrium. Above explanation implies that there is a negative relationship between number of savers and saving rate. In Figure 4, horizontal axis reflects the rate of supply and demand both by private and public sectors, and not cumulative number of savers. Demand and supply rates are flow variables, whereas cumulative number of savers in

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economy is a stock variable.

Following mechansim explains adjustment of number of savers and savings in economy: Suppose there is an equilibrium number of savers and saving rate. If marginal cost of saving goes down, and supply curve regarding number of savers shifts to the right due to entry of new savers in saving industry, saving rate decreases, and savers' demand goes up due to feedback of a lower saving rate. Adjustment of saving rate and number of savers is contingent upon how savers respond to the shock, and the shock's direction and magnitude. Saver's choice is depicted mathematically by taking into consideration saver's profit maximization as given below:

Short Run Problem Short-term problem of saver implies that saver is not doing dynamic optimization and his/her objective is short-term. For discrete time, it can be considered as an objective just for one time period without looking at future times for the sake of an intuitive explanation and to prepare reader for a much more complex dynamic problem presented in next section. Saver's objective function is as given below:

$$\Theta = U_c(c) - \varsigma_A(m_A(c, e_A)), \tag{52}$$

where

Θ = net benefit of saver,

$U_c(c)$ = benefit of the saver by saving,

c = quantum of savings per saver (saving rate in a dynamic setting),

m_A = total number of savers in economy,

e_A = other factors which affect the total number of savers,

$\varsigma_A(m_A(c, e_A))$ = cost as a function of total number of savers in economy (increasing in number of savers).

Taking derivative of Θ with respect to c, we obtain:

$$U'_c(c) - \varsigma'_A(m_A(c, e_A))m'_{A1}(c, e_A) = 0. \tag{53}$$

Suppose a supply shock happens, such as a reduced saving cost, and supply shifts to the right, while demand does not change. New savers in economy will enter saving industry leading number of savers get out of equilibrium. With more savers, the term $\varsigma'_A(m_A(c, e_A))$ is higher at existing value of c. As the term, $m'_{A1}(c, e_A)$ is a function of c, which has not changed yet, therefore, $m'_{A1}(c, e_A)$ is the same as before, and the saver faces the following inequality as a modified profit maximization condition:

$$\frac{\partial \Theta}{\partial c} = U'_c(c) - \varsigma'_A(m_A(c, e_A))m'_{A1}(c, e_A) < 0, \tag{54}$$

therefore, for maximizing profit, saver will reduce the saving rate after supply shock. A plot of profit maximizing pairs of number of savers and respective saving rate is a downward sloping cumulative number of savers curve, with saving rate on y-axis and number of savers on x-axis.

Dynamic Problem For dynamic problem, present discounted value of future stream of profits of saver are maximized, with zero time value being as given below:

$$V(0) = \int_0^\infty \left[U_c(c) - \varsigma_A(m_A(c, e_A)) \right] e^{-\varpi t} dt, \tag{55}$$

ϖ, $c(t)$, **and** $m_A(t)$ **are discount rate, control variable, and state variable respectively. Maximization problem in mathematical notation is depicted below:**

$$\underset{\{c(t)\}}{Max} V(0) = \int_0^\infty \left[U_c(c) - \varsigma_A(m_A(c, e_A)) \right] e^{-\varpi t} dt,$$

subject to the following constraints:

$\dot{m}_A(t) = m'_{A1}(c(t), e_A(c(t), z_A))\dot{c}(t) + m'_{A2}(c(t), e_A(c(t), z_A))\, e'_{A1}(c(t), z_A)\dot{c}(t)$ (state equation, **which describes the change in state variable with time; with** z_A **as exogenous factors**),

$m_A(0) = m_{As}$ (initial condition),

$m_A(t) \geq 0$ (non-negativity constraint on state variable),

$m_A(\infty)$ free (terminal condition).

Following is the expression for current-value Hamiltonian:

$$\widetilde{H} = U_c(c(t)) - \varsigma_A(m_A(c(t), e_A(c(t), z_A))) + \mu_A(t)\dot{c}(t) \left[\begin{array}{c} m'_{A1}(c(t), e_A(c(t), z_A)) + m'_{A2}(c(t), e_A(c(t), z_A)) * \\ e'_{A1}(c(t), z_A) \end{array} \right].$$
$$\tag{56}$$

Conditions for maximization are as given below:

(i) $c^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial c} = 0$,

(ii) $\dot{\mu}_A - \varpi \mu_A = -\frac{\partial \widetilde{H}}{\partial m_A}$,

(iii) $\dot{m}_A{}^* = \frac{\partial \widetilde{H}}{\partial \mu_A}$ (this just gives back the state equation),

(iv) $\lim_{t \to \infty} \mu_A(t)m_A(t)e^{-\varpi t} = 0$ (the transversality condition).

(i) and (ii) conditions are as follows:

$$\frac{\partial \widetilde{H}}{\partial c} = U_c'(c(t)) - \varsigma_A'(m_A(c(t), e_A(c(t), z_A))) \left\{ \begin{array}{c} m_{A1}'(c(t), e_A(c(t), z_A)) + m_{A2}'(c(t), e_A(c(t), z_A))* \\ e_{A1}'(c(t), z_A) \end{array} \right\}$$

$$+ \mu_A(t)\dot{c}(t) * \left[\begin{array}{c} m_{A11}''(c(t), e_A(c(t), z_A)) + m_{A12}''(c(t), e_A(c(t), z_A))e_{A1}'(c(t), z_A) + \\ m_{A21}''(c(t), e_A(c(t), z_A))e_{A1}'(c(t), z_A) + m_{A22}''(c(t), e_A(c(t), z_A))e_{A1}'^2(c(t), z_A) + \\ m_{A2}'(c(t), e_A(c(t), z_A))e_{11}''(c(t), z_A) \end{array} \right]$$

$$= 0. \tag{57}$$

and

$$\dot{\mu}_A - \varpi\mu_A = -\frac{\partial \widetilde{H}}{\partial m_A} = \varsigma_A'(m_A(c(t), e_A(c(t), z_A))). \tag{58}$$

In equilibrium, $\dot{c}(t) = 0$,**substituting which in the expression for $\frac{\partial \widetilde{H}}{\partial c}$, it becomes the following**:

$$U_c'(c(t)) - \varsigma_A'(m_A(c(t), e_A(c(t), z_A))) \left\{ \begin{array}{c} m_{A1}'(c(t), e_A(c(t), z_A)) + m_{A2}'(c(t), e_A(c(t), z_A))* \\ e_{A1}'(c(t), z_A) \end{array} \right\} = 0.$$

Suppose a positive shock shifts supply to the right, then at current saving rate, the number of savers is higher, and the same is the case with the term $\varsigma_A'(m_A(c(t), e_A(c(t), z_A)))$. The term multiplying $\varsigma_A'(m_A(c(t), e_A(c(t), z_A)))$, i.e., $m_{A1}'(c(t), e_A(c(t), z_A)) + m_{A2}'(c(t), e_A(c(t), z_A))e_{A1}'(c(t), z_A)$ is a function of saving rate **which has not changed as yet**. Therefore, saver faces following **inequality** after shock:

$$\frac{\partial \widetilde{H}}{\partial c} < 0.$$

Saver must reduce saving rate for maximizing net benefits in dynamic context after supply shock. This depicts that a negative relationship exists between cumulative number of savers and saving rate. If demand and supply rates are equal, there is a state of equilibrium in market; whereas if a finite difference occurs between their rates, and the other agents do not react to that change, saving rate will get continuously changed by saver until the saturation of market takes place. This behavior of market can be expressed as follows:

Saving rate change $\propto$ *change in number of savers.*

$$C = Saving\ rate\ change.$$

$$M_A = m_A - m_{As} = change\ in\ number\ of\ savers,$$

$$m_A = number\ of\ savers\ at\ time\ t,$$

$$m_{As} = number\ of\ savers\ in\ steady\ state\ equilibrium.$$

$$Input - output = \frac{dm_A}{dt} = \frac{d(m_A - m_{As})}{dt} = \frac{dM_A}{dt},$$

$$or\ M_A = \int (input - output)\, dt.$$

Saving rate change $\propto \int (inflow/supply\ rate - required/demand\ rate)\, dt,$ or

$$C = -K_c \int (inflow/supply\ rate - required/demand\ rate)\, dt,$$

where K_c **is a constant for proportionality.** *Inflow/supply* and *required/demand rates* are inflow of new entrants and demand of number of savers in saving industry respectively. **The sign (negative) reflects when** $(inflow/supply\ rate - required/demand\ rate)$ **is positive,** C **is negative, i.e., saving rate decreases. Above expression can also be written as:**

$$\int (inflow/supply\ rate - required/demand\ rate)\, dt = -\frac{C}{K_c},\ or$$

$$\int (w_{Ai} - wA_0)\, dt = -\frac{C}{K_c}, \tag{59}$$

$$w_{Ai} = inflow/supply\ rate,$$

$$w_{A0} = required/demand\ rate,$$

$$K_c = dimensional\ constant.$$

When $t = 0$**, supply rate = demand rate, and eq. (59) changes into the following expression:**

$$\int (w_{Ais} - w_{A0s})\, dt = 0. \tag{60}$$

Subscript s **denotes a value in a steady state equilibrium and** $C = 0$ **in steady state. Subtracting eq. (60) from (59), we obtain:**

$$\int (w_{Ai} - w_{Ais})\, dt - \int (w_{A0} - wA_{0s})\, dt = -\frac{C}{K_c},\ or$$

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$$\int (W_{Ai} - W_{A0}) \, dt = -\frac{C}{K_c}, \tag{61}$$

where $w_{Ai} - w_{Ais} = W_{Ai} = $ *change in inflow/supply rate*,

$w_{A0} - w_{A0s} = W_{A0} = $ *change in required/demand rate*.

C, W_{Ai} and W_{A0} are deviation variables, i.e., deviation from steady state equilibrium and have zero initial values. Eq. (61) can also be expressed as:

$$C = -K_c \int W_A \, dt = -K_c M_A, \tag{62}$$

where $W_B = W_A = W_{Ai} - W_{A0}$. **If C receives an input other than a change in cumulative number of savers, that can get added to eq. (62) as shown below:**

$$C = -K_c \int W_A \, dt + E_A = -K_c M_A + E_A. \tag{62a}$$

There is an impact on M_A due to saving rate's feedback, however, an exogenous input can also have an impact just like C.

2.6.2 Suppliers

Private Sector Both private and public sectors have a role of supplier and demander of savers in economy, however, we present just one of their roles in this section. Sum of both private and public demand and supply constitute toal demand and supply respectively. Here, we present private sector's role as a supplier of savers. Private sector maximizes present discounted value of series of all future net benefits, and zero time value is shown below:

$$V(0) = \int_0^\infty [U_{pr}(n_{pr}) - \varsigma_{pr}(c(n_{pr}))] e^{-r_{pr}t} dt. \tag{63}$$

$U_{pr}(n_{pr})$ is the benefit for society, a function increasing in number of savers; a higher number of savers implies a higher benefit. $\varsigma_{pr}(c(n_{pr}))$ is the private sector cost; a higher saving rate implies a higher cost. A plot of cost against saving rate is a concave downward curve, i.e., decreasing in slope.

r_{pr} and $n_{pr}(t)$ are discount rate and control variable, and $c(t)$ is a state variable. Maximization problem is as given below:

$$\underset{\{n_{pr}(t)\}}{Max} V(0) = \int_0^\infty [U_{pr}(n_{pr}) - \varsigma_{pr}(c(n_{pr}))] e^{-r_{pr}t} dt,$$

subject to the following constraints:

$\dot{c}(t) = c'(n_{pr}(t))\dot{n}_{pr}(t)$ (state equation, **which describes the change in state variable with time**),

$c(0) = c_s$ (initial condition),

$c(t) \geq 0$ (non-negativity constraint on state variable),

$c(\infty)$ free (terminal condition).

Current-value Hamiltonian is given below:

$$\widetilde{H} = U_{pr}(n_{pr}(t)) - \varsigma_{pr}(c(n_{pr}(t))) + \mu(t) \ c'(n_{pr}(t))\dot{n}_{pr}(t). \tag{64}$$

Maximizing conditions can be expressed as follows:

(i) $n_{pr}{}^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial n_{pr}} = 0$,

(ii) $\dot{\mu}_{pr} - r_{pr}\mu_{pr} = -\frac{\partial \widetilde{H}}{\partial c}$,

(iii) $\dot{c}^* = \frac{\partial \widetilde{H}}{\partial \mu_{pr}}$ (this just gives back the state equation),

(iv) $\lim\limits_{t\to\infty} \mu_{pr}(t)c(t)e^{-r_{pr}t} = 0$ (the transversality condition).

(i) and (ii) are given below:

$$\frac{\partial \widetilde{H}}{\partial n_{pr}} = U'_{pr}(n_{pr}(t)) - \varsigma'_{pr}(c(n_{pr}(t))) \ c'(n_{pr}(t)) + \mu_{pr}(t) \ c''(n_{pr}(t))\dot{n}_{pr}(t) = 0, \tag{65}$$

and

$$\dot{\mu}_{pr} - r_{pr}\mu_{pr} = -\frac{\partial \widetilde{H}}{\partial c} = \varsigma'_{pr}(c(n_{pr}(t))). \tag{66}$$

During equilibrium, $\dot{n}_{pr}(t) = 0$, and we can express $\frac{\partial \widetilde{H}}{\partial n_{pr}}$ as follows:

$$U'_{pr}(n_{pr}(t)) - \varsigma'_{pr}(c(n_{pr}(t))) \ c'(n_{pr}(t)) = 0.$$

If the saving rate rises, the term $\varsigma'_{pr}(c(n_{pr}(t)))$ **decreases, making private sector face the following expression**:

$$\frac{\partial \widetilde{H}}{\partial n_{pr}} > 0.$$

To satisfy condition of dynamic optimization, private sector will increase supply of number of savers. For a linear supply curve (or with linearization around steady state as a reasonable assumption), change in number of savers is proportional to saving rate, and we have the following expression:

$$W_{pr}(t) = K_{pr}C(t), \tag{67}$$

$W_{pr}(t)$ is deviation variable, i.e., a change in number of savers relative to initial equilibrium value, having a zero initial value. Introducing a dead time element due to delay between change in saving rate and change in number of savers supplied, following expression results:

$$W_{pr}(t) = K_{pr}C(t - \tau_{d1}). \tag{68}$$

Public Sector/Government In this section, we present public sector's role as a demander of savers. Public sector maximizes present discounted value of series of all future net benefits, and zero time value is shown below:

$$V(0) = \int_0^\infty [U_{pu}(n_{pu}) - \varsigma_{pu}(c(n_{pu}))] e^{-r_{pu}t} dt, \tag{69}$$

$U_{pu}(n_{pu})$ is the benefit for society, a function increasing in number of savers; a higher number of savers implies a higher benefit. $\varsigma_{pu}(c(n_{pu}))$ is the public sector cost; a higher saving rate implies a higher cost. A plot of cost against saving rate is a concave downward curve, i.e., decreasing in slope.

r_{pu} and $n_{pu}(t)$ are discount rate and control variable, and $c(t)$ is a state variable. Maximization problem is as given below:

$$\underset{\{n_{pu}(t)\}}{Max} \ V(0) = \int_0^\infty [U_{pu}(n_{pu}) - \varsigma_{pu}(c(n_{pu}))] e^{-r_{pu}t} dt,$$

subject to the following constraints:

$\dot{c}(t) = c'(n_{pu}(t))\dot{n}_{pu}(t)$ (state equation, **which describes the change in state variable with time**),

$c(0) = c_s$ (initial condition),

$c(t) \geq 0$ (non-negativity constraint on state variable),

$c(\infty)$ free (terminal condition).

The current-value Hamiltonian for this case is

$$\widetilde{H} = U_{pu}(n_{pu}(t)) - \varsigma_{pu}(c(n_{pu}(t))) + \mu_{pu}(t) \ c'(n_{pu}(t))\dot{n}_{pu}(t). \tag{70}$$

Maximizing conditions can be expressed as follows:

(i) $n_{pu}{}^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial n_{pu}} = 0$,

(ii) $\dot{\mu}_{pu} - r_{pu}\mu_{pu} = -\frac{\partial \widetilde{H}}{\partial c}$,

(iii) $\dot{c}^* = \frac{\partial \widetilde{H}}{\partial \mu_{pu}}$ (this just gives back the state equation),

(iv) $\lim_{t \to \infty} \mu_{pu}(t)c(t)e^{-r_{pu}t} = 0$ (the transversality condition).

(i) and (ii) are given below:

$$\frac{\partial \widetilde{H}}{\partial n_{pu}} = U'_{pu}\left(n_{pu}(t)\right) - \varsigma'_{pu}(c\left(n_{pu}\left(t\right)\right)) \ c'\left(n_{pu}\left(t\right)\right) + \mu_{pu}(t) \ c''(n_{pu}\left(t\right))\dot{n}_{pu}\left(t\right) = 0. \tag{71}$$

and

$$\dot{\mu}_{pu} - r_{pu}\mu_{pu} = -\frac{\partial \widetilde{H}}{\partial c} = \varsigma'_{pu}(c\left(n_{pu}\left(t\right)\right)). \tag{72}$$

During equilibrium, $\dot{n}_{pu}\left(t\right) = 0$, and we can express $\frac{\partial \widetilde{H}}{\partial n_{pu}}$ as follows:

$$U'_{pu}\left(n_{pu}(t)\right) - \varsigma'_{pu}(c\left(n_{pu}\left(t\right)\right)) \ c'\left(n_{pu}\left(t\right)\right) = 0.$$

If the saving rate rises, the term $\varsigma'_{pu}(c\left(n_{pu}\left(t\right)\right))$ **increases, making public sector face the following expression**:

$$\frac{\partial \widetilde{H}}{\partial n_{pu}} < 0.$$

To satisfy condition of dynamic optimization, public sector will decrease demand of savers. For a linear demand curve (or with linearization around steady state as a reasonable assumption), change in number of savers is proportional to saving rate, and we have the following expression:

$$W_{pu}(t) = K_{pu}\left[\epsilon\left(t\right) - C(t)\right] = -K_{pu}\eta(t), \tag{73}$$

where $\epsilon\left(t\right) = e - e_s$; e **is a saving rate as a reference point for comparison with the market saving rate** by public sector for decision making. $W_{pu}(t)$ **is deviation variable, i.e., a change in number of savers relative to initial equilibrium value, having a zero initial value. Introducing a dead time element due to delay between change in saving rate and change in number of savers demanded, following expression results:**

$$W_{pu}(t) = -K_{pu}\eta(t - \tau_{d2}). \tag{74}$$

2.7 Solution of the Model with a Saving Policy

The model has been solved for $\tau_{d1} = \tau_{d2} = 0$. Eq. (62a), (67), and (73) are reproduced as follows:

$$\frac{dC}{dt} = -K_c W_A(t),$$

$$W_{pr}(t) = K_{pr} C(t),$$

$$W_{pu}(t) = K_{pu} \left[\epsilon(t) - C(t) \right],$$

$$W_A(t) = W_1(t) - W_{pu}(t),$$

$$= D(t) + W_{pr}(t) - W_{pu}(t).$$

$$\text{where } D(t) = W_{Ai}(t) - W_{A0}(t).$$

When there is no exogenous shock regarding number of savers or saving rate, $D(t) = 0$. Panel A's policy must be in synchronization with that of panel B, i.e., both policies must be having a direction to achieve one objective, i.e., either an overall increase or decrease in savings in economy. This section presents an example of an optimal policy regarding panel A for a shift in the demand curve, however, to achieve a certain objective, both policies must be aligned. Suppose government shifts demand of savings upward through adopting a policy, i.e.,

$$W_{pu}(t) = K_{pu} \left[A - C(t) \right],$$

where A is the size of the policy. This implies that

$$\frac{dC(t)}{dt} = -K_c \left[W_{pr}(s) - W_{pu}(t) \right]$$

$$= -K_c \left[K_{pr} C(t) - K_{pu} A + K_{pu} C(t) \right]$$

$$= -K_c \left[-K_{pu} A + (K_{pr} + K_{pu}) C(t) \right].$$

After rearranging, we obtain the following expression:

$$\frac{dC(t)}{dt} + K_c(K_{pr} + K_{pu})C(t) = K_c K_{pu} A. \tag{75}$$

Routh-Hurwitz stability criterion, to ensure stability of above differential equation requires $K_c(K_{pr} + K_{pu}) > 0$; and as K_c, K_{pr} and K_{pu} are all positive, the stability criterion holds. This ensures that, away from a given initial equilibrium, every adjustment mechanism will lead to another equilibrium.

The differential equation's characteristic function is as given below:

$$x + K_c(K_{pr} + K_{pu}) = 0,$$

which has a single root as given below:

$$x = -K_c(K_{pr} + K_{pu}).$$

The complementary solution can be written as

$$C_c(t) = C_2 e^{-[K_c(K_{pr}+K_{pu})]t}.$$

The particular solution can be expressed as follows:

$$C_p(t) = C_1.$$

The solution is as given below:

$$C(t) = C_1 + C_2 e^{-[K_c(K_{pr}+K_{pu})]t}. \tag{76}$$

After substitution of above expression into the differential equation, we obtain the following expression:

$$-K_c(K_{pr}+K_{pu})C_2 e^{-[K_c(K_{pr}+K_{pu})]t} + K_c(K_{pr}+K_{pu})C_1 + K_c(K_{pr}+K_{pu})C_2 e^{-[K_c(K_{pr}+K_{pu})]t} = K_c K_{pu} A,$$

$$C_1 = \frac{K_{pu} A}{K_{pr} + K_{pu}}.$$

After substituting the initial conditions and value of C_1 in eq. (76), the following expression results:

$$C(0) = \frac{K_{pu} A}{K_{pr} + K_{pu}} + C_2 = A,$$

$$C_2 = A - \frac{K_{pu} A}{K_{pr} + K_{pu}}$$

$$= \frac{K_{pr} A}{K_{pr} + K_{pu}}.$$

Plugging in the values of C_1 and C_2 in eq. (76) yields:

$$C(t) = \frac{K_{pu} A}{K_{pr} + K_{pu}} + \frac{K_{pr} A}{K_{pr} + K_{pu}} e^{-[K_c(K_{pr}+K_{pu})]t}. \tag{77}$$

$C(0) = 0$ (the initial condition), and $C(\infty) = \frac{K_{pr} A}{K_{pr}+K_{pu}}$ (**final value**). **A change in saving rate, i.e., $C(t)$ is a function of size of policy A, and parameters, K_c, K_{pu}, and K_{pr}.**

2.8 A Dynamic Optimal Savings Policy-Panel A

Equilibrium before adoption of policy can be improved upon as a result of an appropriate savings policy. However, these are not the only efficiency gains or losses, as some efficiency is lost during transient/adjustment to the final equilibrium. Efficiency loss is minimized to derive an optimal policy. When savings policy is implemented, supply of savers either expands or contracts, whereas at the initial saving rate, demand is still is the same as before, hence as a result of the policy, savers market gets out of equilibrium. Saving rate starts adjusting to equalize supply and demand to lead to new savers market equilibrium. Final equilibrium saving rate is a function of elasticities of supply and demand. Social damage is an excessive/short number of savers, i.e., the difference between supply and demand. Total social damage in terms of number of savers is the sum of that during adjustment of the savers market and the one in initial equilibrium as given below:

$$SD = M_A(t) + \int_{-\infty}^{0} W_{pr}(\infty)dt. \tag{78}$$

From eq. (73), a change in number of savers due to implementation of savings policy is given below:

$$W_{pu}(t) = K_{pu}\left[A - C(t)\right],$$

$$\text{or } w_{npu}(t) - w_{ipu}(0) = K_{pu}\left[A - C(t)\right],$$

where $w_{ipu}(0)$ is number of savers in initial equilibrium, and $w_{npu}(t)$ is **the supply after savings policy**. $W_{pu}(t) = w_{npu}(t) - w_{ipu}(0)$, which reflects **deviation from initial value.** A change in quantity of savings per unit time is as given below:

$$IQS = A\left[w_{ipu}(0) + K_{pu}\left\{A - C(t)\right\}\right]. \tag{79}$$

For an optimal policy, efficiency loss needs to be minimized subject to an increase in quantity of savings per unit time being greater than or equal to G_A $\Big($change in quantity of savings/funds per unit time = and can be expressed as follows:

$$\min_{A} SD \quad \text{s.t.} \quad IQS \geq G_A\left(= \frac{dM_B}{dt}\right).$$

Control variable is savings policy A (an initial upward jump in the saving rate chosen by government to shift the demand curve), and constraint is binding at $t = 0$. **We can express Lagrangian for above problem as follows:**

$$\mathcal{L} = M_A(t) + \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - C(t)\right\}\right]\right].$$

Expression from eq. (62a), is given below:

$$C(t) = -K_c M_A + E_A.$$

Through putting initial conditions, value of E_A can be determined as follows:

$$C(0) = -K_c M_A(0) + E_A,$$
$$A = -K_c K_{pr} C(0) + E_A,$$
$$E_A = A\left[1 + K_c K_{pr}\right].$$

This implies that

$$M_A(t) = -\frac{1}{K_c}\left[C(t) - A\left\{1 + K_c K_{pr}\right\}\right].$$

This changes the Lagrangian to the following expression:

$$\mathcal{L} = -\frac{1}{K_c}\left[C(t) - A\left\{1 + K_c K_{pr}\right\}\right] + \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - C(t)\right\}\right]\right]$$

$$= -\frac{1}{K_c}\left[\left\{\frac{K_{pu}A}{K_{pr}+K_{pu}} + \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\} - A\left\{1 + K_c K_{pr}\right\}\right]$$

$$+ \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - \frac{K_{pu}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right]\right].$$

First order derivative with respect to A leads to the following:

$$A = \frac{\lambda w_{ipu}(0) - \frac{1}{K_c}\left[\left\{\frac{-K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\} + \left\{1 + K_c K_{pr}\right\}\right]}{-2\lambda K_{pu}\left\{1 - \frac{K_{pu}}{K_{pr}+K_{pu}} - \frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}}. \qquad (80)$$

Similarly, first order derivative with respect to λ is shown below:

$$G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - \frac{K_{pu}A}{K_{pr}+K_{pu}} - \frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right] = 0. \qquad (81)$$

Putting eq. (80) into (81), we obtain:

$$\lambda = \frac{J_A}{\sqrt{w_{ipu}^2(0) - 4Q_A G_A}}.$$

λ **is positive, as when** G_A **increases, social damage also increases**.

$$Q_A = -K_{pu}\left\{1 - \frac{K_{pu}}{K_{pr} + K_{pu}} - \frac{K_{pr}}{K_{pr} + K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\},$$

$$J_A = \frac{1}{K_c}\left[\left\{\frac{-K_{pu}}{K_{pr} + K_{pu}} - \frac{K_{pr}}{K_{pr} + K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\} + \left\{1 + K_c K_{pr}\right\}\right].$$

Eq. (80) can also be written as

$$A = \frac{\lambda w_{ipu}(0) - J_A}{2\lambda Q_A}. \tag{82}$$

After substituting value of λ **in above equation, the following expression results:**

$$A = \frac{w_{ipu}(0) - \sqrt{w_{ipu}^2(0) - 4Q_A G_A}}{2Q_A}. \tag{83}$$

A **is an expression for a policy regarding** an optimal number of savers in a dynamical setting. **Second order minimization condition has been checked in appendix. By substituting parameters' values based on empirical data in eq.** (83), **an optimal number of savers in the economy can be determined. Both policies** B, **and** A **together vide equations** (41) **and** (83) **determine the optimal quantum of savings in the economy with an optimal number of savers given the constraints.**

3   Results

Without consideration of welfare loss/gain while savings market is adjusting after implementation of a savings policy, welfare picture remains incomplete, and the optimal savings policy based on partial welfare cannot be considered as optimal in true sense. Traditionally, welfare of only producer and consumer is taken into consideration without accounting for welfare of production factors. An expression of efficiency loss/gain as a result of savings policy based on welfare including those of production factors has been presented and optimal savings policies have been derived by minimizing efficiency losses and presented as a final result in the form of mathematical expressions, i.e., (41), or (51) and (83). This paper demonstrates that both supply and demand shocks operate through a common channel, i.e., inventory of funds in savings market as both kinds of shock affect inventory of funds and hence can be categorized just as an inventory shock.

4 Discussion

For an optimal level of savings in an economy, it is important for the government to adopt appropriate policies, i.e., the quantity of savings, and interest rate; and to ensure that there are number of savers and their saving rate consistent with the optimal quantity determined in the policy designed for an optimal quantity and the corresponding interest rate, government needs to adopt a policy to have the right number of savers and the corresponding saving rate in the economy, otherwise, there will be extra efficiency losses due to lack of coordination between government and the number of savers. This is why we have derived two sets of optimal policies to be adopted by the government to ensure the least efficiency loss for achieving an optimal level of savings in an economy given the constraints.

Practical implications for the policy makers are to estimate policies A, and B given by expressions (41), or (51) **and** (83) on the basis of data. As all parameters in the optimal policies expressions can be empirically estimated with the help of data, policy makers can formulate and implement optimal policies regarding quantity of savings, interest rate, number of savers and saving rate in the economy.

5 Conclusion

When a dynamically optimal expansionary/contractionary savings policy is adopted by the government for panel B, by shifting supply curve downward/upward, savings market goes out of equilibrium, with a presemption that it was in equilibrium before the implementation of the policy. Quantity of savings/funds and interest rate adjust over time and the savings market eventually attains the final equilibrium. **Post-policy equilibrium is more efficient in comparison with the pre-policy equilibrium, however, during market adjustment to the final equilibrium, there are some extra efficiency losses.** Eqs (41) and (51) present dynamically optimal expansionary and contractionary savings policies **as a result of minimizing efficiency losses during market adjustment to final equilibrium.** The expressions involve supply of savings/funds, demand, and inventory curves' slopes and initial pre-policy savings/funds quantity.

We develop a savings production model involving number of savers and saving rate which depends on parameters K_c, K_{pr}, K_{pu}, τ_{d1} and τ_{d2} **to derive a dynamically optimal policy for panel A**. **The model has the ability to predict adjustment path and post-policy equilibrium** after a supply/demand or production (of savings) rate shock. Figure 4 depicts how a shift in the supply, demand or both curves determines the saving rate and the number of savers in society. An optimal policy which minimizes social damage in terms of inadequate number of savers in initial

equilibrium as well as on dynamic adjustment path (when number of savers is not in equilibrium) subject to a certain increase in quantity of savings per unit time can be derived on a case by case basis (which shifts either the supply, demand or both curves). Area under the demand curve is the social benefit in terms of quantity of savings per unit time. Eq. (83) presents the expression for the optimal production (of savings) policy depending on parameters $w_{ipu}(0)$, G_A, K_c, K_{pr}, K_{pu}, τ_{d1} and τ_{d2}.

6 Appendix:

6.1 Dynamic Problem of Financial Intermediary/Broker/Commercial Bank

For dynamic problem, present discounted value of future stream of profits of the financial intermediary are maximized, with zero time value being as given below:

$$V(0) = \int_0^\infty \left[rq(r) - \varsigma_B(m_B(r, e_B)) \right] e^{-\sigma t} dt, \tag{84}$$

σ, $r(t)$, and $m_B(t)$ are discount rate, *control variable*, and *state variable* respectively. Maximization problem in mathematical notation is depicted below:

$$\underset{\{r(t)\}}{Max} V(0) = \int_0^\infty \left[rq(r) - \varsigma_B(m_B(r, e_B)) \right] e^{-\sigma t} dt,$$

subject to the constraints that
$\dot{m}_B(t) = m'_{B1}(r(t), e_B(r(t), z_B)) \dot{r}(t) + m'_{B2}(r(t), e_B(r(t), z_B)) e'_{B1}(r(t), z_B) \dot{r}(t)$ (state equation, **which describes the change in state variable with time; with z_B as exogenous factors**),
$m_B(0) = m_{Bs}$ (initial condition),
$m_B(t) \geq 0$ (non-negativity constraint on state variable),
$m_B(\infty)$ free (terminal condition).
Following is the expression for current-value Hamiltonian:

$$\widetilde{H} = r(t)q(r(t)) - \varsigma_B(m_B(r(t), e_B(r(t), z_B))) + \mu_B(t) \dot{r}(t) \left[\begin{array}{c} m'_{B1}(r(t), e_B(r(t), z_B)) + m'_{B2}(r(t), e_B(r(t), z_B)) * \\ e'_{B1}(r(t), z_B) \end{array} \right].$$
$$\tag{85}$$

Conditions for maximization are as given below:
(i) $r^*(t)$ maximizes $\widetilde{H}$ for all t: $\frac{\partial \widetilde{H}}{\partial r} = 0$,
(ii) $\dot{\mu}_B - \sigma \mu_B = -\frac{\partial \widetilde{H}}{\partial m_B}$,
(iii) $\dot{m}_B{}^* = \frac{\partial \widetilde{H}}{\partial \mu_B}$ (this just gives back the state equation),
(iv) $\lim_{t \to \infty} \mu_B(t) m_B(t) e^{-\sigma t} = 0$ (the transversality condition).
Conditions (i) and (ii) are as follows:

$$\frac{\partial \widetilde{H}}{\partial r} = q(r(t)) + r(t)q'(r(t)) - \varsigma'_B(m_B(r(t), e_B(r(t), z_B))) \left\{ \begin{array}{c} m'_{B1}(r(t), e_B(r(t), z_B)) + m'_{B2}(r(t), e_B(r(t), z_B)) * \\ e'_{B1}(r(t), z_B) \end{array} \right\}$$

$$+ \mu_B(t)\dot{r}(t) * \left[\begin{array}{c} m''_{B11}(r(t), e_B(r(t), z_B)) + m''_{B12}(r(t), e_B(r(t), z_B))e'_{B1}(r(t), z_B) + \\ m''_{B21}(r(t), e_B(r(t), z_B))e'_{B1}(r(t), z_B) + m''_{B22}(r(t), e_B(r(t), z_B))e'^2_{B1}(r(t), z_B) + \\ m'_{B2}(r(t), e_B(r(t), z_B))e''_{B11}(r(t), z_B) \end{array} \right]$$

$$= 0, \tag{86}$$

and

$$\dot{\mu}_B - \sigma\mu_B = -\frac{\partial \widetilde{H}}{\partial m_B} = \varsigma'_B(m_B(r(t), e_B(r(t), z_B))). \tag{87}$$

In equilibrium, $\dot{r}(t) = 0$, **substituting which in the expression for** $\frac{\partial \widetilde{H}}{\partial r}$, **it becomes the following:**

$$q(r(t)) + r(t)q'(r(t)) - \varsigma'_B(m_B(r(t), e_B(r(t), z_B))) \left\{ \begin{array}{c} m'_{B1}(r(t), e_B(r(t), z_B)) + m'_{B2}(r(t), e_B(r(t), z_B)) * \\ e'_{B1}(r(t), z_B) \end{array} \right\}$$

$$= 0,$$

$$r(t)q'(r(t)) + q(r(t)) = \varsigma'_B(m_B(r(t), e_B(r(t), z_B))) \left\{ \begin{array}{c} m'_{B1}(r(t), e_B(r(t), z_B)) + m'_{B2}(r(t), e_B(r(t), z_B)) * \\ e'_{B1}(r(t), z_B) \end{array} \right\}$$

$$r(t) \left[1 + \frac{1}{demand\ elasticity} \right] = \varsigma'_B(m_B(r(t), e_B(r(t), z_B))) \left\{ \frac{m'_{B1}(r(t), e_B(r(t), z_B))}{q'(r(t))} + \frac{m'_{B2}(r(t), e_B(r(t), z_B))e'_{B1}(}{q'(r(t))} \right.$$

which implies that interest rate is equal to the marginal cost when demand is infinitely elastic. Marginal cost, i.e., the right hand side expression in the above equation is not the same as that in a myopic problem due to the reason that in dynamic consideration financial intermediary also **considers** impact of market interest rate on his purchase interest rate charged by producers. If a positive supply shock happens, the marginal cost of having another unit of funds in inventory is higher for financial intermediary as $\varsigma'_B(m_B(r(t), e_B(r(t), z_B)))$ term is higher at that point in time at existing interest rate. The term $\frac{m'_{B1}(r(t), e_B(r(t), z_B))}{q'(r(t))} + \frac{m'_{B2}(r(t), e_B(r(t), z_B))e'_{B1}(r(t), z_B)}{q'(r(t))}$ has not changed yet as the interest rate is still the same as before. The financial intermediary faces following inequality at contemporary interest rate:

$$\frac{\partial \widetilde{H}}{\partial r} = q(r(t)) + r(t)q'(r(t)) - \varsigma'_B(m_B(r(t), e_B(r(t), z_B))) \left\{ \begin{array}{c} m'_{B1}(r(t), e_B(r(t), z_B)) + m'_{B2}(r(t), e_B(r(t), z_B)) * \\ e'_{B1}(r(t), z_B) \end{array} \right\}$$

$$+ \mu_B(t)\dot{r}(t) * \left[\begin{array}{c} m''_{B11}(r(t), e_B(r(t), z_B)) + m''_{B12}(r(t), e_B(r(t), z_B))e'_{B1}(r(t), z_B) + \\ m''_{B21}(r(t), e_B(r(t), z_B))e'_{B1}(r(t), z_B) + m''_{B22}(r(t), e_B(r(t), z_B))e'^2_{B1}(r(t), z_B) + \\ m'_{B2}(r(t), e_B(r(t), z_B))e''_{B11}(r(t), z_B) \end{array} \right]$$

$$< 0.$$

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Financial intermediary must reduce interest rate in order to add another unit of funds in inventory for dynamic optimization condition to be satisfied. This depicts that a negative relationship exists between inventory and interest rate. Demand and supply concepts are unified through inventory; if their rates are equal, there is a state of equilibrium in market; whereas if a finite difference occurs between their rates, and the other agents do not react to that change, interest rate will get continuously changed by financial broker until the saturation of market takes place. This behavior of market can be expressed as follows:

$$Interest\ rate\ change \propto change\ in\ market\ inventory\ of\ funds.$$

$$R = interest\ rate\ change.$$

$$M_B = m_B - m_{Bs} = change\ in\ inventory\ of\ funds\ in\ the\ market,$$

$$m_B = inventory\ of\ funds\ at\ time\ t,$$

$$m_{Bs} = inventory\ of\ funds\ in\ steady\ state\ equilibrium.$$

$$Input - output = \frac{dm_B}{dt} = \frac{d(m_B - m_{Bs})}{dt} = \frac{dM_B}{dt},$$

$$or\ M_B = \int (input - output)\,dt.$$

$$Interest\ rate\ change \propto \int (supply\ rate - demand\ rate)\,dt,\ or$$

$$R = -K_m \int (supply\ rate - demand\ rate)\,dt,$$

where K_m is a constant for proportionality. The sign (negative) reflects when $(supply\ rate - demand\ rate)$ is positive, R is negative, i.e., interest rate decreases. Above expression can also be written as:

$$\int (supply\ rate - demand\ rate)\,dt = -\frac{R}{K_m},\ or$$

$$\int (w_{Bi} - w_{B0})\,dt = -\frac{R}{K_m}, \tag{88}$$

$$w_{Bi} = supply\ rate,$$

$$w_{B0} = demand\ rate,$$

$$K_m = dimensional\ constant.$$

When $t = 0$, $supply\ rate = demand\ rate$, and eq. (88) **changes into the following expression:**

$$\int (w_{Bis} - w_{B0s})\,dt = 0. \tag{89}$$

Subscript s denotes a value in a steady state equilibrium and $R = 0$ in steady state. Subtracting eq. (89) from (88), we obtain:

$$\int (w_{Bi} - w_{Bis})\,dt - \int (w_{B0} - w_{B0s})\,dt = -\frac{R}{K_m},\ or$$

$$\int (W_{Bi} - W_{B0})\,dt = -\frac{R}{K_m}, \tag{90}$$

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$$\text{where } w_{Bi} - w_{Bis} = W_{Bi} = \textit{change in supply rate,}$$
$$w_{B0} - w_{B0s} = W_{B0} = \textit{change in demand rate.}$$

R, W_{Bi} and W_{B0} are deviation variables, i.e., deviation from steady state equilibrium and have zero initial values. Eq. (90) can also be expressed as:

$$R = -K_m \int W_B dt = -K_m M_B, \tag{91}$$

where $W_B = W_{Bi} - W_{B0}$. **If R receives an input other than an inventory change, that can get added to** eq. (91) **as shown below:**

$$R = -K_m \int W_B dt + E_B = -K_m M_B + E_B. \tag{91a}$$

Funds inventory can also undergo an exogenous shock in addition to feedback effect of interest rate.

6.2   Solution of the Model with an Expansionary Savings Policy

Expressions from eqs. $(11a), (18), (23)$ and (29) respectively are as follows:

$$\frac{dR(t)}{dt} = -K_m W_B(t),$$
$$W_d(t) = -K_d R(t),$$
$$W_{mp} = -K_{sp}\left(C_p - R\right),$$
$$W_{mc} = -K_{sc}\left(C_c - R\right),$$

and

$$W_B(t) = W_m(t) - W_d(t),$$

in the absence of an exogenous demand/supply shock. $W_m(t)$ is the total funds supply including the supply from the public sector, households and firms, etc., and can be expressed as a sum of two sources of funds supply, i.e., the public sector, and the private sector (households, firms, etc.) as follows:

$$W_m(t) = -K_{sp}\left[C_p(t) - R(t)\right] - K_{sc}\left[C_c(t) - R(t)\right]. \tag{92}$$

Subscripts p and c denote private and public sector respectively. After combining above expressions, the following differential equation results:

$$\begin{aligned}
\frac{dR(t)}{dt} &= -K_m \left[W_m(t) - W_d(t)\right] \\
&= -K_m \left[-K_{sp}\left\{C_p(t) - R(t)\right\} - K_{sc}\left\{C_c(t) - R(t)\right\} + K_d R(t)\right] \\
&= -K_m \left[-K_{sp}C_p(t) - K_{sc}C_c(t) + (K_{sp} + K_{sc} + K_d)R(t)\right].
\end{aligned}$$

Rearranging the above expression gives:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) = K_m\left[K_{sp}C_p(t) + K_{sc}C_c(t)\right]. \tag{93}$$

Routh-Hurwitz stability criterion, **to ensure stability of above differential equation** requires $K_m(K_{sp} + K_{sc} + K_d) > 0$; and as K_m, K_{sp}, K_{sc}, and K_d are all positive, **the stability criterion holds**. This ensures that, away from a given initial equilibrium, every adjustment mechanism will lead to another equilibrium.

In the savings market in comparison with a goods market, the public sector has the role of the government as well as that of a producer of funds. When government reduces the interest rate on funds for investors but not for depositors, the public sector's supply curve shifts downward, and it incurs a cost in the role of the government. We need to keep the role of the public sector as a producer separate from its role as government for clarity of what is exactly happening in the savings/funds market when the government exercises a policy. Suppose the change in the cost of public sector as a result of reducing the interest rate on funds for investors is B, whereas the cost of private sector's supply of funds remains the same, the above equation can be written as:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) = -K_m K_{sc} B. \tag{94}$$

The solution is given by the following expression:

$$R(t) = -\frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}. \tag{95}$$

$R(0) = 0$ (the initial condition), and $R(\infty) = -\frac{K_{sc}B}{(K_{sp}+K_{sc}+K_d)}$ (**final value**). In response to a policy, the interest rate dynamics depends on the parameters K_{sp}, K_{sc}, K_d, K_m and B.

6.3 A Dynamic Optimal Expansionary Savings Policy-Panel B

Depending upon whether the savings policy is expansionary or contractionary, there could either be efficiency gains or losses in the **equilibrium (post policy) as compared to the one which existed before policy.** However, these are not the only efficiency gains or losses, **as some efficiency is lost during transient/adjustment to the final equilibrium**. **When savings policy is implemented, savings supply** either expands or contracts, **whereas at the initial interest rate, demand is still is the same as before, hence as a result of the policy, savings market gets out of equilibrium. Interest rate starts adjusting to equalize supply and demand to lead to new savings market equilibrium. Final equilibrium interest rate is a function of elasticities of supply and demand. As illustrated in the previous section, change in total supply as a result of a savings policy is as given below:**

$$W_m(0) = -K_{sp}\left[C_p(0) - R(0)\right] - K_{sc}\left[C_c(0) - R(0)\right] = K_{sc}B, \tag{96}$$
$$\text{as } R(0) = 0.$$

Due to savings policy, supply of savings either goes up or down by $K_{sc}B$. As demand **is the same as before**, therefore, the inventory of savings also changes by $K_{sc}B$, in the direction of the change

in supply. **Savings market is out of equilibrium with market forces in action pushing savings market to post policy final equilibrium through an interest rate adjustment**. As interest rate changes, demand and supply of funds also change through feedback. If the savings inventory goes up, **it reflects supply is greater than demand** and vice versa. **If supply and demand are equal with savings market in equilibrium, there is no efficiency loss. For an out of equilibrium market,** either supply and/or consumption of savings/funds is getting wasted at that point in time implying that total efficiency loss during market adjustment is sum of gaps in supply and demand at all points in time. After accounting for **a loss in efficiency** in post-policy equilibrium **in comparison with initial quilibrium** (if any), total efficiency loss for an expansionary and contractionary savings policy respectively can be expressed as:

$$EL \text{ (expansionary)} = \int_{-\infty}^{0} W_m(\infty)dt + \int_{0}^{\infty} [W_m(t) - W_d(t)] \, dt$$

$$= \int_{-\infty}^{0} W_m(\infty)dt + M_B(t). \tag{97}$$

$$EL \text{ (contractionary)} = \left| \int_{t}^{\infty} W_m(\infty)dt + \int_{0}^{t} W_d(t)dt + M_B(t) \right|$$

$$= \left| \int_{0}^{t} W_d(t)dt + \int_{t}^{\infty} W_d(\infty)dt + M_B(t) \right|$$

$$= \left| \int_{0}^{\infty} W_d(t)dt + \int_{0}^{\infty} [W_m(t) - W_d(t)] \, dt \right|$$

$$= \left| \int_{0}^{\infty} W_m(t)dt \right|. \tag{98}$$

Eq. $(91a)$ states the following:

$$R(t) = -K_m M_B(t) + E_B.$$

Through putting initial conditions, value of E_B **can be determined** (for expansionary savings policy) as follows:

$$R(0) = -K_m M_B(0) + E_B,$$
$$0 = -K_m K_{sc} B + E_B,$$
$$E_B = K_m K_{sc} B.$$

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Putting above expression in eq. $(91a)$, **the following equation results:**

$$R(t) = -K_m M_B(t) + K_m K_{sc} B, \text{ or}$$

$$M_B(t) = -\frac{1}{K_m} [R(t) - K_m K_{sc} B].$$

With Expansionary Savings Policy Cost Constraint:
According to eq. (30), change **in funds supply** due to **a market interest rate change is as given below:**

$$W_m(t) = -K_{sp} [C_p(t) - R(t)] - K_{sc} [C_c(t) - R(t)].$$

The part of supply coming from public sector is $-K_{sc} [C_c(t) - R(t)]$, and may be written as follows:

$$W_{mc}(t) = -K_{sc} [C_c(t) - R(t)],$$
$$w_{nmc}(t) - w_{imc}(0) = -K_{sc} [C_c(t) - R(t)],$$

where $w_{imc}(0)$ is the initial funds supply from the public sector, and $w_{nmc}(t)$ is **the supply after savings policy.** $W_{mc}(t) = w_{nmc}(t) - w_{imc}(0)$, which reflects **deviation from initial value. Cost to exercise savings policy is as given below:**

$$SPC = B [w_{imc}(0) + K_{sc} \{B + R(t)\}]. \tag{99}$$

For an optimal policy, efficiency loss needs to be minimized subject to the cost constraint as follows:

$$\min_B EL \quad \text{s.t.} \quad SPC \le G_B.$$

G_B is **savings policy cost to the government. Control variable is savings policy** B, and constraint is binding at $t = 0$. **We can express Lagrangian for above problem as follows:**

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$$\mathcal{L} = \int\limits_{-\infty}^{0} W_m(\infty)dt + M_B(t) + \lambda\left[G_B - B\left[w_{imc}(0) + K_{sc}\{B + R(t)\}\right]\right]$$

$$= \int\limits_{-\infty}^{0} \left[K_{sc}B - \frac{K_{sc}(K_{sp} + K_{sc})B}{(K_{sp} + K_{sc} + K_d)}\right] dt$$

$$- \frac{1}{K_m}\left[-\frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_mK_{sc}B\right]$$

$$+ \lambda\left[G_B - B\left[w_{imc}(0) + K_{sc}\left\{B - \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right]\right]$$

$$= \int\limits_{-\infty}^{0} \frac{K_{sc}K_dB}{(K_{sp} + K_{sc} + K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_mK_{sc}B\right]$$

$$+ \lambda\left[G_B - B\left[w_{imc}(0) + K_{sc}\left\{B - \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right]\right].$$

First order derivative with respect to B leads to the following:

$$\int\limits_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp} + K_{sc} + K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_mK_{sc}\right]$$

$$- \lambda\left[w_{imc}(0) + K_{sc}\left\{B - \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right]$$

$$- \lambda BK_{sc}\left[1 - \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right] = 0.$$

Rearranging this, we get:

$$\int\limits_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp} + K_{sc} + K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_mK_{sc}\right]$$

$$- 2\lambda BK_{sc}\left[1 - \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]$$

$$= \lambda w_{imc}(0),$$

or

$$B = -\frac{\lambda w_{imc}(0) - \left[\int_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp}+K_{sc}+K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}\right]\right]}{2\lambda K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]}.$$

(100)

First order derivative with respect to λ is shown below:

$$G_B - B\left[w_{imc}(0) + K_{sc}\left\{B - \frac{K_{sc}B}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}B}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}\right] = 0.$$

(101)

Putting eq. (100) into (101), we get:

$$G_B =$$

$$-w_{imc}(0).\frac{\lambda w_{imc}(0) - \left[\int_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp}+K_{sc}+K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}\right]\right]}{2\lambda K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]}$$

$$+K_{sc}\left\{1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right\}$$

$$*\left[-\frac{\lambda w_{imc}(0) - \left[\int_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp}+K_{sc}+K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}\right]\right]}{2\lambda K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right]}\right]^2,$$

$$\text{or } 4\lambda^2 Q_B G_B = -2\lambda^2 w_{imc}^2(0) + 2\lambda w_{imc}(0)J_B + \lambda^2 w_{imc}^2(0) + J_B^2 - 2\lambda w_{imc}(0)J_B,$$

$$\text{where } Q_B = K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}\right],$$

$$J_B = \int_{-\infty}^{0} \frac{K_{sc}K_d}{(K_{sp}+K_{sc}+K_d)}dt - \frac{1}{K_m}\left[-\frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} - K_m K_{sc}\right].$$

or

$$\left\{ w_{imc}^2(0) + 4Q_B G_B \right\} \lambda^2 - J_B^2 = 0.$$

$$\lambda = \frac{J_B}{\sqrt{w_{imc}^2(0) + 4Q_B G_B}}.$$

λ **is positive, as when** G_B **increases, minimum efficiency loss also increases**. From eq. (100):

$$B = -\frac{\lambda w_{imc}(0) - J_B}{2\lambda Q_B}. \tag{102}$$

Substituting λ **in above equation, following expression for** B **results:**

$$B = -\frac{\frac{w_{imc}(0) J_B}{\sqrt{w_{imc}^2(0) + 4Q_B G_B}} - J_B}{\frac{2Q_B J_B}{\sqrt{w_{imc}^2(0) + 4Q_B G_B}}},$$

$$B = -\frac{w_{imc}(0) - \sqrt{w_{imc}^2(0) + 4Q_B G_B}}{2Q_B}. \tag{103}$$

For checking second order minimization condition, Lagrangian can be expressed as given below:

$$\mathcal{L} = J_B B + \lambda \left[G_B - B \left(w_{ime}(0) + Q_B B \right) \right].$$

Expression for Bordered Hessian matrix of the Lagrange function is as given below:

$$BH = \begin{bmatrix} 0 & w_{imc}(0) + 2Q_B B \\ w_{imc}(0) + 2Q_B B & \frac{-2Q_B J_B}{\sqrt{w_{imc}^2(0) + 4Q_B G_B}} \end{bmatrix}.$$

As $-\left(w_{imc}(0) + 2Q_B B \right)^2 < 0$, **i.e., the determinant of above matrix, efficiency loss is minimized.**

6.4 Solution of the Model with a Contractionary Savings Policy

Expressions from eqs. $(11a), (18), (23)$ and (29) respectively are as follows:

$$\frac{dR(t)}{dt} = -K_m W_B(t),$$
$$W_d(t) = -K_d R(t),$$
$$W_{mp} = -K_{sp} \left(C_p - R \right),$$
$$W_{mc} = -K_{sc} \left(C_c - R \right),$$

and

$$W_B(t) = W_m(t) - W_d(t),$$

in the absence of an exogenous demand/supply shock. $W_m(t)$ is the total funds supply including the supply from public sector, households and firms, etc., and can be expressed as a sum of two sources of supply, i.e., public sector; and private sector (households, firms, etc.) as follows:

$$W_m(t) = -K_{sp}\left[C_p(t) - R(t)\right] - K_{sc}\left[C_c(t) - R(t)\right], \tag{104}$$

Subscripts p and c denote private and public sector respectively. After combining above expressions, the following differential equation results:

$$
\begin{aligned}
\frac{dR(t)}{dt} &= -K_m\left[W_m(t) - W_d(t)\right] \\
&= -K_m\left[-K_{sp}\left\{C_p(t) - R(t)\right\} - K_{sc}\left\{C_c(t) - R(t)\right\} + K_d R(t)\right] \\
&= -K_m\left[-K_{sp}C_p(t) - K_{sc}C_c(t) + (K_{sp} + K_{sc} + K_d)R(t)\right].
\end{aligned}
$$

Rearranging the above expression gives:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) = K_m\left[K_{sp}C_p(t) + K_{sc}C_c(t)\right]. \tag{105}$$

When government increases interest rate (market interest rate which the investor pays) on savings, the public sector's supply curve shifts to left, and it gets some revenue in the role of government. Suppose a change in the cost of public sector as a supplier as a result of increasing interest rate on savings is B, whereas the cost of private supply of funds remains the same, the above equation can be written as:

$$\frac{dR(t)}{dt} + K_m(K_{sp} + K_{sc} + K_d)R(t) = K_m K_{sc} B. \tag{106}$$

The solution is given by the following expression:

$$R(t) = \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} - \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)}e^{-[K_m(K_{sp}+K_{sc}+K_d)]t}. \tag{107}$$

$R(0) = 0$ (initial condition), and $R(\infty) = \frac{K_{sc}B}{(K_{sp}+K_{sc}+K_d)}$ (**final value**).

6.5 A Dynamic Optimal Contractionary Savings Policy-Panel B

Eq. $(11a)$ states the following:

$$R(t) = -K_m M_B + E_B.$$

Value of E_B (for contractionary savings policy) can be determined through imposing the initial conditions as follows:

$$R(0) = -K_m M_B(0) + E_B,$$

$$0 = K_m K_{sc} B + E_B,$$

$$E_B = -K_m K_{sc} B.$$

Putting above expression in eq. $(11a)$, **the following equation results:**

$$R(t) = -K_m M_B(t) - K_m K_{sc} B, \text{ or}$$

$$M_B(t) = -\frac{1}{K_m} [R(t) + K_m K_{sc} B]. \tag{108}$$

With Contractionary Savings Policy Revenue Constraint:
According to eq. (30), change **in funds supply** due to **a market interest rate change is as given below:**

$$W_m(t) = -K_{sp} [C_p(t) - R(t)] - K_{sc} [C_c(t) - R(t)].$$

The part of the supply coming from the public sector is $-K_{sc} [C_c(t) - R(t)]$, and can be written as follows:

$$W_{mc}(t) = -K_{sc} [C_c(t) - R(t)],$$

$$w_{nmc}(t) - w_{imc}(0) = -K_{sc} [C_c(t) - R(t)],$$

where $w_{imc}(0)$ is the initial funds supply from the public sector, and $w_{nmc}(t)$ is **the supply after savings policy.** $W_{mc}(t) = w_{nmc}(t) - w_{imc}(0)$, which reflects **deviation from initial value.** Revenue received by government as a result of a contractionary savings policy can be expressed as:

$$SPR = B [w_{imc}(0) - K_{sc} \{B - R(t)\}]. \tag{109}$$

For an optimal policy, efficiency loss needs to be minimized subject to the constraint as follows:

$$\min_T EL \quad \text{s.t.} \quad SPR \geq G_{BR}.$$

G_{BR} is the government revenue as a result of exercising the savings policy. **Control variable is** savings policy B, and constraint is binding at $t = 0$. **We can express Lagrangian for above problem as follows:**

$$\mathcal{L} = -\int_0^\infty [K_{sp} \{C_p(t) - R(t)\} + K_{sc} \{C_c(t) - R(t)\}] \, dt + \lambda [G_{BR} - B [w_{imc}(0) - K_{sc} \{B - R(t)\}]]$$

$$= \int_0^\infty \left[-K_{sc} B + \frac{K_{sc}(K_{sp} + K_{sc})B}{(K_{sp} + K_{sc} + K_d)} - \frac{K_{sc}(K_{sp} + K_{sc})B}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp} + K_{sc} + K_d)]t} \right] dt$$

$$+ \lambda \left[G_{BR} - B \left[w_{imc}(0) - K_{sc} \left\{ B - \frac{K_{sc} B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc} B}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp} + K_{sc} + K_d)]t} \right\} \right] \right].$$

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Taking first order condition with respect to B, the following expression is obtained:

$$\int_0^\infty \left[-K_{sc} + \frac{K_{sc}(K_{sp} + K_{sc})}{(K_{sp} + K_{sc} + K_d)} - \frac{K_{sc}(K_{sp} + K_{sc})}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right] dt$$

$$- \lambda \left[w_{imc}(0) - K_{sc} \left\{ B - \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right\} \right]$$

$$+ \lambda B K_{sc} \left[1 - \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right] = 0.$$

Rearranging this, we get:

$$\int_0^\infty \left[-K_{sc} + \frac{K_{sc}(K_{sp} + K_{sc})}{(K_{sp} + K_{sc} + K_d)} - \frac{K_{sc}(K_{sp} + K_{sc})}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right] dt$$

$$+ 2\lambda B K_{sc} \left[1 - \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right]$$

$$= \lambda w_{imc}(0),$$

or

$$B = \frac{\lambda w_{imc}(0) - \int_0^\infty \left[-K_{sc} + \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} - \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right] dt}{2\lambda K_{sc} \left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right]}. \tag{110}$$

First order derivative with respect to λ is as given below:

$$G_{BR} - B \left[w_{imc}(0) - K_{sc} \left\{ B - \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} + \frac{K_{sc}B}{(K_{sp} + K_{sc} + K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right\} \right] = 0. \tag{111}$$

Putting eq. (110) into (111), we obtain:

$$G_{BR} = w_{imc}(0) \cdot \frac{\lambda w_{imc}(0) - \int\limits_0^\infty \left[-K_{sc} + \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} - \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right] dt}{2\lambda K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right]}$$

$$- K_{sc}\left\{ 1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right\}$$

$$* \left[\frac{\lambda w_{imc}(0) - \int\limits_0^\infty \left[-K_{sc} + \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} - \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right] dt}{2\lambda K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right]} \right]^2,$$

or $4\lambda^2 Q_B G_{BR} = 2\lambda^2 w_{imc}^2(0) - 2\lambda w_{imc}(0) J_B - \lambda^2 w_{imc}^2(0) - J_B^2 + 2\lambda w_{imc}(0) J_B$,

where $Q_B = K_{sc}\left[1 - \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} + \frac{K_{sc}}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right]$,

$$J_B = \int\limits_0^\infty \left[-K_{sc} + \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} - \frac{K_{sc}(K_{sp}+K_{sc})}{(K_{sp}+K_{sc}+K_d)} e^{-[K_m(K_{sp}+K_{sc}+K_d)]t} \right] dt.$$

or

$$\left\{ w_{imc}^2(0) - 4Q_B G_{BR} \right\} \lambda^2 - J_B^2 = 0.$$

$$\lambda = \frac{J_B}{\sqrt{w_{imc}^2(0) - 4Q_B G_{BR}}}.$$

λ **is positive, as when** G_{BR} **increases, minimum efficiency loss also increases**. From eq. (110):

$$B = \frac{\lambda w_{imc}(0) - J_B}{2\lambda Q_B}. \tag{112}$$

Substituting λ **in above equation, following expression for** B **results:**

$$B = \frac{\frac{w_{imc}(0) J_B}{\sqrt{w_{imc}^2(0) - 4Q_B G_{BR}}} - J_B}{\frac{2Q_B J_B}{\sqrt{w_{imc}^2(0) - 4Q_B G_{BR}}}},$$

$$B = \frac{w_{imc}(0) - \sqrt{w_{imc}^2(0) - 4Q_B G_{BR}}}{2Q_B}. \tag{113}$$

For checking second order minimization condition, Lagrangian can be expressed as given below:

$$\mathcal{L} = J_B B + \lambda \left[G_{BR} - B \left(w_{imc}(0) - Q_B B \right) \right].$$

Expression for Bordered Hessian matrix of the Lagrange function is as given below:

$$BH = \begin{bmatrix} 0 & w_{imc}(0) - 2Q_B B \\ w_{imc}(0) - 2Q_B B & \frac{2Q_B J_B}{\sqrt{w_{imc}^2(0) - 4Q_B G_B}} \end{bmatrix}.$$

As $-\left(w_{imc}(0) - 2Q_B B \right)^2 < 0$, **i.e., the determinant of above matrix, efficiency loss is minimized.**

6.6 A Dynamic Optimal Savings Policy-Panel A

Equilibrium before adoption of policy can be improved upon as a result of an appropriate savings policy. However, these are not the only efficiency gains or losses, as some efficiency is lost during transient/adjustment to the final equilibrium. Efficiency loss is minimized to derive an optimal policy. When savings policy is implemented, supply of savers either expands or contracts, whereas at the initial saving rate, demand is still is the same as before, hence as a result of the policy, savers market gets out of equilibrium. Saving rate starts adjusting to equalize supply and demand to lead to new savers market equilibrium. Final equilibrium saving rate is a function of elasticities of supply and demand. Social damage is an excessive/short number of savers, i.e., the difference between supply and demand. Total social damage in terms of number of savers is the sum of that during adjustment of the savers market and the one in initial equilibrium as given below:

$$SD = M_A(t) + \int_{-\infty}^{0} W_{pr}(\infty) dt. \tag{114}$$

From eq. (73), a change in number of savers due to implementation of savings policy is given below:

$$W_{pu}(t) = K_{pu} \left[A - C(t) \right],$$
$$\text{or } w_{npu}(t) - w_{ipu}(0) = K_{pu} \left[A - C(t) \right],$$

where $w_{ipu}(0)$ is number of savers in initial equilibrium, and $w_{npu}(t)$ is **the supply after savings policy.** $W_{pu}(t) = w_{npu}(t) - w_{ipu}(0)$, which reflects **deviation from initial value.** A change in quantity of savings per unit time is as given below:

$$IQS = \frac{dM_B}{dt} = A \left[w_{ipu}(0) + K_{pu} \left\{ A - C(t) \right\} \right]. \tag{115}$$

For an optimal policy, efficiency loss needs to be minimized subject to an increase in quantity of savings per unit time being greater than or equal to G_A $\Big($change in quantity of savings/funds per unit time = and can be expressed as follows:

$$\min_A SD \quad \text{s.t.} \quad IQS \geq G_A.$$

If we want to minimize the social damage subject to the constraint that an increase in quantity of savings per unit time is greater than or equal to G_A $\left(\text{change in quantity of savings/funds per unit time} = \frac{dM_B}{dt}\right)$ in a given time period, our problem is as follows:

$$\min_A SD \quad \text{s.t.} \quad IQS \geq G_A \left(= \frac{dM_B}{dt}\right).$$

Control variable is savings policy A (an initial upward jump in the saving rate chosen by government to shift the demand curve), and constraint is binding at $t = 0$. **We can express Lagrangian for above problem as follows:**

$$\mathcal{L} = M_A(t) + \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - C(t)\right\}\right]\right].$$

Expression from eq. (62a), is given below:

$$C(t) = -K_c M_A + E_A.$$

Through putting initial conditions, value of E_A **can be determined as follows:**

$$C(0) = -K_c M_A(0) + E_A,$$
$$A = -K_c K_{pr} C(0) + E_A,$$
$$E_A = A\left[1 + K_c K_{pr}\right].$$

This implies that

$$M_A(t) = -\frac{1}{K_c}\left[C(t) - A\left\{1 + K_c K_{pr}\right\}\right].$$

This changes the Lagrangian to the following expression:

$$\mathcal{L} = -\frac{1}{K_c}\left[C(t) - A\left\{1 + K_c K_{pr}\right\}\right] + \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - C(t)\right\}\right]\right]$$

$$= -\frac{1}{K_c}\left[\left\{\frac{K_{pu}A}{K_{pr} + K_{pu}} + \frac{K_{pr}A}{K_{pr} + K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\} - A\left\{1 + K_c K_{pr}\right\}\right]$$

$$+ \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left[w_{ipu}(0) + K_{pu}\left\{A - \frac{K_{pu}A}{K_{pr} + K_{pu}} - \frac{K_{pr}A}{K_{pr} + K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right]\right].$$

First order derivative with respect to A **leads to the following:**

$$-\frac{1}{K_c}\left[\left\{\frac{K_{pu}}{K_{pr}+K_{pu}}+\frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}-\left\{1+K_cK_{pr}\right\}\right]$$

$$-\lambda\left[w_{ipu}(0)+K_{pu}\left\{A-\frac{K_{pu}A}{K_{pr}+K_{pu}}-\frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right]$$

$$-\lambda AK_{pu}\left\{1-\frac{K_{pu}}{K_{pr}+K_{pu}}-\frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\},$$

which implies that

$$-\frac{1}{K_c}\left[\left\{\frac{K_{pu}}{K_{pr}+K_{pu}}+\frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}-\left\{1+K_cK_{pr}\right\}\right]$$

$$-2\lambda AK_{pu}\left\{1-\frac{K_{pu}}{K_{pr}+K_{pu}}-\frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}$$

$$=\lambda w_{ipu}(0).$$

or

$$A=\frac{\lambda w_{ipu}(0)-\frac{1}{K_c}\left[\left\{\frac{-K_{pu}}{K_{pr}+K_{pu}}-\frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}+\left\{1+K_cK_{pr}\right\}\right]}{-2\lambda K_{pu}\left\{1-\frac{K_{pu}}{K_{pr}+K_{pu}}-\frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}}. \tag{116}$$

Similarly, first order derivative with respect to λ is shown below:

$$G_A-A\left[w_{ipu}(0)+K_{pu}\left\{A-\frac{K_{pu}A}{K_{pr}+K_{pu}}-\frac{K_{pr}A}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}\right]=0. \tag{117}$$

Putting eq. (116) into (117), we obtain:

$$G_A=w_{ipu}(0).\frac{\lambda w_{ipu}(0)-\frac{1}{K_c}\left[\left\{\frac{-K_{pu}}{K_{pr}+K_{pu}}-\frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}+\left\{1+K_cK_{pr}\right\}\right]}{-2\lambda K_{pu}\left\{1-\frac{K_{pu}}{K_{pr}+K_{pu}}-\frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}}$$

$$+K_{pu}\left\{1-\frac{K_{pu}}{K_{pr}+K_{pu}}-\frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}$$

$$*\left[\frac{\lambda w_{ipu}(0)-\frac{1}{K_c}\left[\left\{\frac{-K_{pu}}{K_{pr}+K_{pu}}-\frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}+\left\{1+K_cK_{pr}\right\}\right]}{-2\lambda K_{pu}\left\{1-\frac{K_{pu}}{K_{pr}+K_{pu}}-\frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}}\right]^2.$$

or $4\lambda^2 Q_A G_A=2\lambda^2 w_{ipu}^2(0)-2\lambda w_{ipu}(0)J_A-\lambda^2 w_{ipu}^2(0)-J_A^2+2\lambda w_{ipu}(0)J_A,$

where $Q_A=-K_{pu}\left\{1-\frac{K_{pu}}{K_{pr}+K_{pu}}-\frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\},$

$$J_A=\frac{1}{K_c}\left[\left\{\frac{-K_{pu}}{K_{pr}+K_{pu}}-\frac{K_{pr}}{K_{pr}+K_{pu}}e^{-[K_c(K_{pr}+K_{pu})]t}\right\}+\left\{1+K_cK_{pr}\right\}\right].$$

This implies that

$$\{w_{ipu}^2(0) - 4Q_A G_A\}\lambda^2 - J_A^2 = 0,$$

$$\lambda = \frac{J_A}{\sqrt{w_{ipu}^2(0) - 4Q_A G_A}}.$$

λ **is positive, as when** G_A **increases, social damage also increases**.
Eq. (116) can also be written as

$$A = \frac{\lambda w_{ipu}(0) - J_A}{2\lambda Q_A}. \tag{118}$$

After substituting value of λ **in above equation, the following expression results:**

$$A = \frac{\frac{w_{ipu}(0)J_A}{\sqrt{w_{ipu}^2(0) - 4Q_A G_A}} - J_A}{\frac{2Q_A J_A}{\sqrt{w_{ipu}^2(0) - 4Q_A G_A}}},$$

$$A = \frac{w_{ipu}(0) - \sqrt{w_{ipu}^2(0) - 4Q_A G_A}}{2Q_A}. \tag{119}$$

For checking second order minimization condition, Lagrangian can be expressed as given below:

$$\mathcal{L} = J_A A + \int_{-\infty}^{0} W_{pr}(\infty)dt + \lambda\left[G_A - A\left[w_{ipu}(0) - Q_A A\right]\right].$$

Expression for Bordered Hessian matrix of the Lagrange function is as given below:

$$BH = \begin{bmatrix} 0 & w_{ipu}(0) - 2Q_A A \\ w_{ipu}(0) - 2Q_A A & \frac{2Q_A J_A}{\sqrt{w_{ipu}^2(0) - 4Q_A G_A}} \end{bmatrix}.$$

As $-(w_{ipu}(0) - 2Q_A A)^2 < 0$, **i.e., the determinant of above matrix, efficiency loss is minimized.**

7 General Solution of Model-Panel A

Input and output blocks of agents are joined together to form a dynamic savings model in figure 5. To solve differential equations, Laplace transform can be used as a convenient tool, as a result of which figure 5 has been transformed to figure 6. In figure 6 (part A), transfer function relating $C(s)$ to $W_1(s)$ has been evaluated as follows: Assuming $\epsilon(s) = 0$, for blocks in A, we have following equations relating outputs with their respective inputs:

$$C(s) = -\frac{K_c}{s}.W(s),$$

$$W_{pu}(s) = -K_{pu}e^{-s\tau_{d2}}C(s),$$

$$W_B(s) = W_1(s) - W_{pu}(s).$$

Solving above equations simultaneously, we obtain:

$$C(s) = -\frac{K_c}{s}\left[W_1(s) - W_{pu}(s)\right],$$

$$C(s) = -\frac{K_c}{s}\left[W_1(s) + K_{pu}e^{-s\tau_{d2}}C(s)\right],$$

$$C(s)\left[1 + \frac{K_cK_{pu}e^{-s\tau_{d2}}}{s}\right] = -\frac{K_c}{s}W_1(s),$$

$$\frac{C(s)}{W_1(s)} = \frac{-K_c}{s + K_cK_{pu}e^{-s\tau_{d2}}}.$$

Part A in figure 6 gets reduced to one block using above expression. Now shifting $W_0(s)$ in backward direction leads to figure 7, from which overall transfer function for $D(s)$ can be evaluated as follows:

$$C(s) = \frac{-K_c}{s + K_cK_{pu}e^{-s\tau_{d2}}}\left[D(s) + W_{pr}(s)\right],$$

$$\text{where } D(s) = W_i(s) - W_0(s),$$

$$W_{pr}(s) = K_{pr}e^{-s\tau_{d1}}C(s).$$

$C(s)$ can be solved in terms of $D(s)$ as follows:

$$C(s) = \frac{-K_c}{s + K_cK_{pu}e^{-s\tau_{d2}}}\left[D(s) + W_{pr}(s)\right],$$

$$C(s) = \frac{-K_c}{s + K_cK_{pu}e^{-s\tau_{d2}}}\left[D(s) + K_{pr}e^{-s\tau_{d1}}C(s)\right],$$

$$C(s)\left[1 + \frac{K_cK_{pr}e^{-s\tau_{d1}}}{s + K_cK_{pu}e^{-s\tau_{d2}}}\right] = \frac{-K_c}{s + K_cK_{pu}e^{-s\tau_{d2}}}D(s),$$

$$\frac{C(s)}{D(s)} = \frac{\frac{-K_c}{s+K_cK_{pu}e^{-s\tau_{d2}}}}{1 + \frac{K_cK_{pr}e^{-s\tau_{d1}}}{s+K_cK_{pu}e^{-s\tau_{d2}}}},$$

$$\frac{C(s)}{D(s)} = \frac{-K_c}{s + K_cK_{pu}e^{-s\tau_{d2}} + K_cK_{pr}e^{-s\tau_{d1}}}. \tag{120}$$

Saving rate depends on structural parameters, i.e., K_c, K_{pu}, K_{pr}, τ_{d1} and τ_{d2}, which are all positive. Solution and inversion of eq. (120) into time domain can provide useful results and conclusions. Following approximation can be made if inversion of eq. (120) is to be done by partial fractions:

$$e^{-\tau s} \approx 1 - \tau s. \tag{121}$$

Second better approximation is:

$$e^{-\tau s} \approx \frac{1 - (\tau/2)s}{1 + (\tau/2)s}. \tag{122}$$

A better approximation than both above is as follows:

$$e^{-\tau s} \approx \frac{1 - \tau s/2 + \tau^2 s^2/12}{1 + \tau s/2 + \tau^2 s^2/12}. \tag{123}$$

There is a trade off between approximations of eq. (122) and (123) in terms of simplicity and accuracy. If $D(t)$ gets a step input, i.e., A, an exogenous shift in either supply or demand (of number of savers), then after Laplace transform

$$D(s) = \frac{A}{s}.$$

Applying final value theorem of Laplace transform leads to the following expression:

$$C(\infty) = \frac{-A}{K_{pu} + K_{pr}}, \tag{124}$$

$$C(\infty) = C(t) \mid_{t=\infty}.$$

Eq. (122) is rearranged as follows:

$$e^{-\tau s} \approx \frac{2 - \tau s}{2 + \tau s}.$$

Plugging in above approximation in eq. (120) leads to the following expression:

$$\frac{C(s)}{D(s)} = \frac{-K_c}{s + K_c K_{pu}\left(\frac{2 - s\tau_{d2}}{2 + s\tau_{d2}}\right) + K_c K_{pr}\left(\frac{2 - s\tau_{d1}}{2 + s\tau_{d1}}\right)},$$

$$\frac{C(s)}{D(s)} = \frac{-K_c\left(2 + s\tau_{d1}\right)\left(2 + s\tau_{d2}\right)}{s\left(2 + s\tau_{d1}\right)\left(2 + s\tau_{d2}\right) + K_c K_{pu}\left(2 + s\tau_{d1}\right)\left(2 - s\tau_{d2}\right) + K_c K_{pr}\left(2 - s\tau_{d1}\right)\left(2 + s\tau_{d2}\right)},$$

$$= \frac{-K_c\left\{\tau_{d1}\tau_{d2}s^2 + 2\left(\tau_{d1} + \tau_{d2}\right)s + 4\right\}}{\left[\begin{array}{c}\tau_{d1}\tau_{d2}s^3 + 2\left(\tau_{d1} + \tau_{d2}\right)s^2 + 4s + K_c K_{pu}\left\{-\tau_{d1}\tau_{d2}s^2 + 2\left(\tau_{d1} - \tau_{d2}\right)s + 4\right\} \\ + K_c K_{pr}\left\{-\tau_{d1}\tau_{d2}s^2 + 2\left(\tau_{d2} - \tau_{d1}\right)s + 4\right\}\end{array}\right]},$$

$$= \frac{-K_c\left\{\tau_{d1}\tau_{d2}s^2 + 2\left(\tau_{d1} + \tau_{d2}\right)s + 4\right\}}{\left[\begin{array}{c}\tau_{d1}\tau_{d2}s^3 + \left[2\left(\tau_{d1} + \tau_{d2}\right) - K_c K_{pu}\tau_{d1}\tau_{d2} - K_c K_{pr}\tau_{d1}\tau_{d2}\right]s^2 + \\ \left[2K_c K_{pu}\left(\tau_{d1} - \tau_{d2}\right) + 2K_c K_{pr}\left(\tau_{d2} - \tau_{d1}\right) + 4\right]s + 4K_c K_{pu} + 4K_c K_{pr}\end{array}\right]}.$$

Denominator of above expression is as given below:

$$as^3 + bs^2 + cs + d,$$

where

$$a = \tau_{d1}\tau_{d2},$$
$$b = 2\left(\tau_{d1} + \tau_{d2}\right) - K_c\tau_{d1}\tau_{d2}(K_{pu} + K_{pr}),$$
$$c = 2\left[K_c\left(\tau_{d1} - \tau_{d2}\right)\left(K_{pu} - K_{pr}\right) + 2\right],$$
$$d = 4K_c(K_{pu} + K_{pr}).$$

This implies that

$$\frac{C(s)}{D(s)} = \frac{-K_c\left\{as^2 + 2\left(\tau_{d1} + \tau_{d2}\right)s + 4\right\}}{as^3 + bs^2 + cs + d}. \tag{125}$$

The roots of denominator (given below) of eq. (125) depict the qualitative response of saving rate.

$$as^3 + bs^2 + cs + d = 0. \tag{126}$$

τ_{d1} and τ_{d2} and have dimensions of time; dimensions of other parameters are given below:

$$Dimensions\ of\ K_c = (Dimensions\ of\ C)/(time \times Dimensions\ of\ W_A)$$
$$= \frac{Quantity\ of\ new\ savings}{time \times No.\ of\ new\ savers}$$
$$Dimensions\ of\ K_{pu} = (Dimensions\ of\ W_{pu})/(Dimensions\ of\ C)$$
$$= \frac{No.\ of\ new\ savers\ required\ by\ government}{Quantity\ of\ new\ savings},$$
$$Dimensions\ of\ K_{pr} = (Dimensions\ of\ W_{pr})/(Dimensions\ of\ C)$$
$$= \frac{No.\ of\ new\ savers\ due\ to\ private\ sector\ response}{Quantity\ of\ new\ savings}.$$

Therefore K_cK_{pu} and K_cK_{pr} have dimensions of $1/time$, which implies that a has dimensions of $time^2$; b has dimensions of $time$; c is dimensionless and d has dimensions of $1/time$. It can be seen that eq. (126) is dimensionally consistent (as s has dimensions of $1/time$).

Method to Solve eq. (125):

Suppose D gets a step input of magnitude A, then

$$D(s) = \frac{A}{s}. \tag{127}$$

Substituting above in eq. (125), gives the following:

$$C(s) = \frac{-AK_c\left\{as^2 + 2\left(\tau_{d1} + \tau_{d2}\right)s + 4\right\}}{s(as^3 + bs^2 + cs + d)}. \tag{128}$$

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For solution of above equation, structural parameters, i.e., K_c, K_{pr}, K_{pu}, τ_{d1} and τ_{d2} need to be empirically estimated, on the basis of which values of a, b, c and d are calculated. Find roots of eq. (126) to invert eq. (128) to time function of C through partial fractions and Laplace transform table. Applying Final Value Theorem of Laplace transform on eq. (128), following final value is obtained:

$$C(\infty) = -AK_c \times \frac{4}{d}. \tag{129}$$

Putting value of $d = 4K_c(K_{pu} + K_{pr})$, in above expression, we obtain:

$$C(\infty) = \frac{-A}{K_{pu} + K_{pr}}. \tag{130}$$

$C(\infty)$ from eq. (128) is the same as that from eq. (120). Similarly applying the Initial Value Theorem of Laplace transform on eq. (128), we obtain the following initial value:

$$C(0) = 0. \tag{131}$$

Qualitative nature of solution $C(t)$ depends on location of roots of the denominator of $C(s)$ in the complex plane. Several roots are located in figure 8, and Table 1 provides the form of the terms in the expression for $C(t)$ corresponding to those roots. $X1, X2, \ldots, Y1, Y2, \ldots$ are all positive.

A dynamically optimal policy minimizing social damage in terms of inadequate number of savers in pre-policy equilibrium, and also the social loss during adjustment of the savers market to final equilibrium involves a shift of either supply, demand or both curves subject to a certain increase in quantity of savings per unit time, and can be derived on a case by case basis. In equilibrium, the area under the demand curve is the social benefit in terms of quantity of savings per unit time. Eq. (83) presents the expression for the optimal supply of savers policy depending on parameters $w_{ipu}(0)$, G_A, K_c, K_{pr}, K_{pu}, τ_{d1} and τ_{d2}.

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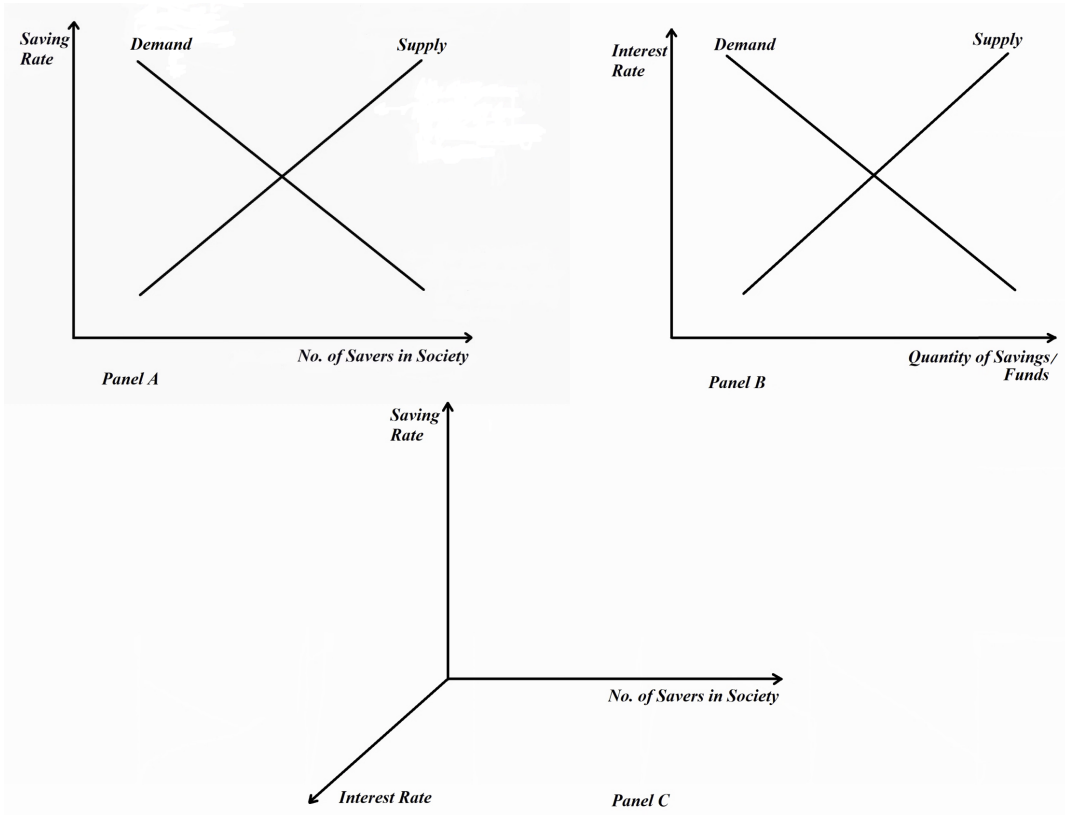
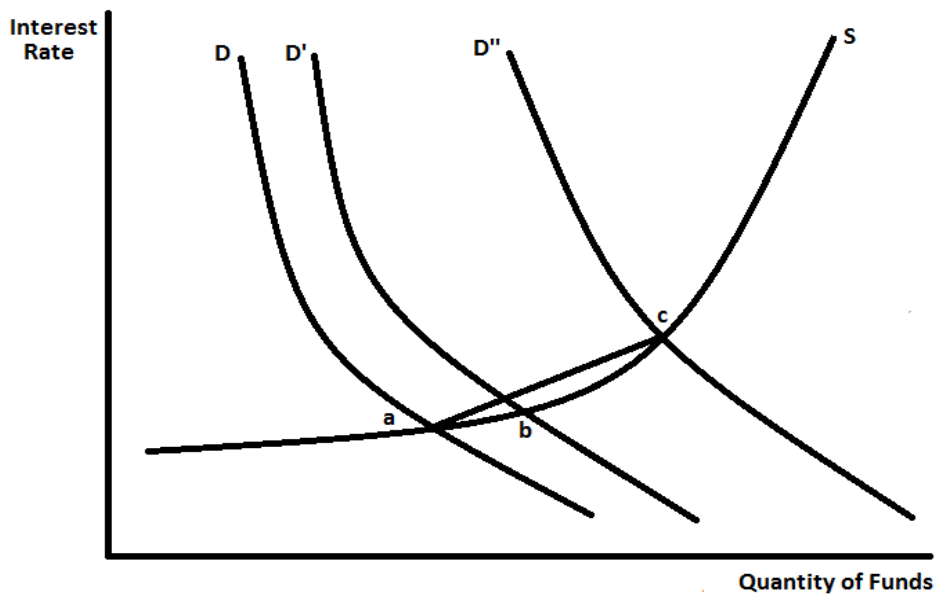Figure 1: Theoretical concept of savings model.

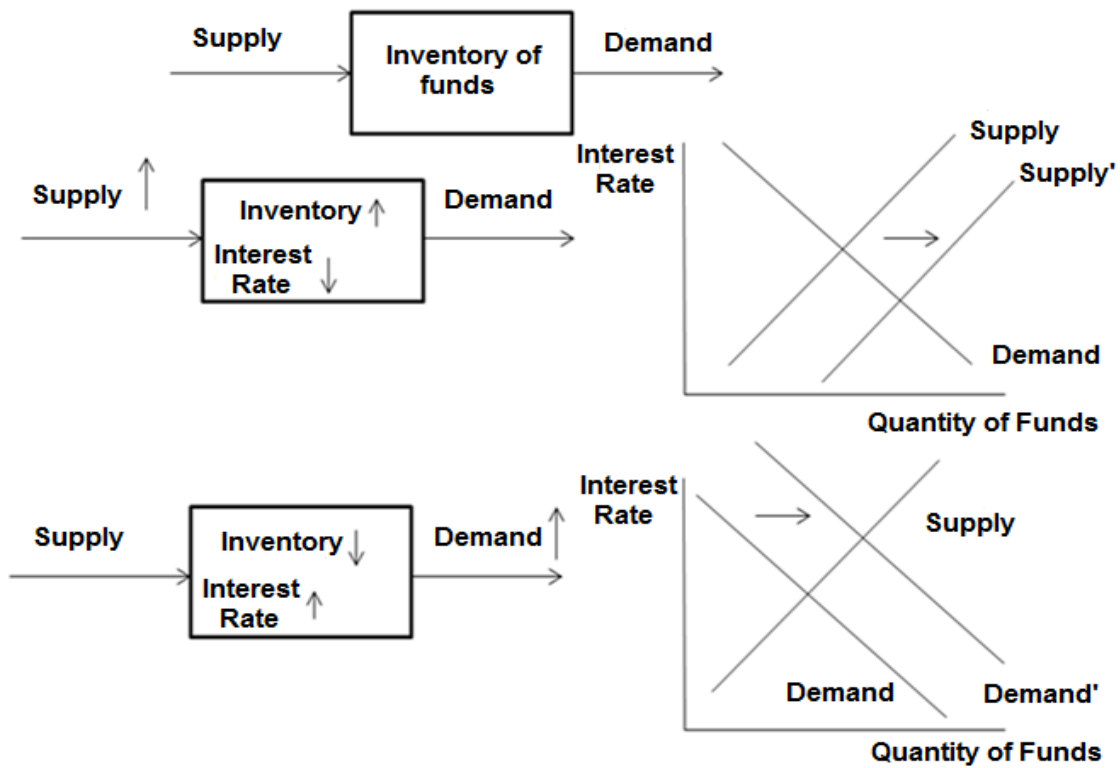Figure 2: When is linearity a reasonable assumption?

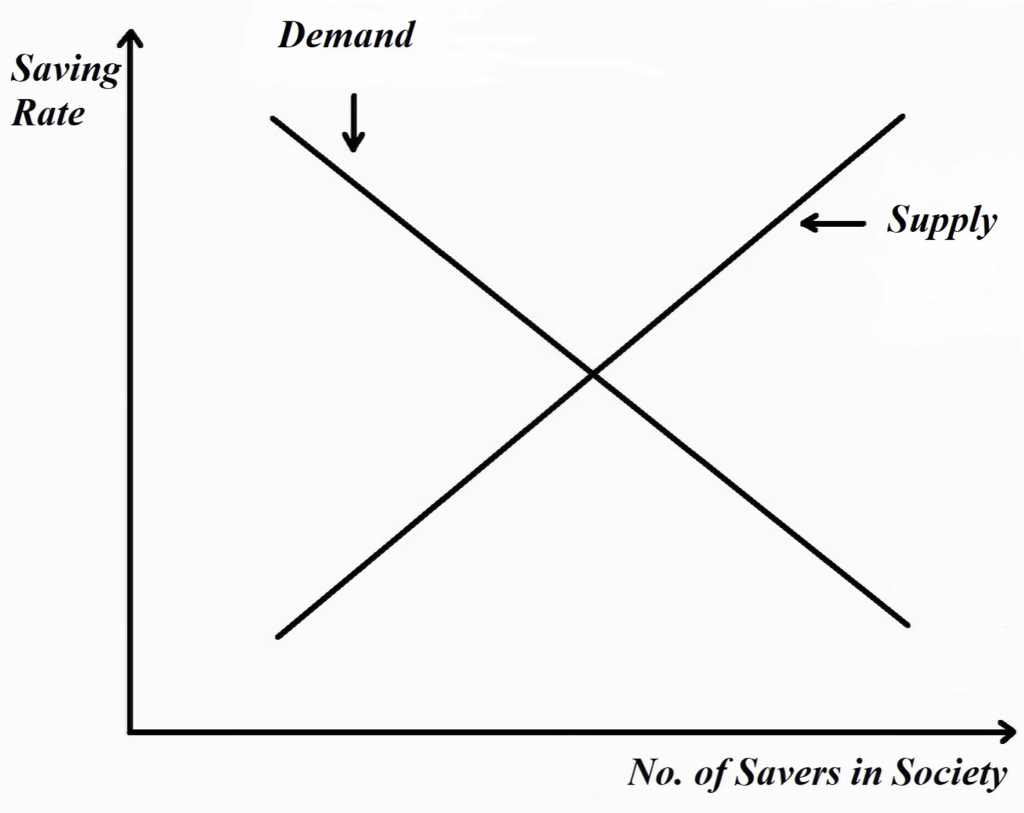Figure 3: Movement of interest rate with stock/inventory of funds.

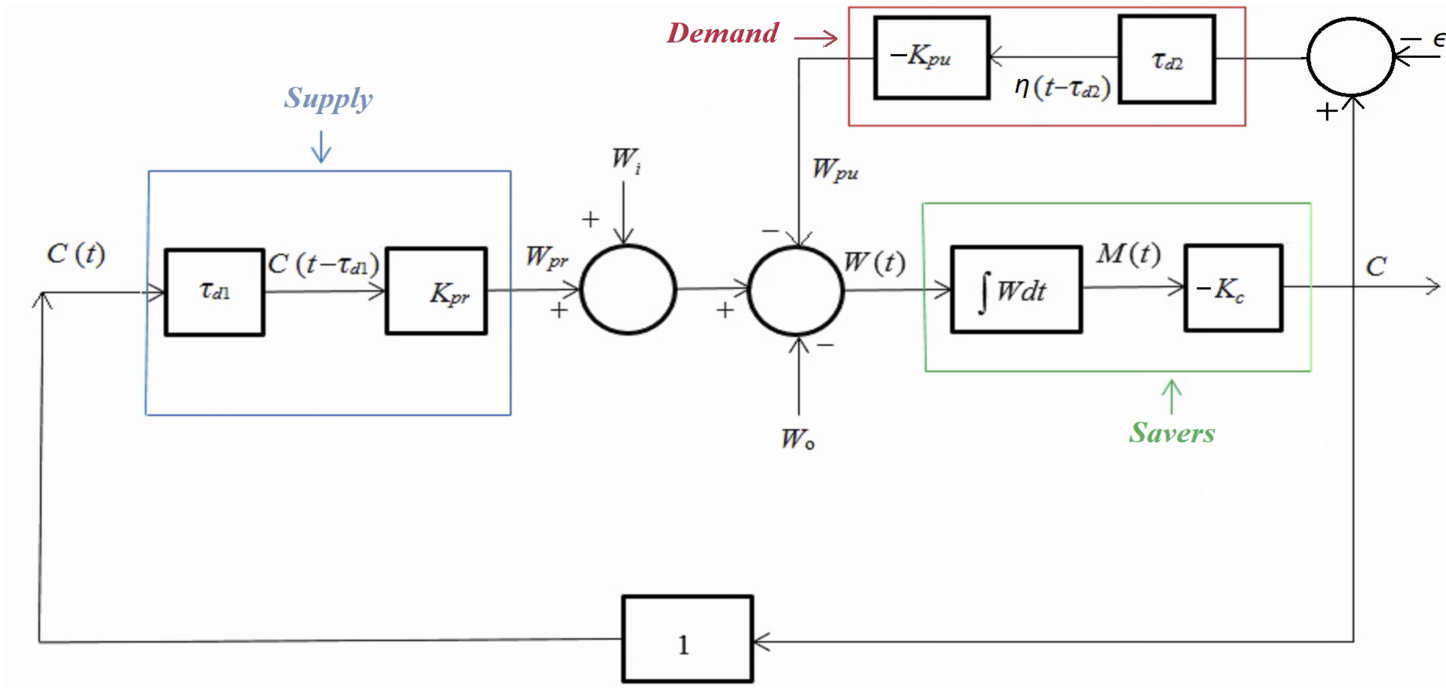Figure 4: Theoretical concept of savings model in panel A.

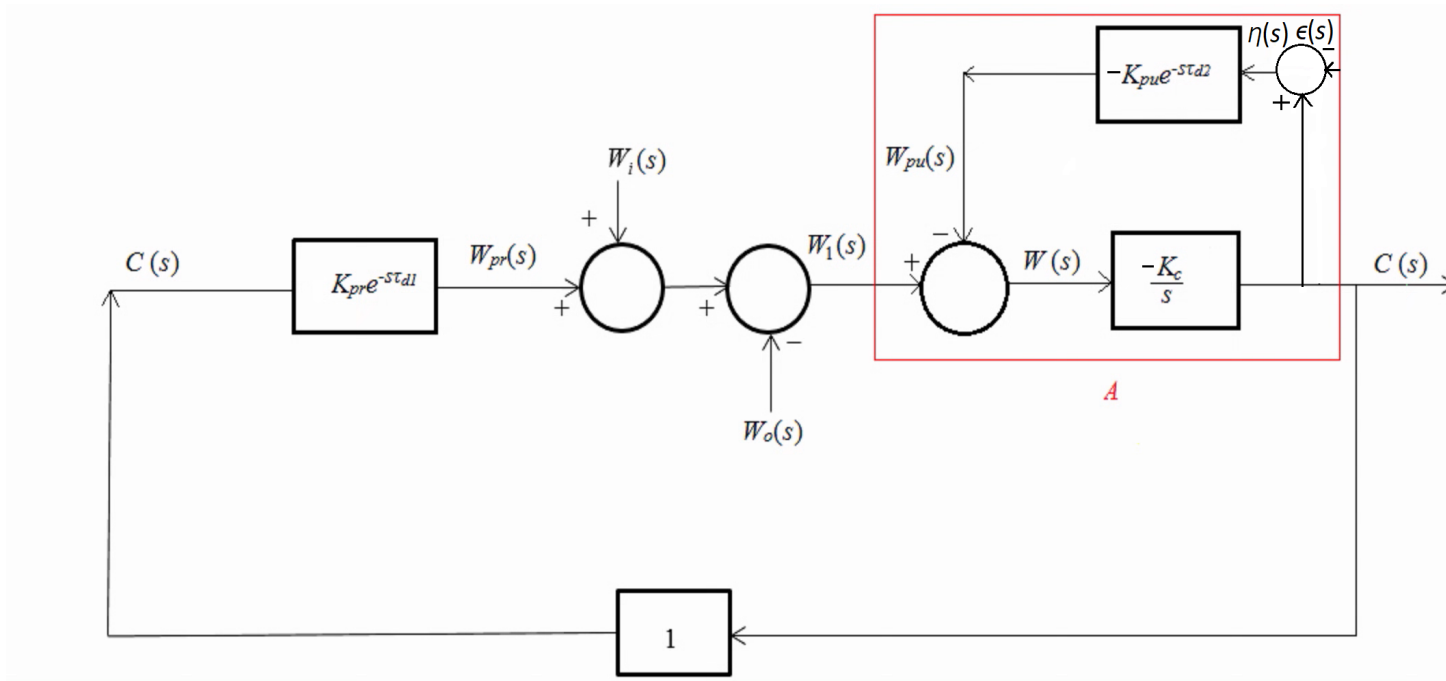Figure 5: A theoretical dynamic savings model in panel A.

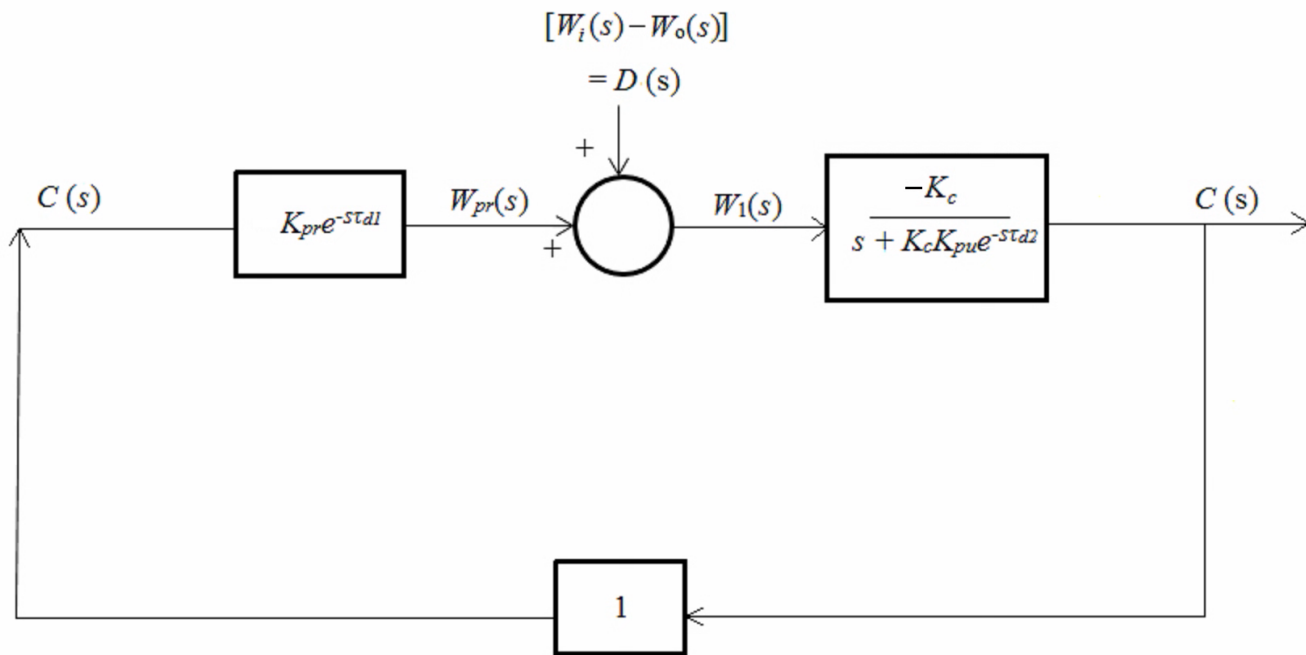Figure 6: Savings model in panel A after Laplace Transform.

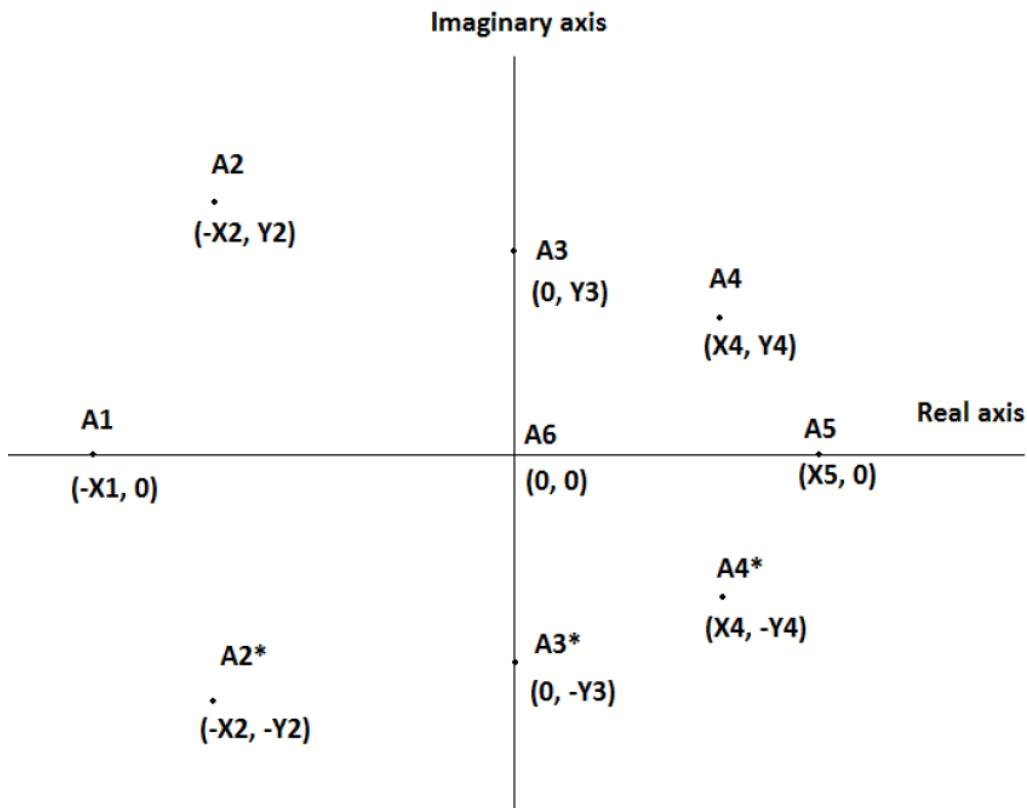Figure 7: Savings model after solution of block A in Figure 6.

Figure 8: Location of roots in a complex plane corresponding to table 1.

Table 1:

S. No	Roots	Terms in P(t) for $t > 0$	Description of Response
1	A1	$C_1 e^{-X_1 t}$	Bounded non-cyclic
2	A2, A2*	$e^{-X_2 t}(C_1 \cos Y_2 t + C_2 \sin Y_2 t)$	Bounded cyclic
3	A3, A3*	$C_1 \cos Y_3 t + C_2 \sin Y_3 t$	Cyclic (constant amplitude)
4	A4, A4*	$e^{X_4 t}(C_1 \cos Y_4 t + C_2 \sin Y_4 t)$	Unbounded cyclic
5	A5	$C_1 e^{X_5 t}$	Unbounded non-cyclic
6	A6	C_1	Constant