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Geometric fractional Brownian motion model for commodity market simulation

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Abstract The geometric Brownian motion (GBM) model is a mathematical model that has been used to model asset price paths. By incorporating Hurst parameter to GBM to characterize long-memory phenomenon, the geometric fractional Brownian motion (GFBM) model was introduced, which allows its disjoint increments to be correlated. This paper investigates the accuracy of GBM and GFBM in modelling Malaysia's crude palm oil price simulation, and to see display of persistent or anti-persistent behaviour across different periods. Results show that the GFBM model is more accurate than the GBM model in simulating future price path for the given data set.

1. Introduction

According to [5], financial assets can be divided into risk-free assets, such as government-issued bonds, and risky assets, such as stocks and commodities. The efficient market hypothesis (EMH) states that current price of a stock reflects the historical prices of the stocks [6]. Hence, the price of a stock is uncertain and unpredictable. This property of a stock price attracts studies on modeling of stock prices, such as the regression method [7,8]. Prediction models reflect the behaviour of the stock prices and conform to historical prices in order to generate future price paths that a stock may follow. Modeling and predicting stock price paths have been an important topic in financial studies. Prediction models are important for investors to guarantee minimum investment risk and to provide with an overlook of the price paths to make financial and investment decision. This is the motivation behind many studies that have developed and constructed prediction models.

Geometric Brownian motion (GBM) model is a stochastic process that assumes normally distributed and independent stock returns. The GBM model is known for its application in stock price modeling [4], and option pricing [1]. In the former application, many studies have modelled stock price paths using the GBM model, such as [11] simulates stock price paths for Australian companies and shows that the simulated prices are aligned with actual prices. While most of these studies model stock prices, others discovered that commodity prices also exhibit randomness which can be explained mathematically via the GBM model, such as oil, petroleum product

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and natural gas [9], rice and coffee [10], rubber [13], gold [53], and iron ores [41]. However, the major discrepancy in GBM modeling is that it assumes real stocks increments to be mutually independent which contradicts practical investigation of real data that mostly exhibit dependency known as self–similarity or long memory [2]. Although the GBM model is proven to be cheaper and an efficient method in time series modelling [12], this assumption cannot be relaxed. In order to characterize this behaviour, the geometric fractional Brownian motion (GFBM) was introduced by [2] by incorporating the Hurst exponent in the stock price dynamics to improve the accuracy of stock price modelling by providing information on the level of self-similarity in a given time series. It has been used prior to stock price modelling in other fields, such as hydrology [17,30] and communication technology [14–16]. Recently, the work of [44] associated the SIRS model with fractional Brownian motion to model epidemic diseases such as the Coronavirus.

The Hurst exponent indicates the intensity of long-range dependence in a time series, and it can be estimated heuristically using various methods as mentioned in [20], while [39] considered GFBM to develop a method for Hurst parameter estimation in high frequency financial time series data. Moreover, [37] using real data on a variety of stock prices to estimate Hurst parameter and obtained the option price driven by fractional Brownian motion. The most straightforward method used to estimate the Hurst exponent is the rescaled range (R/S) [42,19] analysis method because no assumptions are needed for the underlying process of the times series, and it uses simple statistics. Other methods are the Higuchi method [26], the periodogram method [28], and the variance method [27]. The work of [51,52] introduced the complete maximum likelihood estimation method to estimate the unknown parameters in GFBM for constant volatility case, while [51] extended this method for stochastic volatility case.

The GFBM model is more general than the GBM model, and it can explain more behaviours of stock price changes. Past studies have tried to analytically estimate the parameters of this model, however, the adaptation of such estimates proved to be mathematically and timely expensive [53,35,36], thus in this paper, readily available heuristic methods to estimate the Hurst exponent will be employed. The GFBM model is analyzed in the stock market [3,4,29], and many considers the GFBM model for option pricing [32–34]. In another instance, [38] used quadratic variation coupled with maximum likelihood approach to solve the problem with estimating unknown parameter of GFBM in pricing the option based on Chinese financial market performance. Given its tractability and similarity with the classical Brownian motion, where both are self-similar with similar Gaussian structure, the GFBM model is suitable to capture volatility persistence via long memory process to keep the pricing framework largely intact. Moreover, [21,22] overcame arbitrage opportunity in GFBM models, and [19,23] developed stochastic calculus for GFBM which enables wider applications of the GFBM model, such as in [2,24,25].

The fractional differential equations can be used to describe various fields of natural science than the differential equations with integral order [49,49]. There are many mathematical methods for solving nonlinear fractional differential equations, see [45]. Other methods that have been applied on nonlinear models can be found in [44,46,47].

In this study, the authors apply the GBM and GFBM to simulate and evaluate the crude palm oil price paths in Malaysia market. To the authors' best knowledge, no studies on the application of GFBM model can be found for Malaysian commodity market. The focus is on simulating and testing the model to assess the accuracy of the simulation with comparison to the actual prices by computing the mean absolute percentage error (MAPE) taken relative to the simulated prices, as well as to see display of persistent and anti-persistent behaviour of the given data set.

The organization of the study is as follows: Section 2 describes the GBM and GFBM models. Section 3 reviews some of the existing preliminary methods for estimating the Hurst exponent, the mean and volatility. Section 4 presents the numerical results and analysis, and Section 5 concludes the study.

2. Modeling commodity price by using GBM and GFBM

In this section, we give a brief description of the geometric Brownian motion (GBM) and geometric fractional Brownian motion (GFBM) models, and the price path dynamics driven by these two models. Some definitions used in this study can be found in [2,20,19], which are as follows:

Definition 1. A stochastic process $B(t)$ is a Brownian motion if it satisfies the following:

(1) For any $t > s, v > u$ and $u > t$, the increments $B(t) - B(s)$ and $B(v) - B(u)$ are independent.
(2) Each increment is a zero-mean Gaussian random variable such that for all $t > s, B(t) - B(s) \sim \mathcal{N}(0, t - s)$.
(3) $B(0) = 0$.

Definition 2. Let H be a constant belonging to $(0,1)$. A geometric fractional Brownian motion (GFBM), $(B_H(t))_{t \geqslant 0}$, with the Hurst index H is a continuous and centred Gaussian process with covariance function:

$$\mathbf{E}[B_H(t)B_H(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right), \tag{1}$$

where the following properties hold:

(i) $B_H(0) = 0$ and $\mathbf{E}[B_H(t)] = 0$ for all $t \geqslant 0$.
(ii) B_H has homogeneous increments, i.e., for $s, t \geqslant 0, B_H(t+s) - B_H(s)$ has the same law as that of $B_H(t)$.
(iii) B_H is a Gaussian process and $\mathbf{E}[B_H(t)]^2 = t^{2H}, t \geqslant 0$, for all $H \in (0,1)$.
(iv) B_H has continuous trajectories.

The following theorems present the asset price dynamics that follows GBM and GFBM [19, see].

Theorem 1. A stochastic process $S(t)$ follows a geometric Brownian motion (GBM) if the asset price follows the following dynamics:

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t), \tag{2}$$

where $B(t)$ is a Brownian motion, μ is the constant drift, and σ is the constant volatility. The solution to Eq. (2) for any arbitrary initial value $S(0)$ is given as follows:

$$S(t) = S(0)\exp\left(\mu t - \frac{1}{2}\sigma^2 t + \sigma B(t)\right). \tag{3}$$

Theorem 2. A stochastic process $S(t)$ follows a geometric fractional Brownian motion (GFBM) if the asset price follows the following dynamics:

$$dS(t) = \mu S(t)dt + \sigma S(t)dB_H(t), \tag{4}$$

where $B_H(t)$ is an FBM with $H \in (0,1)$, μ is the constant drift, and σ is the constant volatility. By using the Wick Itô Skorohod integrals for GFBM as in [19], the solution to Eq. (4) for any arbitrary initial value $S(0)$ is given as follows:

$$S(t) = S(0)\exp\left(\mu t - \frac{1}{2}\sigma^2 t^{2H} + \sigma B_H(t)\right). \tag{5}$$

3. Parameters estimation

In this section, we briefly describe the methods used in this study to estimate the parameters required to simulate the asset price paths under GBM and GFBM models.

Given a series of logarithmic returns $r_{\Delta t}(t_i)$ for $i, = 1,\ldots,N$ for the GBM model, we need to estimate the sample mean and volatility, respectively, as follows:

$$\mu = \frac{1}{N}\sum_{i=1}^{N} r_{\Delta t}(t_i),$$

and

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(r_{\Delta t}(t_i) - \mu)^2}.$$

On the other hand, for the GFBM model, we need to first estimate the Hurst parameter in order to obtain the sample mean and volatility. The value of the Hurst parameter may explain three situations [19, see]:

(i) If $H \in \left(0, \frac{1}{2}\right)$, this implies that the disjoint increments are positively correlated, and is known as exhibiting short memory dependency.
(ii) If $H = \frac{1}{2}$, this is the classical Brownian motion.
(iii) If $H \in \left(\frac{1}{2}, 1\right)$, this implies that the disjoint increments are negatively correlated, and is known as exhibiting long memory dependency.

In this study, the aggregated variance method, the absolute moment method, and the Higuchi method are implemented to estimate the Hurst parameter, H, which are briefly described here. For more details, one may refer to [20,19].

The aggregated variance method is based on the self-similarity property of a sample. Following [20], the logarithmic returns X are divided into $\frac{N}{m}$ blocks of size m, then calculates:

$$X^{(m)}(k) = \frac{1}{m}\sum_{i=(k-1)m+1}^{km} X(i)$$

for $k = 1,2,\ldots,N/m$, and its sample variance is obtained as such:

$$Var\left(X^{(m)}\right) = \frac{1}{N/m}\sum_{k=1}^{N/m}\left(X^{(m)}(k)\right)^2 - \left(\frac{1}{N/m}\sum_{k=1}^{N/m}X^{(m)}(k)\right)^2, \tag{6}$$

where a straight line with slope, $\beta = 2H - 2$, is formed. Meanwhile, following [19], the absolute moment method similarly divides the logarithmic returns X into $\frac{N}{m}$ blocks of size m, then calculates:

$$X^{(m)}(k) = \frac{1}{m}\sum_{i=(k-1)m}^{km} X(i)$$

for $i = 1,2,\ldots,N/m$, and

$$\widehat{X}_N = \frac{1}{N}\sum_{i=1}^{N} X(i).$$

Then it calculates the absolute moment of the series as such:

$$AM\left(X^{(m)}(k)\right) = \frac{1}{N/m}\sum_{k=1}^{N/m}\left|X^{(m)}(k) - \widehat{X}_N\right|. \tag{7}$$

On the other hand, the Higuchi method [26,20] takes the partial sums of the logarithmic returns X as such:

$$Y(n) = \sum_{i=1}^{n} X_i,$$

and find the normalized length of the curve as follows:

$$L(m) = \frac{N-1}{m^3}\sum_{i=1}^{m}\left[\frac{N-i}{m}\right]^{-1}\sum_{k=1}^{[(N-i)/m]}|Y(i+km) - Y(i+(k-1)m)|, \tag{8}$$

where N is the length of the logarithmic returns, m is a block size and $\left[\frac{N-i}{m}\right]$ is the greatest integer function. A straight line with slope, $D = 2 - H$, is formed, and the estimation of parameter H is obtained by plotting $L(m)$ versus m on a log–log scale.

Using the estimated Hurst parameter and adapting to [29], the sample volatility and mean of the returns under GFBM can be estimated, respectively, as follows:

$$\hat{\sigma} = \frac{\sigma}{\sqrt{|\Delta t|^{2H}}},$$

and

$$\hat{\mu} = \frac{\mu}{\Delta t} + \frac{\hat{\sigma}^2}{2},$$

where $\Delta t = t_2 - t_1 = \ldots = t_N - t_{N-1}$.

4. Numerical results

This section illustrates the modeling of the commodity prices using the GBM and GFBM models that are described in Section 2 and parameters estimation using methods presented in Section 3.

According to [20], the fractional Gaussian noise (FGN) is the increment process of a GFBM, such that:

$$X(t) = B_H(t+1) - B_H(t).$$

Table 1 Mean μ and volatility σ.

Period	GBM		GFBM	
	μ	σ	$\hat{\mu}$	$\hat{\sigma}$
3 years	0.0067	0.0606	0.0086	0.0606
4 years	−0.0010	0.0587	0.0007	0.0587
5 years	−0.0058	0.0558	−0.0043	0.0558

For the simulation of the fractional Gaussian noise (FGN), we choose an exact method, which is the Cholesky decomposition that decomposes the covariance matrix into a product of a lower triangular matrix and its conjugate-transpose $\Gamma(n) = L(n)L(n)^*$. If the covariance matrix is proven to be positive-definite, then $L(n)$ will have real entries and $\Gamma(n) = L(n)L(n)'$. A detailed description can be found in [18].

In 2014–2018, the economy of Malaysia was recovering from the world economic downturn and there was a fall in global demands of palm oil. The data that is used in this study is the monthly average price (MYR/tonne) for crude palm oil (local delivered) from January 2014 until December 2019, which was obtained from the official website of the Economics and Industry Development Division, Malaysian Palm Oil Board [31].

We firstly estimate the Hurst parameter using the methods described in the previous section, which then be used to estimate the standard deviation and mean of the crude palm oil prices under GBM and GFBM models from January 2014 to January 2017 (3 years), January 2014 to January 2018 (4 years), and January 2014 to January 2019 (5 years)[1]. Then, using Eqs. (3) and (5), we simulate three corresponding price paths with the estimated parameters and compare these with the actual price paths.

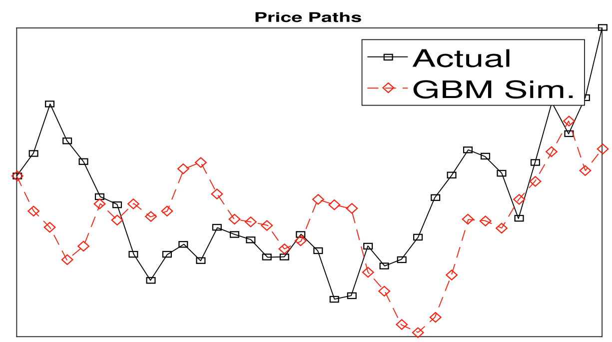
Given three periods with sample size, $N = 36, 48, 60$, respectively, the estimated values of the sample mean μ and volatility σ for the GBM dan GFBM models are documented in Table 1.

Fig. 1 plot the actual prices and simulated prices using GBM model for a period of 3, 4 and 5 years, respectively.

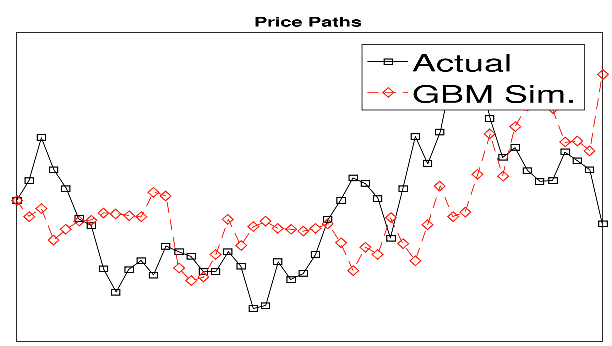
The estimated values of the Hurst parameter for the periods are given in Table 2. Anti-persistence in the series implies that the series is mean-reverting; hence future values have a tendency to return to a long-term mean. By the absolute moment analysis, the Hurst values are smaller than 0.5 for all three periods which indicate that the logarithmic returns are negatively correlated. For the aggregated variance and Higuchi analysis, the Hurst values are higher than 0.5 for the 4-year period which indicate positively correlated logarithmic returns, whereas the 3-year period and 5-year period show negatively correlated logarithmic returns for the estimated Hurst values.

Moreover, by the absolute moment method, the series of the logarithmic returns of the given data set displays slight anti-persistent behaviour for the 3-year period and 4-year period, where $H = 0.4043$ and $H = 0.3657$, respectively. The plots are as shown in Figs. 2(a) and 3(a). Meanwhile, strong anti-persistent behaviour is displayed for the 5-year period, where $H = 0.2343$, as depicted in Fig. 4(a).

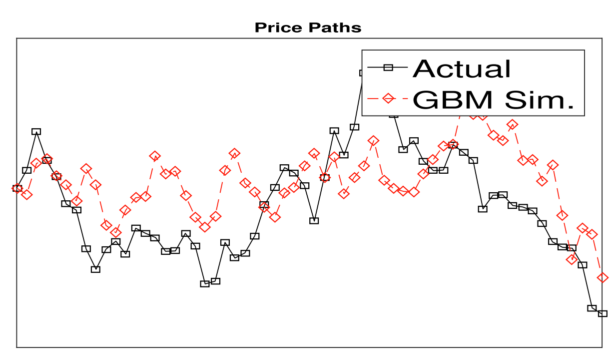
Following that, by using the aggregated variance method, the series shows slight anti-persistent behaviour for the 3-year period, where $H = 0.4286$. The plot can be seen in



(a) 3-year period



(b) 4-year period



(c) 5-year period

Fig. 1 Price path simulation using GBM.

[1] The estimations were implemented in MATLAB and conducted on an Intel(R) Core(TM) i5-4590T CPU @ 2.00 GHz machine running under Windows 10 with 8 GB RAM

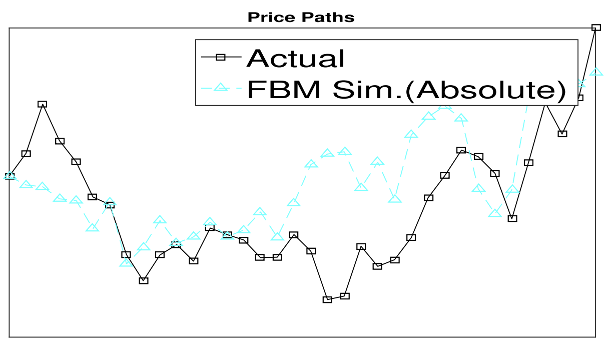
Table 2 Estimated Hurst values, H.

Period	Hurst parameter, H		
	Absolute Moment	Aggregated Variance	Higuchi
3 years	0.4043	0.4286	0.4574
4 years	0.3657	0.5203	0.5001
5 years	0.2343	0.4999	0.4643

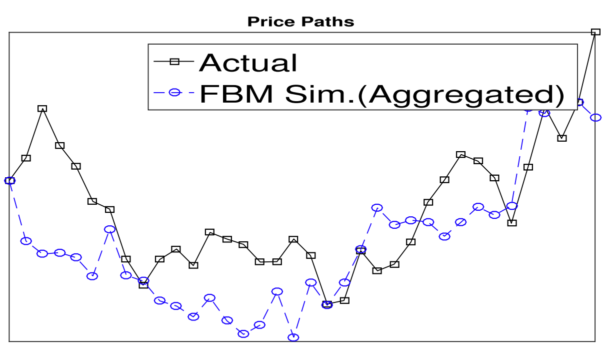
Fig. 2(b). As for the rest of the simulation, the Hurst parameters estimated using the aggregated variance method, are close to half for the 4-year period and 5-year period, as well as using the Higuchi method for all three periods. The plots are depicted in Figs. 3(b), 4(b), 2(c), 3(c) and 4(c).

To compare the accuracy of the GBM and GFBM models in terms of forecasting or simulating the price path of the monthly palm oil price, we compute the mean-average percentage error (MAPE). The results are provided in Table 4.

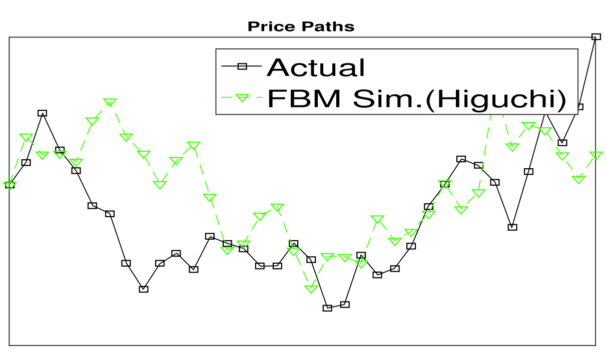
From Table 4, the MAPE value obtained via the GBM model is 10.6587, on average. As for the GFBM models, the MAPE values are obtained separately correspond to the Hurst estimator method that was used. Using the absolute moment method to estimate the Hurst parameter, the MAPE value



(a) Hurst Estimator: Absolute Moment, $H = 0.4043$

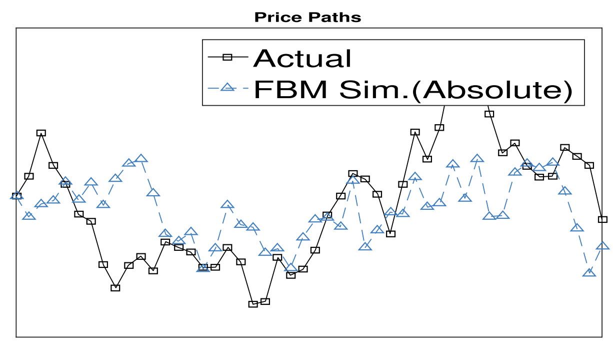


(b) Hurst Estimator: Aggregated Variance, $H = 0.4286$

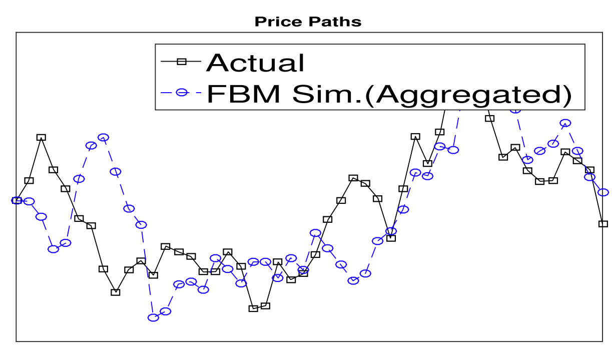


(c) Hurst Estimator: Higuchi, $H = 0.4574$

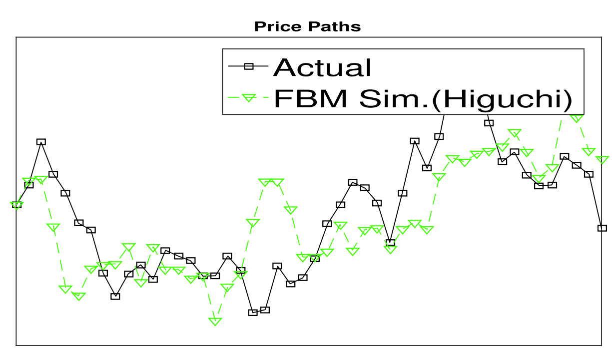
Fig. 2 Price path simulation for a 3-year period using GFBM.



(a) Hurst Estimator: Absolute Moment, $H = 0.3657$

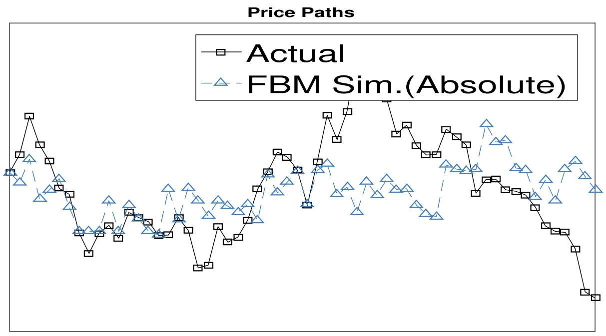


(b) Hurst Estimator: Aggregated Variance, $H = 0.5203$

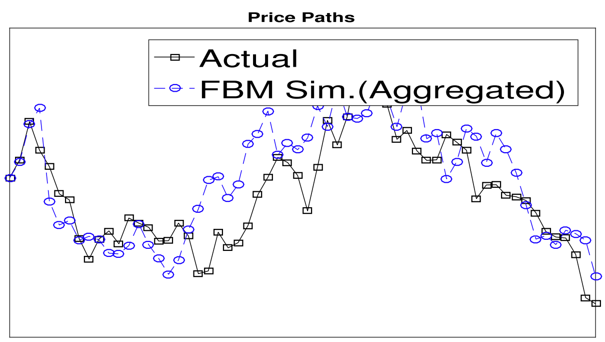


(c) Hurst Estimator: Higuchi, $H = 0.5001$

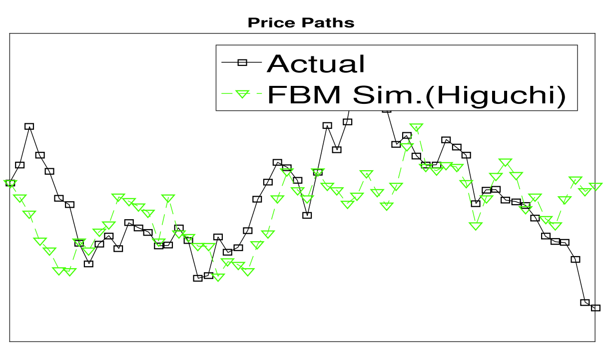
Fig. 3 Price path simulation for a 4-year period using GFBM.



(a) Hurst Estimator: Absolute Moment, $H = 0.2343$



(b) Hurst Estimator: Aggregated Variance, $H = 0.4999$



(c) Hurst Estimator: Higuchi, $H = 0.4643$

Fig. 4 Price path simulation for a 5-year period using GFBM.

Table 3 Forecast Accuracy Judgement Scale.

ϵ	Forecast Accuracy
$< 10\%$	Very accurate
$11 - 20\%$	Accurate
$21 - 50\%$	Inaccurate
$> 50\%$	Very inaccurate

Source: [40]

Table 4 MAPE values, ϵ.

Period	GBM	GFBM		
		Absolute Moment	Aggregated Variance	Higuchi
3 years	10.4471	9.0557	8.9717	9.5434
4 years	10.5903	9.2198	8.4206	8.3658
5 years	10.9386	9.5844	7.6137	9.0929

for the GFBM model is 9.2866, on average. While using the aggregated variance method, it is 8.3353, on average. Lastly, the Higuchi method returns an average of 9.0007 MAPE value. Therefore, using the judgement scale given in Table 3, it can be seen that overall, the GBM and GFBM models produce highly accurate forecast price. However, the results show higher accuracy in the forecast that is produced by the GFBM models.

5. Conclusions

This study tests the accuracy of two mathematical models, geometric Brownian motion (GBM) and geometric fractional Brownian motion (GFBM) models in simulating the future price paths of Malaysia's crude palm oil prices. In order to use these models, unknown parameters need to be estimated using historical prices of the commodity. The accuracy of the simulations is determined by computing the mean-average percentage error (MAPE), where we conclude that both models are able to produce highly accurate forecast prices. However, the GFBM model is more accurate than the GBM models for three different Hurst estimators.

Future work can consider to simulate the fractional Gaussian noise using methods such as fast Fourier transform (FFT) which reduces the computation time taken by the Cholesky method, $O(N^3)$, even though the Cholesky method is more straightforward to implement. Moreover, more factors can be incorporated to the GBM and GFBM models such as mean-reversion with jumps models, or seasonality to replicate the commodity market more closely.

Declaration of Competing Interest

None.

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