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Hermite-Hadamard and Simpson's type of inequalities for first and second ordered derivatives using product of (h_1, h_2, s)-convex and m-convex function

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Abstract

This article is dedicated to find the extensions for Hermite-Hadamard (H-H) and Simpson's type of inequalities. By combining multiple existing convex functions by placing specific restrictions on them is the most effective in many approaches to find a new convex function. Here, to find the new function (h_1, h_2, s)-Convex and m-Convex Function are used. Because of the product of two or even more convex functions does not necessarily have to be convex, we decided to investigate merging distinct convex functions. Combining more

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than two convex functions in a novel adaptive way advances to new applications in a range of disciplines, including mathematics and other fields. In this paper, some extensions for Hermite-Hadamard and Simpson's inequalities is explored. The newly constructed extensions of these inequalities will be considered as the improvements and refinements of previously obtained results.

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1. Introduction

Convex functions are very important because of their geometric interpretation and wide area of application [1-3]. They provides the basis for construction of mathematical inequalities [4-5]. In addition, it provides the basic idea for making a linear comparison between the composition of two points for a function and the composition of image of these two points [5]. At the beginning of the 19^{th} century, Jensen (1906) introduced the mostly used definition of the convex function for real values as follows

A function $f:\mathbb{I}\to\mathbb{R}$, *is convex for* $a_1\in[0,1]$, *if*

$$f(a_1x+(1-a_1)y)\leq a_1f(x)+(1-a_1)f(y), x,y\in\mathbb{I}\subset\mathbb{R} \quad (1)$$

Remarks: If strict inequality (<) holds on Inequality (1), then it is called strictly convex, whenever $x\neq y$ and $\alpha\in(0,1)$. The definition of convex can also be explained using geometry.

Several mathematical inequalities and extensions are the results of convex functions. The first inequality in literature for convex function is Hermite-Hadamard inequality. Different extensions of Inequality (1) have been made in past for different uses [13-15, 17-22].

Gabriela, Noor and Uzair (2015) introduced h_1h_2-convex function [16]. It is a generalized form of h-convex function by using two different positive real-valued functions named h and h_2 as given.

Definition 1: A function $f:\mathbb{I}\to(0,\propto)$ and $h_1,h_2:\to\mathbb{R}$, then f is called (h_1h_2)-convex for $a_1\in[0,1]$, if

$$f(a_1x+(1-a_1)y)\leq h_1(a_1)f(x)+h_2(1-a_1)f(y), x,y\in\mathbb{I}. \quad (2)$$

Here, $h\not\equiv 0$ is a positive function. Furthermore, Özdemir et al. (2016) extended the (h_1, h_2)-convex function by combining it with m-convex function (Toader 1988) as follows.

Definition 2: A function $f:\mathbb{I}\to(0,\propto)$ and $h_1,h_2:\to\mathbb{R}$, then *f* is called (m, h_1, h_2)-convex for $a_1\in[0,1]$, if

$$f(a_1x+m(1-a_1)y)\le h_1(a_1)f(x)+mh_2(1-a_1)f(y),x,y\in\mathbb{I}. \tag{3}$$

Here, $h\not\equiv 0$ is a positive function.

On the basis of these mentioned definitions the most famous H-H inequality was extended in different time.

Let a function $f:\mathbb{I}\to\mathbb{R}_+$ *be convex, then*

$$f\left[\frac{p+q}{2}\right]\le\frac{1}{q-p}\int_p^q f(x)\,dx\le\frac{f(q)+f(p)}{2},p,q\in\mathbb{I}\in\mathbb{R}, \tag{4}$$

is called Hermite-Hadamard (H-H) inequality.

Main Results

In this section, we combined (h_1, h_2)-*convex*, s-convex, and m-convex functions to form (h_1, h_2, s)-m-convex function.

Definition 3: A function $f:\mathbb{I}\to(0,\propto)$ and $h_1,h_2:\to\mathbb{R}$, then *f* is called (h_1, h_2, s)-m-convex for $a_1\in[0,1]$, if

$$f(a_1x+m(1-a_1)y)\le h_1^s(a_1)f(x)+mh_2^s(1-a_1)f(y),x,y\in\mathbb{I}. \tag{5}$$

Here, $h\not\equiv 0$ is a positive function and $s\in[0,1]$.

Lemma 1*: Let a function* $f:\mathbb{I}\to\mathbb{R}$ *be differentiable on* $\mathbb{I}$, *where* $p,q\in\mathbb{I}\subset\mathbb{R}$ *then*

$$\frac{1}{q-p}\int_p^q f(x)\,dx-f\left(\frac{p+q}{2}\right)=\frac{(q-p)^2}{16}\left[\int_0^1 a_1^{\,2}f''\left(a_1\left(\frac{p+q}{2}\right)+p(1-a_1)\right)da_1\right.$$
$$\left.+\int_0^1(a_1-1)^2f''\left(a_1p+(1-a_1)\left(\frac{p+q}{2}\right)da_1\right)\right]. \tag{6}$$

Theorem 1*: Let a function* $f:\mathbb{I}\to\mathbb{R}$ *be differentiable on* $\mathbb{I}$, *where* $p,q\in\mathbb{I}\subset\mathbb{R}$. *If* $|f''|$ *be a* (m, h_1, h_2, s)*-convex function then*

$$f\left(\frac{p+q}{2}\right)-\frac{1}{q-p}\int_p^q f(x)\,dx\le\left[\frac{(q-p)^2}{16}f''\left(\frac{p+q}{2}\right)\int_0^1 a_1^{\,2}h_1^s(a_1)\,da_1\right.$$
$$\left.+m\left(f''\left(\frac{p}{m}\right)\right)\int_0^1 a_1^{\,2}h_2^s(a_1)\,da_1\right]$$

$$+\frac{(q-p)^2}{16}\left[\int_0^1 a_1^{\ 2} f''\left(a_1\left(\frac{p+q}{2}\right)+(1-a_1)p\right)da_1\right.$$
$$\left.+\int_0^1 (a_1-1)^2 f''\left(a_1 p+(1-a_1)\left(\frac{p+q}{2}\right)da_1\right)\right]. \quad (7)$$

***Proof*:**

From Lemma (1), we know that

$$\frac{1}{q-p}\int_p^q f(x)\,dx - f\left(\frac{q+p}{2}\right) \le \frac{(q-p)^2}{16}\left[\int_0^1 a_1^{\ 2} f''\left(a_1\left(\frac{p+q}{2}\right)+(1-a_1)p\right)da_1\right.$$
$$\left.+\int_0^1 (a_1-1)^2 f''\left(a_1 p+(1-a_1)\left(\frac{p+q}{2}\right)da_1\right)\right]. \quad (8)$$

As $|f''|$ is a (h_1, h_2, s, m)-convex function for any $a_1 \in [0,1]$, we have

$$f''\left(a_1\left(\frac{p+q}{2}\right)+m(1-a_1)p\right) \le h_1^{\ s}(a_1) f''\left(\frac{p+q}{2}\right)+mh_2^{\ s}(1-a_1) f''\left(\frac{p}{m}\right), \quad (9)$$

also

$$f''\left(a_1 q+m(1-a_1)\left(\frac{p+q}{2m}\right)\right) \le h_1^{\ s}(a_1) f''(q)+mh_2^{\ s}(1-a_1) f''\left(\frac{p+q}{2m}\right). \quad (10)$$

By using Inequality (9) and Inequality (10) in Inequality (8), we get

$$f\left(\frac{p+q}{2}\right)-\frac{1}{q-p}\int_p^q f(x)\,dx \le \frac{(q-p)^2}{16}\left[\int_0^1 a_1^{\ 2}\left[f''\left(h_1^{s}(a_1)\left(\frac{p+q}{2}\right)\right)\right.\right.$$
$$\left.\left.+mh_2^{\ s}(1-a_1) f''\left(\frac{p}{m}\right)\right]da_1\right]$$
$$+\frac{(q-p)^2}{16}\left[\int_0^1 (a_1-1)^2\left[f''(h_1^{\ s}(a_1)(q))\right.\right.$$
$$\left.\left.+mh_2^{\ s}(1-a_1) f''\left(\frac{p+q}{2m}\right)\right]da_1\right], \quad (11)$$

$$f\left(\frac{p+q}{2}\right)-\frac{1}{q-p}\int_p^q f(x)\,dx = \frac{(q-p)^2}{16}\left[f''\left(\frac{q+p}{2}\right)\int_0^1 a_1^{\ 2}\, h_1^{s}(a_1)\,da_1\right.$$
$$\left.+mf''\left(\frac{p}{m}\right)\int_0^1 a_1^{\ 2}\, h_2^{s}(1-a_1)\,da_1\right]$$
$$+\frac{(q-p)^2}{16}\left[f''(q)\int_0^1 (a_1-1)^2\, h_1^{\ s}(a_1)\,da_1\right.$$
$$\left.+mf''\left(\frac{p+q}{2m}\right)\int_0^1 (a_1-1)^2\, h_2^{s}(1-a_1)\,da_1\right]. \quad (12)$$

This Inequality (12) completes the proof.

Remark:

(1) By using $m = 1$ in Theorem (1), then the results will be converted to H-H inequality for (h_1, h_2, s)-convex function.

(2) By using $h_1^s(a_1) = h^s(a_1)$ and $h_2^s(1-a_1) = h^s(1-a_1)$ in Theorem (1), then results will be converted to (h, s)-convex function.

(3) By using $h_1^s(a_1) = a_1$ and $h_2^s(1-a_1) = 1-a_1$ in Theorem (1), then results will be converted to convex function.

Theorem 2: *Let a function* $f : \mathbb{I} \to \mathbb{R}$ *be differentiable on* $\mathbb{I}$, *where* $p, q \in \mathbb{I} \subset \mathbb{R}$. *If* $|f''|$ *be a* (h_1, h_2, s, m)*-convex function and* $\frac{1}{a} + \frac{1}{b} = 1$ *with* $b > 1$ *then*

$$f\left(\frac{p+q}{2}\right) - \frac{1}{q-p}\int_p^q f(x)\,dx \le \frac{(q-p)^2}{16}\left[\frac{1}{2a_1+1}\right]^{\frac{1}{a}}\left[\left(h_1^s(a_1)\left|f''\left(\frac{p+q}{2}\right)\right|^b + mh_2^s(1-a_1)\left|f''\left(\frac{p}{m}\right)\right|^b\right)^{\frac{1}{b}} + \left(h_1^s(a_1)\,|f''(q)|^b + mh_2^s(1-a_1)\left|f''\left(\frac{p+q}{2m}\right)\right|^b\right)^{\frac{1}{b}}\right]. \tag{13}$$

***Proof*:**

From Lemma (1) and $b > 1$, we have

$$f\left(\frac{p+q}{2}\right) - \frac{1}{q-p}\int_p^q f(x)\,dx \le \frac{(q-p)^2}{16}\left[\int_0^1 a_1^2 f''\left((1-a_1)p + a_1\left(\frac{p+q}{2}\right)\right)da_1 + \int_0^1 (a_1-1)^2 f''\left(a_1 p + \left(\frac{q+p}{2}\right)(1-a_1)\,da_1\right)\right]. \tag{14}$$

By applying Holder's inequality $\frac{1}{a} + \frac{1}{b} = 1$ on Inequality (14), we get

$$f\left(\frac{q+p}{2}\right) - \frac{1}{q-p}\int_p^q f(x)\,dx$$
$$\le \frac{(q-p)^2}{16}\left[\int_0^1 a_1^{2a}\,da_1\right]^{\frac{1}{a}}\left(\int_0^1 f''\left(a_1\left(\frac{p+q}{2}\right) + (1-a_1)p\right)^b da_1\right)^{\frac{1}{b}}$$
$$+ \frac{(q-p)^2}{16}\left[\int_0^1 (a_1-1)^{2a}\,da_1\right]^{\frac{1}{a}}\left(\int_0^1 f''\left(a_1 q + (1-a_1)\left(\frac{p+q}{2}\right)^b\right)da_1\right)^{\frac{1}{b}}. \tag{15}$$

As $|f''|$ is a (h_1, h_2, s, m)-convex function, we have

$$\int_0^1 \left| f''\left(a_1\left(\frac{p+q}{2}\right) + m(1-a_1)q \right) \right|^b da_1 \le h_1^s(a_1) \left| f''\left(\frac{p+q}{2}\right) \right|^b$$
$$+ m h_2^s(1-a_1) \left| f''\left(\frac{p}{m}\right) \right|^b, \tag{16}$$

and

$$\int_0^1 \left| f''\left(a_1(q) + m(1-a_1)\left(\frac{p+q}{2}\right) \right) \right|^b da_1 \le h_1^s(a_1) \,|\, f''(q)\,|^b$$
$$+ m h_2^s(1-a_1) \left| f''\left(\frac{p+q}{2m}\right) \right|^b. \tag{17}$$

As we know that $\int_0^1 a_1^{2a} da_1 = \int_0^1 (1-a_1)^{2a} da_1 = \frac{1}{2a_1+1}$, then Inequality (15) will be reduced to

$$f\left(\frac{p+q}{2}\right) - \frac{1}{q-p}\int_p^q f(x)\,dx$$
$$\le \frac{(q-p)^2}{16}\left[\frac{1}{2a_1+1}\right]^{\frac{1}{a}} \left(\int_0^1 f''\left(a_1\left(\frac{q+p}{2}\right) + (1-a_1)p \right)^b da_1 \right)^{\frac{1}{b}}$$
$$+ \frac{(q-p)^2}{16}\left[\frac{1}{2a_1+1}\right]^{\frac{1}{a}} \left(\int_0^1 f''\left(a_1 q + (1-a_1)\left(\frac{p+q}{2}\right) \right)^b da_1 \right)^{\frac{1}{b}}, \tag{18}$$

$$f\left(\frac{p+q}{2}\right) - \frac{1}{q-p}\int_p^q f(x)\,dx$$
$$\le \frac{(q-p)^2}{16}\left[\frac{1}{2a_1+1}\right]^{\frac{1}{a}} \left[\left(h_1^s(a_1)\left| f''\left(\frac{p+q}{2}\right) \right|^b + m h_2^s(1-a_1)\left| f''\left(\frac{p}{m}\right) \right|^b \right)^{\frac{1}{b}} \right.$$
$$\left. + \left(h_1^s(a_1)|f''(q)|^b + m h_2^s(1-a_1)\left| f''\left(\frac{p+q}{2m}\right) \right|^b \right)^{\frac{1}{b}} \right]. \tag{19}$$

This Inequality (19) completes the proof.

Remarks:

(1) By using $m = 1$ in Theorem (2), then the results will be converted to H-H inequality for (h_1, h_2, s)-convex function.

(2) By using $h_1^s(a_1) = h^s(a_1)$ and $h_2^s(1-a_1) = h^s(1-a_1)$ in Theorem (2), then results will be converted to (h, s)-convex function.

(3) By using $h_1^s(a_1) = a_1$ and $h_2^s(1-a_1) = 1-a_1$ in Theorem (2), then results will be converted to convex function.

Theorem 3*: Let a function* $f:\mathbb{I}\to\mathbb{R}$ *be differentiable on* $\mathbb{I}$*, where* $p,q\in\mathbb{I}\subset\mathbb{R}$. *If* $|f''|$ *be a* (h_1, h_2, s, m)*-convex function and* $\frac{1}{a}+\frac{1}{b}=1$ *with* $b>1$ *then*

$$f\left(\frac{p+q}{2}\right)-\frac{1}{q-p}\int_p^q f(x)\,dx$$

$$\leq \frac{(q-p)^2}{16}\left[\frac{1}{3}\right]^{1-\frac{1}{b}}\left[\left(\left|f''\left(\frac{p+q}{2}\right)\right|^b\int_0^1(a_1)^2\,h_1^s(a_1)\right.\right.$$

$$\left.+m\left|f''\left(\frac{p}{m}\right)\right|^b\int_0^1(a_1)^2\,h_2^s(1-a_1)\right)^{\frac{1}{b}} \tag{20}$$

$$\left.+\left(|f''(q)|^b\int_0^1(a_1^{\ 2}-1)\,h_1^s(a_1)+m\left|f''\left(\frac{p+q}{2m}\right)\right|^b\int_0^1(a_1^{\ 2}-1)\,h_2^s(1-a_1)\right)^{\frac{1}{b}}\right].$$

***Proof*:**

From Lemma (1) and $b>1$, we have

$$f\left(\frac{p+q}{2}\right)-\frac{1}{q-p}\int_p^q f(x)\,dx$$

$$\leq \frac{(q-p)^2}{16}\left[\int_0^1 a_1^{\ 2}\left|f''\left(\left(\frac{p+q}{2}\right)a_1+(1-a_1)p\right)\right|da_1\right.$$

$$\left.+\int_0^1(a_1-1)^2\left|f''\left(a_1p+(1-a_1)\left(\frac{p+q}{2}\right)da_1\right)\right|\right], \tag{21}$$

$$=\frac{(q-p)^2}{16}\left(\int_0^1 a_1^{\ 2}da_1\right)^{\left(1-\frac{1}{b}\right)}\left[\left(\int_0^1 a_1^{\ 2}\left|f''\left(a_1\left(\frac{p+q}{2}\right)+(1-a_1)p\right)\right|^b da_1\right)^{\frac{1}{b}}\right.$$

$$\left.+\int_0^1(a_1-1)^2\left|f''\left(a_1p+(1-a_1)\left(\frac{p+q}{2}\right)da_1\right)^b\right|^{\frac{1}{b}}\right]. \tag{22}$$

As $|f''|$ is a (h_1, h_2, s, m)-convex function, we have

$$\int_0^1(a_1)^2\left|f''\left(a_1\left(\frac{p+q}{2}\right)+(1-a_1)p\right)\right|^b da_1$$

$$\leq\left|f''\left(\frac{p+q}{2}\right)\right|^b\int_0^1(a_1)^2\,h_1^s(a_1)\,da_1$$

$$+m\left|f''\left(\frac{p}{m}\right)\right|^b\int_0^1(a_1)^2h_2^s(1-a_1)\,da_1, \tag{23}$$

and

$$\int_0^1 (a_1)^2 \left| f''\left(a_1 q+(1-a_1)\left(\frac{p+q}{2}\right)\right)\right|^b da_1$$
$$\le |f''(q)|^b \int_0^1 (a_1{}^2-1)h_1{}^s(a_1)\,da_1$$
$$+m\left|f''\left(\frac{p+q}{2m}\right)\right|^b \int_0^1 (a_1{}^2-1)h_2{}^s(1-a_1)\,da_1. \tag{24}$$

Now using the Inequality (23) and Inequality (24) in Inequality (22), we get

$$f\left(\frac{p+q}{2}\right)-\frac{1}{q-p}\int_p^q f(x)\,dx$$
$$\le \frac{(q-p)^2}{16}\left[\frac{1}{3}\right]^{1-\frac{1}{b}}\left[\left(\left|f''\left(\frac{p+q}{2}\right)\right|^b \int_0^1 (a_1)^2 h_1^s(a_1)\right.\right.$$
$$\left.+m\left|f''\left(\frac{p}{m}\right)\right|^b \int_0^1 (a_1)^2 h_2^s(1-a_1)\right)^{\frac{1}{b}} +\left(|f''(q)|^b \int_0^1 (a_1{}^2-1)h_1^s(a_1)\right.$$
$$\left.\left.+m\left|f''\left(\frac{p+q}{2m}\right)\right|^b \int_0^1 (a_1{}^2-1)h_2^s(1-a_1)\right)^{\frac{1}{b}}\right]. \tag{25}$$

This inequality (25) completes the proof.

Remarks:

(1) By using $m = 1$ in Theorem (3), then the results will be converted to H-H inequality for (h_1, h_2, s)-convex function.

(2) By using $h_1{}^s(a_1)=h^s(a_1)$ and $h_2{}^s(1-a_1)=h^s(1-a_1)$ in Theorem (3), then results will be converted to -convex function.

(3) By using $h_1{}^s(a_1)=a_1$ and $h_2{}^s(1-a_1)=1-a_1$ in Theorem (3), then results will be converted to convex function.

Theorem 4: *Let a function $f:\mathbb{I}\to\mathbb{R}$ be differentiable on $\mathbb{I}$, where $p,q\in\mathbb{I}\subset\mathbb{R}$. If $|f''|$ be a (h_1, h_2, s, m)-convex function and $\frac{1}{a}+\frac{1}{b}=1$ with $b > 1$ then*

$$f\left(\frac{p+q}{2}\right)-\frac{1}{q-p}\int_p^q f(x)\,dx$$
$$\le \frac{(q-p)^2}{16}\left(\frac{1}{a(2a+1)}+\frac{1}{b}\left(\left|f''\left(\frac{p+q}{2}\right)\right|^b \int_0^1 h_1{}^s(a_1)\right.\right.$$

$$+m\left|f''\left(\frac{p}{m}\right)\right|^{b}\int_0^1 h_2^{\,s}(1-a_1)\Bigg)\Bigg)$$

$$+\frac{(q-p)^2}{16}\left(\frac{1}{a\left(2a+1\right)}\right.$$

$$\left.+\frac{1}{b}\left(|f''(q)|^{b}\int_0^1 h_1^{\,s}(a_1)+m\left|f''\left(\frac{p}{m}\right)\right|^{b}\int_0^1 h_2^{\,s}(1-a_1)\right)\right). \tag{26}$$

***Proof*:**

From Lemma (1) and $b > 1$, we have

$$f\left(\frac{q+p}{2}\right)-\frac{1}{q-p}\int_p^q f(x)\,dx$$

$$\leq \frac{(q-p)^2}{16}\left[\int_0^1 a_1^{\,2}\left|f''\left(\left(\frac{q+p}{2}\right)a_1+(1-a_1)p\right)\right|da_1\right.$$

$$\left.+\int_0^1 (a_1-1)^2\left|f''\left(a_1p+(1-a_1)\left(\frac{p+q}{2}\right)da_1\right)\right|\right]. \tag{27}$$

$$=\frac{(q-p)^2}{16}\int_0^1\left(\frac{a_1^{\,2a}}{a}+\frac{\left|f''\left(a_1\left(\frac{p+q}{2}\right)+(1-a_1)p\right)\right|^{b}}{b}\right)da_1$$

$$+\frac{(q-p)^2}{16}\int_0^1\left(\frac{(a_1-1)^{2a}}{a}+\frac{\left|f''\left(a_1q+(1-a_1)\left(\frac{p+q}{2}\right)\right)\right|^{b}}{b}\right)da_1. \tag{28}$$

As $|f''|$ is a (h_1, h_2, s, m)-convex function, we have

$$f\left(\frac{p+q}{2}\right)-\frac{1}{q-p}\int_p^q f(x)\,dx$$

$$\leq \frac{(q-p)^2}{16}\left(\frac{1}{a(2a+1)}+\frac{1}{b}\left(\left|f''\left(\frac{p+q}{2}\right)\right|^{b}\int_0^1 h_1^{\,s}(a_1)\right.\right.$$

$$\left.\left.+m\left|f''\left(\frac{p}{m}\right)\right|^{b}\int_0^1 h_2^{\,s}(1-a_1)\right)\right)+\frac{(q-p)^2}{16}\left(\frac{1}{a(2a+1)}\right.$$

$$\left.+\frac{1}{b}\left(|f''(q)|^{b}\int_0^1 h_1^{\,s}(a_1)+m\left|f''\left(\frac{p}{m}\right)\right|^{b}\int_0^1 h_2^{\,s}(1-a_1)\right)\right). \tag{29}$$

This Inequality (29) completes the proof.

Remarks:

(1) By using $m = 1$ in Theorem (4), then the results will be converted to H-H inequality for (h_1, h_2, s)-convex function.

(2) By using $h_1^s(a_1) = h^s(a_1)$ and $h_2^s(1-a_1) = h^s(1-a_1)$ in Theorem (4), then results will be converted to (*h*, *s*)-convex function.

(3) By using $h_1^s(a_1) = a_1$ and $h_2^s(1-a_1) = 1-a_1$ in Theorem (4), then results will be converted to convex function.

Application for Mean (Average)

As we know that for all $p, q \in \mathbb{R}$, some means (average) are as follows

(1) The geometric mean

$$G(p,q) = \sqrt{pq}.$$

(2) The harmonic mean

$$H(p,q) = \frac{2pq}{p+q}.$$

(3) The α -logarithmic mean

$$L_n(p,q) = \left[\frac{q^{\alpha+1} - p^{\alpha+1}}{(\alpha+1)(q-p)}\right]^{\frac{1}{\alpha}}.$$

Here, $\alpha \in \mathbb{R} \setminus \{0\}$ and $p < q$.

Let $\frac{1}{q-p}\int_p^q f(x) \quad dx = L_r^r(p,q)$ and $f\left(\frac{p+q}{2}\right) = A(p,q)$ then the whole expressions of above theorems can be written in the form of applications of special means (averages.)

Conclusion

This paper utilized the procedure of combining more than two functions to extend the convex function. Different types of convex functions are used to extend the previous results and to investigate the H-H inequalities. For some specific value of *h*, h^s, *m* and α, almost all the previous results for discussed functions are derived. The comparison between new as well as old results reflects that all the previous results can be obtained for these new results by only choosing some specific conditions. H-H and Fejer's inequalities are also used to produce the results and original H-H inequalities can also be found by using these results. On the basis of discussed inequalities, mathematical means (averages) are also applied in calculating the required results.

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