An overview on the local limit of non-local conservation laws, and a new proof of a compactness estimate

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AN OVERVIEW OF THE LOCAL LIMIT OF NON-LOCAL CONSERVATION LAWS, AND A NEW PROOF OF A COMPACTNESS ESTIMATE

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Consider a non-local (i.e., involving a convolution term) conservation law: when the convolution term converges to a Dirac delta, in the limit we formally recover a classical (or "local") conservation law. In this note we overview recent progress on this so-called non-local to local limit and in particular we discuss the case of anistropic kernels, which is extremely relevant in view of applications to traffic models. We also provide a new proof of a related compactness estimate.

1. INTRODUCTION

In recent years, the analysis of non-local conservation laws in the form

$$\partial_t u + \partial_x [V(u * \eta) u] = 0 \tag{1.1}$$

has attracted considerable attention. In the previous expression, the unknown u is real-valued and depends on the variables $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. The Lipschitz continuous function $V : \mathbb{R} \to \mathbb{R}$ is assigned and so is the convolution kernel $\eta : \mathbb{R} \to \mathbb{R}_+$. The symbol $*$ denotes the convolution with respect to the space variable only. The analysis of (1.1) and of related equations is motivated by several applications concerning sedimentation [2], supply chains [17], pedestrian [16] and vehicular [7] traffic models, and others. Existence and uniqueness results have been obtained under suitable assumptions in various works, see for instance [3, 16, 18, 22].

In the present contribution we focus on the so-called non-local to local limit problem, which we now introduce. Fix $\varepsilon > 0$ and consider the family of Cauchy problems

$$\begin{cases} \partial_t u_\varepsilon + \partial_x [V(u_\varepsilon * \eta_\varepsilon) u_\varepsilon] = 0 \\ u_\varepsilon(0, \cdot) = u_0 \end{cases} \quad \text{with } \eta_\varepsilon(x) := \frac{1}{\varepsilon} \eta\left(\frac{x}{\varepsilon}\right), \tag{1.2}$$

where the initial datum $u_0 : \mathbb{R} \to \mathbb{R}$ satisfies suitable assumptions discussed in the following. In the non-local to local limit $\varepsilon \to 0^+$ the convolution kernel $\eta_\varepsilon \overset{*}{\rightharpoonup} \delta_{x=0}$ weakly-$*$ in the sense of measures. Consequently, what one *formally* recovers in the vanishing ε limit is the scalar conservation law

$$\begin{cases} \partial_t u + \partial_x [V(u) u] = 0 \\ u(0, \cdot) = u_0. \end{cases} \tag{1.3}$$

Whether or not one can rigorously establish the convergence of u_ε to the entropy admissible solution of the conservation law (1.3) has been the target of recent investigations, that we briefly overview in the next section. We refer instead to the very classical references [19, 25] for the definition of entropy admissible solutions of a conservation law.

This note is organized as follows: in §2 we discuss recent results on the non-local to local limit. In §3 and §4 we provide an alternative proof of the main compactness estimate in [14] concerning the non-local to local limit, which is estimate (2.11) below. The proof of (2.11) we discuss here is longer and less direct then the original one given in [14], but we think it is of interest as we feel it is more transparent as it more clearly elucidates the basic mechanism yielding compactness. In §3 we discuss an heuristic argument, whereas in §4 we provide the complete proof.

2. THE NON-LOCAL TO LOCAL LIMIT: AN OVERVIEW

Let us consider the family of Cauchy problems (1.2): one of the main challenges in studying the vanishing ε limit is that, owing to the presence of the non-local term, it is fairly hard to gain explicit insights on the precise behavior of the solution, and henceforth useful compactness estimates. Note furthermore that, if $u_0 \in L^1(\mathbb{R})$, it is fairly easy to establish bounds on $\|u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})}$ that are uniform in ε and t. This in turn implies that, up to subsequences, the family u_ε converges to some limit measure weakly* in the sense of measures on $[0, T] \times \mathbb{R}$, for every $T > 0$. However, weak convergence alone does not suffice to pass to the limit in the nonlinear term $V(u_\varepsilon * \eta_\varepsilon)u_\varepsilon$ and henceforth to pass from (1.2) to (1.3).

To the best of our knowledge, one of the very first results concerning the vanishing ε limit is the paper by Zumbrun [29], where, among other things, the author establishes convergence of u_ε to the entropy admissible solution u of (1.3) under the assumptions that η is even, that is $\eta(x) = \eta(-x)$, and (quite restrictively) that u is smooth.

The non-local to local limit problem is again addressed in the more recent paper [1] and investigated by relying on numerical experiments. The numerical simulations exhibited in [1] suggest that, as ε gets closer and closer to 0, the solution u_ε of (1.2) approaches the entropy admissible solution of (1.3). In [15] the authors provide counter-examples showing that actually this is not always the case. More precisely, the counter-examples dictate that in general u_ε does not converge to the entropy admissible solution, not even weakly or up to subsequences. This is very loosely speaking achieved by singling out a property (a different one in each counter-example) which is satisfied by the solution u_ε of (1.2) for every ε, passes to the limit, but is *not* satisfied by the entropy admissible solution of (1.3). Another result in [15] focuses on the "viscous counter-part" of the non-local to local limit problem. More precisely, fix $\nu > 0$ and consider the family of viscous non-local problems

$$(2.1) \qquad \begin{cases} \partial_t u_{\varepsilon\nu} + \partial_x[V(u_{\varepsilon\nu} * \eta_\varepsilon)u_{\varepsilon\nu}] = \nu \partial_{xx} u_{\varepsilon\nu} \\ u_{\varepsilon\nu}(0, \cdot) = u_0 \end{cases} \qquad \text{with } \eta_\varepsilon(x) := \frac{1}{\varepsilon}\eta\left(\frac{x}{\varepsilon}\right).$$

Existence and uniqueness results for (2.1) can be established by relying on fairly standard techniques, see §2 in [15]. Extending previous results by Calderoni and Pulvirenti [6], Theorem 1.1 in [15] states that, under quite general assumptions on V, η and u_0, the solutions $u_{\varepsilon\nu}$ converge as $\varepsilon \to 0^+$ to the solution of the viscous conservation law

$$(2.2) \qquad \begin{cases} \partial_t u_\nu + \partial_x[V(u_\nu)u_\nu] = \nu \partial_{xx} u_\nu \\ u_\nu(0, \cdot) = u_0. \end{cases}$$

This result is also relevant from the numerical viewpoint, see the related discussion in [12]. Wrapping up, one has the following diagram:

$$(2.3) \qquad \begin{array}{ccc} \partial_t u_{\varepsilon\nu} + \partial_x\big[u_{\varepsilon\nu}V(u_{\varepsilon\nu} * \eta_\varepsilon)\big] = \nu\partial_{xx}u_{\varepsilon\nu} & \xrightarrow[\text{[15, Theorem 1.1]}]{\varepsilon \to 0^+} & \partial_t u_\nu + \partial_x\big[u_\nu V(u_\nu)\big] = \nu\partial_{xx}u_\nu \\[2mm] {\scriptstyle \nu \to 0^+} \Big\downarrow {\scriptstyle \text{[15, Proposition 1.2]}} & & {\scriptstyle \nu \to 0^+} \Big\downarrow {\scriptstyle \text{Kružkov's Theorem}} \\[2mm] \partial_t u_\varepsilon + \partial_x\big[u_\varepsilon V(u_\varepsilon * \eta_\varepsilon)\big] = 0 & \xrightarrow[\text{False in general}]{\varepsilon \to 0^+} & \partial_t u + \partial_x\big[uV(u)\big] = 0 \end{array}$$

Note that the "vertical" convergence at the left of the previous diagram follows from Proposition 1.2 in [15] and relies on the extension of classical parabolic compactenss estimates to the nonlocal setting, whereas the convergence of the solutions of the viscous conservation law to the entropy admissible solution of (1.3) dates back to the by-now classical work of Kružkov [25]. See also [10] for more recent results on the "diagonal" convergence in the previous diagram.

Going back to the original (non-viscous) non-local to local limit problem, the above mentioned counter-examples in [15] left open the possibility of establishing convergence in a more specific setting. As a matter of fact, in the last very few years several results have been obtained under assumptions that are fairly natural in view of applications to traffic flow models, which we now discuss.

The archetype of fluido-dynamic traffic models is the celebrated LWR model, introduced in the works by Lighthill and Whitham [27] and Richards [28], which involves a conservation law like the one at the first line of (1.3). In the LWR model the unknown u represents the density of vehicles, and V their speed. The model postulates that drivers tune their speed based on the pointwise traffic density, and, since the most common reaction to an increase in the car density is deceleration, the assumptions usually imposed on V are

$$(2.4) \qquad\qquad V \in \mathrm{Lip}(\mathbb{R}), \qquad V' \leq 0.$$

We now turn to the assumptions satisfied by the datum u_0: since it represents the initial density, one assumes $u_0 \geq 0$. The datum u_0 should also not exceed the maximum possible density (corresponding to bumper-to-bumper packing), which with no loss of generality we can assume normalized to 1. Wrapping up,

$$(2.5) \qquad\qquad u_0 \in L^\infty(\mathbb{R}), \quad 0 \leq u_0(x) \leq 1 \text{ for a.e. } x \in \mathbb{R}.$$

Let us now turn to the non-local LWR model (1.1): compared to the classical one, this version of the model aims at taking into account that drivers tune their speed according to the car density in a suitable neighborhood of their position, rather than to the pointwise density only. Also, it turns out that the presence of the convolution term in (1.1) rules out the possible presence of infinite acceleration, a well-known drawback of the classical LWR model. The hypotheses most commonly imposed on η in non-local LWR models are then

$$(2.6) \quad \eta \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad \eta \geq 0, \quad \int_{\mathbb{R}} \eta(x)dx = 1, \quad \mathrm{supp}\,\eta \subseteq \mathbb{R}_-, \qquad \eta \text{ non-decreasing on } \mathbb{R}_-.$$

The first three conditions in (2.6) are fairly standard assumptions for convolution kernels. More interesting is the second-last condition, a "look-ahead-only" assumption that takes into account that drivers tune their speed according to the downstream traffic density only. Finally, the last condition in (2.6) expresses the fact that drivers are more deeply influenced by closer vehicles rather than by those that are further away.

The analysis on the non-local LWR model (1.2) under (2.4),(2.5) and (2.6) was initiated by Blandin and Goatin [3], who in particular established the maximum principle

$$(2.7) \qquad\qquad 0 \leq u_\varepsilon(t,x) \leq 1 \text{ for a.e. } (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

a remarkable property in view of applications that is not satisfied by general non-local conservation laws. Keimer and Pflug [23] were instead the first, to the best of our knowledge, to investigate, under the further assumption that the initial datum u_0 is a monotone function, the local limit of the non-local LWR model. In [4, 5] Bressan and Shen established convergence for general initial data of bounded total variation and bounded away from 0. The analysis in [4, 5] relies on a change of variables that allows to rewrite the equation at the first line of (1.2) as a system with relaxation, and requires that $\eta(x) = e^x \mathbb{1}_{\mathbb{R}_-}(x)$, where $\mathbb{1}_{\mathbb{R}_-}$ denotes the characteristic function of the negative real axis. In [13] we established convergence under quite general assumptions on the convolution kernel η, but requiring more restrictive assumptions than in [4, 5] on the initial datum.

Without entering the details, an important technical point is that a key issue in all the above works [4, 5, 13, 23] is showing that, under the different assumptions considered in each paper, the total variation $\mathrm{TotVar}\,u_\varepsilon(t,\cdot)$ is uniformly bounded in ε and t. Owing to the Helly-Kolmogorov-Fréchet Theorem, this yields compactness of the family $\{u_\varepsilon(t,\cdot)\}$ (some more work is then required to show that u_ε converges to the entropy admissible solution of (1.3)). As a matter of fact, the prevailing feeling in the nonlocal conservation laws community was that the assumptions (2.4),(2.5) and (2.6) should suffice to establish a uniform bound on $\mathrm{TotVar}\,u_\varepsilon(t,\cdot)$: this was also supported by numerical evidence, see [1, 3]. However, in [13] we exhibit a counter-example showing that actually this is not always the case: in [13, §4] we construct an initial datum u_0 such that $\mathrm{TotVar}\,u_0 < +\infty$, but the solution of (1.2) satisfies

$$(2.8) \qquad\qquad \sup_{\varepsilon > 0} \mathrm{TotVar}\,u_\varepsilon(t,\cdot) = +\infty, \quad \text{for every } t > 0.$$

This happens when $V(u) = 1 - u$ and for a fairly large class of convolution kernels satisfying (2.6). In particular, the choice $\eta(x) = e^x \mathbb{1}_{\mathbb{R}_-}(x)$ is possible, and the reason why the counter-example does not contradict the uniform bounds established in [4, 13] is because the initial datum u_0 triggering the total variation blow-up attains the value 0, which is not allowed in [4, 13]. And indeed, by looking at the construction of the counter-example, one realizes that the fact that u_0 vanishes on suitable intervals is essential in the proof of (2.8).

We remark in passing that the maximum principle (2.7) yields compactnes in the L^∞ weak-$*$ topology, but, as pointed out before, owing to non-linearity weak convergence alone does not suffice to pass to the limit in (1.2) to get (1.3). Wrapping up, the total variation blow up (2.8) apparently sets severe constraints on the possibility of establishing convergence in the non-local to local limit. An elegant way out this obstruction was found in the paper [9]: rather than u_ε, the authors consider the convolution term

$$(2.9) \qquad w_\varepsilon(t, x) := u_\varepsilon * \eta_\varepsilon(t, x) = \frac{1}{\varepsilon} \int_x^{+\infty} \eta\left(\frac{x - y}{\varepsilon}\right) u_\varepsilon(t, y) dy$$

and prove that, under suitable assumptions, the total variation $\mathrm{TotVar}\, w_\varepsilon(t, \cdot)$ is a monotone non-increasing function of time, and as such uniformly bounded in ε and $t > 0$. In other words, the total variation of $u_\varepsilon(t, \cdot)$ may blow up, but considering the convolution term allows to gain a bit of regularity sufficient to establish total variation bounds. As a drawback, the analysis in [9], as the one in [4, 5], is restricted to the case $\eta(x) = e^x \mathbb{1}_{\mathbb{R}_-}(x)$, which entails the algebraic identity

$$u_\varepsilon = w_\varepsilon - \varepsilon \partial_x w_\varepsilon.$$

The above identity, in turn, allows the authors of [9] to find an equation for w_ε which does not contain terms involving u_ε (every time the authors have one such term, they replace it by using the above identity), and this is crucially used in the proof of the total variation bound. The convergence argument is then quickly concluded by relying on the argument in [4, 5]. See also [8] for other recent results on the non-local to local limit in the case of exponential kernels.

In the recent paper [14] we extend the analysis in [9] by removing the assumption that $\eta(x) = e^x \mathbb{1}_{\mathbb{R}_-}(x)$. More precisely, in [14] we only impose conditions (2.6), entirely natural in view of applications to traffic models, plus the convexity assumption

$$(2.10) \qquad\qquad\qquad\qquad \eta \text{ is convex on } \mathbb{R}_-,$$

that we discuss in the following. We now quote [14, Theorem 1.1].

Theorem 2.1. *Assume that u_0, V and η satisfy (2.4), (2.5), and (2.6), (2.10), respectively. Assume furthermore that $\mathrm{TotVar}\, u_0 < +\infty$. Then*

$$(2.11) \qquad\qquad \mathrm{TotVar}\, w_\varepsilon(t, \cdot) \leq \mathrm{TotVar}\, w_\varepsilon(0, \cdot) \quad \text{for every } \varepsilon > 0 \text{ and a.e. } t > 0,$$

where w_ε is the same as in (2.9).

Once more, we remark that all assumptions (2.4), (2.5), and (2.6) are entirely consistent with the traffic models framework. In the next two sections we provide a proof of Theorem 2.1 alternative to the original one in [14]. For the time being we point out that by relying on (2.11) and some further elementary arguments it is fairly easy to see that, up to subsequences,

$$(2.12) \qquad w_\varepsilon \to u \text{ strongly in } L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}), \qquad u_\varepsilon \overset{*}{\rightharpoonup} u \text{ weakly}^* \text{ in } L^\infty(\mathbb{R}_+ \times \mathbb{R}),$$

where u is a *distributional* solution of the Cauchy problem (1.3). What a priori is not at all clear is that u is the *entropy admissible* solution of (1.3): to get this, we introduce a new and fairly general entropy admissibility criterion for non-local to local limits, see [14, Theorem 1.2], which eventually concludes the convergence proof. By relying on an argument due to Kuznetsov [26] we also get a convergence rate, see [14, Theorem 1.3]. What is left to be discussed is (2.10), the only one among of our assumptions that is not entirely natural in view of the applications to traffic models: for instance, a convolution kernel commonly used in applications is $\eta(x) = \mathbb{1}_{[-1,0]}$, which violates (2.10). It turns

out that, if η does not satisfy (2.10), then our main compactness estimate (2.11) fails in general: an explicit counter-example is constructed in [14, §6].

To conclude our overview, we touch upon a recent work concerning a problem closely related to (1.2). In [21] the authors consider the "non-local in velocity" problem

$$(2.13) \qquad \begin{cases} \partial_t u_\varepsilon + \partial_x[(V(u_\varepsilon) * \eta_\varepsilon)u_\varepsilon] = 0 \\ u_\varepsilon(0, \cdot) = u_0 \end{cases} \qquad \text{with } \eta_\varepsilon(x) := \frac{1}{\varepsilon}\eta\left(\frac{x}{\varepsilon}\right),$$

where the convolution is applied to the velocity function directly, rather than on the solution u_ε. Note that (2.13) coincides with (1.2) in the linear case $V(u) = 1 - u$, but differs in general. In [21] the authors show that the solution u_ε of (1.2) converge to the entropy admissible solution of (1.2) if either u_0 is monotone or $\eta(x) = e^x \mathbb{1}_{\mathbb{R}_-}(x)$. See also [11, 20, 24] for other recent work concerning problems related to (1.2).

3. Proof of Theorem 2.1: heuristic argument

In this section we discuss an handwaving argument for the total variation estimate (2.11), which we hope provides the main idea underpinning the rigorous argument. The complete proof is then established in the next section.

3.1. **Preliminary results.** We briefly recall some known results (see for instance [14]) that we need in the following. First, owing to (2.7) we have

$$(3.1) \qquad\qquad 0 \leq w_\varepsilon(t, x) \leq 1 \text{ for a.e. } (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Second, w_ε is a Lipschitz continuous function, $w_\varepsilon \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R})$. To fix the notation, we also introduce the characteristic line $X_\varepsilon(\cdot, x, s)$, which is the solution of the Cauchy problem

$$(3.2) \qquad\qquad \begin{cases} \dfrac{dX_\varepsilon}{dt} = V(w_\varepsilon(t, X_\varepsilon)) \\[2mm] X_\varepsilon(s, x, s) = x. \end{cases}$$

A straightforward computation yields the expression for the material derivative of w_ε, namely

$$(3.3) \qquad \partial_t w_\varepsilon + V(w_\varepsilon)\partial_x w_\varepsilon = \frac{1}{\varepsilon^2}\int_x^{+\infty} \eta'\left(\frac{x - y}{\varepsilon}\right)[V(w_\varepsilon(t, x)) - V(w_\varepsilon(t, y))]u_\varepsilon(t, y)dy.$$

Also, one can also show that the map $g : \mathbb{R}_+ \to \mathbb{R}$ define by setting

$$(3.4) \qquad\qquad g(t) := \text{TotVar}\, w_\varepsilon(t, \cdot)$$

is Lipschitz continuous. To conclude, we state a known and elementary result on Lipschitz continuous functions that we need in the following, and provide the proof for the sake of completeness.

Lemma 3.1. *Let $I \subseteq \mathbb{R}$ be an open interval and $f, g : I \to \mathbb{R}$ two Lipschitz continuous functions such that $f \leq g$ on I. If f and g are both differentiable at the point $t_* \in I$ and $f(t_*) = g(t_*)$ then $f'(t_*) = g'(t_*)$.*

Proof. Up to replacing f by $f - g$, we can assume with no loss of generality that $g = 0$. Since $f \leq 0$ and $f(t_*) = 0$ then t_* is a point of maximum for f, which yields $f'(t_*) = 0$. $\qquad\square$

3.2. **The heuristic argument.** Recall (3.4) and assume for a moment that we have shown that

$$(3.5) \qquad\qquad g'(t_*) \leq 0$$

at every point t_* at which the map (3.4) is differentiable. An integration in time then yields (2.11).

In our heuristic argument we establish (3.5) at every point t_* of differentiability for g that satisfies some further suitable conditions. More precisely, we assume that at time t_* the map $w_\varepsilon(t_*, \cdot)$ is compactly supported and has a finite number of local maxima and local minima. To focus on the main issues and make the other details as simple as possible, we perform the explicit computations in the

FIGURE 1. In blue the function $w_\varepsilon(t_*, \cdot)$ in the heuristic argument.
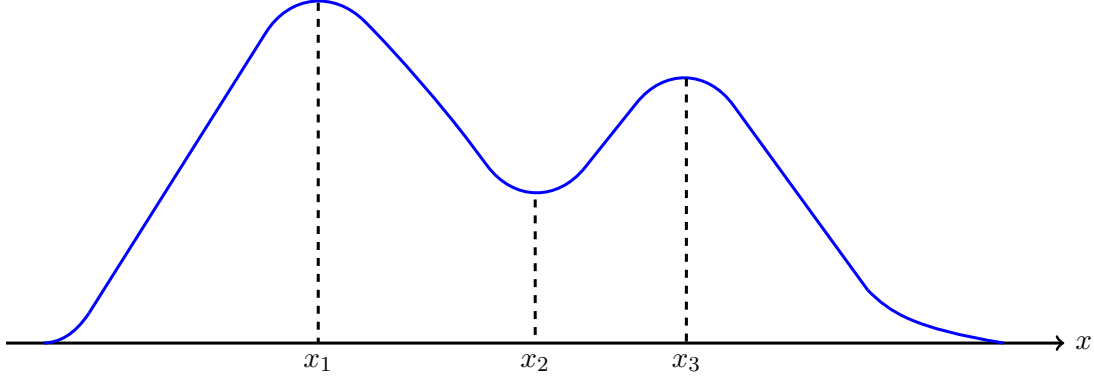
case where $w_\varepsilon(t_*, \cdot)$ has exactly two local maxima, which we term x_1 and x_3, and one local minimum, which we term x_2, satisfying

$$(3.6) \qquad\qquad x_1 < x_2 < x_3.$$

See Figure 1 for a representation. The analysis extends to the case of finitely many extrema, at the price of some elementary but rather tedious computations (see also the argument in §4). We now set

$$(3.7) \qquad f(t) := 2\big[w_\varepsilon(t_*, X_\varepsilon(t, x_1, t_*)) - w_\varepsilon(t_*, X_\varepsilon(t, x_2, t_*)) + w_\varepsilon(t_*, X_\varepsilon(t, x_3, t_*))\big],$$

where X_ε is the same as in (3.2), and recall (3.4). Note that $f(t) \le g(t)$ for every $t \ge 0$. Under our assumptions on $w_\varepsilon(t_*, \cdot)$ (see Figure 1) we also have

$$(3.8) \qquad g(t_*) = \mathrm{TotVar}\, w_\varepsilon(t_*, \cdot) = 2\big[w_\varepsilon(t_*, x_1) - w_\varepsilon(t_*, x_2) + w_\varepsilon(t_*, x_3)\big] = f(t_*).$$

By applying Lemma 3.1 we conclude that to establish (3.5) it suffices to show that

$$(3.9) \qquad\qquad f'(t_*) \le 0.$$

Towards this end, we recall the expression (3.3) for the material derivative, and conclude that

$$
\begin{aligned}
f'(t_*) =2\bigg[& \frac{1}{\varepsilon^2} \int_{x_1}^{+\infty} \eta'\left(\frac{x_1 - y}{\varepsilon}\right) [V(w_\varepsilon(t_*, x_1)) - V(w_\varepsilon(t_*, y))]u_\varepsilon(t_*, y)dy \\
& - \frac{1}{\varepsilon^2} \int_{x_2}^{+\infty} \eta'\left(\frac{x_2 - y}{\varepsilon}\right) [V(w_\varepsilon(t_*, x_2)) - V(w_\varepsilon(t_*, y))]u_\varepsilon(t_*, y)dy \\
& + \frac{1}{\varepsilon^2} \int_{x_3}^{+\infty} \eta'\left(\frac{x_3 - y}{\varepsilon}\right) [V(w_\varepsilon(t_*, x_3)) - V(w_\varepsilon(t_*, y))]u_\varepsilon(t_*, y)dy\bigg].
\end{aligned}
$$

We now recall (3.6) and rearrange the above terms as

$$f'(t_*) = 2\Bigg[\frac{1}{\varepsilon^2}\underbrace{\int_{x_1}^{x_2}\eta'\left(\frac{x_1-y}{\varepsilon}\right)[V(w_\varepsilon(t_*,x_1))-V(w_\varepsilon(t_*,y))]u_\varepsilon(t_*,y)dy}_{:=I_1}$$

$$+\frac{1}{\varepsilon^2}\underbrace{\int_{x_2}^{+\infty}\eta'\left(\frac{x_1-y}{\varepsilon}\right)[V(w_\varepsilon(t_*,x_1))-V(w_\varepsilon(t_*,x_2))]u_\varepsilon(t_*,y)dy}_{:=I_2}$$

$$+\frac{1}{\varepsilon^2}\underbrace{\int_{x_2}^{x_3}\left[\eta'\left(\frac{x_1-y}{\varepsilon}\right)-\eta'\left(\frac{x_2-y}{\varepsilon}\right)\right][V(w_\varepsilon(t_*,x_2))-V(w_\varepsilon(t_*,y))]u_\varepsilon(t_*,y)dy}_{:=I_3}$$

$$+\frac{1}{\varepsilon^2}\underbrace{\int_{x_3}^{+\infty}\left[\eta'\left(\frac{x_1-y}{\varepsilon}\right)-\eta'\left(\frac{x_2-y}{\varepsilon}\right)\right][V(w_\varepsilon(t_*,x_2))-V(w_\varepsilon(t_*,x_3))]u_\varepsilon(t_*,y)dy}_{:=I_4}$$

$$+\frac{1}{\varepsilon^2}\underbrace{\int_{x_3}^{+\infty}\left[\eta'\left(\frac{x_1-y}{\varepsilon}\right)-\eta'\left(\frac{x_2-y}{\varepsilon}\right)+\eta'\left(\frac{x_3-y}{\varepsilon}\right)\right][V(w_\varepsilon(t_*,x_3))-V(w_\varepsilon(t_*,y))]u_\varepsilon(t_*,y)dy}_{:=I_5}\Bigg].$$

To control I_1 and I_2, we recall that by assumption (see Figure 1)

$$w_\varepsilon(t_*,x_1) \geq w_\varepsilon(t_*,y) \text{ for every } y \in [x_1,x_2]$$

$$\overset{V'\leq 0}{\Longrightarrow} V(w_\varepsilon(t_*,x_1)) \leq V(w_\varepsilon(t_*,y)) \text{ for every } y \in [x_1,x_2].$$

Since $\eta' \geq 0$ owing to (2.6) and $u_\varepsilon \geq 0$ by (3.1), this yields $I_1 \leq 0$, $I_2 \leq 0$. To control I_3 and I_4, we point out that

$$w_\varepsilon(t_*,x_2) \leq w_\varepsilon(t_*,y) \text{ for every } y \in [x_2,x_3]$$

$$\overset{V'\leq 0}{\Longrightarrow} V(w_\varepsilon(t_*,x_2)) \geq V(w_\varepsilon(t_*,y)) \text{ for every } y \in [x_2,x_3].$$

Since by assumption (2.10) the function η' is monotone non-decreasing on $\mathbb{R}_-$, we have

$$\eta'\left(\frac{x_1-y}{\varepsilon}\right) \leq \eta'\left(\frac{x_2-y}{\varepsilon}\right)$$

and hence $I_3 \leq 0$, $I_4 \leq 0$. We are left to control I_5. By arguing as before, we have

$$w_\varepsilon(t_*,x_3) \geq w_\varepsilon(t_*,y) \text{ for every } y \in [x_3,+\infty[$$

$$\overset{V'\leq 0}{\Longrightarrow} V(w_\varepsilon(t_*,x_3)) \leq V(w_\varepsilon(t_*,y)) \text{ for every } y \in [x_3,+\infty[$$

and

$$\eta'\left(\frac{x_1-y}{\varepsilon}\right) - \eta'\left(\frac{x_2-y}{\varepsilon}\right) + \eta'\left(\frac{x_3-y}{\varepsilon}\right) \overset{\eta'\geq 0}{\geq} -\eta'\left(\frac{x_2-y}{\varepsilon}\right) + \eta'\left(\frac{x_3-y}{\varepsilon}\right) \overset{(2.10)}{\geq} 0.$$

This yields $I_5 \leq 0$ and hence concludes the proof of (3.9).

4. Rigorous proof of Theorem 2.1

In this section we provide the complete proof of (2.11). In §4.1 we establish it under some further assumptions on the data, which are then removed through a fairly standard approximation argument in §4.2.

4.1. **Proof under further assumptions on the data.** Since $\mathrm{TotVar}\, u_0 < +\infty$, then there are $u_-, u_+ \in [0, 1]$ such that

(4.1)
$$\lim_{x \to -\infty} u_0(x) = u_-, \qquad \lim_{x \to +\infty} u_0(x) = u_+.$$

In this paragraph we establish (2.11) under the further assumptions

(4.2)
$$\eta \in C^2(]-\infty, 0[), \qquad \eta'' \in L^1(]-\infty, 0[)$$

and

(4.3)
$$u_0 \in \mathrm{Lip}(\mathbb{R}), \qquad u_0(x) = u_+ \text{ for every } x \geq R,$$

for some suitable $R > 0$. We now fix $T > 0$ once and for all and establish the inequality in (2.11) for every $t \in [0, T]$. By the arbitrariness of T, this implies that (2.11) holds for every $t > 0$. We proceed according to the following steps.

STEP 1: we point out that (4.3) yields

(4.4)
$$w_\varepsilon(t, x) = u_+ \text{ for every } t > 0 \text{ and } x \geq R + V(u_+)t.$$

Very loosely speaking, this is due to the fact that, by the kernel anisotropy, along the characteristic line $X_\varepsilon(\cdot, R, 0)$ the convolution term only "sees" the value u_+, which in turn owing to (3.2) implies that the speed of this characteristic line is exactly $V(u_+)$. Property (4.4) can then be rigorously established by following the same argument sketched after the statement of [13, Lemma 13]. Once can also show that

(4.5)
$$\lim_{x \to -\infty} w_\varepsilon(t, x) = u_- \quad \text{for every } t \in [0, T],$$

where u_- is the same as in (4.1).

STEP 2: we recall that owing to (2.4) $V(0)$ is the highest possible value attained by $V(w_\varepsilon)$, we fix $N \in \mathbb{N}$ such that $N \geq R$ and set

$$\mathrm{TotVar}_N w_\varepsilon(t, \cdot) :=$$

(4.6)
$$\sup\left\{ |w_\varepsilon(t, x_1) - u_-| + \sum_{i=1}^{N-1} |w_\varepsilon(t, x_{i+1}) - w_\varepsilon(t, x_i)| : X_\varepsilon(t, -N, 0) \leq x_1 \leq \ldots \leq x_N = N + V(u_+)t \right\},$$

where X_ε is the same as in (3.2) and, owing to (4.4), $N + V(u_+)t = X_\varepsilon(t, N, 0)$. Owing to (4.5) we have

(4.7)
$$\mathrm{TotVar}\, w_\varepsilon(t, \cdot) = \lim_{N \to +\infty} \mathrm{TotVar}_N w_\varepsilon(t, \cdot) = \sup_{N \in \mathbb{N}} \mathrm{TotVar}_N w_\varepsilon(t, \cdot).$$

By STEP 2 in the proof of [14, Theorem 1.1], we have $\mathrm{TotVar}\, w_\varepsilon(t, \cdot) < +\infty$ for every $t \geq 0$, and owing to (4.7) this implies $\mathrm{TotVar}_N w_\varepsilon(t, \cdot) < +\infty$ for every $t \geq 0$. We now establish some further properties of $\mathrm{TotVar}_N w_\varepsilon(t, \cdot)$.

STEP 2A: we show that the sup in (4.6) is attained, i.e. for every $t \in [0, T]$ and every $N \geq R$ there are N points $x_1 \leq x_2 \cdots \leq x_N$ such that

(4.8)
$$\mathrm{TotVar}_N w_\varepsilon(t, \cdot) = |w_\varepsilon(t, x_1) - u_-| + \sum_{i=1}^{N-1} |w_\varepsilon(t, x_{i+1}) - w_\varepsilon(t, x_i)|.$$

Fix $t \in [0, T]$, $N \geq R$ and a maximizing sequence $x_{1k}, \ldots, x_{Nk}$ for $|w_\varepsilon(t, x_1) - u_-| + \sum_{i=1}^{N-1} |w_\varepsilon(t, x_{i+1}) - w_\varepsilon(t, x_i)|$. We now point out that the points are all confined in the compact set $[X_\varepsilon(t, -N, 0), N + V(u_+)t]$, and conclude that up to subsequences the maximizing sequences converge to some limit points $x_1, \ldots, x_N$ satisfying (4.8).

STEP 2B: we show that the map $t \mapsto \mathrm{TotVar}_N w_\varepsilon(t, \cdot)$ is Lipschitz continuous, and henceforth a.e. differentiable. To this end we fix $t_1, t_2 \in [0, T]$ and assume without loss of generality that $\mathrm{TotVar}_N w_\varepsilon(t_1, \cdot) \leq$

$\text{TotVar}_N w_\varepsilon(t_2, \cdot)$. We rely on STEP 2A and find $x_1, \ldots, x_N$, such that (4.8) holds at $t = t_1$. By recalling definition (4.6) we infer that

$$\text{TotVar}_N w_\varepsilon(t_2, \cdot) \leq |w_\varepsilon(t_2, X_\varepsilon(t_2, x_1, t_1)) - u_-|$$
$$+ \sum_{i=1}^{N-1} |w_\varepsilon(t_2, X_\varepsilon(t_2, x_{i+1}, t_1)) - w_\varepsilon(t_2, X_\varepsilon(t_2, x_i, t_1))|,$$

which implies that

$$|\text{TotVar}_N w_\varepsilon(t_2, \cdot) - \text{TotVar}_N w_\varepsilon(t_1, \cdot)| = \text{TotVar}_N w_\varepsilon(t_2, \cdot) - \text{TotVar}_N w_\varepsilon(t_1, \cdot)$$

$$\leq |w_\varepsilon(t_2, X_\varepsilon(t_2, x_1, t_1)) - u_-| - |w_\varepsilon(t_1, x_1) - u_-|$$

$$+ \sum_{i=1}^{N-1} |w_\varepsilon(t_2, X_\varepsilon(t_2, x_{i+1}, t_1)) - w_\varepsilon(t_2, X_\varepsilon(t_2, x_i, t_1))| - |w_\varepsilon(t_1, x_{i+1}) - w_\varepsilon(t_1, x_i)|$$

$$\leq |w_\varepsilon(t_2, X_\varepsilon(t_2, x_1, t_1)) - w_\varepsilon(t_1, x_1)| + \sum_{i=1}^{N-1} |w_\varepsilon(t_2, X_\varepsilon(t_2, x_{i+1}, t_1)) - w_\varepsilon(t_1, x_{i+1})|$$

$$+ \sum_{i=1}^{N-1} |w_\varepsilon(t_2, X_\varepsilon(t_2, x_i, t_1)) - w_\varepsilon(t_1, x_i)|.$$

Since the functions X_ε and w_ε are both Lipschitz continuous this concludes STEP 2C.
STEP 3: we now fix $N \geq R$ and $t_* \in]0, T[$ such that the map $t \mapsto \text{TotVar}_N w_\varepsilon(t, \cdot)$ is differentiable at t_*. Assume for a moment that we have shown that

$$(4.9) \qquad \frac{d}{dt} \text{TotVar}_N w_\varepsilon(t, \cdot) \bigg|_{t=t_*} \leq 0.$$

Owing to the arbitrariness of t_* by integrating in time we get

$$\text{TotVar}_N w_\varepsilon(t, \cdot) \leq \text{TotVar}_N w_\varepsilon(0, \cdot) \overset{(4.7)}{\leq} \text{TotVar } w_\varepsilon(0, \cdot)$$

and owing to (4.7) this yields (2.11). To establish (4.9) we set

$$(4.10) \qquad g(t) := \text{TotVar}_N w_\varepsilon(t, \cdot)$$

and point out that, owing to STEP 2B, the function $g : [0, T] \to \mathbb{R}_+$ is Lipschitz continuous. Next, we fix t_* at which g is differentiable, recall STEP 2A and fix $x_1, \ldots, x_N = N + V(u_+)t_*$ such that

$$(4.11) \qquad \text{TotVar}_N w_\varepsilon(t_*, \cdot) = |w_\varepsilon(t_*, x_1) - u_-| + \sum_{i=1}^{N-1} |w_\varepsilon(t_*, x_{i+1}) - w_\varepsilon(t_*, x_i)|.$$

Up to removing some intermediate points and relabeling the points (if needed) we find $m \leq N$ such that $x_1 < x_2 < \cdots < x_m = N + V(u_+)t$ are all distinct points and one of the following two cases is verified:

    i) either

$$(4.12) \qquad w_\varepsilon(t_*, x_1) \geq u_-, \quad w_\varepsilon(t_*, x_i) \geq w_\varepsilon(t_*, y) \text{ for every } y \in [x_i, x_{i+1}], i = 1, \ldots, m-1 \text{ odd}$$

    and

$$(4.13) \qquad w_\varepsilon(t_*, x_i) \leq w_\varepsilon(t_*, y) \text{ for every } y \in [x_i, x_{i+1}] \text{ and } i = 2, \ldots, m-1 \text{ even},$$

    ii) or

$$w_\varepsilon(t_*, x_1) \leq u_-, \quad w_\varepsilon(t_*, x_i) \leq w_\varepsilon(t_*, y) \text{ for every } y \in [x_i, x_{i+1}], i = 1, \ldots, m-1 \text{ odd}$$

and

$$w_\varepsilon(t_*, x_i) \geq w_\varepsilon(t_*, y) \text{ for every } y \in [x_i, x_{i+1}] \text{ and } i = 2, \ldots, m - 1 \text{ even,}$$

We now focus on case i) above. The proof in case ii) is similar. Note that in case i), recalling that $w_\varepsilon(t_*, x_m) = u_+$, we have

$$g(t_*) \overset{(4.10)}{=} \mathrm{TotVar}_N w_\varepsilon(t_*, \cdot) = w_\varepsilon(t_*, x_1) - u_- + \sum_{i=1}^{m-1} |w_\varepsilon(t_*, x_{i+1}) - w_\varepsilon(t_*, x_i)|$$

(4.14)
$$= -u_- + \begin{cases} -u_+ + 2 \sum_{i=1}^{m-1} (-1)^{i+1} w_\varepsilon(t_*, x_i) & m \text{ even} \\ u_+ + 2 \sum_{i=1}^{m-1} (-1)^{i+1} w_\varepsilon(t_*, x_i) & m \text{ odd.} \end{cases}$$

We recall the notation (3.2) and set

(4.15)
$$f(t) := -u_- + \begin{cases} -u_+ + 2 \sum_{i=1}^{m-1} (-1)^{i+1} w_\varepsilon(t_*, X_\varepsilon(t, x_i, t_*)) & m \text{ even} \\ u_+ + 2 \sum_{i=1}^{m-1} (-1)^{i+1} w_\varepsilon(t_*, X_\varepsilon(t, x_i, t_*)) & m \text{ odd.} \end{cases}$$

Note that $f(t) \leq g(t)$ for every $t \in [0, T]$, where g is the same as in (4.10). By recalling (4.14) and Lemma 3.1 we conclude that to establish (4.9) it suffices to show that $f'(t_*) \leq 0$.

STEP 4: by combining (3.3) and (4.15) we get

(4.16)
$$f'(t_*) = 2 \sum_{i=1}^{m-1} \frac{(-1)^{i+1}}{\varepsilon^2} \int_{x_i}^{+\infty} \eta'\left(\frac{x-y}{\varepsilon}\right) [V(w_\varepsilon(t_*, x_i)) - V(w_\varepsilon(t, y))] u_\varepsilon(t_*, y) dy$$
$$= \frac{2}{\varepsilon^2} \sum_{i=1}^{m-1} \int_{x_i}^{x_{i+1}} \sigma_i(t_*, y) u_\varepsilon(t_*, y) dy + \frac{2}{\varepsilon^2} \int_{x_m}^{+\infty} \sigma_m(t_*, y) u_\varepsilon(t_*, y) dy$$

where

(4.17)
$$\sigma_1(t_*, y) = \eta'\left(\frac{x_1 - y}{\varepsilon}\right) [V(w_\varepsilon(t_*, x_1)) - V(w_\varepsilon(t_*, y))]$$

and

(4.18)
$$\sigma_i(t_*, y) = \sum_{j=1}^{i-1} \left[\sum_{k=1}^{j} (-1)^{k+1} \eta'\left(\frac{x_k - y}{\varepsilon}\right) \right] [V(w_\varepsilon(t_*, x_j)) - V(w_\varepsilon(t_*, x_{j+1}))]$$
$$+ \left[\sum_{k=1}^{i} (-1)^{k+1} \eta'\left(\frac{x_k - y}{\varepsilon}\right) \right] [V(w_\varepsilon(t_*, x_i)) - V(w_\varepsilon(t_*, y))] \quad i \geq 2.$$

To verify the above formula we rely on an induction argument. By a straightforward computation, the formula is verified for $i = 2$. Let us now assume that it holds at the i-th step, and check that it is also

verified at the $(i+1)$-th. We have

$$\sigma_{i+1}(t_*, y) = \sigma_i(t_*, y) + (-1)^{i+2}\eta'\left(\frac{x_{i+1} - y}{\varepsilon}\right)[V(w_\varepsilon(t_*, x_{i+1})) - V(w_\varepsilon(t_*, y))]$$

$$\stackrel{(4.18)}{=} \sum_{j=1}^{i-1}\left[\sum_{k=1}^{j}(-1)^{k+1}\eta'\left(\frac{x_k - y}{\varepsilon}\right)\right][V(w_\varepsilon(t_*, x_j)) - V(w_\varepsilon(t_*, x_{j+1}))]$$

$$+ \left[\sum_{k=1}^{i}(-1)^{k+1}\eta'\left(\frac{x_k - y}{\varepsilon}\right)\right][V(w_\varepsilon(t_*, x_i)) - V(w_\varepsilon(t_*, y))]$$

$$+ (-1)^{i+2}\eta'\left(\frac{x_{i+1} - y}{\varepsilon}\right)[V(w_\varepsilon(t_*, x_{i+1})) - V(w_\varepsilon(t_*, y))]$$

and hence

$$\sigma_{i+1}(t_*, y) = \sum_{j=1}^{i-1}\left[\sum_{k=1}^{j}(-1)^{k+1}\eta'\left(\frac{x_k - y}{\varepsilon}\right)\right][V(w_\varepsilon(t_*, x_j)) - V(w_\varepsilon(t_*, x_{j+1}))]$$

$$+ \left[\sum_{k=1}^{i}(-1)^{k+1}\eta'\left(\frac{x_k - y}{\varepsilon}\right)\right][V(w_\varepsilon(t_*, x_i)) - V(w_\varepsilon(t_*, x_{i+1}))]$$

$$+ \sum_{k=1}^{i+1}(-1)^{k+1}\eta'\left(\frac{x_k - y}{\varepsilon}\right)[V(w_\varepsilon(t_*, x_{i+1})) - V(w_\varepsilon(t_*, y))],$$

which is consistent with (4.18).

STEP 5: we recall that $u_\varepsilon \geq 0$ owing to (3.1), and by plugging this inequality in (4.16) we conclude that to show that $f'(t_*) \leq 0$ it suffices to show that

$$(4.19) \qquad \sigma_i(t_*, \cdot) \leq 0 \text{ a.e. on }]x_i, x_{i+1}[\text{ for } i = 1, \ldots, m-1, \qquad \sigma_m(t_*, \cdot) \leq 0 \text{ on }]x_m, +\infty[.$$

To establish (4.19) we first focus on $i = 1$. Note that (4.12) yields $w_\varepsilon(t_*, y) \leq w_\varepsilon(t_*, x_1)$ for every $y \in [x_1, x_2]$ and by using the inequality $V' \leq 0$ we conclude that $V(w_\varepsilon(t_*, x_1)) \leq V(w_\varepsilon(t_*, y))$ for every $y \in [x_1, x_2]$. By recalling the inequality $\eta' \geq 0$ we conclude that $\sigma_1(t_*, \cdot) \leq 0$ on $[x_1, x_2]$.

We now consider the case $i > 1$. Assume for a moment that we have established the inequalities

$$(4.20) \qquad \sum_{k=1}^{j}(-1)^{k+1}\eta'\left(\frac{x_k - y}{\varepsilon}\right) \geq 0 \quad \text{if } j = 1, \ldots, m, \ j \text{ odd}$$

and

$$(4.21) \qquad \sum_{k=1}^{j}(-1)^{k+1}\eta'\left(\frac{x_k - y}{\varepsilon}\right) \leq 0 \quad \text{if } j = 2, \ldots, m, \ j \text{ even}.$$

By combining (4.20) with (4.12), (4.21) with (4.13) and recalling the inequality $V' \leq 0$ we get that the first and (choosing $j = i$) the second term in (4.18) are nonpositive. Hence, to establish (4.19) we are left to establish (4.20) and (4.21). We first establish (4.20). If $j = 1$, then

$$(4.22) \qquad \sum_{k=1}^{1}(-1)^{k+1}\eta'\left(\frac{x_k - y}{\varepsilon}\right) = \eta'\left(\frac{x_1 - y}{\varepsilon}\right) \stackrel{(2.6)}{\geq} 0.$$

If $j \geq 3$ is odd, then we use the equality

$$(4.23) \qquad \sum_{k=1}^{j}(-1)^{k+1}\eta'\left(\frac{x_k - y}{\varepsilon}\right) = \eta'\left(\frac{x_1 - y}{\varepsilon}\right) + \sum_{h=2,\,h \text{ even}}^{j-1}\left[\eta'\left(\frac{x_{h+1} - y}{\varepsilon}\right) - \eta'\left(\frac{x_h - y}{\varepsilon}\right)\right]$$

$$\text{for every } j = 3, \ldots, m, \ j \text{ odd}.$$

Since $\eta' \geq 0$ and η' is a a non-decreasing function, each of the terms in the above sum is non-negative, which combined with (4.22) establishes (4.20). We now turn to (4.21). We have

$$\sum_{k=1}^{j}(-1)^{k+1}\eta'\left(\frac{x_k - y}{\varepsilon}\right) = \sum_{h=1,\, h \text{ odd}}^{j-1}\left[\eta'\left(\frac{x_h - y}{\varepsilon}\right) - \eta'\left(\frac{x_{h+1} - y}{\varepsilon}\right)\right] \ \forall\, j = 2,\ldots,m,\ j \text{ even.}$$

Owing to the assumption that η' is a non-decreasing function, each of the terms in the above sum is non-positive and this establishes (4.21). This concludes the proof of the inequality $f'(t_*) \leq 0$ and hence of (2.11).

4.2. **Conclusion of the proof of** (2.11)**.** To complete the proof of (2.11) we are left to remove the assumptions (4.2) and (4.3). We can use the same approximation argument as in STEP 3 of the proof of [14, Theorem 1.1], the only difference is that we also have to get the second condition in (4.3). To achieve this, we can define u_{0n} as the convolution of the compactly supported kernel ρ_n with

$$\widetilde{u}_{0n} := \left\{ \begin{array}{ll} u_0(x) & x \leq n \\ u_+ & x \geq n \end{array} \right.$$

In this way (4.3) is verified by u_{0n} for some R depending on n and on the size of the support of ρ_n.

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REFERENCES

[1] P. Amorim, R. M. Colombo, and A. Teixeira. On the numerical integration of scalar nonlocal conservation laws. *ESAIM Math. Model. Numer. Anal.*, 49(1):19–37, 2015.

[2] F. Betancourt, R. Bürger, K. H. Karlsen, and E. M. Tory. On nonlocal conservation laws modelling sedimentation. *Nonlinearity*, 24(3):855–885, 2011.

[3] S. Blandin and P. Goatin. Well-posedness of a conservation law with non-local flux arising in traffic flow modeling. *Numer. Math.*, 132(2):217–241, 2016.

[4] A. Bressan and W. Shen. On traffic flow with nonlocal flux: a relaxation representation. *Arch. Ration. Mech. Anal.*, 237(3):1213–1236, 2020.

[5] A. Bressan and W. Shen. Entropy admissibility of the limit solution for a nonlocal model of traffic flow. *Commun. Math. Sci.*, 19(5):1447–1450, 2021.

[6] P. Calderoni and M. Pulvirenti. Propagation of chaos for Burgers' equation. *Ann. Inst. H. Poincaré Sect. A (N.S.)*, 39(1):85–97, 1983.

[7] F. A. Chiarello. An overview of non-local traffic flow models. In *Mathematical descriptions of traffic flow: micro, macro and kinetic models. Selected papers based on the presentations of the mini-symposium at ICIAM 2019, Valencia, Spain, July 2019*, pages 79–91. Cham: Springer, 2021.

[8] G.M. Coclite, M Colombo, G Crippa, N. De Nitti, A. Keimer, E. Marconi, L. Pflug, and L.V. Spinolo. Oleĭnik-type estimates for nonlocal conservation laws and applications to the nonlocal-to-local limit. *Preprint ArXiv:2304.01309*, 2023.

[9] G.M. Coclite, J.-M. Coron, N. De Nitti, A. Keimer, and L. Pflug. A general result on the approximation of local conservation laws by nonlocal conservation laws: The singular limit problem for exponential kernels. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 2022.

[10] G.M. Coclite, N. De Nitti, A. Keimer, and L. Pflug. Singular limits with vanishing viscosity for nonlocal conservation laws. *Nonlinear Anal.*, 211, 2021.

[11] G.M. Coclite, K.H. Karlsen, and N.H. Risebro. A nonlocal lagrangian traffic flow model and the zero-filter limit. *Preprint ArXiv:2302.03889*, 2023.

[12] M. Colombo, G. Crippa, M. Graff, and L. V. Spinolo. On the role of numerical viscosity in the study of the local limit of nonlocal conservation laws. *ESAIM, Math. Model. Numer. Anal.*, 55(6):2705–2723, 2021.

[13] M. Colombo, G. Crippa, E. Marconi, and L. V. Spinolo. Local limit of nonlocal traffic models: convergence results and total variation blow-up. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 38(5):1653–1666, 2021.

[14] M. Colombo, G. Crippa, E. Marconi, and L. V. Spinolo. Nonlocal traffic models with general kernels: singular limit, entropy admissibility, and convergence rate. *Arch. Ration. Mech. Anal.*, 247(2):32, 2023.

[15] M. Colombo, G. Crippa, and L. V. Spinolo. On the singular local limit for conservation laws with nonlocal fluxes. *Arch. Rat. Mech. Anal.*, 233(3):1131–1167, 2019.

[16] R. M. Colombo, M. Garavello, and M. Lécureux-Mercier. A class of nonlocal models for pedestrian traffic. *Math. Models Methods Appl. Sci.*, 22(4):1150023, 34, 2012.

[17] R. M. Colombo, M. Herty, and M. Mercier. Control of the continuity equation with a non local flow. *ESAIM Control Optim. Calc. Var.*, 17(2):353–379, 2011.

[18] G. Crippa and M. Lécureux-Mercier. Existence and uniqueness of measure solutions for a system of continuity equations with non-local flow. *NoDEA Nonlinear Differential Equations Appl.*, 20(3):523–537, 2013.

[19] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, fourth edition, 2016.

[20] Q. Du, K. Huang, J. Scott, and W. Shen. A space-time nonlocal traffic flow model: relaxation representation and local limit. *Discrete Contin. Dyn. Syst.*, 43(9):3456–3484, 2023.

[21] J. Friedrich, S. Göttlich, A. Keimer, and L. Pflug. Conservation laws with nonlocal velocity - the singular limit problem. *Preprint ArXiv:2210.12141*, 2022.

[22] A. Keimer and L. Pflug. Existence, uniqueness and regularity results on nonlocal balance laws. *J. Differential Equations*, 263(7):4023–4069, 2017.

[23] A. Keimer and L. Pflug. On approximation of local conservation laws by nonlocal conservation laws. *J. Math. Anal. Appl.*, 475(2):1927–1955, 2019.

[24] A. Keimer and L. Pflug. On the singular limit problem for nonlocal conservation laws: A general approximation result for kernels with fixed support. *Preprint ArXiv:2310.09041*, 2023.

[25] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[26] N. N. Kuznetsov. Accuracy of some approximate methods for computing the weak solutions of a first-order quasi-linear equation. *U.S.S.R. Comput. Math. Math. Phys.*, 16(6):105–119, 1978.

[27] M. Lighthill and G. Whitham. On kinematic waves. II. A theory of traffic flow on long crowded roads. *Proceedings of the Royal Society of London: Series A.*, 229:317–345, 1955.

[28] P. I. Richards. Shock waves on the highway. *Operations Res.*, 4:42–51, 1956.

[29] K. Zumbrun. On a nonlocal dispersive equation modeling particle suspensions. *Quart. Appl. Math.*, 57(3):573–600, 1999.

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