The Ramanujan Journal (2024) 63:839–872
https://doi.org/10.1007/s11139-023-00790-6

Infinite product representations of some q-series

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Received: 11 February 2023 / Accepted: 8 September 2023 / Published online: 16 October 2023

Abstract

For integers a and b (not both 0) we define the integers $c(a, b; n)$ $(n = 0, 1, 2, \ldots)$ by

$$\sum_{n=0}^{\infty} c(a, b; n) q^n = \prod_{n=1}^{\infty} \left(1 - q^n\right)^a (1 - q^{2n})^b \quad (|q| < 1).$$

These integers include the numbers $t_k(n) = c(-k, 2k; n)$, which count the number of representations of n as a sum of k triangular numbers, and the numbers $(-1)^n r_k(n) = c(2k, -k; n)$, where $r_k(n)$ counts the number of representations of n as a sum of k squares. A computer search was carried out for integers a and b, satisfying $-24 \leq a, b \leq 24$, such that at least one of the sums

$$\sum_{n=0}^{\infty} c(a, b; 3n + j) q^n, \quad j = 0, 1, 2, \tag{0.1}$$

is either zero or can be expressed as a nonzero constant multiple of the product of a power of q and a single infinite product of factors involving powers of $1 - q^{rn}$ with $r \in \{1, 2, 3, 4, 6, 8, 12, 24\}$ for all powers of q up to q^{1000}. A total of 84 such

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candidate identities involving 56 pairs of integers (a, b) all satisfying $a \equiv b \pmod 3$ were found and proved in a uniform manner. The proof of these identities is extended to establish general formulas for the sums (0.1). These formulas are used to determine formulas for the sums

$$\sum_{n=0}^{\infty} t_k(3n+j)q^n, \quad \sum_{n=0}^{\infty} r_k(3n+j)q^n, \quad j = 0, 1, 2.$$

Keywords Generating functions · Infinite products · q-Series · Sum of squares · Sums of triangular numbers

Mathematics Subject Classification 05A15 · 05A19 · 05A30 · 11D85 · 11E25

1 Introduction

Throughout this paper q denotes a complex variable satisfying $|q| < 1$. For arbitrary integers a and b (not both 0), we define the integers $c(a, b; n)$ $(n = 0, 1, 2, \ldots)$ by

$$\sum_{n=0}^{\infty} c(a, b; n)q^n = \prod_{n=1}^{\infty}(1-q^n)^a(1-q^{2n})^b, \tag{1.1}$$

so that $c(a, b; 0) = 1$, $c(a, b; 1) = -a$, $c(a, b; 2) = (a^2 - 3a - 2b)/2$.

Our first interest lies in determining those integers a and b for which at least one of the sums

$$\sum_{n=0}^{\infty} c(a, b; 3n)q^n, \quad \sum_{n=0}^{\infty} c(a, b; 3n+1)q^n, \quad \sum_{n=0}^{\infty} c(a, b; 3n+2)q^n \tag{1.2}$$

is either zero or can be expressed as a nonzero constant multiple of the product of a power of q and a single infinite product of factors involving powers of $1 - q^{rn}$ with $r \in \{1, 2, 3, 4, 6, 8, 12, 24\}$. A computer search was carried out by the first author for integers a and b in the range $-24 \le a, b \le 24$. The search was conducted utilizing the Maple `qseries` package by Garvan [1]. Eighty-four possible identities were found. These include four candidate "zero" identities, namely

$$\begin{cases} \sum_{n=0}^{\infty} c(0, 3; 3n+1)q^n = 0, & \sum_{n=0}^{\infty} c(-1, 2; 3n+2)q^n = 0, \\ \sum_{n=0}^{\infty} c(2, -1; 3n+2)q^n = 0, & \sum_{n=0}^{\infty} c(3, 0; 3n+2)q^n = 0. \end{cases}$$

The 84 identities involve 56 pairs (a, b) of integers. All of these integers a and b satisfy $a \equiv b \pmod 3$ and the 80 "nonzero" identities involve only products of powers of $1 - q^{rn}$ with $r \in \{1, 2, 3, 6\}$. The 84 identities are listed in Theorem 1.3. Our proof of theses identities is given in Sect. 5 and makes use of two eta quotient identities due to Kac [2] as well as the (p, k)-method introduced by Alaca et al. [3].

Our second interest lies in determining explicit evaluations of the three sums in (1.2). This is carried out in Sect. 6, see Theorem 6.1. This theorem is applied to the evaluation of $\sum_{n=0}^{\infty} t_k(3n+j)q^n \quad (j=0,1,2)$ (Theorem 6.2) and $\sum_{n=0}^{\infty} r_k(3n+j)q^n \quad (j=0,1,2)$ (Theorem 6.3), where $t_k(n)$ and $r_k(n)$ were defined in the abstract.

We begin by introducing a little notation and stating the two Kac identities that we use, see Theorem 1.1. For z belonging to the Poincaré upper half plane $\mathcal{H}$ of complex numbers with imaginary part of z greater than zero, the Dedekind eta function is defined by

$$\eta(z) := e^{\frac{2\pi i z}{24}} \prod_{n=1}^{\infty}(1-e^{2\pi i n z}). \tag{1.3}$$

The function $\eta(z)$ is holomorphic on $\mathcal{H}$ and has no zeros on $\mathcal{H}$. It is a modular form of weight $\frac{1}{2}$ and level 1 for a certain character of order 24. If we set $q=e^{2\pi i z}$ for $z \in \mathcal{H}$, we have $|q|<1$ and (1.3) becomes

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n). \tag{1.4}$$

Introducing the shorthand

$$E_k(q) = E_k := \prod_{n=1}^{\infty}(1-q^{kn}), \tag{1.5}$$

where k is an arbitrary positive integer, we deduce from (1.4) and (1.5) that

$$\eta(kz) = q^{\frac{k}{24}} E_k. \tag{1.6}$$

An eta quotient is a product $\prod_k \eta^{a_k}(kz)$, where k runs through a finite set of positive integers and the a_k are integers (positive, negative, or zero). Arising out of his work on the Macdonald identities [4], Kac [2, p. 122] (see also [5, Theorem 8.2, pp. 115–116]) has given four eta quotients explicitly as theta series. Of these we require the following two identities, namely

$$E_1 E_2^{-1} E_3^{-1} E_6^2 = \sum_{k=-\infty}^{\infty} (-1)^k q^{3k^2+2k} \tag{1.7}$$

and

$$E_1^{-1} E_2 E_3^2 E_6^{-1} = \sum_{k=-\infty}^{\infty} q^{\frac{3k^2+k}{2}}. \tag{1.8}$$

In Sect. 2 we use (1.7) and (1.8) to prove the Kac identities stated in Theorem 1.1.

Theorem 1.1 (Kac Identities)

$$\text{(i)}\quad E_1^{-1}E_2^2 = E_3^{-1}E_6E_9^2E_{18}^{-1} + qE_9^{-1}E_{18}^2.$$
$$\text{(ii)}\quad E_1^2E_2^{-1} = E_9^2E_{18}^{-1} - 2qE_3E_6^{-1}E_9^{-1}E_{18}^2.$$

In addition to these formulas, which express $E_1^{-1}E_2^2$ and $E_1^2E_2^{-1}$ as linear polynomials in q whose coefficients only involve E_3, E_6, E_9, and E_{18}, we require the companion formulas for $E_1E_2^{-2}$ and $E_1^{-2}E_2$. (We recognize that in regarding (i) and (ii) as linear polynomials in q we are ignoring the dependence of E_3, E_6, E_9, and E_{18} upon q.) These formulas are stated in Theorem 1.2.

Theorem 1.2 (Companion Formulas to Kac's Identities)

$$\text{(i)}\quad E_1E_2^{-2} = E_3^2E_6^{-6}E_9^3 - qE_3^3E_6^{-7}E_{18}^3 + q^2E_3^4E_6^{-8}E_9^{-3}E_{18}^6.$$
$$\text{(ii)}\quad E_1^{-2}E_2 = E_3^{-8}E_6^4E_9^6E_{18}^{-3} + 2qE_3^{-7}E_6^3E_9^3 + 4q^2E_3^{-6}E_6^2E_{18}^3.$$

Theorem 1.2 is proved in Sect. 3. In the course of the proof two identities involving E_1, E_2, E_3, and E_6 are needed. These identities are given in Proposition 1.1, and proved in Sect. 4 using the (p, k)-method.

Proposition 1.1 *The following identities hold*

$$\text{(i)}\quad E_1E_2^3E_3^9 + qE_1^4E_6^9 = E_2^8E_3^4E_6.$$
$$\text{(ii)}\quad E_2^4E_3^9 - 8qE_1^3E_2E_6^9 = E_1^8E_3E_6^4.$$

We next describe our strategy for proving the 84 identities of Theorem 1.3. Let a and b be integers such that $a \equiv b \pmod 3$ so that $(a+2b)/3$ and $(2a+b)/3$ are integers (positive, negative, or zero). Guided by Theorems 1.1 and 1.2, we define $r_{a,b}(q)$ and $s_{a,b}(q)$ by

$$r_{a,b}(q) := \begin{cases} E_3^{-1}E_6E_9^2E_{18}^{-1} + qE_9^{-1}E_{18}^2 & \text{if } a+2b \geq 0, \\ E_3^2E_6^{-6}E_9^3 - qE_3^3E_6^{-7}E_{18}^3 + q^2E_3^4E_6^{-8}E_9^{-3}E_{18}^6 & \text{if } a+2b < 0, \end{cases} \tag{1.9}$$

and

$$s_{a,b}(q) := \begin{cases} E_9^2E_{18}^{-1} - 2qE_3E_6^{-1}E_9^{-1}E_{18}^2 & \text{if } 2a+b \geq 0, \\ E_3^{-8}E_6^4E_9^6E_{18}^{-3} + 2qE_3^{-7}E_6^3E_9^3 + 4q^2E_3^{-6}E_6^2E_{18}^3 & \text{if } 2a+b < 0. \end{cases} \tag{1.10}$$

We regard $r_{a,b}(q)$ and $s_{a,b}(q)$ as polynomials in q of degree 1 or 2 with coefficients depending upon E_3, E_6, E_9 and E_{18}. As $E_m \neq 0$ for any positive integer m, we have

$$\deg r_{a,b}(q) = \begin{cases} 1 & \text{if } a+2b \geq 0, \\ 2 & \text{if } a+2b < 0, \end{cases} \tag{1.11}$$

and

$$\deg s_{a,b}(q) = \begin{cases} 1 & \text{if } 2a+b \geq 0, \\ 2 & \text{if } 2a+b < 0. \end{cases} \tag{1.12}$$

Then, by Theorems 1.1 and 1.2, we have

$$r_{a,b}(q) = \left(E_1^{-1}E_2^2\right)^{\operatorname{sgn}(a+2b)} \text{ and } s_{a,b}(q) = \left(E_1^2E_2^{-1}\right)^{\operatorname{sgn}(2a+b)}, \tag{1.13}$$

where $\operatorname{sgn}(x) = 1$ if $x \geq 0$ and $\operatorname{sgn}(x) = -1$ if $x < 0$. Hence, as $x = \operatorname{sgn}(x)|x|$, we have

$$\begin{aligned} E_1^a E_2^b &= \left(E_1^{-1}E_2^2\right)^{(a+2b)/3}\left(E_1^2E_2^{-1}\right)^{(2a+b)/3} \\ &= \left(E_1^{-1}E_2^2\right)^{|a+2b|\operatorname{sgn}(a+2b)/3}\left(E_1^2E_2^{-1}\right)^{|2a+b|\operatorname{sgn}(2a+b)/3} \end{aligned}$$

and thus by (1.13)

$$E_1^a E_2^b = r_{a,b}(q)^{|a+2b|/3} s_{a,b}(q)^{|2a+b|/3}. \tag{1.14}$$

Equation (1.14) expresses $E_1^a E_2^b$ as a polynomial in q with coefficients depending upon a, b, E_3, E_6, E_9, and E_{18}. The degree d of this polynomial is by (1.11), (1.12) and (1.14)

$$d = \begin{cases} \frac{a+2b}{3} + \frac{2a+b}{3} = a+b & \text{if } a+2b \geq 0,\ 2a+b \geq 0, \\ 2(\frac{-a-2b}{3}) + \frac{2a+b}{3} = -b & \text{if } a+2b < 0,\ 2a+b \geq 0, \\ \frac{a+2b}{3} + 2(\frac{-2a-b}{3}) = -a & \text{if } a+2b \geq 0,\ 2a+b < 0, \\ 2(\frac{-a-2b}{3}) + 2(\frac{-2a-b}{3}) = -2a-2b & \text{if } a+2b < 0,\ 2a+b < 0. \end{cases} \tag{1.15}$$

Thus

$$E_1^a E_2^b = A_0 + A_1 q + \cdots + A_d q^d, \tag{1.16}$$

where

$$A_r = A_r(a, b, E_3, E_6, E_9, E_{18}), \quad r = 0, 1, \ldots, d, \tag{1.17}$$

depends only on r, a, b, E_3, E_6, E_9, and E_{18}. Let ω be a cube root of unity with $\omega \neq 1$. We note that if we replace q with ωq in A_r, as $E_{3m}(\omega q) = E_{3m}(q)$, A_r remains unchanged. Then, appealing to (1.1), (1.16), and (1.17), we deduce

$$3\sum_{n=0}^{\infty} c(a, b; 3n)q^{3n} = \sum_{n=0}^{\infty} c(a, b; n)\left(1 + \omega^n + \omega^{2n}\right)q^n$$

$$
\begin{aligned}
&= E_1^a E_2^b + E_1^a E_2^b(\omega q) + E_1^a E_2^b\left(\omega^2 q\right) \\
&= \sum_{r=0}^{d} A_r \left(q^r + (\omega q)^r + (\omega^2 q)^r\right) \\
&= 3 \sum_{\substack{r=0 \\ r \equiv 0 \pmod 3}}^{d} A_r q^r
\end{aligned}
$$

so that

$$
\sum_{n=0}^{\infty} c(a, b; 3n) q^{3n} = \sum_{0 \le m \le d/3} A_{3m} q^{3m}. \tag{1.18}
$$

Similarly, we have

$$
\sum_{n=0}^{\infty} c(a, b; 3n+1) q^{3n+1} = \sum_{0 \le m \le (d-1)/3} A_{3m+1} q^{3m+1} \tag{1.19}
$$

and

$$
\sum_{n=0}^{\infty} c(a, b; 3n+2) q^{3n+2} = \sum_{0 \le m \le (d-2)/3} A_{3m+2} q^{3m+2}. \tag{1.20}
$$

Next, replacing E_3 by E_1, E_6 by E_2, E_9 by E_3, and E_{18} by E_6 in (1.17), we define

$$
B_r := A_r(a, b, E_1, E_2, E_3, E_6), \quad r = 0, 1, \ldots, d, \tag{1.21}
$$

so that under the transformation $q^3 \to q$ we have $A_r \to B_r$ for $r = 0, 1, \ldots, d$. Thus, from (1.18)–(1.20), we deduce

$$
\sum_{n=0}^{\infty} c(a, b; 3n) q^{n} = \sum_{0 \le m \le d/3} B_{3m} q^{m}, \tag{1.22}
$$

$$
\sum_{n=0}^{\infty} c(a, b; 3n+1) q^{n} = \sum_{0 \le m \le (d-1)/3} B_{3m+1} q^{m}, \tag{1.23}
$$

$$
\sum_{n=0}^{\infty} c(a, b; 3n+2) q^{n} = \sum_{0 \le m \le (d-2)/3} B_{3m+2} q^{m}. \tag{1.24}
$$

Thus to prove for given integers a and b an identity in Theorem 1.3 of the type

$$
\sum_{n=0}^{\infty} c(a, b; 3n) q^{n} = E_1^{r_1} E_2^{r_2} E_3^{r_3} E_6^{r_6},
$$

(recall $c(a, b; 0) = 1$), where r_1, r_2, r_3, and r_6 are integers, we have by (1.22) only to prove the identity

$$\sum_{0\le m\le d/3} B_{3m}q^m = E_1^{r_1}E_2^{r_2}E_3^{r_3}E_6^{r_6}$$

using the (p, k)-method as $B_0, B_3, \ldots, B_{3[d/3]}$ only depend upon E_1, E_2, E_3, and E_6, and similarly for the identities of types (1.23) and (1.24). Thus the proof of Theorem 1.3 consists of determining $B_0, B_1, \ldots, B_d$ for the 56 relevant values of (a, b) and then proving the appropriate identities (1.22), (1.23), (1.24) using Proposition 1.1.

We now state Theorem 1.3, which is proved in Sect. 5 using the strategy just described.

Theorem 1.3 (a) *The following* 16 *identities for q-series of the form* $\sum_{n=0}^{\infty} c(a, b; 3n)q^n$ *hold:*

(i) $\sum_{n=0}^{\infty} c(-3, 6; 3n)q^n = E_1^{-4}E_2^8E_3E_6^{-2}$.
(ii) $\sum_{n=0}^{\infty} c(-2, 1; 3n)q^n = E_1^{-8}E_2^4E_3^6E_6^{-3}$.
(iii) $\sum_{n=0}^{\infty} c(-2, 4; 3n)q^n = E_1^{-2}E_2^2E_3^4E_6^{-2}$.
(iv) $\sum_{n=0}^{\infty} c(-2, 7; 3n)q^n = E_1^4E_3^2E_6^{-1}$.
(v) $\sum_{n=0}^{\infty} c(-1, 2; 3n)q^n = E_1^{-1}E_2E_3^2E_6^{-1}$.
(vi) $\sum_{n=0}^{\infty} c(0, 9; 3n)q^n = E_2^{12}E_6^{-3}$.
(vii) $\sum_{n=0}^{\infty} c(1, -2; 3n)q^n = E_1^2E_2^{-6}E_3^3$.
(viii) $\sum_{n=0}^{\infty} c(1, 1; 3n)q^n = E_1^{-1}E_2E_3^4E_6^{-2}$.
(ix) $\sum_{n=0}^{\infty} c(1, 4; 3n)q^n = E_1^{-4}E_2^8E_3^5E_6^{-4}$.
(x) $\sum_{n=0}^{\infty} c(2, -1; 3n)q^n = E_3^2E_6^{-1}$.
(xi) $\sum_{n=0}^{\infty} c(4, -5; 3n)q^n = E_1^{12}E_2^{-16}E_6^3$.
(xii) $\sum_{n=0}^{\infty} c(4, -2; 3n)q^n = E_3^4E_6^{-2}$.
(xiii) $\sum_{n=0}^{\infty} c(5, -4; 3n)q^n = E_1^{10}E_2^{-10}E_3^{-1}E_6^2$.
(xiv) $\sum_{n=0}^{\infty} c(6, -3; 3n)q^n = E_1^8E_2^{-4}E_3^{-2}E_6$.
(xv) $\sum_{n=0}^{\infty} c(7, -2; 3n)q^n = E_1^7E_2^{-3}E_3^2E_6^{-1}$.
(xvi) $\sum_{n=0}^{\infty} c(9, 0; 3n)q^n = E_1^{12}E_3^{-3}$.

(b) *The following* 25 *identities for q-series of the form* $\sum_{n=0}^{\infty} c(a, b; 3n+1)q^n$ *hold:*

(i) $\sum_{n=0}^{\infty} c(-6, 15; 3n+1)q^n = 6E_2^8E_3^2E_6^{-1}$.
(ii) $\sum_{n=0}^{\infty} c(-5, 10; 3n+1)q^n = 5E_1^{-5}E_2^9E_3^2E_6^{-1}$.
(iii) $\sum_{n=0}^{\infty} c(-4, 5; 3n+1)q^n = 4E_1^{-10}E_2^{10}E_3^2E_6^{-1}$.
(iv) $\sum_{n=0}^{\infty} c(-3, 3; 3n+1)q^n = 3E_1^{-8}E_2^4E_3^5E_6^{-1}$.
(v) $\sum_{n=0}^{\infty} c(-3, 6; 3n+1)q^n = 3E_1^{-2}E_2^2E_3^3$.
(vi) $\sum_{n=0}^{\infty} c(-2, 1; 3n+1)q^n = 2E_1^{-7}E_2^3E_3^3$.
(vii) $\sum_{n=0}^{\infty} c(-2, 4; 3n+1)q^n = 2E_1^{-1}E_2E_3E_6$.
(viii) $\sum_{n=0}^{\infty} c(-1, 2; 3n+1)q^n = E_3^{-1}E_6^2$.
(ix) $\sum_{n=0}^{\infty} c(0, -3; 3n+1)q^n = 9qE_2^{-12}E_6^9$.
(x) $\sum_{n=0}^{\infty} c(0, 3; 3n+1)q^n = 0$.

(xi) $\sum_{n=0}^{\infty} c(0, 6; 3n+1)q^n = 9qE_6^6$.
(xii) $\sum_{n=0}^{\infty} c(1, -2; 3n+1)q^n = -E_1^3 E_2^{-7} E_6^3$.
(xiii) $\sum_{n=0}^{\infty} c(1, 1; 3n+1)q^n = -E_3 E_6$.
(xiv) $\sum_{n=0}^{\infty} c(1, 4; 3n+1)q^n = -E_1^6 E_2^{-2} E_3^{-1} E_6^2$.
(xv) $\sum_{n=0}^{\infty} c(2, -1; 3n+1)q^n = -2E_1 E_2^{-1} E_3^{-1} E_6^2$.
(xvi) $\sum_{n=0}^{\infty} c(3, -3; 3n+1)q^n = -3E_1^3 E_2^{-7} E_3^2 E_6^2$.
(xvii) $\sum_{n=0}^{\infty} c(3, 0; 3n+1)q^n = -3E_3^3$.
(xviii) $\sum_{n=0}^{\infty} c(3, 6; 3n+1)q^n = -3E_1^3 E_2^5 E_3^2 E_6^{-1}$.
(xix) $\sum_{n=0}^{\infty} c(4, -2; 3n+1)q^n = -4E_1 E_2^{-1} E_3 E_6$.
(xx) $\sum_{n=0}^{\infty} c(4, 1; 3n+1)q^n = -4E_1^{-2} E_2^6 E_3^2 E_6^{-1}$.
(xxi) $\sum_{n=0}^{\infty} c(6, -3; 3n+1)q^n = -6E_1 E_2^{-1} E_3^3$.
(xxii) $\sum_{n=0}^{\infty} c(9, -6; 3n+1)q^n = -9E_1^{11} E_2^{-11} E_6^3$.
(xxiii) $\sum_{n=0}^{\infty} c(9, -3; 3n+1)q^n = -9E_3^9 E_6^{-3}$.
(xxiv) $\sum_{n=0}^{\infty} c(10, -5; 3n+1)q^n = -10E_1^9 E_2^{-5} E_3^{-1} E_6^2$.
(xxv) $\sum_{n=0}^{\infty} c(12, -3; 3n+1)q^n = -12E_1^6 E_2^2 E_3^2 E_6^{-1}$.

(c) *The following* 43 *identities for q-series of the form* $\sum_{n=0}^{\infty} c(a, b; 3n+2)q^n$ *hold:*

(i) $\sum_{n=0}^{\infty} c(-7, 14; 3n+2)q^n = 21E_1^{-6} E_2^{10} E_3^3$.
(ii) $\sum_{n=0}^{\infty} c(-6, 9; 3n+2)q^n = 18E_1^{-11} E_2^{11} E_3^3$.
(iii) $\sum_{n=0}^{\infty} c(-5, 4; 3n+2)q^n = 16E_1^{-16} E_2^{12} E_3^3$.
(iv) $\sum_{n=0}^{\infty} c(-4, 2; 3n+2)q^n = 12E_1^{-14} E_2^6 E_3^6$.
(v) $\sum_{n=0}^{\infty} c(-4, 5; 3n+2)q^n = 9E_1^{-8} E_2^4 E_3^4 E_6$.
(vi) $\sum_{n=0}^{\infty} c(-4, 8; 3n+2)q^n = 6E_1^{-2} E_2^2 E_3^2 E_6^2$.
(vii) $\sum_{n=0}^{\infty} c(-4, 11; 3n+2)q^n = 3E_1^4 E_6^3$.
(viii) $\sum_{n=0}^{\infty} c(-4, 14; 3n+2)q^n = 81q^2 E_3^{-4} E_6^{14}$.
(ix) $\sum_{n=0}^{\infty} c(-3, 0; 3n+2)q^n = 9E_1^{-12} E_3^9$.
(x) $\sum_{n=0}^{\infty} c(-3, 3; 3n+2)q^n = 6E_1^{-7} E_2^3 E_3^2 E_6^2$.
(xi) $\sum_{n=0}^{\infty} c(-3, 6; 3n+2)q^n = 3E_1^{-1} E_2 E_6^3$.
(xii) $\sum_{n=0}^{\infty} c(-3, 9; 3n+2)q^n = -9qE_3^{-3} E_6^9$.
(xiii) $\sum_{n=0}^{\infty} c(-3, 12; 3n+2)q^n = -3E_1^2 E_2^6 E_3^{-1} E_6^2$.
(xiv) $\sum_{n=0}^{\infty} c(-2, 1; 3n+2)q^n = 4E_1^{-6} E_2^2 E_6^3$.
(xv) $\sum_{n=0}^{\infty} c(-2, 4; 3n+2)q^n = E_3^{-2} E_6^4$.
(xvi) $\sum_{n=0}^{\infty} c(-2, 7; 3n+2)q^n = -2E_1^{-3} E_2^7 E_3^{-1} E_6^2$.
(xvii) $\sum_{n=0}^{\infty} c(-1, -1; 3n+2)q^n = 3E_1^{-4} E_2^{-4} E_3^3 E_6^3$.
(xviii) $\sum_{n=0}^{\infty} c(-1, 2; 3n+2)q^n = 0$.
(xix) $\sum_{n=0}^{\infty} c(-1, 5; 3n+2)q^n = -3E_1^{-1} E_2 E_3^2 E_6^2$.
(xx) $\sum_{n=0}^{\infty} c(0, 3; 3n+2)q^n = -3E_6^3$.
(xxi) $\sum_{n=0}^{\infty} c(1, -2; 3n+2)q^n = E_1^4 E_2^{-8} E_3^{-3} E_6^6$.
(xxii) $\sum_{n=0}^{\infty} c(1, 1; 3n+2)q^n = -2E_1 E_2^{-1} E_3^{-2} E_6^4$.
(xxiii) $\sum_{n=0}^{\infty} c(2, -4; 3n+2)q^n = 3E_1^6 E_2^{-14} E_6^6$.
(xxiv) $\sum_{n=0}^{\infty} c(2, -1; 3n+2)q^n = 0$.
(xxv) $\sum_{n=0}^{\infty} c(2, 2; 3n+2)q^n = -3E_3^2 E_6^2$.

(xxvi) $\sum_{n=0}^{\infty} c(2,5;3n+2)q^n = -6E_1^{-3}E_2^7E_3^3.$
(xxvii) $\sum_{n=0}^{\infty} c(3,-3;3n+2)q^n = 3E_1^4E_2^{-8}E_3^{-1}E_6^5.$
(xxviii) $\sum_{n=0}^{\infty} c(3,0;3n+2)q^n = 0.$
(xxix) $\sum_{n=0}^{\infty} c(4,-2;3n+2)q^n = 4E_1^2E_2^{-2}E_3^{-2}E_6^4.$
(xxx) $\sum_{n=0}^{\infty} c(4,1;3n+2)q^n = E_1^8E_2^{-4}E_3^{-4}E_6^5.$
(xxxi) $\sum_{n=0}^{\infty} c(5,-4;3n+2)q^n = 9E_1^4E_2^{-8}E_3E_6^4.$
(xxxii) $\sum_{n=0}^{\infty} c(5,-1;3n+2)q^n = 6E_1E_2^{-1}E_3^2E_6^2.$
(xxxiii) $\sum_{n=0}^{\infty} c(5,2;3n+2)q^n = 3E_1^7E_2^{-3}E_6^3.$
(xxxiv) $\sum_{n=0}^{\infty} c(5,5;3n+2)q^n = -81qE_3^5E_6^5.$
(xxxv) $\sum_{n=0}^{\infty} c(6,-3;3n+2)q^n = 12E_1^2E_2^{-2}E_6^3.$
(xxxvi) $\sum_{n=0}^{\infty} c(6,0;3n+2)q^n = 9E_3^6.$
(xxxvii) $\sum_{n=0}^{\infty} c(6,3;3n+2)q^n = 6E_1^5E_2^3E_3^{-1}E_6^2.$
(xxxviii) $\sum_{n=0}^{\infty} c(7,-2;3n+2)q^n = 16E_2^4E_3^{-1}E_6^2.$
(xxxix) $\sum_{n=0}^{\infty} c(8,-4;3n+2)q^n = 24E_1^2E_2^{-2}E_3^2E_6^2.$
(xl) $\sum_{n=0}^{\infty} c(11,-4;3n+2)q^n = 48E_2^4E_3^3.$
(xli) $\sum_{n=0}^{\infty} c(14,-7;3n+2)q^n = 84E_1^{10}E_2^{-6}E_6^3.$
(xlii) $\sum_{n=0}^{\infty} c(14,-4;3n+2)q^n = 81E_3^{14}E_6^{-4}.$
(xliii) $\sum_{n=0}^{\infty} c(15,-6;3n+2)q^n = 96E_1^8E_3^{-1}E_6^2.$

If (a,b) is one of the 56 pairs of integers to which Theorem 1.3 applies and $c(a,b;n)$ is known explicitly for all n, then Theorem 1.3 gives an explicit q-series expansion of a certain product of the form $E_1^{a_1}E_2^{a_2}E_3^{a_3}E_6^{a_6}$.

We illustrate this in the case $(a,b)=(1,1)$. Theorem 1.3 (c) applies to this case and $c(1,1;n)$ is known explicitly [5, Eq. (10.1), p. 133], namely

$$c(1,1;n) = \sum_{\substack{x^2+2y^2=24n+3\\ x>0,\ y>0}} \left(\frac{12}{xy}\right),$$

where

$$\left(\frac{12}{k}\right) = \begin{cases} 1 & \text{if } k \equiv 1, 11 \pmod{12},\\ -1 & \text{if } k \equiv 5, 7 \pmod{12},\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$c(1,1;3n+2) = \sum_{\substack{x^2+2y^2=72n+51\\ x>0,\ y>0}} \left(\frac{12}{xy}\right),$$

so, by Theorem 1.3 (c)(xxii), we obtain

$$E_1 E_2^{-1} E_3^{-2} E_6^4 = \sum_{n=0}^{\infty} \left(-\frac{1}{2} \sum_{\substack{x^2+2y^2=72n+51 \\ x>0,\ y>0}} \left(\frac{12}{xy}\right) \right) q^n. \tag{1.25}$$

The authors have not found the identity (1.25) in the literature.

2 Two identities of Kac: proof of Theorem 1.1

In this section we prove the two identities of Kac stated in Theorem 1.1 as these are not given explicitly in Kac's paper [2] nor proved in Köhler's book [5, pp. 115–116].

Proof of Theorem 1.1 (i) We have

$$\sum_{r=0}^{\infty} q^{\frac{r(r+1)}{2}} = \sum_{r=0}^{\infty} q^{\frac{3r(3r+1)}{2}} + \sum_{r=0}^{\infty} q^{\frac{(3r+1)(3r+2)}{2}} + \sum_{r=0}^{\infty} q^{\frac{(3r+2)(3r+3)}{2}}. \tag{2.1}$$

First, we observe that

$$\sum_{r=0}^{\infty} q^{\frac{r(r+1)}{2}} = \sum_{r=0}^{\infty} q^{\frac{(2r+1)^2-1}{8}} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} q^{\frac{n^2-1}{8}}$$

so that

$$\sum_{r=0}^{\infty} q^{\frac{r(r+1)}{2}} = q^{-\frac{1}{8}} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} q^{\frac{n^2}{8}}. \tag{2.2}$$

Secondly, we have

$$\sum_{r=0}^{\infty} q^{\frac{(3r+1)(3r+2)}{2}} = \sum_{r=0}^{\infty} q^{\frac{9r^2+9r+2}{2}} = \sum_{r=0}^{\infty} q^{\frac{36r^2+36r+8}{8}} = \sum_{r=0}^{\infty} q^{\frac{9(2r+1)^2-1}{8}} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} q^{\frac{9n^2-1}{8}},$$

so that

$$\sum_{r=0}^{\infty} q^{\frac{(3r+1)(3r+2)}{2}} = q^{-\frac{1}{8}} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} q^{\frac{9n^2}{8}}. \tag{2.3}$$

Thirdly, we have

$$\sum_{r=0}^{\infty} q^{\frac{(3r+2)(3r+3)}{2}} \overset{r=-t-1}{=} \sum_{t=-\infty}^{-1} q^{\frac{9t^2+3t}{2}}. \tag{2.4}$$

Thus, appealing to (2.4), we have

$$\sum_{r=0}^{\infty} q^{\frac{3r(3r+1)}{2}} + \sum_{r=0}^{\infty} q^{\frac{(3r+2)(3r+3)}{2}} = \sum_{r=-\infty}^{\infty} q^{\frac{9r^2+3r}{2}}. \tag{2.5}$$

Hence, from (2.1)–(2.3) and (2.5), we deduce

$$q^{-\frac{1}{8}} \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} q^{\frac{n^2}{8}} = q^{-\frac{1}{8}} \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} q^{\frac{9n^2}{8}} + \sum_{r=-\infty}^{\infty} q^{\frac{9r^2+3r}{2}}. \tag{2.6}$$

The following identity is a classical result of Gauss [6, Theorem 8.1, p. 114]

$$q^{\frac{1}{8}} \frac{E_2^2}{E_1} = \frac{\eta^2(2z)}{\eta(z)} = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} q^{\frac{n^2}{8}}, \tag{2.7}$$

where the first equality follows from (1.6). Mapping $q \to q^9$ (so $z \to 9z$) in (2.7), we obtain

$$q^{\frac{9}{8}} \frac{E_{18}^2}{E_9} = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} q^{\frac{9n^2}{8}}. \tag{2.8}$$

Mapping $q \to q^3$ (so $z \to 3z$) in (1.8), we deduce

$$\frac{E_6 E_9^2}{E_3 E_{18}} = \sum_{r=-\infty}^{\infty} q^{\frac{9r^2+3r}{2}}. \tag{2.9}$$

From (2.6)–(2.9), we obtain

$$\frac{E_2^2}{E_1} = q\frac{E_{18}^2}{E_9} + \frac{E_6 E_9^2}{E_3 E_{18}},$$

as asserted.

(ii) We have

$$\sum_{r=-\infty}^{\infty} (-1)^r q^{r^2} = \sum_{r=-\infty}^{\infty} (-1)^{3r} q^{(3r)^2} + \sum_{r=-\infty}^{\infty} (-1)^{3r+1} q^{(3r+1)^2}$$

$$+\sum_{r=-\infty}^{\infty}(-1)^{3r-1}q^{(3r-1)^2}$$
$$=\sum_{r=-\infty}^{\infty}(-1)^r q^{9r^2}-q\sum_{r=-\infty}^{\infty}(-1)^r q^{9r^2+6r}-q\sum_{r=-\infty}^{\infty}(-1)^r q^{9r^2-6r}.$$

Changing r to $-r$ in the third sum, we deduce

$$\sum_{r=-\infty}^{\infty}(-1)^r q^{r^2}=\sum_{r=-\infty}^{\infty}(-1)^r q^{9r^2}-2q\sum_{r=-\infty}^{\infty}(-1)^r q^{9r^2+6r}. \tag{2.10}$$

A classical result of Jacobi [6, Eq. (8.7), p. 114] asserts that

$$\frac{E_1^2}{E_2}=\sum_{r=-\infty}^{\infty}(-1)^r q^{r^2}. \tag{2.11}$$

Mapping q to q^9 in (2.11), we deduce

$$\frac{E_9^2}{E_{18}}=\sum_{r=-\infty}^{\infty}(-1)^r q^{9r^2}. \tag{2.12}$$

Mapping $q\to q^3$ in Kac's identity (1.7), we obtain

$$\frac{E_3E_{18}^2}{E_6E_9}=\sum_{r=-\infty}^{\infty}(-1)^r q^{9r^2+6r}. \tag{2.13}$$

Putting (2.11), (2.12) and (2.13) into (2.10), we deduce

$$\frac{E_1^2}{E_2}=\frac{E_9^2}{E_{18}}-2q\frac{E_3E_{18}^2}{E_6E_9},$$

as claimed. □

3 Two companion formulas to Kac's identities: proof of Theorem 1.2.

Proof (i) We have

$$(E_3^{-1}E_6E_9^2E_{18}^{-1}+qE_9^{-1}E_{18}^2)(E_3^2E_6^{-6}E_9^3-qE_3^3E_6^{-7}E_{18}^3+q^2E_3^4E_6^{-8}E_9^{-3}E_{18}^6)$$
$$=E_3E_6^{-5}E_9^5E_{18}^{-1}+q^3E_3^4E_6^{-8}E_9^{-4}E_{18}^8,$$

so, by Theorem 1.1(i), we obtain

$$E_3^2E_6^{-6}E_9^3-qE_3^3E_6^{-7}E_{18}^3+q^2E_3^4E_6^{-8}E_9^{-3}E_{18}^6=E_1E_2^{-2}(E_3E_6^{-5}E_9^5E_{18}^{-1}$$

$$+ q^3 E_3^4 E_6^{-8} E_9^{-4} E_{18}^8).$$

Multiplying identity (i) of Proposition 1.1 by $E_2^{-8} E_3^{-4} E_6^{-1}$, we obtain

$$E_1 E_2^{-5} E_3^5 E_6^{-1} + q E_1^4 E_2^{-8} E_3^{-4} E_6^8 = 1.$$

Mapping q to q^3, we see that

$$E_3 E_6^{-5} E_9^5 E_{18}^{-1} + q^3 E_3^4 E_6^{-8} E_9^{-4} E_{18}^8 = 1,$$

which proves that

$$E_1 E_2^{-2} = E_3^2 E_6^{-6} E_9^3 - q E_3^3 E_6^{-7} E_{18}^3 + q^2 E_3^4 E_6^{-8} E_9^{-3} E_{18}^6.$$

(ii) We have

$$\begin{aligned}(E_9^2 E_{18}^{-1} - 2q E_3 E_6^{-1} E_9^{-1} E_{18}^2)(E_3^{-8} E_6^4 E_9^6 E_{18}^{-3} + 2q E_3^{-7} E_6^3 E_9^3 + 4q^2 E_3^{-6} E_6^2 E_{18}^3)\\ = E_3^{-8} E_6^4 E_9^8 E_{18}^{-4} - 8q^3 E_3^{-5} E_6 E_9^{-1} E_{18}^5,\end{aligned}$$

so, by Theorem 1.1(ii), we deduce

$$\begin{aligned}E_3^{-8} E_6^4 E_9^6 E_{18}^{-3} + 2q E_3^{-7} E_6^3 E_9^3 + 4q^2 E_3^{-6} E_6^2 E_{18}^3 = E_1^{-2} E_2 (E_3^{-8} E_6^4 E_9^8 E_{18}^{-4}\\ - 8q^3 E_3^{-5} E_6 E_9^{-1} E_{18}^5).\end{aligned}$$

Multiplying identity (ii) of Proposition 1.1 by $E_1^{-8} E_3^{-1} E_6^{-4}$, we obtain

$$E_1^{-8} E_2^4 E_3^8 E_6^{-4} - 8q E_1^{-5} E_2 E_3^{-1} E_6^5 = 1.$$

Mapping q to q^3, we deduce

$$E_3^{-8} E_6^4 E_9^8 E_{18}^{-4} - 8q^3 E_3^{-5} E_6 E_9^{-1} E_{18}^5 = 1,$$

which proves

$$E_1^{-2} E_2 = E_3^{-8} E_6^4 E_9^6 E_{18}^{-3} + 2q E_3^{-7} E_6^3 E_9^3 + 4q^2 E_3^{-6} E_6^2 E_{18}^3.$$

□

4 Identities involving E_1, E_2, E_3, and E_6: proof of Proposition 1.1

Ramanujan's theta function $\varphi(q)$ [7, p. 6] is defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}.$$

Following [3, p. 178], we define

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}$$

and

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}.$$

It is known that E_1, E_2, E_3, E_4, E_6, and E_{12} can be given explicitly in terms of q, p, and k [8, p. 997].

We have

$$E_1 = \prod_{n=1}^{\infty}(1-q^n) = q^{-\frac{1}{24}} 2^{-\frac{1}{6}} p^{\frac{1}{24}} (1-p)^{\frac{1}{2}} (1+p)^{\frac{1}{6}} (1+2p)^{\frac{1}{8}} (2+p)^{\frac{1}{8}} k^{\frac{1}{2}}, \tag{4.1}$$

$$E_2 = \prod_{n=1}^{\infty}(1-q^{2n}) = q^{-\frac{1}{12}} 2^{-\frac{1}{3}} p^{\frac{1}{12}} (1-p)^{\frac{1}{4}} (1+p)^{\frac{1}{12}} (1+2p)^{\frac{1}{4}} (2+p)^{\frac{1}{4}} k^{\frac{1}{2}}, \tag{4.2}$$

$$E_3 = \prod_{n=1}^{\infty}(1-q^{3n}) = q^{-\frac{1}{8}} 2^{-\frac{1}{6}} p^{\frac{1}{8}} (1-p)^{\frac{1}{6}} (1+p)^{\frac{1}{2}} (1+2p)^{\frac{1}{24}} (2+p)^{\frac{1}{24}} k^{\frac{1}{2}}, \tag{4.3}$$

$$E_4 = \prod_{n=1}^{\infty}(1-q^{4n}) = q^{-\frac{1}{6}} 2^{-\frac{2}{3}} p^{\frac{1}{6}} (1-p)^{\frac{1}{8}} (1+p)^{\frac{1}{24}} (1+2p)^{\frac{1}{8}} (2+p)^{\frac{1}{2}} k^{\frac{1}{2}}, \tag{4.4}$$

$$E_6 = \prod_{n=1}^{\infty}(1-q^{6n}) = q^{-\frac{1}{4}} 2^{-\frac{1}{3}} p^{\frac{1}{4}} (1-p)^{\frac{1}{12}} (1+p)^{\frac{1}{4}} (1+2p)^{\frac{1}{12}} (2+p)^{\frac{1}{12}} k^{\frac{1}{2}}, \tag{4.5}$$

$$E_{12} = \prod_{n=1}^{\infty}(1-q^{12n}) = q^{-\frac{1}{2}} 2^{-\frac{2}{3}} p^{\frac{1}{2}} (1-p)^{\frac{1}{24}} (1+p)^{\frac{1}{8}} (1+2p)^{\frac{1}{24}} (2+p)^{\frac{1}{6}} k^{\frac{1}{2}}. \tag{4.6}$$

The six Eqs. (4.1)–(4.6) can be solved for the six quantities p, $1-p$, $1+p$, $1+2p$, $2+p$, and k in terms of q, E_1, E_2, E_3, E_4, E_6, and E_{12} [8, pp. 997–998]. We have

$$p = 2q \prod_{n=1}^{\infty} \frac{(1-q^{2n})^3(1-q^{3n})^3(1-q^{12n})^6}{(1-q^n)(1-q^{4n})^2(1-q^{6n})^9} = 2q E_1^{-1} E_2^3 E_3^3 E_4^{-2} E_6^{-9} E_{12}^6, \tag{4.7}$$

$$1-p=\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-q^{2n})(1-q^{3n})^2(1-q^{12n})^3}{(1-q^{4n})(1-q^{6n})^7}=E_1^2E_2E_3^2E_4^{-1}E_6^{-7}E_{12}^3, \tag{4.8}$$

$$1+p=\prod_{n=1}^{\infty}\frac{(1-q^{2n})^3(1-q^{3n})^6(1-q^{12n})^3}{(1-q^n)^2(1-q^{4n})(1-q^{6n})^9}=E_1^{-2}E_2^3E_3^6E_4^{-1}E_6^{-9}E_{12}^3, \tag{4.9}$$

$$1+2p=\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{10}(1-q^{3n})^4(1-q^{12n})^4}{(1-q^n)^4(1-q^{4n})^4(1-q^{6n})^{10}}=E_1^{-4}E_2^{10}E_3^4E_4^{-4}E_6^{-10}E_{12}^4, \tag{4.10}$$

$$2+p=2\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{3n})^3(1-q^{4n})^2(1-q^{12n})^2}{(1-q^n)(1-q^{6n})^7}=2E_1^{-1}E_2E_3^3E_4^2E_6^{-7}E_{12}^2, \tag{4.11}$$

$$k=\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-q^{4n})^2(1-q^{6n})^{15}}{(1-q^{2n})^5(1-q^{3n})^6(1-q^{12n})^6}=E_1^2E_2^{-5}E_3^{-6}E_4^2E_6^{15}E_{12}^{-6}. \tag{4.12}$$

We now prove Proposition 1.1 using the formulas (4.1)–(4.12).

Proof of Proposition 1.1 (i) Appealing to (4.1)–(4.3), (4.5), and (4.7)–(4.12), we obtain

$$\begin{aligned}
E_1E_2^3E_3^9+qE_1^4E_6^9 &= q^{-\frac{17}{12}}2^{-\frac{8}{3}}p^{\frac{17}{12}}(1-p)^{\frac{11}{4}}(1+p)^{\frac{59}{12}}(1+2p)^{\frac{5}{4}}(2+p)^{\frac{5}{4}}k^{\frac{13}{2}}\\
&\quad+q^{-\frac{17}{12}}2^{-\frac{11}{3}}p^{\frac{29}{12}}(1-p)^{\frac{11}{4}}(1+p)^{\frac{35}{12}}(1+2p)^{\frac{5}{4}}(2+p)^{\frac{5}{4}}k^{\frac{13}{2}}\\
&= q^{-\frac{17}{12}}2^{-\frac{11}{3}}p^{\frac{17}{12}}(1-p)^{\frac{11}{4}}(1+p)^{\frac{35}{12}}(1+2p)^{\frac{5}{4}}\\
&\quad(2+p)^{\frac{5}{4}}k^{\frac{13}{2}}(2(1+p)^2+p)\\
&= q^{-\frac{17}{12}}2^{-\frac{11}{3}}p^{\frac{17}{12}}(1-p)^{\frac{11}{4}}(1+p)^{\frac{35}{12}}(1+2p)^{\frac{9}{4}}(2+p)^{\frac{9}{4}}k^{\frac{13}{2}}\\
&= E_2^8E_3^4E_6.
\end{aligned}$$

(ii) Appealing to (4.1)–(4.3), (4.5), and (4.7)–(4.12), we deduce

$$\begin{aligned}
E_2^4E_3^9-8qE_1^3E_2E_6^9 &= q^{-\frac{35}{24}}2^{-\frac{17}{6}}p^{\frac{35}{24}}(1-p)^{\frac{5}{2}}(1+p)^{\frac{29}{6}}(1+2p)^{\frac{11}{8}}(2+p)^{\frac{11}{8}}k^{\frac{13}{2}}\\
&\quad-q^{-\frac{35}{24}}2^{-\frac{5}{6}}p^{\frac{59}{24}}(1-p)^{\frac{5}{2}}(1+p)^{\frac{17}{6}}(1+2p)^{\frac{11}{8}}(2+p)^{\frac{11}{8}}k^{\frac{13}{2}}\\
&= q^{-\frac{35}{24}}2^{-\frac{17}{6}}p^{\frac{35}{24}}(1-p)^{\frac{5}{2}}(1+p)^{\frac{17}{6}}(1+2p)^{\frac{11}{8}}\\
&\quad(2+p)^{\frac{11}{8}}k^{\frac{13}{2}}((1+p)^2-4p)\\
&= q^{-\frac{35}{24}}2^{-\frac{17}{6}}p^{\frac{35}{24}}(1-p)^{\frac{9}{2}}(1+p)^{\frac{17}{6}}(1+2p)^{\frac{11}{8}}(2+p)^{\frac{11}{8}}k^{\frac{13}{2}}\\
&= E_1^8E_3E_6^4.
\end{aligned}$$

□

5 Eighty-four identities: proof of Theorem 1.3

In preparation for the proof of Theorem 1.3, we give three identities which are simple consequences of Proposition 1.1.

Proposition 5.1

(i) $E_2^6E_3^{18} - 7qE_1^3E_2^3E_3^9E_6^9 - 8q^2E_1^6E_6^{18} = E_1^7E_2^7E_3^5E_6^5.$

(ii) $E_2^9E_3^{27} - 6qE_1^3E_2^6E_3^{18}E_6^9 - 15q^2E_1^6E_2^3E_3^9E_6^{18} - 8q^3E_1^9E_6^{27} = E_1^6E_2^{15}E_3^9E_6^6.$

(iii) $E_2^9E_3^{27} - 15qE_1^3E_2^6E_3^{18}E_6^9 + 48q^2E_1^6E_2^3E_3^9E_6^{18} + 64q^3E_1^9E_6^{27} = E_1^{15}E_2^6E_3^6E_6^9.$

Proof (i) We have

$$\begin{aligned}
&E_2^6E_3^{18} - 7qE_1^3E_2^3E_3^9E_6^9 - 8q^2E_1^6E_6^{18}\\
&= (E_2^3E_3^9 + qE_1^3E_6^9)(E_2^3E_3^9 - 8qE_1^3E_6^9)\\
&= (E_1^{-1}E_2^8E_3^4E_6)(E_2^{-1}E_1^8E_3E_6^4)\\
&= E_1^7E_2^7E_3^5E_6^5,
\end{aligned}$$

by Proposition 1.1(i)(ii).

(ii) We have

$$\begin{aligned}
&E_2^9E_3^{27} - 6qE_1^3E_2^6E_3^{18}E_6^9 - 15q^2E_1^6E_2^3E_3^9E_6^{18} - 8q^3E_1^9E_6^{27}\\
&= (E_2^3E_3^9 + qE_1^3E_6^9)^2(E_2^3E_3^9 - 8qE_1^3E_6^9)\\
&= (E_1^{-1}E_2^8E_3^4E_6)^2(E_2^{-1}E_1^8E_3E_6^4)\\
&= E_1^6E_2^{15}E_3^9E_6^6,
\end{aligned}$$

by Proposition 1.1(i)(ii).

(iii) We have

$$\begin{aligned}
&E_2^9E_3^{27} - 15qE_1^3E_2^6E_3^{18}E_6^9 + 48q^2E_1^6E_2^3E_3^9E_6^{18} + 64q^3E_1^9E_6^{27}\\
&= (E_2^3E_3^9 + qE_1^3E_6^9)(E_2^3E_3^9 - 8qE_1^3E_6^9)^2\\
&= (E_1^{-1}E_2^8E_3^4E_6)(E_2^{-1}E_1^8E_3E_6^4)^2\\
&= E_1^{15}E_2^6E_3^6E_6^9,
\end{aligned}$$

by Proposition 1.1(i)(ii). □

Proof of Theorem 1.3 For the 56 pairs of integers (a, b) listed in the first column of Table 1, we determined the values of d listed in the second column by means of (1.15), as well as the 84 relevant values of

$$C_r := \sum_{0\le m\le (d-r)/3} B_{3m+r}q^m, \quad r = 0, 1, 2,$$

listed in the third column using (1.22)–(1.24). For each such (a, b) the part of Theorem 1.3 to which the value of C_r applies is listed in the fourth column.

If the C_r entry is just cq^s (c = constant, $s = 0, 1, 2$) times a product of powers of E_1, E_2, E_3, and E_6, then Theorem 1.3 is immediately established in such a case. For example, when $(a, b) = (-3, 9)$ the table gives $C_2 = -9qE_3^{-3}E_6^9$ and, by (1.24), part (c)(xii) of Theorem 1.3 is proved. Fifty-two of the identities of Theorem 1.3 are proved in this way.

If the entry $C_r = U + Vq$, where U and V are nonzero multiples of products of powers of E_1, E_2, E_3, and E_6, then Proposition 1.1 must be used to convert C_r into a single multiple of a product of powers of E_1, E_2, E_3, and E_6. Part (i) of Proposition 1.1 is used for the 12 parts

(a)(i)(ix); (b)(ii)(iii)(xx); (c)(i)(ii)(iii)(xvi)(xxvi)(xxxviii)(xl)

and part (ii) for the 12 parts

(a)(iv)(xi)(xiii)(xiv)(xv); (b)(xiv)(xxii)(xxiv); (c)(vii)(xxx)(xxxiii)(xli).

For example, if $(a, b) = (5, 2)$ the table gives $C_2 = 3E_1^{-1}E_2E_3^8E_6^{-1} - 24qE_1^2E_2^{-2}E_3^{-1}E_6^8$ and by Proposition 1.1(ii) we have

$$\begin{aligned}C_2 &= 3E_1^{-1}E_2^{-3}E_3^{-1}E_6^{-1}(E_2^4E_3^9 - 8qE_1^3E_2E_6^9)\\ &= 3E_1^{-1}E_2^{-3}E_3^{-1}E_6^{-1} \cdot E_1^8E_3E_6^4 = 3E_1^7E_2^{-3}E_6^3,\end{aligned}$$

which is Theorem 1.3(c)(xxxiii). This completes the proof of twenty-four identities.

If the entry $C_r = U + Vq + Wq^2$, where U, V, and W are multiples of products of powers of E_1, E_2, E_3, and E_6 with $(U, V) \neq (0, 0)$ and $W \neq 0$ (this occurs for the 6 identities (b)(i) (xviii) (xxv) and (c)(xiii) (xxxvii) (xliii)), then Proposition 5.1(i) is used to convert C_r into a single multiple of powers of E_1, E_2, E_3, and E_6. For example, if $(a, b) = (-3, 12)$ then

$$C_2 = -3E_1^{-5}E_2^5E_3^{12}E_6^{-3} + 21qE_1^{-2}E_2^2E_3^3E_6^6 + 24q^2E_1E_2^{-1}E_3^{-6}E_6^{15}$$

and by Proposition 5.1(i) we obtain

$$\begin{aligned}C_2 &= -3E_1^{-5}E_2^{-1}E_3^{-6}E_6^{-3}(E_2^6E_3^{18} - 7qE_1^3E_2^3E_3^9E_6^9 - 8q^2E_1^6E_6^{18})\\ &= -3E_1^{-5}E_2^{-1}E_3^{-6}E_6^{-3} \cdot E_1^7E_2^7E_3^5E_6^5 = -3E_1^2E_2^6E_3^{-1}E_6^2,\end{aligned}$$

which is Theorem 1.3(c)(xiii). This proves six of the identities.

Finally, the entry C_r is cubic in q only for the 2 identities (a)(vi)(xvi). Proposition 5.1(ii) is used to convert C_r into a single product for the identity (a)(vi) and Proposition 5.1(iii) for the identity (a)(xvi). This proves the remaining two identities.

This completes the proof of all $52 + 24 + 6 + 2 = 84$ identities of Theorem 1.3. □

Table 1 Values of $\sum_{0\le m\le (d-r)/3} B_{3m+r}q^m$ $(r=0,1,2)$ for 56 pairs (a,b) of integers

a, b	d	$C_r := \sum_{0\le m\le (d-r)/3} B_{3m+r}q^m, \quad r=0,1,2$	Theorem 1.3
−7, 14	7	$C_2 = 21E_1^{-5}E_2^5E_3^8E_6^{-1} + 21E_1^{-2}E_2^2E_3^{-1}E_6^8q$	(c)(i)
−6, 9	6	$C_2 = 18E_1^{-10}E_2^6E_3^8E_6^{-1} + 18E_1^{-7}E_2^3E_3^{-1}E_6^8q$	(c)(ii)
−6, 15	9	$C_1 = 6E_1^{-7}E_2^7E_3^{15}E_6^{-6} - 42E_1^{-4}E_2^4E_3^6E_6^3q - 48E_1^{-1}E_2E_3^{-3}E_6^{12}q^2$	(b)(i)
−5, 4	5	$C_2 = 16E_1^{-15}E_2^7E_3^8E_6^{-1} + 16E_1^{-12}E_2^4E_3^{-1}E_6^8q$	(c)(iii)
−5, 10	5	$C_1 = 5E_1^{-4}E_2^4E_3^7E_6^{-2} + 5E_1^{-1}E_2E_3^{-2}E_6^7q$	(b)(ii)
−4, 2	4	$C_2 = 12E_1^{-14}E_2^6E_3^6$	(c)(iv)
−4, 5	4	$C_1 = 4E_1^{-9}E_2^5E_3^7E_6^{-2} + 4E_1^{-6}E_2^2E_3^{-2}E_6^7q$	(b)(iii)
		$C_2 = 9E_1^{-8}E_2^4E_3^4E_6$	(c)(v)
−4, 8	4	$C_2 = 6E_1^{-2}E_2^2E_3^2E_6^2$	(c)(vi)
−4, 11	7	$C_2 = 3E_1^{-4}E_2^4E_3^8E_6^{-1} - 24E_1^{-1}E_2E_3^{-1}E_6^8q$	(c)(vii)
−4, 14	10	$C_2 = 81E_3^{-4}E_6^{14}q^2$	(c)(viii)
−3, 0	6	$C_2 = 9E_1^{-12}E_3^9$	(c)(ix)
−3, 3	3	$C_1 = 3E_1^{-8}E_2^4E_3^5E_6^{-1}$	(b)(iv)
		$C_2 = 6E_1^{-7}E_2^3E_3^2E_6^2$	(c)(x)
−3, 6	3	$C_0 = E_1^{-3}E_2^3E_3^6E_6^{-3} + E_3^{-3}E_6^6q$	(a)(i)
		$C_1 = 3E_1^{-2}E_2^2E_3^3$	(b)(v)
		$C_2 = 3E_1^{-1}E_2E_6^3$	(c)(xi)
−3, 9	6	$C_2 = -9E_3^{-3}E_6^9q$	(c)(xii)
−3, 12	9	$C_2 = -3E_1^{-5}E_2^5E_3^{12}E_6^{-3} + 21E_1^{-2}E_2^2E_3^3E_6^6q + 24E_1E_2^{-1}E_3^{-6}E_6^{15}q^2$	(c)(xiii)

Table 1 continued

a, b	d	$C_r := \sum_{0\le m\le (d-r)/3} B_{3m+r}q^m, \quad r = 0, 1, 2$	Theorem 1.3
$-2, 1$	2	$C_0 = E_1^{-8}E_2^4E_3^6E_6^{-3}$	(a)(ii)
		$C_1 = 2E_1^{-7}E_2^3E_3^3$	(b)(vi)
		$C_2 = 4E_1^{-6}E_2^2E_6^3$	(c)(xiv)
$-2, 4$	2	$C_0 = E_1^{-2}E_2^2E_3^4E_6^{-2}$	(a)(iii)
		$C_1 = 2E_1^{-1}E_2E_3E_6$	(b)(vii)
		$C_2 = E_3^{-2}E_6^4$	(c)(xv)
$-2, 7$	5	$C_0 = E_1^{-4}E_2^4E_3^{10}E_6^{-5} - 8E_1^{-1}E_2E_3E_6^4q$	(a)(iv)
		$C_2 = -2E_1^{-2}E_2^2E_3^4E_6 - 2E_1E_2^{-1}E_3^{-5}E_6^{10}q$	(c)(xvi)
$-1, -1$	4	$C_2 = 3E_1^{-4}E_2^{-4}E_3^3E_6^3$	(c)(xvii)
$-1, 2$	1	$C_0 = E_1^{-1}E_2E_3^2E_6^{-1}$	(a)(v)
		$C_1 = E_3^{-1}E_6^2$	(b)(viii)
		$C_2 = 0$	(c)(xviii)
$-1, 5$	4	$C_2 = -3E_1^{-1}E_2E_3^2E_6^2$	(c)(xix)
$0, -3$	6	$C_1 = 9E_2^{-12}E_6^9q$	(b)(ix)
$0, 3$	3	$C_1 = 0$	(b)(x)
		$C_2 = -3E_6^3$	(c)(xx)
$0, 6$	6	$C_1 = 9E_6^6q$	(b)(xi)
$0, 9$	9	$C_0 = E_1^{-6}E_2^6E_3^{18}E_6^{-9} - 6E_1^{-3}E_2^3E_3^9q - 15E_6^9q^2 - 8E_1^3E_2^{-3}E_3^{-9}E_6^{18}q^3$	(a)(vi)

Table 1 continued

a, b	d	$C_r := \sum_{0\leq m\leq (d-r)/3} B_{3m+r}q^m, \quad r=0,1,2$	Theorem 1.3
$1, -2$	2	$C_0 = E_1^2E_2^{-6}E_3^3$	(a)(vii)
		$C_1 = -E_1^3E_2^{-7}E_6^3$	(b)(xii)
		$C_2 = E_1^4E_2^{-8}E_3^{-3}E_6^6$	(c)(xxi)
$1, 1$	2	$C_0 = E_1^{-1}E_2E_3^4E_6^{-2}$	(a)(viii)
		$C_1 = -E_3E_6$	(b)(xiii)
		$C_2 = -2E_1E_2^{-1}E_3^{-2}E_6^4$	(c)(xxii)
$1, 4$	5	$C_0 = E_1^{-3}E_2^3E_3^{10}E_6^{-5} + E_3E_6^4q$	(a)(ix)
		$C_1 = -E_1^{-2}E_2^2E_3^7E_6^{-2} + 8E_1E_2^{-1}E_3^{-2}E_6^7q$	(b)(xiv)
$2, -4$	4	$C_2 = 3E_1^6E_2^{-14}E_6^6$	(c)(xxiii)
$2, -1$	1	$C_0 = E_3^2E_6^{-1}$	(a)(x)
		$C_1 = -2E_1E_2^{-1}E_3^{-1}E_6^2$	(b)(xv)
		$C_2 = 0$	(c)(xxiv)
$2, 2$	4	$C_2 = -3E_3^2E_6^2$	(c)(xxv)
$2, 5$	7	$C_2 = -6E_1^{-2}E_2^2E_3^8E_6^{-1} - 6E_1E_2^{-1}E_3^{-1}E_6^8q$	(c)(xxvi)
$3, -3$	3	$C_1 = -3E_1^3E_2^{-7}E_3^2E_6^2$	(b)(xvi)
		$C_2 = 3E_1^4E_2^{-8}E_3^{-1}E_6^5$	(c)(xxvii)
$3, 0$	3	$C_1 = -3E_3^3$	(b)(xvii)
		$C_2 = 0$	(c)(xxviii)
$3, 6$	9	$C_1 = -3E_1^{-4}E_2^4E_3^{15}E_6^{-6} + 21E_1^{-1}E_2E_3^6E_6^3q + 24E_1^2E_2^{-2}E_3^{-3}E_6^{12}q^2$	(b)(xviii)

Table 1 continued

a, b	d	$C_r := \sum_{0\le m\le (d-r)/3} B_{3m+r}q^m, \quad r=0,1,2$	Theorem 1.3
4, −5	5	$C_0 = E_1^4E_2^{-12}E_3^8E_6^{-1} - 8E_1^7E_2^{-15}E_3^{-1}E_6^8q$	(a)(xi)
4, −2	2	$C_0 = E_3^4E_6^{-2}$	(a)(xii)
		$C_1 = -4E_1E_2^{-1}E_3E_6$	(b)(xix)
		$C_2 = 4E_1^2E_2^{-2}E_3^{-2}E_6^4$	(c)(xxix)
4, 1	5	$C_1 = -4E_1^{-1}E_2E_3^7E_6^{-2} - 4E_1^2E_2^{-2}E_3^{-2}E_6^7q$	(b)(xx)
		$C_2 = E_3^4E_6 - 8E_1^3E_2^{-3}E_3^{-5}E_6^{10}q$	(c)(xxx)
5, −4	4	$C_0 = E_1^2E_2^{-6}E_3^7E_6^{-2} - 8E_1^5E_2^{-9}E_3^{-2}E_6^7q$	(a)(xiii)
		$C_2 = 9E_1^4E_2^{-8}E_3E_6^4$	(c)(xxxi)
5, −1	4	$C_2 = 6E_1E_2^{-1}E_3^2E_6^2$	(c)(xxxii)
5, 2	7	$C_2 = 3E_1^{-1}E_2E_3^8E_6^{-1} - 24E_1^2E_2^{-2}E_3^{-1}E_6^8q$	(c)(xxxiii)
5, 5	10	$C_2 = -81E_3^5E_6^5q$	(c)(xxxiv)
6, −3	3	$C_0 = E_3^6E_6^{-3} - 8E_1^3E_2^{-3}E_3^{-3}E_6^6q$	(a)(xiv)
		$C_1 = -6E_1E_2^{-1}E_3^3$	(b)(xxi)
		$C_2 = 12E_1^2E_2^{-2}E_6^3$	(c)(xxxv)
6, 0	6	$C_2 = 9E_3^6$	(c)(xxxvi)
6, 3	9	$C_2 = 6E_1^{-2}E_2^2E_3^{12}E_6^{-3} - 42E_1E_2^{-1}E_3^3E_6^6q - 48E_1^4E_2^{-4}E_3^{-6}E_6^{15}q^2$	(c)(xxxvii)
7, −2	5	$C_0 = E_1^{-1}E_2E_3^{10}E_6^{-5} - 8E_1^2E_2^{-2}E_3E_6^4q$	(a)(xv)
		$C_2 = 16E_1E_2^{-1}E_3^4E_6 + 16E_1^4E_2^{-4}E_3^{-5}E_6^{10}q$	(c)(xxxviii)
8, −4	4	$C_2 = 24E_1^2E_2^{-2}E_3^2E_6^2$	(c)(xxxix)
9, −6	6	$C_1 = -9E_1^3E_2^{-7}E_3^8E_6^{-1} + 72E_1^6E_2^{-10}E_3^{-1}E_6^8q$	(b)(xxii)
9, −3	6	$C_1 = -9E_3^9E_6^{-3}$	(b)(xxiii)

Table 1 continued

a, b	d	$C_r := \sum_{0\le m\le (d-r)/3} B_{3m+r} q^m, \quad r = 0, 1, 2$	Theorem 1.3
9, 0	9	$C_0 = E_1^{-3}E_2^3E_3^{18}E_6^{-9} - 15E_3^9q + 48E_1^3E_2^{-3}E_6^9q^2 + 64E_1^6E_2^{-6}E_3^{-9}E_6^{18}q^3$	(a)(xvi)
10, −5	5	$C_1 = -10E_1E_2^{-1}E_3^7E_6^{-2} + 80E_1^4E_2^{-4}E_3^{-2}E_6^7q$	(b)(xxiv)
11, −4	7	$C_2 = 48E_1E_2^{-1}E_3^8E_6^{-1} + 48E_1^4E_2^{-4}E_3^{-1}E_6^8q$	(c)(xl)
12, −3	9	$C_1 = -12E_1^{-1}E_2E_3^{15}E_6^{-6} + 84E_1^2E_2^{-2}E_3^6E_6^3q + 96E_1^5E_2^{-5}E_3^{-3}E_6^{12}q^2$	(b)(xxv)
14, −7	7	$C_2 = 84E_1^2E_2^{-2}E_3^8E_6^{-1} - 672E_1^5E_2^{-5}E_3^{-1}E_6^8q$	(c)(xli)
14, −4	10	$C_2 = 81E_3^{14}E_6^{-4}$	(c)(xlii)
15, −6	9	$C_2 = 96E_1E_2^{-1}E_3^{12}E_6^{-3} - 672E_1^4E_2^{-4}E_3^3E_6^6q - 768E_1^7E_2^{-7}E_3^{-6}E_6^{15}q^2$	(c)(xliii)

6 Formulas for $\sum_{n=0}^{\infty} c(a, b; 3n + j)q^n$, $\quad j = 0, 1, 2$.

In this section we generalize the method of proof of Theorem 1.3 in order to determine formulas for $\sum_{n=0}^{\infty} c(a, b; 3n + j)q^n \quad (j = 0, 1, 2)$ valid for all integers a and b (not both 0) satisfying $a \equiv b \pmod 3$. These identities are given in Theorem 6.1. After the completion of the proof of Theorem 6.1, we apply it to sums of the type $\sum_{n=0}^{\infty} t_k(3n+j)q^n$ and $\sum_{n=0}^{\infty} r_k(3n+j)q^n$, where k is a positive integer, $j \in \{0, 1, 2\}$, and $t_k(m)$ (resp. $r_k(m)$) counts the number of representations of the nonnegative integer m as a sum of k triangular numbers (resp. squares). Letting $\mathbb{Z}$ (resp. $\mathbb{N}_0$) denote the set of integers (resp. nonnegative integers), we have

$$t_k(m) := \text{card}\{(x_1, \ldots, x_k) \in \mathbb{N}_0^k \mid m = \frac{1}{2}x_1(x_1 + 1) + \cdots + \frac{1}{2}x_k(x_k + 1)\}$$

and

$$r_k(m) := \text{card}\{(x_1, \ldots, x_k) \in \mathbb{Z}^k \mid m = x_1^2 + \cdots + x_k^2\}.$$

Theorem 6.1 *Let a and b be integers not both 0 satisfying $a \equiv b \pmod 3$. Define integers R and S by*

$$R := \frac{a + 2b}{3}, \quad S := \frac{2a + b}{3}.$$

(i) *If $a + 2b \geq 0$ and $2a + b \geq 0$, then $R \geq 0$, $S \geq 0$, $d = a + b$ (by (1.15)), and*

$$\sum_{n=0}^{\infty} c(a, b; n)q^n = E_3^{-R} E_6^{R} E_9^{2R+2S} E_{18}^{-R-S} \sum_{t=0}^{a+b} \alpha_{++}(a, b; t) E_3^{t} E_6^{-t} E_9^{-3t} E_{18}^{3t} q^t, \tag{6.1}$$

and for $j \in \{0, 1, 2\}$ we have

$$\sum_{n=0}^{\infty} c(a, b; 3n + j)q^n = E_1^{-R+j} E_2^{R-j} E_3^{2R+2S-3j} E_6^{-R-S+3j} \sum_{0 \leq t \leq (a+b-j)/3} \alpha_{++}(a, b; 3t + j) E_1^{3t} E_2^{-3t} E_3^{-9t} E_6^{9t} q^t, \tag{6.2}$$

where

$$\alpha_{++}(a, b; t) := \sum_{\substack{n_1, n_2 = 0 \\ n_1 + n_2 = t}}^{R, S} (-1)^{n_2} \binom{R}{n_1} \binom{S}{n_2} 2^{n_2}, \quad t = 0, 1, \ldots, a + b. \tag{6.3}$$

(ii) *If $a + 2b < 0$ and $2a + b \geq 0$, then $R < 0$, $S \geq 0$, $b < 0$, $d = -b = |b|$ (by (1.15)), and*

$$\sum_{n=0}^{\infty} c(a,b;n)q^n = E_3^{2|R|} E_6^{-6|R|} E_9^{3|R|+2S} E_{18}^{-S} \sum_{t=0}^{|b|} \alpha_{-+}(a,b;t) E_3^t E_6^{-t} E_9^{-3t} E_{18}^{3t} q^t, \tag{6.4}$$

and for $j \in \{0,1,2\}$ we have

$$\sum_{n=0}^{\infty} c(a,b;3n+j)q^n = E_1^{2|R|+j} E_2^{-6|R|-j} E_3^{3|R|+2S-3j} E_6^{-S+3j} \sum_{0 \leq t \leq (|b|-j)/3} \alpha_{-+}(a,b;3t+j) E_1^{3t} E_2^{-3t} E_3^{-9t} E_6^{9t} q^t, \tag{6.5}$$

where

$$\alpha_{-+}(a,b;t) := \sum_{\substack{n_1,n_2,n_3,n_4=0 \\ n_1+n_2+n_3=|R|,\ n_2+2n_3+n_4=t}}^{|R|,|R|,|R|,S} (-1)^{n_2+n_4} \binom{|R|}{n_1,n_2,n_3} \binom{S}{n_4} 2^{n_4}, \quad t = 0,1,\ldots,|b|. \tag{6.6}$$

(iii) *If $a + 2b \geq 0$ and $2a + b < 0$, then $R \geq 0$, $S < 0$, $a < 0$, $d = -a = |a|$ (by (1.15)), and*

$$\sum_{n=0}^{\infty} c(a,b;n)q^n = E_3^{-R-8|S|} E_6^{R+4|S|} E_9^{2R+6|S|} E_{18}^{-R-3|S|} \sum_{t=0}^{|a|} \alpha_{+-}(a,b;t) E_3^t E_6^{-t} E_9^{-3t} E_{18}^{3t} q^t, \tag{6.7}$$

and for $j \in \{0,1,2\}$ we have

$$\sum_{n=0}^{\infty} c(a,b;3n+j)q^n = E_1^{-R-8|S|+j} E_2^{R+4|S|-j} E_3^{2R+6|S|-3j} E_6^{-R-3|S|+3j} \sum_{0 \leq t \leq (|a|-j)/3} \alpha_{+-}(a,b;3t+j) E_1^{3t} E_2^{-3t} E_3^{-9t} E_6^{9t} q^t, \tag{6.8}$$

where

$$\alpha_{+-}(a,b;t) := \sum_{\substack{n_1,n_2,n_3,n_4=0\\ n_2+n_3+n_4=|S|,\ n_1+n_3+2n_4=t}}^{R,|S|,|S|,|S|} \binom{R}{n_1}\binom{|S|}{n_2,n_3,n_4} 2^{n_3+2n_4}, \quad t=0,1,\ldots,|a|. \tag{6.9}$$

(iv) *If* $a+2b<0$ *and* $2a+b<0$, *then* $R<0$, $S<0$, $a+b<0$, $d=-2a-2b=2|a+b|$ *(by (1.15)), and*

$$\sum_{n=0}^{\infty} c(a,b;n)q^n = E_3^{2|R|-8|S|} E_6^{-6|R|+4|S|} E_9^{3|R|+6|S|} E_{18}^{-3|S|} \sum_{t=0}^{2|a+b|} \alpha_{--}(a,b;t) E_3^{t} E_6^{-t} E_9^{-3t} E_{18}^{3t} q^t, \tag{6.10}$$

and for $j\in\{0,1,2\}$ *we have*

$$\sum_{n=0}^{\infty} c(a,b;3n+j)q^n = E_1^{2|R|-8|S|+j} E_2^{-6|R|+4|S|-j} E_3^{3|R|+6|S|-3j} E_6^{-3|S|+3j} \sum_{0\le t\le (2|a+b|-j)/3} \alpha_{--}(a,b;3t+j) E_1^{3t} E_2^{-3t} E_3^{-9t} E_6^{9t} q^t, \tag{6.11}$$

where

$$\alpha_{--}(a,b;t) := \sum_{\substack{n_1,n_2,n_3,n_4,n_5,n_6=0,\ n_1+n_2+n_3=|R|\\ n_4+n_5+n_6=|S|,\ n_2+2n_3+n_5+2n_6=t}}^{|R|,|R|,|R|,|S|,|S|,|S|} (-1)^{n_2}\binom{|R|}{n_1,n_2,n_3}\binom{|S|}{n_4,n_5,n_6} 2^{n_5+2n_6}, \quad t=0,1,\ldots,2|a+b|. \tag{6.12}$$

Proof We just prove cases (i) and (iv). Equations (6.4)–(6.9) of cases (ii) and (iii) can be deduced in a similar manner.

(i) $a+2b\ge 0$, $2a+b\ge 0$:

From (1.9), (1.10) and (1.14), we obtain using the binomial theorem and the relation $R+S=a+b$

$$\begin{aligned} E_1^a E_2^b &= \left(E_3^{-1}E_6E_9^2E_{18}^{-1} + qE_9^{-1}E_{18}^2\right)^R \left(E_9^2E_{18}^{-1} - 2qE_3E_6^{-1}E_9^{-1}E_{18}^2\right)^S \\ &= \sum_{n_1=0}^{R}\binom{R}{n_1}\left(E_3^{-1}E_6E_9^2E_{18}^{-1}\right)^{R-n_1}\left(qE_9^{-1}E_{18}^2\right)^{n_1} \\ &\quad\times \sum_{n_2=0}^{S}\binom{S}{n_2}\left(E_9^2E_{18}^{-1}\right)^{S-n_2}\left(-2qE_3E_6^{-1}E_9^{-1}E_{18}^2\right)^{n_2} \end{aligned}$$

$$= \sum_{t=0}^{a+b} \left(\sum_{\substack{n_1,n_2=0 \\ n_1+n_2=t}}^{R,S} (-1)^{n_2} \binom{R}{n_1} \binom{S}{n_2} 2^{n_2} E_3^{t-R} E_6^{R-t} E_9^{2R+2S-3t} E_{18}^{-R-S+3t} \right) q^t$$

$$= E_3^{-R} E_6^R E_9^{2R+2S} E_{18}^{-R-S} \sum_{t=0}^{a+b} \alpha_{++}(a,b;t) E_3^t E_6^{-t} E_9^{-3t} E_{18}^{3t} q^t,$$

which is (6.1).

From (1.16) and (6.1) we have

$$A_t = E_3^{-R} E_6^R E_9^{2R+2S} E_{18}^{-R-S} \alpha_{++}(a,b;t) E_3^t E_6^{-t} E_9^{-3t} E_{18}^{3t}, \quad t = 0, 1, \ldots, a+b.$$

Then, by (1.21), we have

$$\begin{aligned} B_t &= A_t(a,b,E_1,E_2,E_3,E_6) \\ &= E_1^{-R} E_2^R E_3^{2R+2S} E_6^{-R-S} \alpha_{++}(a,b;t) E_1^t E_2^{-t} E_3^{-3t} E_6^{3t}, \end{aligned}$$

so that for $j \in \{0, 1, 2\}$, appealing to (1.22)–(1.24), we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} c(a,b;3n+j) q^n \\ &= \sum_{0 \le t \le (a+b-j)/3} B_{3t+j} q^t \\ &= E_1^{-R} E_2^R E_3^{2R+2S} E_6^{-R-S} \sum_{0 \le t \le (a+b-j)/3} \alpha_{++}(a,b;3t+j) E_1^{3t+j} E_2^{-3t-j} \\ &\qquad \times E_3^{-3(3t+j)} E_6^{3(3t+j)} q^t \\ &= E_1^{-R+j} E_2^{R-j} E_3^{2R+2S-3j} E_6^{-R-S+3j} \sum_{0 \le t \le (a+b-j)/3} \alpha_{++}(a,b;3t+j) E_1^{3t} E_2^{-3t} \\ &\qquad \times E_3^{-9t} E_6^{9t} q^t, \end{aligned}$$

which is (6.2).

(iv) $a + 2b < 0, 2a + b < 0$:

In this case we have $a + b < 0$ so by (1.15) we deduce $d = -2a - 2b = 2|a + b|$. From (1.9), (1.10) and (1.14), we obtain using the multinomial theorem

$$\begin{aligned} E_1^a E_2^b &= \left(E_3^2 E_6^{-6} E_9^3 - q E_3^3 E_6^{-7} E_{18}^3 + q^2 E_3^4 E_6^{-8} E_9^{-3} E_{18}^6 \right)^{|R|} \\ &\quad \times \left(E_3^{-8} E_6^4 E_9^6 E_{18}^{-3} + 2q E_3^{-7} E_6^3 E_9^3 + 4q^2 E_3^{-6} E_6^2 E_{18}^3 \right)^{|S|} \\ &= \sum_{\substack{n_1,n_2,n_3=0 \\ n_1+n_2+n_3=|R|}}^{|R|} \binom{|R|}{n_1, n_2, n_3} \left(E_3^2 E_6^{-6} E_9^3 \right)^{n_1} \left(-q E_3^3 E_6^{-7} E_{18}^3 \right)^{n_2} \end{aligned}$$

$$\times \left(q^2 E_3^4 E_6^{-8} E_9^{-3} E_{18}^6\right)^{n_3}$$
$$\times \sum_{\substack{n_4,n_5,n_6=0 \\ n_4+n_5+n_6=|S|}}^{|S|} \binom{|S|}{n_4, n_5, n_6} \left(E_3^{-8} E_6^4 E_9^6 E_{18}^{-3}\right)^{n_4} \left(2q E_3^{-7} E_6^3 E_9^3\right)^{n_5}$$
$$\times \left(4q^2 E_3^{-6} E_6^2 E_{18}^3\right)^{n_6}$$
$$= \sum_{t=0}^{2|a+b|} \left(\sum_{\substack{n_1,n_2,n_3,n_4,n_5,n_6=0,\ n_1+n_2+n_3=|R| \\ n_4+n_5+n_6=|S|,\ n_2+2n_3+n_5+2n_6=t}}^{|R|,|R|,|R|,|S|,|S|,|S|} (-1)^{n_2} \binom{|R|}{n_1, n_2, n_3} \binom{|S|}{n_4, n_5, n_6} \right.$$
$$\left. \times 2^{n_5+2n_6} E_3^{2|R|-8|S|+t} E_6^{-6|R|+4|S|-t} E_9^{3|R|+6|S|-3t} E_{18}^{-3|S|+3t} \right) q^t$$
$$= E_3^{2|R|-8|S|} E_6^{-6|R|+4|S|} E_9^{3|R|+6|S|} E_{18}^{-3|S|}$$
$$\times \sum_{t=0}^{2|a+b|} \alpha_{--}(a, b; t) E_3^t E_6^{-t} E_9^{-3t} E_{18}^{3t} q^t, \qquad \text{(by (6.12))}$$

which is (6.10).

By (1.16) and (6.10) we have

$$A_t = E_3^{2|R|-8|S|} E_6^{-6|R|+4|S|} E_9^{3|R|+6|S|} E_{18}^{-3|S|} \alpha_{--}(a, b; t) E_3^t E_6^{-t} E_9^{-3t} E_{18}^{3t},$$
$$t = 0, 1, \ldots, 2|a+b|.$$

Thus, by (1.21), we have

$$B_t = E_1^{2|R|-8|S|} E_2^{-6|R|+4|S|} E_3^{3|R|+6|S|} E_6^{-3|S|} \alpha_{--}(a, b; t) E_1^t E_2^{-t} E_3^{-3t} E_6^{3t},$$

so that by (1.22)–(1.24) we obtain for $j \in \{0, 1, 2\}$

$$\sum_{n=0}^{\infty} c(a, b; 3n+j) q^n = \sum_{0 \le t \le (2|a+b|-j)/3} B_{3t+j} q^t$$
$$= E_1^{2|R|-8|S|} E_2^{-6|R|+4|S|} E_3^{3|R|+6|S|} E_6^{-3|S|}$$
$$\sum_{0 \le t \le (2|a+b|-j)/3} \alpha_{--}(a, b; 3t+j) E_1^{3t+j} E_2^{-3t-j} E_3^{-9t-3j} E_6^{9t+3j} q^t$$
$$= E_1^{2|R|-8|S|+j} E_2^{-6|R|+4|S|-j} E_3^{3|R|+6|S|-3j} E_6^{-3|S|+3j}$$
$$\sum_{0 \le t \le (2|a+b|-j)/3} \alpha_{--}(a, b; 3t+j) E_1^{3t} E_2^{-3t} E_3^{-9t} E_6^{9t} q^t,$$

which is (6.11). □

In our next theorem (Theorem 6.2), we apply Theorem 6.1 to the evaluation of the sums

$$\sum_{n=0}^{\infty} t_k(3n+j)q^n, \quad j=0,1,2,$$

where k is a positive integer.

Theorem 6.2 *Let k be a positive integer. Then the following identities hold*

$$\sum_{n=0}^{\infty} t_k(n)q^n = \sum_{r=0}^{k} \binom{k}{r} E_3^{r-k} E_6^{k-r} E_9^{2k-3r} E_{18}^{3r-k} q^r, \tag{6.13}$$

$$\sum_{n=0}^{\infty} t_k(3n)q^n = \sum_{0\le r\le k/3} \binom{k}{3r} E_1^{3r-k} E_2^{k-3r} E_3^{2k-9r} E_6^{9r-k} q^r, \tag{6.14}$$

$$\sum_{n=0}^{\infty} t_k(3n+1)q^n = \sum_{0\le r\le (k-1)/3} \binom{k}{3r+1} E_1^{3r+1-k} E_2^{k-3r-1} E_3^{2k-9r-3} E_6^{9r+3-k} q^r, \tag{6.15}$$

$$\sum_{n=0}^{\infty} t_k(3n+2)q^n = \sum_{0\le r\le (k-2)/3} \binom{k}{3r+2} E_1^{3r+2-k} E_2^{k-3r-2} E_3^{2k-9r-6} E_6^{9r+6-k} q^r. \tag{6.16}$$

Proof As

$$E_1^{-1}E_2^2 = \sum_{m=0}^{\infty} q^{m(m+1)/2},$$

see for example [6, Theorem 354, p. 284], we have by (1.1)

$$\sum_{n=0}^{\infty} c(-k,2k;n)q^n = E_1^{-k}E_2^{2k} = \left(E_1^{-1}E_2^2\right)^k = \left(\sum_{m=0}^{\infty} q^{m(m+1)/2}\right)^k = \sum_{n=0}^{\infty} t_k(n)q^n,$$

so $c(-k,2k;n)=t_k(n)$. Applying Theorem 6.1(i) with $a=-k$ and $b=2k$, so that $a+2b=3k>0$ and $2a+b=0$, we obtain from (6.3)

$$\alpha_{++}(-k,2k;t) = \binom{k}{t}, \quad t=0,1,\ldots,k,$$

and the asserted identities follow from (6.1) and (6.2). □

Thus, for example, we have from Eq. (6.14) of Theorem 6.2 with $k=3$ the identity

$$\sum_{n=0}^{\infty} t_3(3n)q^n = E_1^{-3}E_2^3E_3^6E_6^{-3} + qE_3^{-3}E_6^6,$$

which can be expressed as a single product since

$$\begin{aligned} E_1^{-3}E_2^3E_3^6E_6^{-3} + qE_3^{-3}E_6^6 &= E_1^{-4}E_3^{-3}E_6^{-3}(E_1E_2^3E_3^9 + qE_1^4E_6^9) \\ &= E_1^{-4}E_3^{-3}E_6^{-3} \cdot E_2^8E_3^4E_6 \qquad \text{(by Prop. 1.1(i))} \\ &= E_1^{-4}E_2^8E_3E_6^{-2}, \end{aligned}$$

and from Eq. (6.16) of Theorem 6.2 with $k = 5$

$$\sum_{n=0}^{\infty} t_5(3n+2)q^n = 10E_1^{-3}E_2^3E_3^4E_6 + qE_3^{-5}E_6^{10},$$

which cannot be expressed in the form $10E_1^{a_1}E_2^{a_2}E_3^{a_3}E_6^{a_6}$ for any integers a_1, a_2, a_3, and a_6.

Cooper and Hirschhorn [9] (see also [10, 11]) have shown in an elementary combinatorial way that

$$r_k(8n+k) = c_k t_k(n) \tag{6.17}$$

for all nonnegative integers n and $k \in \{1, 2, 3, 4, 5, 6, 7\}$, where

$$c_1 = 2,\ c_2 = 4,\ c_3 = 8,\ c_4 = 24,\ c_5 = 112,\ c_6 = 544,\ c_7 = 2368. \tag{6.18}$$

Using (6.17) and (6.18) in Eqs. (6.13)–(6.16) of Theorem 6.2, we obtain the following results: For $k \in \{1, 2, 3, 4, 5, 6, 7\}$ we have

$$\sum_{n=0}^{\infty} r_k(8n+k)q^n = c_k \sum_{r=0}^{k} \binom{k}{r} E_3^{r-k}E_6^{k-r}E_9^{2k-3r}E_{18}^{3r-k}q^r, \tag{6.19}$$

$$\sum_{n=0}^{\infty} r_k(24n+k)q^n = c_k \sum_{0\le r\le k/3} \binom{k}{3r} E_1^{3r-k}E_2^{k-3r}E_3^{2k-9r}E_6^{9r-k}q^r, \tag{6.20}$$

$$\sum_{n=0}^{\infty} r_k(24n+8+k)q^n = c_k \sum_{0\le r\le (k-1)/3} \binom{k}{3r+1} \times E_1^{3r+1-k}E_2^{k-3r-1}E_3^{2k-9r-3}E_6^{9r+3-k}q^r, \tag{6.21}$$

$$\sum_{n=0}^{\infty} r_k(24n+16+k)q^n = c_k \sum_{0\le r\le (k-2)/3} \binom{k}{3r+2} \times E_1^{3r+2-k}E_2^{k-3r-2}E_3^{2k-9r-6}E_6^{9r+6-k}q^r. \tag{6.22}$$

If the right hand side of one of the identities (6.19)–(6.22) can be reduced for a particular value of k to a constant multiple of a finite product of powers of E_1, E_2, E_3, and E_6 then that identity is known and can be found in different notation in one of the three papers Hirschhorn and McGowan [12] ($k = 2, 4$), Cooper and Hirschhorn [13] ($k = 3$), Barruncand et al. [14] ($k = 5, 6, 7$). This occurs for example for (6.22) when

$k = 3$. In this case we have

$$\sum_{n=0}^{\infty} r_3(24n+19)q^n = 24E_1^{-1}E_2E_6^3.$$

Cooper and Hirschhorn [13, Eq. (2.23), p. 13] give

$$\sum_{n=0}^{\infty} r_3(24n+19)q^n = 24\psi(q)\{c(q^2)/3\}.$$

These two formulas agree as $\psi(q) = E_1^{-1}E_2^2$ and $c(q^2)/3 = E_2^{-1}E_6^3$. Another example occurs with (6.20) when $k = 3$. In this case we have

$$\begin{aligned}\sum_{n=0}^{\infty} r_3(24n+3)q^n &= 8\left(E_1^{-3}E_2^3E_3^6E_6^{-3} + qE_3^{-3}E_6^6\right)\\ &= 8E_1^{-4}E_3^{-3}E_6^{-3}\left(E_1E_2^3E_3^9 + qE_1^4E_6^9\right)\\ &= 8E_1^{-4}E_3^{-3}E_6^{-3}\cdot E_2^8E_3^4E_6 \qquad \text{(by Prop. 1.1(i))}\\ &= 8E_1^{-4}E_2^8E_3E_6^{-2}\\ &= 8\psi^4(q)/\psi(q^3),\end{aligned}$$

which is Eq. (1.19) on page 11 of [13]. A particularly nice example is (6.22) for $k = 7$. In this case (6.22) gives

$$\begin{aligned}\sum_{n=0}^{\infty} r_7(24n+23)q^n &= 2368\left(21E_1^{-5}E_2^5E_3^8E_6^{-1} + 21E_1^{-2}E_2^2E_3^{-1}E_6^8q\right)\\ &= 49728E_1^{-6}E_2^2E_3^{-1}E_6^{-1}\left(E_1E_2^3E_3^9 + qE_1^4E_6^9\right)\\ &= 49728E_1^{-6}E_2^2E_3^{-1}E_6^{-1}\cdot E_2^8E_3^4E_6 \qquad \text{(by Prop. 1.1(i))}\\ &= 49728E_1^{-6}E_2^{10}E_3^3,\end{aligned}$$

which Barrucand, Cooper and Hirschhorn give in the abstract of their paper [14].

The identities (6.19)–(6.22) which cannot be reduced to a single product are new. One such example is (6.20) when $k = 4$. In this case the identity asserts

$$\sum_{n=0}^{\infty} r_4(24n+4)q^n = 24\left(E_1^{-4}E_2^4E_3^8E_6^{-4} + 4qE_1^{-1}E_2E_3^{-1}E_6^5\right)$$

or equivalently (as $r_4(24n+4) = 3r_4(6n+1)$)

$$\sum_{n=0}^{\infty} r_4(6n+1)q^n = 8E_1^{-4}E_2^4E_3^8E_6^{-4} + 32qE_1^{-1}E_2E_3^{-1}E_6^5,$$

where the right hand side cannot be expressed as $8E_1^{a_1}E_2^{a_2}E_3^{a_3}E_6^{a_6}$ for any integers a_1, a_2, a_3, and a_6.

Our next theorem gives the analogous theorem to Theorem 6.2 for $r_k(n)$.

Theorem 6.3 *Let k be a positive integer. Then the following identities hold*

$$\sum_{n=0}^{\infty} r_k(n)q^n = \sum_{r=0}^{k} 2^r \binom{k}{r} E_3^{-r} E_6^{2r} E_9^{3r-2k} E_{12}^{-r} E_{18}^{5k-6r} E_{36}^{3r-2k} q^r, \tag{6.23}$$

$$\sum_{n=0}^{\infty} r_k(3n)q^n = \sum_{0\le r\le k/3} 2^{3r} \binom{k}{3r} E_1^{-3r} E_2^{6r} E_3^{9r-2k} E_4^{-3r} E_6^{5k-18r} E_{12}^{9r-2k} q^r, \tag{6.24}$$

$$\sum_{n=0}^{\infty} r_k(3n+1)q^n = \sum_{0\le r\le (k-1)/3} 2^{3r+1} \binom{k}{3r+1} E_1^{-3r-1} E_2^{6r+2} E_3^{9r+3-2k} E_4^{-3r-1} E_6^{5k-18r-6} E_{12}^{9r+3-2k} q^r, \tag{6.25}$$

$$\sum_{n=0}^{\infty} r_k(3n+2)q^n = \sum_{0\le r\le (k-2)/3} 2^{3r+2} \binom{k}{3r+2} E_1^{-3r-2} E_2^{6r+4} E_3^{9r+6-2k} E_4^{-3r-2} E_6^{5k-18r-12} E_{12}^{9r+6-2k} q^r. \tag{6.26}$$

Proof We have

$$\left(\sum_{m=-\infty}^{\infty} q^{m^2}\right)^k = \sum_{m_1,\ldots,m_k=-\infty}^{\infty} q^{m_1^2+\cdots+m_k^2} = \sum_{n=0}^{\infty} \sum_{\substack{m_1,\ldots,m_k=-\infty \\ m_1^2+\cdots+m_k^2=n}}^{\infty} q^{m_1^2+\cdots+m_k^2}$$

so that

$$\left(\sum_{m=-\infty}^{\infty} q^{m^2}\right)^k = \sum_{n=0}^{\infty} q^n r_k(n).$$

Replacing q by $-q$, and appealing to (2.11) and (1.1), we obtain

$$\sum_{n=0}^{\infty} r_k(n)(-q)^n = \left(\sum_{m=-\infty}^{\infty} (-q)^{m^2}\right)^k = \left(E_1^2 E_2^{-1}\right)^k$$
$$= E_1^{2k} E_2^{-k} = \sum_{n=0}^{\infty} c(2k, -k; n)q^n$$

so

$$(-1)^n r_k(n) = c(2k, -k; n).$$

Hence, with $a = 2k, b = -k$, so that $R = 0$, $S = k$, we obtain from (6.3)

$$\alpha_{++}(2k, -k; t) = (-1)^t \binom{k}{t} 2^t, \quad t = 0, 1, \ldots, k,$$

so by Eq. (6.1) of Theorem 6.1, we deduce

$$\sum_{n=0}^{\infty} r_k(n)(-q)^n = \sum_{t=0}^{k} 2^t \binom{k}{t} E_3^t E_6^{-t} E_9^{2k-3t} E_{18}^{3t-k} (-q)^t.$$

Mapping q to $-q$, we obtain

$$\sum_{n=0}^{\infty} r_k(n) q^n = \sum_{t=0}^{k} 2^t \binom{k}{t} E_3^t(-q) E_6^{-t}(-q) E_9^{2k-3t}(-q) E_{18}^{3t-k}(-q) q^t.$$

Now

$$E_m(-q) = \begin{cases} E_m & \text{if } m \text{ is even,} \\ E_m^{-1} E_{2m}^3 E_{4m}^{-1} & \text{if } m \text{ is odd,} \end{cases}$$

so

$$\begin{aligned} \sum_{n=0}^{\infty} r_k(n) q^n &= \sum_{t=0}^{k} 2^t \binom{k}{t} \left(E_3^{-1} E_6^3 E_{12}^{-1}\right)^t E_6^{-t} \left(E_9^{-1} E_{18}^3 E_{36}^{-1}\right)^{2k-3t} E_{18}^{3t-k} q^t \\ &= \sum_{t=0}^{k} 2^t \binom{k}{t} E_3^{-t} E_6^{2t} E_9^{3t-2k} E_{12}^{-t} E_{18}^{5k-6t} E_{36}^{3t-2k} q^t, \end{aligned}$$

which is (6.23). Finally, by Eq. (6.2) of Theorem 6.1, we obtain for $j \in \{0, 1, 2\}$

$$\begin{aligned} &\sum_{n=0}^{\infty} (-1)^{3n+j} r_k(3n + j) q^n \\ &\quad = \sum_{n=0}^{\infty} c(2k, -k; 3n + j) q^n \\ &\quad = E_1^j E_2^{-j} E_3^{2k-3j} E_6^{-k+3j} \\ &\qquad \times \sum_{0 \le t \le (k-j)/3} (-1)^{3t+j} 2^{3t+j} \binom{k}{3t+j} E_1^{3t} E_2^{-3t} E_3^{-9t} E_6^{9t} q^t. \end{aligned}$$

Cancelling $(-1)^j$, and then mapping q to $-q$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} r_k(3n+j)q^n \\
&\quad= \sum_{0\le t\le (k-j)/3} 2^{3t+j}\binom{k}{3t+j} E_1^{3t+j}(-q)E_2^{-3t-j}(-q)E_3^{2k-3j-9t}(-q) \\
&\qquad\quad \times E_6^{-k+3j+9t}(-q)q^t \\
&\quad= \sum_{0\le t\le (k-j)/3} 2^{3t+j}\binom{k}{3t+j}\left(E_1^{-1}E_2^3E_4^{-1}\right)^{3t+j} E_2^{-3t-j} \\
&\qquad\quad \times \left(E_3^{-1}E_6^3E_{12}^{-1}\right)^{2k-3j-9t} E_6^{-k+3j+9t} q^t \\
&\quad= \sum_{0\le t\le (k-j)/3} 2^{3t+j}\binom{k}{3t+j} E_1^{-3t-j}E_2^{6t+2j}E_3^{9t+3j-2k} \\
&\qquad\quad \times E_4^{-3t-j}E_6^{5k-6j-18t}E_{12}^{3j+9t-2k}q^t.
\end{aligned}$$

This proves identities (6.24)–(6.26) of Theorem 6.3. □

We conclude this section with an example of Theorem 6.3 with $k > 7$. We take $k = 8$ in Eq. (6.26) of Theorem 6.3. We obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} r_8(3n+2)q^n &= 112\prod_{n=1}^{\infty}\frac{\left(1-q^{2n}\right)^4\left(1-q^{6n}\right)^{28}}{(1-q^n)^2\left(1-q^{3n}\right)^{10}\left(1-q^{4n}\right)^2\left(1-q^{12n}\right)^{10}} \\
&\quad + 1792q\prod_{n=1}^{\infty}\frac{\left(1-q^{2n}\right)^{10}\left(1-q^{6n}\right)^{10}}{(1-q^n)^5\left(1-q^{3n}\right)\left(1-q^{4n}\right)^5\left(1-q^{12n}\right)} \\
&\quad + 256q^2\prod_{n=1}^{\infty}\frac{\left(1-q^{2n}\right)^{16}\left(1-q^{3n}\right)^8\left(1-q^{12n}\right)^8}{(1-q^n)^8\left(1-q^{4n}\right)^8\left(1-q^{6n}\right)^8}.
\end{aligned}$$

Author Contributions FM, LP, KSW were all involved in the research and the writing of the paper describing the research. The first author carried out the computer programming.

Declarations

Conflicts of interest The authors declare no competing interests.

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