ARITHMETIC PROPERTIES OF THE TERNARY QUADRATIC FORM $3x^2 + 6y^2 + 14z^2 + 4yz + 2zx + 2xy$

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Abstract

In this paper we study the arithmetic properties of the ternary quadratic form $3x^2 + 6y^2 + 14z^2 + 4yz + 2zx + 2xy$. This is the ternary quadratic form of least discriminant, which is conjectured, but has not been proven, to be regular.

1. Introduction

We denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{C}$ the sets of positive integers, non-negative integers, all integers, and complex numbers, respectively. Furthermore, we denote the upper-half plane of $\mathbb{C}$ by

$$\mathcal{H} := \{x + iy \in \mathbb{C} \mid y > 0\}.$$

We consider ternary quadratic forms $ax^2+by^2+cz^2+dyz+ezx+fxy$, which are integral ($a, b, c, d, e, f \in \mathbb{Z}$), primitive ($\gcd(a, b, c, d, e, f) = 1$) and positive-definite. The discriminant of such a ternary form is the positive integer

$$\frac{1}{2}\begin{vmatrix} 2a & f & e \\ f & 2b & d \\ e & d & 2c \end{vmatrix}.$$

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The representation number $r(a,b,c,d,e,f;n)$ of a ternary $ax^2+by^2+cz^2+dyz+ezx+fxy$ is defined for all $n \in \mathbb{N}_0$ by

$$r(a,b,c,d,e,f;n) := \operatorname{card}\{(x,y,z) \in \mathbb{Z}^3 \mid ax^2+by^2+cz^2+dyz+ezx+fxy=n\},$$

so that $r(a,b,c,d,e,f;0)=1$. For $n \notin \mathbb{N}_0$ we define $r(a,b,c,d,e,f;n)=0$. We also define

$$\omega_m := e^{\frac{2\pi i}{m}} \quad (m \in \mathbb{N}),$$
$$\theta(w) := \sum_{x=-\infty}^{\infty} e^{2\pi i w x^2} \quad (w \in \mathcal{H}),$$

and

$$\theta(r;w) := \sum_{(x_1,\ldots,,x_k)\in\mathbb{Z}^k} e^{2\pi i r(x_1,\ldots,x_k)w} \quad (w \in \mathcal{H}),$$

where $r(x_1,\ldots,x_k)$ is a positive-definite quadratic form with integral coefficients.

If m is a nonzero integer and p is a prime, then $\nu_p(m)$ is the nonnegative integer such that $p^{\nu_p(m)} \mid m$ and $p^{\nu_p(m)+1} \nmid m$. We write $p^{\nu_p(m)} \mid\mid m$.

A ternary quadratic form that represents all the positive integers represented by its genus is called regular. In 1997, Jagy, Kaplansky and Schiemann [8] gave a list of 913 integral positive-definite ternary quadratic forms. Of these 913 forms, exactly 794 of them belong to a genus of size 1, and thus are regular. Of the remaining 119 forms in genera of size at least 2, the regularity of 97 of them was established by the authors. This left 22 forms conjectured to be regular, but for which no proof existed. Oh [9] proved that 8 of these 22 forms are regular. The remaining 14 forms are conjectured to be regular, but this remains unproven. Olivier [10] has proved that all 14 forms are regular assuming the validity of a form of the Generalized Riemann Hypothesis.

The 14 forms only conditionally proved to be regular comprise 11 forms each in a genus of size 2 and three forms each in a genus of size 3. The form with the smallest discriminant among the 11 forms, each in a genus of size 2, is the form

$$f(x,y,z) := 3x^2+6y^2+14z^2+4yz+2zx+2xy.$$

In this work we use the theory of modular forms to study the arithmetic properties of $f(x,y,z)$ including the explicit determination of $r(f;n)$ for all $n \in \mathbb{N}$. In doing this it is convenient to introduce three other ternary quadratic forms:

$$g(x,y,z) := 6x^2+7y^2+7z^2+2yz+2zx+6xy,$$
$$h(x,y,z) := 3x^2+4y^2+6z^2+4yz+2zx,$$
$$k(x,y,z) := x^2+3y^2+5z^2+2yz.$$

The discriminants of f, g, h, and k are

$$\operatorname{disc}(f) = \frac{1}{2}\begin{vmatrix} 6 & 2 & 2 \\ 2 & 12 & 4 \\ 2 & 4 & 28 \end{vmatrix} = 896, \qquad \operatorname{disc}(g) = \frac{1}{2}\begin{vmatrix} 12 & 6 & 2 \\ 6 & 14 & 2 \\ 2 & 2 & 14 \end{vmatrix} = 896,$$

$$\operatorname{disc}(h) = \frac{1}{2}\begin{vmatrix} 6 & 0 & 2 \\ 0 & 8 & 4 \\ 2 & 4 & 12 \end{vmatrix} = 224, \qquad \operatorname{disc}(k) = \frac{1}{2}\begin{vmatrix} 2 & 0 & 0 \\ 0 & 6 & 2 \\ 0 & 2 & 10 \end{vmatrix} = 56.$$

The form g is the genus mate of the ternary f [3]. The ternary h is alone in its genus [3] and so is regular. Its discriminant differs from that of f by a square (namely 2^2). The ternary k is also alone in its genus [3], thus regular, and its discriminant differs from that of f by a square (namely 4^2).

We are able to prove a necessary and sufficient condition for f to be regular and in doing so we exhibit explicitly the local impediment to proving unconditionally that f is regular. Our arithmetic study of the ternary quadratic form f begins with the determination of the sum $r(f;n) + r(g;n)$ in terms of $r(h;n)$. We prove, using a result about modular forms from our recent paper with Z. S. Aygin and G. Doyle [2], the following theorem in Section 2.

Theorem 1.1. *For all $n \in \mathbb{N}_0$ we have*

$$r(f;n) + r(g;n) = s(n)r(h;n),$$

where

$$s(n) = \begin{cases} 1 & \text{if } n \equiv 1, 2, 3 \pmod 4, \\ 2 & \text{if } n \equiv 0, 12, 28 \pmod{32}, \\ 0 & \text{if } n \equiv 4, 8, 20 \pmod{32}, \\ \frac{2}{3} & \text{if } n \equiv 16 \pmod{32}, \\ \frac{4}{3} & \text{if } n \equiv 24 \pmod{32}. \end{cases}$$

In Section 3 we use the theory of modular forms to determine $r(h;n)$ in terms of $r(k;n)$.

Theorem 1.2. *For all $n \in \mathbb{N}_0$ we have*

$$r(h;n) = t(n)r(k;n),$$

where

$$t(n) = \begin{cases} 1 & \text{if } n \equiv 0, 3, 7 \pmod 8, \\ 0 & \text{if } n \equiv 1, 2, 5 \pmod 8, \\ \frac{1}{3} & \text{if } n \equiv 4 \pmod 8, \\ \frac{2}{3} & \text{if } n \equiv 6 \pmod 8. \end{cases}$$

From Theorems 1.1 and 1.2 we immediately obtain the relationship between $r(f;n)+r(g;n)$ and $r(k;n)$.

Theorem 1.3. *For all $n \in \mathbb{N}_0$ we have*

$$r(f;n)+r(g;n)=u(n)r(k;n), \tag{1.1}$$

where

$$u(n)=s(n)t(n)=\begin{cases} 2 & \text{if } n\equiv 0 \pmod{32},\\ \frac{4}{3} & \text{if } n\equiv 24 \pmod{32},\\ 1 & \text{if } n\equiv 3 \pmod{4},\\ \frac{2}{3} & \text{if } n\equiv 6 \pmod{8}, n\equiv 12 \pmod{16} \text{ or } n\equiv 16 \pmod{32},\\ 0 & \text{if } n\equiv 1 \pmod{4}, n\equiv 2 \pmod{8}, n\equiv 4 \pmod{16} \text{ or}\\ & \quad n\equiv 8 \pmod{32}. \end{cases} \tag{1.2}$$

In Section 4 we use an elementary argument to show that $r(f;n)$ and $r(g;n)$ are equal when $n \not\equiv 3 \pmod 4$.

Theorem 1.4. *For $n\in\mathbb{N}_0$ with $n\not\equiv 3 \pmod 4$ we have $r(f;n)=r(g;n)$.*

As the ternary quadratic form k is alone in its genus, we can apply Siegel's formula to determine $r(k;n)$.

Siegel's formula. (see for example [4, pp. 374–378]) *Let $s := s(x,y,z)$ be an integral, primitive, positive-definite ternary quadratic form, which is alone in its genus. Then for $n\in\mathbb{N}$,*

$$r(s;n)=\frac{4\pi\sqrt{n}}{\sqrt{\operatorname{disc}(s)}}\prod_p d_p(s;n), \tag{1.3}$$

where p runs through primes, and the local density $d_p(s;n)$ is defined by

$$d_p(s;n) := \lim_{t\to\infty}\frac{\text{number of solutions of } s(x,y,z)\equiv n \pmod{p^t}}{p^{2t}}. \tag{1.4}$$

In using Siegel's formula to determine $r(k;n)$ in Section 5, and in subsequent sections, we use the following notation. For $n\in\mathbb{N}$ we define $\alpha,\beta\in\mathbb{N}_0$ and $g,h\in\mathbb{N}$ uniquely in terms of n by

$$n=2^\alpha 7^\beta gh^2, \quad (gh,14)=1, \quad g \text{ squarefree}. \tag{1.5}$$

Further, we define the squarefree positive integer n^* by

$$n^* := 2^{\alpha+1-2[(\alpha+1)/2]}\, 7^{\beta+1-2[(\beta+1)/2]}\, g \tag{1.6}$$

and the positive integer $l(n)$ by

$$l(n) := \prod_{p|h} \left(\sigma(p^{\nu_p(h)}) - \left(\frac{-n^*}{p}\right) \sigma(p^{\nu_p(h)-1}) \right). \tag{1.7}$$

The class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-n^*})$ is denoted by $h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right)$.

Theorem 1.5. *For $n \in \mathbb{N}$ we have*

$$r(k;n) = c_1(n) l(n) h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right),$$

where the values of $c_1(n)$ are given in Table 1.1.

α, β	g	$c_1(n)$
	α*(even)* β*(even)*	
$\alpha = 0, \beta \geq 0$		$\frac{1}{6} \cdot 7^{(\beta+2)/2} - \frac{2}{3}$
$\alpha \geq 2, \beta \geq 0$		$\frac{1}{2} \cdot 7^{(\beta+2)/2} - 2$
	α*(even)* β*(odd)*	
$\alpha = 0, \beta \geq 1$	$g \equiv 1, 2, 4 \pmod 7$	$\frac{4}{3} \cdot 7^{(\beta+1)/2} - \frac{4}{3}$
	$g \equiv 3, 5, 6 \pmod 7$	$7^{(\beta+1)/2}$
$\alpha \geq 2, \beta \geq 1$	$g \equiv 1, 2, 4 \pmod 7$	$4 \cdot 7^{(\beta+1)/2} - 4$
	$g \equiv 3, 5, 6 \pmod 7$	$3 \cdot 7^{(\beta+1)/2}$
	α*(odd)* β*(even)*	
$\alpha \geq 1, \beta \geq 0$	$g \equiv 1 \pmod 8$	0
	$g \equiv 3 \pmod 4$	$\frac{1}{2} \cdot 7^{(\beta+2)/2} - 2$
	$g \equiv 5 \pmod 8$	$7^{(\beta+2)/2} - 4$
	α*(odd)* β*(odd)*	
$\alpha \geq 1, \beta \geq 1$	$g = 1$	$2 \cdot 7^{(\beta+1)/2} - 2$
	$g \equiv 1 \pmod 4, g \equiv 1, 2, 4 \pmod 7, g \neq 1$	$4 \cdot 7^{(\beta+1)/2} - 4$
	$g = 3$	$2 \cdot 7^{(\beta+1)/2}$
	$g \equiv 3 \pmod 8, g \equiv 3, 5, 6 \pmod 7, g \neq 3$	$6 \cdot 7^{(\beta+1)/2}$
	$g \equiv 1 \pmod 4, g \equiv 3, 5, 6 \pmod 7$	$3 \cdot 7^{(\beta+1)/2}$
	$g \equiv 3 \pmod 8, g \equiv 1, 2, 4 \pmod 7$	$8 \cdot 7^{(\beta+1)/2} - 8$
	$g \equiv 7 \pmod 8$	0

Table 1.1: Values of $c_1(n)$

Theorem 1.5 enables us to determine in Section 6 precisely which positive integers n are represented by $k(x, y, z)$.

Theorem 1.6. *Let $n \in \mathbb{N}$. Then n is represented by $k(x,y,z)$ if and only if n is not of the form $4^k(16l+2)$ for any $k, l \in \mathbb{N}_0$.*

We remark that Theorem 1.6 proves the assertion for form no. 28 in [6, Appendix, Table 1]. Note that the discriminant used in [6] differs from the discriminant here by a factor of 4. As $4^k(16l+2)$ is an even positive integer for all $k, l \in \mathbb{N}_0$, Theorem 1.7 is an immediate consequence of Theorem 1.6.

Theorem 1.7. *Every odd positive integer is represented by $k(x,y,z)$.*

From Theorems 1.3, 1.4, and 1.5 we deduce immediately an explicit formula for $r(f;n) = r(g;n)$, when $n \not\equiv 3 \pmod 4$.

Theorem 1.8. *Let $n \in \mathbb{N}$ be such that $n \not\equiv 3 \pmod 4$. Define α, β, g, h, n^*, and $l(n)$ as in (1.5), (1.6), and (1.7). Then*

$$r(f;n) = r(g;n) = c_2(n) l(n) h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right),$$

where the values of $c_2(n)$ are given in Table 1.2.

It is clear from Theorems 1.3, 1.4, and 1.5 that

$$c_2(n) = \frac{1}{2} u(n) c_1(n), \; n \not\equiv 3 \pmod 4.$$

As $n \not\equiv 3 \pmod 4$ in Theorem 1.8 the possibilities

$$\alpha = 0, \; \beta \text{ even}, \; g \equiv 3 \pmod 4$$

and

$$\alpha = 0, \; \beta \text{ odd}, \; g \equiv 1 \pmod 4$$

are excluded in Table 1.2 by (1.5).

α, β	g	$c_2(n)$
	α(even) β(even)	
$\alpha = 0, \beta \geq 0$	$g \equiv 1 \pmod 4$	0
$\alpha = 2, \beta \geq 0$	$g \equiv 1 \pmod 4$	0
	$g \equiv 3 \pmod 4$	$\frac{1}{6} \cdot 7^{(\beta+2)/2} - \frac{2}{3}$
$\alpha = 4, \beta \geq 0$		$\frac{1}{6} \cdot 7^{(\beta+2)/2} - \frac{2}{3}$
$\alpha \geq 6, \beta \geq 0$		$\frac{1}{2} \cdot 7^{(\beta+2)/2} - 2$
		Continued on next page

α, β	g	$c_2(n)$
	α(even) β(odd)	
$\alpha = 0, \beta \geq 1$	$g \equiv 3 \pmod 4$	0
$\alpha = 2, \beta \geq 1$	$g \equiv 1 \pmod 4, g \equiv 1, 2, 4 \pmod 7$	$\frac{4}{3} \cdot 7^{(\beta+1)/2} - \frac{4}{3}$
	$g \equiv 1 \pmod 4, g \equiv 3, 5, 6 \pmod 7$	$7^{(\beta+1)/2}$
	$g \equiv 3 \pmod 4$	0
$\alpha = 4, \beta \geq 1$	$g \equiv 1, 2, 4 \pmod 7$	$\frac{4}{3} \cdot 7^{(\beta+1)/2} - \frac{4}{3}$
	$g \equiv 3, 5, 6 \pmod 7$	$7^{(\beta+1)/2}$
$\alpha \geq 6, \beta \geq 1$	$g \equiv 1, 2, 4 \pmod 7$	$4 \cdot 7^{(\beta+1)/2} - 4$
	$g \equiv 3, 5, 6 \pmod 7$	$3 \cdot 7^{(\beta+1)/2}$
	α(odd) β(even)	
$\alpha = 1, \beta \geq 0$	$g \equiv 1 \pmod 4$	0
	$g \equiv 3 \pmod 4$	$\frac{1}{6} \cdot 7^{(\beta+2)/2} - \frac{2}{3}$
$\alpha = 3, \beta \geq 0$	$g \equiv 1 \pmod 4$	0
	$g \equiv 3 \pmod 4$	$\frac{1}{3} \cdot 7^{(\beta+2)/2} - \frac{4}{3}$
$\alpha \geq 5, \beta \geq 0$	$g \equiv 1 \pmod 8$	0
	$g \equiv 3 \pmod 4$	$\frac{1}{2} \cdot 7^{(\beta+2)/2} - 2$
	$g \equiv 5 \pmod 8$	$7^{(\beta+2)/2} - 4$
	α(odd) β(odd)	
$\alpha = 1, \beta \geq 1$	$g = 1$	$\frac{2}{3} \cdot 7^{(\beta+1)/2} - \frac{2}{3}$
	$g \equiv 1 \pmod 4, g \equiv 1, 2, 4 \pmod 7, g \neq 1$	$\frac{4}{3} \cdot 7^{(\beta+1)/2} - \frac{4}{3}$
	$g \equiv 1 \pmod 4, g \equiv 3, 5, 6 \pmod 7$	$7^{(\beta+1)/2}$
	$g \equiv 3 \pmod 4$	0
$\alpha = 3, \beta \geq 1$	$g = 1$	$\frac{4}{3} \cdot 7^{(\beta+1)/2} - \frac{4}{3}$
	$g \equiv 1 \pmod 4, g \equiv 1, 2, 4 \pmod 7, g \neq 1$	$\frac{8}{3} \cdot 7^{(\beta+1)/2} - \frac{8}{3}$
	$g \equiv 1 \pmod 4, g \equiv 3, 5, 6 \pmod 7$	$2 \cdot 7^{(\beta+1)/2}$
	$g \equiv 3 \pmod 4$	0
$\alpha \geq 5, \beta \geq 1$	$g = 1$	$2 \cdot 7^{(\beta+1)/2} - 2$
	$g \equiv 1 \pmod 4, g \equiv 1, 2, 4 \pmod 7, g \neq 1$	$4 \cdot 7^{(\beta+1)/2} - 4$
	$g = 3$	$2 \cdot 7^{(\beta+1)/2}$
	$g \equiv 3 \pmod 8, g \equiv 3, 5, 6 \pmod 7, g \neq 3$	$6 \cdot 7^{(\beta+1)/2}$
	$g \equiv 1 \pmod 4, g \equiv 3, 5, 6 \pmod 7$	$3 \cdot 7^{(\beta+1)/2}$
	$g \equiv 3 \pmod 8, g \equiv 1, 2, 4 \pmod 7$	$8 \cdot 7^{(\beta+1)/2} - 8$
	$g \equiv 7 \pmod 8$	0

Table 1.2: Values of $c_2(n)$: $n \not\equiv 3 \pmod 4$

In Section 7 we use Theorem 1.8 to determine the positive integers $n \not\equiv 3 \pmod 4$ which are represented by f.

Theorem 1.9. *Let $n \in \mathbb{N}$ be such that $n \not\equiv 3 \pmod 4$. Then n is represented by f if and only if n is not of the form*

$$4l + 1, 16l + 4, 16l + 10, 64l + 40, 4^k(16l + 2),$$

for any $k, l \in \mathbb{N}_0$.

In Section 8 we deduce from Theorems 1.3 and 1.6 the integers represented by the genus containing the forms f and g.

Theorem 1.10. *A positive integer n is represented by the genus $\{f, g\}$ if and only if n is not of the form*

$$4l + 1, 16l + 4, 16l + 10, 64l + 40, 4^k(16l + 2),$$

for any $k, l \in \mathbb{N}_0$.

The form g does not represent 3, so by Theorem 1.10 g is not regular. From Theorems 1.9 and 1.10 we deduce a necessary and sufficient condition for the form f to be regular.

Theorem 1.11. *The form f is regular if and only if f represents every positive integer $n \equiv 3 \pmod 4$.*

Proving that f represents every $n \in \mathbb{N}$ with $n \equiv 3 \pmod 4$ is the local impediment to proving that f is regular. Theorem 1.7 is perhaps a natural starting place to attempt a proof of the regularity of f. By Theorem 1.7 every positive integer $n \equiv 3 \pmod 4$ is represented by $x^2 + 3y^2 + 5z^2 + 2yz$. Thus there are integers u, v, and w such that

$$n = u^2 + 3v^2 + 5w^2 + 2vw. \tag{1.8}$$

Taking equation (1.8) modulo 4, we deduce that

$$u \equiv v \equiv w \equiv 1 \pmod 2, \text{ or} \tag{1.9}$$

$$u \equiv w \equiv 0 \pmod 2, v \equiv 1 \pmod 2, u \equiv w \pmod 4, \text{ or} \tag{1.10}$$

$$u \equiv w \equiv 0 \pmod 2, v \equiv 1 \pmod 2, u \equiv w + 2 \pmod 4. \tag{1.11}$$

If (1.9) holds, as (u, v, w) and $(u, -v, -w)$ are both solutions of (1.8), one of these solutions satisfies $u \equiv w \pmod 4$. Thus, if (1.9) or (1.10) holds, we may define integers x, y, and z such that

$$x = -v,\ y = -\frac{1}{4}(u + 3w),\ z = \frac{1}{4}(u - w). \tag{1.12}$$

These integers satisfy

$$3x^2 + 6y^2 + 14z^2 + 4yz + 2zx + 2xy = u^2 + 3v^2 + 5w^2 + 2vw = n$$

so n is represented by the form f. However, it does not seem possible to create a solution (x, y, z) of $f(x, y, z) = n$ in a similar way from a solution (u, v, w) of type (1.11).

We can prove, however, that an infinite number of the positive integers $n \equiv 3 \pmod 4$ are represented by the form f.

Theorem 1.12. *Let $n \in \mathbb{N}$ be such that $n \equiv 3 \pmod 4$ and $n \equiv 0 \pmod 7$, equivalently $n \equiv 7 \pmod{28}$. Then n is represented by the form f.*

Proof. Let $n \in \mathbb{N}$ be such that $n \equiv 7 \pmod{28}$. As $n \equiv 3 \pmod 4$, by Theorem 1.7, n is represented by the form k. Hence, there exist integers u, v, w such that (1.8) holds. From (1.9) – (1.11) u, v, w satisfy $u \equiv w \pmod 4$ or

$$u \equiv w \equiv 0 \pmod 2, v \equiv 1 \pmod 2, u \equiv w + 2 \pmod 4. \tag{1.13}$$

In the former case the integers x, y, z defined in (1.12) satisfy $n = 3x^2 + 6y^2 + 14z^2 + 4yz + 2zx + 2xy$. If the latter case does not occur then this proves that n is represented by f. Now suppose that the latter case does occur. Then

$$\begin{aligned} 3n &= 3u^2 + 9v^2 + 6vw + 15w^2 \\ &= 3u^2 + (3v + w)^2 + 14w^2 \\ &\equiv 3u^2 + (3v + w)^2 \pmod 7 \\ &\equiv (3v + w)^2 - 4u^2 \pmod 7 \\ &\equiv (3v + w - 2u)(3v + w + 2u) \pmod 7. \end{aligned}$$

As $n \equiv 0 \pmod 7$ we have

$$3v + w - 2u \equiv 0 \pmod 7 \text{ or } 3v + w + 2u \equiv 0 \pmod 7.$$

Changing u to $-u$ does not affect (1.8) or (1.11) (as $-u \equiv -w-2 \equiv w+2 \pmod 4$). Thus we may suppose that $3v + w - 2u \equiv 0 \pmod 7$ so

$$w \equiv 2u + 4v \pmod 7. \tag{1.14}$$

From (1.13) we have

$$w \equiv u + 2v \pmod 4. \tag{1.15}$$

Hence, using (1.14) and (1.15), we obtain

$$8u - 12v + 24w \equiv 0 \pmod 4, \qquad 8u - 12v + 24w \equiv 0 \pmod 7,$$

$$-9u + 10v + 15w \equiv 0 \pmod 4, \qquad -9u + 10v + 15w \equiv 0 \pmod 7,$$
$$5u + 10v + w \equiv 0 \pmod 4, \qquad 5u + 10v + w \equiv 0 \pmod 7,$$

so that

$$x = \frac{8u - 12v + 24w}{28}, \quad y = \frac{-9u + 10v + 15w}{28}, \quad z = \frac{5u + 10v + w}{28},$$

are integers, which satisfy

$$3x^2 + 6y^2 + 14z^2 + 4yz + 2zx + 2xy = u^2 + 3v^2 + 5w^2 + 2vw = n,$$

proving that n is represented by the form f. □

In Theorem 1.8 we determined $r(f;n)$ and $r(g;n)$ for all positive integers $n \not\equiv 3 \pmod 4$. In our next theorem we evaluate $r(f;n)$ and $r(g;n)$ for all $n \equiv 3 \pmod 4$. In this case, by (1.5), we see that $2^\alpha 7^\beta g h^2 \equiv 3 \pmod 4$ so that $\alpha = 0$ and $g \equiv (-1)^{\beta+1} \pmod 4$. Also, by (1.6), we see that

$$n^* = \begin{cases} 2g & \text{if } \beta \equiv 1 \pmod 2 \text{ (equiv. } g \equiv 1 \pmod 4)\text{)}, \\ 14g & \text{if } \beta \equiv 0 \pmod 2 \text{ (equiv. } g \equiv 3 \pmod 4)\text{)}. \end{cases}$$

Theorem 1.13 is proved in Section 9.

Theorem 1.13. *Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 4$. Then*

$$r(f;n) = c_3(n) l(n) h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right) + \nu(n),$$
$$r(g;n) = c_3(n) l(n) h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right) - \nu(n),$$

where $c_3(n)$ is given in Table 1.3 and the integers $\nu(m)$ ($m \in \mathbb{N}$) are defined by

$$\frac{\eta^2(4w)\eta(16w)\eta(56w)}{\eta(8w)} = \sum_{m=1}^{\infty} \nu(m) e^{2\pi i w m}, \ w \in \mathcal{H}. \tag{1.16}$$

β	g	$c_3(n)$
β *even*	$g \equiv 3 \pmod 4$	$\frac{1}{12}\left(7^{(\beta+2)/2} - 4\right)$
β *odd*	$g \equiv 1 \pmod 4, g \equiv 1, 2, 4 \pmod 7$	$\frac{2}{3}\left(7^{(\beta+1)/2} - 1\right)$
β *odd*	$g \equiv 1 \pmod 4, g \equiv 3, 5, 6 \pmod 7$	$\frac{1}{2} 7^{(\beta+1)/2}$

Table 1.3: Values of $c_3(n)$: $n \equiv 3 \pmod 4$

If we set $q := e^{2\pi i w}$ $(w \in \mathcal{H})$ so that $|q| < 1$, then

$$\eta(w) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)$$

so that

$$\frac{\eta^2(4w)\eta(16w)\eta(56w)}{\eta(8w)} = q^3 \prod_{m=1}^{\infty} \frac{(1-q^{4m})^2(1-q^{16m})(1-q^{56m})}{(1-q^{8m})}, \tag{1.17}$$

and thus from (1.16) and (1.17), we deduce that

$$\nu(m) = 0 \text{ if } m \not\equiv 3 \pmod 4. \tag{1.18}$$

As $c_3(n)$, $l(n)$, and $h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right)$ are all positive, in order to prove that $r(f;n) > 0$ for $n \equiv 3 \pmod 4$ (thus proving that such n are represented by f), we must show that

$$c_3(n)l(n)h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right) + \nu(n) > 0$$

for all $n \equiv 3 \pmod 4$. This is obvious, if $\nu(n) \geq 0$. However, if $\nu(n) < 0$, we must show that

$$|\nu(n)| < c_3(n)l(n)h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right).$$

Let us see what this requires when n satisfies

$$n \equiv 3 \pmod 4, \quad \nu(n) < 0, \quad n \not\equiv 0 \pmod 7, \quad n \text{ squarefree}.$$

Such n exist, for example, $n = 19$. In this case we have, appealing to (1.5) - (1.7), and Table 1.3, $n = g \equiv 3 \pmod 4$, $\alpha = 0$, $\beta = 0$, $h = 1$, $n^* = 14g$, $c_3(n) = \frac{1}{4}$, $l(n) = 1$, $h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right) = h\left(\mathbb{Q}\left(\sqrt{-14g}\right)\right)$, so we require

$$|\nu(g)| < \frac{1}{4} h\left(\mathbb{Q}\left(\sqrt{-14g}\right)\right). \tag{1.19}$$

Numerical data for such g up to $5 \cdot 10^6$ suggests that an inequality of the type

$$|\nu(g)| \leq \frac{3}{2} g^{\frac{1}{4}} d(g) \tag{1.20}$$

holds, where $d(g)$ denotes the number of divisors of g. However, the best effective lower bounds for $h\left(\mathbb{Q}\left(\sqrt{D}\right)\right)$, where D is a negative discriminant, are of the type

$$h\left(\mathbb{Q}\left(\sqrt{D}\right)\right) \geq c_\kappa (\log |D|)^\kappa$$

for any $\kappa > 0$ and some computable constant c_κ. This estimate is not strong enough to prove (1.19) using a result of the type (1.20). We conclude with a conjecture.

Conjecture 1.14. Let $n \in \mathbb{N}$ satisfy $n \equiv 3 \pmod 4$. Define

$$\begin{aligned} r_1(k;n) &:= \text{card}\{(x,y,z) \in \mathbb{Z}^3 \mid k(x,y,z) = n,\ x \equiv y \equiv z \equiv 1 \pmod 2\}, \\ r_2(k;n) &:= \text{card}\{(x,y,z) \in \mathbb{Z}^3 \mid k(x,y,z) = n,\ x \equiv z \equiv 0 \pmod 2,\ y \equiv 1 \pmod 2, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \equiv z \pmod 4\}, \\ r_3(k;n) &:= \text{card}\{(x,y,z) \in \mathbb{Z}^3 \mid k(x,y,z) = n,\ x \equiv z \equiv 0 \pmod 2,\ y \equiv 1 \pmod 2, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \equiv z + 2 \pmod 4\}. \end{aligned}$$

Then

$$\begin{aligned} r(f;n) &= \frac{1}{2} r_1(k;n) + r_2(k;n), \\ r(g;n) &= \frac{1}{2} r_1(k;n) + r_3(k;n). \end{aligned}$$

This conjecture has been verified for all $n \equiv 3 \pmod 4$ with $n < 3 \cdot 10^6$. Following the statement of Theorem 1.11, we proved that if $r_1(k;n) > 0$ or $r_2(k;n) > 0$, then $r(f;n) > 0$. The conjecture makes this connection between $r_1(k;n), r_2(k;n)$, and $r(f;n)$ more precise. Note that the second equality of the conjecture follows from the first as

$$\begin{aligned} r(g;n) &= r(k;n) - r(f;n) \\ &= r_1(k;n) + r_2(k;n) + r_3(k;n) - \left(\frac{1}{2} r_1(k;n) + r_2(k;n)\right) \\ &= \frac{1}{2} r_1(k;n) + r_3(k;n). \end{aligned}$$

The difference of the two equalities of the conjecture asserts that

$$r_2(k;n) - r_3(k;n) = 2\nu(n)$$

for all $n \in \mathbb{N}$ with $n \equiv 3 \pmod 4$.

For all $n \in \mathbb{N}$ with $n \equiv 3 \pmod 4$ up to $3 \cdot 10^6$ we have

$r_1(k;n) > 0$ except for 5 cases ($n = 3, 23, 83, 111, 155$),
$r_2(k;n) > 0$ except for 32 cases ($n = 7, 11, 15, 35, 47, 51, 59, 87, 103, 123, 167, 183, 191, 203, 287, 407, 455, 467, 587, 767, 803, 863, 987, 1043, 1187, 1343, 1763, 2843, 3183, 4983, 8207, 11543$),
$r_3(k;n) > 0$ except for 30 cases ($n = 3, 11, 15, 23, 47, 55, 67, 71, 107, 143, 147, 159, 239, 251, 263, 267, 323, 347, 527, 647, 795, 935, 1127, 1167, 1223, 1563, 1823, 2039, 2495, 6083$).

2. Relationship between the Representation Numbers of *f*, *g*, and *h*: Proof of Theorem 1.1

As $r(f;0) = r(g;0) = r(h;0) = 1$ and $s(0) = 2$, Theorem 1.1 is true for $n = 0$, so we may suppose that $n \in \mathbb{N}$.

We define three primitive, positive-definite, integral quaternary quadratic forms F, G, and H in the variables x, y, z, and u by

$$F(x,y,z,u) := f + 32u^2 = 3x^2 + 6y^2 + 14z^2 + 32u^2 + 4yz + 2zx + 2xy,$$
$$G(x,y,z,u) := g + 32u^2 = 6x^2 + 7y^2 + 7z^2 + 32u^2 + 2yz + 2zx + 6xy,$$
$$H(x,y,z,u) := h + 32u^2 = 3x^2 + 4y^2 + 6z^2 + 32u^2 + 4yz + 2zx.$$

The matrices of the forms F, G, and H are respectively the matrices $M(F)$, $M(G)$, and $M(H)$ defined as follows:

$$M(F) = \begin{bmatrix} 6 & 2 & 2 & 0 \\ 2 & 12 & 4 & 0 \\ 2 & 4 & 28 & 0 \\ 0 & 0 & 0 & 64 \end{bmatrix}, \quad M(G) = \begin{bmatrix} 12 & 6 & 2 & 0 \\ 6 & 14 & 2 & 0 \\ 2 & 2 & 14 & 0 \\ 0 & 0 & 0 & 64 \end{bmatrix},$$

$$M(H) = \begin{bmatrix} 6 & 0 & 2 & 0 \\ 0 & 8 & 4 & 0 \\ 2 & 4 & 12 & 0 \\ 0 & 0 & 0 & 64 \end{bmatrix}.$$

We have

$$\det M(F) = \det M(G) = 114688 = 2^{14} \cdot 7, \qquad \det M(H) = 28672 = 2^{12} \cdot 7.$$

Also

$$896M(F)^{-1} = \begin{bmatrix} 160 & -24 & -8 & 0 \\ -24 & 82 & -10 & 0 \\ -8 & -10 & 34 & 0 \\ 0 & 0 & 0 & 14 \end{bmatrix}, \quad 896M(G)^{-1} = \begin{bmatrix} 96 & -40 & -8 & 0 \\ -40 & 82 & -6 & 0 \\ -8 & -6 & 66 & 0 \\ 0 & 0 & 0 & 14 \end{bmatrix},$$

$$896M(H)^{-1} = \begin{bmatrix} 160 & 16 & -32 & 0 \\ 16 & 136 & -48 & 0 \\ -32 & -48 & 96 & 0 \\ 0 & 0 & 0 & 14 \end{bmatrix}.$$

Hence, the character associated with

$$F \text{ is } \left(\frac{(-1)^{4/2}\det M(F)}{*}\right) = \left(\frac{2^{14}\cdot 7}{*}\right) = \left(\frac{2^2\cdot 7}{*}\right) = \left(\frac{28}{*}\right) = \chi_{28},$$

$$G \text{ is } \left(\frac{(-1)^{4/2}\det M(G)}{*}\right) = \left(\frac{2^{14}\cdot 7}{*}\right) = \left(\frac{2^{2}\cdot 7}{*}\right) = \left(\frac{28}{*}\right) = \chi_{28},$$
$$H \text{ is } \left(\frac{(-1)^{4/2}\det M(H)}{*}\right) = \left(\frac{2^{12}\cdot 7}{*}\right) = \left(\frac{2^{2}\cdot 7}{*}\right) = \left(\frac{28}{*}\right) = \chi_{28},$$

and the level of each of F, G, H is 896. By [11, Theorem 10.1, p. 363] we have

$$\theta(F;w) \in M_2(\Gamma_0(896), \chi_{28}),$$
$$\theta(G;w) \in M_2(\Gamma_0(896), \chi_{28}),$$
$$\theta(H;w) \in M_2(\Gamma_0(896), \chi_{28}).$$

Next, we apply [2, Prop. 2.3] with

$$R = 32 = 2^5, \quad N = 896 = 2^7 \cdot 7, \quad k = 2, \quad \chi = \chi_{28},$$
$$N^* = \operatorname{lcm}(R^2, N) = \operatorname{lcm}(2^{10}, 2^7\cdot 7) = 2^{10}\cdot 7 = 7168,$$
$$\frac{N^*}{R} = \frac{2^{10}\cdot 7}{2^5} = 2^5 \cdot 7 = 224,$$

so that

$$\operatorname{cond}(\chi_{28}) = 28 = 2^2 \cdot 7 \,\Big|\, \frac{N^*}{R}$$

and

$$\gcd(\delta, N^*) = \gcd(\delta, 2^{10}\cdot 7) = 1 \iff \gcd(\delta, 14) = 1 \iff \delta \text{ is coprime with 2 and 7.}$$

Define for all $j \in \mathbb{Z}$

$$e_j := \begin{cases} \frac{1}{24} + \frac{1}{24}\left(\frac{-1}{j}\right) i & \text{if } j \equiv 1 \pmod 2, \\ \frac{1}{24} + \frac{1}{8}\left(\frac{-1}{j/2}\right) i & \text{if } j \equiv 2 \pmod 4, \\ 0 & \text{if } j \equiv 0 \pmod 4,\ j \not\equiv 0 \pmod{32}, \\ 1 & \text{if } j \equiv 0 \pmod{32}. \end{cases}$$

If $j \equiv 1 \pmod 2$, then $j\delta^2 \equiv 1 \pmod 2$ and

$$e_{j\delta^2} = \frac{1}{24} + \frac{1}{24}\left(\frac{-1}{j\delta^2}\right) i = \frac{1}{24} + \frac{1}{24}\left(\frac{-1}{j}\right) i = e_j.$$

If $j \equiv 2 \pmod 4$, then $j\delta^2 \equiv 2 \pmod 4$ and

$$e_{j\delta^2} = \frac{1}{24} + \frac{1}{8}\left(\frac{-1}{j/2\ \delta^2}\right) i = \frac{1}{24} + \frac{1}{8}\left(\frac{-1}{j/2}\right) i = e_j.$$

If $j \equiv 0 \pmod 4$ and $j \not\equiv 0 \pmod{32}$, then

$$e_{j\delta^2} = 0 = e_j.$$

If $j \equiv 0 \pmod{32}$, then
$$e_{j\delta^2} = 1 = e_j.$$
Hence, $e_{j\delta^2} = e_j$ for all δ with $\gcd(\delta, N^*) = 1$.

As $\theta(H;w) \in M_2(\Gamma_0(896), \chi_{28})$ we have by [2, Prop. 2.3]
$$\sum_{j=1}^{32} e_j \theta\left(H; w + \frac{j}{32}\right) \in M_2(\Gamma_0(7168), \chi_{28}).$$
As $7168 = 8 \cdot 896$ we have
$$\theta(F;w),\ \theta(G;w),\ \theta(H;w) \in M_2(\Gamma_0(7168), \chi_{28})$$
so
$$T(w) := \theta(F;w) + \theta(G;w) - \sum_{j=1}^{32} e_j \theta\left(H; w + \frac{j}{32}\right) \in M_2(\Gamma_0(7168), \chi_{28}).$$

The Sturm bound [5, Definition 5.6.13, p. 185] in $M_2(\Gamma_0(7168), \chi_{28})$ is
$$\begin{aligned} 1 + \left[\frac{[SL_2(\mathbb{Z}) : \Gamma_0(7168)] \cdot 2}{12}\right] &= 1 + \left[\frac{7168 \prod_{p \mid 7168}\left(1 + \frac{1}{p}\right)}{6}\right] \\ &= 1 + \left[\frac{3584}{3}\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{7}\right)\right] = 2049. \end{aligned}$$
The first $3 \cdot 10^6$ coefficients of $T(w)$ in its q-expansion, where $q = e^{2\pi i w}$, are all zero, so by Sturm's theorem [5, Corollary 5.6.14, p. 185] we have
$$T(w) = 0 \text{ for all } w \in \mathcal{H}.$$
Hence
$$\theta(F;w) + \theta(G;w) = \sum_{j=1}^{32} e_j \theta\left(H; w + \frac{j}{32}\right), \quad w \in \mathcal{H}.$$
Now, for $j = 1, 2, \ldots, 32$, we have
$$\begin{aligned} \theta\left(H; w + \frac{j}{32}\right) &= \sum_{(x,y,z,u) \in \mathbb{Z}^4} e^{2\pi i (w + \frac{j}{32}) H(x,y,z,u)} \\ &= \sum_{(x,y,z,u) \in \mathbb{Z}^4} e^{2\pi i w H(x,y,z,u) + \frac{2\pi i j}{32} H(x,y,z,u)} \\ &= \sum_{(x,y,z,u) \in \mathbb{Z}^4} e^{2\pi i w (h(x,y,z) + 32u^2) + \frac{2\pi i j}{32}(h(x,y,z) + 32u^2)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{(x,y,z,u)\in\mathbb{Z}^4} e^{2\pi i w h(x,y,z)+2\pi i w 32u^2+\frac{2\pi i j}{32}h(x,y,z)+2\pi i j u^2} \\
&= \sum_{(x,y,z)\in\mathbb{Z}^3} e^{2\pi i w h(x,y,z)+\frac{2\pi i j}{32}h(x,y,z)} \sum_{u\in\mathbb{Z}} e^{2\pi i w 32u^2} \quad (\text{as } e^{2\pi i j u^2}=1) \\
&= \sum_{(x,y,z)\in\mathbb{Z}^3} e^{2\pi i (w+\frac{j}{32})h(x,y,z)}\theta(32w) \\
&= \theta\left(h; w+\frac{j}{32}\right)\theta(32w).
\end{aligned}
$$

Similarly, we see that

$$\theta(F;w)=\theta(f;w)\theta(32w)$$

and

$$\theta(G;w)=\theta(g;w)\theta(32w).$$

Hence

$$(\theta(f;w)+\theta(g;w))\,\theta(32w)=\sum_{j=1}^{32} e_j\theta\left(h;w+\frac{j}{32}\right)\theta(32w).$$

But $\theta(32w)\neq 0$ for all $w\in\mathcal{H}$, so

$$\theta(f;w)+\theta(g;w)=\sum_{j=1}^{32} e_j\theta\left(h;w+\frac{j}{32}\right) \quad \text{for all } w\in\mathcal{H}.$$

Hence, with $q=e^{2\pi i w}$ $(w\in\mathcal{H})$ and $\omega_{32}=e^{\frac{2\pi i}{32}}$, we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} r(f;n)q^n+\sum_{n=0}^{\infty} r(g;n)q^n &= \sum_{j=1}^{32} e_j \sum_{(x,y,z)\in\mathbb{Z}^3} e^{2\pi i (w+\frac{j}{32})h(x,y,z)} \\
&= \sum_{j=1}^{32} e_j \sum_{(x,y,z)\in\mathbb{Z}^3} q^{h(x,y,z)} e^{\frac{2\pi i j}{32}h(x,y,z)} = \sum_{j=1}^{32} e_j \sum_{(x,y,z)\in\mathbb{Z}^3} q^{h(x,y,z)}\omega_{32}^{h(x,y,z)j} \\
&= \sum_{j=1}^{32} e_j \sum_{n=0}^{\infty} \sum_{\substack{(x,y,z)\in\mathbb{Z}^3\\ h(x,y,z)=n}} q^n\omega_{32}^{nj} = \sum_{j=1}^{32} e_j \sum_{n=0}^{\infty} q^n \omega_{32}^{nj} r(h;n) \\
&= \sum_{n=0}^{\infty}\left(\sum_{j=1}^{32} e_j\omega_{32}^{nj}\right) r(h;n)q^n.
\end{aligned}
$$

Next

$$\sum_{\substack{j=1\\ j\equiv 1 \pmod 2}}^{32} e_j\omega_{32}^{nj} = \sum_{k=1}^{16}\left(\frac{1}{24}+\frac{1}{24}\left(\frac{-1}{2k-1}\right)i\right)\omega_{32}^{n(2k-1)}$$

$$= \frac{1}{24}\omega_{32}^{-n}\sum_{k=1}^{16}\omega_{16}^{nk} - \frac{1}{24}\omega_{32}^{-n} i\sum_{k=1}^{16}(-1)^k\omega_{16}^{nk}$$
$$= \frac{1}{24}\omega_{32}^{-n}\begin{Bmatrix} 16 & \text{if } n \equiv 0 \pmod{16} \\ 0 & \text{if } n \not\equiv 0 \pmod{16} \end{Bmatrix} - \frac{1}{24}\omega_{32}^{-n} i\sum_{k=1}^{16}\omega_{16}^{(n+8)k}$$
$$= \begin{Bmatrix} \frac{2}{3}(-1)^{n/16} & \text{if } n \equiv 0 \pmod{16} \\ 0 & \text{if } n \not\equiv 0 \pmod{16} \end{Bmatrix} - \frac{1}{24}\omega_{32}^{-n} i \begin{Bmatrix} 16 & \text{if } n \equiv 8 \pmod{16} \\ 0 & \text{if } n \not\equiv 8 \pmod{16} \end{Bmatrix}$$
$$= \begin{Bmatrix} \frac{2}{3}(-1)^{n/16} & \text{if } n \equiv 0 \pmod{16} \\ 0 & \text{if } n \not\equiv 0 \pmod{16} \end{Bmatrix} - \begin{Bmatrix} \frac{2}{3}(-1)^{(n-8)/16} & \text{if } n \equiv 8 \pmod{16} \\ 0 & \text{if } n \not\equiv 8 \pmod{16} \end{Bmatrix}$$
$$= \begin{Bmatrix} \frac{2}{3}(-1)^{n/16} & \text{if } n \equiv 0 \pmod{16} \\ \frac{2}{3}(-1)^{(n+8)/16} & \text{if } n \equiv 8 \pmod{16} \\ 0 & \text{if } n \not\equiv 0 \pmod{8} \end{Bmatrix};$$

$$\sum_{\substack{j=1 \\ j\equiv 2 \pmod 4}}^{32} e_j\omega_{32}^{nj} = \sum_{\substack{j=1 \\ j\equiv 2 \pmod 4}}^{32} \left(\frac{1}{24} + \frac{1}{8}\left(\frac{-1}{j/2}\right)i\right)\omega_{32}^{nj}$$
$$= \sum_{k=1}^{8}\left(\frac{1}{24} + \frac{1}{8}\left(\frac{-1}{2k-1}\right)i\right)\omega_{32}^{n(4k-2)}$$
$$= \frac{1}{24}\omega_{16}^{-n}\sum_{k=1}^{8}\omega_8^{nk} + \frac{i}{8}\omega_{16}^{-n}\sum_{k=1}^{8}(-1)^{k-1}\omega_8^{nk}$$
$$= \frac{1}{24}\omega_{16}^{-n}\begin{Bmatrix} 8 & \text{if } n \equiv 0 \pmod{8} \\ 0 & \text{if } n \not\equiv 0 \pmod{8} \end{Bmatrix} - \frac{i}{8}\omega_{16}^{-n}\sum_{k=1}^{8}\omega_8^{(n+4)k}$$
$$= \begin{Bmatrix} \frac{1}{3}(-1)^{n/8} & \text{if } n \equiv 0 \pmod{8} \\ 0 & \text{if } n \not\equiv 0 \pmod{8} \end{Bmatrix} - \frac{i}{8}\omega_{16}^{-n}\begin{Bmatrix} 8 & \text{if } n \equiv 4 \pmod{8} \\ 0 & \text{if } n \not\equiv 4 \pmod{8} \end{Bmatrix}$$
$$= \begin{Bmatrix} \frac{1}{3}(-1)^{n/8} & \text{if } n \equiv 0 \pmod{8} \\ 0 & \text{if } n \not\equiv 0 \pmod{8} \end{Bmatrix} + \begin{Bmatrix} (-1)^{(n+4)/8} & \text{if } n \equiv 4 \pmod{8} \\ 0 & \text{if } n \not\equiv 4 \pmod{8} \end{Bmatrix}$$
$$= \begin{Bmatrix} \frac{1}{3}(-1)^{n/8} & \text{if } n \equiv 0 \pmod{8} \\ (-1)^{(n+4)/8} & \text{if } n \equiv 4 \pmod{8} \\ 0 & \text{if } n \not\equiv 0 \pmod{4} \end{Bmatrix};$$

$$\sum_{\substack{j=1 \\ j\equiv 0 \pmod 4}}^{32} e_j\omega_{32}^{nj} = e_{32}\omega_{32}^{32n} = 1 \qquad (\text{as } e_4 = e_8 = \ldots = e_{28} = 0 \text{ and } e_{32} = 1);$$

$$\begin{aligned}
\sum_{j=1}^{32} e_j\omega_{32}^{nj} &= \sum_{\substack{j=1\\ j\equiv 1 \pmod 2}}^{32} e_j\omega_{32}^{nj} + \sum_{\substack{j=1\\ j\equiv 2 \pmod 4}}^{32} e_j\omega_{32}^{nj} + \sum_{\substack{j=1\\ j\equiv 0 \pmod 4}}^{32} e_j\omega_{32}^{nj} \\
&= \left\{\begin{matrix} \frac{2}{3}(-1)^{n/16} & \text{if } n \equiv 0 \pmod{16} \\ \frac{2}{3}(-1)^{(n+8)/16} & \text{if } n \equiv 8 \pmod{16} \\ 0 & \text{if } n \not\equiv 0 \pmod 8 \end{matrix}\right\} \\
&+ \left\{\begin{matrix} \frac{1}{3}(-1)^{n/8} & \text{if } n \equiv 0 \pmod 8 \\ (-1)^{(n+4)/8} & \text{if } n \equiv 4 \pmod 8 \\ 0 & \text{if } n \not\equiv 0 \pmod 4 \end{matrix}\right\} + 1 \\
&= \begin{cases} 0+0+1=1 & \text{if } n \equiv 1 \pmod 4, \\ 0+0+1=1 & \text{if } n \equiv 2 \pmod 4, \\ 0+0+1=1 & \text{if } n \equiv 3 \pmod 4, \\ \frac{2}{3}+\frac{1}{3}+1=2 & \text{if } n \equiv 0 \pmod{32}, \\ 0+1+1=2 & \text{if } n \equiv 12 \pmod{32}, \\ 0+1+1=2 & \text{if } n \equiv 28 \pmod{32}, \\ 0-1+1=0 & \text{if } n \equiv 4 \pmod{32}, \\ -\frac{2}{3}-\frac{1}{3}+1=0 & \text{if } n \equiv 8 \pmod{32}, \\ 0-1+1=0 & \text{if } n \equiv 20 \pmod{32}, \\ -\frac{2}{3}+\frac{1}{3}+1=\frac{2}{3} & \text{if } n \equiv 16 \pmod{32}, \\ \frac{2}{3}-\frac{1}{3}+1=\frac{4}{3} & \text{if } n \equiv 24 \pmod{32} \end{cases} \\
&= s(n).
\end{aligned}$$

Finally, we have

$$\sum_{n=0}^{\infty} r(f;n)q^n + \sum_{n=0}^{\infty} r(g;n)q^n = \sum_{n=0}^{\infty} s(n)r(h;n)q^n,$$

so equating coefficients of q^n, we deduce

$$r(f;n) + r(g;n) = s(n)r(h;n)$$

for all $n \in \mathbb{N}$. □

3. Relationship between the Representation Numbers of h and k: Proof of Theorem 1.2

As $r(h;0) = r(k;0) = 1$ and $t(0) = 1$, the theorem is true for $n = 0$, so we may suppose that $n \in \mathbb{N}$.

We define two primitive, positive-definite, integral quaternary quadratic forms H and K in the variables x, y, z, and u by

$$H(x,y,z,u) := h + 8u^2 = 3x^2 + 4y^2 + 6z^2 + 8u^2 + 4yz + 2zx,$$
$$K(x,y,z,u) := k + 8u^2 = x^2 + 3y^2 + 5z^2 + 8u^2 + 2yz.$$

The matrices of the forms H and K are respectively the matrices $M(H)$ and $M(K)$ defined as follows:

$$M(H) = \begin{bmatrix} 6 & 0 & 2 & 0 \\ 0 & 8 & 4 & 0 \\ 2 & 4 & 12 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix}, \quad M(K) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & 2 & 10 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix}.$$

We have

$$\det M(H) = 7168 = 2^{10} \cdot 7, \qquad \det M(K) = 1792 = 2^8 \cdot 7.$$

Also

$$224M(H)^{-1} = \begin{bmatrix} 40 & 4 & -8 & 0 \\ 4 & 34 & -12 & 0 \\ -8 & -12 & 24 & 0 \\ 0 & 0 & 0 & 14 \end{bmatrix}, \quad 224M(K)^{-1} = \begin{bmatrix} 112 & 0 & 0 & 0 \\ 0 & 40 & -8 & 0 \\ 0 & -8 & 24 & 0 \\ 0 & 0 & 0 & 14 \end{bmatrix}.$$

Hence, the character associated with

$$H \text{ is } \left(\frac{(-1)^{4/2}\det M(H)}{*}\right) = \left(\frac{2^{10}\cdot 7}{*}\right) = \left(\frac{2^2\cdot 7}{*}\right) = \left(\frac{28}{*}\right) = \chi_{28},$$

$$K \text{ is } \left(\frac{(-1)^{4/2}\det M(K)}{*}\right) = \left(\frac{2^{8}\cdot 7}{*}\right) = \left(\frac{2^2\cdot 7}{*}\right) = \left(\frac{28}{*}\right) = \chi_{28},$$

and the level of both H and K is 224.

By [11, Theorem 10.1, p. 363] we have

$$\theta(H;w) \in M_2(\Gamma_0(224), \chi_{28}),$$

$$\theta(K;w) \in M_2(\Gamma_0(224), \chi_{28}).$$

Next, we apply [2, Prop. 2.3] with

$$R = 8 = 2^3, \quad N = 224 = 2^5 \cdot 7, \quad k = 2, \quad \chi = \chi_{28},$$
$$N^* = \text{lcm}(R^2, N) = \text{lcm}(2^6, 2^5 \cdot 7) = 2^6 \cdot 7 = 448,$$
$$\frac{N^*}{R} = \frac{2^6 \cdot 7}{2^3} = 2^3 \cdot 7 = 56,$$

so that
$$\mathrm{cond}(\chi_{28}) = 28 = 2^2 \cdot 7 \,\Big|\, \frac{N^*}{R}$$
and
$$\gcd(\delta, N^*) = \gcd(\delta, 2^6 \cdot 7) = 1 \Longleftrightarrow \gcd(\delta, 14) = 1 \Longleftrightarrow \delta \text{ is coprime with 2 and 7.}$$
Define for all $j \in \mathbb{Z}$
$$e_j := \begin{cases} \frac{1}{12} + \frac{1}{12}\left(\frac{-1}{j}\right) i & \text{if } j \equiv 1 \pmod 2, \\ \frac{1}{12} + \frac{1}{4}\left(\frac{-1}{j/2}\right) i & \text{if } j \equiv 2 \pmod 4, \\ 0 & \text{if } j \equiv 4 \pmod 8, \\ \frac{1}{2} & \text{if } j \equiv 0 \pmod 8. \end{cases}$$
If $j \equiv 1 \pmod 2$, then $j\delta^2 \equiv 1 \pmod 2$ and
$$e_{j\delta^2} = \frac{1}{12} + \frac{1}{12}\left(\frac{-1}{j\delta^2}\right) i = \frac{1}{12} + \frac{1}{12}\left(\frac{-1}{j}\right) i = e_j.$$
If $j \equiv 2 \pmod 4$, then $j\delta^2 \equiv 2 \pmod 4$ and
$$e_{j\delta^2} = \frac{1}{12} + \frac{1}{4}\left(\frac{-1}{j/2\ \delta^2}\right) i = \frac{1}{12} + \frac{1}{4}\left(\frac{-1}{j/2}\right) i = e_j.$$
If $j \equiv 4 \pmod 8$, then
$$e_{j\delta^2} = 0 = e_j.$$
If $j \equiv 0 \pmod 8$, then
$$e_{j\delta^2} = \frac{1}{2} = e_j.$$
Hence, $e_{j\delta^2} = e_j$ for all δ with $\gcd(\delta, N^*) = 1$.

As $\theta(K; w) \in M_2(\Gamma_0(224), \chi_{28})$ by [2, Prop. 2.3] we deduce
$$\sum_{j=1}^{8} e_j \theta\left(K; w + \frac{j}{8}\right) \in M_2(\Gamma_0(448), \chi_{28}).$$
As $448 = 2 \cdot 224$, we have
$$\theta(H; w),\ \theta(K; w) \in M_2(\Gamma_0(448), \chi_{28})$$
so
$$T(w) := \theta(H; w) - \sum_{j=1}^{8} e_j \theta\left(K; w + \frac{j}{8}\right) \in M_2(\Gamma_0(448), \chi_{28}).$$

The Sturm bound [5, Definition 5.6.13, p. 185] in $M_2(\Gamma_0(448), \chi_{28})$ is

$$1 + \left[\frac{[SL_2(\mathbb{Z}) : \Gamma_0(448)] \cdot 2}{12} \right] = 1 + \left[\frac{448 \prod_{p|448} \left(1 + \frac{1}{p}\right)}{6} \right]$$
$$= 1 + \left[\frac{224}{3} \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{7}\right) \right] = 129.$$

The first $3 \cdot 10^6$ coefficients of $T(w)$ in its q-expansion, where $q = e^{2\pi i w}$, are all zero, so by Sturm's theorem [5, Corollary 5.6.14, p. 185] we have

$$T(w) = 0 \text{ for all } w \in \mathcal{H}.$$

Hence

$$\theta(H; w) = \sum_{j=1}^{8} e_j \theta\left(K; w + \frac{j}{8}\right), \quad w \in \mathcal{H}.$$

Now, for $j = 1, 2, \ldots, 8$, we have

$$\begin{aligned}
\theta\left(K; w + \frac{j}{8}\right) &= \sum_{(x,y,z,u) \in \mathbb{Z}^4} e^{2\pi i (w + \frac{j}{8}) K(x,y,z,u)} \\
&= \sum_{(x,y,z,u) \in \mathbb{Z}^4} e^{2\pi i w K(x,y,z,u) + \frac{2\pi i j}{8} K(x,y,z,u)} \\
&= \sum_{(x,y,z,u) \in \mathbb{Z}^4} e^{2\pi i w (k(x,y,z) + 8u^2) + \frac{2\pi i j}{8} (k(x,y,z) + 8u^2)} \\
&= \sum_{(x,y,z,u) \in \mathbb{Z}^4} e^{2\pi i w k(x,y,z) + 2\pi i w 8u^2 + \frac{2\pi i j}{8} k(x,y,z) + 2\pi i j u^2} \\
&= \sum_{(x,y,z) \in \mathbb{Z}^3} e^{2\pi i w k(x,y,z) + \frac{2\pi i j}{8} k(x,y,z)} \sum_{u \in \mathbb{Z}} e^{2\pi i w 8u^2} \quad (\text{as } e^{2\pi i j u^2} = 1) \\
&= \sum_{(x,y,z) \in \mathbb{Z}^3} e^{2\pi i (w + \frac{j}{8}) k(x,y,z)} \theta(8w) \\
&= \theta\left(k; w + \frac{j}{8}\right) \theta(8w).
\end{aligned}$$

Similarly, we see that

$$\theta(H; w) = \theta(h; w)\theta(8w).$$

Hence

$$\theta(h; w)\theta(8w) = \sum_{j=1}^{8} e_j \theta\left(k; w + \frac{j}{8}\right) \theta(8w).$$

Since $\theta(8w) \neq 0$ for all $w \in \mathcal{H}$, we have

$$\theta(h; w) = \sum_{j=1}^{8} e_j \theta\left(k; w + \frac{j}{8}\right) \quad \text{for all } w \in \mathcal{H}.$$

Hence, with $q = e^{2\pi i w}$ $(w \in \mathcal{H})$ and $\omega_8 = e^{\frac{2\pi i}{8}}$, we deduce

$$\begin{aligned}\sum_{n=0}^{\infty} r(h;n)q^n &= \sum_{j=1}^{8} e_j \sum_{(x,y,z)\in\mathbb{Z}^3} e^{2\pi i(w+\frac{j}{8})k(x,y,z)} \\ &= \sum_{j=1}^{8} e_j \sum_{(x,y,z)\in\mathbb{Z}^3} q^{k(x,y,z)} e^{\frac{2\pi i j}{8}k(x,y,z)} \\ &= \sum_{j=1}^{8} e_j \sum_{(x,y,z)\in\mathbb{Z}^3} q^{k(x,y,z)} \omega_8^{k(x,y,z)j} \\ &= \sum_{j=1}^{8} e_j \sum_{n=0}^{\infty} \sum_{\substack{(x,y,z)\in\mathbb{Z}^3 \\ k(x,y,z)=n}} q^n \omega_8^{nj} = \sum_{j=1}^{8} e_j \sum_{n=0}^{\infty} q^n \omega_8^{nj} r(k;n) \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=1}^{8} e_j \omega_8^{nj} \right) r(k;n) q^n.\end{aligned}$$

Next

$$\begin{aligned}&\sum_{\substack{j=1 \\ j\equiv 1 \pmod 2}}^{8} e_j \omega_8^{nj} = \sum_{k=1}^{4} \left(\frac{1}{12} + \frac{1}{12} \left(\frac{-1}{2k-1} \right) i \right) \omega_8^{n(2k-1)} \\ &= \frac{1}{12}\omega_8^{-n} \sum_{k=1}^{4} \omega_4^{nk} - \frac{i}{12}\omega_8^{-n} \sum_{k=1}^{4} (-1)^k \omega_4^{nk} \\ &= \frac{1}{12}\omega_8^{-n} \begin{Bmatrix} 4 & \text{if } n \equiv 0 \pmod 4 \\ 0 & \text{if } n \not\equiv 0 \pmod 4 \end{Bmatrix} - \frac{i}{12}\omega_8^{-n} \sum_{k=1}^{4} \omega_4^{(n+2)k} \\ &= \begin{Bmatrix} \frac{1}{3}(-1)^{n/4} & \text{if } n \equiv 0 \pmod 4 \\ 0 & \text{if } n \not\equiv 0 \pmod 4 \end{Bmatrix} - \frac{i}{12}\omega_8^{-n} \begin{Bmatrix} 4 & \text{if } n \equiv 2 \pmod 4 \\ 0 & \text{if } n \not\equiv 2 \pmod 4 \end{Bmatrix} \\ &= \begin{Bmatrix} \frac{1}{3}(-1)^{n/4} & \text{if } n \equiv 0 \pmod 4 \\ 0 & \text{if } n \not\equiv 0 \pmod 4 \end{Bmatrix} - \begin{Bmatrix} \frac{1}{3}(-1)^{(n-2)/4} & \text{if } n \equiv 2 \pmod 4 \\ 0 & \text{if } n \not\equiv 2 \pmod 4 \end{Bmatrix} \\ &= \begin{Bmatrix} \frac{1}{3}(-1)^{n/4} & \text{if } n \equiv 0 \pmod 4 \\ \frac{1}{3}(-1)^{(n+2)/4} & \text{if } n \equiv 2 \pmod 4 \\ 0 & \text{if } n \equiv 1 \pmod 2 \end{Bmatrix};\end{aligned}$$

$$\sum_{\substack{j=1 \\ j\equiv 2 \pmod 4}}^{8} e_j \omega_8^{nj} = \sum_{\substack{j=1 \\ j\equiv 2 \pmod 4}}^{8} \left(\frac{1}{12} + \frac{1}{4} \left(\frac{-1}{j/2} \right) i \right) \omega_8^{nj}$$

$$= \sum_{k=1}^{2} \left(\frac{1}{12} + \frac{1}{4} \left(\frac{-1}{2k-1} \right) i \right) \omega_8^{n(4k-2)}$$
$$= \frac{1}{12}\omega_4^{-n} \sum_{k=1}^{2} (-1)^{nk} + \frac{i}{4}\omega_4^{-n} \sum_{k=1}^{2} (-1)^{k-1}(-1)^{nk}$$
$$= \frac{1}{12}\omega_4^{-n} \begin{Bmatrix} 2 & \text{if } n \equiv 0 \pmod 2 \\ 0 & \text{if } n \equiv 1 \pmod 2 \end{Bmatrix} - \frac{i}{4}\omega_4^{-n} \sum_{k=1}^{2} (-1)^{(n+1)k}$$
$$= \begin{Bmatrix} \frac{1}{6}(-1)^{n/2} & \text{if } n \equiv 0 \pmod 2 \\ 0 & \text{if } n \equiv 1 \pmod 2 \end{Bmatrix} - \frac{i}{4}\omega_4^{-n} \begin{Bmatrix} 0 & \text{if } n \equiv 0 \pmod 2 \\ 2 & \text{if } n \equiv 1 \pmod 2 \end{Bmatrix}$$
$$= \begin{Bmatrix} \frac{1}{6}(-1)^{n/2} & \text{if } n \equiv 0 \pmod 2 \\ 0 & \text{if } n \equiv 1 \pmod 2 \end{Bmatrix} + \begin{Bmatrix} 0 & \text{if } n \equiv 0 \pmod 2 \\ \frac{1}{2}(-1)^{(n+1)/2} & \text{if } n \equiv 1 \pmod 2 \end{Bmatrix}$$
$$= \begin{Bmatrix} \frac{1}{6}(-1)^{n/2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{1}{2}(-1)^{(n+1)/2} & \text{if } n \equiv 1 \pmod 2 \end{Bmatrix};$$

$$\sum_{\substack{j=1 \\ j \equiv 0 \pmod 4}}^{8} e_j \omega_8^{nj} = e_4 \omega_8^{4n} + e_8 \omega_8^{8n} = 0 + \frac{1}{2} = \frac{1}{2}.$$

Thus

$$\sum_{j=1}^{8} e_j \omega_8^{nj} = \sum_{\substack{j=1 \\ j \equiv 1 \pmod 2}}^{8} e_j \omega_8^{nj} + \sum_{\substack{j=1 \\ j \equiv 2 \pmod 4}}^{8} e_j \omega_8^{nj} + \sum_{\substack{j=1 \\ j \equiv 0 \pmod 4}}^{8} e_j \omega_8^{nj}$$
$$= \begin{Bmatrix} \frac{1}{3}(-1)^{n/4} & \text{if } n \equiv 0 \pmod 4 \\ \frac{1}{3}(-1)^{(n+2)/4} & \text{if } n \equiv 2 \pmod 4 \\ 0 & \text{if } n \equiv 1 \pmod 2 \end{Bmatrix}$$
$$+ \begin{Bmatrix} \frac{1}{6}(-1)^{n/2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{1}{2}(-1)^{(n+1)/2} & \text{if } n \equiv 1 \pmod 2 \end{Bmatrix} + \frac{1}{2}$$
$$= \begin{cases} 0 - \frac{1}{2} + \frac{1}{2} = 0 & \text{if } n \equiv 1 \pmod 8, \\ -\frac{1}{3} - \frac{1}{6} + \frac{1}{2} = 0 & \text{if } n \equiv 2 \pmod 8, \\ 0 + \frac{1}{2} + \frac{1}{2} = 1 & \text{if } n \equiv 3 \pmod 8, \\ -\frac{1}{3} + \frac{1}{6} + \frac{1}{2} = \frac{1}{3} & \text{if } n \equiv 4 \pmod 8, \\ 0 - \frac{1}{2} + \frac{1}{2} = 0 & \text{if } n \equiv 5 \pmod 8, \\ \frac{1}{3} - \frac{1}{6} + \frac{1}{2} = \frac{2}{3} & \text{if } n \equiv 6 \pmod 8, \\ 0 + \frac{1}{2} + \frac{1}{2} = 1 & \text{if } n \equiv 7 \pmod 8, \\ \frac{1}{3} + \frac{1}{6} + \frac{1}{2} = 1 & \text{if } n \equiv 0 \pmod 8 \end{cases}$$
$$= t(n).$$

Finally, we have

$$\sum_{n=0}^{\infty} r(h;n)q^n = \sum_{n=0}^{\infty} t(n)r(k;n)q^n,$$

so equating coefficients of q^n, we deduce

$$r(h;n) = t(n)r(k;n)$$

for all $n \in \mathbb{N}$. □

4. Relationship between the Representation Numbers of f and g for $n \not\equiv 3 \pmod 4$: Proof of Theorem 1.4

Let $n \in \mathbb{N}$ satisfy $n \equiv 0 \pmod 2$. Let

$$A := \{(x,y,z) \in \mathbb{Z}^3 \mid f(x,y,z) = n\} \quad \text{and} \quad B := \{(u,v,w) \in \mathbb{Z}^3 \mid g(u,v,w) = n\}.$$

Suppose $(x,y,z) \in A$. Then

$$3x^2 + 6y^2 + 14z^2 + 4yz + 2zx + 2xy = n.$$

As $n \equiv 0 \pmod 2$ we have $x \equiv 0 \pmod 2$ so $\frac{x}{2} \in \mathbb{Z}$. Also

$$g\left(y+z, \frac{x}{2} - z, -\frac{x}{2} - z\right) = f(x,y,z) = n.$$

Thus we can define $\lambda : A \to B$ by

$$\lambda(x,y,z) = \left(y+z, \frac{x}{2} - z, -\frac{x}{2} - z\right).$$

Let $(u,v,w) \in B$. Then $g(u,v,w) = n$ so

$$6u^2 + 7v^2 + 7w^2 + 2vw + 2wu + 6uv = n.$$

As $n \equiv 0 \pmod 2$, we have $v \equiv w \pmod 2$. Thus $\frac{v+w}{2} \in \mathbb{Z}$ and

$$\left(v - w, u + \frac{v+w}{2}, -\frac{v+w}{2}\right) \in \mathbb{Z}^3.$$

Also

$$f\left(v - w, u + \frac{v+w}{2}, -\frac{v+w}{2}\right) = g(u,v,w) = n,$$

so

$$\left(v - w, u + \frac{v+w}{2}, -\frac{v+w}{2}\right) \in A.$$

Moreover,

$$\lambda\left(v-w, u+\frac{v+w}{2}, -\frac{v+w}{2}\right) = (u,v,w),$$

so λ is a surjection. As λ is an invertible linear transformation, λ is an injection. This proves λ is a bijection, so $r(f,n) = \text{card } A = \text{card } \lambda(A) = \text{card } B = r(g;n)$ for $n \equiv 0 \pmod 2$.

Now we consider the case when $n \in \mathbb{N}$ satisfies $n \equiv 1 \pmod 4$. By Theorems 1.1 and 1.2, we have $r(f;n) + r(g;n) = r(h;n)$ and $r(h;n) = 0$. Hence, we have $r(f;n) = r(g;n) = 0$. □

5. Evaluation of the Representation Number $r(k;n)$: Proof of Theorem 1.5

As the ternary quadratic form k is alone in its genus, we can apply Siegel's formula (1.3) to determine $r(k;n)$. In order to use this formula we require the local densities $d_p(k;n)$. To calculate these we determine the number of solutions of the congruence

$$k(x,y,z) \equiv n \pmod{p^t},$$

for $n, t \in \mathbb{N}$ and a prime p. We prove

Theorem 5.1. *Let $n \in \mathbb{N}$. Let p be a prime. Let $t \in \mathbb{N}$ be such that*

$$t \geq \begin{cases} \nu_p(n)+3 & \text{if } p=2, \\ \nu_p(n)+1 & \text{if } p \neq 2, \end{cases}$$

where $p^{\nu_p(n)} \,||\, n$. Let $N(n,p^t)$ denote the number of solutions of the congruence

$$x^2+3y^2+5z^2+2yz \equiv n \pmod{p^t}.$$

Then

$$N(n,2^t) = \begin{cases} 2^{2t} & \text{if } \nu_2(n)=0, \\ 2^{2t-\frac{\nu_2(n)-1}{2}} - \left(\frac{-1}{\frac{n}{2^{\nu_2(n)}}}\right) 2^{2t-\frac{\nu_2(n)+1}{2}} & \\ \quad - \left(\frac{2}{\frac{n}{2^{\nu_2(n)}}}\right) 2^{2t-\frac{\nu_2(n)+3}{2}} - \left(\frac{-2}{\frac{n}{2^{\nu_2(n)}}}\right) 2^{2t-\frac{\nu_2(n)+3}{2}} & \text{if } \nu_2(n) \text{ (odd)} \geq 1, \\ 3 \cdot 2^{2t-\frac{\nu_2(n)}{2}} & \text{if } \nu_2(n) \text{ (even)} \geq 2; \end{cases}$$

$$N(n,7^t) = \begin{cases} 2\cdot 7^{2t} - 7^{2t-\frac{\nu_7(n)}{2}} - 7^{2t-\frac{\nu_7(n)}{2}-1} & \text{if } \nu_7(n) \text{ (even)} \geq 0, \\ 2\cdot 7^{2t} - 7^{2t-\frac{\nu_7(n)+1}{2}} - \left(\frac{\frac{n}{7^{\nu_7(n)}}}{7}\right) 7^{2t-\frac{\nu_7(n)+1}{2}} & \text{if } \nu_7(n) \text{ (odd)} \geq 1; \end{cases}$$

and, for $p \neq 2, 7$,

$$N(n,p^t) = \\ = \begin{cases} p^{2t} + p^{2t-1} - p^{2t-\frac{1}{2}\nu_p(n)-1} + \left(\frac{\frac{-14n}{p^{\nu_p(n)}}}{p}\right) p^{2t-\frac{1}{2}\nu_p(n)-1} & \textit{if } \nu_p(n) \textit{ (even)} \geq 0, \\ p^{2t} + p^{2t-1} - p^{2t-\frac{1}{2}(\nu_p(n)+1)} - p^{2t-\frac{1}{2}(\nu_p(n)+3)} & \textit{if } \nu_p(n) \textit{ (odd)} \geq 1. \end{cases}$$

Proof. We just prove the formula for $N(n, 2^t)$ as the formulas for $N(n, 7^t)$ and $N(n, p^t)$ $(p \neq 2, 7)$ can be proved in a similar (and easier) manner.

To keep the notation simple we set $k := \nu_2(n)$, so that $2^k \mid\mid n$, and $m := n/2^k$ so that m is an odd positive integer. We write N for $N(n, 2^t)$, where $t \in \mathbb{N}$ is such that $t \geq k + 3$. For $s \in \mathbb{Z}$ we have

$$\sum_{u=0}^{2^t-1} e^{2\pi i s u/2^t} = \begin{cases} 2^t & \text{if } s \equiv 0 \pmod{2^t}, \\ 0 & \text{if } s \not\equiv 0 \pmod{2^t}. \end{cases}$$

Hence

$$N = \frac{1}{2^t} \sum_{x,y,z=0}^{2^t-1} \sum_{u=0}^{2^t-1} e^{2\pi i (x^2+3y^2+5z^2+2yz-n)u/2^t}.$$

The term with $u = 0$ contributes

$$\frac{1}{2^t} \sum_{x,y,z=0}^{2^t-1} 1 = \frac{1}{2^t}(2^t)^3 = 2^{2t}$$

to the sum. Taking this term to the left hand side, and then multiplying both sides by 2^t, we obtain

$$2^t N - 2^{3t} = \sum_{u=1}^{2^t-1} e^{-2\pi i n u/2^t} G(u; 2^t)\, G(3, 2, 5; u; 2^t),$$

where $G(u; 2^t)$ is the single Gauss sum

$$G(u; 2^t) := \sum_{x=0}^{2^t-1} e^{2\pi i u x^2/2^t}$$

and $G(3, 2, 5; u; 2^t)$ is the double Gauss sum

$$G(3, 2, 5; u; 2^t) := \sum_{y,z=0}^{2^t-1} e^{2\pi i u(3y^2+2yz+5z^2)/2^t}.$$

Expressing

$$u = 2^v w,\ 0 \leq v \leq t-1,\ 1 \leq w \leq 2^{t-v} - 1,\ w \equiv 1 \pmod 2,$$

we obtain

$$2^tN - 2^{3t} = \sum_{v=0}^{t-1} \sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} e^{-2\pi i n w/2^{t-v}} G(2^v w; 2^t)\, G(3,2,5; 2^v w; 2^t).$$

Now for $0 \le v \le t-1$ and $w \equiv 1 \pmod 2$, we have

$$G(2^v w; 2^t) = \begin{cases} \left(\frac{2}{w}\right)^{t-v} (1+i^w) 2^{(t+v)/2} & \text{if } 0 \le v \le t-2, \\ 0 & \text{if } v = t-1; \end{cases}$$

see [1, pp. 127–128]. Also we have

$$\begin{aligned} G(3,2,5;2^v w;2^t) &= 2^{2v} G(3,2,5;w;2^{t-v}) \\ &= \begin{cases} -\left(\frac{2}{w}\right)(1+i^w)(1+i^{3w}) 2^{t+v}\sqrt{2} & \text{if } v \le t-3, \\ 0 & \text{if } v = t-2,\ t-1, \end{cases} \end{aligned}$$

by [1, Theorem 1.2, p. 131]. Hence, as

$$(1+i^w)^2(1+i^{3w}) = 2\left(1 + \left(\frac{-1}{w}\right) i\right),$$

we deduce, on replacing n by $2^k m$,

$$2^tN - 2^{3t} = -\sum_{v=0}^{t-3} \sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} e^{-2\pi i m w/2^{t-v-k}} \left(\frac{2}{w}\right)^{t-v+1} \left(1 + \left(\frac{-1}{w}\right) i\right) 2^{\frac{3}{2}(t+v+1)}.$$

Now let

$$S_1 := -\sum_{\substack{v=0 \\ v\equiv t+1 \pmod 2}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} e^{-2\pi i m w/2^{t-v-k}},$$

$$S_2 := -i\sum_{\substack{v=0 \\ v\equiv t+1 \pmod 2}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{-1}{w}\right) e^{-2\pi i m w/2^{t-v-k}},$$

$$S_3 := -\sum_{\substack{v=0 \\ v\equiv t \pmod 2}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{2}{w}\right) e^{-2\pi i m w/2^{t-v-k}},$$

$$S_4 := -i\sum_{\substack{v=0 \\ v\equiv t \pmod 2}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{-2}{w}\right) e^{-2\pi i m w/2^{t-v-k}},$$

so that

$$2^t N - 2^{3t} = S_1 + S_2 + S_3 + S_4.$$

We examine each of S_1, S_2, S_3, and S_4 in turn.

First, we consider S_1. We set

$$S_1 = S_1' + S_1'',$$

where

$$S_1' := - \sum_{\substack{v=0 \\ v \equiv t+1 \pmod 2 \\ t-v-k \le 0}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w \equiv 1 \pmod 2}}^{2^{t-v}-1} e^{-2\pi i m w / 2^{t-v-k}}$$

and

$$S_1'' := - \sum_{\substack{v=0 \\ v \equiv t+1 \pmod 2 \\ t-v-k \ge 1}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w \equiv 1 \pmod 2}}^{2^{t-v}-1} e^{-2\pi i m w / 2^{t-v-k}}.$$

If $k = 0, 1, 2$, the condition $t - v - k \le 0$ requires $v \ge t - k \ge t - 2$, which is outside of the range of summation in S_1', so

$$S_1' = 0 \quad \text{if } k = 0, 1, 2.$$

If $k \ge 3$, the condition $t - v - k \le 0$ requires $v \ge t - k$ so

$$\begin{aligned} S_1' &= - \sum_{\substack{v=t-k \\ v \equiv t+1 \pmod 2}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w \equiv 1 \pmod 2}}^{2^{t-v}-1} 1 \\ &= - \sum_{\substack{v=t-k \\ v \equiv t+1 \pmod 2}}^{t-3} 2^{\frac{1}{2}(5t+v+1)} \\ &\overset{(h=t-v)}{=} -2^{3t} \sum_{\substack{h=3 \\ h \equiv 1 \pmod 2}}^{k} \frac{1}{2^{(h-1)/2}} \\ &= \begin{cases} -2^{3t}\left(1 - \frac{1}{2^{(k-1)/2}}\right) & \text{if } k \text{ (odd)} \ge 3, \\ -2^{3t}\left(1 - \frac{1}{2^{k/2-1}}\right) & \text{if } k \text{ (even)} \ge 4, \end{cases} \end{aligned}$$

that is,

$$S_1' = \begin{cases} -2^{3t} + 2^{3t - \left(\frac{k-1}{2}\right)} & \text{if } k \text{ (odd)} \ge 3, \\ -2^{3t} + 2^{3t - \frac{k}{2} + 1} & \text{if } k \text{ (even)} \ge 4. \end{cases}$$

Putting the values of S_1' together, we obtain

$$S_1' = \begin{cases} 0 & \text{if } k = 0, \\ -2^{3t} + 2^{3t-(\frac{k-1}{2})} & \text{if } k \text{ (odd)} \geq 1, \\ -2^{3t} + 2^{3t-\frac{k}{2}+1} & \text{if } k \text{ (even)} \geq 2. \end{cases}$$

Next

$$S_1'' = -\sum_{\substack{v=0 \\ v\equiv t+1 \pmod 2}}^{\min(t-3,t-k-1)} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} e^{-2\pi imw/2^{t-v-k}}.$$

Now

$$\sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} e^{-2\pi imw/2^{t-v-k}} = \begin{cases} -2^k & \text{if } t-v-k = 1, \\ 0 & \text{if } t-v-k \geq 2, \end{cases}$$

so

$$S_1'' = -\sum_{\substack{v=0 \\ v\equiv t+1 \pmod 2}}^{\min(t-3,t-k-1)} 2^{\frac{3}{2}(t+v+1)} \begin{cases} -2^k & \text{if } v = t-k-1, \\ 0 & \text{if } v \leq t-k-2. \end{cases}$$

Thus, the only possible term contributing to S_1'' is $v = t-k-1$, and this only contributes provided $t-k-1 \leq t-3$ and $t-k-1 \equiv t+1 \pmod 2$, that is, when $k \geq 2$ and k is even, and its contribution in this case is

$$-2^{\frac{3}{2}(t+t-k-1+1)} \cdot -2^k = 2^{3t-\frac{k}{2}}.$$

Hence, we deduce

$$S_1'' = \begin{cases} 0 & \text{if } k = 0 \text{ or } k \text{ (odd)} \geq 1, \\ 2^{3t-\frac{k}{2}} & \text{if } k \text{ (even)} \geq 2. \end{cases}$$

Adding the values of S_1' and S_1'', we deduce

$$S_1 = \begin{cases} 0 & \text{if } k = 0, \\ -2^{3t} + 2^{3t-\frac{k-1}{2}} & \text{if } k \text{ (odd)} \geq 1, \\ -2^{3t} + 3 \cdot 2^{3t-\frac{k}{2}} & \text{if } k \text{ (even)} \geq 2. \end{cases}$$

Secondly, we consider S_2. We set

$$S_2 = S_2' + S_2'',$$

where

$$S_2' := -i \sum_{\substack{v=0 \\ v\equiv t+1 \pmod 2 \\ t-v-k\leq 0}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{-1}{w}\right) e^{-2\pi imw/2^{t-v-k}}$$

and

$$S_2'' := -i \sum_{\substack{v=0 \\ v\equiv t+1 \pmod 2 \\ t-v-k\geq 1}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{-1}{w}\right) e^{-2\pi imw/2^{t-v-k}}.$$

If $k = 0, 1, 2$, the condition $t - v - k \leq 0$ requires $v \geq t - k \geq t - 2$, which is outside of the range of summation in S_2', so

$$S_2' = 0 \quad \text{if } k = 0, 1, 2.$$

If $k \geq 3$, the condition $t - v - k \leq 0$ requires $v \geq t - k$ so

$$S_2' = -i \sum_{\substack{v=t-k \\ v\equiv t+1 \pmod 2}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{-1}{w}\right).$$

Now $v \leq t - 3$ so $t - v > 2$ and thus $2^{t-v} - 1 \equiv 3 \pmod 4$. Hence, as $\left(\frac{-1}{4l+1}\right) + \left(\frac{-1}{4l+3}\right) = 1 - 1 = 0$, we have

$$\sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{-1}{w}\right) = 0$$

and

$$S_2' = 0 \quad \text{if } k \geq 3.$$

Hence, we have

$$S_2' = 0 \quad \text{for all } k \geq 0.$$

Next

$$S_2'' = -i \sum_{\substack{v=0 \\ v\equiv t+1 \pmod 2}}^{\min(t-3,t-k-1)} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{-1}{w}\right) e^{-2\pi imw/2^{t-v-k}}.$$

Now

$$\sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{-1}{w}\right) e^{-2\pi imw/2^{t-v-k}} = \begin{cases} 2^{k+1} i^{-m} & \text{if } t-v-k = 2, \\ 0 & \text{otherwise,} \end{cases}$$

so

$$S_2'' = -i \sum_{\substack{v=0 \\ v\equiv t+1 \pmod 2}}^{\min(t-3,t-k-1)} 2^{\frac{3}{2}(t+v+1)} \begin{cases} 2^{k+1} i^{-m} & \text{if } t-v-k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the only possible term contributing to S_2'' is $v = t-k-2$, and this only contributes provided $0 \le t-k-2 \le t-3$ and $t-k-2 \equiv t+1 \pmod 2$, that is, when $k \ge 1$ and k is odd, and its contribution in this case is

$$-i2^{\frac{3}{2}(t+t-k-2+1)} \cdot 2^{k+1} i^{-m} = (-1)^{(m+1)/2} 2^{3t-\frac{k}{2}-\frac{1}{2}}.$$

Hence,

$$S_2 = \begin{cases} 0 & \text{if } k \text{ (even)} \ge 0, \\ -\left(\frac{-1}{m}\right) 2^{3t-\frac{k}{2}-\frac{1}{2}} & \text{if } k \text{ (odd)} \ge 1. \end{cases}$$

Thirdly, we evaluate S_3. We set

$$S_3 = S_3' + S_3'',$$

where

$$S_3' := -\sum_{\substack{v=0 \\ v\equiv t \pmod 2 \\ t-v-k\le 0}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{2}{w}\right) e^{-2\pi i m w/2^{t-v-k}}$$

and

$$S_3'' := -\sum_{\substack{v=0 \\ v\equiv t \pmod 2 \\ t-v-k\ge 1}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{2}{w}\right) e^{-2\pi i m w/2^{t-v-k}}.$$

If $k = 0, 1, 2$, the condition $t-v-k \le 0$ requires $v \ge t-k \ge t-2$, which is outside of the range of summation in S_3', so

$$S_3' = 0 \quad \text{if } k = 0, 1, 2.$$

If $k \ge 3$, the condition $t-v-k \le 0$ requires $v \ge t-k$ so

$$S_3' = -\sum_{\substack{v=t-k \\ v\equiv t \pmod 2}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{2}{w}\right).$$

Now $v \le t-3$ so $t-v \ge 3$ and thus $2^{t-v}-1 \equiv 7 \pmod 8$. Hence, as

$$\left(\frac{2}{8l+1}\right) + \left(\frac{2}{8l+3}\right) + \left(\frac{2}{8l+5}\right) + \left(\frac{2}{8l+7}\right) = 1-1-1+1 = 0,$$

we have

$$\sum_{\substack{w=1 \\ w\equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{2}{w}\right) = 0,$$

so

$$S_3' = 0 \quad \text{if } k \geq 3.$$

Hence, we have

$$S_3' = 0 \quad \text{for all } k \geq 0.$$

Next

$$S_3'' = -\sum_{\substack{v=0 \\ v \equiv t \pmod 2}}^{\min(t-3,t-k-1)} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w \equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{2}{w}\right) e^{-2\pi i m w/2^{t-v-k}}.$$

For $0 \leq v \leq \min(t-3, t-k-1)$, we have by a short calculation

$$\sum_{\substack{w=1 \\ w \equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{2}{w}\right) e^{-2\pi i m w/2^{t-v-k}} = \begin{cases} \left(\frac{2}{m}\right) 2^{k+\frac{3}{2}} & \text{if } v = t-k-3, \\ 0 & \text{otherwise.} \end{cases}$$

Here we used $t \geq k+3$ so that $t-k-3 \geq 0$, and also $k \geq 0$ so that $t-k-3 \leq t-3$. Hence

$$\begin{aligned} S_3'' &= \begin{cases} -2^{\frac{3}{2}(t+t-k-3+1)} \left(\frac{2}{m}\right) 2^{k+\frac{3}{2}} & \text{if } k \text{ (odd)} \geq 1, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} -\left(\frac{2}{m}\right) 2^{3t-\frac{k}{2}-\frac{3}{2}} & \text{if } k \text{ (odd)} \geq 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Putting the values of S_3' and S_3'' together, we deduce

$$S_3 = \begin{cases} -\left(\frac{2}{m}\right) 2^{3t-\frac{k}{2}-\frac{3}{2}} & \text{if } k \text{ (odd)} \geq 1, \\ 0 & \text{if } k \text{ (even)} \geq 0. \end{cases}$$

Fourthly, we evaluate S_4. We set

$$S_4' := -i \sum_{\substack{v=0 \\ v \equiv t \pmod 2 \\ t-v-k \leq 0}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w \equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{-2}{w}\right) e^{-2\pi i m w/2^{t-v-k}}$$

and

$$S_4'' := -i \sum_{\substack{v=0 \\ v \equiv t \pmod 2 \\ t-v-k \geq 1}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w \equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{-2}{w}\right) e^{-2\pi i m w/2^{t-v-k}},$$

so that

$$S_4 = S_4' + S_4''.$$

If $k = 0, 1, 2$, the condition $t - v - k \leq 0$ requires $v \geq t - k \geq t - 2$, which is outside of the range of summation in S_4', so

$$S_4' = 0 \quad \text{if } k = 0, 1, 2.$$

If $k \geq 3$, the condition $t - v - k \leq 0$ requires $v \geq t - k$ so

$$S_4' = -i \sum_{\substack{v=t-k \\ v \equiv t \pmod 2}}^{t-3} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w \equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{-2}{w}\right).$$

Now $v \leq t - 3$ so $t - v \geq 3$ and thus $2^{t-v} - 1 \equiv 7 \pmod 8$. Hence, as

$$\left(\frac{-2}{8l+1}\right) + \left(\frac{-2}{8l+3}\right) + \left(\frac{-2}{8l+5}\right) + \left(\frac{-2}{8l+7}\right) = 1 + 1 - 1 - 1 = 0,$$

we have

$$\sum_{\substack{w=1 \\ w \equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{-2}{w}\right) = 0,$$

so

$$S_4' = 0 \quad \text{if } k \geq 3.$$

Hence, we have

$$S_4' = 0 \quad \text{for all } k \geq 0.$$

Next

$$S_4'' = -i \sum_{\substack{v=0 \\ v \equiv t \pmod 2}}^{\min(t-3, t-k-1)} 2^{\frac{3}{2}(t+v+1)} \sum_{\substack{w=1 \\ w \equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{-2}{w}\right) e^{-2\pi i m w / 2^{t-v-k}}.$$

For $0 \leq v \leq \min(t - 3, t - k - 1)$, we have

$$\sum_{\substack{w=1 \\ w \equiv 1 \pmod 2}}^{2^{t-v}-1} \left(\frac{-2}{w}\right) e^{-2\pi i m w / 2^{t-v-k}} = \begin{cases} -i \left(\frac{-2}{m}\right) 2^{k+\frac{3}{2}} & \text{if } v = t - k - 3, \\ 0 & \text{otherwise.} \end{cases}$$

Here we used $t \geq k+3$ so that $v = t-k-3 \geq 0$, and $k \geq 0$ so that $v = t-k-3 \leq t-3$. Hence, we have

$$\begin{aligned} S_4'' &= \begin{cases} -i 2^{\frac{3}{2}(t+t-k-3+1)} (-i) \left(\frac{-2}{m}\right) 2^{k+\frac{3}{2}} & \text{if } k \text{ (odd)} \geq 1, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} -\left(\frac{-2}{m}\right) 2^{3t - \frac{k}{2} - \frac{3}{2}} & \text{if } k \text{ (odd)} \geq 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Putting the values of S_4' and S_4'' together, we obtain

$$S_4 = \begin{cases} -\left(\frac{-2}{m}\right) 2^{3t-\frac{k}{2}-\frac{1}{2}} & \text{if } k \text{ (odd)} \geq 1, \\ 0 & \text{if } k \text{ (even)} \geq 0. \end{cases}$$

Adding the values of S_1, S_2, S_3, and S_4, we obtain

$$S_1 + S_2 + S_3 + S_4 = \begin{cases} 0 & \text{if } k = 0, \\ -2^{3t} + 2^{3t-\frac{k}{2}+\frac{1}{2}} - \left(\frac{-1}{m}\right) 2^{3t-\frac{k}{2}-\frac{1}{2}} & \\ \qquad - \left(\frac{2}{m}\right) 2^{3t-\frac{k}{2}-\frac{3}{2}} - \left(\frac{-2}{m}\right) 2^{3t-\frac{k}{2}-\frac{3}{2}} & \text{if } k \text{ (odd)} \geq 1, \\ -2^{3t} + 3 \cdot 2^{3t-\frac{k}{2}} & \text{if } k \text{ (even)} \geq 2. \end{cases}$$

Finally, we have

$$\begin{aligned} N(n,2^t) &= 2^{2t} + \frac{1}{2^t}(S_1 + S_2 + S_3 + S_4) \\ &= \begin{cases} 2^{2t} & \text{if } k = 0, \\ 2^{2t-\frac{k-1}{2}} - \left(\frac{-1}{m}\right) 2^{2t-\frac{k+1}{2}} - \left(\frac{2}{m}\right) 2^{2t-\frac{k+3}{2}} - \left(\frac{-2}{m}\right) 2^{2t-\frac{k+3}{2}} & \text{if } k \text{ (odd)} \geq 1, \\ 3 \cdot 2^{2t-\frac{k}{2}} & \text{if } k \text{ (even)} \geq 2. \end{cases} \end{aligned}$$

□

Our next task is to determine the local density $d_p(k;n)$ defined in (1.4) for $k = x^2 + 3y^2 + 5z^2 + 2yz$, that is,

$$d_p(n) := d_p(k;n) = \lim_{t\to\infty} \frac{N(n,p^t)}{p^{2t}}.$$

Theorem 5.2. *We have for $n \in \mathbb{N}$ and p a prime*

(i)

$$d_2(n) = \begin{cases} 1 & \textit{if } \alpha = 0, \\ 0 & \textit{if } \alpha \textit{ (odd)} \geq 1, \beta \textit{ (even)} \geq 0, g \equiv 1 \pmod 8 \textit{ or} \\ & \quad \alpha \textit{ (odd)} \geq 1, \beta \textit{ (odd)} \geq 1, \ g \equiv 7 \pmod 8, \\ \frac{3}{2^{(\alpha+1)/2}} & \textit{if } \alpha \textit{ (odd)} \geq 1, \beta \textit{ (even)} \geq 0, g \equiv 3 \pmod 4 \textit{ or} \\ & \quad \alpha \textit{ (odd)} \geq 1, \beta \textit{ (odd)} \geq 1, \ g \equiv 1 \pmod 4, \\ \frac{1}{2^{(\alpha-1)/2}} & \textit{if } \alpha \textit{ (odd)} \geq 1, \beta \textit{ (even)} \geq 0, g \equiv 5 \pmod 8 \textit{ or} \\ & \quad \alpha \textit{ (odd)} \geq 1, \beta \textit{ (odd)} \geq 1, \ g \equiv 3 \pmod 8, \\ \frac{3}{2^{\alpha/2}} & \textit{if } \alpha \textit{ (even)} \geq 2; \end{cases}$$

(ii)

$$d_7(n) = \begin{cases} \frac{6}{7} & \textit{if } \beta = 0, \\ 2 - \frac{2}{7^{(\beta+1)/2}} & \textit{if } \beta \textit{ (odd)} \geq 1, g \equiv 1,2,4 \pmod 7, \\ 2 & \textit{if } \beta \textit{ (odd)} \geq 1, g \equiv 3,5,6 \pmod 7, \\ 2 - \frac{8}{7^{\beta/2+1}} & \textit{if } \beta \textit{ (even)} \geq 2; \end{cases}$$

(iii) *if* $p \neq 2, 7$,

$$d_p(n) = \begin{cases} \left(1 + \frac{1}{p}\right)\left(1 - \frac{1}{p^{\nu_p(h)+1}}\right) & \textit{if } p \mid g,\ p \mid h, \\ 1 - \frac{1}{p^2} & \textit{if } p \mid g,\ p \nmid h, \\ 1 + \frac{1}{p} & \textit{if } p \nmid g,\ p \mid h,\ \left(\frac{-14n/p^{\nu_p(n)}}{p}\right) = 1, \\ 1 + \frac{1}{p} - \frac{2}{p^{\nu_p(h)+1}} & \textit{if } p \nmid g,\ p \mid h,\ \left(\frac{-14n/p^{\nu_p(n)}}{p}\right) = -1, \\ 1 + \left(\frac{-14n}{p}\right)\frac{1}{p} & \textit{if } p \nmid g,\ p \nmid h. \end{cases}$$

Proof. (i) By Theorem 5.1 we have for $t \geq \nu_2(n) + 3$

$$N(n, 2^t) = \begin{cases} 2^{2t} & \text{if } \nu_2(n) = 0, \\ 2^{2t - \frac{\nu_2(n)-1}{2}} - \left(\frac{-1}{\frac{n}{2^{\nu_2(n)}}}\right) 2^{2t - \frac{\nu_2(n)+1}{2}} & \\ - \left(\frac{2}{\frac{n}{2^{\nu_2(n)}}}\right) 2^{2t - \frac{\nu_2(n)+3}{2}} - \left(\frac{-2}{\frac{n}{2^{\nu_2(n)}}}\right) 2^{2t - \frac{\nu_2(n)+3}{2}} & \text{if } \nu_2(n) \text{ (odd)} \geq 1, \\ 3 \cdot 2^{2t - \frac{\nu_2(n)}{2}} & \text{if } \nu_2(n) \text{ (even)} \geq 2. \end{cases}$$

This implies that

1) For $\alpha = \nu_2(n) = 0$,
$$\frac{N(n, 2^t)}{2^{2t}} = \frac{2^{2t}}{2^{2t}} = 1.$$

2) For $\alpha = \nu_2(n)$ (even) ≥ 2,
$$\frac{N(n, 2^t)}{2^{2t}} = \frac{3 \cdot 2^{2t - \frac{\alpha}{2}}}{2^{2t}} = \frac{3}{2^{\alpha/2}}.$$

3) For $\alpha = \nu_2(n)$ (odd) ≥ 1, and either $\beta = \nu_7(n)$ (even) ≥ 0, $g \equiv 1 \pmod 8$, or $\beta = \nu_7(n)$ (odd) ≥ 1, $g \equiv 7 \pmod 8$, we have
$$\left(\frac{-1}{\frac{n}{2^\alpha}}\right) = \left(\frac{-1}{7^\beta g h^2}\right) = \left(\frac{-1}{7^\beta g}\right) = 1,$$
$$\left(\frac{2}{\frac{n}{2^\alpha}}\right) = \left(\frac{2}{7^\beta g h^2}\right) = \left(\frac{2}{7^\beta g}\right) = 1, \text{ and so } \left(\frac{-2}{\frac{n}{2^\alpha}}\right) = 1.$$

Hence, we deduce

$$\frac{N(n,2^t)}{2^{2t}} = \frac{2^{2t-\frac{\alpha-1}{2}} - 2^{2t-\frac{\alpha+1}{2}} - 2^{2t-\frac{\alpha+3}{2}} - 2^{2t-\frac{\alpha+3}{2}}}{2^{2t}} = 0.$$

4) For $\alpha = \nu_2(n)$ (odd) ≥ 1, $\beta = \nu_7(n)$ (even) ≥ 0, and $g \equiv 3 \pmod 4$, we have

$$\left(\frac{-1}{\frac{n}{2^\alpha}}\right) = \left(\frac{-1}{7^\beta g h^2}\right) = \left(\frac{-1}{g}\right) = -1,$$
$$\left(\frac{2}{\frac{n}{2^\alpha}}\right) = \left(\frac{2}{7^\beta g h^2}\right) = \left(\frac{2}{g}\right) = (-1)^{(g+1)/4} \text{ and so } \left(\frac{-2}{\frac{n}{2^\alpha}}\right) = -(-1)^{(g+1)/4}.$$

Hence, we obtain

$$\begin{aligned}\frac{N(n,2^t)}{2^{2t}} &= \frac{2^{2t-\frac{\alpha-1}{2}} + 2^{2t-\frac{\alpha+1}{2}} - (-1)^{(g+1)/4}2^{2t-\frac{\alpha+3}{2}} + (-1)^{(g+1)/4}2^{2t-\frac{\alpha+3}{2}}}{2^{2t}} \\ &= \frac{3}{2^{(\alpha+1)/2}}.\end{aligned}$$

5) For $\alpha = \nu_2(n)$ (odd) $\geq 1, \beta = \nu_7(n)$ (odd) ≥ 1, and $g \equiv 1 \pmod 4$, we have

$$\left(\frac{-1}{\frac{n}{2^\alpha}}\right) = \left(\frac{-1}{7^\beta g h^2}\right) = -1,$$
$$\left(\frac{2}{\frac{n}{2^\alpha}}\right) = \left(\frac{2}{7^\beta g h^2}\right) = \left(\frac{2}{g}\right) = (-1)^{(g-1)/4} \text{ and so } \left(\frac{-2}{\frac{n}{2^\alpha}}\right) = -(-1)^{(g-1)/4}.$$

Hence, we obtain

$$\begin{aligned}\frac{N(n,2^t)}{2^{2t}} &= \frac{2^{2t-\frac{\alpha-1}{2}} + 2^{2t-\frac{\alpha+1}{2}} - (-1)^{(g-1)/4}2^{2t-\frac{\alpha+3}{2}} + (-1)^{(g-1)/4}2^{2t-\frac{\alpha+3}{2}}}{2^{2t}} \\ &= \frac{3}{2^{(\alpha+1)/2}}.\end{aligned}$$

6) For $\alpha = \nu_2(n)$ (odd) ≥ 1, and either $\beta = \nu_7(n)$ (even) ≥ 0, $g \equiv 5 \pmod 8$, or $\beta = \nu_7(n)$ (odd) ≥ 1, $g \equiv 3 \pmod 8$, we have

$$\left(\frac{-1}{\frac{n}{2^\alpha}}\right) = \left(\frac{-1}{7^\beta g h^2}\right) = \left(\frac{-1}{7^\beta g}\right) = 1,$$
$$\left(\frac{2}{\frac{n}{2^\alpha}}\right) = \left(\frac{2}{7^\beta g h^2}\right) = \left(\frac{2}{7^\beta g}\right) = -1 \text{ and so } \left(\frac{-2}{\frac{n}{2^\alpha}}\right) = -1.$$

Hence

$$\frac{N(n,2^t)}{2^{2t}} = \frac{2^{2t-\frac{\alpha-1}{2}} - 2^{2t-\frac{\alpha+1}{2}} + 2^{2t-\frac{\alpha+3}{2}} + 2^{2t-\frac{\alpha+3}{2}}}{2^{2t}} = \frac{1}{2^{(\alpha-1)/2}}.$$

(ii) By Theorem 5.1 we have, for $t \geq \nu_7(n)+1$,

$$N(n,7^t)=\begin{cases}2\cdot 7^{2t}-7^{2t-\frac{\nu_7(n)}{2}}-7^{2t-\frac{\nu_7(n)}{2}-1} & \text{if } \nu_7(n) \text{ (even)} \geq 0,\\ 2\cdot 7^{2t}-7^{2t-\frac{\nu_2(n)+1}{2}}-\left(\frac{\frac{n}{7^{\nu_7(n)}}}{7}\right)7^{2t-\frac{\nu_7(n)+1}{2}} & \text{if } \nu_7(n) \text{ (odd)} \geq 1.\end{cases}$$

This implies that

1) for $\beta=\nu_7(n)=0$, we have

$$\frac{N(n,7^t)}{7^{2t}}=\frac{2\cdot 7^{2t}-7^{2t}-7^{2t-1}}{7^{2t}}=\frac{6}{7},$$

2) for $\beta=\nu_7(n)$ (odd) ≥ 1, and $g\equiv 1,2,4 \pmod 7$, we have

$$\begin{aligned}\frac{N(n,7^t)}{7^{2t}}&=\frac{2\cdot 7^{2t}-7^{2t-\frac{\beta+1}{2}}-\left(\frac{\frac{n}{7^\beta}}{7}\right)7^{2t-\frac{\beta+1}{2}}}{7^{2t}}\\&=2-\frac{1}{7^{(\beta+1)/2}}-\left(\frac{\frac{n}{7^\beta}}{7}\right)\cdot\frac{1}{7^{(\beta+1)/2}}\\&=2-\frac{1}{7^{(\beta+1)/2}}-\left(\frac{2^\alpha gh^2}{7}\right)\cdot\frac{1}{7^{(\beta+1)/2}}\\&=2-\frac{2}{7^{(\beta+1)/2}},\end{aligned}$$

3) for $\beta=\nu_7(n)$ (odd) ≥ 1, and $g\equiv 3,5,6 \pmod 7$, we have

$$\begin{aligned}\frac{N(n,7^t)}{7^{2t}}&=\frac{2\cdot 7^{2t}-7^{2t-\frac{\beta+1}{2}}-\left(\frac{\frac{n}{7^\beta}}{7}\right)7^{2t-\frac{\beta+1}{2}}}{7^{2t}}\\&=2-\frac{1}{7^{(\beta+1)/2}}-\left(\frac{\frac{n}{7^\beta}}{7}\right)\cdot\frac{1}{7^{(\beta+1)/2}}\\&=2-\frac{1}{7^{(\beta+1)/2}}-\left(\frac{2^\alpha gh^2}{7}\right)\cdot\frac{1}{7^{(\beta+1)/2}}\\&=2,\end{aligned}$$

4) for $\beta=\nu_7(n)$ (even) ≥ 2, we have

$$\frac{N(n,7^t)}{7^{2t}}=\frac{2\cdot 7^{2t}-7^{2t-\beta/2}-7^{2t-\beta/2-1}}{7^{2t}}=2-\frac{8}{7^{\beta/2+1}}.$$

(iii) By Theorem 5.1, for $p\neq 2,7$ and $t\geq \nu_p(n)+1$, we have

$$N(n,p^t)$$
$$=\begin{cases}p^{2t}+p^{2t-1}-p^{2t-\frac{1}{2}\nu_p(n)-1}+\left(\frac{\frac{-14n}{p^{\nu_p(n)}}}{p}\right)p^{2t-\frac{1}{2}\nu_p(n)-1} & \text{if } \nu_p(n) \text{ (even)} \geq 0,\\ p^{2t}+p^{2t-1}-p^{2t-\frac{1}{2}(\nu_p(n)+1)}-p^{2t-\frac{1}{2}(\nu_p(n)+3)} & \text{if } \nu_p(n) \text{ (odd)} \geq 1.\end{cases}$$

1) Let $p \mid g$, $p \mid h$. Then $\nu_p(g) = 1$, $\nu_p(h) > 0$, $\nu_p(n) = 1 + 2\nu_p(h)$ and we have

$$\frac{N(n, p^t)}{p^{2t}} = \frac{p^{2t} + p^{2t-1} - p^{2t-\frac{1}{2}(1+2\nu_p(h)+1)} - p^{2t-\frac{1}{2}(1+2\nu_p(h)+3)}}{p^{2t}} = \left(1 + \frac{1}{p}\right)\left(1 - \frac{1}{p^{\nu_p(h)+1}}\right).$$

2) Let $p \mid g$, $p \nmid h$. Then $\nu_p(h) = 0$ and $\nu_p(g) = 1$. Therefore, $\nu_p(n) = 1$. Hence

$$\frac{N(n, p^t)}{p^{2t}} = \frac{p^{2t} + p^{2t-1} - p^{2t-1} - p^{2t-2}}{p^{2t}} = 1 - \frac{1}{p^2}.$$

3) Let $p \nmid g$, $p \mid h$ and $\left(\frac{\frac{-14n}{p^{\nu_p(n)}}}{p}\right) = 1$. Then $\nu_p(g) = 0$ and $\nu_p(n) = 2\nu_p(h)$. Hence

$$\frac{N(n, p^t)}{p^{2t}} = \frac{p^{2t} + p^{2t-1} - p^{2t-\nu_p(h)-1} + p^{2t-\nu_p(h)-1}}{p^{2t}} = 1 + \frac{1}{p}.$$

4) Let $p \nmid g$, $p \mid h$ and $\left(\frac{\frac{-14n}{p^{\nu_p(n)}}}{p}\right) = -1$. Then $\nu_p(g) = 0$ and $\nu_p(n) = 2\nu_p(h)$. Hence

$$\frac{N(n, p^t)}{p^{2t}} = \frac{p^{2t} + p^{2t-1} - p^{2t-\nu_p(h)-1} - p^{2t-\nu_p(h)-1}}{p^{2t}} = 1 + \frac{1}{p} - \frac{2}{p^{\nu_p(h)+1}}.$$

5) Let $p \nmid g$, $p \nmid h$. Then $\nu_p(g) = \nu_p(h) = 0$ and $\nu_p(n) = 0$.

$$\frac{N(n, p^t)}{p^{2t}} = \frac{p^{2t} + p^{2t-1} - p^{2t-1} + \left(\frac{-14n}{p}\right) p^{2t-1}}{p^{2t}} = 1 + \left(\frac{-14n}{p}\right)\frac{1}{p}.$$

□

We are now in a position to prove Theorem 1.5.

Proof. From (1.5) and (1.6) we observe that

$$n^* = \begin{cases} g & \text{if } \alpha \equiv 1 \pmod 2,\ \beta \equiv 1 \pmod 2, \\ 2g & \text{if } \alpha \equiv 0 \pmod 2,\ \beta \equiv 1 \pmod 2, \\ 7g & \text{if } \alpha \equiv 1 \pmod 2,\ \beta \equiv 0 \pmod 2, \\ 14g & \text{if } \alpha \equiv 0 \pmod 2,\ \beta \equiv 0 \pmod 2. \end{cases} \tag{5.1}$$

From the definition of $l(n)$ given in (1.7), we have

$$l(n) = \prod_{\substack{p \mid g \\ p \mid h}} \frac{p^{\nu_p(h)+1} - 1}{p - 1} \prod_{\substack{p \nmid g \\ p \mid h \\ \left(\frac{-n^*}{p}\right) = 1}} p^{\nu_p(h)} \prod_{\substack{p \nmid g \\ p \mid h \\ \left(\frac{-n^*}{p}\right) = -1}} \frac{p^{\nu_p(h)+1} + p^{\nu_p(h)} - 2}{p - 1}. \tag{5.2}$$

By Theorem 5.2 (iii) we have

$$\prod_{p|h} d_p(n) = \prod_{\substack{p|g\\p|h}} \left(1+\frac{1}{p}\right)\left(1-\frac{1}{p^{\nu_p(h)+1}}\right) \prod_{\substack{p\nmid g\\p|h\\ \left(\frac{-14n/p^{\nu_p(n)}}{p}\right)=1}} \left(1+\frac{1}{p}\right)$$
$$\prod_{\substack{p\nmid g\\p|h\\ \left(\frac{-14n/p^{\nu_p(n)}}{p}\right)=-1}} \left(1+\frac{1}{p}-\frac{2}{p^{\nu_p(n)+1}}\right).$$

If $p \nmid g$ and $p \mid h$ then $p \neq 2, 7$ and $\nu_p(n) = 2\nu_p(h)$ so

$$14n/p^{\nu_p(n)} = n^* j^2,$$

where

$$j = 2^{\left[\frac{\alpha+1}{2}\right]} 7^{\left[\frac{\beta+1}{2}\right]} \frac{h}{p^{\nu_p(h)}} \not\equiv 0 \pmod{p},$$

and so

$$\left(\frac{-14n/p^{\nu_p(n)}}{p}\right) = \left(\frac{-n^* j^2}{p}\right) = \left(\frac{-n^*}{p}\right).$$

Thus

$$\prod_{p|h} d_p(n) = \prod_{\substack{p|g\\p|h}} \left(\frac{p+1}{p}\right)\left(\frac{p^{\nu_p(h)+1}-1}{p^{\nu_p(h)+1}}\right) \prod_{\substack{p\nmid g\\p|h\\ \left(\frac{n^*}{p}\right)=1}} \left(\frac{p+1}{p}\right) \tag{5.3}$$
$$\prod_{\substack{p\nmid g\\p|h\\ \left(\frac{n^*}{p}\right)=-1}} \left(\frac{p^{\nu_p(h)+1}+p^{\nu_p(h)}-2}{p^{\nu_p(n)+1}}\right).$$

From (5.2) and (5.3), we deduce (after a short calculation)

$$\frac{h}{l(n)} \prod_{p|h} d_p(n) = \prod_{\substack{p|g\\p|h}} \left(1-\frac{1}{p^2}\right) \prod_{\substack{p\nmid g\\p|h}} \left(1+\left(\frac{-n^*}{p}\right)\frac{1}{p}\right). \tag{5.4}$$

Next, by Theorem 5.2 (iii), we obtain

$$\prod_{\substack{p\nmid h\\p\neq 2,7}} d_p(n) = \prod_{\substack{p|g\\p\nmid h\\p\neq 2,7}} \left(1-\frac{1}{p^2}\right) \prod_{\substack{p\nmid g\\p\nmid h\\p\neq 2,7}} \left(1+\left(\frac{-n^*}{p}\right)\frac{1}{p}\right), \tag{5.5}$$

as $\left(\frac{-14n}{p}\right) = \left(\frac{-n^*}{p}\right)$ for $p \neq 2, 7, p \nmid g, p \nmid h$. Further, as $p \nmid n^*$ for $p \neq 2, 7, p \nmid g$, we have

$$1 + \left(\frac{-n^*}{p}\right)\frac{1}{p} = \frac{1 - \frac{1}{p^2}}{1 - \left(\frac{-n^*}{p}\right)\frac{1}{p}},$$

so that (5.5) becomes

$$\prod_{\substack{p \nmid h \\ p \neq 2,7}} d_p(n) = \prod_{\substack{p \neq 2,7 \\ p \nmid h}} \left(1 - \frac{1}{p^2}\right) \prod_{\substack{p \neq 2,7 \\ p \nmid g \\ p \nmid h}} \left(1 - \left(\frac{-n^*}{p}\right)\frac{1}{p}\right)^{-1}. \tag{5.6}$$

We now let K denote the imaginary quadratic field $\mathbb{Q}(\sqrt{-n^*})$. We denote the class number of K by $h(K)$ and the number of roots of unity in the ring of integers of K by $w(K)$. We have

$$w(K) = \begin{cases} 6 & \text{if } n^* = 3 \quad (\text{equiv. } K = \mathbb{Q}(\sqrt{-3})), \\ 4 & \text{if } n^* = 1 \quad (\text{equiv. } K = \mathbb{Q}(\sqrt{-1})), \\ 2 & \text{otherwise.} \end{cases} \tag{5.7}$$

The discriminant $d(K)$ of K is given by

$$d(K) = \begin{cases} -n^* & \text{if } -n^* \equiv 1 \pmod 4, \\ -4n^* & \text{if } -n^* \equiv 2, 3 \pmod 4. \end{cases}$$

By Dirichlet's class number formula, we have

$$h(K) = \begin{cases} \frac{w(K)\sqrt{n^*}}{2\pi} \prod_p \left(1 - \left(\frac{-n^*}{p}\right)\frac{1}{p}\right)^{-1} & \text{if } d(K) = -n^*, \\ \frac{w(K)\sqrt{n^*}}{\pi} \prod_{p \neq 2} \left(1 - \left(\frac{-n^*}{p}\right)\frac{1}{p}\right)^{-1} & \text{if } d(K) = -4n^*. \end{cases} \tag{5.8}$$

We combine the two formulas in (5.8) into one formula by defining

$$r(n^*) := \begin{cases} \frac{w(K)\sqrt{n^*}}{2\pi} \left(1 - \left(\frac{-n^*}{2}\right)\frac{1}{2}\right)^{-1} \left(1 - \left(\frac{-n^*}{7}\right)\frac{1}{7}\right)^{-1} & \text{if } d(K) = -n^*, \\ \frac{w(K)\sqrt{n^*}}{\pi} \left(1 - \left(\frac{-n^*}{7}\right)\frac{1}{7}\right)^{-1} & \text{if } d(K) = -4n^*, \end{cases} \tag{5.9}$$

so that

$$h(K) = r(n^*) \prod_{p \neq 2,7} \left(1 - \left(\frac{-n^*}{p}\right)\frac{1}{p}\right)^{-1}. \tag{5.10}$$

From (5.5) and (5.10) we deduce

$$\frac{1}{h(K)} \prod_{\substack{p \nmid h \\ p \neq 2,7}} d_p(n) = \frac{1}{r(n^*)} \prod_{\substack{p \neq 2,7 \\ p \nmid h}} \left(1 - \frac{1}{p^2}\right) \frac{\prod_{p \neq 2,7} \left(1 - \left(\frac{-n^*}{p}\right)\frac{1}{p}\right)}{\prod_{\substack{p \neq 2,7 \\ p \nmid g \\ p \nmid h}} \left(1 - \left(\frac{-n^*}{p}\right)\frac{1}{p}\right)}.$$

As $\left(\frac{-n^*}{p}\right) = 0$ for $p \neq 2, 7, p \mid g$, we obtain

$$\prod_{\substack{p \nmid h \\ p \neq 2,7}} d_p(n) = \frac{h(K)}{r(n^*)} \prod_{\substack{p \neq 2,7 \\ p \nmid h}} \left(1 - \frac{1}{p^2}\right) \prod_{\substack{p \neq 2,7 \\ p \nmid g \\ p \mid h}} \left(1 - \left(\frac{-n^*}{p}\right) \frac{1}{p}\right). \tag{5.11}$$

Rearranging (5.4) slightly, we have

$$\prod_{\substack{p \mid h \\ p \neq 2,7}} d_p(n) = \frac{l(n)}{h} \prod_{\substack{p \neq 2,7 \\ p \mid g \\ p \mid h}} \left(1 - \frac{1}{p^2}\right) \prod_{\substack{p \neq 2,7 \\ p \nmid g \\ p \mid h}} \left(1 + \left(\frac{-n^*}{p}\right) \frac{1}{p}\right). \tag{5.12}$$

Multiplying (5.11) and (5.12) together, we deduce (recalling that $\left(\frac{-n^*}{p}\right) = \pm 1$ for $p \neq 2, 7, p \nmid g$)

$$\prod_{p \neq 2,7} d_p(n) = \frac{l(n)h(K)}{hr(n^*)} \prod_{\substack{p \neq 2,7 \\ p \nmid h}} \left(1 - \frac{1}{p^2}\right) \prod_{\substack{p \neq 2,7 \\ p \mid g \\ p \mid h}} \left(1 - \frac{1}{p^2}\right) \prod_{\substack{p \neq 2,7 \\ p \nmid g \\ p \mid h}} \left(1 - \frac{1}{p^2}\right);$$

that is,

$$\prod_{p \neq 2,7} d_p(n) = \frac{l(n)h(K)}{hr(n^*)} \prod_{p \neq 2,7} \left(1 - \frac{1}{p^2}\right).$$

A simple calculation starting from the well-known result

$$\prod_p \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2}$$

yields

$$\prod_{p \neq 2,7} \left(1 - \frac{1}{p^2}\right) = \frac{49}{6\pi^2}$$

so

$$\prod_p d_p(n) = \frac{49 l(n)h(K)d_2(n)d_7(n)}{6\pi^2 hr(n^*)}.$$

By Siegel's formula, which is given in (1.3), we have

$$r(1, 3, 5, 2, 0, 0; n) = \frac{2\pi\sqrt{n}}{\sqrt{14}} \prod_p d_p(n)$$

so that

$$r(1, 3, 5, 2, 0, 0; n) = c_1(n)l(n)h(K),$$

where

$$c_1(n) := \frac{49\sqrt{n}\ d_2(n)\ d_7(n)}{3\sqrt{14}\,\pi\, h\, r(n^*)}.$$

Using (1.5) and (1.6), we see that

$$\sqrt{\frac{n}{n^*}} = \frac{2^{[\frac{\alpha+1}{2}]}7^{[\frac{\beta+1}{2}]}h}{\sqrt{14}},$$

so that $c_1(n)$ can be rewritten as

$$c_1(n) = \frac{2^{[\frac{\alpha-1}{2}]}7^{[\frac{\beta+3}{2}]}h}{3\pi}\sqrt{n^*}\frac{d_2(n)d_7(n)}{r(n^*)}.$$

Putting the value of $r(n^*)$ from (5.9) into this formula, we obtain

$$c_1(n) = \begin{cases} \frac{1}{3w(K)}2^{[\frac{\alpha+1}{2}]}\left(1-\left(\frac{-n^*}{2}\right)\frac{1}{2}\right)d_2(n) & \\ \quad\cdot 7^{[\frac{\beta+3}{2}]}\left(1-\left(\frac{-n^*}{7}\right)\frac{1}{7}\right)d_7(n) & \text{if } n^* \equiv 3 \pmod 4, \\ \frac{1}{3w(K)}2^{[\frac{\alpha-1}{2}]}d_2(n)7^{[\frac{\beta+3}{2}]}\left(1-\left(\frac{-n^*}{7}\right)\frac{1}{7}\right)d_7(n) & \text{if } n^* \equiv 1,2 \pmod 4. \end{cases}$$

Finally, we calculate $c_1(n)$ explicitly for each of the 16 cases in Table 1.1. We make use of (5.1) for the value of n^*, (5.7) for the value of $w(K)$, and Theorem 5.2 for the values of $d_2(n)$ and $d_7(n)$.

`Case 1.` $\alpha = 0, \beta$ (even) ≥ 0.

$$n^* = 14g \equiv 2 \pmod 4, \quad K = \mathbb{Q}(\sqrt{-14g}) \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3}), \quad w(K) = 2,$$
$$\left(\frac{-n^*}{7}\right) = \left(\frac{-14g}{7}\right) = 0, \quad \frac{1}{3w(K)} = \frac{1}{6},$$
$$2^{[\frac{\alpha-1}{2}]}d_2(n) = 2^{-1}\cdot 1 = \frac{1}{2},$$
$$7^{[\frac{\beta+3}{2}]}\left(1-\left(\frac{-n^*}{7}\right)\frac{1}{7}\right)d_7(n) = 7^{\frac{\beta}{2}+1}\left(2-\frac{8}{7^{\frac{\beta}{2}+1}}\right),$$
$$c_1(n) = \frac{1}{6}7^{\frac{\beta+2}{2}} - \frac{2}{3}.$$

`Case 2.` α (even) $\geq 2, \beta$ (even) ≥ 0.

$$n^* = 14g \equiv 2 \pmod 4, \quad K = \mathbb{Q}(\sqrt{-14g}) \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3}), \quad w(K) = 2,$$
$$\left(\frac{-n^*}{7}\right) = \left(\frac{-14g}{7}\right) = 0, \quad \frac{1}{3w(K)} = \frac{1}{6},$$
$$2^{[\frac{\alpha-1}{2}]}d_2(n) = 2^{\frac{\alpha-2}{2}}\cdot\frac{3}{2^{\alpha/2}} = \frac{3}{2},$$

$$7^{[\frac{\beta+3}{2}]}\left(1-\left(\frac{-n^*}{7}\right)\frac{1}{7}\right)d_7(n)=7^{\frac{\beta+2}{2}}\left(2-\frac{8}{7^{\frac{\beta+2}{2}}}\right),$$
$$c_1(n)=\frac{1}{6}\cdot\frac{3}{2}\cdot\left(2\cdot 7^{\frac{\beta+2}{2}}-8\right)=\frac{1}{2}7^{\frac{\beta+2}{2}}-2.$$

`Case 3.` $\alpha=0,\beta\text{ (odd)}\geq 1, g\equiv 1,2,4 \pmod 7$.

$$n^*=2g\equiv 2 \pmod 4,\quad K=\mathbb{Q}(\sqrt{-2g})\neq\mathbb{Q}(\sqrt{-1}),\mathbb{Q}(\sqrt{-3}),\quad w(K)=2,$$
$$\left(\frac{-n^*}{7}\right)=\left(\frac{-2g}{7}\right)=-\left(\frac{g}{7}\right)=-1,\quad \frac{1}{3w(K)}=\frac{1}{6},$$
$$2^{[\frac{\alpha-1}{2}]}d_2(n)=2^{-1}\cdot 1=\frac{1}{2},$$
$$7^{[\frac{\beta+3}{2}]}\left(1-\left(\frac{-n^*}{7}\right)\frac{1}{7}\right)d_7(n)=7^{\frac{\beta+3}{2}}\cdot\frac{8}{7}\left(2-\frac{2}{7^{\frac{\beta+1}{2}}}\right)=16\left(7^{\frac{\beta+1}{2}}-1\right),$$
$$c_1(n)=\frac{1}{6}\cdot\frac{1}{2}\cdot 16\left(7^{\frac{\beta+1}{2}}-1\right)=\frac{4}{3}\left(7^{\frac{\beta+1}{2}}-1\right).$$

`Case 4.` $\alpha=0,\beta\text{ (odd)}\geq 1, g\equiv 3,5,6 \pmod 7$.

$$n^*=2g\equiv 2 \pmod 4,\quad K=\mathbb{Q}(\sqrt{-2g})\neq\mathbb{Q}(\sqrt{-1}),\mathbb{Q}(\sqrt{-3}),\quad w(K)=2,$$
$$\left(\frac{-n^*}{7}\right)=\left(\frac{-2g}{7}\right)=-\left(\frac{g}{7}\right)=1,\quad \frac{1}{3w(K)}=\frac{1}{6},$$
$$2^{[\frac{\alpha-1}{2}]}d_2(n)=2^{-1}\cdot 1=\frac{1}{2},$$
$$7^{[\frac{\beta+3}{2}]}\left(1-\left(\frac{-n^*}{7}\right)\frac{1}{7}\right)d_7(n)=7^{\frac{\beta+3}{2}}\cdot\frac{6}{7}\cdot 2=12\cdot 7^{\frac{\beta+1}{2}},$$
$$c_1(n)=\frac{1}{6}\cdot\frac{1}{2}\cdot 12\cdot 7^{\frac{\beta+1}{2}}=7^{\frac{\beta+1}{2}}.$$

`Case 5.` $\alpha\text{ (even)}\geq 2,\beta\text{ (odd)}\geq 1, g\equiv 1,2,4 \pmod 7$.

$$n^*=2g\equiv 2 \pmod 4,\quad K=\mathbb{Q}(\sqrt{-2g})\neq\mathbb{Q}(\sqrt{-1}),\mathbb{Q}(\sqrt{-3}),\quad w(K)=2,$$
$$\left(\frac{-n^*}{7}\right)=\left(\frac{-2g}{7}\right)=-\left(\frac{g}{7}\right)=-1,\quad \frac{1}{3w(K)}=\frac{1}{6},$$
$$2^{[\frac{\alpha-1}{2}]}d_2(n)=2^{\frac{\alpha-2}{2}}\cdot\frac{3}{2^{\alpha/2}}=\frac{3}{2},$$
$$7^{[\frac{\beta+3}{2}]}\left(1-\left(\frac{-n^*}{7}\right)\frac{1}{7}\right)d_7(n)=7^{\frac{\beta+3}{2}}\cdot\frac{8}{7}\left(2-\frac{2}{7^{\frac{\beta+1}{2}}}\right)=16\left(7^{\frac{\beta+1}{2}}-1\right),$$
$$c_1(n)=\frac{1}{6}\cdot\frac{3}{2}\cdot 16\left(7^{\frac{\beta+1}{2}}-1\right)=4\cdot 7^{\frac{\beta+1}{2}}-4.$$

`Case 6.` $\alpha\text{ (even)}\geq 2,\beta\text{ (odd)}\geq 1, g\equiv 3,5,6 \pmod 7$.

$$n^*=2g\equiv 2 \pmod 4,\quad K=\mathbb{Q}(\sqrt{-2g})\neq\mathbb{Q}(\sqrt{-1}),\mathbb{Q}(\sqrt{-3}),\quad w(K)=2,$$

$$\left(\frac{-n^*}{7}\right) = \left(\frac{-2g}{7}\right) = -\left(\frac{g}{7}\right) = 1, \quad \frac{1}{3w(K)} = \frac{1}{6},$$
$$2^{[\frac{\alpha-1}{2}]} d_2(n) = 2^{\frac{\alpha-2}{2}} \cdot \frac{3}{2^{\alpha/2}} = \frac{3}{2},$$
$$7^{[\frac{\beta+3}{2}]}\left(1 - \left(\frac{-n^*}{7}\right)\frac{1}{7}\right) d_7(n) = 7^{\frac{\beta+3}{2}} \cdot \frac{6}{7} \cdot 2 = 12 \cdot 7^{\frac{\beta+1}{2}},$$
$$c_1(n) = \frac{1}{6} \cdot \frac{3}{2} \cdot 12 \cdot 7^{\frac{\beta+1}{2}} = 3 \cdot 7^{\frac{\beta+1}{2}}.$$

Case 7. α (odd) $\geq 1, \beta$ (even) $\geq 0, g \equiv 1 \pmod 8$.

In this case $d_2(n) = 0$ so $c_1(n) = 0$.

Case 8. α (odd) $\geq 1, \beta$ (even) $\geq 0, g \equiv 3 \pmod 4$.

$$n^* = 7g \equiv 1 \pmod 4, \quad K = \mathbb{Q}(\sqrt{-7g}) \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3}), \quad w(K) = 2,$$
$$\left(\frac{-n^*}{7}\right) = \left(\frac{-7g}{7}\right) = 0, \quad \frac{1}{3w(K)} = \frac{1}{6},$$
$$2^{[\frac{\alpha-1}{2}]} d_2(n) = 2^{\frac{\alpha-1}{2}} \cdot \frac{3}{2^{\frac{\alpha+1}{2}}} = \frac{3}{2},$$
$$7^{[\frac{\beta+3}{2}]}\left(1 - \left(\frac{-n^*}{7}\right)\frac{1}{7}\right) d_7(n) = 7^{\frac{\beta+2}{2}}\left(2 - \frac{8}{7^{\frac{\beta+2}{2}}}\right) = 2 \cdot 7^{\frac{\beta+2}{2}} - 8,$$
$$c_1(n) = \frac{1}{6} \cdot \frac{3}{2}\left(2 \cdot 7^{\frac{\beta+2}{2}} - 8\right) = \frac{1}{2} 7^{\frac{\beta+2}{2}} - 2.$$

Case 9. α (odd) $\geq 1, \beta$ (even) $\geq 0, g \equiv 5 \pmod 8$.

$$n^* = 7g \equiv 3 \pmod 8, \quad -n^* \equiv 5 \pmod 8, \quad K = \mathbb{Q}(\sqrt{-7g}) \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3}),$$
$$w(K) = 2, \quad \left(\frac{-n^*}{7}\right) = \left(\frac{-7g}{7}\right) = 0, \quad \frac{1}{3w(K)} = \frac{1}{6},$$
$$2^{[\frac{\alpha+1}{2}]}\left(1 - \left(\frac{-n^*}{2}\right)\frac{1}{2}\right) d_2(n) = 2^{\frac{\alpha+1}{2}} \cdot \frac{3}{2} \cdot \frac{1}{2^{\frac{\alpha-1}{2}}} = 3,$$
$$7^{[\frac{\beta+3}{2}]}\left(1 - \left(\frac{-n^*}{7}\right)\frac{1}{7}\right) d_7(n) = 7^{\frac{\beta+2}{2}}\left(2 - \frac{8}{7^{\frac{\beta+2}{2}}}\right) = 2 \cdot 7^{\frac{\beta+2}{2}} - 8,$$
$$c_1(n) = \frac{1}{6} \cdot 3\left(2 \cdot 7^{\frac{\beta+2}{2}} - 8\right) = 7^{\frac{\beta+2}{2}} - 4.$$

Case 10. α (odd) $\geq 1, \beta$ (odd) $\geq 1, g = 1$.

$$n^* = g = 1, \quad K = \mathbb{Q}(\sqrt{-1}), \quad w(K) = 4,$$
$$\left(\frac{-n^*}{7}\right) = \left(\frac{-1}{7}\right) = -1, \quad \frac{1}{3w(K)} = \frac{1}{12},$$
$$2^{[\frac{\alpha-1}{2}]} d_2(n) = 2^{\frac{\alpha-1}{2}} \cdot \frac{3}{2^{\frac{\alpha+1}{2}}} = \frac{3}{2},$$

$$7^{[\frac{\beta+3}{2}]}\left(1-\left(\frac{-n^*}{7}\right)\frac{1}{7}\right)d_7(n)=7^{\frac{\beta+3}{2}}\cdot\frac{8}{7}\left(2-\frac{2}{7^{\frac{\beta+1}{2}}}\right)=16\left(7^{\frac{\beta+1}{2}}-1\right),$$
$$c_1(n)=\frac{1}{12}\cdot\frac{3}{2}\cdot 16\left(7^{\frac{\beta+1}{2}}-1\right)=2\cdot 7^{\frac{\beta+1}{2}}-2.$$

Case 11. α (odd) $\geq 1, \beta$ (odd) $\geq 1, g\equiv 1 \pmod 4, g\equiv 1,2,4 \pmod 7, g\neq 1$.

$$n^*=g\neq 1,\quad K=\mathbb{Q}(\sqrt{-g})\neq\mathbb{Q}(\sqrt{-1}),\mathbb{Q}(\sqrt{-3}),\quad w(K)=2,$$
$$\left(\frac{-n^*}{7}\right)=\left(\frac{-g}{7}\right)=-\left(\frac{g}{7}\right)=-1,\quad \frac{1}{3w(K)}=\frac{1}{6},$$
$$2^{[\frac{\alpha-1}{2}]}d_2(n)=2^{\frac{\alpha-1}{2}}\cdot\frac{3}{2^{\frac{\alpha+1}{2}}}=\frac{3}{2},$$
$$7^{[\frac{\beta+3}{2}]}\left(1-\left(\frac{-n^*}{7}\right)\frac{1}{7}\right)d_7(n)=7^{\frac{\beta+3}{2}}\cdot\frac{8}{7}\left(2-\frac{2}{7^{\frac{\beta+1}{2}}}\right)=16\left(7^{\frac{\beta+1}{2}}-1\right),$$
$$c_1(n)=\frac{1}{6}\cdot\frac{3}{2}\cdot 16\left(7^{\frac{\beta+1}{2}}-1\right)=4\cdot 7^{\frac{\beta+1}{2}}-4.$$

Case 12. α (odd) $\geq 1, \beta$ (odd) $\geq 1, g=3$.

$$n^*=g=3,\quad K=\mathbb{Q}(\sqrt{-3}),\quad w(K)=6,$$
$$\left(\frac{-n^*}{2}\right)=\left(\frac{-3}{2}\right)=\left(\frac{5}{2}\right)=-1,\left(\frac{-n^*}{7}\right)=\left(\frac{-3}{7}\right)=\left(\frac{4}{7}\right)=1,\quad \frac{1}{3w(K)}=\frac{1}{18},$$
$$2^{[\frac{\alpha+1}{2}]}\left(1-\left(\frac{-n^*}{2}\right)\frac{1}{2}\right)d_2(n)=2^{\frac{\alpha+1}{2}}\cdot\frac{3}{2}\cdot\frac{1}{2^{\frac{\alpha-1}{2}}}=3,$$
$$7^{[\frac{\beta+3}{2}]}\left(1-\left(\frac{-n^*}{7}\right)\frac{1}{7}\right)d_7(n)=7^{\frac{\beta+3}{2}}\cdot\frac{6}{7}\cdot 2=12\cdot 7^{\frac{\beta+1}{2}},$$
$$c_1(n)=\frac{1}{18}\cdot 3\cdot 12\cdot 7^{\frac{\beta+1}{2}}=2\cdot 7^{\frac{\beta+1}{2}}.$$

Case 13. α (odd) $\geq 1, \beta$ (odd) $\geq 1, g\equiv 3 \pmod 8, g\equiv 3,5,6 \pmod 7, g\neq 3$.

$$n^*=g,\quad K=\mathbb{Q}(\sqrt{-g})\neq\mathbb{Q}(\sqrt{-1}),\mathbb{Q}(\sqrt{-3}),\quad w(K)=2,$$
$$\left(\frac{-n^*}{2}\right)=\left(\frac{-g}{2}\right)=\left(\frac{5}{2}\right)=-1,\left(\frac{-n^*}{7}\right)=\left(\frac{-g}{7}\right)=-\left(\frac{g}{7}\right)=1,\quad \frac{1}{3w(K)}=\frac{1}{6},$$
$$2^{[\frac{\alpha+1}{2}]}\left(1-\left(\frac{-n^*}{2}\right)\frac{1}{2}\right)d_2(n)=2^{\frac{\alpha+1}{2}}\cdot\frac{3}{2}\cdot\frac{1}{2^{\frac{\alpha-1}{2}}}=3,$$
$$7^{[\frac{\beta+3}{2}]}\left(1-\left(\frac{-n^*}{7}\right)\frac{1}{7}\right)d_7(n)=7^{\frac{\beta+3}{2}}\cdot\frac{6}{7}\cdot 2=12\cdot 7^{\frac{\beta+1}{2}},$$
$$c_1(n)=\frac{1}{6}\cdot 3\cdot 12\cdot 7^{\frac{\beta+1}{2}}=6\cdot 7^{\frac{\beta+1}{2}}.$$

Case 14. α (odd) $\geq 1, \beta$ (odd) $\geq 1, g\equiv 1 \pmod 4, g\equiv 3,5,6 \pmod 7$.

$$n^*=g,\quad K=\mathbb{Q}(\sqrt{-g})\neq\mathbb{Q}(\sqrt{-1}),\mathbb{Q}(\sqrt{-3}),\quad w(K)=2,$$

$$\left(\frac{-n^*}{7}\right) = \left(\frac{-g}{7}\right) = -\left(\frac{g}{7}\right) = 1, \quad \frac{1}{3w(K)} = \frac{1}{6},$$
$$2^{[\frac{\alpha-1}{2}]}d_2(n) = 2^{\frac{\alpha-1}{2}} \cdot \frac{3}{2^{\frac{\alpha+1}{2}}} = \frac{3}{2},$$
$$7^{[\frac{\beta+3}{2}]}\left(1-\left(\frac{-n^*}{7}\right)\frac{1}{7}\right)d_7(n) = 7^{\frac{\beta+3}{2}} \cdot \frac{6}{7} \cdot 2 = 12 \cdot 7^{\frac{\beta+1}{2}},$$
$$c_1(n) = \frac{1}{6} \cdot \frac{3}{2} \cdot 12 \cdot 7^{\frac{\beta+1}{2}} = 3 \cdot 7^{\frac{\beta+1}{2}}.$$

Case 15. α (odd) $\geq 1, \beta$ (odd) $\geq 1, g \equiv 3 \pmod 8), g \equiv 1, 2, 4 \pmod 7$.

$$n^* = g, \quad K = \mathbb{Q}(\sqrt{-g}) \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3}), \quad w(K) = 2,$$
$$\left(\frac{-n^*}{2}\right) = \left(\frac{-g}{2}\right) = \left(\frac{5}{2}\right) = -1, \left(\frac{-n^*}{7}\right) = \left(\frac{-g}{7}\right) = -\left(\frac{g}{7}\right) = -1, \quad \frac{1}{3w(K)} = \frac{1}{6},$$
$$2^{[\frac{\alpha+1}{2}]}\left(1-\left(\frac{-n^*}{2}\right)\frac{1}{2}\right)d_2(n) = 2^{\frac{\alpha+1}{2}} \cdot \frac{3}{2} \cdot \frac{1}{2^{\frac{\alpha-1}{2}}} = 3,$$
$$7^{[\frac{\beta+3}{2}]}\left(1-\left(\frac{-n^*}{7}\right)\frac{1}{7}\right)d_7(n) = 7^{\frac{\beta+3}{2}} \cdot \frac{8}{7}\left(2-\frac{2}{7^{\frac{\beta+1}{2}}}\right) = 16\left(7^{\frac{\beta+1}{2}}-1\right),$$
$$c_1(n) = \frac{1}{6} \cdot 3 \cdot 16\left(7^{\frac{\beta+1}{2}}-1\right) = 8 \cdot 7^{\frac{\beta+1}{2}} - 8.$$

Case 16. α (odd) $\geq 1, \beta$ (odd) $\geq 1, g \equiv 7 \pmod 8$.

In this case we have $d_2(n) = 0$ so $c_1(n) = 0$.

This completes the proof of Theorem 1.5. □

6. Integers Represented by k: Proof of Theorem 1.6

Let $n \in \mathbb{N}$. We have $l(n) > 0$ and $h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right) \geq 1$. Hence, by Theorem 1.5, we deduce

$$\begin{aligned} n \text{ is not represented by } k(x,y,z) = x^2+3y^2+5z^2+2yz \\ \iff & c_1(n) = 0 \\ \iff & \alpha \text{ (odd)} \geq 1, \beta \text{ (even)} \geq 0, g \equiv 1 \pmod 8 \text{ or} \\ & \alpha \text{ (odd)} \geq 1, \beta \text{ (odd)} \geq 1, g \equiv 7 \pmod 8 \\ \iff & n = 4^k(16l+2) \text{ for some } k, l \in \mathbb{N}_0. \end{aligned}$$

□

7. Integers $n \not\equiv 3 \pmod 4$ Represented by f: Proof of Theorem 1.9

Throughout this section n denotes a positive integer satisfying $n \not\equiv 3 \pmod 4$. As $l(n) > 0$ and $h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right) \geq 1$, by Theorem 1.8 the integer n is not represented by f if and only if $c_2(n) = 0$. By Table 1.2 these are precisely those $n = 2^\alpha 7^\beta g h^2$ satisfying

$$\begin{aligned}
(A) &\quad \alpha = 0, \beta \text{ (even)} \geq 0, g \equiv 1 \pmod 4 \text{ or}\\
(B) &\quad \alpha = 2, \beta \text{ (even)} \geq 0, g \equiv 1 \pmod 4 \text{ or}\\
(C) &\quad \alpha = 0, \beta \text{ (odd)} \geq 1, g \equiv 3 \pmod 4 \text{ or}\\
(D) &\quad \alpha = 2, \beta \text{ (odd)} \geq 1, g \equiv 3 \pmod 4 \text{ or}\\
(E) &\quad \alpha = 1, \beta \text{ (even)} \geq 0, g \equiv 1 \pmod 4 \text{ or}\\
(F) &\quad \alpha = 3, \beta \text{ (even)} \geq 0, g \equiv 1 \pmod 4 \text{ or}\\
(G) &\quad \alpha \text{ (odd)} \geq 5, \beta \text{ (even)} \geq 0, g \equiv 1 \pmod 8 \text{ or}\\
(H) &\quad \alpha = 1, \beta \text{ (odd)} \geq 1, g \equiv 3 \pmod 4 \text{ or}\\
(I) &\quad \alpha = 3, \beta \text{ (odd)} \geq 1, g \equiv 3 \pmod 4 \text{ or}\\
(J) &\quad \alpha \text{ (odd)} \geq 5, \beta \text{ (odd)} \geq 1, g \equiv 7 \pmod 8.
\end{aligned}$$

Now

$$\begin{aligned}
(A) &\Rightarrow n = 4l+1,\\
(B) &\Rightarrow n = 16l+4,\\
(C) &\Rightarrow n = 4l+1,\\
(D) &\Rightarrow n = 16l+4,\\
(E) &\Rightarrow n = 8m+2 \Rightarrow n = 4^0(16l+2) \text{ or } n = 16l+10,\\
(F) &\Rightarrow n = 32m+8 \Rightarrow n = 4^1(16l+2) \text{ or } n = 64l+40,\\
(G) &\Rightarrow n = 4^k(16l+2)\ (k \geq 2),\\
(H) &\Rightarrow n = 8m+2 \Rightarrow n = 4^0(16l+2) \text{ or } n = 16l+10,\\
(I) &\Rightarrow n = 32m+8 \Rightarrow n = 4^1(16l+2) \text{ or } n = 64l+40,\\
(J) &\Rightarrow n = 4^k(16l+2)\ (k \geq 2),
\end{aligned}$$

and

$$\begin{aligned}
n = 4l+1 &\Rightarrow (A) \text{ or } (C),\\
n = 16l+4 &\Rightarrow (B) \text{ or } (D),\\
n = 16l+10 &\Rightarrow (E)(\text{with } g \equiv 5 \pmod 8) \text{ or } (H)(\text{with } g \equiv 3 \pmod 8)),\\
n = 64l+40 &\Rightarrow (F)(\text{with } g \equiv 5 \pmod 8) \text{ or } (I)(\text{with } g \equiv 3 \pmod 8)),\\
n = 4^k(16l+2) &\Rightarrow (E)(\text{with } g \equiv 1 \pmod 8)) \text{ or } (F)(\text{with } g \equiv 1 \pmod 8)) \text{ or}
\end{aligned}$$

(G) or (J).

Thus

$$\text{(A) or (B) or } \dots \text{ or (J)} \iff n = 4l+1, 16l+4, 16l+10, 64l+40, 4^k(16l+2)$$

for some k, $l \in \mathbb{N}_0$. This completes the proof of Theorem 1.9. □

8. Integers Represented by the Genus $\{f, g\}$: Proof of Theorem 1.10

The positive integers n not represented by the genus $\{f, g\}$ are precisely those satisfying $r(f; n) = 0$ and $r(g; n) = 0$, that is those satisfying $r(f; n) + r(g; n) = 0$. By (1.1) of Theorem 1.3 these n are given by $u(n)r(k; n) = 0$, that is $u(n) = 0$ or $r(k; n) = 0$, that is (by (1.2) of Theorem 1.3 and Theorem 1.6)

$$n \equiv 1 \pmod 4,\ n \equiv 2 \pmod 8,\ n \equiv 4 \pmod{16},\ n \equiv 8 \pmod{32} \text{ or } n = 4^k(16l+2)$$

for any k, $l \in \mathbb{N}_0$. Now

$$n \equiv 2 \pmod 8 \iff n \equiv 2 \pmod{16} \text{ or } n \equiv 10 \pmod{16}$$

and

$$n \equiv 8 \pmod{32} \iff n \equiv 8 \pmod{64} \text{ or } n \equiv 40 \pmod{64},$$

so n is not represented by the genus $\{f, g\}$ if and only if

$$n \equiv 1 \pmod 4,\ n \equiv 4 \pmod{16},\ n \equiv 10 \pmod{16},\ n \equiv 40 \pmod{64} \text{ or}$$
$$n = 4^k(16l+2),$$

for some $k, l \in \mathbb{N}_0$. □

9. Evaluation of the Representation Numbers of f and g for $n \equiv 3 \pmod 4$: Proof of Theorem 1.13

We define the primitive, positive-definite, integral quaternary quadratic forms F and G by

$$F(x, y, z, t) := f + t^2 = 3x^2 + 6y^2 + 14z^2 + t^2 + 4yz + 2zx + 2xy,$$

$$G(x, y, z, t) := g + t^2 = 6x^2 + 7y^2 + 7z^2 + t^2 + 2yz + 2zx + 6xy.$$

The matrices of F and G are

$$M(F) = \begin{bmatrix} 6 & 2 & 2 & 0 \\ 2 & 12 & 4 & 0 \\ 2 & 4 & 28 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad M(G) = \begin{bmatrix} 12 & 6 & 2 & 0 \\ 6 & 14 & 2 & 0 \\ 2 & 2 & 14 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

The determinant of F is

$$\det M(F) = 2 \begin{vmatrix} 6 & 2 & 2 \\ 2 & 12 & 4 \\ 2 & 4 & 28 \end{vmatrix} = 2 \cdot 1792 = 2^9 \cdot 7$$

and the determinant of G is

$$\det M(G) = 2 \begin{vmatrix} 12 & 6 & 2 \\ 6 & 14 & 2 \\ 2 & 2 & 14 \end{vmatrix} = 2 \cdot 1792 = 2^9 \cdot 7.$$

The inverses of $M(F)$ and $M(G)$ are

$$M(F)^{-1} = \frac{1}{448} \begin{bmatrix} 80 & -12 & -4 & 0 \\ -12 & 41 & -5 & 0 \\ -4 & -5 & 17 & 0 \\ 0 & 0 & 0 & 224 \end{bmatrix}$$

and

$$M(G)^{-1} = \frac{1}{448} \begin{bmatrix} 48 & -20 & -4 & 0 \\ -20 & 41 & -3 & 0 \\ -4 & -3 & 33 & 0 \\ 0 & 0 & 0 & 224 \end{bmatrix}.$$

The smallest positive integer N such that $NM(F)^{-1}$ is an integral matrix with even diagonal entries is 896. The same is true for G. Thus the level of both F and G is 896; see [11, p. 363]. The character associated with both F and G is

$$\left(\frac{\det M(F)}{*}\right) = \left(\frac{\det M(G)}{*}\right) = \left(\frac{2^9 \cdot 7}{*}\right) = \chi_{56};$$

see [11, p. 363]. Hence, by [11, Theorem 10.1, p. 363] we have

$$\theta(F; w),\ \theta(G; w) \in M_2(\Gamma_0(896), \chi_{56}).$$

Thus

$$\theta(F; w) - \theta(G; w) \in M_2(\Gamma_0(896), \chi_{56}). \tag{9.1}$$

As a simple consequence of Jacobi's triple product identity (see [7, pp. 282–284]), we have

$$\theta(w) = \frac{\eta^5(2w)}{\eta^2(w)\eta^2(4w)}, \quad w \in \mathcal{H}.$$

Hence, for $w \in \mathcal{H}$, we have

$$\begin{aligned}\theta(w)\sum_{m=1}^{\infty}\nu(m)e^{2\pi iwm} &= \frac{\eta^5(2w)}{\eta^2(w)\eta^2(4w)}\frac{\eta^2(4w)\eta(16w)\eta(56w)}{\eta(8w)}\\ &= \eta^{-2}(w)\eta^5(2w)\eta^{-1}(8w)\eta(16w)\eta(56w)\\ &= \prod_{m\mid 896}\eta^{r_m}(mw),\end{aligned}$$

where

$$r_1 = -2, r_2 = 5, r_8 = -1, r_{16} = 1, r_{56} = 1,$$

and $r_m = 0$ for all the other divisors m of 896. Now

$$\begin{aligned}k &:= \sum_{m\mid 896}\frac{r_m}{2} = -\frac{2}{2}+\frac{5}{2}-\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 2,\\ &\sum_{m\mid 896} mr_m = 1(-2)+2(5)+8(-1)+16(1)+56(1) = 72 \equiv 0 \pmod{24},\\ &\sum_{m\mid 896}\frac{896}{m}r_m = 896(-2)+448(5)+112(-1)+56(1)+16(1) = 408 \equiv 0 \pmod{24},\\ &\sum_{m\mid 896}\frac{r_m}{24m} = \frac{408}{896\cdot 24} = \frac{17}{896},\end{aligned}$$

so, by [5, Prop. 5.9.2, p. 193], $\theta(w)\sum_{m=1}^{\infty}\nu(m)e^{2\pi iwm}$ is a modular function of weight $k = 2$ for $\Gamma_0(M)$, where $M = \operatorname{lcm}(1, 2, 8, 16, 56, 896) = 896$, with character

$$\left(\frac{(-1)^k\prod_{m\mid 896}m^{r_m}}{*}\right) = \left(\frac{1^{-2}2^5 8^{-1}16^1 56^1}{*}\right) = \left(\frac{2^9\cdot 7}{*}\right) = \left(\frac{2^3\cdot 7}{*}\right) = \chi_{56}.$$

Thus

$$\theta(w)\sum_{m=1}^{\infty}\nu(m)e^{2\pi iwm} \in M_2(\Gamma_0(896), \chi_{56}). \tag{9.2}$$

From (9.1) and (9.2), we deduce

$$\theta(F; w) - \theta(G; w) - 2\theta(w)\sum_{m=1}^{\infty}\nu(m)e^{2\pi iwm} \in M_2\left(\Gamma_0(896), \chi_{56}\right).$$

Numerically we found that the q-series expansion

$$\theta(F;w)-\theta(G;w)-2\theta(w)\sum_{m=1}^{\infty}\nu(m)e^{2\pi iwm}=\sum_{n=0}^{\infty}a(n)q^n \quad (q=e^{2\pi iw})$$

satisfies $a(n)=0$ for $0\le n\le 1+\left[\frac{mk}{12}\right]=257$, where $k=2$ and

$$m=[SL_2(\mathbb{Z}):\Gamma_0(896)]=896\prod_{p\mid 896}\left(1+\frac{1}{p}\right)=896\left(1+\frac{1}{2}\right)\left(1+\frac{1}{7}\right)=1536.$$

Hence, by the Sturm bound; see for example [5, Cor. 5.6.14, p. 185], we have

$$\theta(F;w)-\theta(G;w)-2\theta(w)\sum_{m=1}^{\infty}\nu(m)e^{2\pi iwm}=0$$

for all $w\in\mathcal{H}$. Now for $w\in\mathcal{H}$ we have

$$\theta(F;w)=\theta(w)\theta(f;w) \qquad \text{and} \qquad \theta(G;w)=\theta(w)\theta(g;w)$$

so

$$\theta(w)\left(\theta(f;w)-\theta(g;w)-2\sum_{m=1}^{\infty}\nu(m)e^{2\pi iwm}\right)=0.$$

But $\theta(w)\neq 0$ for $w\in\mathcal{H}$ thus

$$\theta(f;w)-\theta(g;w)-2\sum_{m=1}^{\infty}\nu(m)e^{2\pi iwm}=0.$$

Hence, we have

$$\sum_{n=0}^{\infty}r(f;n)q^n-\sum_{n=0}^{\infty}r(g;n)q^n-2\sum_{n=1}^{\infty}\nu(n)q^n=0.$$

Equating coefficients of q^n $(n\in\mathbb{N}_0)$, we obtain

$$r(f;n)-r(g;n)-2\nu(n)=0.$$

For $n\not\equiv 3 \pmod 4$ this statement is trivial, as in this case $r(f;n)=r(g;n)$ by Theorem 1.4 and $\nu(n)=0$ by (1.18). For our purposes we need

$$r(f;n)-r(g;n)=2\nu(n), \quad n\equiv 3 \pmod 4. \tag{9.3}$$

For $n\equiv 3 \pmod 4$, by Theorem 1.3 we have

$$r(f;n)+r(g;n)=r(k;n). \tag{9.4}$$

As

$$n \equiv 3 \pmod 4 \iff \alpha = 0,\ \beta \text{ even},\ g \equiv 3 \pmod 4 \text{ or}$$
$$\alpha = 0,\ \beta \text{ odd},\ g \equiv 1 \pmod 4,$$

by Theorem 1.5 we have

$$r(k;n) = 2c_3(n)l(n)h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right), \tag{9.5}$$

where

$$c_3(n) := \frac{1}{2}c_1(n) = \begin{cases} \frac{1}{12}\cdot 7^{(\beta+2)/2} - \frac{1}{3} & \text{if } \alpha = 0,\ \beta \text{ even},\ g \equiv 3 \pmod 4, \\ \frac{2}{3}\cdot 7^{(\beta+1)/2} - \frac{2}{3} & \text{if } \alpha = 0,\ \beta \text{ odd},\ g \equiv 1 \pmod 4,\ g \equiv 1,2,4 \pmod 7, \\ \frac{1}{2}\cdot 7^{(\beta+1)/2} & \text{if } \alpha = 0,\ \beta \text{ odd},\ g \equiv 1 \pmod 4,\ g \equiv 3,5,6 \pmod 7. \end{cases} \tag{9.6}$$

Hence, by (9.4) and (9.5), we obtain

$$r(f;n) + r(g;n) = 2c_3(n)l(n)h\left(\mathbb{Q}\left(\sqrt{-n^*}\right)\right),\ n \equiv 3 \pmod 4, \tag{9.7}$$

where $c_3(n)$ is given in (9.6). Adding and subtracting (9.3) and (9.7), we obtain the formulas of Theorem 1.13 for $r(f;n)$ and $r(g;n)$. □

10. Concluding Remarks

In this paper we studied the arithmetic properties of the ternary quadratic form $f := 3x^2 + 6y^2 + 14z^2 + 4yz + 2zx + 2xy$. The ternary quadratic form f is of special interest as it is the form of least discriminant which is conjectured to be regular, but which has not been proven to be regular. We have given explicit formulas for the representation numbers of both f and its genus mate g; see Theorem 1.8 for $n \not\equiv 3 \pmod 4$ and Theorem 1.13 for $n \equiv 3 \pmod 4$. We determined the positive integers n represented by the genus $\{f, g\}$ in Theorem 1.10 and the integers $n \not\equiv 3 \pmod 4$ represented by the form f in Theorem 1.9. This enabled us to give a simple, necessary and sufficient condition for f to be regular in Theorem 1.11 and to identify that proving that f represents every positive integer $n \equiv 3 \pmod 4$ is the local impediment to proving that f is regular. We hope that in the future it will be proved that f represents every $n \equiv 3 \pmod 4$ thus establishing the regularity of f.

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